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Introduction en français

1 Combinatoire énumérative

La combinatoire énumérative [107] est un domaine des mathématiques qui cherche principalement à déterminer des méthodes efficaces pour compter les éléments d'ensembles finis. Un problème typique est de prendre une collection d'ensembles finis $(\mathcal{C}_n)_{n\geq 0}$ appelée une *classe combinatoire*, et de chercher sa *suite de comptage* $(c_n)_{n\geq 0}$, où c_n désigne le nombre d'éléments de $\mathcal{C}_n$, l'ensemble des objets de *taille* n. Les solutions d'un tel problème de comptage (aussi appelé problème de dénombrement, ou problème d'énumération) peuvent prendre des formes diverses. Nous présentons ci-dessous les plus courantes; pour cet exemple, $\mathcal{C}_n$ est constitué des ensembles de trois nombres à valeurs dans $\{1, 2, \ldots, n\}$.

- La forme la plus élégante qu'une solution peut prendre est une *formule close*, qui se définit généralement comme une formule explicite faisant intervenir un nombre borné de termes et de fonctions connues.

  Pour notre exemple, on peut choisir un nombre dans $\{1, \ldots, n\}$ de n manières possibles; il reste alors $n-1$ choix pour le second nombre, et $n-2$ pour le troisième. On peut remarquer que chaque sous-ensemble de trois nombres peut être obtenu par ce processus de 6 manières possibles, selon l'ordre dans lequel ces nombres sont choisis (puisque les ensembles $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, et $\{3,2,1\}$ sont tous identiques), et l'on obtient la formule close suivante.

  $$c_n = \frac{n(n-1)(n-2)}{6}$$

- Certaines formules explicites peuvent contenir un nombre arbitrairement grand de termes, qui grandit avec le nombre n. Ces termes sont le plus souvent liés par une somme ou un produit.

  Dans notre exemple, il est possible d'éviter le problème de l'ordre de parcours des nombres en les choisissant dans l'ordre décroissant. Le plus grand nombre i doit valoir au moins 3, le deuxième plus grand j doit être compris entre 2 et $i-1$, et le plus petit nombre k peut prendre n'importe quelle valeur entre 1 et $j-1$, donc il y a $j-1$ choix possibles pour k. Ce processus correspond à la formule explicite ci-dessous.

  $$c_n = \sum_{i=3}^{n} \sum_{j=2}^{i-1} (j-1)$$

- Une suite de comptage peut également être décrite à l'aide d'une relation de récurrence, qui nécessite de connaître les premiers termes de la suite pour calculer le suivant.

Dans notre exemple, on peut compter les éléments de $\mathcal{C}_{n+1}$ en distinguant deux cas. Les éléments de $\mathcal{C}_{n+1}$ qui ne contiennent pas $n+1$ forment l'ensemble $\mathcal{C}_n$. Ceux qui contiennent $n+1$ sont en bijection avec les ensembles de deux nombres à valeurs dans $\{1, \ldots, n\}$, qui sont comptés par $\frac{n^2-n}{2}$ (cela se montre de la même manière que la formule close pour c_n énoncée précédemment). Puisque $\mathcal{C}_0$ est vide, on peut calculer les nombres c_n à partir des équations ci-dessous.

$$c_0 = 0, \quad c_{n+1} = c_n + \frac{n^2 - n}{2}$$

Dans des problèmes de dénombrement plus difficiles, il arrive souvent que la connaissance des premiers termes d'une suite de comptage ne suffise pas à calculer le terme suivant à l'aide d'une telle formule. Néanmoins, connaître davantage d'information sur les objets combinatoires étudiés permet parfois d'obtenir une relation de récurrence à plusieurs variables, où les nouvelles variables indiquent certains paramètres des objets en plus de leur taille. Il s'agit d'une technique que nous utiliserons souvent dans cette thèse.

- Il est également possible d'exprimer une suite de comptage $(c_n)_{n\geq 0}$ à travers sa *fonction génératrice*, qui est la série formelle dont les coefficients sont les nombres c_n.

$$C(x) = \sum_{n\geq 0} c_n x^n$$

Il existe deux manières communes de décrire une fonction génératrice. La première est de l'écrire sous une forme explicite. Pour notre exemple, la méthode symbolique [53] permet d'obtenir la formule ci-dessous.

$$C(x) = \frac{x^3}{(1-x)^4}$$

La seconde manière est de définir la fonction implicitement à partir d'équations qu'elle vérifie. Une telle équation peut prendre de nombreuses formes selon la nature d'une fonction génératrice $F(x)$. Parmi les exemples communs, on trouve les fonctions *algébriques* [79, 43], qui vérifient une équation

$$P(x, F(x)) = 0$$

pour un polynôme (non nul) à deux variables $P(x, y)$, et les fonctions *holonomes* (ou *D-finies*) [106, 62], qui satisfont

$$P_0(x)F(x) + P_1(x)F'(x) + \cdots + P_k(x)F^{(k)}(x) = 0$$

pour certains polynômes (pas tous nuls) $P_0(x), \ldots, P_k(x)$.

Tout comme les relations de récurrence, les fonctions génératrices peuvent aussi prendre plusieurs variables pour encapsuler davantage d'information.

Les exemples précédents montrent de nombreuses formules pour compter les mêmes objets. Cela était possible car nous avions choisi un problème très simple; dans des problèmes de comptage plus difficiles, les solutions que l'on peut trouver sont beaucoup plus restreintes. Dans cette thèse, nous utiliserons principalement des formules de récurrence à plusieurs variables, ou bien des équations fonctionnelles qui sont leur homologue pour les fonctions génératrices. Dans quelques rares cas, nous pouvons les résoudre pour obtenir des formules explicites.

Un outil essentiel pour la recherche en combinatoire énumérative est l'encyclopédie en ligne des suites de nombres entiers (OEIS) [96]. Cette encyclopédie contient un très grand nombre

de suites (près de 400 000 en avril 2026), et conduit souvent à la découverte de liens inattendus entre des classes combinatoires qui partagent la même suite de comptage.

Enfin, il est important de définir précisément en quoi consiste une solution à un problème de comptage. La réponse peut changer en fonction du contexte; dans cette thèse nous suivons la définition de Wilf [112] selon laquelle le dénombrement d'une classe combinatoire est résolu lorsqu'un algorithme permet de calculer sa suite de comptage en *temps polynomial*; autrement dit, le nombre d'opérations requises pour calculer chaque nombre c_n (en binaire, ou en base dix) est majoré par l'évaluation d'un certain polynôme en n. Remarquons qu'une telle formule n'est pas toujours optimale, au sens où une meilleure formule nécessitant moins de calculs (avec une majoration par un polynôme de plus petit degré, par exemple) peut exister. Nous considérons néanmoins qu'une telle formule est une solution suffisamment bonne. En effet, même si une approche générale permet de calculer systématiquement le type de suite de comptage que nous étudions en temps exponentiel ou pire, trouver un algorithme en temps polynomial est une tâche plus difficile (l'existence d'un tel algorithme est souvent une question ouverte).

2 Motifs de permutations

Dans le contexte des motifs de permutations [25, 64, 109], une permutation se définit naturellement comme une suite de nombres entiers: étant donné un entier naturel n, une *permutation* de longueur n est une suite contenant une occurrence de chaque entier entre 1 et n, dans n'importe quel ordre. On note $\mathcal{S}_n$ l'ensemble des permutations de longueur n, qui sont au nombre de $n! = 1 \times 2 \times \cdots \times n$. Plus généralement, en mathématiques, le terme permutation est utilisé pour désigner une bijection d'un ensemble vers lui-même. De ce point de vue, les éléments de $\mathcal{S}_n$ sont les permutations de l'ensemble $\{1, \ldots, n\}$, que l'on écrit comme la suite σ des images de chaque entier de 1 à n, autrement dit l'image de i est le i-ème terme de σ, noté σ_i.

Une permutation de longueur n peut être représentée graphiquement par un diagramme en grille de dimensions $n \times n$ qui contient un point dans chaque ligne et dans chaque colonne. Un point aux coordonnées (i, j) indique que le i-ème terme de la permutation a pour valeur j, voir Figure 1. En dehors de la communauté étudiant les motifs de permutations, cette représentation est plus connue comme une *matrice de permutation*, c'est-à-dire une matrice $n \times n$ qui contient uniquement les valeurs 0 et 1, avec un unique 1 par ligne et par colonne.

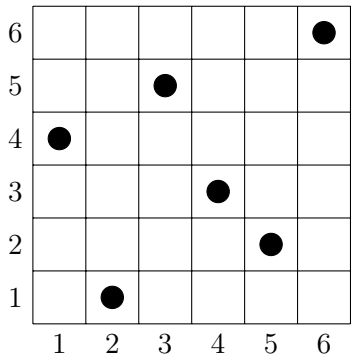

Figure 1: Le diagramme de la permutation 415326.

Étant donné deux permutations σ et π, on dit que σ *contient* le *motif* (classique) π si σ admet une sous-suite dont les termes ont le même ordre relatif que π. De manière équivalente, σ contient le motif π si le diagramme de π peut être obtenu en supprimant des points du diagramme

de σ, ainsi que leur ligne et leur colonne. Si σ ne contient pas π, on dit que σ *évite* le motif π. Par exemple, la permutation 415326 de la Figure 1 contient quatre occurrences du motif 123, et elle évite le motif 1423. On dit que deux motifs sont *Wilf-équivalents* s'il existe le même nombre de permutations de longueur n évitant l'un ou l'autre motif, pour n'importe quel entier naturel n.

Les plus anciens résultats sur les motifs classiques de permutations datent du travail de MacMahon, qui a été le premier à montrer que les permutations évitant le motif 123 sont comptées par les nombres de Catalan [81], et qui a également étudié le nombre d'*inversions* (les occurrences du motif 21) dans les permutations [82]. L'introduction de la notion de motif classique est généralement attribuée à Knuth [66]. Knuth a montré que les permutations triables par une pile sont celles qui évitent le motif 231, et qu'elles sont aussi comptées par les nombres de Catalan. Puisque les permutations évitant n'importe quel motif parmi $\{132, 213, 231, 312\}$ sont trivialement en bijection (par une réflexion de leur diagramme par rapport à un axe vertical ou horizontal), ainsi que celles évitant 123 ou 321, cela implique que toutes les six permutations de longueur 3 sont Wilf-équivalentes. Knuth a également décrit la première de nombreuses bijections entre les permutations évitant 321 et celles évitant 132. La première étude systématique des motifs de permutations est le travail fondateur de Simion et Schmidt [105], qui ont dénombré les permutations évitant n'importe quel ensemble de motifs de longueur 3. Les motifs de permutations sont encore un domaine de recherche très actif aujourd'hui, et une conférence dédiée à ce sujet a lieu chaque année depuis 2003.

3 Séquences d'inversions

La *séquence d'inversions* (ou table d'inversions) d'une permutation σ de longueur n est une suite d'entiers naturels τ de longueur n telle que $\tau_i = |\{j \ : \ \sigma_j > \sigma_i \text{ et } j < i\}|$ compte le nombre d'inversions de σ dont le second élément est σ_i, pour tout $i \in \{1, \ldots, n\}$. La fonction qui associe à une permutation sa séquence d'inversions est appelée le *code de Lehmer* en hommage à D.H. Lehmer [72], bien qu'elle soit apparue précédemment dans un article de Laisant [71] dès le 19e siècle. Le code de Lehmer est une bijection entre l'ensemble $\mathcal{S}_n$ des permutations de longueur n, et l'ensemble $\mathcal{I}_n$ des séquences d'inversions de longueur n, c'est-à-dire les suites d'entiers τ telles que $0 \leq \tau_i < i$ pour tout $i \in \{1, \ldots, n\}$.

Les séquences d'inversions peuvent se représenter par des points dans un diagramme en escalier de la même manière que les permutations; ce diagramme contient toujours un unique point par colonne, mais les lignes peuvent avoir un nombre de points variable, voir Figure 2.

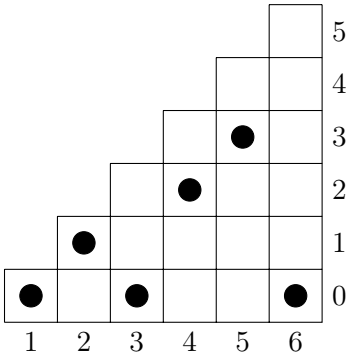

Figure 2: Le diagramme de la séquence d'inversions 010230.

La notion de motif définie sur les permutations s'étend aux séquences d'inversions, mais ces motifs ne sont pas nécessairement des permutations, ni des séquences d'inversions. Pour que chaque motif soit défini de manière unique, on considère maintenant qu'un motif est une suite non vide d'entiers naturels qui contient chaque entier entre 0 et son maximum. Par exemple 0212 est un motif, mais 4520 n'en est pas un puisque les nombres 1 et 3 sont manquants. Étant donné un motif ρ, on note $\mathcal{I}(\rho)$ l'ensemble des séquences d'inversions qui évitent ρ, et $\mathcal{I}_n(\rho)$ le sous-ensemble des séquences de longueur n.

Dans cette thèse, les séquences d'inversions sont l'objet principal de notre étude. Leur lien avec les permutations n'est pas exploité, car l'évitement de motif dans les séquences d'inversions se traduit rarement en des contraintes simples à décrire sur les permutations.

Contrairement aux permutations, l'étude des motifs dans les séquences d'inversions a commencé il y a seulement une dizaine d'années, dans deux articles indépendants [85, 46] qui dénombrent des séquences d'inversions évitant un motif de longueur 3. Ces travaux fondateurs ont montré des connexions avec de nombreuses suites de nombres bien connues, telles que les puissances de 2, les nombres de Fibonacci, les nombres de Schröder, les nombres zigzag d'Euler, ou encore les nombres de Bell. La Table 1 présente une liste complète (qui comprend des résultats plus récents) des suites de comptage des séquences d'inversions évitant un motif de longueur 3.

Motif ρ	$\lvert\mathcal{I}_n(\rho)\rvert$ pour $n = 1, \ldots, 7$	Dénombré dans	OEIS [96]
000	1, 2, 5, 16, 61, 272, 1385	[46]	A000111
001	1, 2, 4, 8, 16, 32, 64	[46]	A000079
010	1, 2, 5, 15, 53, 215, 979	[A]	A263779
011	1, 2, 5, 15, 52, 203, 877	[46]	A000110
012	1, 2, 5, 13, 34, 89, 233	[46] et [85]	A001519
021	1, 2, 6, 22, 90, 394, 1806	[46] et [85]	A006318
100	1, 2, 6, 23, 106, 565, 3399	[69]	A263780
101 ou 110	1, 2, 6, 23, 105, 549, 3207	[46]	A113227
102	1, 2, 6, 22, 89, 381, 1694	[85]	A200753
120	1, 2, 6, 23, 103, 515, 2803	[85]	A263778
201 ou 210	1, 2, 6, 24, 118, 674, 4306	[85]	A263777

Table 1: Suites de comptage des séquences d'inversions évitant un seul motif de longueur 3.

Ces résultats ont éveillé de l'intérêt pour l'étude de différents types de motifs dans les séquences d'inversions, et de nombreux articles ont suivi au cours des dix dernières années, à propos de

- motifs classiques [26, 73, 113, 55, 41, 31, 32, 88, 87, 69, 39, 99, 89, 57, 91, 5, 58, 63],
- motifs de relations, dont l'évitement généralise celui d'un seul motif classique, mais correspond à l'évitement d'un ensemble de motifs classiques [93, 19, 35, 74, 115, 3, 38, 42, 27],
- motifs consécutifs, qui sont définis de même manière que les motifs classiques, avec la contrainte supplémentaire que les occurrences doivent utiliser des termes consécutifs [7, 8, 50, 52],

- les motifs "vincular", dans lesquels seulement certains termes doivent être consécutifs [75, 9, 77, 21].

L'évitement de motifs a aussi été étudié récemment dans plusieurs sous-ensembles remarquables des séquences d'inversions, comme les mots de Catalan [13, 14, 15, 16, 1], les fonctions à croissance restreinte [59, 103, 60, 76, 34, 102, 18], les "ascent sequences" [47, 40, 114, 84, 17, 74, 77, 45, 48, 33, 90, 80], et certaines variantes de ces dernières [65, 36, 22, 78, 116, 37].

Yan et Lin [113] ont mené une étude systématique des séquences d'inversions qui évitent une paire de motifs de longueur 3, laissant ouvert leur dénombrement pour 32 des 78 paires possibles. Huit de ces cas ont été résolus dans [69], et un autre dans [39] et [99] indépendamment. Dans la plupart des cas, ces problèmes ont été résolus à l'aide de bijections avec d'autres objets combinatoires connus, ou avec des arbres de génération (une méthode que nous utiliserons également). Nous complétons cette tâche en apportant des formules pour les 23 cas restants dans [A, B].

4 Sommaire

Dans cette thèse, nous étudions l'évitement de motifs dans les séquences d'inversions, et principalement leur dénombrement. Les deux premiers chapitres présentent des constructions pour des séquences d'inversions évitant certains ensembles de motifs fixés. Plus précisément, ils complètent le comptage des séquences d'inversions qui évitent un ou deux motifs de longueur 3. Ces résultats sont résumés dans la Table 2. Le troisième chapitre résout de manière systématique le dénombrement des séquences d'inversions qui évitent 021 et n'importe quel ensemble fini de motifs. Le quatrième introduit la notion de séquence d'inversions *minimale* contenant un motif, et apporte quelques résultats sur leur structure et sur leur comptage.

Le Chapitre 1 correspond aux Sections 4 et 5 de [A]. Les méthodes employées dans ce chapitre comptent les séquences d'inversions à l'aide d'un découpage en deux facteurs.

- La Section 1.1 utilise un découpage des séquences d'inversions autour de leur premier maximum, qui a d'abord été utilisé dans [85], et qui engendre des relations de récurrence pour compter les séquences d'inversions. Dans ce cas, une séquence d'inversions est vue comme la concaténation d'une séquence d'inversions plus petite et d'un mot. Puisque la plupart des familles de mots qui apparaissent dans la Section 1.1 n'ont pas été dénombrées précédemment, nous donnons aussi des formules pour compter des mots évitant des motifs à l'aide du même découpage au premier maximum. La première partie de la Section 1.1 s'intéresse à l'évitement du motif 120, et les ensembles de motifs étudiés sont $\{120, \tau\}$ pour $\tau \in \{000, 010, 011, 100, 101, 110, 201\}$. La seconde partie s'intéresse à l'évitement de 010, et plus précisément aux ensembles de motifs $\{010\}$, et $\{010, \tau\}$ pour $\tau \in \{000, 201, 210, 110\}$.

- La Section 1.2 introduit les séquences d'inversions *décalées*, et les compte à l'aide d'un découpage autour de leur premier minimum, similaire à celui de la Section 1.1. Il est parfois plus simple de construire des séquences d'inversions évitant des motifs en les voyant comme un cas particulier des séquences d'inversions décalées. Nous appliquons cette approche aux séquences d'inversions évitant les paires de motifs $\{010, 102\}$ et $\{100, 102\}$, et nous obtenons des formules de récurrence pour les compter.

Dans le Chapitre 1, les formules de comptage prennent toujours la forme de relations de récurrence à plusieurs variables, qui contiennent souvent plusieurs sommes imbriquées. De telles formules sont suffisamment efficaces pour calculer des centaines de termes des suites de comptage (voir

Ensemble de motifs P	$\lvert\mathcal{I}_n(P)\rvert$ pour $n = 1, \ldots, 7$	Résolu dans	Performances	OEIS [96]
{011, 120}	1, 2, 5, 14, 42, 132, 431	Section 1.1.3	> 400	A374548
{100, 120}	1, 2, 6, 22, 92, 421, 2062	Section 1.1.4	> 300	A374549
{120, 201}	1, 2, 6, 23, 102, 498, 2607	Section 1.1.5	> 300	A374550
{110, 120}	1, 2, 6, 22, 92, 423, 2091	Section 1.1.6	> 300	A279570
{010, 120}	1, 2, 5, 15, 52, 201, 845	Section 1.1.7	> 300	A279559
{101, 120}	1, 2, 6, 22, 90, 397, 1859	Section 1.1.8	> 200	A374551
{000, 120}	1, 2, 5, 15, 50, 185, 737	Section 1.1.9	> 300	A374552
{010}	1, 2, 5, 15, 53, 215, 979	Section 1.1.10	> 180	A263779
{000, 010}	1, 2, 4, 10, 29, 95, 345	Section 1.1.11	> 200	A279552
{010, 201} ou {010, 210}	1, 2, 5, 15, 53, 214, 958	Section 1.1.13	> 180	A360052
{010, 110}	1, 2, 5, 15, 52, 201, 847	Section 1.1.14	> 140	A359191
{010, 102}	1, 2, 5, 15, 51, 186, 707	Section 1.2.2	> 200	A374553
{100, 102}	1, 2, 6, 21, 80, 318, 1305	Section 1.2.3	> 300	A374554
{000, 100}	1, 2, 5, 16, 60, 260, 1267	Section 2.2.2	> 900	A279564
{102, 201}	1, 2, 6, 22, 87, 354, 1465	Section 2.2.3	$> 10\,000$	A279566
{000, 102}	1, 2, 5, 14, 40, 121, 373	Section 2.3.2	$> 10\,000$	A374541
{102, 210}	1, 2, 6, 22, 87, 351, 1416	Section 2.3.3	$> 10\,000$	A374542
{000, 201} ou {000, 210}	1, 2, 5, 16, 60, 257, 1218	Section 2.3.5	> 700	A374543
{100, 110}	1, 2, 6, 22, 93, 437, 2233	Section 2.3.6	> 800	A374544
{100, 101}	1, 2, 6, 22, 93, 439, 2267	Section 2.3.7	> 800	A374545
{110, 201}	1, 2, 6, 23, 103, 512, 2739	Section 2.3.8	> 800	A374546
{101, 210}	1, 2, 6, 23, 103, 513, 2763	Section 2.3.9	> 800	A374547
{201, 210}	1, 2, 6, 24, 116, 632, 3720	[39], [99], Section 2.4.2	$> 10\,000$	A212198
{010, 102} (bis)	1, 2, 5, 15, 51, 186, 707	Section 2.4.3	$> 10\,000$	A374553

Table 2: Suites de comptage des séquences d'inversions évitant des motifs étudiées aux Chapitres 1 et 2. Chaque section de cette table correspond à une construction différente. La colonne "performances" indique le nombre de termes de chaque suite que nous avons pu calculer en une minute, avec Maple ou des programmes C++, sur un ordinateur personnel (une approche naïve permet de calculer environ 20 termes au plus).

Table 2), mais elles n'aboutissent pas à des équations simples pour les fonctions génératrices. Les fonctions génératrices des classes de séquences d'inversions étudiées au Chapitre 1 sont toutes conjecturées non holonomes, sauf dans le cas $\{010, 102\}$ dont nous prouvons l'algébricité au Chapitre 2.

Le Chapitre 2 correspond à [A, Sections 2, 3, 6] et [B], et s'intéresse aux constructions par arbre de génération. Un *arbre de génération* [110, 10] est un arbre enraciné étiqueté tel que l'étiquette de chaque nœud détermine ses enfants. Les arbres de génération sont utilisés pour dénombrer des objets combinatoires en assignant un nœud à chaque objet d'une classe combinatoire, de manière à ce que chaque objet de taille n corresponde à un nœud au niveau n. Chaque nœud est étiqueté non pas par l'objet lui-même mais par une étiquette plus simple qui représente son "état" dans une construction récursive de la classe combinatoire. Ces étiquettes sont généralement des nombres entiers, ou des uplets de nombres entiers. L'évolution des étiquettes dans un arbre de génération est décrite par une *règle de succession*, qui indique l'étiquette de la racine et

définit la manière dont l'étiquette de chaque nœud détermine celles de ses enfants. Les règles de succession correspondent à des formules de récurrence qui peuvent être utilisées pour compter des objets combinatoires. Une introduction plus détaillée aux arbres de génération est présentée en Section 2.1.

Le Chapitre 2 utilise trois différentes constructions par arbres de génération pour les séquences d'inversions évitant des motifs.

- La Section 2.2 est l'approche la plus évidente, celle qui construit les suites de gauche à droite : les enfants d'une séquence d'inversions de longueur n sont obtenus en ajoutant une valeur entre 0 et n à la fin de la séquence. Puisque cette construction est très naturelle, elle a déjà été beaucoup utilisée (en particulier avec l'approche algorithmique de [69]). Par conséquent, il ne restait qu'un seul cas ouvert dans lequel l'application de cette méthode "telle quelle" engendre une règle de succession simple, à savoir $\mathcal{I}(000, 100)$. Cependant, enrichir cette construction avec des *engagements* [99] ("commitments" en anglais) conduit à une bonne règle de succession pour $\mathcal{I}(102, 201)$, à partir de laquelle nous calculons sa fonction génératrice algébrique.

- La Section 2.3 utilise une construction de "bas en haut" : les termes d'une séquence d'inversions sont insérés par *valeurs* croissantes. Au sens strict, il ne s'agit pas d'une seule construction, mais plutôt d'une famille de constructions qui dépendent de l'ordre d'insertion des termes de même valeur (nous utiliserons des ordres différents selon les motifs à éviter). Avec cette approche, nous obtenons des règles de successions pour $\mathcal{I}(000, 102)$ et $\mathcal{I}(102, 210)$. Dans le cas $\mathcal{I}(000, 102)$, nous donnons une règle de succession avec des *sauts* [51], qui décrit un arbre de génération dans lequel des objets de taille n peuvent avoir pour enfants des objets de taille $n + 1$ ou $n + 2$. Pour $\mathcal{I}(102, 210)$, nous nous tenons à la définition "standard" d'un arbre de génération, mais notre construction insère les termes dans un ordre inhabituel afin d'aboutir à une règle de succession efficace. À partir des ces deux règles de succession, nous calculons les fonctions génératrices de $\mathcal{I}(000, 102)$ et $\mathcal{I}(102, 210)$, qui sont toutes deux algébriques.

  Nous utilisons ensuite une décomposition des séquences d'inversions en retirant leurs maximums. Cette décomposition est très similaire à l'approche par arbre de génération qui précède, mais prend un point de vue différent pour simplifier la présentation; ce lien est discuté en Section 2.3.4. Nous appliquons cette décomposition pour compter les séquences d'inversions qui évitent les motifs $\{000, 201\}$, $\{000, 210\}$, $\{100, 110\}$, $\{100, 101\}$, $\{110, 201\}$, et $\{101, 210\}$, obtenant ainsi des formules de récurrence avec des sommes imbriquées.

- La Section 2.4 emploie une construction par arbre de génération "grandissant par la gauche" des séquences d'inversions. Cette construction est apparue dans [46] pour dénombrer les classes $\mathcal{I}(101)$ et $\mathcal{I}(110)$. Contrairement aux deux constructions précédentes, celle-ci n'insère pas seulement un nouveau terme à chaque étape, mais elle modifie également des termes existants. Avec cette approche, nous donnons des constructions détaillées des séquences d'inversions évitant les paires de motifs $\{201, 210\}$ et $\{010, 102\}$.

  La classe $\mathcal{I}(201, 210)$ a une fonction génératrice algébrique, qui avait déjà été trouvée dans [39] et [99] indépendamment. Notre arbre de génération grandissant sur la gauche conduit à une nouvelle règle de succession pour $\mathcal{I}(201, 210)$, qui rend le calcul de sa fonction génératrice très simple.

  La seconde classe que nous examinons est $\mathcal{I}(010, 102)$. Après que nous avions trouvé une formule de récurrence pour celle-ci avec la décomposition du Chapitre 1 (voir Section 1.2.2),

Pantone a examiné sa suite de comptage avec son logiciel [98] et a conjecturé [99] que sa fonction génératrice est algébrique. Ni notre décomposition précédente, ni une autre par Pantone ne pouvait montrer cette conjecture, mais nous la démontrons enfin à l'aide d'un arbre de génération grandissant sur la gauche modifié, en Section 2.4.3.

Ensuite, nous explorons en Section 2.4.4 à quel point l'arbre de génération grandissant sur la gauche peut s'appliquer à la résolution d'autres problèmes similaires. Pour illustrer la robustesse de cette approche, nous présentons des règles de succession pour les séquences d'inversions évitant n'importe quel motif de longueur 3, dont certaines nécessitent d'effectuer des modifications à l'arbre de génération grandissant sur la gauche standard.

Notre construction de $\mathcal{I}(102, 201)$ et notre approche en Section 2.4 ont toutes deux été réutilisées avec succès dans [27] pour dénombrer des séquences d'inversions évitant des motifs de relations.

Nous concluons les Chapitres 1 et 2 avec une brève discussion sur le comportement asymptotique des suites de comptage des séquences d'inversions évitant des motifs. En outre, nous donnons des conditions suffisantes sur un ensemble de motifs P pour que la croissance de la suite de comptage de $\mathcal{I}(P)$ soit majorée par une fonction exponentielle, ou au contraire qu'elle soit sur-exponentielle.

Le Chapitre 3 étudie les séquences d'inversions évitant 021 et d'autres motifs. En utilisant un algorithme développé dans [69], Mansour et Yıldırım [87, 86] ont calculé les fonctions génératrices des séquences d'inversions qui évitent 021 et n'importe quel motif de longueur 4 ou 5. Leurs résultats donnent lieu à plusieurs observations intéressantes. En particulier, ces fonctions génératrices sont toutes algébriques, ce qui conduit naturellement à conjecturer que $\mathcal{I}(021, \rho)$ a une fonction génératrice algébrique pour n'importe quel motif ρ. Dans le Chapitre 3, nous montrons un résultat plus général : les séquences d'inversions qui évitent n'importe quel ensemble fini de motifs P qui contient 021 ont toujours une fonction génératrice algébrique. Notre résultat reste vrai sur les "ascent sequences" [29] et sur les fonctions à croissance restreinte [94]; pour ces dernières, les fonctions génératrices sont même *rationnelles*. Notre approche utilise un encodage des séquences d'inversions évitant 021 sous forme de chemins de Dyck, qui est déjà apparu dans [73]. Cette famille de chemins de Dyck peut également être définie comme un langage sur un alphabet à trois lettres, reconnu par un automate à compteur. À travers cet encodage, les motifs des séquences d'inversions évitant 021 deviennent un certain type de motifs sur les motifs, dont l'évitement peut être décidé par un automate fini. Par conséquent, les séquences d'inversions évitant P (un ensemble fini de motifs qui contient 021) sont en bijection avec un langage reconnu par un automate à compteur (déterministe), et elles ont donc une fonction génératrice algébrique d'après le théorème d'énumération de Chomsky–Schützenberger [43]. Nous montrons également que les fonctions génératrices sont rationnelles dans le cas des séquences d'inversions (ou des "ascent sequences") qui évitent un ensemble fini de motifs contenant à la fois 021 et un motif strictement croissant. Ce résultat s'obtient de manière analogue, en trouvant une bijection qui préserve la longueur entre ces suites et des langages réguliers. Pour décrire une telle bijection, nous utilisons un encodage différent qui utilise des engagements [99].

Le Chapitre 4 introduit la notion de séquence d'inversions ρ-minimale pour un motif ρ. Pour faire simple, les séquences d'inversions ρ-minimales sont les plus petites (au sens de l'inclusion de motif) séquences d'inversions contenant le motif ρ. En particulier, une séquence d'inversions évite ρ si et seulement si elle évite toutes les séquences d'inversions ρ-minimales.

- Dans la Section 4.3, nous nous intéressons à des propriétés structurelles des séquences d'inversions minimales, et nous trouvons deux résultats notables. Le premier affirme qu'une

séquence d'inversions est ρ-minimale si et seulement si elle contient une occurrence de ρ satisfaisant certaines conditions qui sont faciles à vérifier. Le second est une borne supérieure sur la longueur des séquences d'inversions ρ-minimales qui dépend de certaines statistiques de ρ, et qui implique en particulier que le nombre de séquences d'inversions ρ-minimales est toujours fini. Plus spécifiquement, une borne moins précise garantit qu'un motif de longueur k ne possède que des séquences d'inversions minimales de longueur inférieure à $3k$.

- Dans la Section 4.4, nous montrons que la borne supérieure de la section précédente est atteinte pour la plupart des motifs. Nous montrons également que même pour les motifs restants, il existe des séquences d'inversions minimales dont la longueur est très proche de cette borne.

- Dans la Section 4.5, nous décrivons précisément et comptons les séquences d'inversions ρ-minimales pour n'importe quel motif ρ qui commence par une valeur suffisamment grande, à l'aide d'une bijection entre les séquences d'inversions et les arbres croissants. Nous donnons également une manière de générer exhaustivement les séquences d'inversions minimales pour n'importe quel motif, et présentons des données sur les séquences d'inversions minimales pour des petits motifs obtenues par cette méthode.

- Dans la Section 4.6, nous dénombrons l'intersection entre les séquences d'inversions et les motifs, car cette classe a un rôle important dans notre étude des séquences d'inversions minimales. Nous trouvons une connexion à un carré de nombres intéressant appelés les nombres de poly-Bernoulli.

- Nous concluons le Chapitre 4 avec quelques questions ouvertes pour des travaux futurs en Section 4.7.

Introduction

1 Enumerative combinatorics

Enumerative combinatorics [107] is an area of mathematics that is primarily concerned with finding efficient methods to count the elements of finite sets. A typical problem involves a collection of finite sets $(\mathcal{C}_n)_{n\geq 0}$ called a *combinatorial class*, and consists in finding its *counting sequence* $(c_n)_{n\geq 0}$, where c_n denotes the number of elements in $\mathcal{C}_n$, the objects of *size* n. The solution to such a counting (also called enumeration) problem can take various forms. We present some of the most common ones below; for the sake of this example, $\mathcal{C}_n$ consists of all subsets of three numbers in $\{1, 2, \ldots, n\}$.

- The nicest form that a solution can take is a *closed formula*, generally defined as an explicit formula which involves only a bounded number of terms and well-known functions.

  For our example, we can pick a number in $\{1, \ldots, n\}$ in n ways; we then have $n-1$ remaining candidates to pick a second number from, and $n-2$ possible choices for a third number. We observe that each subset of three numbers can be obtained by this process in exactly 6 different ways depending on the order they are picked (since $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, and $\{3,2,1\}$ all describe the same set), and we obtain the following closed formula.

  $$c_n = \frac{n(n-1)(n-2)}{6}$$

- Some explicit formulas may instead contain an arbitrarily large number of terms which grows with n. These terms are usually related by summation or product.

  In our example, we can avoid the ordering problem by picking three numbers in decreasing order. The largest number i must be at least 3, the second largest j must be between 2 and $i-1$, and the smallest number k can take any value between 1 and $j-1$, so there are $j-1$ possible choices for k. This yields the explicit formula below.

  $$c_n = \sum_{i=3}^{n} \sum_{j=2}^{i-1} (j-1)$$

- A counting sequence may also be described by a recurrence relation which requires knowing some initial terms of the sequence to compute the next one.

  In our example, we can count elements of $\mathcal{C}_{n+1}$ by distinguishing two cases. Elements of $\mathcal{C}_{n+1}$ which do not contain the value $n+1$ form the set $\mathcal{C}_n$. Those which do contain $n+1$ are in bijection with subsets of two numbers in $\{1, \ldots, n\}$, which are counted by $\frac{n^2-n}{2}$ (this

can be shown in the same way as the closed formula for c_n above). Since $\mathcal{C}_0$ is empty, we may compute the numbers c_n from the equations below.

$$c_0 = 0, \quad c_{n+1} = c_n + \frac{n^2 - n}{2}$$

In more difficult counting problems, knowledge of some initial terms of a counting sequence is often not enough to compute the next one through recurrence relations. Nevertheless, knowing some additional information about the studied combinatorial objects may be enough to derive a recurrence relation in several variables, where the added variables keep track of some parameters of the objects in addition to their size. This is a technique we will often use in this thesis.

- It is also possible to express a counting sequence $(c_n)_{n\geq 0}$ through its *generating function*, that is, the formal power series whose coefficients are the numbers c_n.

$$C(x) = \sum_{n\geq 0} c_n x^n$$

There are two common ways to describe a generating function. The first is to write it in an explicit form. For our example, the symbolic method [53] gives the formula below.

$$C(x) = \frac{x^3}{(1-x)^4}$$

The second way is to define the function implicitly by equations it satisfies. There are many forms that such equations can take depending on the nature of a generating function $F(x)$. Some common examples include *algebraic* functions [79, 43], which satisfy an equation

$$P(x, F(x)) = 0$$

for a (nonzero) bivariate polynomial $P(x, y)$, and *D-finite* (or *holonomic*) functions [106, 62], which satisfy

$$P_0(x)F(x) + P_1(x)F'(x) + \cdots + P_k(x)F^{(k)}(x) = 0$$

for some (not all zero) polynomials $P_0(x), \ldots, P_k(x)$.

Much like recurrence relations, generating functions may also involve several variables to carry additional information.

The previous examples showed many different formulas to count the same objects. This was possible because we chose a very simple problem; in more difficult counting problems, the solutions we can find are much more restricted. In this thesis, we mainly use recurrence relations in several variables, or functional equations which are their counterparts for generating functions. Only in a few rare cases can we solve them to obtain explicit formulas.

An essential tool for research in enumerative combinatorics is the On-Line Encyclopedia of Integer Sequences (OEIS) [96]. This encyclopedia contains a very large number of sequences (nearly 400 000 as of April 2026), and often leads to the discovery of unexpected links between combinatorial classes which share the same counting sequence.

Lastly, it is important to define precisely what constitutes a solution to a counting problem. While the answer may change depending on the context, in this thesis we choose to follow

Wilf [112], and consider that the enumeration of a combinatorial class is solved once we have an algorithm which can compute its counting sequence in *polynomial time*; in other words, the number of operations required to compute each number c_n (in binary, or in base ten) is bounded above by the evaluation of some polynomial in n. We remark that such a formula is not always optimal, in the sense that a better formula requiring fewer computations (bounded by a polynomial of smaller degree, for instance) may exist. We consider that such a formula is a good enough solution nonetheless. Indeed, while a general approach can provide a way to systematically compute the kind of counting sequences we study in exponential time or worse, finding polynomial time solutions is a more difficult task (the existence of such a solution is often an open question).

2 Permutation patterns

In the context of permutation patterns [25, 64, 109], a permutation is most naturally defined as an integer sequence: given a nonnegative integer n, a *permutation* of length n is a sequence consisting of one occurrence of each integer between 1 and n, in any order. We denote by $\mathcal{S}_n$ the set of permutations of length n, counted by $n! = 1 \times 2 \times \cdots \times n$. More generally, in mathematics, the term permutation is used to describe a bijection from a set to itself. From this point of view, elements of $\mathcal{S}_n$ are permutations of the set $\{1, \ldots, n\}$, written as the sequence σ of images of each integer from 1 to n, i.e. the image of i is the i-th term of σ, denoted σ_i.

A length n permutation can be represented by a plot in the form of a $n \times n$ grid containing exactly one point in each row and each column. A point at coordinates (i, j) indicates that the i-th term of the permutation has value j, see Figure 1. Outside of the permutation patterns community, this representation is more commonly known as a *permutation matrix*, that is a $n \times n$ matrix containing only the values 0 and 1, and having exactly one 1 in each row and each column.

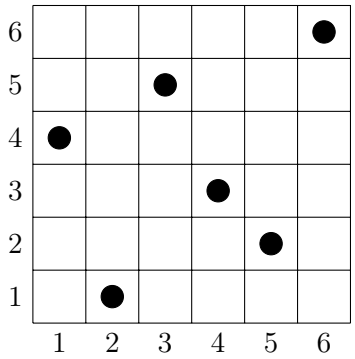

Figure 1: The plot of the permutation 415326.

Given two permutations σ and π, σ is said to *contain* the (classical) *pattern* π if σ contains a subsequence whose terms have the same relative ordering as π. Equivalently, σ contains the pattern π if the plot of π can be obtained by erasing some points from the plot of σ and rescaling the remaining ones. If σ does not contain π, then σ is said to *avoid* the pattern π. For instance, the permutation 415326 of Figure 1 contains four occurrences of the pattern 123, and avoids the pattern 1423. Two permutation patterns are said to be *Wilf-equivalent* if there are the same number of permutations of length n avoiding either pattern, for any nonnegative integer n.

The oldest results on classical permutation patterns come from the work of MacMahon, who

first showed that permutations avoiding the pattern 123 are counted by the Catalan numbers [81], and also studied the number of *inversions* (occurrences of the pattern 21) in permutations [82]. The introduction of classical permutation patterns is usually attributed to Knuth [66]. Knuth showed that the stack-sortable permutations are those which avoid the pattern 231, and that they are also counted by the Catalan numbers. Since permutations avoiding any pattern among $\{132, 213, 231, 312\}$ are trivially in bijection (by reflection of their plot with respect to a vertical or horizontal axis), as well as those avoiding 123 or 321, this implies that all six permutation patterns of length 3 are Wilf-equivalent. Knuth also described the first of many bijections between permutations avoiding 321 and those avoiding 132. The first systematic study of patterns in permutations was the seminal work of Simion and Schmidt [105], who enumerated permutations avoiding any set of patterns of length 3. Permutation patterns are still a very active research area to this day, and a conference dedicated to the subject has been held each year since 2003.

3 Inversion sequences

The *inversion sequence* of a permutation σ of length n is the sequence of nonnegative integers τ of length n such that $\tau_i = |\{j \,:\, \sigma_j > \sigma_i \text{ and } j < i\}|$ counts the number of inversions of σ whose second element is σ_i, for all $i \in \{1, \ldots, n\}$. The mapping from a permutation to its inversion sequence is called the *Lehmer code* after D.H. Lehmer [72], although it previously appeared in the work of Laisant [71] in the 19th century. The Lehmer code is a bijection between the set $\mathcal{S}_n$ of permutations of length n, and the set $\mathcal{I}_n$ of inversion sequences of length n, that is, integer sequences τ such that $0 \leq \tau_i < i$ for all $i \in \{1, \ldots, n\}$.

Inversion sequences can be represented by points in a staircase-shaped plot in much the same way as permutations; this plot also contains a single point in each column, but rows may now contain any number of points, see Figure 2.

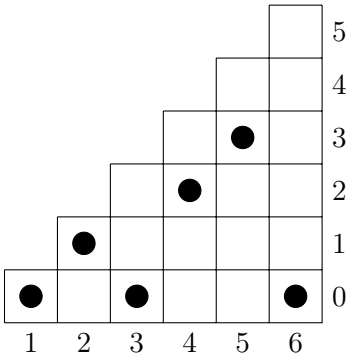

Figure 2: The plot of the inversion sequence 010230.

The notion of pattern defined on permutations extends to inversion sequences, although these patterns are not necessarily permutations, nor inversion sequences. To define each pattern uniquely, we now consider that a pattern is a nonempty sequence of nonnegative integers which contains every integer between 0 and its maximum. For instance 0212 is a pattern, but 4520 is not since the numbers 1 and 3 are missing. Given a pattern ρ, we denote by $\mathcal{I}(\rho)$ the set of inversion sequences avoiding ρ, and by $\mathcal{I}_n(\rho)$ its subset of sequences of length n.

In this thesis, inversion sequences are the main object studied. Their link to permutations is not exploited, since pattern avoidance in inversion sequences rarely translates to nice constraints

on permutations.

Unlike patterns in permutations, the study of patterns in inversion sequences began only a decade ago, in two independent articles [85, 46] enumerating inversion sequences avoiding a pattern of length 3. These foundational works showed connections to many well-known sequences of numbers, such as powers of 2, Fibonacci numbers, Schröder numbers, Euler up/down numbers, and Bell numbers. See Table 1 for a complete list (including some more recent results) of counting sequences of inversion sequences avoiding a pattern of length 3.

Pattern ρ	$\lvert\mathcal{I}_n(\rho)\rvert$ for $n = 1, \ldots, 7$	Enumerated in	OEIS [96]
000	1, 2, 5, 16, 61, 272, 1385	[46]	A000111
001	1, 2, 4, 8, 16, 32, 64	[46]	A000079
010	1, 2, 5, 15, 53, 215, 979	[A]	A263779
011	1, 2, 5, 15, 52, 203, 877	[46]	A000110
012	1, 2, 5, 13, 34, 89, 233	[46] and [85]	A001519
021	1, 2, 6, 22, 90, 394, 1806	[46] and [85]	A006318
100	1, 2, 6, 23, 106, 565, 3399	[69]	A263780
101 or 110	1, 2, 6, 23, 105, 549, 3207	[46]	A113227
102	1, 2, 6, 22, 89, 381, 1694	[85]	A200753
120	1, 2, 6, 23, 103, 515, 2803	[85]	A263778
201 or 210	1, 2, 6, 24, 118, 674, 4306	[85]	A263777

Table 1: Counting sequences of inversion sequences avoiding a single pattern of length 3.

These results sparked significant interest in the study of various types of patterns in inversion sequences, and many articles have followed over the past ten years, covering

- classical patterns [26, 73, 113, 55, 41, 31, 32, 88, 87, 69, 39, 99, 89, 57, 91, 5, 58, 63],
- patterns of relations, whose avoidance generalizes that of a single classical pattern, but corresponds to avoiding some set of classical patterns [93, 19, 35, 74, 115, 3, 38, 42, 27],
- consecutive patterns, which are defined similarly to classical patterns, with the added restriction that occurrences must use consecutive terms [7, 8, 50, 52],
- vincular patterns, in which only some terms must be consecutive [75, 9, 77, 21].

Pattern avoidance was also recently studied in several remarkable subsets of inversion sequences, such as Catalan words [13, 14, 15, 16, 1], restricted growth functions [59, 103, 60, 76, 34, 102, 18], ascent sequences [47, 40, 114, 84, 17, 74, 77, 45, 48, 33, 90, 80], as well as some variants of ascent sequences [65, 36, 22, 78, 116, 37].

Yan and Lin [113] conducted a systematic study of inversion sequences avoiding a pair of patterns of length 3, leaving open the enumeration of inversion sequences avoiding 32 of the 78 possible pairs. Eight of these were solved in [69], and one additional case was solved in [39] and [99] independently. Most of these problems were solved using bijections with other known combinatorial objects, or through generating trees (a method we will also employ). We completed this task by providing formulas for the remaining 23 cases in [A, B].

4 Summary of contents

In this thesis, we study pattern avoidance in inversion sequences, from an enumerative point of view. The first two chapters are about constructions of inversion sequences avoiding specific sets of patterns. More precisely, they complete the enumeration of inversion sequences avoiding one or two patterns of length 3. These results are summarized in Table 2. The third chapter systematically solves the enumeration of inversion sequences avoiding 021 and any finite set of patterns. The fourth chapter introduces *minimal* inversion sequences containing a pattern, and provides some results on the structure and the enumeration of these objects.

Pattern set P	$\lvert\mathcal{I}_n(P)\rvert$ for $n = 1, \ldots, 7$	Solved in	Performance	OEIS [96]
{011, 120}	1, 2, 5, 14, 42, 132, 431	Section 1.1.3	> 400	A374548
{100, 120}	1, 2, 6, 22, 92, 421, 2062	Section 1.1.4	> 300	A374549
{120, 201}	1, 2, 6, 23, 102, 498, 2607	Section 1.1.5	> 300	A374550
{110, 120}	1, 2, 6, 22, 92, 423, 2091	Section 1.1.6	> 300	A279570
{010, 120}	1, 2, 5, 15, 52, 201, 845	Section 1.1.7	> 300	A279559
{101, 120}	1, 2, 6, 22, 90, 397, 1859	Section 1.1.8	> 200	A374551
{000, 120}	1, 2, 5, 15, 50, 185, 737	Section 1.1.9	> 300	A374552
{010}	1, 2, 5, 15, 53, 215, 979	Section 1.1.10	> 180	A263779
{000, 010}	1, 2, 4, 10, 29, 95, 345	Section 1.1.11	> 200	A279552
{010, 201} or {010, 210}	1, 2, 5, 15, 53, 214, 958	Section 1.1.13	> 180	A360052
{010, 110}	1, 2, 5, 15, 52, 201, 847	Section 1.1.14	> 140	A359191
{010, 102}	1, 2, 5, 15, 51, 186, 707	Section 1.2.2	> 200	A374553
{100, 102}	1, 2, 6, 21, 80, 318, 1305	Section 1.2.3	> 300	A374554
{000, 100}	1, 2, 5, 16, 60, 260, 1267	Section 2.2.2	> 900	A279564
{102, 201}	1, 2, 6, 22, 87, 354, 1465	Section 2.2.3	$> 10\,000$	A279566
{000, 102}	1, 2, 5, 14, 40, 121, 373	Section 2.3.2	$> 10\,000$	A374541
{102, 210}	1, 2, 6, 22, 87, 351, 1416	Section 2.3.3	$> 10\,000$	A374542
{000, 201} or {000, 210}	1, 2, 5, 16, 60, 257, 1218	Section 2.3.5	> 700	A374543
{100, 110}	1, 2, 6, 22, 93, 437, 2233	Section 2.3.6	> 800	A374544
{100, 101}	1, 2, 6, 22, 93, 439, 2267	Section 2.3.7	> 800	A374545
{110, 201}	1, 2, 6, 23, 103, 512, 2739	Section 2.3.8	> 800	A374546
{101, 210}	1, 2, 6, 23, 103, 513, 2763	Section 2.3.9	> 800	A374547
{201, 210}	1, 2, 6, 24, 116, 632, 3720	[39], [99], Section 2.4.2	$> 10\,000$	A212198
{010, 102} (again)	1, 2, 5, 15, 51, 186, 707	Section 2.4.3	$> 10\,000$	A374553

Table 2: Enumeration of pattern-avoiding inversion sequences studied in Chapters 1 and 2. Each section of the table corresponds to a different construction. The "performance" column indicates how many terms of each counting sequence we were able to compute in one minute, with Maple or C++ programs running on a personal computer (naive methods can compute around 20 terms at most).

Chapter 1 corresponds to Sections 4 and 5 of [A]. The methods used in this chapter enumerate pattern-avoiding inversion sequences by splitting them in two smaller factors.

- Section 1.1 uses a decomposition of inversion sequences around their leftmost maximum, first employed in [85], which yields recurrence relations for counting these sequences. In that

case, an inversion sequence is seen as the concatenation of a smaller inversion sequence and a word. Since most of the families of words occurring in Section 1.1 were not previously enumerated, we also provide formulas for counting pattern-avoiding words through the same decomposition around the first maximum. The first part of Section 1.1 examines the avoidance of the pattern 120, and the sets of patterns we study are $\{120, \tau\}$ for $\tau \in \{000, 010, 011, 100, 101, 110, 201\}$. The second part examines the avoidance of the pattern 010, specifically the sets of patterns $\{010\}$, and $\{010, \tau\}$ for $\tau \in \{000, 201, 210, 110\}$.

- Section 1.2 introduces *shifted* inversion sequences and counts them using a decomposition around their first minimum, similar to that of Section 1.1. It is sometimes easier to construct pattern-avoiding inversion sequences by seeing them as a particular case of shifted inversion sequences. We apply this method to inversion sequences avoiding the pairs of patterns $\{010, 102\}$ and $\{100, 102\}$, and obtain recurrence formulas for counting them.

In Chapter 1, the formulas counting pattern-avoiding inversion sequences always take the form of recurrence relations in several variables, usually involving several nested sums. Such formulas are good enough to compute hundreds of terms of counting sequences (see Table 2), but they do not lead to nice equations for generating functions. The generating functions of the classes of inversion sequences studied in Chapter 1 are all conjectured non D-finite, except for the case $\{010, 102\}$ which is proved to be algebraic in Chapter 2.

Chapter 2 corresponds to [A, Sections 2, 3, 6] and [B], and it is about generating tree constructions. A *generating tree* [110, 10] is a rooted labelled tree such that the label of each node determines its children. Generating trees are used for enumerating combinatorial objects by assigning a node to each object of a combinatorial class, in such a way that objects of size n correspond to nodes at level n. Each node is labelled not by the object itself, but by a simpler label that represents its "state" in a recursive construction of the combinatorial class. These labels are usually integers or tuples of integers. The evolution of labels in a generating tree is described by a *succession rule*, which indicates the root's label and defines how the label of each node determines the labels of its children. Succession rules correspond to recurrence relations that can be used for counting objects. A more thorough introduction to generating trees is given in Section 2.1.

Chapter 2 makes use of three different generating constructions of pattern-avoiding inversion sequences.

- Section 2.2 is the most obvious approach, building sequences from left to right: the children of an inversion sequence of length n are obtained by appending to it a value between 0 and n. Since this is a very natural construction, it has been used extensively before (especially with the algorithm of [69]). Consequently, there was only one open case in which applying this approach "as is" yields a simple succession rule, namely $\mathcal{I}(000, 100)$. Nevertheless, using it together with *commitments* [99] leads to a nice succession rule for $\mathcal{I}(102, 201)$, from which we compute its algebraic generating function.

- Section 2.3 uses a construction from "bottom to top": entries of an inversion sequence are inserted by increasing *values*. Strictly speaking, this is not a single construction, but a collection of constructions since they also depend on the order of insertion of entries having the same value (we use different orders according to the avoided patterns). With this approach, we find succession rules for $\mathcal{I}(000, 102)$ and $\mathcal{I}(102, 210)$. For $\mathcal{I}(000, 102)$, we give a *jumping* succession rule [51] to describe a generating tree in which sequences of length n may have children of length $n + 1$ and $n + 2$. For $\mathcal{I}(102, 210)$, we keep to the

"standard" definition of generating trees, but our construction inserts entries in an unusual order to result in a nice succession rule. From their succession rules, we compute the generating functions of $\mathcal{I}(000, 102)$ and $\mathcal{I}(102, 210)$, both of which are algebraic.

We then use a decomposition of inversion sequences by removing their maxima. This decomposition is similar to the generating tree approach we described, although viewed from a different perspective to simplify presentation, as discussed in Section 2.3.4. We apply this decomposition to count inversion sequences avoiding $\{000, 201\}$, $\{000, 210\}$, $\{100, 110\}$, $\{100, 101\}$, $\{110, 201\}$, and $\{101, 210\}$, and we obtain recurrence formulas with some nested sums.

- Section 2.4 is about a construction of inversion sequences that we call the generating tree "growing on the left". This construction first appeared in [46] to enumerate the classes $\mathcal{I}(101)$ and $\mathcal{I}(110)$. Unlike the previous two constructions, it does not only insert a new entry at each step, but also modifies some existing entries. With this approach, we give detailed constructions of inversion sequences avoiding the pairs of patterns $\{201, 210\}$ and $\{010, 102\}$.

  The class $\mathcal{I}(201, 210)$ has an algebraic generating function, which was already found in [39] and [99] independently. Our generating tree growing on the left yields a new succession rule for $\mathcal{I}(201, 210)$, which makes it very easy to compute its generating function.

  The second class we examine is $\mathcal{I}(010, 102)$. After we found a recurrence formula for it with the decomposition of Chapter 1 (see Section 1.2.2), Pantone investigated its counting sequence with his software [98] and conjectured [99] that it had an algebraic generating function. Neither our earlier decomposition, nor a different one by Pantone could prove this conjecture, but we manage to prove it at last with a modified generating tree growing on the left in Section 2.4.3.

  Afterwards, we explore in Section 2.4.4 to what extent the generating tree growing on the left could be applied to other similar problems. To illustrate the robustness of this approach, we present some succession rules for inversion sequences avoiding any pattern of length 3, some of which require modifications of the standard generating tree growing on the left.

Our construction of $\mathcal{I}(102, 201)$ and the approach of Section 2.4 were both successfully reused in [27] to enumerate inversion sequences avoiding patterns of relations.

We conclude Chapters 1 and 2 with a brief discussion on the asymptotic behavior of the counting sequences of pattern-avoiding inversion sequences we studied. In addition, we give sufficient conditions on a set of patterns P to show that the growth of the counting sequence of $\mathcal{I}(P)$ is bounded above by an exponential function, or to show that it is super-exponential.

Chapter 3 investigates inversion sequences avoiding 021 and other patterns. Using an algorithm developed in [69], Mansour and Yıldırım [87, 86] computed the generating functions of inversion sequences avoiding 021 and any pattern of length 4 or 5. Their results lead to several interesting observations. In particular, these generating functions are all algebraic, which naturally leads to the conjecture that $\mathcal{I}(021, \rho)$ has an algebraic generating function for any pattern ρ. In Chapter 3, we prove a more general result: inversion sequences avoiding any finite set of patterns P which contains 021 always have an algebraic generating function. Our result also holds on ascent sequences [29] and restricted growth functions [94]; for the latter, generating functions are even *rational*. Our approach uses an encoding of 021-avoiding inversion sequences as coloured Dyck paths, which first appeared in [73]. This family of Dyck paths can equivalently

be defined as a language over an alphabet of three letters, recognized by a counter automaton. Under this encoding, patterns in 021-avoiding inversion sequences correspond to a certain type of patterns on words, whose avoidance can be decided by a finite automaton. Consequently, P-avoiding inversion sequences (for a finite set of patterns P containing 021) are in bijection with a language recognized by a (deterministic) counter automaton; they therefore have an algebraic generating function, according to the Chomsky–Schützenberger enumeration theorem [43]. We also show that generating functions are always rational for inversion (or ascent) sequences avoiding a finite set of patterns which contains both 021 and a strictly increasing pattern. This result is obtained in a similar manner, by finding a length-preserving bijection between these sequences and regular languages. To describe this bijection, we introduce a different encoding which uses commitments [99].

Chapter 4 introduces the notion of ρ-minimal inversion sequences for a pattern ρ. Simply put, ρ-minimal inversion sequences are the smallest (in the sense of pattern containment) inversion sequences which contain the pattern ρ. In particular, an inversion sequence avoids ρ if and only if it avoids every ρ-minimal inversion sequence.

- In Section 4.3, we examine structural properties of minimal inversion sequences, and find two notable results. The first states that an inversion sequence is ρ-minimal if and only if it contains an occurrence of ρ which satisfies some conditions that are easy to check. The second is an upper bound on the length of ρ-minimal inversion sequences depending on some statistics of ρ, which implies in particular that the number of ρ-minimal inversion sequences is always finite. More specifically, a less precise bound guarantees that a pattern of length k only has minimal inversion sequences of length less than $3k$.

- In Section 4.4, we show that the upper bound given in the previous section is reached for most patterns. We also show that even for the remaining patterns, there always exist some minimal inversion sequences whose length is very close to this bound.

- In Section 4.5, we precisely describe and count ρ-minimal inversion sequences for any pattern ρ that begins with a large enough value, using a bijection between inversion sequences and increasing trees. We also provide a way to exhaustively generate minimal inversion sequences for any pattern, and present some data on minimal inversion sequences for small patterns obtained with this method.

- In Section 4.6, we enumerate the intersection of inversion sequences and patterns, as this class has an important role in our study of minimal inversion sequences. We find a connection to an interesting array of integers called the poly-Bernoulli numbers.

- We conclude Chapter 4 with some open questions for future work in Section 4.7.

Preliminaries

Integer sequences and patterns

Let $\mathbb{N}$ be the set of natural numbers, including 0, and let $\mathbb{Z}$ be the set of all integers. A *combinatorial class* is a set of objects $\mathcal{C}$ together with a size function $|\cdot| : \mathcal{C} \to \mathbb{N}$ such that there is a finite number of objects of size n for each $n \in \mathbb{N}$. We also use the vertical bar notation $|A|$ for the cardinality of a set A; this is consistent in the sense that the "size" of a set is its number of elements. If $\mathcal{C}$ is a combinatorial class, and $\mathcal{C}_n$ its subset of objects of size n for each $n \in \mathbb{N}$, we call $(|\mathcal{C}_n|)_{n\in\mathbb{N}}$ the *counting sequence* of $\mathcal{C}$ (here, the vertical bars denote set cardinality).

Given a set A and a natural number $n \in \mathbb{N}$, we call *sequences* of *length* (or *size*) n over A the elements of A^n. When the set A is finite, we also call these sequences *words* over the *alphabet* A. We always index entries of a sequence starting at 1, e.g. $\sigma = (\sigma_1, \ldots, \sigma_n) \in A^n$. We denote by ε the *empty sequence*, that is, the only sequence of length 0. We call *integer sequences* the elements of $\mathbb{Z}^n$ for any $n \in \mathbb{N}$. We denote by $\mathcal{I}_n$ the set of *inversion sequences* of length n, that is the set of sequences $\sigma \in \mathbb{N}^n$ such that $\sigma_i < i$ for all $i \in \{1, \ldots, n\}$. We denote by $\mathcal{I} = \sqcup_{n\geq 0}\mathcal{I}_n$ the set of all inversion sequences.

Given two integer sequences σ of length n, and ρ of length k such that $1 \leq k \leq n$, we say that σ *contains* the *pattern* ρ if there exists a subsequence of k entries of σ whose values appear in the same order as ρ. Such a subsequence is said to be *order-isomorphic* to ρ, and it is called an *occurrence* of the pattern ρ in σ. We say that σ *avoids* the pattern ρ if σ does not contain ρ. We write patterns without commas or parenthesis for simplicity (and we also adopt this notation for all integer sequences in Chapter 4). For instance, the sequence $(0, 1, 0, 3, 0, 2, 3)$ avoids the pattern 110, but it contains the pattern 021 since the subsequence $(0, 3, 2)$ is an occurrence of 021.

Given a set of integer sequences $\mathcal{S}$ and a set of patterns P, we always denote by $\mathcal{S}(P)$ the subset of sequences of $\mathcal{S}$ which avoid every pattern in P. Assume now that the subset $\mathcal{S}_n$ of sequences of length n in $\mathcal{S}$ is finite for all $n \in \mathbb{N}$ (so that $(\mathcal{S}_n)_{n\in\mathbb{N}}$ forms a combinatorial class).

- If ρ and ξ are two patterns such that ρ contains the pattern ξ, then $\mathcal{S}(\xi) \subseteq \mathcal{S}(\rho)$
- For any two sets of patterns P and Q, we have $\mathcal{S}(P \cup Q) = \mathcal{S}(P) \cap \mathcal{S}(Q)$.
- We say that two sets of patterns P and Q are *Wilf-equivalent* for $\mathcal{S}$ if $|\mathcal{S}_n(P)| = |\mathcal{S}_n(Q)|$ for all $n \in \mathbb{N}$.

The *reduction* of an integer sequence is the sequence obtained by replacing its smallest value by 0, its second smallest value by 1, and so on. The reduction function **red** is idempotent, i.e. $\mathbf{red} \circ \mathbf{red} = \mathbf{red}$. It is easy to see that containing a pattern is equivalent to containing its reduction, since their values keep the same relative ordering. In order to avoid redundancy, we now consider as patterns only nonempty *reduced* sequences i.e. fixed points of the function

red. Equivalently, a pattern is now a nonempty sequence of nonnegative integers σ which contains every value between 0 and its maximum at least once. These sequences have many names in the literature, such as packed words [70], surjective words [54], initial words [100], Cayley permutations [95], or preferential arrangements [24]. In this thesis, we usually call these sequences "patterns" for simplicity, with some exceptions as follows:

- in Chapter 1, they are sometimes seen as (subsets of) words on a fixed alphabet,
- in Chapter 3, they are occasionally called "integer patterns" to distinguish them from a different kind of pattern,
- in Chapter 4, they are sometimes called "Cayley permutations" to emphasize their structure as a combinatorial object, rather than their role as a pattern contained or avoided by another sequence.

For any integers $a, b \in \mathbb{Z}$, we denote the integer interval $[a, b] = \{k \in \mathbb{Z} \,:\, a \leq k \leq b\}$. Given two integer sequences $\sigma \in \mathbb{Z}^n$ and $\tau \in \mathbb{Z}^m$, we say that τ is a *factor* of σ if τ is a subsequence of consecutive terms of σ, i.e. if there exists two integers $a \leq b \in [1, n]$ such that $\tau = (\sigma_i)_{i\in[a,b]}$, or $\tau = \varepsilon$. In particular, a factor $\tau = (\sigma_i)_{i\in[a,b]}$ is called a *prefix* of σ if $a = 1$, and a *suffix* if $b = n$.

Let $n, m \in \mathbb{N}$, and let $\sigma \in \mathbb{Z}^n, \tau \in \mathbb{Z}^m$ be two integer sequences. We define the *concatenation* of σ and τ as $\sigma \cdot \tau = (\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m) \in \mathbb{Z}^{n+m}$. For all $k \in \mathbb{N}$, we denote by σ^k the concatenation of k copies of σ (in particular, $\sigma^0 = \varepsilon$). For any $k \in \mathbb{Z}$, we denote by $\sigma + k = (\sigma_i + k)_{i\in[1,n]}$ the sequence obtained by adding k to each term of σ.

We now define some simple statistics on integer sequences which will be used throughout this thesis. For all $n \in \mathbb{N}$ and $\sigma \in \mathbb{Z}^n$, let

- $|\sigma| = n$ be *length* (or *size*) of σ,
- $\mathbf{Vals}(\sigma) = \{\sigma_i \,:\, i \in [1, n]\}$ be the *set of values* of σ,
- $\mathbf{min}(\sigma) = \mathbf{min}(\mathbf{Vals}(\sigma))$ be the *minimum* of σ, with the convention $\mathbf{min}(\varepsilon) = +\infty$,
- $\mathbf{max}(\sigma) = \mathbf{max}(\mathbf{Vals}(\sigma))$ be the *maximum* of σ, with the convention $\mathbf{max}(\varepsilon) = -1$ (we mainly consider sequences with values in $\mathbb{N}$, and this is a natural convention in that case),
- $\mathbf{dist}(\sigma) = |\mathbf{Vals}(\sigma)|$ be the number of *distinct values* of σ.

For instance, if $\sigma = (0, 7, 2, 3, 2, 0, 4, 7)$, then $|\sigma| = 8$, $\mathbf{Vals}(\sigma) = \{0, 2, 3, 4, 7\}$, $\mathbf{min}(\sigma) = 0$, $\mathbf{max}(\sigma) = 7$, and $\mathbf{dist}(\sigma) = 5$.

Generating functions

Let $\mathcal{C}$ be a combinatorial class, and for all $n \in \mathbb{N}$, let $\mathcal{C}_n$ be its subset of objects of size n. The *ordinary generating function* of $\mathcal{C}$ is the formal power series $\sum_{n\in\mathbb{N}} |\mathcal{C}_n| x^n$ in the indeterminate x. Similarly, the *exponential generating function* of $\mathcal{C}$ is the formal power series $\sum_{n\in\mathbb{N}} \frac{|\mathcal{C}_n|}{n!} x^n$. These two kinds of generating functions naturally appear in different types of constructions (ordinary generating functions are used for unlabelled objects, while exponential generating functions are used for labelled objects).

Sometimes, constructions of combinatorial objects require tracking some parameters in addition to size. For instance, we might want to partition each set $\mathcal{C}_n$ into subsets $\mathcal{C}_{n,k}$ for some $k \in \mathbb{N}$.

In this case, it is natural to consider *multivariate* generating functions, defined in a similar manner. For example, the bivariate ordinary generating function of $(\mathcal{C}_{n,k})_{n,k\in\mathbb{N}}$ is the formal power series $\sum_{n,k\in\mathbb{N}} |\mathcal{C}_{n,k}| x^n y^k$ in the indeterminates x and y, and its bivariate exponential generating function is $\sum_{n,k\in\mathbb{N}} \frac{|\mathcal{C}_{n,k}|}{n!k!} x^n y^k$. In some cases, it is useful to define *mixed* generating functions which are ordinary in some parameters and exponential in others, for instance $\sum_{n,k\in\mathbb{N}} \frac{|\mathcal{C}_{n,k}|}{n!} x^n y^k$ is exponential in x and ordinary in y.

Let $\mathbb{K}$ be a field, and let $F(x) = \sum_{n\geq 0} c_n x^n$ be a formal power series with coefficients in $\mathbb{K}$.

- The series $F(x)$ is *rational* over $\mathbb{K}$ if it can be written as

$$F(x) = \frac{P(x)}{Q(x)},$$

  where $P(x)$ and $Q(x)$ are polynomials with coefficients in $\mathbb{K}$, and $Q(x)$ has a nonzero constant term. Equivalently, $F(x)$ is rational if and only if its coefficients satisfy a linear recurrence equation with constant coefficients:

$$a_0 c_n + a_1 c_{n+1} + \cdots + a_k c_{n+k} = 0$$

  for some $k \geq 0$, some (not all zero) constants $a_0, \ldots, a_k \in \mathbb{K}$, and any $n \geq 0$.

- The series $F(x)$ is *algebraic* [79, 43] over $\mathbb{K}$ if it satisfies an equation of the form

$$P(x, F(x)) = 0$$

  for a nonzero bivariate polynomial $P(x, y)$ with coefficients in $\mathbb{K}$. The (algebraic) *degree* of $F(x)$ is the smallest degree in y among all such polynomials $P(x, y)$.

- The series $F(x)$ is *D-finite* [106, 62] over $\mathbb{K}$ if it satisfies an equation of the form

$$P_0(x)F(x) + P_1(x)F'(x) + \cdots + P_k(x)F^{(k)}(x) = 0$$

  for some $k \geq 0$ and some (not all zero) polynomials $P_0(x), \ldots, P_k(x)$ with coefficients in $\mathbb{K}$. Equivalently, $F(x)$ is D-finite if and only if its coefficients satisfy a linear recurrence equation with polynomial coefficients:

$$a_0(n)c_n + a_1(n)c_{n+1} + \cdots + a_k(n)c_{n+k} = 0$$

  for some (not all zero) polynomials $a_0(x), \ldots, a_k(x)$ with coefficients in $\mathbb{K}$, and any $n \geq 0$.

For our purposes, $\mathbb{K}$ is always the field of rational numbers $\mathbb{Q}$, so we can consider polynomials with *integer* coefficients without loss of generality.

Clearly, a rational function is algebraic (of degree 1). It is also well known that algebraic functions are always D-finite [44]. The computations required to go from an algebraic equation to a D-finite equation, and from a D-finite equation to a linear recurrence are implemented in most computer algebra systems (for instance, we use the Maple package `gfun`). In practice, this means that finding an algebraic or D-finite equation satisfied by a generating function makes it possible to compute its coefficients very fast (by iterating the associated linear recurrence relation).

Chapter 1

Splitting sequences in two factors

In this chapter, we count pattern-avoiding inversion sequences by splitting them in two factors. Section 1.1 splits inversion sequences at their leftmost maximum; we use this approach for 11 families of pattern-avoiding inversion sequences. Section 1.2 introduces shifted inversion sequences and splits them at their first minimum; we apply it on inversion sequences avoiding the patterns $\{010, 102\}$ and $\{100, 102\}$.

For any integer sequence σ of length n, let

- $\mathbf{firstmax}(\sigma) = \mathbf{min}(i \in [1, n] \ : \ \sigma_i = \mathbf{max}(\sigma))$ be the position of the *first maximum* of σ, with the convention $\mathbf{firstmax}(\varepsilon) = 0$,
- $\mathbf{lastmax}(\sigma) = \mathbf{max}(i \in [1, n] \ : \ \sigma_i = \mathbf{max}(\sigma))$ be the position of the *last maximum* of σ, with the convention $\mathbf{lastmax}(\varepsilon) = 0$.

We denote by $\delta_{a,b} = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b \end{cases}$ the Kronecker delta function.

For all $n, k \in \mathbb{N}$, we denote by $\mathcal{W}_{n,k} = [0, k-1]^n$ the set of words of length n over the alphabet $[0, k-1]$, and by $\overline{\mathcal{W}}_{n,k} = \{\omega \in \mathcal{W}_{n,k} \ : \ \mathbf{dist}(\omega) = k\}$ the subset of words in which each letter of the alphabet $[0, k-1]$ appears at least once[1]. For any set of patterns P, we denote by $\mathcal{W}_{n,k}(P)$ and $\overline{\mathcal{W}}_{n,k}(P)$ the subsets of P-avoiding words of $\mathcal{W}_{n,k}$ and $\overline{\mathcal{W}}_{n,k}$. The following proposition shows that it is essentially equivalent to solve the enumeration of $\mathcal{W}_{n,k}(P)$ or that of $\overline{\mathcal{W}}_{n,k}(P)$.

Proposition 1.0.1. *For any set of patterns P, for all $n, k \in \mathbb{N}$,*

$$|\mathcal{W}_{n,k}(P)| = \sum_{d=0}^{\mathbf{min}(n,k)} \binom{k}{d} |\overline{\mathcal{W}}_{n,d}(P)|.$$

Proof. Let $n, k \geq 0$, $d \in [0, \mathbf{min}(n, k)]$, and let $\mathfrak{W}_{n,k,d} = \{\omega \in \mathcal{W}_{n,k} \ : \ d = \mathbf{dist}(\omega)\}$ be the subset of $\mathcal{W}_{n,k}$ of words containing exactly d distinct letters. Let $F_{d,k}$ be the set of strictly increasing functions from $[0, d-1]$ to $[0, k-1]$. It is easy to see that the function $F_{d,k} \times \overline{\mathcal{W}}_{n,d} \to \mathfrak{W}_{n,k,d}$, $(\varphi, \omega) \mapsto (\varphi(\omega_i))_{i \in [1,n]}$ is a bijection. For any set of patterns P, restricting this function to the domain $F_{d,k} \times \overline{\mathcal{W}}_{n,d}(P)$ yields a bijection with the set of P-avoiding words in $\mathfrak{W}_{n,k,d}$ (the image of (φ, ω) is order-isomorphic to ω, so it contains the same patterns). This concludes the proof, since $\mathcal{W}_{n,k} = \coprod_{d=0}^{\mathbf{min}(n,k)} \mathfrak{W}_{n,k,d}$, and $|F_{d,k}| = \binom{k}{d}$. □

[1] Equivalently, $\mathcal{W}_{n,k}$ is the set of maps $[1, n] \to [0, k-1]$, and $\overline{\mathcal{W}}_{n,k}$ is the subset of surjective maps. The set of surjective maps $[1, n] \to [1, k]$ is also known as the set of *Cayley permutations* [95] of length n and maximum k.

1.1 Splitting at the first maximum

1.1.1 Method

The method described in this section is a generalization of the construction used in [85] for 120-avoiding inversion sequences.

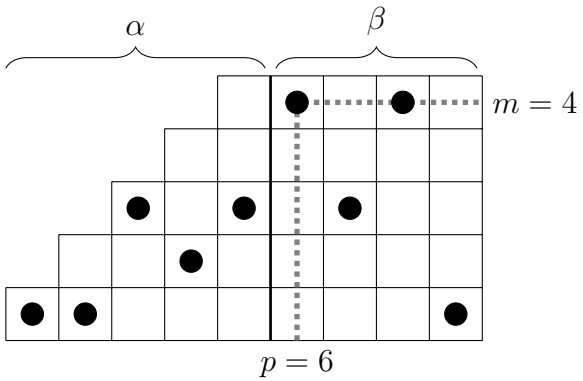

Figure 1.1: The inversion sequence $\sigma = (0, 0, 2, 1, 2, 4, 2, 4, 0)$ is split into the inversion sequence $\alpha = (0, 0, 2, 1, 2)$, and the word $\beta = (4, 2, 4, 0)$.

Let $\sigma \in \mathcal{I}_n$ be an inversion sequence of size $n \geq 1$, $m = \mathbf{max}(\sigma)$, and $p = \mathbf{firstmax}(\sigma)$. By definition of an inversion sequence, $p > m$. Let $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\beta = (\sigma_i)_{i\in[p,n]}$ be two factors of σ which satisfy $\alpha \cdot \beta = \sigma$, see Figure 1.1. The case $m = 0$ corresponds to constant inversion sequences, whose enumeration is trivial. When $m > 0$, we use this decomposition to count inversion sequences σ from the number of possible choices for α and β. Note that α is an inversion sequence of size $p - 1$ and maximum less than m, and β is a word of length $n - p + 1$ over the alphabet $[0, m]$ such that $\beta_1 = m$. More precisely, the following holds.

Remark 1.1.1. For all $0 \leq m < p < n$, the set $\{\sigma \in \mathcal{I}_n \ : \ m = \mathbf{max}(\sigma), p = \mathbf{firstmax}(\sigma)\}$ is equal to $\{\alpha \cdot \beta \ : \ \alpha \in \mathcal{I}_{p-1}, \mathbf{max}(\alpha) < m, \beta \in \mathcal{W}_{n-p+1,m+1}, \beta_1 = m\}$.

Now we can express pattern avoidance on σ in terms of necessary and sufficient conditions on α and β.

Remark 1.1.2. Let τ be a pattern of length $k \geq 2$. Then σ avoids τ if and only if all three following conditions are satisfied:

1. α avoids τ,
2. β avoids τ,
3. There are no $i_1 < i_2 < \cdots < i_k \in [1, n]$ such that $i_1 < p$, $i_k \geq p$ and $(\sigma_{i_j})_{j\in[1,k]}$ is an occurrence of τ.

Condition 3 is difficult to work with in general, but it can be turned into simple conditions on α and β for certain patterns, such as 120 or 010.

Proposition 1.1.3. *If $\tau = 120$, condition 3 is equivalent to* $\mathbf{max}(\alpha) \leq \mathbf{min}(\beta)$.

Proof. If $\mathbf{max}(\alpha) > \mathbf{min}(\beta)$, then $(\mathbf{max}(\alpha), m, \mathbf{min}(\beta))$ is an occurrence of 120. Conversely, if there are three integers $i_1 < i_2 < i_3$ such that $i_1 < p$, $i_3 \geq p$ and $(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3})$ is an occurrence of 120, then $\mathbf{max}(\alpha) \geq \sigma_{i_1} > \sigma_{i_3} \geq \mathbf{min}(\beta)$. □

Proposition 1.1.4. *If $\tau = 010$, condition 3 is equivalent to* $\mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta) = \emptyset$.

Proof. If $v \in \mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta)$, then (v, m, v) is an occurrence of 010. Conversely, if there are three integers $i_1 < i_2 < i_3$ such that $i_1 < p$, $i_3 \geq p$ and $(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3})$ is an occurrence of 010, then $\sigma_{i_1} = \sigma_{i_3} \in \mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta)$. □

Sections 1.1.2 to 1.1.9 are focused on the pattern 120, while Sections 1.1.10 to 1.1.14 are focused on the pattern 010.

We will always count the possible choices for α by recurrence, since α is an inversion sequence that is smaller than σ and must avoid the same patterns. On the other hand, the words β form a different family of objects which must be enumerated independently. This could be solved by applying the same decomposition around the first maximum to words, since Remark 1.1.1 has an analogue for words[2]. We use generating trees instead, since it results in simpler and more efficient recurrence formulas.

Depending on the pair of patterns studied, we consider different types of words in our decomposition. We always denote β the word defined as above, $\gamma = (\beta_i)_{i \in [2, n-p]}$ the same word without the first letter m (so that $\sigma = \alpha \cdot m \cdot \gamma$), and $\gamma' = (\beta_i)_{i \in [1, n-p], \beta_i \neq m}$ the word obtained by removing all letters m from β.

1.1.2 A useful symmetry

Given two integers $n, k \in \mathbb{N}$, a *composition* of n into k (possibly empty) *parts* is an integer sequence $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} \lambda_i = n$. Let $n, k \in \mathbb{N}$ and let λ be a composition of n into k parts. We denote by $\mathcal{W}_\lambda \subseteq \mathcal{W}_{n,k}$ the set of words in which each letter $\ell \in [0, k-1]$ appears exactly $\lambda_{\ell+1}$ times (i.e. permutations of the multiset $\{\ell^{\lambda_{\ell+1}} \ : \ \ell \in [0, k-1]\}$). As before, we denote by $\mathcal{W}_\lambda(P)$ the subset of words of $\mathcal{W}_\lambda(P)$ avoiding some set of patterns P. The following theorem is a result of [6] (up to symmetry).

Theorem 1.1.5. *Let $k \in \mathbb{N}$. The function $\mathbb{N}^k \to \mathbb{N}$, $(\lambda_1, \ldots, \lambda_k) \mapsto |\mathcal{W}_\lambda(120)|$ is symmetric.*

In other words, if $\lambda, \mu \in \mathbb{N}^k$ are two sequences such that there exists a permutation π of $[1, k]$ which satisfies $\mu_i = \lambda_{\pi(i)}$ for all $i \in [1, k]$, then the identity $|\mathcal{W}_\lambda(120)| = |\mathcal{W}_\mu(120)|$ holds. This result was generalized in [2] and [104] to show that the number $|\mathcal{W}_\lambda(120)|$ remains unchanged when the pattern 120 is replaced by any permutation of $\{0, 1, 2\}$.

1.1.3 Avoiding {011, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(011, 120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{011, 120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \ : \ \omega_i = \omega_j \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any repeated letter. Let $\mathfrak{C}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \ : \ \omega_i = \omega_j \neq 0 \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any nonzero repeated letter. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$, and $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$.

Theorem 1.1.6. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \mathfrak{c}_{n-p,m} + \sum_{j=1}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j-1}.$$

[2] More precisely, the set of words $\omega \in \mathcal{W}_{n,k}$ such that $m = \mathbf{max}(\omega)$ and $p = \mathbf{firstmax}(\omega)$ (note that p does not have to be greater than m now) is in bijection with the set of ordered pairs (α, β) such that $\alpha \in \mathcal{W}_{p-1,m}$ and $\beta \in \mathcal{W}_{n-p+1,m+1}$ satisfies $\beta_1 = m$. Remark 1.1.2 and Propositions 1.1.3 and 1.1.4 apply to these words as well.

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, and let $j = \mathbf{max}(\alpha)$. In particular, $j < m$.

- If $j = 0$, then $\alpha = (0)^{p-1}$, and $\gamma \in \mathfrak{C}_{n-p,m}$ since $\sigma_1 = 0$ implies that any nonzero repeated letter creates an occurrence of the pattern 011.
- If $j \geq 1$, then $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 1.1.3, γ is a 120-avoiding word of length $n-p$ over the alphabet $[j+1, m-1]$ (since repeating j or m would create an occurrence of 011) in which all letters are distinct (to avoid 011). Subtracting $j+1$ from every letter in γ yields a word in $\mathfrak{B}_{n-p,m-j-1}$.

Observing that these two constructions are bijective concludes the proof. □

Let us now enumerate the families of words $(\mathfrak{B}_{n,k})_{n,k\in\mathbb{N}}$ and $(\mathfrak{C}_{n,k})_{n,k\in\mathbb{N}}$.

Proposition 1.1.7. *For all $n, k \geq 0$,*

$$\mathfrak{b}_{n,k} = \frac{\binom{k}{n}\binom{2n}{n}}{n+1}.$$

Proof. There are $\binom{k}{n}$ possible choices for the set of n distinct letters in $[0, k-1]$ which appear in a word of $\mathfrak{B}_{n,k}$. Once this set of letters is decided, choosing their order amounts to choosing a 120-avoiding permutation of size n, counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. □

Lemma 1.1.8. *For all $n \geq 1, k \geq 2$,*

$$\mathfrak{c}_{n,k} = 2\mathfrak{c}_{n,k-1} + \mathfrak{c}_{n-1,k} - \mathfrak{c}_{n-1,k-1} - \frac{\binom{k-2}{n}\binom{2n}{n}}{n+1}.$$

Proof. Let $n \geq 0, k \geq 2$, and $\omega \in \mathfrak{C}_{n,k}$. We consider three different cases, based on the position of the letter 1 in ω.

- If ω does not contain the letter 1, then subtracting 1 from every nonzero letter in ω yields a word in $\mathfrak{C}_{n,k-1}$, and this is a bijection. As such, the words of $\mathfrak{C}_{n,k}$ which do not contain the letter 1 are counted by $\mathfrak{c}_{n,k-1}$.
- Otherwise, ω contains exactly one occurrence of the letter 1. This implies that $n \geq 1$ and $k \geq 2$.
  - If there is any occurrence of the letter 0 appearing to the right of the letter 1 in ω, then every such occurrence must be in a single factor adjacent to the letter 1 (otherwise, ω would contain the pattern 120). Hence, removing the last letter 0 in ω can yield any word in $\mathfrak{C}_{n-1,k}$ which contains the letter 1, and this is a bijection. We know from the previous case that the words of $\mathfrak{C}_{n-1,k}$ which do not contain the letter 1 are counted by $\mathfrak{c}_{n-1,k-1}$, hence the number of possible words ω in this case is $\mathfrak{c}_{n-1,k} - \mathfrak{c}_{n-1,k-1}$.
  - Otherwise, every letter 0 in ω appears to the left of the letter 1. Subtracting 1 from every nonzero letter in ω then yields a word in $\mathfrak{C}_{n,k-1}$ which contains the letter 0. Note this defines a bijection: its inverse takes any word in $\mathfrak{C}_{n,k-1}$ which contains the letter 0, adds 1 to every nonzero letter, and replaces the last letter 0 by the letter 1. Since the words of $\mathfrak{C}_{n,k-1}$ which do not contain the letter 0 are trivially in bijection with $\mathfrak{B}_{n,k-2}$, we find there are $\mathfrak{c}_{n,k-1} - \mathfrak{b}_{n,k-2}$ possible choices for the word ω in this case. □

We also give an explicit expression for $\mathfrak{c}_{n,k}$ in Lemma 1.1.11.

1.1.4 Avoiding {100, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(100,120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{100,120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \ : \ \omega_i = \omega_j \neq k-1 \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0,k-1]$ which do not contain any non-maximal repeated letter. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.9. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j+1}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, and let $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 1.1.3, $\gamma \in [j,m]^{n-p}$. Since σ avoids 100, γ cannot contain any repeated value lower than m. Subtracting j from each letter of γ yields a word in $\mathfrak{B}_{n-p,m-j+1}$. □

As in Section 1.1.3, let $\mathfrak{C}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \ : \ \omega_i = \omega_j \neq 0 \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0,k-1]$ which do not contain any nonzero repeated letter, and let $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. The number of words $\mathfrak{c}_{n,k}$ was counted in Lemma 1.1.8.

Lemma 1.1.10. *For all $n,k \geq 0$, $\mathfrak{b}_{n,k} = \mathfrak{c}_{n,k}$.*

Proof. Let $\mathfrak{B}_{n,k,r} = \{\omega \in \mathfrak{B}_{n,k} \ : \ r = |\{i \in [1,n] \ : \ \omega_i = k-1\}|\}$ be the subset of $\mathfrak{B}_{n,k}$ of words in which the largest letter $k-1$ appears exactly r times. Likewise, let $\mathfrak{C}_{n,k,r} = \{\omega \in \mathfrak{C}_{n,k} \ : \ r = |\{i \in [1,n] \ : \ \omega_i = 0\}|\}$ be the subset of $\mathfrak{C}_{n,k}$ of words in which the letter 0 appears exactly r times. By definition, we have

$$\mathfrak{B}_{n,k} = \coprod_{r=0}^{n} \mathfrak{B}_{n,k,r} \quad \text{and} \quad \mathfrak{C}_{n,k} = \coprod_{r=0}^{n} \mathfrak{C}_{n,k,r}.$$

Let $\mathfrak{D}_{n,k} = \{\lambda \in \{0,1\}^k \ : \ n = \sum_{i=1}^{k} \lambda_i\}$ be the set of compositions of n into k parts of size 0 or 1. In the notation of Section 1.1.2, for all $r \geq 0$,

$$\mathfrak{B}_{n,k,r} = \coprod_{\lambda \in \mathfrak{D}_{n-r,k-1}} \mathcal{W}_{\lambda\cdot r}(120) \quad \text{and} \quad \mathfrak{C}_{n,k,r} = \coprod_{\lambda \in \mathfrak{D}_{n-r,k-1}} \mathcal{W}_{r\cdot\lambda}(120).$$

By Theorem 1.1.5, for all $r \geq 0$ and $\lambda \in \mathfrak{D}_{n-r,k-1}$, $|\mathcal{W}_{\lambda\cdot r}(120)| = |\mathcal{W}_{r\cdot\lambda}(120)|$. This implies that $|\mathfrak{B}_{n,k,r}| = |\mathfrak{C}_{n,k,r}|$ for all $n,k,r \geq 0$, hence $\mathfrak{b}_{n,k} = \mathfrak{c}_{n,k}$. □

We also provide an explicit expression of $\mathfrak{b}_{n,k}$.

Lemma 1.1.11. *For all $n \geq 0, k \geq 1$,*

$$\mathfrak{b}_{n,k} = \frac{1}{n+1} \sum_{d=0}^{\mathbf{min}(n,k)} \binom{k-1}{d} \binom{n+d}{n} (n-d+1).$$

Proof. Let $\mathfrak{E}_{n,k} = \{\omega \in \mathfrak{B}_{n,k+1} \ : \ [0,k-1] \subseteq \mathbf{Vals}(\omega)\}$ be the subset of words of $\mathfrak{B}_{n,k+1}$ which contain all letters $[0,k-1]$ (and may contain the letter k). In particular, for all $n \geq 0$, $\mathfrak{E}_{n,k}$ is empty if $k > n$, and $\mathfrak{E}_{n,0} = \{(0)^n\}$. Note that any word of $\mathfrak{E}_{n,k}$ contains exactly one occurrence of each letter in $[0,k-1]$, and $n-k$ occurrences of the letter k. Let $\mathfrak{e}_{n,k} = |\mathfrak{E}_{n,k}|$.

The same construction as in Proposition 1.0.1 gives the following identity, seeing that $\binom{k-1}{d}$ is the number of increasing functions $[0,d] \to [0,k-1]$ such that $d \mapsto k-1$. For all $n \geq 0, k \geq 1$,

$$\mathfrak{b}_{n,k} = \sum_{d=0}^{\mathbf{min}(n,k)} \binom{k-1}{d} \mathfrak{e}_{n,d}.$$

Let us now count the words of $\mathfrak{E}_{n,k}$. Let $1 \leq k \leq n$, and $\omega \in \mathfrak{E}_{n,k}$.

- If $\omega_n = k$, then removing ω_n yields a word in $\mathfrak{E}_{n-1,k}$ and this is a bijection.
- If $\omega_n < k$, then no letter k may appear after the letter $k-1$ in ω, otherwise the subsequence $(k-1, k, \omega_n)$ would be an occurrence of 120. Replacing every letter k in ω by $k-1$ yields a word $\omega' \in \mathfrak{E}_{n,k-1}$ (since each letter in $[0,k-2]$ still appears exactly once in ω'), and this is a bijection. Indeed, it can be reversed by taking any word $\omega' \in \mathfrak{E}_{n,k-1}$ and replacing every letter $k-1$ except the last one by the letter k (notice that $\omega' \in \mathfrak{E}_{n,k-1}$ must contain the letter $k-1$ since $k \leq n$).

Hence, $(\mathfrak{e}_{n,k})_{n,k\geq 0}$ satisfies the recurrence relation $\mathfrak{e}_{n,k} = \mathfrak{e}_{n-1,k} + \mathfrak{e}_{n,k-1}$ for all $1 \leq k \leq n$, with initial conditions $\mathfrak{e}_{n,0} = 1$ for all $n \geq 0$ and $\mathfrak{e}_{0,k} = 0$ for all $k \geq 1$. From this recurrence, we can easily prove by induction that for all $0 \leq k \leq n$,

$$\mathfrak{e}_{n,k} = \frac{\binom{n+k}{n}(n-k+1)}{n+1},$$

which shows that $(\mathfrak{e}_{n,k})_{n,k\geq 0}$ is Catalan's triangle (entry A009766 of [96]). □

1.1.5 Avoiding {120, 201}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(120, 201) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{120, 201\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k+1}(120) \ : \ \omega_i < \omega_j \neq k \implies j < i\}$ be the set of 120-avoiding words of length n over the alphabet $[0,k]$ whose non-maximal letters appear in nonincreasing order. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.12. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, and let $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 1.1.3, $\gamma \in [j,m]^{n-p}$. Since σ avoids 201 and $\sigma_p = m$, all letters different from m in γ appear in nonincreasing order. By subtracting j from all letters in γ, we obtain a word in the set $\mathfrak{B}_{n-p,m-j}$. We easily observe that this construction is a bijection. □

Remark 1.1.13. For all $n \geq 0$ and $k \geq 1$, there are $\binom{k+n-1}{n}$ nonincreasing words of length n over the alphabet $[0, k-1]$. Indeed, the set of nonincreasing words of length n over the alphabet $[0, k-1]$ is in bijection with the set of compositions of n into k (possibly empty) parts. More precisely, for each composition of n into k parts λ, there is a single nonincreasing word in the set $\mathcal{W}_\lambda$ (using the notation from Section 1.1.2).

Lemma 1.1.14. *For all $n, k \geq 1$,*

$$\mathfrak{b}_{n,k} = \binom{k+n+1}{n} + k\left(2^n - 1 - \frac{n(n+1)}{2}\right) - n.$$

Proof. Let $n, k \geq 1$, and $\omega \in \mathfrak{B}_{n,k}$.

- If the letter k does not appear in ω, then ω can be any nonincreasing word of length n over the alphabet $[0, k-1]$. By Remark 1.1.13, there are $\binom{k+n-1}{n}$ such words.
- Otherwise, let $p = \mathbf{firstmax}(\omega)$ be the position of the first letter k in ω.
  - If $p = 1$, then $\omega_1 = k$ and $(\omega_i)_{i\in[2,n]}$ can be any word of $\mathfrak{B}_{n-1,k}$.
  - If $p \geq 2$, let $\alpha = (\omega_i)_{i\in[1,p-1]}$ and $\gamma = (\omega_i)_{i\in[p+1,n]}$. In particular, α is a nonincreasing word of length $p-1$ over the alphabet $[0, k-1]$. By Remark 1.1.13, there are $\binom{k+p-2}{p-1}$ possible words α. Note that any letter $v \in \mathbf{Vals}(\gamma)$ such that $v \neq k$ must satisfy both $v \leq \mathbf{min}(\alpha)$ (by the nonincreasing property) and $v \geq \mathbf{max}(\alpha)$ (to avoid 120). We distinguish two cases:
    - If α is constant, then $\alpha = (v)^{p-1}$ for some $v \in [0, k-1]$, and $\gamma \in \{v, k\}^{n-p}$. In that case, there are k possible choices for α, and for each α there are 2^{n-p} possible choices for γ.
    - Otherwise, there are $\binom{k+p-2}{p-1} - k$ remaining choices for α. Since α is not constant, $\mathbf{min}(\alpha) < \mathbf{max}(\alpha)$, hence no letter in $[0, k-1]$ may appear in γ, so $\gamma = (k)^{n-p}$.

We obtain the following equation for all $n, k \geq 1$:

$$\begin{aligned}
\mathfrak{b}_{n,k} &= \binom{k+n-1}{n} + \mathfrak{b}_{n-1,k} + \sum_{p=2}^{n} \left(k \cdot 2^{n-p} + \binom{k+p-2}{p-1} - k \right) \\
&= \binom{k+n-1}{n} + \mathfrak{b}_{n-1,k} + k(2^{n-1} - 1) + \binom{k+n-1}{n-1} - 1 - k(n-1) \\
&= \mathfrak{b}_{n-1,k} + \binom{k+n}{n} + k(2^{n-1} - n) - 1.
\end{aligned}$$

We conclude the proof with a telescoping sum. For all $n, k \geq 1$,

$$\begin{aligned}
\mathfrak{b}_{n,k} &= \mathfrak{b}_{0,k} + \sum_{i=1}^{n} (\mathfrak{b}_{i,k} - \mathfrak{b}_{i-1,k}) \\
&= 1 + \sum_{i=1}^{n} \left(\binom{k+i}{i} + k(2^{i-1} - i) - 1 \right) \\
&= \binom{k+n+1}{n} + k \left(2^n - 1 - \frac{n(n+1)}{2} \right) - n.
\end{aligned}$$

□

1.1.6 Avoiding {110, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(110, 120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{110, 120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \coprod_{\ell\in[0,n]} \mathcal{W}_{\ell,k}(110, 120)$ be the set of $\{110, 120\}$-avoiding words of length at most n over the alphabet $[0, k-1]$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.15. *For all* $0 < m < n$,

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 1.1.3, $\gamma \in [j,m]^{n-p}$. Since σ avoids 110, if $\gamma_q = m$ for some $q \in [1, n-p]$ then $\gamma_i = m$ for all $i \in [q, n-p]$. In other words, all occurrences of m in γ (if any) must appear in a single factor at the end of γ. Let $k \in [0, n-p]$ be the number of occurrences of m in γ, and $\gamma' = (\gamma_i)_{i\in[1,n-p-k]}$ be the word obtained by removing all letters m from γ. By subtracting j from all letters in γ', we obtain a word in the set $\mathfrak{B}_{n-p,m-j}$. We easily observe that this construction is a bijection. □

Let $\mathfrak{C}_{n,k} = \overline{\mathcal{W}}_{n,k}(110, 120)$ be the set of $\{110, 120\}$-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$. Let $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. By definition of $\mathfrak{B}_{n,k}$ and Proposition 1.0.1, for all $n, k \geq 0$,

$$\mathfrak{b}_{n,k} = \sum_{\ell=0}^{n} \sum_{d=0}^{\mathbf{min}(\ell,k)} \binom{k}{d} \mathfrak{c}_{\ell,d}.$$

Lemma 1.1.16. *For all* $1 \leq k \leq n$,

$$\mathfrak{c}_{n,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.$$

Proof. For all $0 \leq k \leq n$ and $\omega \in \mathfrak{C}_{n,k}$, let $\mathbf{dec}(\omega)$ be the size of the longest strictly decreasing factor at the beginning of ω. In particular, if $n > 0$ then $\omega_{\mathbf{dec}(\omega)}$ is the first letter 0 in ω (otherwise the first 0 would appear after a weak increase, creating an occurrence of 110 or 120). Also note that $\mathbf{dec}(\omega) \leq k$ since all letters of a strictly decreasing factor are distinct. Let $\mathfrak{C}_{n,k,s} = \{\omega \in \mathfrak{C}_{n,k} \ : \ s = \mathbf{dec}(\omega)\}$, and $\mathfrak{c}_{n,k,s} = |\mathfrak{C}_{n,k,s}|$.

We consider the combinatorial generating tree for $\coprod_{n\geq k\geq 0} \mathfrak{C}_{n,k}$ obtained by inserting letters in decreasing order of value, and inserting repeats of a letter from right to left. This amounts to inserting the letter 0 before the first 0 in a word $\omega \in \mathfrak{C}_{n,k}$, or inserting the letter 0 in $\omega + 1$. This indeed defines a generating tree since any nonempty word ω has a unique parent of size $|\omega| - 1$, obtained by removing the first 0 from ω, and subtracting 1 from every letter if the resulting word does not contain the letter 0.

For any word $\omega \in \mathfrak{C}_{n,k}$, the positions where the letter 0 may be inserted before the first 0 in ω are $[1, \mathbf{dec}(\omega)]$, and this clearly cannot create an occurrence of 110 or 120. The positions where the letter 0 may be inserted in $\omega + 1$ without creating an occurrence of 110 or 120 are $[1, \mathbf{dec}(\omega)+1]$. This implies that each word of $\mathfrak{C}_{n,k,s}$ has one child in $\mathfrak{C}_{n+1,k,i}$ for all $i \in [1, s]$, and one child in $\mathfrak{C}_{n+1,k+1,i}$ for all $i \in [1, s+1]$. We obtain the following equation for all $2 \leq i \leq k \leq n$:

$$\mathfrak{c}_{n,k,i} = \sum_{s=i}^{n-1} \mathfrak{c}_{n-1,k,s} + \sum_{s=i-1}^{n-1} \mathfrak{c}_{n-1,k-1,s}.$$

In particular, $\mathfrak{c}_{n,k,i} - \mathfrak{c}_{n,k,i+1} = \mathfrak{c}_{n-1,k,i} + \mathfrak{c}_{n-1,k-1,i-1}$, so a simpler recurrence relation holds: for all $2 \leq s \leq k \leq n$,

$$\mathfrak{c}_{n,k,s} = \mathfrak{c}_{n,k,s+1} + \mathfrak{c}_{n-1,k,s} + \mathfrak{c}_{n-1,k-1,s-1}.$$

From this equation, we can easily prove by induction that for all $1 \leq s \leq k \leq n$,

$$\mathfrak{c}_{n,k,s} = \frac{s(n+k-s-1)!}{(n-k)!k!(k-s)!}.$$

Finally, the expression of $\mathfrak{c}_{n,k}$ is obtained by summing over s. For all $1 \leq k \leq n$,

$$\begin{aligned}
\mathfrak{c}_{n,k} &= \sum_{s=1}^{k} \mathfrak{c}_{n,k,s} \\
&= \sum_{s=1}^{k} \frac{s(n+k-s-1)!}{(n-k)!k!(k-s)!} \\
&= \frac{1}{k}\binom{n-1}{k-1} \sum_{s=1}^{k} s\binom{n+k-s-1}{n-1} \\
&= \frac{1}{k}\binom{n-1}{k-1} \sum_{i=1}^{k}\sum_{s=i}^{k} \binom{n+k-s-1}{n-1} \\
&= \frac{1}{k}\binom{n-1}{k-1} \sum_{i=1}^{k} \binom{n+k-i}{n} \\
&= \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.
\end{aligned}$$

□

1.1.7 Avoiding {010, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(010, 120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{010, 120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \mathcal{W}_{n,k}(010, 120)$ be the set of $\{010,\ 120\}$-avoiding words of length n over the alphabet $[0, k-1]$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.17. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j}.$$

Proof. It follows from Remarks 1.1.1 and 1.1.2 and Propositions 1.1.3 and 1.1.4 that for $0 < m < n$, $\mathfrak{A}_{n,m}$ is the set of inversion sequences $\alpha \cdot m \cdot \gamma$ such that:

- $\alpha \in \mathfrak{a}_{p-1,j}$ for some $p \in [m+1, n]$ and $j \in [0, m-1]$,
- $\gamma \in [0, m]^{n-p}$ avoids 010 and 120 and satisfies $\mathbf{min}(\gamma) > \mathbf{max}(\alpha) = j$.

In other words, γ is a $\{010,\ 120\}$-avoiding word of length $n-p$ over the alphabet $[j+1, m]$, and the set of such words is clearly in bijection with $\mathfrak{B}_{n-p,m-j}$. □

Let $\mathfrak{C}_{n,k} = \overline{\mathcal{W}}_{n,k}(010, 120)$ be the set of $\{010,\ 120\}$-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$. Let $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. By Proposition 1.0.1, for all $n, k \geq 0$,

$$\mathfrak{b}_{n,k} = \sum_{d=0}^{\mathbf{min}(n,k)} \binom{k}{d} \mathfrak{c}_{n,d}.$$

Lemma 1.1.18. *For all $n, k \geq 1$,*

$$\mathfrak{c}_{n,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.$$

Proof. We consider the combinatorial generating tree for $\coprod_{n\geq k\geq 0} \mathfrak{C}_{n,k}$ obtained by inserting letters in increasing order of value, and inserting repeats of a letter from right to left. In other words, at each step of the construction, we either insert a letter k in a word $\omega \in \mathfrak{C}_{n,k}$, or insert a letter $k-1$ before the first $k-1$ in ω.

For all $1 \leq k \leq n$ and $\omega \in \mathfrak{C}_{n,k}$, let

$$\mathbf{Sites}(\omega) = \{q \in [1, \mathbf{firstmax}(\omega)] \ : \ \mathbf{max}((\omega_i)_{i\in[1,q-1]}) < \mathbf{min}((\omega_i)_{i\in[q,n]})\}$$

be the set of positions where the letter $k-1$ may be inserted before the first letter $k-1$ in ω without creating an occurrence of 010 or 120. The set of positions where the letter k may be inserted in ω without creating an occurrence of 010 or 120 is $\mathbf{Sites}(\omega) \sqcup \{n+1\}$. For all $1 \leq s \leq k \leq n$, let $\mathfrak{C}_{n,k,s} = \{\omega \in \mathfrak{C}_{n,k} \ : \ s = |\mathbf{Sites}(\omega)|\}$.

Let $1 \leq s \leq k \leq n$ and $\omega \in \mathfrak{C}_{n,k,s}$. Let $q_1 < q_2 < \cdots < q_s$ be the elements of $\mathbf{Sites}(\omega)$. Then for all $i \in [1, s]$, inserting the letter $k-1$ at position q_i yields a word of $\mathfrak{C}_{n+1,k,i}$, and inserting the letter k at position q_i yields a word of $\mathfrak{C}_{n+1,k+1,i}$. Additionally, inserting the letter k at position $n+1$ yields a word in $\mathfrak{C}_{n+1,k+1,s+1}$. Hence the following equation holds: for all $2 \leq i \leq k \leq n$,

$$\mathfrak{c}_{n,k,i} = \sum_{s=i}^{n-1} \mathfrak{c}_{n-1,k,s} + \sum_{s=i-1}^{n-1} \mathfrak{c}_{n-1,k-1,s}.$$

The same recurrence relation was found in the proof of Lemma 1.1.16, with the same initial condition $\mathfrak{c}_{1,1,1} = 1$. This exhibits a bijection between the sets of words $\overline{\mathcal{W}}_{n,k}(010, 120)$ and $\overline{\mathcal{W}}_{n,k}(110, 120)$, which trivially extends to a bijection between $\mathcal{W}_{n,k}(010, 120)$ and $\mathcal{W}_{n,k}(110, 120)$, and concludes our proof. □

1.1.8 Avoiding {101, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(101, 120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{101, 120\}$-avoiding inversion sequences of size n and maximum m. Since the sequences in $\mathfrak{A}_{n,m}$ avoid 101, all occurrences of their maximum m are consecutive. Let $\mathfrak{A}'_{n,m} = \{\sigma \in \mathfrak{A}_{n,m} \ : \ \sigma_n = m\}$ be the subset of $\mathfrak{A}_{n,m}$ of sequences whose last value is their maximum. Let $\mathfrak{B}_{n,k} = \coprod_{\ell\in[0,n]} \mathcal{W}_{\ell,k}(101, 120)$ be the set of $\{101, 120\}$-avoiding words of length at most n over the alphabet $[0, k-1]$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, $\mathfrak{a}'_{n,m} = |\mathfrak{A}'_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.19. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}'_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j} + (\mathfrak{a}_{p-1,j} - \mathfrak{a}'_{p-1,j}) \cdot \mathfrak{b}_{n-p,m-j-1},$$

$$\mathfrak{a}'_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{n-1} \mathfrak{a}_{p-1,j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$ and $\gamma \in [j, m]^{n-p}$. Let $\ell = \mathbf{lastmax}(\sigma) \in [p, n]$ be the position of the last value m in σ. Since σ avoids 101, all occurrences of the maximum m are consecutive, hence their positions are $[p, \ell]$. Let $\gamma' = (\sigma_i)_{i\in[\ell+1,n]}$ be the word obtained by removing all letters m from γ.

- If $\alpha \in \mathfrak{A}'_{p-1,j}$, then γ' can be any $\{101, 120\}$-avoiding word of length $n - \ell \in [0, n-p]$ over the alphabet $[j, m-1]$. Subtracting j from all letters in γ' yields any word in the set $\mathfrak{B}_{n-p,m-j}$.

- If $\alpha \in \mathfrak{A}_{p-1,j} \backslash \mathfrak{A}'_{p-1,j}$, then $\sigma_{p-1} < j$. The subsequence (j, σ_{p-1}) is an occurrence of the pattern 10, which implies that the value j does not appear in γ' (otherwise σ would contain 101). In that case, γ' is a word over the alphabet $[j+1, m-1]$, and the number of possible words γ' is $\mathfrak{b}_{n-p,m-j-1}$.

Finally, if $\sigma \in \mathfrak{A}'_{n,m}$ then $\gamma = (m)^{n-p}$ and α can be any sequence in $\mathfrak{A}_{p-1,j}$. This yields the second equation. □

It remains to count the words of $\mathfrak{B}_{n,k}$. Let $F(x,y) = \sum_{n,k\geq 0} |\mathcal{W}_{n,k}(101, 120)| x^n y^k$ be the ordinary generating function of $\{101, 120\}$-avoiding words. The enumeration of $\{101, 120\}$-avoiding words was solved in [83], with the following expression for F:

$$F(x,y) = \frac{(1-x)^2 - (1-2x)y - \sqrt{(1-x)^4 - 2(1-x)^2 y + (1-4x^2+4x^3)y^2}}{2xy(1-y)}.$$

Since $\mathcal{W}_{n,k}(101, 120) = \mathfrak{B}_{n,k} \backslash \mathfrak{B}_{n-1,k}$, the generating function of $\mathfrak{B}_{n,k}$ is

$$\sum_{n,k\geq 0} \mathfrak{b}_{n,k}\, x^n y^k = \frac{F(x,y)}{1-x}.$$

1.1.9 Avoiding {000, 120}

Given a sequence $\sigma \in \mathbb{N}^n$ and an integer $v \in \mathbb{N}$, let $\mathbf{occ}(\sigma, v) = |\{i \in [1,n] \ : \ \sigma_i = v\}|$ be the number of occurrences of the value v in σ.

Let $\mathfrak{A}_{n,m,r} = \{\sigma \in \mathcal{I}_n(000, 120) \ : \ m = \mathbf{max}(\sigma), r = \mathbf{occ}(\sigma, m)\}$ be the set of $\{000, 120\}$-avoiding inversion sequences of size n, maximum m, and r occurrences of m. Let $\mathfrak{B}_{n,k} = \mathcal{W}_{n,k}(000, 120)$ be the set of $\{000, 120\}$-avoiding words of length n over the alphabet $[0, k-1]$, and $\mathfrak{C}_{n,k} = \{\omega \in \mathfrak{B}_{n,k} \ : \ \mathbf{occ}(\omega, k-1) = 1\}$ be the subset of words of $\mathfrak{B}_{n,k}$ in which the largest letter $k-1$ appears exactly once. We also consider the subsets of $\mathfrak{B}_{n,k}$ and $\mathfrak{C}_{n,k}$ of words in which the letter 0 appears less than twice: let $\mathfrak{B}_{n,k,\not 00} = \{\omega \in \mathfrak{B}_{n,k} \ : \ \mathbf{occ}(\omega, 0) < 2\}$ and $\mathfrak{C}_{n,k,\not 00} = \{\omega \in \mathfrak{C}_{n,k} \ : \ \mathbf{occ}(\omega, 0) < 2\}$. Let $\mathfrak{a}_{n,m,r} = |\mathfrak{A}_{n,m,r}|$, $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$, $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$, $\mathfrak{b}_{n,k,\not 00} = |\mathfrak{B}_{n,k,\not 00}|$, and $\mathfrak{c}_{n,k,\not 00} = |\mathfrak{C}_{n,k,\not 00}|$.

Theorem 1.1.20. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m,1} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,1} \cdot \mathfrak{b}_{n-p,m-j,\not 00} + \mathfrak{a}_{p-1,j,2} \cdot \mathfrak{b}_{n-p,m-j-1},$$

$$\mathfrak{a}_{n,m,2} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,1} \cdot \mathfrak{c}_{n-p,m-j+1,\not 00} + \mathfrak{a}_{p-1,j,2} \cdot \mathfrak{c}_{n-p,m-j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$ and $\gamma \in [j,m]^{n-p}$ by Proposition 1.1.3.

- If $\sigma \in \mathfrak{A}_{n,m,1}$, then the letter m cannot appear in γ, so $\gamma \in [j, m-1]^{n-p}$.
  - If $\alpha \in \mathfrak{A}_{p-1,j,1}$ then the letter j appears at most once in γ (to avoid the pattern 000). By subtracting j from every letter in γ, we obtain a word in the set $\mathfrak{B}_{n-p,m-j,\not 00}$.
  - If $\alpha \in \mathfrak{A}_{p-1,j,2}$ then the letter j cannot appear in γ. By subtracting $j+1$ from every letter in γ, we obtain a word in the set $\mathfrak{B}_{n-p,m-j-1}$.

- If $\sigma \in \mathfrak{A}_{n,m,2}$, then the letter m appears exactly once in γ.
  - If $\alpha \in \mathfrak{A}_{p-1,j,1}$ then subtracting j from every letter in γ yields a word in the set $\mathfrak{C}_{n-p,m-j+1,\not{00}}$.
  - If $\alpha \in \mathfrak{A}_{p-1,j,2}$ then subtracting $j+1$ from every letter in γ yields a word in the set $\mathfrak{C}_{n-p,m-j}$.

We easily observe that this construction is a bijection. □

For $a, b \in \{1,2\}$, let $\mathfrak{D}_{n,k,a,b} = \{\omega \in \overline{\mathcal{W}}_{n,k}(000,120) \ : \ a = \mathbf{occ}(\omega,0),\, b = \mathbf{occ}(\omega,k-1)\}$ be the set of $\{000,120\}$-avoiding words of length n which contain every letter of the alphabet $[0,k-1]$ and in which the letter 0 appears a times and the letter $k-1$ appears b times. Let $\mathfrak{d}_{n,k,a,b} = |\mathfrak{D}_{n,k,a,b}|$.

Proposition 1.1.21. *For all $n, k \geq 1$,*

$$\mathfrak{b}_{n,k} = \sum_{d=1}^{\mathbf{min}(n,k)} \binom{k}{d} \sum_{a,b\in\{1,2\}} \mathfrak{d}_{n,d,a,b},$$

$$\mathfrak{c}_{n,k} = \sum_{d=1}^{\mathbf{min}(n,k)} \binom{k-1}{d-1} \sum_{a\in\{1,2\}} \mathfrak{d}_{n,d,a,1},$$

$$\mathfrak{b}_{n,k,\not{00}} = \sum_{d=1}^{\mathbf{min}(n,k)} \left(\binom{k-1}{d-1} \sum_{b\in\{1,2\}} \mathfrak{d}_{n,d,1,b} + \binom{k-1}{d} \sum_{a,b\in\{1,2\}} \mathfrak{d}_{n,d,a,b} \right),$$

$$\mathfrak{c}_{n,k,\not{00}} = \sum_{d=1}^{\mathbf{min}(n,k)} \left(\binom{k-2}{d-2} \mathfrak{d}_{n,d,1,1} + \binom{k-2}{d-1} \sum_{a\in\{1,2\}} \mathfrak{d}_{n,d,a,1} \right).$$

Proof. Let $F_{d,k}$ be the set of increasing functions from $[0,d-1]$ to $[0,k-1]$. The four identities above can be proven using the same construction as in the proof of Proposition 1.0.1, observing that

1. $\binom{k}{d} = |F_{d,k}|$,
2. $\binom{k-1}{d-1} = |\{\varphi \in F_{d,k} \ : \ \varphi(d-1) = k-1\}|$,
3. $\binom{k-1}{d-1} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) = 0\}|$, and $\binom{k-1}{d} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) > 0\}|$,
4. $\binom{k-2}{d-2} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) = 0,\, \varphi(d-1) = k-1\}|$, and $\binom{k-2}{d-1} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) > 0,\, \varphi(d-1) = k-1\}|$. □

Let $\mathfrak{E}_{n,k} = \{\lambda \in \{1,2\}^k \ : \ n = \sum_{i=1}^k \lambda_i\}$ be the set of compositions of n into k parts of size 1 or 2. Equivalently, $\mathfrak{E}_{n,k}$ is the set of words of length k over the alphabet $\{1,2\}$ which contain exactly $2k-n$ occurrences of the letter 1 and $n-k$ occurrences of the letter 2. Observe that $\mathfrak{E}_{n,k}$ is empty if $n < k$ or $n > 2k$.

Remark 1.1.22. In the notation of Section 1.1.2,

$$\mathfrak{D}_{n,k,a,b} = \coprod_{\substack{\lambda\in\mathfrak{E}_{n,k} \\ (\lambda_1,\lambda_k)=(a,b)}} \mathcal{W}_\lambda(120).$$

Let $\mathfrak{F}_{n,k} = \mathcal{W}_{(2)^{n-k}\cdot(1)^{2k-n}}(120)$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$, which contain exactly 2 occurrences of each letter in $[0, n-k-1]$ and 1 occurrence of each letter in $[n-k, k-1]$, and let $\mathfrak{f}_{n,k} = |\mathfrak{F}_{n,k}|$.

Proposition 1.1.23. *For all $k \geq 2, n \in [k, 2k]$ and $a, b \in \{1, 2\}$,*

$$\mathfrak{d}_{n,k,a,b} = \binom{k-2}{n-k+2-a-b}\mathfrak{f}_{n,k}$$

Proof. For all $k \geq 2, n \in [k, 2k]$ and $a, b \in \{1, 2\}$, the set of sequences $\lambda \in \mathfrak{E}_{n,k}$ such that $\lambda_1 = a$ and $\lambda_k = b$ is the set of sequences of the form $a \cdot \lambda' \cdot b$ where λ' is a word of length $k-2$ over the alphabet $\{1, 2\}$ which contains exactly $n-k+2-a-b$ occurrences of the letter 2. Hence, there are $\binom{k-2}{n-k+2-a-b}$ such sequences.

By Theorem 1.1.5, the number of words $|\mathcal{W}_\lambda(120)|$ is independent of the choice of $\lambda \in \mathfrak{E}_{n,k}$, and counted by $\mathfrak{f}_{n,k}$. Remark 1.1.22 concludes the proof. □

Lemma 1.1.24. *The following recurrence relation holds for all $k \geq 1, n \in [k+1, 2k]$:*

$$\mathfrak{f}_{n,k} = \mathfrak{f}_{n,k+1} - \mathfrak{f}_{n-1,k},$$

with initial conditions $\mathfrak{f}_{n,n} = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$.

Proof. Let $k \geq 2, n \in [k, 2k-2]$, and $\omega \in \mathfrak{F}_{n,k}$. In particular, ω contains a single letter $k-1$ and a single letter $k-2$.

- If the letter $k-2$ is on the left of $k-1$, then $k-1$ is the last letter of ω (because ω avoids the pattern 120). In that case, removing the letter $k-1$ yields a word in $\mathfrak{F}_{n-1,k-1}$, and this is a bijection.

- Otherwise, the letter $k-2$ is on the right of $k-1$. In that case, replacing the letter $k-1$ by $k-2$ yields a word in $\mathcal{W}_{(2)^{n-k}\cdot(1)^{2k-n-2}\cdot(2)}(120)$, and this is a bijection. By Theorem 1.1.5, this set is equinumerous with $\mathcal{W}_{(2)^{n-k+1}\cdot(1)^{2k-n-2}}(120) = \mathfrak{F}_{n,k-1}$.

This shows that for all $k \geq 2$ and $n \in [k, 2k-2]$, $\mathfrak{f}_{n,k} = \mathfrak{f}_{n,k-1} + \mathfrak{f}_{n-1,k-1}$, which is equivalent to the recurrence relation of the lemma.

For all $n \geq 0$, $\mathfrak{F}_{n,n}$ is trivially in bijection with the set of 120-avoiding permutations of size n, which is known to be counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. □

The numbers $\mathfrak{f}_{n,k}$ can be found in entry A059346 of [96].

1.1.10 The pattern 010

We denote by $\begin{bmatrix} n \\ k \end{bmatrix}$ the unsigned Stirling numbers of the first kind, which count the number of permutations of size n with k cycles (among other combinatorial interpretations, cf. entry A132393 of [96]).

Let $\mathfrak{A}_{n,m,d} = \{\sigma \in \mathcal{I}_n(010) \ : \ m = \mathbf{max}(\sigma), d = \mathbf{dist}(\sigma)\}$ be the set of 010-avoiding inversion sequences of size n, maximum m, and having exactly d distinct values. Let also $\mathfrak{B}_{n,k} = \{\omega \in \overline{\mathcal{W}}_{n,k}(010) \ : \ \omega_1 = k-1\}$ be the set of 010-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$ and begin with their largest letter. Let $\mathfrak{a}_{n,m,d} = |\mathfrak{A}_{n,m,d}|$ and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.25. *For all* $2 \leq d \leq m+1 \leq n$,

$$\mathfrak{a}_{n,m,d} = \sum_{i=0}^{d-1} \binom{m-i}{d-i-1} \sum_{p=m+1}^{n} \mathfrak{b}_{n-p+1,d-i} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m,d}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, and $\beta = (\sigma_i)_{i\in[p,n]}$.

Since σ avoids the pattern 010, α and β avoid 010, and $\mathbf{Vals}(\alpha)\cap\mathbf{Vals}(\beta) = \emptyset$ by Proposition 1.1.4. In particular, $\alpha \in \mathfrak{A}_{p-1,j,i}$ for some $j < m$ and $i < d$, and β is a 010-avoiding word of length $n-p+1$, which contains exactly $d-i$ distinct values chosen from the remaining $m+1-i$ (that is, all values in $[0,m]$ except for the i values in α), and such that $\beta_1 = m$. Since m is always in β, there are $\binom{m-i}{d-i-1}$ possible choices for the set $\mathbf{Vals}(\beta)$. Once the values of β are chosen, there are $\mathfrak{b}_{n-p+1,d-i}$ ways to arrange them into a word avoiding 010 and beginning with its largest letter. □

Lemma 1.1.26. *For all* $n, k \geq 1$,

$$\mathfrak{b}_{n,k} = \begin{bmatrix} n \\ n+1-k \end{bmatrix}.$$

Proof. Let $n \geq k \geq 2$, and $\omega \in \mathfrak{B}_{n,k}$. Since ω avoids the pattern 010, all letters 0 in ω are consecutive.

- If ω contains several letters 0, then removing one of them yields a word $\omega' \in \mathfrak{B}_{n-1,k}$, and this is clearly a bijection.
- If ω contains a single letter 0, then removing it and subtracting 1 from all other letters yields a word $\omega' \in \mathfrak{B}_{n-1,k-1}$. Since $\omega_1 = k-1 > 0$, there are $n-1$ possible positions for the letter 0 in ω, so exactly $n-1$ words $\omega \in \mathfrak{B}_{n,k}$ yield the same ω'.

Hence the following recurrence relation holds for all $n \geq k \geq 2$:

$$\mathfrak{b}_{n,k} = \mathfrak{b}_{n-1,k} + (n-1)\mathfrak{b}_{n-1,k-1}.$$

This recurrence relation is also satisfied by the Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ n+1-k \end{smallmatrix}\right]$, and we can easily verify that initial conditions also match. □

1.1.11 Avoiding {000, 010}

Let $\mathfrak{A}_{n,m,d} = \{\sigma \in \mathcal{I}_n(000,010) \,:\, m = \mathbf{max}(\sigma),\, d = \mathbf{dist}(\sigma)\}$ be the set of $\{000,010\}$-avoiding inversion sequences of size n, maximum m, and having d distinct values. Let $\mathfrak{B}_{n,k} = \{\omega \in \overline{\mathcal{W}}_{n,k}(000,010) \,:\, \omega_1 = k-1\}$ be the set of $\{000,010\}$-avoiding words of length n which contain all letters of the alphabet $[0,k-1]$ and begin with their largest letter. Let $\mathfrak{a}_{n,m,d} = |\mathfrak{A}_{n,m,d}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$. Since the sequences of $\mathfrak{A}_{n,m,d}$ and the words of $\mathfrak{B}_{n,k}$ avoid the pattern 000, we have $\mathfrak{a}_{n,m,d} = 0$ if $n > 2d$, and $\mathfrak{b}_{n,k} = 0$ if $n > 2k$.

Theorem 1.1.27. *For all* $2 \leq d \leq m+1 \leq n$,

$$\mathfrak{a}_{n,m,d} = \sum_{i=0}^{d-1} \binom{m-i}{d-i-1} \sum_{p=m+1}^{n} \mathfrak{b}_{n-p+1,d-i} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i}.$$

Proof. Identical to the proof of Theorem 1.1.25: the pattern 000 could not spread over α and β, since the avoidance of 010 already implies that α and β do not share any values. □

Lemma 1.1.28. *For all $n, k \geq 2$,*

$$\mathfrak{b}_{n,k} = (n-1)\mathfrak{b}_{n-1,k-1} + (n-2)\mathfrak{b}_{n-2,k-1}.$$

Proof. Let $n \geq k \geq 2$, and $\omega \in \mathfrak{B}_{n,k}$. Since ω avoids 010, all letters 0 in ω are consecutive. Since ω avoids 000, ω has at most two letters 0.

- If ω contains a single letter 0, then removing it and subtracting 1 from all other letters yields a word $\omega' \in \mathfrak{B}_{n-1,k-1}$. Since $\omega_1 = k - 1 > 0$, there are $n-1$ possible positions for the letter 0 in ω, so exactly $n-1$ words $\omega \in \mathfrak{B}_{n,k}$ yield the same ω'.

- If ω contains two letters 0, then removing them and subtracting 1 from all other letters yields a word $\omega' \in \mathfrak{B}_{n-2,k-1}$. There are $n-2$ possible positions for the two consecutive letters 0 in ω, so exactly $n-2$ words $\omega \in \mathfrak{B}_{n,k}$ yield the same ω'. □

1.1.12 Forbidden values

In the remainder of Section 1.1, if α is a sequence of nonnegative integers avoiding a set of patterns P, we say that a value $v \in \{0, \ldots, \mathbf{max}(\alpha)\}$ is *forbidden* by P in α if $\alpha \cdot mv$ contains a pattern in P when $m > \mathbf{max}(\alpha)$. We denote by $\mathbf{Forb}(\alpha, P)$ the set of values forbidden by P in α, or simply $\mathbf{Forb}(\alpha)$ when there is no ambiguity.

In our decomposition around the first maximum of an inversion sequence $\sigma = \alpha \cdot m \cdot \gamma$, if γ contains a value in $\mathbf{Forb}(\alpha, \tau)$, then σ contains τ. This means that in order for σ to avoid τ, γ must be a word over the alphabet $[0, m]\backslash\mathbf{Forb}(\alpha)$. This is in fact a weaker version of condition 3 from Remark 1.1.2, since it only ensures that σ does not contain any occurrence of τ whose last entry only is in γ.

Note that Propositions 1.1.3 and 1.1.4 can be expressed in terms of forbidden values:

$$\mathbf{Forb}(\alpha, 120) = [0, \mathbf{max}(\alpha) - 1], \qquad \mathbf{Forb}(\alpha, 010) = \mathbf{Vals}(\alpha),$$

and these identities hold for any nonnegative integer sequence α. In particular, if α is an inversion sequence such that $\mathbf{max}(\alpha) > 0$, then $|\mathbf{Forb}(\alpha, 010)| \geq 2$.

Earlier, we refined the enumeration of pattern-avoiding inversion sequences σ according to one or two parameters (in addition to their size). The first one was the maximum m of σ, and is required for our decomposition around the first maximum (so that the maximum of the left part α is less than m). For every pair of pattern which contained 120, the number of forbidden values was redundant with the maximum of σ, and therefore unnecessary. In the previous two cases (the patterns 010 and {000, 010}), the second parameter counting the number of distinct values d of σ was in fact the number of forbidden values $|\mathbf{Forb}(\sigma, 010)|$.

For each pair of patterns P which follows, there is no simple equivalent description of $|\mathbf{Forb}(\sigma, P)|$, and introducing this parameter allows us to solve the enumeration of P-avoiding inversion sequences.

1.1.13 Avoiding {010, 201} or {010, 210}

A bijection between $\mathcal{I}(010, 201)$ and $\mathcal{I}(010, 210)$ was established in [113]. In this section we work with the pair of patterns $\{010, 210\}$, although our construction can also be applied to inversion sequences avoiding the pair $\{010, 201\}$, resulting in the same equations.

Remark 1.1.29. Let σ be a $\{010, 210\}$-avoiding sequence of nonnegative integers. Let q be the largest value of σ such that a larger value appears to its left, or $q = 0$ if there is no such value (i.e. if σ is nondecreasing). Then we have $\mathbf{Forb}(\sigma, 210) = [0, q-1]$. Recalling also that $\mathbf{Forb}(\sigma, 010) = \mathbf{Vals}(\sigma)$, we find $\mathbf{Forb}(\sigma, \{010, 210\}) = [0, q-1] \sqcup \{i \in \mathbf{Vals}(\sigma) \ : \ i \geq q\}$.

Let $\mathfrak{A}_{n,m,f} = \{\sigma \in \mathcal{I}_n(010, 210) \ : \ m = \mathbf{max}(\sigma),\ f = |\mathbf{Forb}(\sigma)|\}$ be the set of $\{010, 210\}$-avoiding inversion sequences of size n, maximum m, and having f forbidden values. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k+1}(010) \ : \ \omega_i < \omega_j < k \implies i < j \ \text{ and } \ k-1 \in \mathbf{Vals}(\omega)\}$ if $k > 0$, or $\mathfrak{B}_{n,k} = \{(0)^n\}$ if $k = 0$, be the set of 010-avoiding words ω of length n over the alphabet $[0, k]$ such that the subword ω' obtained by removing all letters k from ω is nondecreasing, and $k-1$ is the maximum of ω'. Let $\mathfrak{a}_{n,m,f} = |\mathfrak{A}_{n,m,f}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 1.1.30. *For all* $2 \leq f \leq m+1 \leq n$,

$$\mathfrak{a}_{n,m,f} = \sum_{p=m+1}^{n} \sum_{i=0}^{f-1} \mathfrak{b}_{n-p,f-i-1} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m,f}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i \in [1,p-1]}$, and $\gamma = (\sigma_i)_{i \in [p+1,n]}$. Let γ' be the subsequence of γ obtained by removing all values m from γ. Since σ avoids the pattern 010, α and γ avoid 010, and $\mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\gamma) = \emptyset$ by Proposition 1.1.4. Since σ avoids the pattern 210, α avoids 210, and γ' is nondecreasing.

The subsequence α is in $\mathfrak{A}_{p-1,j,i}$ for some $j < m$ and $i \leq f$. Notice that we actually have $i < f$ since m is a forbidden value for σ (because of the avoidance of 010), but not for α (since $m > \mathbf{max}(\alpha)$). The subsequence γ is a 010-avoiding word of length $n-p$ over the alphabet $\Sigma = [0, m] \backslash \mathbf{Forb}(\alpha)$ of size $m+1-i$, and such that γ' is nondecreasing.

The maximum of γ' is the largest letter of $m \cdot \gamma$ such that a larger letter appears to its left, and m is the only value in $m \cdot \gamma$ which is greater than $\mathbf{max}(\gamma')$. Hence, by Remark 1.1.29, $\mathbf{Forb}(\sigma) = \mathbf{Forb}(\alpha) \sqcup \{\ell \in \Sigma \ : \ \ell \leq \mathbf{max}(\gamma')\} \sqcup \{m\}$ (this still holds if γ' is empty, i.e. $\mathbf{max}(\gamma') = -1$). Since $|\{\ell \in \Sigma \ : \ \ell \leq \mathbf{max}(\gamma')\}| = |\mathbf{Forb}(\sigma)| - |\mathbf{Forb}(\alpha)| - |\{m\}| = f-i-1$, replacing the letters $\{\ell \in \Sigma \ : \ \ell \leq \mathbf{max}(\gamma')\}$ by $[0, f-i-2]$ and the letter m by $f-i-1$ yields a bijection between the words γ and the words of $\mathfrak{B}_{n-p,f-i-1}$. □

Let $\mathfrak{C}_{n,k}$ be the set of 010-avoiding words ω of length n over the alphabet $[0, k-1] \sqcup \{\infty\}$ (where ∞ is the largest letter) such that the subword ω' defined by removing all letters ∞ from ω is nondecreasing, and $k-1$ is the maximum of ω'. Clearly, taking any word $\omega \in \mathfrak{C}_{n,k}$ and replacing each letter ∞ in ω by the letter k yields a word in $\mathfrak{B}_{n,k}$, and this is a bijection. It is more convenient for the proof of the following lemma to count the words of $\mathfrak{C}_{n,k}$ rather than those of $\mathfrak{B}_{n,k}$.

Let $\mathfrak{C}^{(1)}_{n,k} = \{\omega \in \mathfrak{C}_{n,k} \mid \omega_n = \infty\}$, and $\mathfrak{C}^{(2)}_{n,k} = \{\omega \in \mathfrak{C}_{n,k} \mid \omega_n \neq \infty\}$, so that $\mathfrak{C}_{n,k} = \mathfrak{C}^{(1)}_{n,k} \sqcup \mathfrak{C}^{(2)}_{n,k}$. Let $\mathfrak{c}^{(1)}_{n,k} = |\mathfrak{C}^{(1)}_{n,k}|$, and $\mathfrak{c}^{(2)}_{n,k} = |\mathfrak{C}^{(2)}_{n,k}|$. In particular, $\mathfrak{b}_{n,k} = \mathfrak{c}^{(1)}_{n,k} + \mathfrak{c}^{(2)}_{n,k}$.

Lemma 1.1.31. *For all* $n \geq 2, k \geq 1$,

$$\mathfrak{b}_{n,k} = \mathfrak{b}_{n,k-1} + 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} - \mathfrak{b}_{n-1,k-1} + \mathfrak{b}_{n-2,k-1}.$$

Proof. Let $n \geq 1, k \geq 0$, and $\omega \in \mathfrak{C}_{n,k}$.

- If $\omega \in \mathfrak{C}^{(1)}_{n,k}$, then $\omega_n = \infty$. Removing ω_n yields a word in $\mathfrak{C}_{n-1,k}$.
- If $\omega \in \mathfrak{C}^{(2)}_{n,k}$, then by the nondecreasing property, $\omega_n = k-1$.

– If $\omega_{n-1} = k-1$, then removing ω_n yields a word in $\mathfrak{C}^{(2)}_{n-1,k}$.
– If $\omega_{n-1} = \infty$, then the avoidance of the pattern 010 ensures ω cannot contain another letter $k-1$, therefore removing ω_n yields a word in $\mathfrak{C}^{(1)}_{n-1,i}$ for some $i < k$.
– Otherwise, $\omega_{n-1} = i$ for some $i < k$, and removing ω_n yields a word in $\mathfrak{C}^{(2)}_{n-1,i}$.

The maps described above are all bijections, hence for all $n \geq 1, k \geq 0$,

$$\mathfrak{c}^{(1)}_{n,k} = \mathfrak{b}_{n-1,k},$$

and for all $n \geq 2, k \geq 0$,

$$\begin{aligned}\mathfrak{c}^{(2)}_{n,k} &= \mathfrak{c}^{(2)}_{n-1,k} + \sum_{i=0}^{k-1} \mathfrak{c}^{(1)}_{n-1,i} + \mathfrak{c}^{(2)}_{n-1,i} \\ &= \mathfrak{b}_{n-1,k} - \mathfrak{c}^{(1)}_{n-1,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i} \\ &= \mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i}.\end{aligned}$$

By summing $\mathfrak{c}^{(1)}_{n,k}$ and $\mathfrak{c}^{(2)}_{n,k}$, we have for all $n \geq 2, k \geq 0$,

$$\mathfrak{b}_{n,k} = 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i}.$$

We conclude by telescoping the sum over i. For all $n \geq 2, k \geq 1$,

$$\begin{aligned}\mathfrak{b}_{n,k} - \mathfrak{b}_{n,k-1} &= 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i} - (2\mathfrak{b}_{n-1,k-1} - \mathfrak{b}_{n-2,k-1} + \sum_{i=0}^{k-2} \mathfrak{b}_{n-1,i}) \\ &= 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} - \mathfrak{b}_{n-1,k-1} + \mathfrak{b}_{n-2,k-1}.\end{aligned}$$

□

1.1.14 Avoiding {010, 110}

Remark 1.1.32. Let σ be a $\{010, 110\}$-avoiding sequence of nonnegative integers. Let q be the largest repeated value in σ, or $q = 0$ if there is no such value. Then $\mathbf{Forb}(\sigma, 110) = [0, q-1]$. Recalling that $\mathbf{Forb}(\sigma, 010) = \mathbf{Vals}(\sigma)$, we find $\mathbf{Forb}(\sigma, \{010, 110\}) = [0, q-1] \sqcup \{i \in \mathbf{Vals}(\sigma) : i \geq q\}$.

Let $\mathfrak{A}_{n,m,f} = \{\sigma \in \mathcal{I}_n(010, 110) : m = \mathbf{max}(\sigma), f = |\mathbf{Forb}(\sigma)|\}$ be the set of $\{010, 110\}$-avoiding inversion sequences of size n, maximum m, and having f forbidden values. Let $\mathfrak{B}_{n,k,f} = \{\omega \in \mathcal{W}_{n,k}(010, 110) : f = |\mathbf{Forb}(\omega)|\}$ be the set of $\{010, 110\}$-avoiding words of length n over the alphabet $[0, k-1]$ having f forbidden values. Let $\mathfrak{C}_{n,k} = \coprod_{\ell \in [0,n]} \mathcal{W}_{\ell,k}(010, 110)$ be the set of $\{010, 110\}$-avoiding words of length at most n over the alphabet $[0, k-1]$. Let $\mathfrak{a}_{n,m,f} = |\mathfrak{A}_{n,m,f}|$, $\mathfrak{b}_{n,k,f} = |\mathfrak{B}_{n,k,f}|$, and $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$.

Theorem 1.1.33. *For all* $2 \leq f \leq m+1 \leq n$,

$$\mathfrak{a}_{n,m,f} = \sum_{p=m+1}^{n} \sum_{i=0}^{f-1} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i} \cdot (\mathfrak{b}_{n-p,m-i,f-i-1} + \delta_{f,m+1} \cdot \mathfrak{c}_{n-p-1,m-i}).$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m,f}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, and $\gamma = (\sigma_i)_{i\in[p+1,n]}$. Since σ avoids the pattern 010, α and γ avoid 010, and $\mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\gamma) = \emptyset$ by Proposition 1.1.4. Since σ avoids the pattern 110, α and γ avoid 110, and all letters of γ are greater than any repeated letter of α.

The subsequence α is in $\mathfrak{A}_{p-1,j,i}$ for some $j < m$ and $i < f$ (for the same reason as in the proof of Theorem 1.1.30). The subsequence γ is a $\{010, 110\}$-avoiding word of length $n-p$ over the alphabet $\Sigma = [0, m]\backslash\mathbf{Forb}(\alpha)$ of size $m+1-i$. Additionally, if a letter m appears in γ, then all letters to its right are also m; otherwise σ would contain the pattern 110. We distinguish two cases for γ.

- If γ does not contain the letter m, then γ is a $\{010, 110\}$-avoiding word of length $n-p$ over an alphabet of size $m-i$. Let q be the largest repeated letter in γ, or $q = 0$ if there is no such letter. From Remark 1.1.32, we observe that the forbidden values of σ are $\mathbf{Forb}(\sigma) = \mathbf{Forb}(\alpha) \sqcup \{\ell \in \Sigma \ : \ \ell < q\} \sqcup \{\ell \in \mathbf{Vals}(\gamma) \ : \ \ell \geq q\} \sqcup \{m\}$. In particular, $|\{\ell \in \Sigma \ : \ \ell < q\} \sqcup \{\ell \in \mathbf{Vals}(\gamma) \ : \ \ell \geq q\}| = f - i - 1$. Replacing the letters of Σ by $[0, m-i]$ (while preserving their order) yields a bijection between the words γ and the words of $\mathfrak{B}_{n-p,m-i,f-i-1}$.
- If γ contains the letter m, then γ can be written as $\gamma' \cdot (m)^k$, where γ' does not contain the letter m, and $k > 0$. The number of such words γ' is $\mathfrak{c}_{n-p-1,m-i}$. In this case, m is the largest repeated letter in σ as well as the maximum of σ. This occurs if and only if $f = m+1$ (i.e. all values $[0, m]$ are forbidden by σ). □

By definition,

$$\mathfrak{C}_{n,k} = \coprod_{\ell=0}^{n} \coprod_{f=0}^{k} \mathfrak{B}_{\ell,k,f}, \qquad \text{hence} \qquad \mathfrak{c}_{n,k} = \sum_{\ell=0}^{n} \sum_{f=0}^{k} \mathfrak{b}_{\ell,k,f}.$$

We count the words of $\mathfrak{B}_{n,k,f}$ in Lemma 1.1.35. In order to do so, we first construct the words of $\overline{\mathcal{W}}_{n,k}(010, 110)$ by inserting their letters in increasing order from 0 to $k-1$ (all such letters necessarily appearing), and inserting repeats of a letter from left to right. For all $\omega \in \overline{\mathcal{W}}_{n,k}(010, 110)$, let $\mathbf{Sites}(\omega) = \{i \in [1, n+1] \ : \ \mathbf{Vals}((\omega_j)_{j<i}) \cap \mathbf{Vals}((\omega_j)_{j\geq i}) = \emptyset\}$ be the set of positions (called *active sites*) where the letter k may be inserted in ω without creating an occurrence of the pattern 010. In particular, 1 and $n+1$ are always active sites of ω. Note also that inserting the letter k in ω cannot create an occurrence of the pattern 110: the letter k cannot take the role of the letter 0 in the pattern 110 since it is greater than all letters of ω, and it cannot take the role of the letter 1 since it only appears once.

Let $\mathfrak{D}_{n,k,s} = \{\omega \in \overline{\mathcal{W}}_{n,k}(010, 110) \ : \ s = |\mathbf{Sites}(\omega)|\}$ be the set of $\{010, 110\}$-avoiding words of length n containing all letters of the alphabet $[0, k-1]$ and having s active sites. Let $\mathfrak{D}^{(1)}_{n,k,s}$ be the subset of $\mathfrak{D}_{n,k,s}$ of words containing exactly one letter $k-1$, and $\mathfrak{D}^{(2)}_{n,k,s}$ be the subset of remaining words (i.e. words containing at least two letters $k-1$ if $k \geq 1$, or the empty word if $(n, k, s) = (0, 0, 1)$), so that $\mathfrak{D}_{n,k,s} = \mathfrak{D}^{(1)}_{n,k,s} \sqcup \mathfrak{D}^{(2)}_{n,k,s}$. Let $\mathfrak{d}_{n,k,s} = |\mathfrak{D}_{n,k,s}|$, $\mathfrak{d}^{(1)}_{n,k,s} = |\mathfrak{D}^{(1)}_{n,k,s}|$, and $\mathfrak{d}^{(2)}_{n,k,s} = |\mathfrak{D}^{(2)}_{n,k,s}|$.

Lemma 1.1.34. *For all $n, k, s \geq 1$,*

$$\mathfrak{d}^{(1)}_{n,k,s} = (s-1)\mathfrak{d}_{n-1,k-1,s-1}.$$

For all $n, s \geq 2, k \geq 1$,

$$\mathfrak{d}^{(2)}_{n,k,s} = \mathfrak{d}^{(2)}_{n,k,s+1} + \mathfrak{d}^{(2)}_{n-1,k,s} - \mathfrak{d}^{(2)}_{n-1,k,s+1} + \mathfrak{d}_{n-2,k-1,s-1}.$$

Proof. Let $n, k, s \geq 0$, $\omega \in \mathfrak{D}_{n,k,s}$, and let $(p_1, \ldots, p_s)$ be the values of $\mathbf{Sites}(\omega)$ in increasing order (in particular, $p_1 = 1$ and $p_s = n+1$). We consider three different ways in which the word ω can "grow".

1. Inserting a letter k. Let $i \in [1, s]$, and let ω' be the word obtained by inserting the letter k at position p_i in ω. Then $\mathbf{Sites}(\omega') = \{p_j \ : \ j \in [1, i]\} \sqcup \{p_j + 1 \ : \ j \in [i, s]\}$, so $\omega' \in \mathfrak{D}^{(1)}_{n+1,k+1,s+1}$.
2. Inserting two occurrences of the letter k. This is similar to the previous case, since the rightmost letter k can only be inserted at the end of the word in order to avoid the pattern 110. Let $i \in [1, s]$, and let ω' be the word obtained by inserting one letter k at position p_i in ω, and one letter k at the end of the resulting word (at position $n+2$). Then $\mathbf{Sites}(\omega') = \{p_j \ : \ j \in [1, i]\} \sqcup \{n+3\}$, so $\omega' \in \mathfrak{D}^{(2)}_{n+2,k+1,i+1}$.
3. Inserting a repeat of letter $k-1$, to the right of the rightmost letter $k-1$, and only if ω already contains at least two occurrences of the letter $k-1$ (i.e. $\omega \in \mathfrak{D}^{(2)}_{n,k,s}$ and $k > 0$). In that case, every letter $k-1$ in ω except (possibly) the leftmost $k-1$ must be in a single factor at the end of ω, in order to avoid 110. Let ω' be the word obtained by inserting the letter $k-1$ at position $n+1$ in ω. Then $\mathbf{Sites}(\omega') = \big(\mathbf{Sites}(\omega) \backslash \{n+1\}\big) \sqcup \{n+2\}$, so $\omega' \in \mathfrak{D}^{(2)}_{n+1,k,s}$.

It can be seen that any word in a set $\mathfrak{D}_{n,k,s}$ for some $n, k, s \geq 0$ can be obtained in exactly one way from the construction described above, starting from the empty word ε. In other words, we described a combinatorial generating tree for the class $\coprod_{n,k \geq 0} \overline{\mathcal{W}}_{n,k}(010, 110)$.

From item 1, we have for all $n, k, s \geq 1$,

$$\mathfrak{d}^{(1)}_{n,k,s} = (s-1)\mathfrak{d}_{n-1,k-1,s-1}.$$

From items 2 and 3, we have for all $n, s \geq 2, k \geq 1$,

$$\mathfrak{d}^{(2)}_{n,k,s} = \mathfrak{d}^{(2)}_{n-1,k,s} + \sum_{j=s-1}^{n-1} \mathfrak{d}_{n-2,k-1,j},$$

which may be rewritten

$$\mathfrak{d}^{(2)}_{n,k,s} - \mathfrak{d}^{(2)}_{n,k,s+1} = \mathfrak{d}^{(2)}_{n-1,k,s} - \mathfrak{d}^{(2)}_{n-1,k,s+1} + \mathfrak{d}_{n-2,k-1,s-1}. \qquad \square$$

For any nonnegative integer sequence σ, let $\mathbf{rep}(\sigma) = \mathbf{max}(v \in \mathbf{Vals}(\sigma) \ : \ \exists i \neq j, \ \sigma_i = \sigma_j = v)$ be the largest repeated value in σ, with the convention $\mathbf{rep}(\sigma) = -1$ if σ does not contain any repeated value. Let $\mathbf{top}(\sigma) = |\{i \in \mathbf{Vals}(\sigma) \ : \ i > \mathbf{rep}(\sigma)\}|$ be the number of values of σ which are greater than its largest repeated value (if no value is repeated, then $\mathbf{top}(\sigma)$ is the number of distinct values of σ, or equivalently the size of σ).

We call *unused letters* of a word $\omega \in \mathcal{W}_{n,k}$ the letters in the set $[0, k-1] \backslash \mathbf{Vals}(\omega)$. Note this definition relies not only on ω, but also on the alphabet considered (e.g. each word of $\mathcal{W}_{n,k}$ is also in $\mathcal{W}_{n,k+1}$, but has different unused letters).

Lemma 1.1.35. *For all $n, k, f \geq 0$,*

$$\mathfrak{b}_{n,k,f} = \sum_{t=0}^{f} \binom{t+k-f}{k-f} \sum_{b=0}^{f-t} \binom{f-t-1}{b} \sum_{s=1}^{n-t+1} \frac{(s+t-1)!}{(s-1)!} \mathfrak{d}^{(2)}_{n-t,f-b-t,s}.$$

Proof. Let $\omega \in \mathfrak{B}_{n,k,f}$. In particular, by Remark 1.1.32, $f = \mathbf{rep}(\omega) + \mathbf{top}(\omega) + 1$, and $k - f$ is the number of unused letters of ω greater than $\mathbf{rep}(\omega)$. Let $t = \mathbf{top}(\omega)$, $d = \mathbf{dist}(\omega)$, and $b = f - d$ be the number of unused letters in ω less than $\mathbf{rep}(\omega)$. To summarize, over the alphabet $[0, k-1]$, ω has:

- t letters greater than $\mathbf{rep}(\omega)$,
- $k - f$ unused letters greater than $\mathbf{rep}(\omega)$,
- b unused letters less than $\mathbf{rep}(\omega)$,
- $\mathbf{rep}(\omega) - b = f - t - 1 - b$ letters less than $\mathbf{rep}(\omega)$, if $\omega \neq \varepsilon$.

Replacing the d letters of $\mathbf{Vals}(\omega)$ by $[0, d-1]$ while preserving their order (by shifting the values so that there are no more unused letters) yields a word $\omega' \in \overline{\mathcal{W}}_{n,d}$. Further removing all letters greater than $\mathbf{rep}(\omega')$ from ω' yields a word $\omega'' \in \mathfrak{D}^{(2)}_{n-t,f-b-t,s}$ for some $s \in [1, n-t+1]$.

For any n, k, f, t, b, s, and $\omega'' \in \mathfrak{D}^{(2)}_{n-t,f-b-t,s}$, there are $\binom{t+k-f}{k-f}\binom{f-t-1}{b}\frac{(s+t-1)!}{(s-1)!}$ words $\omega \in \mathfrak{B}_{n,k,f}$ whose image under the above construction is ω''. Indeed, there are

- $\binom{t+k-f}{k-f}$ possible sets of unused letters greater than $\mathbf{rep}(\omega)$,
- $\binom{f-t-1}{b}$ possible sets of unused letters less than $\mathbf{rep}(\omega)$ (this still holds if $\omega = \varepsilon$),
- $\frac{(s+t-1)!}{(s-1)!}$ possible placements for the t letters greater than $\mathbf{rep}(\omega)$. This can be seen by inserting each of those t letters in increasing order: inserting a letter greater than the maximum in a word corresponds to item 1 in the proof of Lemma 1.1.34, so there are s possible positions for the smallest inserted letter, $s+1$ for the next letter, and so on. □

1.2 Shifted inversion sequences

1.2.1 Method

For all $n, s \in \mathbb{N}$, let $\mathcal{I}_n^s = \{\sigma \in \mathbb{N}^n \ : \ \sigma_i < i + s \quad \forall i \in [1, n]\}$ be the set of *s-shifted* inversion sequences of size n. In particular, $\mathcal{I}_n^s \subseteq \mathcal{I}_n^{s+1}$, and $\mathcal{I}_n^0 = \mathcal{I}_n$. An s-shifted inversion sequence of size n can be seen as an inversion sequence of size $n + s$ whose first s entries were removed. More precisely, for all $n, s \geq 0$, we have $\mathcal{I}_{n+s} \cong \mathcal{I}_s \times \mathcal{I}_n^s$. For any set of patterns P, we denote by $\mathcal{I}_n^s(P)$ the set of P-avoiding s-shifted inversion sequences of size n.

In this section, we study some patterns for which it is easier to split sequences around their minimum. We use a decomposition of sequences around their first minimum, similar to that of Section 1.1, although a shifted inversion sequence now naturally appears on the right side when this decomposition is applied to an inversion sequence.

In Section 1.1, the right side of the decomposition was a word, since the value of the maximum was fixed. A similar event occurs in the upcoming cases, when decomposing sequences around their first minimum: the sequences of Sections 1.2.2 and 1.2.3 avoid the pattern 102, so when the left side of the decomposition is nonempty (i.e. the first value of the sequence is not the minimum), all values of the right side must be less than or equal to all values of the left side, hence the right side is a word on a fixed alphabet, once again. It follows that shifted inversion sequences only appear on the right side of this decomposition when the left side is empty. Naturally, the left side of the decomposition around the first minimum of an inversion sequence is always empty,

since the leftmost value of a nonempty inversion sequence is always 0. It follows that words only appear when this decomposition is recursively applied to a shifted inversion sequence.

For instance, applying this decomposition to $(0,0,0,0,4,4,5,0,3,4,2,3,0) \in \mathcal{I}_{13}(102)$ will first remove each leading zero, yielding the sequence $(4,4,5,0,3,4,2,3,0) \in \mathcal{I}_9^4(102)$, then split it into a shifted inversion sequence $(4,4,5)$ and a word $(3,4,2,3,0)$.

1.2.2 Avoiding {010, 102}

Let $\mathfrak{A}_{n,s} = \mathcal{I}_n^s(010,102)$ be the set of $\{010,102\}$-avoiding s-shifted inversion sequences of size n. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(010,102) \ : \ \mathbf{max}(\omega) = k-1\}$ be the set of $\{010,102\}$-avoiding words of length n over the alphabet $[0,k-1]$ which contain the letter $k-1$. Let $\mathfrak{a}_{n,s} = |\mathfrak{A}_{n,s}|$ and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$. In particular, $|\mathcal{I}_n(010,102)| = \mathfrak{a}_{n,0}$.

Theorem 1.2.1. *For all $n \geq 1, s \geq 0$,*

$$\mathfrak{a}_{n,s} = (\sum_{z=0}^{n} \mathfrak{a}_{n-z,s+z-1}) + (\sum_{z=1}^{n-1} \mathfrak{a}_{n-z,s-1}) + \sum_{r=1}^{n-2} \sum_{m=1}^{s} \mathfrak{b}_{r,m} \sum_{\ell=0}^{n-r-1} (n-r-\ell-\delta_{\ell,0}) \mathfrak{a}_{\ell,s-m-1}.$$

Proof. Let $n \geq 1, s \geq 0$, and $\sigma \in \mathfrak{A}_{n,s}$. Since σ avoids the pattern 010, all occurrences of the value 0 in σ must be consecutive. Let $z = |\{i \in [1,n] \ : \ \sigma_i = 0\}|$ be the number of occurrences of the value 0 in σ. Let σ' be the sequence obtained by removing every occurrence of the value 0 from σ, and subtracting 1 from all remaining values.

- If $z = 0$ or $\sigma_1 = 0$, then $\sigma' \in \mathfrak{A}_{n-z,s+z-1}$. This is a bijection since $\sigma = (0)^z \cdot (\sigma'+1)$.
- If $\sigma_1 \neq 0$ and $\sigma_n = 0$, then $\sigma' \in \mathfrak{A}_{n-z,s-1}$. This is a bijection since $\sigma = (\sigma'+1) \cdot (0)^z$.
- If $z > 0$, $\sigma_1 \neq 0$, and $\sigma_n \neq 0$, let α and β be the (uniquely defined) integer sequences such that $\sigma = \alpha \cdot (0)^z \cdot \beta$. In particular, α and β are both nonempty, and their values are positive. Let r be the size of β, m be the maximum of β, and $\beta' \in \mathfrak{B}_{r,m}$ be the word obtained by subtracting 1 from each letter of β.

  Since σ avoids 102, every value of α is greater than or equal to m (in particular, $m \leq \alpha_1 \leq s$). All occurrences of the value m in α must be in a single factor at the end of α; otherwise, a value greater than m would appear to the right of a value m in α, creating an occurrence of the pattern 010 in σ (since β also contains m). Let q be the number of occurrences of the value m in α, and let $\ell = |\alpha| - q$ be the number of entries of α of value greater than m. Let $\alpha' \in \mathfrak{A}_{\ell,s-m-1}$ be the sequence obtained by removing every occurrence of the value m from α and subtracting $m+1$ from the remaining values.

  Note that any choice of q and z such that $q+z = n-\ell-r$ does not affect α' and β'. We have $q \in [0, n-r-\ell-1]$ if α' is nonempty, and $q \in [1, n-r-\ell-1]$ if α' is empty (since α is not empty), so there are $n-r-\ell-\delta_{\ell,0}$ possible values for q. For each possible value of q, there is one value of z such that $q+z = n-\ell-r$.

  This decomposition is bijective, since for any choice of

  - $m \in [1,s]$,
  - $\ell, q, z, r \in \mathbb{N}$ such that $\ell+q, z, r \geq 1$ and $\ell+q+z+r = n$,
  - $\alpha' \in \mathfrak{A}_{\ell,s-m-1}$,
  - $\beta' \in \mathfrak{B}_{r,m}$,

we have $(\alpha' + m + 1) \cdot (m)^q \cdot (0)^z \cdot (\beta' + 1) \in \mathfrak{A}_{n,s}$. Specifically, this construction cannot create any occurrence of a pattern 010 or 102 since all values of $\alpha' + m + 1$ are greater than all values of $\beta' + 1$. □

Let $F : (x, y) \mapsto \sum_{n,k\geq 0} |\mathcal{W}_{n,k}(010, 102)| x^n y^k$ be the ordinary generating function of words avoiding 010 and 102. The enumeration of $\{010, 102\}$-avoiding words was solved in [83], with the following expression for F:

$$F(x, y) = \frac{(1-x)^2 - (1-2x)y - \sqrt{(1-x)^4 - 2(1-x)^2 y + (1-4x^2+4x^3)y^2}}{2xy(1-y)}.$$

Note that the function F is the same as in Section 1.1.8, since $\{010, 102\}$-avoiding words and $\{101, 120\}$-avoiding words are in bijection via the complement map

$$\mathcal{W}_{n,k}(010, 102) \to \mathcal{W}_{n,k}(101, 120),\ \omega \mapsto (k - 1 - \omega_i)_{i\in[1,n]}.$$

Since $\mathfrak{B}_{n,k} = \mathcal{W}_{n,k}(010, 102) \backslash \mathcal{W}_{n,k-1}(010, 102)$, the generating function of $\mathfrak{B}_{n,k}$ is

$$\sum_{n,k\geq 0} \mathfrak{b}_{n,k}\, x^n y^k = (1-y)F(x, y).$$

1.2.3 Avoiding {100, 102}

Let $\mathfrak{A}_{n,s} = \mathcal{I}_n^s(100, 102)$ be the set of $\{100, 102\}$-avoiding s-shifted inversion sequences of size n. Let $\mathfrak{A}'_{n,s} = \{\sigma \in \mathfrak{A}_{n,s} \ :\ 0 \in \mathbf{Vals}(\sigma) \text{ and } \sigma_1 \neq 0\}$ be the subset of sequences which contain a 0 but do not begin by 0. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(102) \ :\ \omega_i = \omega_j \implies i = j\}$ be the set of 102-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any repeated letter. Let $\mathfrak{C}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(102) \ :\ \omega_i = \omega_j \neq k-1 \implies i = j\}$ be the set of 102-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any non-maximal repeated letter. Let $\mathfrak{a}_{n,s} = |\mathfrak{A}_{n,s}|$, $\mathfrak{a}'_{n,s} = |\mathfrak{A}'_{n,s}|$, $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$, and $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. In particular, $\mathcal{I}_n(100, 102) = \mathfrak{a}_{n,0}$.

Theorem 1.2.2. *For all $n \geq 1, s \geq 0$,*

$$\mathfrak{a}_{n,s} = \mathfrak{a}_{n,s-1} + \mathfrak{a}_{n-1,s+1} + \mathfrak{a}'_{n,s},$$

$$\mathfrak{a}'_{n,s} = \sum_{p=2}^{n} \sum_{m=1}^{s} \left(\mathfrak{c}_{n-p,m} + \sum_{r=1}^{p-2} \mathfrak{a}_{p-1-r,s+r-m-1} \cdot \mathfrak{b}_{n-p,m} + \sum_{r=0}^{p-2} \mathfrak{a}'_{p-1-r,s+r-m} \cdot \mathfrak{b}_{n-p,m-1} \right).$$

Proof. Let $n \geq 1, s \geq 0$, and $\sigma \in \mathfrak{A}_{n,s}$.

- If σ does not contain the value 0, then subtracting 1 from every value of σ yields a sequence in $\mathfrak{A}_{n,s-1}$, and this is a bijection.
- If $\sigma_1 = 0$, then removing the first term from σ yields a sequence in $\mathfrak{A}_{n-1,s+1}$, and this is a bijection.
- Otherwise, $\sigma \in \mathfrak{A}'_{n,s}$.

This proves the first equality. Let us now turn to the second one.

Let $n \geq 1, s \geq 0$, and $\sigma \in \mathfrak{A}'_{n,s}$. Since σ avoids the pattern 100 and $\sigma_1 > 0$, there is exactly one term of value 0 in σ. Let p be the position of the only value 0 in σ. Let $\alpha = (\sigma_i)_{i\in[1,p-1]}$ and $\beta = (\sigma_i)_{i\in[p+1,n]}$ be the two subsequences such that $\sigma = \alpha \cdot 0 \cdot \beta$. In particular, all values of α and

β are positive. By construction, α is nonempty, but β may be empty. Let m be the minimum of α. Since σ avoids the pattern 102, all values of β must be less than or equal to m, so β is a word of length $n-p$ over the alphabet $[1,m]$. Since σ avoids the pattern 100, each value in $[1,m-1]$ appears at most once in β. Let β' be the word obtained by subtracting 1 from every letter of β.

- If $\alpha = (m)^{p-1}$ is constant, then β' can be any word of $\mathfrak{C}_{n-p,m}$.
- Otherwise, α contains at least one value greater than m. Since σ avoids the pattern 100, the letter m can appear at most once in β. In other words, all letters of β must be distinct, so $\beta' \in \mathfrak{B}_{n-p,m}$. Let $r \in [0, p-2]$ be the number of consecutive occurrences of the value m at the beginning of α.
  - If all values m in α appear in the factor $(m)^r$ at the beginning of α, let $\alpha' \in \mathfrak{A}_{p-1-r,s+r-m-1}$ be the sequence obtained by removing this factor from α, and subtracting $m+1$ from all remaining values. Recall α must contain the value m at least once, so $r > 0$. This defines a bijection, since for any $\alpha' \in \mathfrak{A}_{p-1-r,s+r-m-1}$ and $\beta' \in \mathfrak{B}_{n-p,m}$, the sequence $(m)^r \cdot (\alpha' + m + 1) \cdot 0 \cdot (\beta' + 1)$ is in $\mathfrak{A}'_{n,s}$.
  - Otherwise, there is exactly one value m in α outside of the factor $(m)^r$ at the beginning of α, since having a second value m outside of this factor would imply that the subsequence (α_{r+1}, m, m) is an occurrence of the pattern 100. The same reasoning shows that β cannot contain the letter m, so $\beta' \in \mathfrak{B}_{n-p,m-1}$. Let $\alpha' \in \mathfrak{A}'_{p-1-r,s+r-m}$ be the sequence obtained by removing the factor $(m)^r$ at the beginning of α, and subtracting m from all remaining values. This defines a bijection, since for any $\alpha' \in \mathfrak{A}'_{p-1-r,s+r-m}$ and $\beta' \in \mathfrak{B}_{n-p,m-1}$, the sequence $(m)^r \cdot (\alpha' + m) \cdot 0 \cdot (\beta' + 1)$ is in $\mathfrak{A}'_{n,s}$. □

Lemma 1.2.3. *For all $n, k \geq 0$,*

$$\mathfrak{b}_{n,k} = \frac{\binom{k}{n}\binom{2n}{n}}{n+1}.$$

For all $n \geq 1, k \geq 2$,

$$\mathfrak{c}_{n,k} = 2\mathfrak{c}_{n,k-1} + \mathfrak{c}_{n-1,k} - \mathfrak{c}_{n-1,k-1} - \frac{\binom{k-2}{n}\binom{2n}{n}}{n+1}.$$

Proof. Words avoiding the pairs of patterns $\{100, 102\}$ or $\{011, 120\}$ are trivially in bijection via the complement map $\mathcal{W}_{n,k}(100, 102) \to \mathcal{W}_{n,k}(011, 120)$, $\omega \mapsto (k-1-\omega_i)_{i\in[1,n]}$. We studied the pair of patterns $\{011, 120\}$ in Section 1.1.3, and it can easily be seen that the complement map defines bijections between the sets $\mathfrak{B}_{n,k}$, resp. $\mathfrak{C}_{n,k}$, and their counterparts of Section 1.1.3. Both identities immediately follow from Proposition 1.1.7 and Lemma 1.1.8. □

Chapter 2

Generating trees

2.1 Introduction

A *generating tree* is a rooted, labelled tree such that the label of each node determines its number of children, and their labels. Generating trees were first introduced in the context of pattern-avoiding permutations in [110, 111]. In this thesis, we consider that generating trees are *non-plane* trees (the children of a node form a set rather than a sequence); while this is the most common definition, some authors have also used plane generating trees [69].

We call *combinatorial generating tree* a generating tree labelled by the objects of a combinatorial class and such that each object of size n labels exactly one node, at level n. The root is usually at level 0 or 1, depending on the presence of an empty object in the class. A combinatorial generating tree can be defined using a map, called *ECO operator* [12], which maps each object of size n to some set of children objects of size $n+1$ each, in such a way that each non-root object has a unique parent. More generally, a generating tree can always be defined by a *succession rule*

$$\Omega = \begin{cases} a \\ \ell \quad \rightsquigarrow \quad i^{m(\ell,i)} \quad \text{for} \quad i \in C(\ell) \end{cases}$$

composed of an *axiom*, which labels the root of the tree, and a *production* which associates to each label ℓ the labels of the children of any node labelled ℓ in the tree. In the rule Ω above, a denotes the axiom, $C(\ell)$ the set of labels of children of ℓ, and $m(\ell, i)$ the number of such children having label i, for each $i \in C(\ell)$.

Most generating trees used in the literature to solve enumeration problems are not combinatorial generating trees, but instead use labels that are much simpler (typically, integers or tuples of integers). We define a generating tree to be *concise* if it does not contain two isomorphic subtrees rooted in nodes having different labels. Informally, a generating tree is concise if it involves the minimal number of labels required to describe the "shape" of the tree.

We say that two nodes of a tree $\mathcal{T}$ are $\mathcal{T}$*-equivalent* if they are roots of isomorphic subtrees of $\mathcal{T}$. In particular, two nodes of $\mathcal{T}$ which have the same label are always $\mathcal{T}$-equivalent: the label of a node determines its number of children and their labels, hence a label also determines the entire subtree rooted in its node, by induction. Note that $\mathcal{T}$ is concise if and only if two $\mathcal{T}$-equivalent nodes always have the same label. We now explain how to go (in favorable cases) from a combinatorial generating tree to a concise generating tree.

Assume we are given a combinatorial generating tree $\mathcal{T}$ for a combinatorial class $\mathcal{C}$ and a statistic $s : \mathcal{C} \to L$ for some set L, such that for each $\sigma \in \mathcal{C}$, the value $s(\sigma)$ determines the number of children of the node labelled σ in $\mathcal{T}$, and the image under s of each child. Formally,

this means there exists a function $f : L \to \mathbb{N}^L$ (where $\mathbb{N}^L$ is the set of multisets of elements in L) such that for all $\sigma \in \mathcal{C}$, $f(s(\sigma))$ is the finite multiset $s(\tau_1), \ldots, s(\tau_q)$, where $\tau_1, \ldots, \tau_q$ are the labels of the children of the node labelled σ in $\mathcal{T}$. Taking our tree $\mathcal{T}$ and replacing each label σ by $s(\sigma)$ yields an isomorphic tree $\mathcal{T}'$. Notice that $\mathcal{T}'$ is a generating tree defined by a succession rule which only involves the values of the statistic s: the axiom of this rule is the image under s of the label of the root of $\mathcal{T}$, and its production is the function we denoted f.

By keeping only the value $s(\sigma)$ rather than the "complete" object $\sigma \in \mathcal{C}$ as a label, we can retain less information about objects, which is still sufficient to describe the tree (up to isomorphism). We say that two objects $\sigma, \tau \in \mathcal{C}$ such that $s(\sigma) = s(\tau)$ are *s-equivalent*. Each label $\ell \in L$ of $\mathcal{T}'$ corresponds to the *s-equivalence class* $s^{-1}(\ell) \subseteq \mathcal{C}$. Since s-equivalent objects of $\mathcal{C}$ are always labels of $\mathcal{T}$-equivalent nodes, the coarsest s-equivalence relation is obtained by defining s as the statistic which maps each object of $\mathcal{C}$ to the $\mathcal{T}$-equivalence class of its node. In particular, for any generating tree $\mathcal{T}$, a concise generating tree $\mathcal{T}'$ isomorphic to $\mathcal{T}$ can be obtained in this manner.

In practice, finding this tree $\mathcal{T}'$ requires some way of knowing whether two nodes are $\mathcal{T}$-equivalent, which is not always obvious. For slightly different definitions[3] of generating trees and equivalence, [69] presents an algorithm which can test whether two nodes are equivalent in some combinatorial generating trees for pattern-avoiding inversion sequence. This algorithm then labels each node of the tree by the smallest inversion sequence in lexicographic order which corresponds to a node in the same equivalence class.

Classically, generating trees are used to solve enumeration problems because they induce a recurrence relation satisfied by some refinement of the counting sequence of a combinatorial class (which may also be turned into equations satisfied by its generating function). Here we formalize how a succession rule can be turned into such a recurrence relation. Assume we have a combinatorial generating tree $\mathcal{T}$ for a combinatorial class $\mathcal{C}$, a generating tree $\mathcal{T}'$ labelled by elements in some set L, and a function $s : \mathcal{C} \to L$ such that replacing each label σ of $\mathcal{T}$ by its image $s(\sigma)$ yields $\mathcal{T}'$. For all $(n, \ell) \in \mathbb{N} \times L$, let $\mathfrak{c}_{n,\ell} = |\{\sigma \in \mathcal{C} \ : \ |\sigma| = n, s(\sigma) = \ell\}|$, so that the number of objects of size n in $\mathcal{C}$ is $\sum_\ell \mathfrak{c}_{n,\ell}$. Let $f : L \to \mathbb{N}^L$ be the production of $\mathcal{T}'$. For all $n \geq 1$ and $\ell \in L$, we have

$$\mathfrak{c}_{n,\ell} = \sum_{k \in L} f(k)(\ell) \cdot \mathfrak{c}_{n-1,k},$$

where $f(k)(\ell)$ counts the multiplicity of the label ℓ among the children of a node labelled k in $\mathcal{T}'$.

In the present chapter, we make use of three different generating tree constructions for (pattern-avoiding) inversion sequences. The first one is the most obvious choice, and simply inserts each entry of an inversion sequence by increasing positions (from left to right). The second one inserts entries by increasing *values*. Strictly speaking, it is not a single generating tree construction, since the order of insertion of entries of identical value is not always the same, depending on the avoided patterns. The third one is more unusual since it does not only insert a new entry at each step, but also modifies some existing entries. It can be briefly described as a construction which insets new entries by decreasing positions (from right to left), and sets "permanent" values in decreasing order, while entries with "transient" values are temporarily set at 0.

[3] In [69], generating trees are defined as plane trees, and two nodes are equivalent if they are roots of isomorphic plane subtrees. This relation relies on an order on the children of each node, and it is finer than the $\mathcal{T}$-equivalence we defined.

2.2 Generating trees growing on the right

2.2.1 Method

The simplest way to construct an inversion sequence is to insert each of its terms from right to left. This describes a combinatorial generating tree for the class of inversion sequences $\mathcal{I}$, defined by the succession rule

$$\Omega_{\text{right}} = \begin{cases} \varepsilon \\ \sigma \rightsquigarrow \sigma \cdot i \quad \text{for} \quad i \in [0, |\sigma|]. \end{cases}$$

Replacing each label σ by the length $|\sigma|$ yields a concise generating tree described by the succession rule below.

$$\Omega_{\text{factorial}} = \begin{cases} (0) \\ (n) \rightsquigarrow (n+1)^{n+1} \end{cases}$$

This succession rule corresponds to the recurrence relation $|\mathcal{I}_{n+1}| = (n+1)|\mathcal{I}_n|$, which makes it obvious there are $n!$ nodes at level n.

For any set of patterns P, a combinatorial generating tree for the class of P-avoiding inversion sequences $\mathcal{I}(P)$ can be obtained by restricting the production of Ω_{right} to values of i such that $\sigma \cdot i$ avoids the patterns in P. This does define a generating tree for $\mathcal{I}(P)$: indeed, if $\sigma \cdot i$ avoids P, then σ must also avoid P, so each nonempty sequence of $\mathcal{I}(P)$ has a parent sequence. We call this tree the generating tree *growing on the right* for inversion sequences avoiding the patterns P. The present section is dedicated to this generating tree, which is the most common construction of pattern-avoiding inversion sequences. In all of Section 2.2, we call i a *forbidden* value for (σ, P) (or simply for σ, when P is implicit) if $\sigma \in \mathcal{I}(P)$ and $\sigma \cdot i$ contains a pattern in P.

For instance, the following succession rule describes the generating tree growing on the right for inversion sequences avoiding the pattern 10 (i.e. nondecreasing inversion sequences), pictured on Figure 2.1.

$$\Omega_{10} = \begin{cases} \varepsilon \\ \sigma \rightsquigarrow \sigma \cdot i \quad \text{for} \quad i \in [\mathbf{max}(\sigma), |\sigma|]. \end{cases}$$

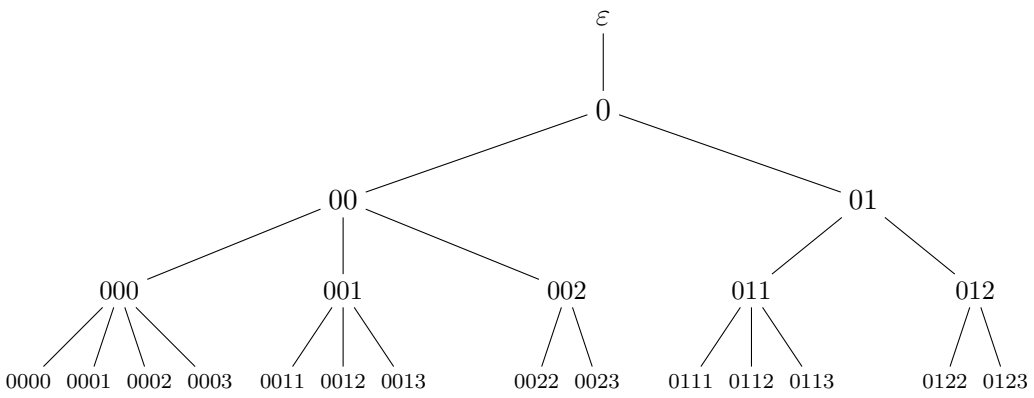

Figure 2.1: First five levels of the combinatorial generating tree growing on the right for $\mathcal{I}(10)$.

For each $\sigma \in \mathcal{I}(10)$, let $s(\sigma)$ be the number of children of the node labelled σ in this tree. It can be seen that $s(\varepsilon) = 1$, and $s(\sigma) = 1 + |\sigma| - \mathbf{max}(\sigma)$ if σ is nonempty. By replacing each

label σ by the value $s(\sigma)$, we obtain an isomorphic generating tree, pictured on Figure 2.2, and described by the succession rule

$$\Omega_{\mathrm{Cat}} = \begin{cases} (1) \\ (k) \rightsquigarrow (i) \quad \text{for} \quad i \in [2, k+1]. \end{cases}$$

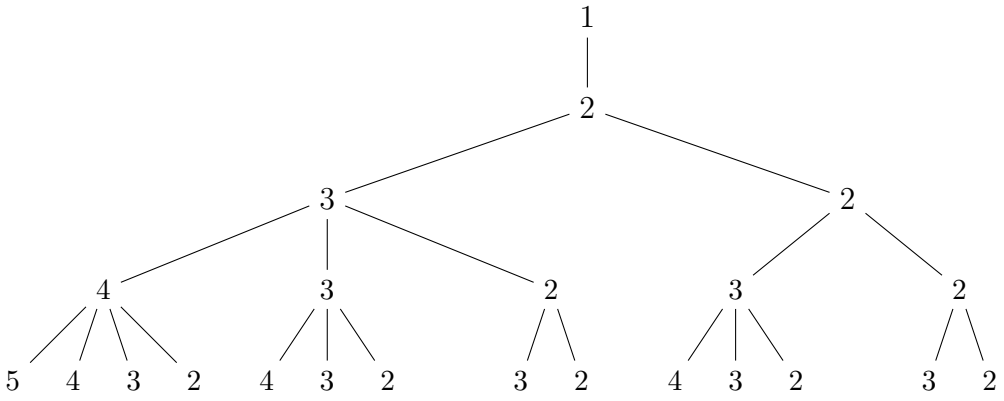

Figure 2.2: First five levels of the generating tree described by the succession rule Ω_{Cat}.

It is well known [110] that the succession rule Ω_{Cat} describes a tree in which the number of nodes at each level n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is easy to see that this succession rule describes a concise generating tree: the label of each node counts its number of children, so if two nodes are roots of isomorphic subtrees, they must have the same label.

From now on, we will often represent a combinatorial generating tree and an isomorphic concise generating tree on the same figure, by placing two labels on each node.

2.2.2 Avoiding {000, 100}

Theorem 2.2.1. *The enumeration of inversion sequences avoiding the patterns 000 and 100 is given by the succession rule*

$$\Omega_{\{000,100\}} = \begin{cases} (1,0) \\ (a,b) \rightsquigarrow (a+1, b-1)^b \\ \qquad\qquad (a+1-j, b+j) \quad \textit{for} \quad j \in [1,a]. \end{cases}$$

Proof. For all $n \in \mathbb{N}$ and $\sigma \in \mathcal{I}_n(000, 100)$, let

- $A(\sigma) = \{i > \mathbf{max}(\sigma) \ : \ \sigma \cdot i \in \mathcal{I}(000, 100)\}$ be the set of values which can be inserted at the end of σ "strictly above" its maximum,
- $B(\sigma) = \{i \leq \mathbf{max}(\sigma) \ : \ \sigma \cdot i \in \mathcal{I}(000, 100)\}$ be the set of values which can be inserted at the end of σ "weakly below" its maximum.

Inserting any value greater than $\mathbf{max}(\sigma)$ at the end of σ cannot create an occurrence of the patterns 000 or 100. As a result, $A(\sigma) = [\mathbf{max}(\sigma)+1, n]$. Let $a(\sigma) = |A(\sigma)|$, and $b(\sigma) = |B(\sigma)|$. In particular, $(a(\varepsilon), b(\varepsilon)) = (1, 0)$.

We show that the combinatorial generating tree growing on the right for $\mathcal{I}(000, 100)$ is isomorphic to the tree described by $\Omega_{\{000,100\}}$. This isomorphism relabels each node σ by $(a(\sigma), b(\sigma))$.

Let $n \in \mathbb{N}$, $\sigma \in \mathcal{I}_n(000, 100)$, $i \in A(\sigma) \sqcup B(\sigma)$, and $\sigma' = \sigma \cdot i$.

- If $i \in B(\sigma)$, then $A(\sigma') = A(\sigma) \sqcup \{n+1\}$ and $B(\sigma') = B(\sigma) \backslash \{i\}$, since if $i < \mathbf{max}(\sigma)$, then i becomes forbidden to avoid 100, and if $i = \mathbf{max}(\sigma)$, then i becomes forbidden to avoid 000. In this case, σ' satisfies $(a(\sigma'), b(\sigma')) = (a(\sigma) + 1, b(\sigma) - 1)$.
- If $i \in A(\sigma)$, then no additional values are forbidden, and the values $[\mathbf{max}(\sigma)+1, i]$ are now less than or equal to $\mathbf{max}(\sigma')$. Let $j = i - \mathbf{max}(\sigma)$. In this case, σ' satisfies $(a(\sigma'), b(\sigma')) = (a(\sigma) + 1 - j, b(\sigma) + j)$. As i ranges over $A(\sigma)$, j varies from 1 to $a(\sigma)$. □

2.2.3 Avoiding {102, 201}

A core idea in our construction of inversion sequences avoiding 102 and 201 is that we may restrict the "future" of combinatorial objects in a generating tree construction; we discuss this further in Remark 2.2.12. This idea comes from Pantone, who calls this a "commitment", see [99].

We begin by establishing some general properties of integer sequences avoiding 102 and 201. For any $\sigma \in \mathbb{N}^n$ such that $\mathbf{dist}(\sigma) \geq 2$, let $\mathbf{sec}(\sigma) = \mathbf{max}(\mathbf{Vals}(\sigma) \backslash \{\mathbf{max}(\sigma)\})$ be the second largest value of σ.

Proposition 2.2.2. *Let σ be a length n integer sequence such that $\mathbf{dist}(\sigma) \geq 2$, let $m = \mathbf{max}(\sigma)$ and $s = \mathbf{sec}(\sigma)$. Then σ avoids 102 and 201 if and only if all three following conditions are satisfied:*

1. *$(\sigma_i)_{i \in [1, \mathbf{firstmax}(\sigma)]}$ is nondecreasing,*
2. *$(\sigma_i)_{i \in [\mathbf{lastmax}(\sigma), n]}$ is nonincreasing,*
3. *$\forall i \in [\mathbf{firstmax}(\sigma), \mathbf{lastmax}(\sigma)]$, $\sigma_i \in \{m, s\}$.*

Proof. First, we show that each condition is necessary:

1. If $i < j \in [1, \mathbf{firstmax}(\sigma)]$ satisfy $\sigma_i > \sigma_j$, then $\sigma_j < m$. In particular, we have $j \neq \mathbf{firstmax}(\sigma)$ hence (σ_i, σ_j, m) is an occurrence of 102.
2. If $i < j \in [\mathbf{lastmax}(\sigma), n]$ satisfy $\sigma_i < \sigma_j$, then $\sigma_i < m$. In particular, we have $i \neq \mathbf{lastmax}(\sigma)$ hence (m, σ_i, σ_j) is an occurrence of 201.
3. If $i \in [\mathbf{firstmax}(\sigma), \mathbf{lastmax}(\sigma)]$ satisfies $\sigma_i < s$, let j be an integer such that $\sigma_j = s$. If $j < i$, then (σ_j, σ_i, m) is an occurrence of 102. If $i < j$, then (m, σ_i, σ_j) is an occurrence of 201.

Assume, for the sake of contradiction, that σ satisfies all three conditions, and that $(\sigma_i, \sigma_j, \sigma_k)$ is an occurrence of 102 for some $i < j < k \in [1, n]$. Since $\sigma_i > \sigma_j$, condition 1 implies that $j > \mathbf{firstmax}(\sigma)$. Since $\sigma_j < \sigma_k$, condition 2 implies that $j < \mathbf{lastmax}(\sigma)$. Hence by condition 3, $\sigma_j \geq s$, which implies that $(\sigma_i, \sigma_j) = (m, s)$ because $\sigma_i > \sigma_j$. Finally $\sigma_k > \sigma_i = m$, a contradiction. The same reasoning holds for the pattern 201, by the reverse symmetry. □

Proposition 2.2.2 has two immediate corollaries.

Corollary 2.2.3. *Let σ be a {102, 201}-avoiding integer sequence such that $m = \mathbf{max}(\sigma)$, and $s = \mathbf{sec}(\sigma)$. Any occurrence of the pattern 101 in σ is a subsequence (m, s, m).*

Corollary 2.2.4. *An integer sequence avoids {101, 102, 201} if and only if it is unimodal.*

We first study inversion sequences avoiding {101, 102, 201}, then those which avoid {102, 201} and contain the pattern 101. The enumeration of inversion sequences avoiding 101, 102 and 201 was already solved and their generating function is given in [32][4]. We nevertheless describe a generating tree for this class, for the sake of completeness, and in order to better introduce the more difficult case of sequences containing 101.

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(10) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of nondecreasing inversion sequences of length n and maximum m for $n \geq 1$, and let $\mathfrak{A}_{0,0} = \{\varepsilon\}$ by convention. Let $\mathfrak{C}^{(1)}_{n,\ell} = \{\sigma \in \mathcal{I}_n(101, 102, 201) \ : \ \ell = \sigma_n < \mathbf{max}(\sigma)\}$ be the set of $\{101, 102, 201\}$-avoiding inversion sequences of length n, last value ℓ, and such that ℓ is not the maximum. Let $\mathfrak{A} = \coprod_{n,m\geq 0} \mathfrak{A}_{n,m}$, and $\mathfrak{C}^{(1)} = \coprod_{n>\ell\geq 0} \mathfrak{C}^{(1)}_{n,\ell}$. From Corollary 2.2.4, $\mathfrak{C}^{(1)}$ is the set of unimodal inversion sequences which decrease after reaching their maximum, and we have $\mathcal{I}(101, 102, 201) = \mathfrak{A} \sqcup \mathfrak{C}^{(1)}$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{c}^{(1)}_{n,\ell} = |\mathfrak{C}^{(1)}_{n,\ell}|$.

Lemma 2.2.5. *For all $n > m \geq 0$,*

$$\mathfrak{a}_{n,m} = \frac{\binom{n+m-1}{m}(n-m)}{n}.$$

For all $n \geq 3, \ell \geq 0$,

$$\mathfrak{c}^{(1)}_{n,\ell} = \sum_{i=\ell}^{n-2} \mathfrak{c}^{(1)}_{n-1,i} + \sum_{i=\ell+1}^{n-2} \mathfrak{a}_{n-1,i}.$$

Proof. Let $n \geq 1$, $m \in [0, n-1]$, and $\sigma \in \mathfrak{A}_{n,m}$. By definition of $\mathfrak{A}_{n,m}$, we have $\sigma_n = m$. Removing the last entry σ_n from σ yields a sequence in $\mathfrak{A}_{n-1,i}$ for some $i \in [0, m]$. Conversely, σ can be obtained by appending the value m to some sequence in $\mathfrak{A}_{n-1,i}$. Hence, there is a bijection between the sets $\mathfrak{A}_{n,m}$ and $\coprod_{i=0}^{m} \mathfrak{A}_{n-1,i}$. This yields the equation $\mathfrak{a}_{n,m} = \sum_{i=0}^{m} \mathfrak{a}_{n-1,i}$, from which we can easily prove that $\mathfrak{a}_{n,m} = \frac{\binom{n+m-1}{m}(n-m)}{n}$ for all $n > m \geq 0$. This corresponds to Catalan's Triangle, entry A009766 of [96].

Let $n \geq 3, \ell \geq 0$, $\sigma \in \mathfrak{C}^{(1)}_{n,\ell}$, and $i = \sigma_{n-1} \geq \ell$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ. If σ' is nondecreasing, then $i = \mathbf{max}(\sigma) > \ell$, and $\sigma' \in \mathfrak{A}_{n-1,i}$. Otherwise, $\sigma' \in \mathfrak{C}^{(1)}_{n,\ell}$. As in the previous case, this is a bijection between $\mathfrak{C}^{(1)}_{n,\ell}$ and $\left(\coprod_{i=\ell}^{n-2} \mathfrak{C}^{(1)}_{n-1,i}\right) \sqcup \left(\coprod_{i=\ell+1}^{n-2} \mathfrak{A}_{n-1,i}\right)$. □

Remark 2.2.6. The bijections from the proof of Lemma 2.2.5 can be turned into a succession rule, by labelling (n, m) each sequence in $\mathfrak{A}_{n,m}$, and labelling (ℓ) each sequence in $\mathfrak{C}^{(1)}_{n,\ell}$:

$$\Omega_{\{101,102,201\}} = \begin{cases} (0,0) & & & \\ (n,m) & \rightsquigarrow & (n+1,i) & \text{for} \quad i \in [m,n] \\ & & (i) & \text{for} \quad i \in [0,m-1] \\ (\ell) & \rightsquigarrow & (i) & \text{for} \quad i \in [0,\ell]. \end{cases}$$

It remains to count inversion sequences in $\mathcal{I}(102, 201)$ which contain 101. By Proposition 2.2.2 and Corollary 2.2.3, an inversion sequence σ avoids {102, 201} and contains 101 if and only if it can be split into three factors $\alpha \cdot \beta \cdot \gamma = \sigma$ such that:

[4] [32] employs the same generating tree construction (that is, inserting each entry of a sequence from left to right). However, it does not present an explicit description of the labels in terms of statistics of inversion sequences, since the algorithmic approach from [69] was used to derive the succession rule.

1. α is a nondecreasing inversion sequence such that $\mathbf{max}(\alpha) < \mathbf{max}(\sigma)$,
2. β is a word over the alphabet $\{\mathbf{max}(\sigma), \mathbf{sec}(\sigma)\}$ which contains 101, and such that $\beta_1 = \mathbf{max}(\sigma)$,
3. γ is a (possibly empty) nonincreasing word over the alphabet $[0, \mathbf{sec}(\sigma) - 1]$.

We distinguish several sets of sequences according to how much of the first occurrence of the pattern 101 has already appeared. Let

- $\mathfrak{B}^{(1)}_{n,s} = \{\sigma \in \mathcal{I}_n(10) \ : \ s \in [\mathbf{sec}(\sigma), \mathbf{max}(\sigma) - 1]\}$, which can be seen as the sequences in which only the first 1 in a pattern 101 has yet appeared.
- $\mathfrak{B}^{(2)}_{n,s} = \{\sigma \cdot (s)^i \ : \ i \in [1, n-2], \sigma \in \mathfrak{B}^{(1)}_{n-i,s}\}$, which can be seen as the sequences in which only the first 10 in a pattern 101 has yet appeared.
- $\mathfrak{B}^{(3)}_{n,s} = \{\sigma \in \mathcal{I}_n(102, 201) \ : \ s = \mathbf{sec}(\sigma) \leq \sigma_n, \sigma \text{ contains } 101\}$ be the set of $\{102, 201\}$-avoiding inversion sequences of length n and second largest value s which contain 101, and for which the factor denoted γ earlier is empty.
- $\mathfrak{C}^{(2)}_{n,\ell} = \{\sigma \in \mathcal{I}_n(102, 201\} \ : \ \ell = \sigma_n < \mathbf{sec}(\sigma), \sigma \text{ contains } 101\}$ be the set of $\{102, 201)$-avoiding inversion sequences of length n and last value ℓ which contain 101, and for which the factor denoted γ earlier is nonempty.

Let $\mathfrak{B}^{(2)} = \coprod_{n,s\geq 0} \mathfrak{B}^{(2)}_{n,s}$, $\mathfrak{B}^{(3)} = \coprod_{n,s\geq 0} \mathfrak{B}^{(3)}_{n,s}$, and $\mathfrak{C}^{(2)} = \coprod_{n,\ell\geq 0} \mathfrak{C}^{(2)}_{n,\ell}$. In particular, $\{\sigma \in \mathcal{I}(102, 201) \ : \ \sigma \text{ contains } 101\} = \mathfrak{B}^{(3)} \sqcup \mathfrak{C}^{(2)}$.

Remark 2.2.7. The subsets of $\mathcal{I}(102, 201)$ we have defined are not disjoint. More precisely, the following holds.

1. The sets $\mathfrak{B}^{(1)}_{n,s}$ intersect for different values of s. For instance, the sequence $(0, 0, 2)$ belongs to both $\mathfrak{B}^{(1)}_{3,0}$ and $\mathfrak{B}^{(1)}_{3,1}$.
2. Each sequence in a set $\mathfrak{B}^{(1)}_{n,s}$ also belongs to a set $\mathfrak{A}_{n,m}$ for some $m > s$.
3. Each set $\mathfrak{B}^{(2)}_{n,s}$ is a subset of $\mathfrak{C}^{(1)}_{n,s}$.

Let $\mathfrak{b}^{(i)}_{n,s} = |\mathfrak{B}^{(i)}_{n,s}|$ for $i \in \{1, 2, 3\}$, and $\mathfrak{c}^{(2)}_{n,s} = |\mathfrak{C}^{(2)}_{n,s}|$.

Lemma 2.2.8. *For all* $0 \leq s \leq n-2$,

$$\mathfrak{b}^{(1)}_{n,s} = \mathfrak{b}^{(1)}_{n-1,s} + (n-1-s)\mathfrak{a}_{n,s}.$$

For all $0 \leq s \leq n-3$,

$$\mathfrak{b}^{(2)}_{n,s} = \mathfrak{b}^{(2)}_{n-1,s} + \mathfrak{b}^{(1)}_{n-1,s}.$$

For all $0 \leq s \leq n-4$,

$$\mathfrak{b}^{(3)}_{n,s} = 2\mathfrak{b}^{(3)}_{n-1,s} + \mathfrak{b}^{(2)}_{n-1,s}.$$

For all $0 \leq \ell \leq n-6$,

$$\mathfrak{c}^{(2)}_{n,\ell} = \sum_{i=\ell}^{n-5} \mathfrak{c}^{(2)}_{n-1,i} + \sum_{i=\ell+1}^{n-5} \mathfrak{b}^{(3)}_{n-1,i}.$$

Proof. We give a bijective proof of each identity by removing the last entry of an inversion sequence.

Let $n \geq 2, s \in [0, n-2]$, $\sigma \in \mathfrak{B}^{(1)}_{n,s}$. In particular, $\sigma_n = \mathbf{max}(\sigma)$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ, and let $i = \sigma_{n-1}$. If $i = \sigma_n$, then $\sigma' \in \mathfrak{B}^{(1)}_{n-1,s}$. Otherwise, $i = \mathbf{sec}(\sigma) \leq s$, $\sigma' \in \mathfrak{A}_{n-1,i}$, and $\sigma_n \in [s+1, n-1]$. This yields a bijection between $\mathfrak{B}^{(1)}_{n,s}$ and $\mathfrak{B}^{(1)}_{n-1,s} \coprod \left([s+1, n-1] \times \coprod_{i=0}^{s} \mathfrak{A}_{n-1,i}\right)$. We know from the proof of Lemma 2.2.5 that $\coprod_{i=0}^{s} \mathfrak{A}_{n-1,i}$ is in bijection with $\mathfrak{A}_{n,s}$.

Let $n \geq 3, s \in [0, n-3]$, and $\sigma \in \mathfrak{B}^{(2)}_{n,s}$. In particular, $\sigma_n = s$, and $\sigma_{n-1} \in \{\mathbf{max}(\sigma), s\}$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ, and let $i = \sigma_{n-1}$. If $i = s$, then $\sigma' \in \mathfrak{B}^{(2)}_{n-1,s}$. Otherwise $i = \mathbf{max}(\sigma)$, and $\sigma' \in \mathfrak{B}^{(1)}_{n-1,s}$.

Let $n \geq 4, s \in [0, n-4]$, and $\sigma \in \mathfrak{B}^{(3)}_{n,s}$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ. If σ' contains 101, then $\sigma' \in \mathfrak{B}^{(3)}_{n-1,s}$, and $\sigma_n \in \{\mathbf{max}(\sigma), s\}$. Otherwise $\sigma' \in \mathfrak{B}^{(2)}_{n-1,s}$, and $\sigma_n = \mathbf{max}(\sigma)$.

Let $n \geq 6, \ell \in [0, n-6]$, and $\sigma \in \mathfrak{C}^{(2)}_{n,\ell}$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ, and let $i = \sigma_{n-1} \geq \ell$. If $i < \mathbf{sec}(\sigma)$, then $\sigma' \in \mathfrak{C}^{(2)}_{n-1,i}$. Otherwise $i > \ell$, and $\sigma' \in \mathfrak{B}^{(3)}_{n-1,i}$. □

Let $\mathfrak{C}_{n,\ell} = \mathfrak{C}^{(1)}_{n,\ell} \sqcup \mathfrak{C}^{(2)}_{n,\ell}$ and $\mathfrak{c}_{n,\ell} = |\mathfrak{C}_{n,\ell}| = \mathfrak{c}^{(1)}_{n,\ell} + \mathfrak{c}^{(2)}_{n,\ell}$. Let also $\mathfrak{C} = \mathfrak{C}^{(1)} \sqcup \mathfrak{C}^{(2)}$.

Proposition 2.2.9. *For all $n \geq 1$,*

$$|\mathcal{I}_n(102, 201)| = \left(\sum_{m=0}^{n-1} \mathfrak{a}_{n,m}\right) + \left(\sum_{s=0}^{n-4} \mathfrak{b}^{(3)}_{n,s}\right) + \left(\sum_{\ell=0}^{n-3} \mathfrak{c}_{n,\ell}\right).$$

Proof. Merging our results for inversion sequences which avoid or contain 101, we obtain

$$\mathcal{I}(102, 201) = \left(\mathfrak{A} \sqcup \mathfrak{C}^{(1)}\right) \sqcup \left(\mathfrak{B}^{(3)} \sqcup \mathfrak{C}^{(2)}\right) = \mathfrak{A} \sqcup \mathfrak{B}^{(3)} \sqcup \mathfrak{C}.$$ □

Remark 2.2.10. The equations for counting $\mathfrak{c}^{(1)}_{n,\ell}$ and $\mathfrak{c}^{(2)}_{n,\ell}$ from Lemmas 2.2.5 and 2.2.8 can be merged into a single recurrence relation for $\mathfrak{c}_{n,\ell}$. For all $0 \leq \ell \leq n-3$,

$$\mathfrak{c}_{n,\ell} = \mathfrak{c}_{n-1,\ell} + \sum_{i=\ell+1}^{n-2} \mathfrak{a}_{n-1,i} + \mathfrak{b}^{(3)}_{n-1,i} + \mathfrak{c}_{n-1,i}.$$

Remark 2.2.11. By labelling

- (a, n, m) each sequence of $\mathfrak{A}_{n,m}$,
- $(b^{(i)}, s)$ each sequence of $\mathfrak{B}^{(i)}_{n,s}$ for $i \in \{1, 2, 3\}$
- (c, ℓ) each sequence of $\mathfrak{C}_{n,\ell}$,

the recurrence relations from Lemmas 2.2.5, 2.2.8, and Remark 2.2.10 correspond to the succession rule

$$\Omega_{\{102,201\}} = \begin{cases} (a,0,0) & & & \\ (a,n,m) & \rightsquigarrow & (a,n+1,i) & \text{for } i \in [m,n] \\ & & (b^{(1)},i)^{n-i} & \text{for } i \in [m,n-1] \\ & & (c,i) & \text{for } i \in [0,m-1] \\ (b^{(1)},s) & \rightsquigarrow & (b^{(1)},s)\,(b^{(2)},s) & \\ (b^{(2)},s) & \rightsquigarrow & (b^{(2)},s)\,(b^{(3)},s) & \\ (b^{(3)},s) & \rightsquigarrow & (b^{(3)},s)^2 & \\ & & (c,i) & \text{for } i \in [0,s-1] \\ (c,\ell) & \rightsquigarrow & (c,i) & \text{for } i \in [0,\ell]. \end{cases}$$

Let $\mathcal{T}$ be the generating tree described by the succession rule $\Omega_{\{102,201\}}$. We now explain how to relate $\mathcal{T}$ with the definitions of Section 2.1. First, note that $\mathcal{T}$ is not isomorphic to a combinatorial generating tree for the class $\mathcal{I}(102,201)$, because of the intersections mentioned in Remark 2.2.7. By construction, $\mathcal{T}$ is isomorphic to a combinatorial generating tree for the disjoint union $\mathcal{U}$ of the sets $\mathfrak{A}$, $\{\coprod_{n\geq 0}\mathfrak{B}^{(1)}_{n,s}\}_{s\geq 0}$, $\mathfrak{B}^{(2)}$, $\mathfrak{B}^{(3)}$ and $\mathfrak{C}$. We can view the combinatorial class $\mathcal{U}$ as a subset of $\mathbb{N}\times\mathcal{I}(102,201)$:

$$\mathcal{U} \cong \Big(\{0\}\times\big(\mathfrak{A}\sqcup\mathfrak{B}^{(3)}\sqcup\mathfrak{C}\big)\Big)\sqcup\Big(\{1\}\times\mathfrak{B}^{(2)}\Big)\sqcup\left(\coprod_{s\geq 0}\left(\{s+2\}\times\coprod_{n\geq 0}\mathfrak{B}^{(1)}_{n,s}\right)\right),$$

and define the size of each object $(k,\sigma)\in\mathcal{U}$ as the length of σ. By the partition used in Proposition 2.2.9, the objects of $\mathcal{U}$ of the form $(0,\sigma)$ are trivially in bijection with $\mathcal{I}(102,201)$. The remaining objects of $\mathcal{U}$, of the form (k,σ) for $k\geq 1$, can be called *phantom objects* as they are ultimately not counted.

These phantom objects are nevertheless necessary in our construction, since removing all the nodes of $\mathcal{T}$ corresponding to phantom objects would disconnect[5] $\mathcal{T}$. The tree $\mathcal{T}$ is pictured on Figure 2.3, on page 48.

Remark 2.2.12. The phantom objects can be understood as follows. In an object $(1,\sigma)\in\mathcal{U}$, $\sigma\in\mathcal{I}(101,102,201)$ is a prefix of an inversion sequence avoiding 102 and 201, whose maximum m and second maximum s form an occurrence (m,s) of the pattern 10 which must be completed into an occurrence (m,s,m) of 101 in the future. In particular, we may only insert the values m or s at the end of σ; otherwise, the subsequence (m,s,m) could never appear, by Corollary 2.2.3. Similarly, in an object $(s+2,\sigma)\in\mathcal{U}$, $\sigma\in\mathcal{I}(10)$ is a prefix of an inversion sequence avoiding 102 and 201, that has maximum $m>s$ and which must contain an occurrence (m,s,m) of 101 in the future. In particular, we may only insert the values m or s at the end of σ; otherwise, the subsequence (m,s,m) could never appear, by Corollary 2.2.3.

This construction may appear more complicated than necessary. Indeed, we could simply consider the generating tree growing on the right for $\mathcal{I}(102,201)$ instead, and this would not involve phantom objects. However, that would be less efficient, since this tree requires more labels. Specifically, the number of distinct labels appearing at level n in this tree would be quadratic in n (we would need to record both the values of the maximum and the second maximum of sequences in $\mathfrak{A}$), whereas our construction only involves a linear number of labels.

[5]More precisely, each node of $\mathcal{T}$ corresponding to an object in $\{0\}\times\mathfrak{B}^{(3)}$ or $\{0\}\times\mathfrak{C}^{(2)}$ has an ancestor which corresponds to a phantom object in $\{1\}\times\mathfrak{B}^{(2)}$.

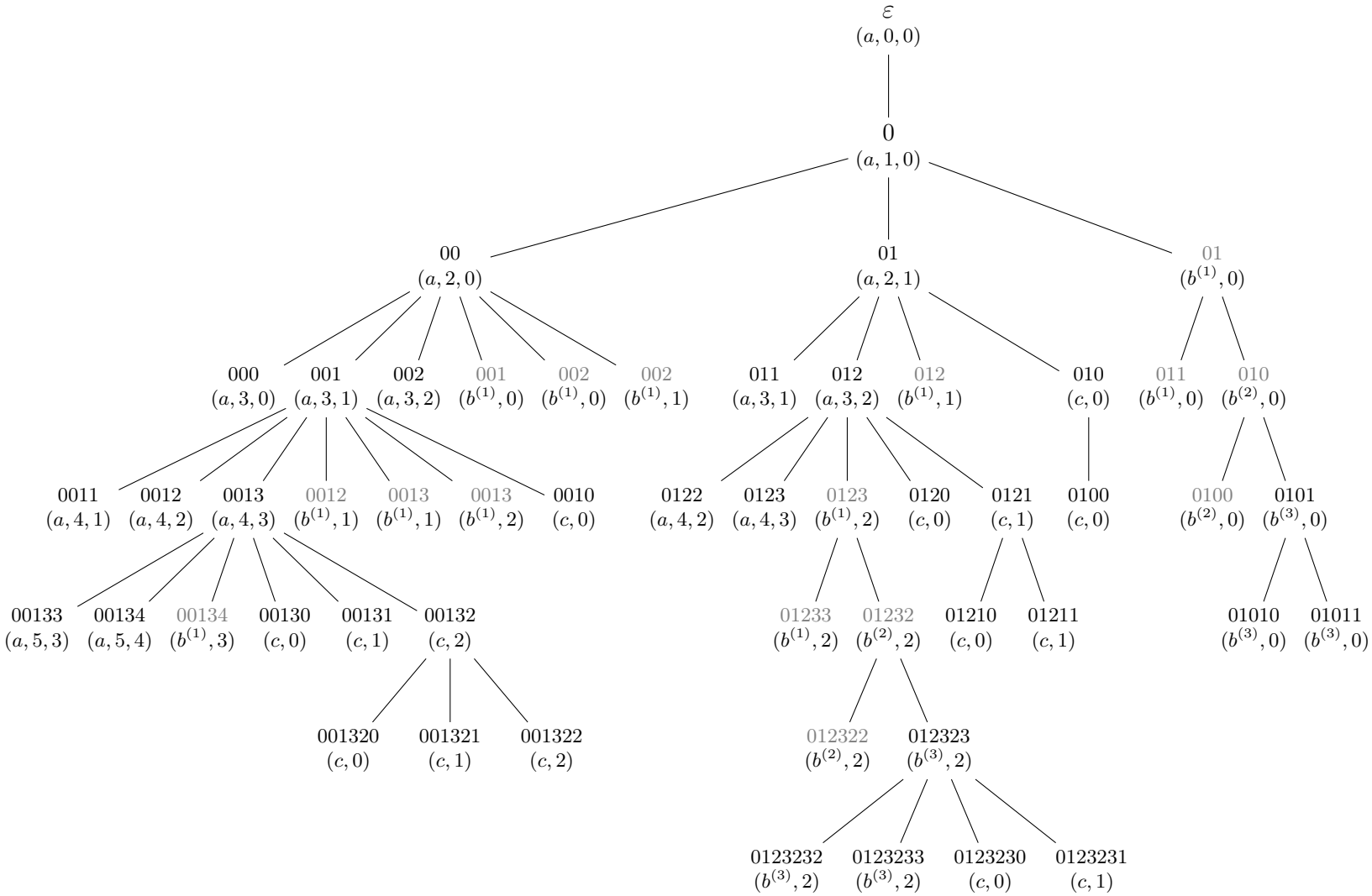

Figure 2.3: Part of the generating tree described by the succession rule $\Omega_{\{102,201\}}$. Phantom objects are colored in gray.

Remark 2.2.13. We could directly obtain an explicit expression for $|\mathcal{I}_n(102,201)|$ from the observations of Proposition 2.2.2 and Corollaries 2.2.3 and 2.2.4, although using it for enumeration is not as efficient as our generating tree approach. The sequences of $\mathcal{I}_n(101,102,201)$ are counted by

$$1+\sum_{m=1}^{n-1}\sum_{\ell=m+1}^{n}\frac{\binom{\ell+m-1}{m}(\ell-m)}{\ell}\cdot\binom{n-\ell+m-1}{n-\ell},$$

as shown in [32]. The sequences of $\mathcal{I}_n(102,201)$ which contain 101 are counted by

$$\sum_{f=2}^{n-2}\sum_{\ell=f+2}^{n}\sum_{s=0}^{f-2}(f-s-1)\cdot\frac{\binom{f+s-1}{s}(f-s)}{f}\cdot(2^{\ell-f-1}-1)\cdot\binom{n-\ell+s}{n-\ell}.$$

The summands count the number of sequences $\sigma\in\mathcal{I}_n(102,201)$ such that $m=\mathbf{max}(\sigma)$, $s=\mathbf{sec}(\sigma)$, $f=\mathbf{firstmax}(\sigma)$, and $\ell=\mathbf{lastmax}(\sigma)$.

2.2.3.1 The algebraic generating function of $\mathcal{I}$(102,201)

Let $F(x)=\sum_{n\geq 0}|\mathcal{I}_n(102,201)|x^n$ be the ordinary generating function of $\mathcal{I}(102,201)$, and let

$$\begin{aligned}A(x,y)&=\sum_{n,m\geq 0}\mathfrak{a}_{n,m}x^n y^m,\\ B^{(i)}(x,y)&=\sum_{n,s\geq 0}\mathfrak{b}^{(i)}_{n,s}x^n y^s\quad\text{for}\quad i\in\{1,2,3\},\\ C(x,y)&=\sum_{n,\ell\geq 0}\mathfrak{c}_{n,\ell}x^n y^\ell.\end{aligned}$$

Proposition 2.2.14. *The functions $A, B^{(1)}, B^{(2)}, B^{(3)}, C$, and F satisfy the following equations.*

$$\begin{aligned}A(x,y)&=1+\frac{x}{1-y}\big(A(x,y)-yA(xy,1)\big)\\ B^{(1)}(x,y)&=xB^{(1)}(x,y)+\frac{x}{(1-y)}\left(x\frac{\partial A}{\partial x}(x,y)-y\frac{\partial A}{\partial y}(x,y)+\frac{y}{1-y}\big(A(xy,1)-A(x,y)\big)\right)\\ B^{(2)}(x,y)&=x\big(B^{(1)}(x,y)+B^{(2)}(x,y)\big)\\ B^{(3)}(x,y)&=x\big(B^{(2)}(x,y)+2B^{(3)}(x,y)\big)\\ C(x,y)&=\frac{x}{1-y}\big(C(x,1)-yC(x,y)+A(x,1)-A(x,y)+B^{(3)}(x,1)-B^{(3)}(x,y)\big)\\ F(x)&=A(x,1)+B^{(3)}(x,1)+C(x,1)\end{aligned}$$

Proof. We turn the succession rule $\Omega_{\{102,201\}}$ into equations relating the generating functions.

$$\begin{aligned}A(x,y)&=1+\sum_{n,m\geq 0}\mathfrak{a}_{n,m}x^{n+1}\sum_{i=m}^{n}y^i\\ &=1+\sum_{n,m\geq 0}\mathfrak{a}_{n,m}x^{n+1}\frac{y^m-y^{n+1}}{1-y}\\ &=1+\frac{x}{1-y}\big(A(x,y)-yA(xy,1)\big)\end{aligned}$$

$$\begin{aligned}
B^{(1)}(x,y) &= \left(\sum_{n,s\geq 0} \mathfrak{b}^{(1)}_{n,s} x^{n+1} y^s\right) + \left(\sum_{n,m\geq 0} \mathfrak{a}_{n,m} x^{n+1} \sum_{i=m}^{n-1} (n-i) y^i\right) \\
&= xB^{(1)}(x,y) + \sum_{n,m\geq 0} \mathfrak{a}_{n,m} \frac{x^{n+1}}{1-y}\left(ny^m - my^m + \frac{y^{n+1}-y^{m+1}}{1-y}\right) \\
&= xB^{(1)}(x,y) + \frac{x}{(1-y)}\left(x\frac{\partial A}{\partial x}(x,y) - y\frac{\partial A}{\partial y}(x,y) + \frac{y}{1-y}\big(A(xy,1) - A(x,y)\big)\right)
\end{aligned}$$

$$\begin{aligned}
B^{(2)}(x,y) &= \sum_{n,s\geq 0} (\mathfrak{b}^{(1)}_{n,s} + \mathfrak{b}^{(2)}_{n,s}) x^{n+1} y^s \\
&= x\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)
\end{aligned}$$

$$\begin{aligned}
B^{(3)}(x,y) &= \sum_{n,s\geq 0} (\mathfrak{b}^{(2)}_{n,s} + 2\mathfrak{b}^{(3)}_{n,s}) x^{n+1} y^s \\
&= x\big(B^{(2)}(x,y) + 2B^{(3)}(x,y)\big)
\end{aligned}$$

$$\begin{aligned}
C(x,y) &= \left(\sum_{n,\ell\geq 0} \mathfrak{c}_{n,\ell} x^{n+1} \sum_{i=0}^{\ell} y^i\right) + \left(\sum_{n,k\geq 0} (\mathfrak{a}_{n,k} + \mathfrak{b}^{(3)}_{n,k}) x^{n+1} \sum_{i=0}^{k-1} y^i\right) \\
&= \left(\sum_{n,\ell\geq 0} \mathfrak{c}_{n,\ell} x^{n+1} \frac{1-y^{\ell+1}}{1-y}\right) + \left(\sum_{n,k\geq 0} (\mathfrak{a}_{n,k} + \mathfrak{b}^{(3)}_{n,k}) x^{n+1} \frac{1-y^k}{1-y}\right) \\
&= \frac{x}{1-y}\big(C(x,1) - yC(x,y) + A(x,1) - A(x,y) + B^{(3)}(x,1) - B^{(3)}(x,y)\big)
\end{aligned}$$

The equation $F(x) = A(x,1) + B^{(3)}(x,1) + C(x,1)$ is an immediate consequence of Proposition 2.2.9. □

Theorem 2.2.15. *The generating function of* $\mathcal{I}(102,201)$ *is*

$$F(x) = \frac{(2x^3 - 5x^2 + 6x - 2)\sqrt{(1-x)(1-5x)} - 4x^4 + 9x^3 - 5x^2 - 4x + 2}{2x(2-x)^2(1-2x)(1-x)}.$$

Proof. From Proposition 2.2.14, we have

$$(1-y-x)A(x,y) = 1 - y - xyA(xy,1). \tag{2.1}$$

We cancel the kernel $(1-y-x)$ by defining $Y(x) = 1-x$ and replacing y by $Y(x)$ in (2.1):

$$0 = 1 - (1-x) - x(1-x)A(x(1-x),1),$$

and obtain

$$A(x-x^2,1) = \frac{1}{1-x}.$$

This uniquely determines the formal power series $A(x,1)$:

$$A(x,1) = \frac{1-\sqrt{1-4x}}{2x}.$$

Replacing $A(x,1)$ by this expression in (2.1) yields

$$A(x,y) = \frac{1-2y+\sqrt{1-4xy}}{2-2x-2y}.$$

We now compute explicit expressions for $B^{(1)}(x,y)$, $B^{(2)}(x,y)$, and $B^{(3)}(x,y)$ from that of $A(x,y)$, with the formulas of Proposition 2.2.14.

$$\begin{aligned} B^{(1)}(x,y) &= \frac{x}{(1-x)(1-y)}\left(x\frac{\partial A}{\partial x}(x,y) - y\frac{\partial A}{\partial y}(x,y) + \frac{y}{1-y}\big(A(xy,1)-A(x,y)\big)\right) \\ &= \frac{(2x-1)\sqrt{-4xy+1} + 2x^2 - 2xy - 2x + 1}{2(1-x)(1-x-y)^2} \end{aligned}$$

$$\begin{aligned} B^{(2)}(x,y) &= \frac{x}{1-x}B^{(1)}(x,y) \\ &= \frac{x(2x-1)\sqrt{-4xy+1} + x(2x^2-2xy-2x+1)}{2(1-x)^2(1-x-y)^2} \end{aligned}$$

$$\begin{aligned} B^{(3)}(x,y) &= \frac{x}{1-2x}B^{(2)}(x,y) \\ &= \frac{x^2(2x-1)\sqrt{-4xy+1} + x^2(2x^2-2xy-2x+1)}{2(1-2x)(1-x)^2(1-x-y)^2} \end{aligned}$$

As for $C(x,y)$, we apply the kernel method again. From Proposition 2.2.14, we have

$$(1-y+xy)C(x,y) = x\big(C(x,1) + A(x,1) - A(x,y) + B^{(3)}(x,1) - B^{(3)}(x,y)\big).$$

The kernel is $1-y+xy$, and we cancel it by setting $y = \frac{1}{1-x}$. This leads to the following expression for $C(x,1)$:

$$\begin{aligned} C(x,1) &= A\left(x, \frac{1}{1-x}\right) - A(x,1) + B^{(3)}\left(x, \frac{1}{1-x}\right) - B^{(3)}(x,1) \\ &= \frac{(x^2-x+1)\sqrt{1-4x}}{2x(1-x)^2} - \frac{(x^2-2x+2)\sqrt{(1-x)(1-5x)}}{2x(2-x)^2(1-x)} \\ &\quad + \frac{-4x^5+14x^4-21x^3+18x^2-10x+2}{2x(2-x)^2(2x-1)(1-x)^2}. \end{aligned}$$

Finally, we can write an expression for $F(x)$:

$$\begin{aligned} F(x) &= A(x,1) + B^{(3)}(x,1) + C(x,1) \\ &= \frac{(2x^3-5x^2+6x-2)\sqrt{(1-x)(1-5x)} - 4x^4 + 9x^3 - 5x^2 - 4x + 2}{2x(2-x)^2(1-2x)(1-x)}. \end{aligned}$$

□

2.3 Inserting or removing a maximum

2.3.1 Method

Generally speaking, removing a term from an inversion sequence does not always create another inversion sequence. In fact, removing the i-th term σ_i from an inversion sequence $\sigma \in \mathcal{I}_n$ for some $i \in [1, n]$ yields an inversion sequence if and only if $\sigma_j < j - 1$ for all $j \in [i + 1, n]$. In particular if $\sigma_i = \mathbf{max}(\sigma)$, then for all $j \in [i + 1, n]$ we have $\sigma_j \leq \sigma_i < i \leq j - 1$, and the above condition is satisfied. Hence, removing a maximum from an inversion sequence always yields an inversion sequence.

Following this idea, we can decompose inversion sequences by removing their maxima one by one. For this decomposition to be unambiguous, we must choose the order of removal of the different occurrences of the maximum (this order is not always trivial). Observe that this decomposition defines a combinatorial generating tree $\mathcal{T}$ for inversion sequences. In that tree, the parent of each nonempty inversion sequence is obtained by removing its "next" maximum in the chosen order.

For any set of patterns P, a combinatorial generating tree for P-avoiding inversion sequences can be obtained by removing every node of $\mathcal{T}$ which corresponds to an inversion sequence that contains a pattern in P. Indeed, this procedure does not disconnect $\mathcal{T}$ since the parent of any nonempty P-avoiding inversion sequence also avoids P.

Let us quickly go over each pattern ρ that appears in this section, and establish necessary and sufficient conditions under which the maximum of an integer sequence never appears in any occurrence of ρ.

- 000: The maximum does not appear more than twice.
- 100: All values appearing after the first maximum are distinct, or equal to the maximum.
- 101: All occurrences of the maximum are consecutive.
- 110: All maxima after the first one appear in a single factor, at the end of the sequence.
- 102: The subsequence to the left of the last maximum is nondecreasing, ignoring other occurrences of the maximum.
- 201: The subsequence to the right of the first maximum is nonincreasing, ignoring other occurrences of the maximum.
- 210: The subsequence to the right of the first maximum is nondecreasing, ignoring other occurrences of the maximum.

Since our generating tree construction inserts a maximum, the children of a node labelled σ in $\mathcal{T}'$ are the children of σ in $\mathcal{T}$ which satisfy the above conditions (for the patterns in P). Such facts will be used in proofs throughout this section without explicit reference.

2.3.2 Avoiding {000, 102}

In this subsection, we use a generalization of succession rules called *doubled succession rules*, introduced in [51], which allows a node at level n in a generating tree to have children at levels $n + 1$ or $n + 2$, denoted by $\overset{1}{\rightsquigarrow}$ and $\overset{2}{\rightsquigarrow}$ in the succession rule.

For all $\sigma \in \mathcal{I}(000, 102)$, let $\mathbf{inc}(\sigma)$ be the length of the longest nondecreasing factor at the beginning of σ, and let $\mathbf{sites}(\sigma) = \mathbf{inc}(\sigma) - \mathbf{max}(\sigma)$. In particular, $\mathbf{sites}(\varepsilon) = 1$.

Remark 2.3.1. Let $\sigma \in \mathcal{I}(000,102)\backslash\{\varepsilon\}$ be a nonempty inversion sequence. Since σ avoids the pattern 102, the factor $(\sigma_i)_{i\in[1,\mathbf{firstmax}(\sigma)]}$ is nondecreasing. In other words, the maximum of σ is reached before its first descent, and therefore $\sigma_{\mathbf{inc}(\sigma)} = \mathbf{max}(\sigma)$. This also implies that $\mathbf{inc}(\sigma) > \mathbf{max}(\sigma)$, hence $\mathbf{sites}(\sigma)$ is positive.

Given $n \geq 1, \sigma \in \mathbb{N}^n$, $p \in [1, n+1]$ and $v \in \mathbb{N}$, we denote by $[\sigma]_p^v$ the sequence obtained from σ by inserting the value v at position p, that is $[\sigma]_p^v = (\sigma_i)_{i\in[1,p-1]} \cdot v \cdot (\sigma_i)_{i\in[p,n]}$.

Proposition 2.3.2. *Let $n \geq 0$, $\sigma \in \mathcal{I}_n(000,102)$, $m = \mathbf{max}(\sigma)$, $i \in [1, n+1]$, and $k > 0$. Let also $\sigma' = [\sigma]_i^{m+k}$ be the sequence obtained by inserting the value $m+k$ at position i in σ. Then $\sigma' \in \mathcal{I}_{n+1}(000,102)$ if and only if $i \in [m+k+1, \mathbf{inc}(\sigma)+1]$.*

Proof. Since σ does not contain the value $m+k$, inserting it cannot create an occurrence of the pattern 000. The value $m+k$ must be inserted in the interval $[m+k+1, n+1]$ in order to create an inversion sequence. By definition of $\mathbf{inc}(\sigma)$, $(\sigma_i)_{i\in[1,\mathbf{inc}(\sigma)]}$ is nondecreasing, hence inserting $m+k$ in any position $[m+k+1, \mathbf{inc}(\sigma)+1]$ does not create an occurrence of 102. On the contrary, inserting $m+k$ in any position $[\mathbf{inc}(\sigma)+2, n+1]$ creates an occurrence of 102, since $(\sigma_{\mathbf{inc}(\sigma)}, \sigma_{\mathbf{inc}(\sigma)+1})$ is an occurrence of 10 and $m+k > \sigma_{\mathbf{inc}(\sigma)}$. □

It follows from Proposition 2.3.2 that the statistic $\mathbf{sites}(\sigma)$ can be interpreted either as the number of values (denoted k earlier) greater than m which could be inserted in σ, or the number of positions (denoted i earlier) where a value greater than m can be inserted in σ, so that the resulting sequence is in $\mathcal{I}_{n+1}(000,102)$.

Lemma 2.3.3. *The enumeration of inversion sequences avoiding the patterns 000 and 102 is given by the doubled succession rule*

$$\Omega_{\{000,102\}} = \begin{cases} (1) & \\ (s) \overset{1}{\rightsquigarrow} (j)^{s+1-j} & \text{for} \quad j \in [1,s] \\ (s) \overset{2}{\rightsquigarrow} (j+1)^{s+1-j}(j)^{\binom{s+1-j}{2}} & \text{for} \quad j \in [1,s]. \end{cases}$$

The generating tree described by $\Omega_{\{000,102\}}$ is depicted on Figure 2.4.

Proof. We consider the (jumping) combinatorial generating tree for $\mathcal{I}(000,102)$ such that the parent of any $\sigma \in \mathcal{I}(000,102)\backslash\{\varepsilon\}$ is obtained from σ by removing all occurrences of $\mathbf{max}(\sigma)$. Due to the avoidance of 000, $\mathbf{max}(\sigma)$ appears either once or twice in σ, so the difference between the level of σ and that of its parent is either 1 or 2. We then replace each label σ by the value $\mathbf{sites}(\sigma)$ defined above. The axiom of the resulting succession rule is (1) since $\mathbf{sites}(\varepsilon) = 1$.

In the rest of this proof, let $n \geq 0$, $\sigma \in \mathcal{I}_n(000,102)$ and $m = \mathbf{max}(\sigma)$.

Let $i, k \geq 1$ be two integers, and $\sigma' = [\sigma]_i^{m+k}$. From Proposition 2.3.2, $\sigma' \in \mathcal{I}_{n+1}(000,102)$ if and only if $i \in [m+2, \mathbf{inc}(\sigma)+1]$ and $k \in [1, i-m-1]$. Let us now assume that is the case. By definition of σ', we have $\mathbf{inc}(\sigma') = i$ and $\mathbf{max}(\sigma') = m+k$. In particular, $\mathbf{sites}(\sigma') = i - (m+k) \in [1, \mathbf{sites}(\sigma)]$.

Let $j \in [1, \mathbf{sites}(\sigma)]$. Let us count how many choices of i and k satisfy the identity $\mathbf{sites}([\sigma]_i^{m+k}) = j$. The ordered pair (i,k) is a solution whenever $j = i - (m+k)$, i.e. whenever $k = i - m - j$. Since k is positive, there is one solution for each value of i such that $i > m + j$. The number of solutions (i,k) is the number of values i in the interval $[m+j+1, \mathbf{inc}(\sigma)+1]$, so there are $\mathbf{inc}(\sigma) + 1 - (m+j) = \mathbf{sites}(\sigma) + 1 - j$ solutions. This corresponds to $(s) \overset{1}{\rightsquigarrow} (j)^{s+1-j}$ in the succession rule. Now let us count in how many ways we may insert two occurrences of a value greater than m in σ. In the previous case, inserting a second $m+k$ adjacent to the first

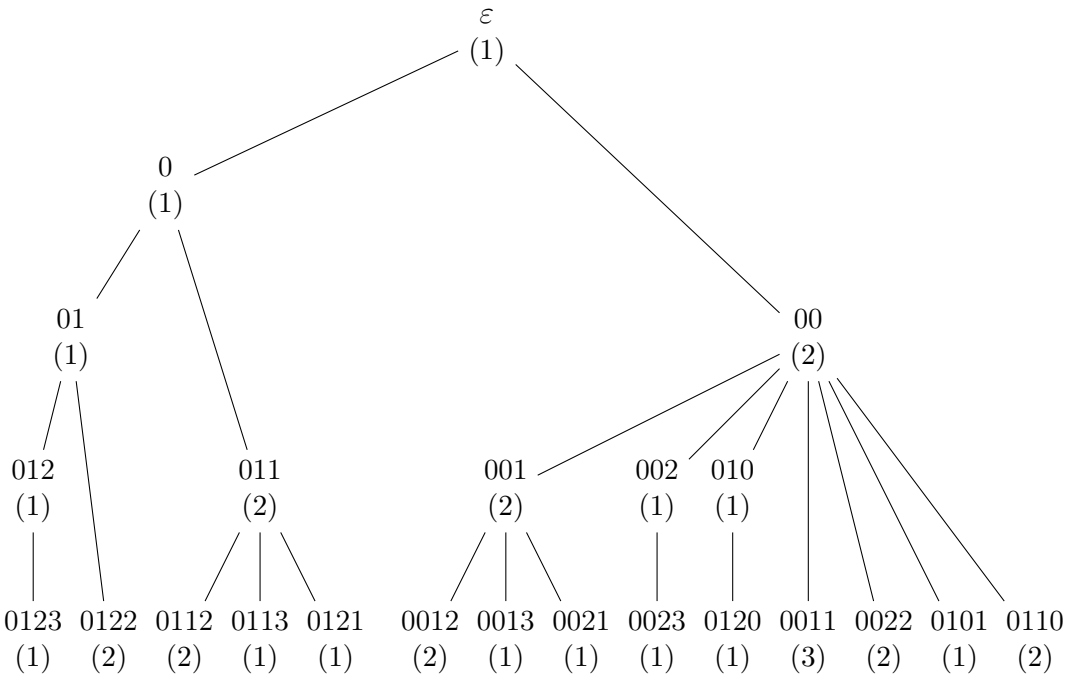

Figure 2.4: First five levels of the generating tree described by the succession rule $\Omega_{\{000,102\}}$.

one yields a sequence $\sigma'' = [\sigma']_i^{m+k}$ which satisfies $\mathbf{sites}(\sigma'') = i + 1 - (m + k) = \mathbf{sites}(\sigma') + 1$. We obtain $\mathbf{sites}(\sigma) + 1 - j$ sequences of length $n + 2$ and label $j + 1$ for each $j \in [1, \mathbf{sites}(\sigma)]$. This corresponds to $(s) \overset{2}{\rightsquigarrow} (j+1)^{s+1-j}$ in the succession rule.

In order to insert a value $m + k > m$ in two non adjacent positions in σ, we choose two positions $i_1 < i_2 \in [m + k + 1, \mathbf{inc}(\sigma) + 1]$ and define $\sigma'' = [[\sigma]_{i_2}^{m+k}]_{i_1}^{m+k}$, which satisfies $\mathbf{sites}(\sigma'') = i_1 - (m + k) \in [1, \mathbf{sites}(\sigma)]$. Let $j \in [1, \mathbf{sites}(\sigma)]$, and let us count how many choices of i_1, i_2 and k satisfy $\mathbf{sites}(\sigma'') = j$. The triple (i_1, i_2, k) is a solution whenever $j = i_1 - (m+k)$, i.e. whenever $k = i_1 - m - j$. Since k is positive, there is one solution for each (i_1, i_2) such that $i_1 > m + j$. The number of solutions is the number of ordered pairs (i_1, i_2) such that $i_1 < i_2$ and $i_1, i_2 \in [m + j + 1, \mathbf{inc}(\sigma) + 1]$, so there are $\binom{\mathbf{sites}(\sigma)+1-j}{2}$ solutions. This corresponds to $(s) \overset{2}{\rightsquigarrow} (j)^{\binom{s+1-j}{2}}$ in the succession rule. □

Let $F(x) = \sum_{n\geq 0} |\mathcal{I}_n(000, 102)| x^n$ be the ordinary generating function of $\mathcal{I}(000, 102)$.

Theorem 2.3.4. *The function F is algebraic of degree 4, and satisfies*

$$x^4F(x)^4 - 2x^3(x-1)F(x)^3 + x(x^3 - 2x^2 + 4x - 1)F(x)^2 - (2x^2 - 2x + 1)F(x) + 1 = 0.$$

Theorem 2.3.4 was proved by Jay Pantone in a personal communication, by solving a system of equations we derived from the succession rule $\Omega_{\{000,102\}}$ using methods from [28]. We present this proof in the appendix of [A].

2.3.3 Avoiding {102, 210}

For any $n \geq 0$ and any $\sigma \in \mathbb{N}^n$, let $\mathbf{Des}(\sigma) = \{i \in [1, n-1] \ : \ \sigma_i > \sigma_{i+1}\}$ be the set of *descent tops* of σ.

Proposition 2.3.5. *Let σ be a $\{102, 210\}$-avoiding integer sequence. Then any descent top of σ is the position of a maximum of σ, i.e. if $d \in \mathbf{Des}(\sigma)$ then $\sigma_d = \mathbf{max}(\sigma)$.*

Proof. Assume, for the sake of contradiction, that $i \in \mathbf{Des}(\sigma)$ and $\sigma_i < \mathbf{max}(\sigma)$. Since $\mathbf{Des}(\sigma)$ is nonempty, $|\sigma| \geq 2$. Let $j \in [1, |\sigma|]$ be such that $\sigma_j = \mathbf{max}(\sigma)$. In particular, $j \notin \{i, i+1\}$. If $j < i$, then $(\sigma_j, \sigma_i, \sigma_{i+1})$ is an occurrence of 210. If $j > i+1$, then $(\sigma_i, \sigma_{i+1}, \sigma_j)$ is an occurrence of 102. □

Proposition 2.3.6. *Let σ be a $\{102, 210\}$-avoiding integer sequence, and let $v > \mathbf{max}(\sigma)$.*

- *If σ is nondecreasing, then inserting the value v anywhere in σ does not create any occurrence of 102 or 210.*
- *If σ has exactly one descent $\sigma_d > \sigma_{d+1}$, then inserting the value v at some position p in σ yields a $\{102, 210\}$-avoiding sequence if and only if $p = d + 1$.*
- *If σ has at least two descents, then inserting the value v anywhere in σ creates an occurrence of 102 or 210.*

Proof. We go case-by-case.

- Since v is greater than all values of σ, the value v could only take the role of the 2 in a pattern 102 or 210. Hence, in order for the insertion of the value v to create an occurrence of either pattern, σ must already contain the pattern 10.
- If d is the only descent top of σ, then all occurrences (σ_i, σ_j) of the pattern 10 in σ must satisfy $i \leq d$ and $j > d$. In particular, inserting v at position $d + 1$ cannot create an occurrence of either pattern 102 or 210. If v is inserted at any other position, the resulting sequence still contains a descent top of value $\sigma_d < v$, so it contains 102 or 210 by Proposition 2.3.5.
- Inserting v anywhere in σ yields a sequence which still contains a descent top of value $\mathbf{max}(\sigma) < v$, and therefore contains 102 or 210 by Proposition 2.3.5. □

It is now clear we want to partition $\mathcal{I}(102, 210)$ into three subsets. For all $n \geq 0$, let

- $\mathfrak{A}_n = \{\sigma \in \mathcal{I}_n(102, 210) \ : \ |\mathbf{Des}(\sigma)| = 0\} = \mathcal{I}_n(10)$ be the set of nondecreasing inversion sequences of length n,
- $\mathfrak{B}_n = \{\sigma \in \mathcal{I}_n(102, 210) \ : \ |\mathbf{Des}(\sigma)| = 1\}$ be the set of $\{102, 210\}$-avoiding inversion sequences of length n having exactly one descent,
- $\mathfrak{C}_n = \{\sigma \in \mathcal{I}_n(102, 210) \ : \ |\mathbf{Des}(\sigma)| \geq 2\}$ be the set of $\{102, 210\}$-avoiding inversion sequences of length n having at least two descents.

We define a statistic $\mathbf{bounce} : \mathcal{I}(102, 210) \to \mathbb{N}$, which behaves differently on each subset.

- For all $\sigma \in \mathfrak{A}_n$, let $\mathbf{bounce}(\sigma) = n - \sigma_n$, with the convention $\mathbf{bounce}(\varepsilon) = 0$.
- For all $\sigma \in \mathfrak{B}_n$, let $\mathbf{bounce}(\sigma) = d - \sigma_d$, where d is the only element of $\mathbf{Des}(\sigma)$.
- For all $\sigma \in \mathfrak{C}_n$, let $\mathbf{bounce}(\sigma) = \mathbf{firstmax}(\sigma) - \mathbf{max}(\sigma)$.

Note that when σ is nonempty, $\mathbf{bounce}(\sigma)$ counts the difference between some position of the maximum of σ and the value of this maximum; however, the exact position varies according to the number of descents of σ. In particular, $\mathbf{bounce}(\sigma) > 0$ if $\sigma \neq \varepsilon$.

Let $\mathfrak{A}_{n,k} = \{\sigma \in \mathfrak{A}_n \ : \ k = \mathbf{bounce}(\sigma)\}$, $\mathfrak{B}_{n,k} = \{\sigma \in \mathfrak{B}_n \ : \ k = \mathbf{bounce}(\sigma)\}$, and $\mathfrak{C}_{n,k} = \{\sigma \in \mathfrak{C}_n \ : \ k = \mathbf{bounce}(\sigma)\}$. We also consider some subsets of $\mathfrak{B}_{n,k}$ and $\mathfrak{C}_{n,k}$.

- Let $\mathfrak{B}^{(1)}_{n,k} = \{\sigma \in \mathfrak{B}_{n,k} \ : \ (\sigma_i)_{i\in[1,n]\setminus \mathbf{firstmax}(\sigma)} \in \mathfrak{A}_{n-1}\}$ be the subset of sequences of $\mathfrak{B}_{n,k}$ such that removing the first maximum yields a sequence in $\mathfrak{A}_{n-1}$. Let $\mathfrak{B}^{(2)}_{n,k} = \mathfrak{B}_{n,k}\setminus\mathfrak{B}^{(1)}_{n,k}$.
- Let $\mathfrak{C}^{(1)}_{n,k} = \{\sigma \in \mathfrak{C}_{n,k} \ : \ (\sigma_i)_{i\in[1,n]\setminus \mathbf{firstmax}(\sigma)} \in \mathfrak{B}_{n-1}\}$ be the subset of sequences of $\mathfrak{C}_{n,k}$ such that removing the first maximum yields a sequence in $\mathfrak{B}_{n-1}$. Let $\mathfrak{C}^{(2)}_{n,k} = \mathfrak{C}_{n,k}\setminus\mathfrak{C}^{(1)}_{n,k}$.

We give an example for each of the five cases.

- $\alpha = (0,0,1,3,3,3,4,6,6) \in \mathfrak{A}_{9,3}$, since $\mathbf{Des}(\alpha) = \emptyset$ and $\mathbf{bounce}(\alpha) = 9-6$.
- $\beta = (0,0,0,1,1,4,1,3,4) \in \mathfrak{B}^{(1)}_{9,2}$, since $\mathbf{Des}(\beta) = \{6\}$ and $\mathbf{bounce}(\beta) = 6-4$.
- $\beta' = (0,0,0,1,2,2,2,0,1) \in \mathfrak{B}^{(2)}_{9,5}$, since $\mathbf{Des}(\beta') = \{7\}$ and $\mathbf{bounce}(\beta') = 7-2$.
- $\gamma = (0,0,1,1,2,4,3,4,3) \in \mathfrak{C}^{(1)}_{9,2}$, since $\mathbf{Des}(\gamma) = \{6,8\}$ and $\mathbf{bounce}(\gamma) = 6-4$.
- $\gamma' = (0,1,1,3,3,1,3,2,3) \in \mathfrak{C}^{(2)}_{9,1}$, since $\mathbf{Des}(\gamma') = \{5,7\}$ and $\mathbf{bounce}(\gamma') = 4-3$.

Lemma 2.3.7. *The enumeration of inversion sequences avoiding the patterns 102 and 210 is given by the succession rule*

$$\Omega_{\{102,210\}} = \begin{cases} (a,0) & & & \\ (a,k) & \rightsquigarrow & (a,i) & \text{for } i \in [1,k+1] \\ & & (b^{(1)},i)^{k-i} & \text{for } i \in [1,k-1] \\ (b^{(1)},k) & \rightsquigarrow & (b^{(1)},k) & \\ & & (b^{(2)},i) & \text{for } i \in [1,k+1] \\ & & (c^{(1)},i) & \text{for } i \in [1,k-1] \\ (b^{(2)},k) & \rightsquigarrow & (b^{(2)},i) & \text{for } i \in [1,k+1] \\ (c^{(1)},k) & \rightsquigarrow & (c^{(1)},k) & \\ & & (c^{(2)},i) & \text{for } i \in [1,k] \\ (c^{(2)},k) & \rightsquigarrow & (c^{(2)},i) & \text{for } i \in [1,k]. \end{cases}$$

The generating tree described by $\Omega_{\{102,210\}}$ is depicted on Figure 2.5, on page 57.

Proof. We describe a combinatorial generating tree for $\mathcal{I}(102,210)$ which "grows" inversion sequences by inserting entries in increasing order of value. We then label the sequences of $\mathcal{I}(102,210)$ according to which set they belong to: each sequence in a set $\mathfrak{A}_{n,k}$ is labelled (a,k), each sequence in a set $\mathfrak{B}^{(i)}_{n,k}$ is labelled $(b^{(i)},k)$, and each sequence in a set $\mathfrak{C}^{(i)}_{n,k}$ is labelled $(c^{(i)},k)$, for $i \in \{1,2\}$. In particular, the empty sequence ε is labelled $(a,0)$, which is the axiom of the succession rule.

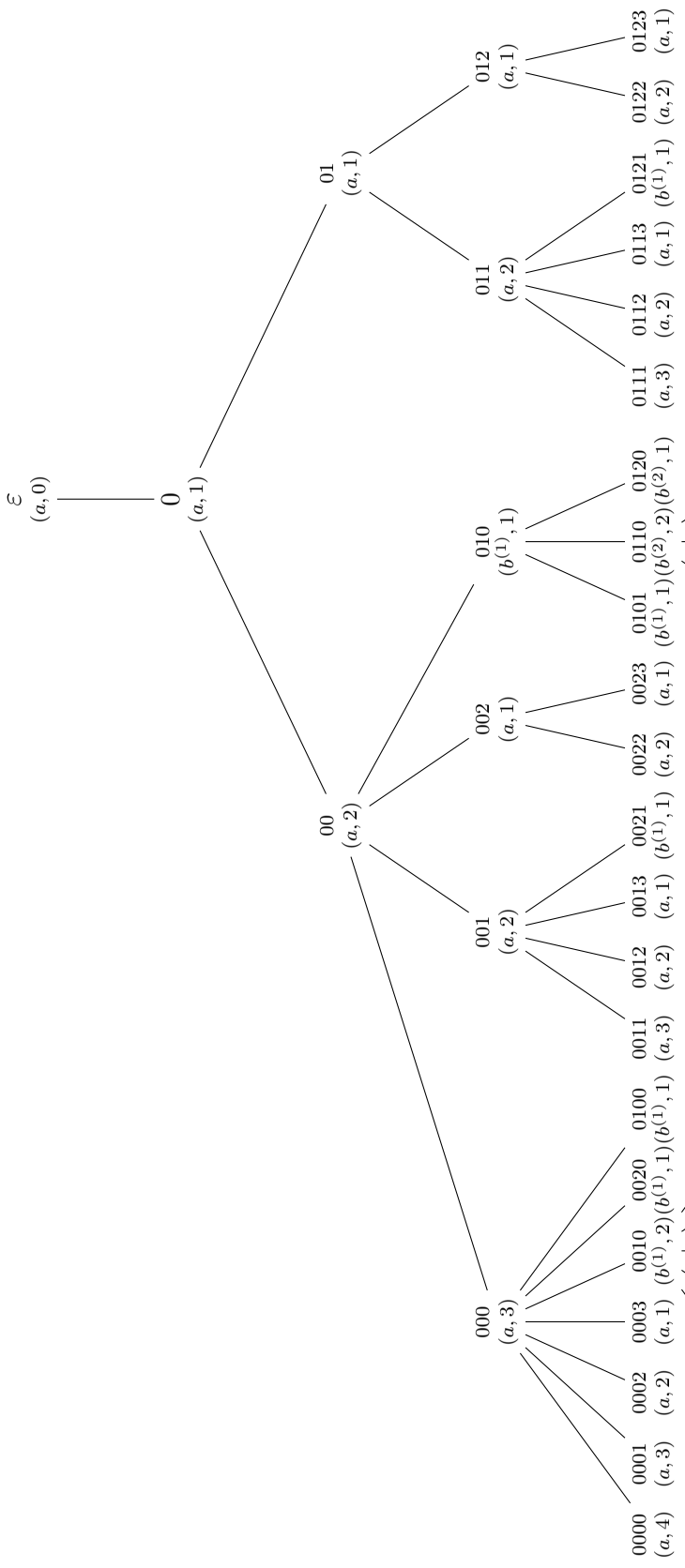

Figure 2.5: Part of the generating tree described by the succession rule $\Omega_{\{102,210\}}$.

The order of insertion of the different occurrences of each value is not simple. We first explain it informally, before describing the full construction in detail. Let $\sigma \in \mathcal{I}(102, 210)$ be an inversion sequence of length $n \geq 1$ and maximum m. Let $\sigma' \in \mathcal{I}(102, 210)$ be the subsequence obtained by removing all values m from σ. We generate σ from σ' by inserting entries of value m in the following order.

1. The rightmost descent top of σ (if σ has at least one descent).
2. Any entries of value m appearing in a single factor at the end of σ, from left to right.
3. The second rightmost descent top of σ (if σ has at least two descents).
4. Any entries appearing in a constant factor of value m which contains the rightmost descent top of σ (if σ has at least one descent), inserted from left to right, and to the right of the entry inserted at step 1.
5. Any remaining entries of value m, from right to left.

For instance if $\sigma = (0, 0, 0, 1, 2, 2, 6, 6, 2, 3, 6, 4, 6, 6, 6, 4, 4, 6, 6, 6, 4, 6, 6) \in \mathcal{I}_{23}(102, 210)$, then the order of insertion of the 11 entries of maximal value 6 is shown on the bottom row of the example below.

0	0	0	1	2	2	6	6	2	3	6	4	6	6	6	4	4	6	6	6	4	6	6
						11	10			9		8	7	4			1	5	6		2	3

This order of insertion may not be intuitive, but the resulting generating tree construction has some advantages compared to a simpler order:

- most importantly, it only requires one parameter (the value of the **bounce** statistic),
- it does not involve "phantom objects" as in Section 2.2.3,
- the resulting succession rule does not involve the length n of the sequences.

We now turn to a step-by-step description of the generating tree, which matches the succession rule $\Omega_{\{102,210\}}$. Let $\sigma \in \mathcal{I}(102, 210)$, $n = |\sigma|$, $k = \mathbf{bounce}(\sigma)$, and $m = \mathbf{max}(\sigma)$.

- Suppose $\sigma \in \mathfrak{A}_{n,k}$. In particular, if σ is nonempty then $\mathbf{max}(\sigma) = \sigma_n = n - k$.
  - Inserting any value $v \in [n - k, n]$ at the end of σ yields a sequence in $\mathfrak{A}_{n+1,n+1-v}$, where $n+1-v \in [1, k+1]$. This production alone generates all nondecreasing inversion sequences.
  - If σ is nonempty, inserting any value $v \in [n-k+1, n-1]$ at any position $p \in [v+1, n]$ creates a sequence in $\mathfrak{B}^{(1)}_{n+1,p-v}$. For each $i = p-v \in [1, k-1]$, there are $k-i$ sequences in $\mathfrak{B}^{(1)}_{n+1,i}$ generated by σ (one sequence for each value $v \in [n-k+1, n-i]$, for $p = v+i$).
- Suppose $\sigma \in \mathfrak{B}^{(1)}_{n,k}$. In particular, $\mathbf{Des}(\sigma) = \{k + m\}$.
  - Inserting the value m at the end of σ yields a sequence in $\mathfrak{B}^{(1)}_{n+1,k}$.
  - Inserting any value $v \in [m, m + k]$ between the top and bottom of the only descent of σ (i.e. at position $k + m + 1$) yields a sequence in $\mathfrak{B}^{(2)}_{n+1,k-v+m+1}$.
  - Inserting the value m at any position $p \in [m + 1, k + m - 1]$ yields a sequence in $\mathfrak{C}^{(1)}_{n+1,p-m}$.

- Suppose $\sigma \in \mathfrak{B}^{(2)}_{n,k}$. In particular, $\mathbf{Des}(\sigma) = \{k+m\}$. As before, inserting any value $v \in [m, m+k]$ between the top and bottom of the only descent of σ (i.e. at position $k+m+1$) yields a sequence in $\mathfrak{B}^{(2)}_{n+1,k-v+m+1}$.

- Suppose $\sigma \in \mathfrak{C}^{(1)}_{n,k}$. In particular, $\mathbf{firstmax}(\sigma) = k+m$.

  - Inserting the value m right-adjacent to the rightmost descent top of σ (i.e. at position $\mathbf{max}(\mathbf{Des}(\sigma))+1$) yields a sequence in $\mathfrak{C}^{(1)}_{n+1,k}$.
  - Inserting the value m at any position $p \in [m+1, \mathbf{firstmax}(\sigma)]$ yields a sequence in $\mathfrak{C}^{(2)}_{n+1,p-m}$.

- Suppose $\sigma \in \mathfrak{C}^{(2)}_{n,k}$. In particular, $\mathbf{firstmax}(\sigma) = k+m$. As before, inserting the value m at any position $p \in [m+1, \mathbf{firstmax}(\sigma)]$ yields a sequence in $\mathfrak{C}^{(2)}_{n+1,p-m}$.

Notice that inserting some value $v \geq m$ at any other position would either yield a sequence which already appears in this construction, or create an occurrence of a pattern 102 or 210. □

2.3.3.1 The algebraic generating function of $\mathcal{I}$(102,210)

Let $F(x) = \sum_{n\geq 0} |\mathcal{I}_n(102,210)|x^n$ be the ordinary generating function of $\mathcal{I}(102,210)$, and let

$$\begin{aligned} A(x,y) &= \sum_{n,k\geq 0} \mathfrak{a}_{n,k} x^n y^k, \\ B^{(i)}(x,y) &= \sum_{n,k\geq 0} \mathfrak{b}^{(i)}_{n,k} x^n y^k \quad \text{for} \quad i \in \{1,2\}, \\ C^{(i)}(x,y) &= \sum_{n,k\geq 0} \mathfrak{c}^{(i)}_{n,k} x^n y^k \quad \text{for} \quad i \in \{1,2\}. \end{aligned}$$

Proposition 2.3.8. *The functions $A, B^{(1)}, B^{(2)}, C^{(1)}, C^{(2)}$, and F satisfy the following equations.*

$$\begin{aligned} A(x,y) &= 1 + \frac{xy}{1-y}\big(A(x,1) - yA(x,y)\big) \\ B^{(1)}(x,y) &= xB^{(1)}(x,y) + \frac{xy}{1-y}\left(\frac{\partial A}{\partial y}(x,1) + \frac{1}{1-y}\big(A(x,y) - A(x,1)\big)\right) \\ B^{(2)}(x,y) &= \frac{xy}{1-y}\Big(B^{(1)}(x,1) + B^{(2)}(x,1) - y\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)\Big) \\ C^{(1)}(x,y) &= xC^{(1)}(x,y) + \frac{x}{1-y}\Big(yB^{(1)}(x,1) - B^{(1)}(x,y)\Big) \\ C^{(2)}(x,y) &= \frac{xy}{1-y}\Big(C^{(1)}(x,1) + C^{(2)}(x,1) - C^{(1)}(x,y) - C^{(2)}(x,y)\Big). \\ F(x) &= A(x,1) + B^{(1)}(x,1) + B^{(2)}(x,1) + C^{(1)}(x,1) + C^{(2)}(x,1) \end{aligned}$$

Proof. We turn the succession rule $\Omega_{\{102,210\}}$ into equations relating the generating functions.

$$\begin{aligned} A(x,y) &= 1 + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} x^{n+1} \sum_{i=1}^{k+1} y^i \\ &= 1 + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} x^{n+1} \frac{y - y^{k+2}}{1-y} \\ &= 1 + \frac{xy}{1-y}\big(A(x,1) - yA(x,y)\big) \end{aligned}$$

$$\begin{aligned} B^{(1)}(x,y) &= \left(\sum_{n,k\geq 0} \mathfrak{b}^{(1)}_{n,k} x^{n+1} y^k\right) + \left(\sum_{n,k\geq 0} \mathfrak{a}_{n,k} x^{n+1} \sum_{i=1}^{k-1} (k-i) y^i\right) \\ &= xB^{(1)}(x,y) + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} \frac{x^{n+1} y}{1-y} \left(k + \frac{y^k - 1}{1-y}\right) \\ &= xB^{(1)}(x,y) + \frac{xy}{1-y}\left(\frac{\partial A}{\partial y}(x,1) + \frac{1}{1-y}\big(A(x,y) - A(x,1)\big)\right) \end{aligned}$$

$$\begin{aligned} B^{(2)}(x,y) &= \sum_{n,k\geq 0} \left(\mathfrak{b}^{(1)}_{n,k} + \mathfrak{b}^{(2)}_{n,k}\right) x^{n+1} \sum_{i=1}^{k+1} y^i \\ &= \sum_{n,k\geq 0} \left(\mathfrak{b}^{(1)}_{n,k} + \mathfrak{b}^{(2)}_{n,k}\right) x^{n+1} \frac{y - y^{k+2}}{1-y} \\ &= \frac{xy}{1-y}\left(B^{(1)}(x,1) + B^{(2)}(x,1) - y\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)\right) \end{aligned}$$

$$\begin{aligned} C^{(1)}(x,y) &= \left(\sum_{n,k\geq 0} \mathfrak{c}^{(1)}_{n,k} x^{n+1} y^k\right) + \left(\sum_{n,k\geq 0} \mathfrak{b}^{(1)}_{n,k} x^{n+1} \sum_{i=1}^{k-1} y^i\right) \\ &= xC^{(1)}(x,y) + \sum_{n,k\geq 0} \mathfrak{b}^{(1)}_{n,k} x^{n+1} \frac{y - y^k}{1-y} \\ &= xC^{(1)}(x,y) + \frac{x}{1-y}\left(yB^{(1)}(x,1) - B^{(1)}(x,y)\right) \end{aligned}$$

$$\begin{aligned} C^{(2)}(x,y) &= \sum_{n,k\geq 0} \left(\mathfrak{c}^{(1)}_{n,k} + \mathfrak{c}^{(2)}_{n,k}\right) x^{n+1} \sum_{i=1}^{k} y^i \\ &= \sum_{n,k\geq 0} \left(\mathfrak{c}^{(1)}_{n,k} + \mathfrak{c}^{(2)}_{n,k}\right) x^{n+1} \frac{y - y^{k+1}}{1-y} \\ &= \frac{xy}{1-y}\left(C^{(1)}(x,1) + C^{(2)}(x,1) - C^{(1)}(x,y) - C^{(2)}(x,y)\right). \end{aligned}$$

By definition, $F(x) = A(x,1) + B^{(1)}(x,1) + B^{(2)}(x,1) + C^{(1)}(x,1) + C^{(2)}(x,1)$. □

Theorem 2.3.9. *The generating function of* $\mathcal{I}(102, 210)$ *is*

$$F(x) = \frac{(4x-1)(4x^4-22x^3+25x^2-9x+1) - (2x-1)(x^2-5x+1)(2x^2-4x+1)\sqrt{1-4x}}{2x^3(4x-1)(x-1)^2}.$$

Proof. From Proposition 2.3.8, we have

$$(1-y+xy^2)A(x,y) = 1-y+xyA(x,1) \tag{2.2}$$

$$B^{(1)}(x,y) = \frac{xy}{(1-x)(1-y)}\left(\frac{\partial A}{\partial y}(x,1) + \frac{1}{1-y}\big(A(x,y)-A(x,1)\big)\right) \tag{2.3}$$

$$(1-y+xy^2)B^{(2)}(x,y) = xy\left(B^{(1)}(x,1) + B^{(2)}(x,1) - yB^{(1)}(x,y)\right) \tag{2.4}$$

$$C^{(1)}(x,y) = \frac{x}{(1-x)(1-y)}\left(yB^{(1)}(x,1) - B^{(1)}(x,y)\right) \tag{2.5}$$

$$(1-y+xy)C^{(2)}(x,y) = xy\left(C^{(1)}(x,1) + C^{(2)}(x,1) - C^{(1)}(x,y)\right). \tag{2.6}$$

First, we derive an expression of $A(x,y)$ from (2.2). To cancel the kernel $(1-y+xy^2)$, there are two solution in y:

$$Y_1(x) = \frac{1-\sqrt{1-4x}}{2x}, \qquad Y_2(x) = \frac{1+\sqrt{1-4x}}{2x}.$$

Only Y_1 defines a formal power series. We replace y by $Y_1(x)$ in (2.2), and obtain

$$A(x,1) = \frac{1-\sqrt{1-4x}}{2x}.$$

Replacing $A(x,1)$ by this expression in (2.2) yields

$$A(x,y) = \frac{2-y-y\sqrt{1-4x}}{2(1-y+xy^2)}.$$

Having this closed form expression for $A(x,y)$ implies that we also have one for $B^{(1)}(x,y)$ since it depends only on A in (2.3), and one for $C^{(1)}(x,y)$ since it depends only on $B^{(1)}$ in (2.5).

Next, we look for an expression of $B^{(2)}(x,1)$ from (2.4). The kernel is the same as for A. Replacing y by $Y_1(x) = \frac{1-\sqrt{1-4x}}{2x}$ in (2.4) yields

$$B^{(2)}(x,1) = Y_1(x)B^{(1)}(x,Y_1(x)) - B^{(1)}(x,1).$$

Evaluating $B^{(1)}(x,Y_1(x))$ from (2.3) requires an expression of $A(x,Y_1(x))$. Substituting y for $Y_1(x)$ in our closed form expression of $A(x,y)$ symbolically yields the fraction $\frac{0}{0}$. Instead, we compute the limit of $A(x,Y_1(z))$ as z tends to x, and find that

$$A(x,Y_1(x)) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-4x}}\right).$$

We can now obtain an expression of $B^{(1)}(x,Y_1(x))$ from (2.3), and therefore of $B^{(2)}(x,1)$.

Finally, we look for an expression of $C^{(2)}(x,1)$ from (2.6). The kernel is $(1-y+xy)$. We cancel it by setting $y = \frac{1}{1-x}$, and obtain

$$C^{(2)}(x,1) = C^{(1)}\left(x, \frac{1}{1-x}\right) - C^{(1)}(x,1).$$

The generating function of $\mathcal{I}(102,210)$ is

$$\begin{aligned} F(x) &= A(x,1) + B^{(1)}(x,1) + B^{(2)}(x,1) + C^{(1)}(x,1) + C^{(2)}(x,1) \\ &= \frac{(4x-1)(4x^4-22x^3+25x^2-9x+1) - (2x-1)(x^2-5x+1)(2x^2-4x+1)\sqrt{1-4x}}{2x^3(4x-1)(x-1)^2}. \end{aligned}$$ □

2.3.4 A generating tree perspective on the upcoming sections

In Sections 2.3.5 to 2.3.9, we do not provide succession rules for generating trees, although our approach is very similar to a generating tree inserting increasing values. In order to simplify the presentation, we instead count inversion sequences which contain a single maximum (with two parameters recording its position and value). We then easily generalize this result to count pattern-avoiding inversion sequences containing any number of maximal entries.

- The inversion sequences of Section 2.3.5 avoid 000, so only one repetition of the maximum may be inserted.
- The pairs of patterns studied in Sections 2.3.6 to 2.3.9 contain either 101 or 110. In a sequence that avoids 101, all occurrences of the maximum must be consecutive. In a sequence that avoids 110, all occurrences of the maximum after the leftmost one must appear in a single factor at the end of the sequence. In either case, each repetition of the maximum can only be placed at a single position.

In all five cases, inversion sequences could also be generated by inserting their values one by one, in increasing order (and inserting repetitions of a value from left to right), yielding generating trees as in the previous two sections. This would require distinguishing several subsets of inversion sequences to write succession rules, as we did in Sections 2.2.3 and 2.3.3, leading to slightly more complex formulas. These succession rules would always have two integer parameters, which record the number of positions where a new maximum may be inserted, and the number of possible values of a new maximum. The first parameter is influenced by pattern-avoidance, while the second one is used to maintain the "subdiagonal" property characterizing inversion sequences.

Note that Sections 2.3.2 and 2.3.3 use such generating tree constructions, but only have one integer parameter, for the reasons explained below.

- For the pair of patterns $\{000, 102\}$, the two parameters are one and the same, and correspond to the statistic we denoted **sites**, as explained in the paragraph just before Lemma 2.3.3.
- For the pair $\{102, 210\}$:
  - a new maximum cannot be inserted in a sequence with several descents; it follows that such sequences only require one parameter to record the number of positions where repetitions of the current maximum could be inserted.
  - in a sequence which has 0 or 1 descent, the positions where a new maximum may be inserted do not need to be recorded in a parameter (for 0 descents every position is available, and for 1 descent the only available position is between the top and the bottom of the descent), so only the number of possible values of a new maximum is required.

Such simplifications would not occur in the generating trees corresponding to Sections 2.3.5 to 2.3.9, hence our choice of opting for a different presentation.

2.3.5 Avoiding {000, 201} or {000, 210}

A bijection between $\mathcal{I}(000, 201)$ and $\mathcal{I}(000, 210)$ was established in [113, Theorem 8.1]. In this section we enumerate $\mathcal{I}(000, 201)$, although the same construction can also be applied to

$\mathcal{I}(000, 210)$ with a little more work, and results in the same equations. Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(000, 201) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{000, 201\}$-avoiding inversion sequences of length n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. By definition, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leq m$. When $m = 0$, $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$. The numbers $(\mathfrak{a}_{n,m,p})_{n,m,p\in\mathbb{N}}$ can then be calculated using the following lemma.

Lemma 2.3.10. *For all* $1 \leq m < p \leq n$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{p} \mathfrak{a}_{n-1,s,q} + (p - q + \delta_{p,q} - \delta_{p,n})\mathfrak{a}_{n-2,s,q}$$

Proof. Let $m \geq 1, p > m, n \geq p$ be three integers, and let $\sigma \in \mathfrak{A}_{n,m,p}$. Let σ' be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ be the second largest value in σ, and $q = \mathbf{firstmax}(\sigma')$. Note that if $q < p$ then $\sigma_q = s$, and if $q \geq p$ then $\sigma_{q+1} = s$. In particular, if $q > p$ then the subsequence $(\sigma_p, \sigma_{p+1}, \sigma_{q+1})$ is an occurrence of the pattern 201. Hence q is less than or equal to p.

Since σ' avoids the pattern 000, its maximum s appears either once or twice.

- If s appears only once in σ', then $\sigma' \in \mathfrak{A}_{n-1,s,q}$. Note that inserting the value m at position p in any sequence in $\mathfrak{A}_{n-1,s,q}$ for $s < m$ and $q \leq p$ does not create an occurrence of 000 or 201, so this is a bijection.

- If s appears twice, then removing the second occurrence yields an inversion sequence $\sigma'' \in \mathfrak{A}_{n-2,s,q}$. The possible positions for the second s in σ' are:

  - Before p: $\begin{cases} [q+1, p-1] & \text{if} \quad q < p \\ \emptyset & \text{if} \quad q = p \end{cases}$, so there are $p - q - 1 + \delta_{p,q}$ choices.

  - After p: $\begin{cases} \{p+1\} & \text{if} \quad p < n \\ \emptyset & \text{if} \quad p = n \end{cases}$, so there are $1 - \delta_{p,n}$ choices.

  This adds up to $p - q + \delta_{p,q} - \delta_{p,n}$ possible positions for the second s.

Summing over all possible values of m and p completes the proof. □

Theorem 2.3.11. *For all* $n \geq 2$,

$$|\mathcal{I}_n(000, 201)| = \sum_{m=0}^{n-1} \sum_{p=m+1}^{n} \mathfrak{a}_{n,m,p} + (n - p)\mathfrak{a}_{n-1,m,p}.$$

Proof. Let $n \geq 2$, $\sigma \in \mathcal{I}_n(000, 201)$, $m = \mathbf{max}(\sigma)$, and $p = \mathbf{firstmax}(\sigma)$. Since σ avoids 000, the maximum m appears either once or twice in σ.

- If m appears only once, then $\sigma \in \mathfrak{A}_{n,m,p}$.

- If m appears twice, then removing the second m yields a sequence $\sigma' \in \mathfrak{A}_{n-1,m,p}$. There are $n - p$ possible positions for the second m, hence $n - p$ different sequences σ correspond to the same σ'.

Summing over all possible values of m and p completes the proof. □

2.3.6 Avoiding {100, 110}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(100, 110) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{100, 110\}$-avoiding inversion sequences of length n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leq m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 2.3.12. *Let $1 \leq m < p \leq n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} (n-p) \cdot \mathfrak{a}_{n-1,s,p} + \sum_{q=s+1}^{p-1} \mathfrak{a}_{n-1,s,q} + \mathfrak{a}_{n-2,s,q}.$$

Proof. Let $1 \leq m < p \leq n$, and $\sigma \in \mathfrak{A}_{n,m,p}$. Let $\sigma' = (\sigma_i)_{i\in[1,n]\setminus\{p\}}$ be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ and $q = \mathbf{firstmax}(\sigma')$. Let σ'' be the sequence obtained by removing all terms of value s except the first one from σ', so that $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for some $\ell \in [q, n-1]$.

Since σ avoids 100, all terms of σ after position p must have distinct values. In particular, s appears at most once after p. Since σ' avoids 110, every s after the first one must appear in a single factor at the end of σ'. At this point, there are two cases:

1. If $p \in \{n-1, n\}$, then the value s could appear any number of times without creating an occurrence of 100 in σ. In that case, σ'' can be any sequence in $\mathfrak{A}_{\ell,s,q}$ for any $\ell \in [q, n-1]$: starting from any sequence in $\mathfrak{A}_{\ell,s,q}$ for any $s < m$, $q > s$, and $\ell < n$, we can append $n-\ell-1$ terms of value s, then insert m at position $p \in \{n-1, n\}$ without creating an occurrence of 100 or 110, and this is a bijection. Thus, we obtain the first equation of the lemma.

2. If $p < n-1$:

   - If $q < p$:
     - If σ contains a single term of value s, then $\sigma'' = \sigma' \in \mathfrak{A}_{n-1,s,q}$.
     - If σ contains two terms of value s, then the second one is σ_n due to the avoidance of 110. In that case, $\sigma'' \in \mathfrak{A}_{n-2,s,q}$.

     In either case, this is a bijection for the same reason as before. Note that s cannot appear more than twice in σ, otherwise $(\sigma_p, \sigma_{n-1}, \sigma_n) = (m, s, s)$ would be an occurrence of 100.

   - If $q \geq p$, then s appears only once in σ in order to avoid 100. In that case, $\sigma'' = \sigma'$ can be any sequence in $\mathfrak{A}_{n-1,s,q}$ such that all values $(\sigma''_i)_{i\in[p,n-1]}$ are distinct. Moving the value s in σ'' from position q to position p yields a sequence $\tau \in \mathfrak{A}_{n-1,s,p}$. Conversely, starting from any sequence $\tau \in \mathfrak{A}_{n-1,s,p}$ and moving the value s from position p to position $q \in [p, n-1]$ yields a sequence $\sigma'' \in \mathfrak{A}_{n-1,s,q}$ such that all values $(\sigma''_i)_{i\in[p,n-1]}$ are distinct. Since there are $n-p$ possible values of q, each sequence $\tau \in \mathfrak{A}_{n-1,s,p}$ is obtained from $n-p$ different sequences σ''. □

Theorem 2.3.13. *For all $n \geq 2$,*

$$|\mathcal{I}_n(100, 110)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Let $n \geq 2$, $\sigma \in \mathcal{I}_n(100, 110)$, $m = \mathbf{max}(\sigma)$, and $p = \mathbf{firstmax}(\sigma)$. Since σ avoids 110, all terms of value m after the first one σ_p are in a factor $(\sigma_i)_{i\in[\ell+1,n]}$ for some $\ell \in [p, n]$. Let σ' be the sequence obtained by removing all terms of value m from σ, except the first one σ_p. By definition, $\sigma' \in \mathfrak{A}_{\ell,m,p}$. Since we can rebuild σ by appending $n - \ell$ terms of value m to σ', this is a bijection between $\mathcal{I}_n(100, 110)$ and $\coprod_{0\leq m<p\leq\ell\leq n} \mathfrak{A}_{\ell,m,p}$. □

2.3.7 Avoiding {100, 101}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(100, 101) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{100, 101\}$-avoiding inversion sequences of length n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leq m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 2.3.14. *Let $1 \leq m < p \leq n$. If $p \in \{n - 1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n - 1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} (n - p) \cdot \mathfrak{a}_{n-1,s,p} + \sum_{q=s+1}^{p-1} \sum_{\ell=n-p+q-1}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

Proof. This is quite similar to the proof of Lemma 2.3.12, except that now the terms of value s in σ' appear in a single factor starting at position q, instead of being split into σ'_q and a factor at the end of σ'. This makes a difference in only one case: if $p < n - 1$ and $q < p$, then there may be more than two terms of value s. More precisely, the terms of value s in σ' are in a factor $(\sigma_i)_{i\in[q,q+k]}$ for some $k \in [0, p - q]$. In particular $\sigma'' = (\sigma'_i)_{i\in[1,n-1]\setminus[q+1,q+k]} \in \mathfrak{A}_{\ell,s,q}$, so $\ell = n - 1 - k \in [n - p + q - 1, n - 1]$. □

Theorem 2.3.15. *For all $n \geq 2$,*

$$|\mathcal{I}_n(100, 101)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Very similar to the proof of Theorem 2.3.13, but now the factor of repetitions of the maximum is placed immediately after the first maximum, instead of being at the end of the sequence. □

2.3.8 Avoiding {110, 201}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(110, 201) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{110, 201\}$-avoiding inversion sequences of length n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leq m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 2.3.16. *Let $1 \leq m < p \leq n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{p} \mathfrak{a}_{n-1,s,q} + \sum_{\ell=q}^{p-1+\delta_{p,q}} \mathfrak{a}_{\ell,s,q}.$$

Proof. Let $1 \leq m < p \leq n$, and $\sigma \in \mathfrak{A}_{n,m,p}$. Let $\sigma' = (\sigma_i)_{i\in[1,n]\setminus\{p\}}$ be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ and $q = \mathbf{firstmax}(\sigma')$. In particular $q \leq p$, otherwise $(\sigma_p, \sigma_{p+1}, \sigma_{q+1})$ would be an occurrence of 201. Let σ'' be the sequence obtained by removing all terms of value s except the first one from σ', so that $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for some $\ell \in [q, n-1]$.

- If $p \in \{n-1, n\}$, we obtain the same equation as in Lemma 2.3.12, for the same reason.
- If $p < n-1$,
  - If s appears only once in σ', then $\sigma'' = \sigma'$ can be any sequence in $\mathfrak{A}_{n-1,s,q}$
  - If s appears more than once, then every s after the first one must be in a single factor at the end of σ' to avoid 110. Since σ avoids 201, the factor $(\sigma_i)_{i\in[p+1,n]}$ is nonincreasing, which means every term in this factor has value s. The sequence σ'' obtained has length $\ell \in [q, p-1]$ if $q < p$, or $\ell = p$ if $q = p$. □

Theorem 2.3.17. *For all $n \geq 2$,*

$$|\mathcal{I}_n(110, 201)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Identical to the proof of Theorem 2.3.13, since inserting a term of maximal value anywhere after the first maximum cannot create an occurrence of 201. □

2.3.9 Avoiding {101, 210}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(101, 210) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{101, 210\}$-avoiding inversion sequences of length n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leq m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 2.3.18. *Let $1 \leq m < p \leq n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \left(\sum_{q=p}^{n-1} \mathfrak{a}_{q,s,p} \right) + \sum_{q=s+1}^{p-1} \mathfrak{a}_{q,s,q} + \sum_{\ell=n-p+q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

Proof. Let $1 \leq m < p \leq n$, and $\sigma \in \mathfrak{A}_{n,m,p}$. Let $\sigma' = (\sigma_i)_{i\in[1,n]\setminus\{p\}}$ be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ and $q = \mathbf{firstmax}(\sigma')$. Let σ'' be the sequence obtained by removing all terms of value s except the first one from σ', so that $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for some $\ell \in [q, n-1]$.

- If $p \in \{n-1, n\}$, we obtain the same equation as in Lemma 2.3.14, for the same reason.
- If $p < n-1$:
  - If $q \geq p$, then every term in the factor $(\sigma_i)_{i\in[q+1,n]}$ has value s, because σ avoids 210. In that case σ'' is a sequence in $\mathfrak{A}_{q,s,q}$ such that $(\sigma''_i)_{i\in[p,q]}$ is nondecreasing. The set of such sequences σ'' is in bijection with $\mathfrak{A}_{q,s,p}$ (to see that, simply move the value s from position q to position p).
  - If $q < p$, let $k+1$ be the number of terms of value s in σ. In particular, $k = |\sigma'| - |\sigma''| = n-1-\ell$. Note that the terms of value s in σ' are exactly $(\sigma'_i)_{i\in[q,q+k]}$.
    - If $k \geq p-q$, then $\sigma_{p+1} = s$, which implies that every term $(\sigma_i)_{i\in[q,n]\setminus\{p\}}$ has value s because σ avoids 210. In particular $k = n-1-q$, $\ell = q$, and we obtain a sequence $\sigma'' \in \mathfrak{A}_{q,s,q}$.
    - If $k < p-q$, then the value s does not appear after position p, and we obtain a sequence $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for $\ell = n-1-k \in [n-p+q, n-1]$. □

Theorem 2.3.19. *For all $n \geq 2$,*

$$|\mathcal{I}_n(101, 210)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Identical to the proof of Theorem 2.3.15, since inserting a term of maximal value anywhere after the first maximum cannot create an occurrence of 210. □

2.4 Generating trees growing on the left

2.4.1 Introduction

In this section, we present a construction we call the generating tree "growing on the left" for inversion sequences. This construction first appeared in [46] to enumerate $\mathcal{I}(101)$ and $\mathcal{I}(110)$. We apply it to the classes $\mathcal{I}(201, 210)$ and $\mathcal{I}(010, 102)$.

The class $\mathcal{I}(201, 210)$ has an algebraic generating function, which was already found in two articles using two different methods. Chen and Lin [39] use a decomposition around the rightmost saturated[6] entry of inversion sequences, with a single parameter counting the number of saturated entries. Pantone [99] uses a generating tree in one integer parameter and two boolean parameters, involving some commitments to future decisions. In Section 2.4.2, we consider the generating tree growing on the left for $\mathcal{I}(201, 210)$. We find a succession rule which corresponds to a functional equation containing two evaluations $A(t, 1)$ and $A(t, 2)$ of a bivariate series A. This is quite unusual, and it occurs because the succession rule produces labels with multiplicities that are powers of 2. Regardless, this equation can still be solved with the kernel method, resulting in a new and simpler way of computing the algebraic generating function of $\mathcal{I}(201, 210)$.

[6] In an inversion sequence σ, the entry σ_i is said to be "saturated" if $\sigma_i = i-1$ (since $i-1$ is the largest value which could appear at position i). Saturated entries will have an important role in Chapter 4.

The class $\mathcal{I}(010, 102)$ is one we already studied in Section 1.2.2. The recurrence formula we found then can be used to compute hundreds of terms of its counting sequence, from which Pantone [99] conjectured an algebraic generating function using the software [98]. In Section 2.4.3, we modify the generating tree growing on the left construction to apply it to inversion sequences avoiding 010 and 102. In this case, we obtain a functional equation involving two catalytic variables, which can be solved by simply applying the kernel method twice. This allows us to compute the generating function of $\mathcal{I}(010, 102)$, confirming Pantone's conjecture.

For all $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}^n$, let $\mathbf{Zeros}(\sigma) = \{i \in [1, n] : \sigma_i = 0\}$ be the set of positions of zeros of σ. Let χ be the characteristic function of a statement, defined by $\chi(\mathrm{S}) = \begin{cases} 0 & \text{if S is false,} \\ 1 & \text{if S is true.} \end{cases}$

We define the generating tree for inversion sequences *growing on the left* as the only generating tree which has one node labelled by each inversion sequence (we now identify nodes with their labels), and in which the parent of a nonempty sequence is obtained by deleting its leftmost entry (which is always a zero) and subtracting 1 from every nonzero value. In this tree, the children of an inversion sequence are obtained by

1. first adding 1 to every nonzero value in the sequence,
2. then adding 1 to some (or none) of the zero entries,
3. lastly, inserting a zero at the leftmost position.

Observe that this construction is indeed a generating tree (it produces only inversion sequences and, starting from the empty sequence ε, it produces every inversion sequence exactly once). Note that steps 1 and 3 are deterministic, whereas step 2 involves choosing a subset of the zero entries. For any inversion sequence $\sigma \in \mathcal{I}$, and any $Z \subseteq \mathbf{Zeros}(\sigma)$, we denote

$$\mathbf{child}(\sigma, Z) = 0 \cdot (\sigma_i + \chi(\sigma_i > 0) + \chi(i \in Z))_{i \in [1, |\sigma|]}$$

the child of σ corresponding to the subset of zeros Z to which 1 is added. Since each child corresponds to a subset $Z \subseteq \mathbf{Zeros}(\sigma)$, the sequence σ has $2^{|\mathbf{Zeros}(\sigma)|}$ children. For instance, the sequence $\sigma = (0, 1, 1, 3, 0, 2, 5)$ satisfies $\mathbf{Zeros}(\sigma) = \{1, 5\}$, and its four children are

$$\begin{aligned} \mathbf{child}(\sigma, \emptyset) &= (0, 0, 2, 2, 4, 0, 3, 6), \\ \mathbf{child}(\sigma, \{1\}) &= (0, 1, 2, 2, 4, 0, 3, 6), \\ \mathbf{child}(\sigma, \{5\}) &= (0, 0, 2, 2, 4, 1, 3, 6), \\ \mathbf{child}(\sigma, \{1, 5\}) &= (0, 1, 2, 2, 4, 1, 3, 6). \end{aligned}$$

In the generating tree growing on the left, labelling each inversion sequence by its number of zeros yields a concise generating tree described by the succession rule Ω_{Left} below. This generating tree is represented on Figure 2.6.

$$\Omega_{\text{Left}} = \begin{cases} (0) & \\ (k) & \rightsquigarrow \quad (i+1)^{\binom{k}{i}} \quad \text{for} \quad i \in [0, k] \end{cases}$$

In comparison to the generating tree growing on the right of Section 2.2, growing sequences on the left is not the most practical way to construct the set of inversion sequences (it is not even obvious that the tree described by Ω_{Left} has $n!$ nodes at each level n). However, this construction

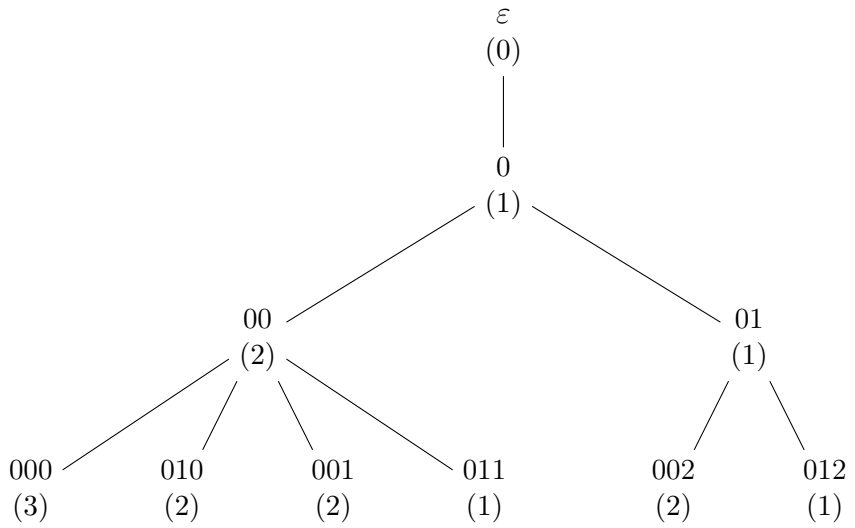

Figure 2.6: First four levels of the generating tree growing on the left for all inversion sequences.

becomes simpler for some classes of pattern-avoiding inversion sequences. For instance, it is shown in [46] that the generating trees growing on the left for the classes $\mathcal{I}(101)$ and $\mathcal{I}(110)$ are both described by the succession rule

$$\Omega_{\mathrm{pCat}} = \begin{cases} (0) & & \\ (k) & \rightsquigarrow & (k+1) \\ & & (i)^i \qquad \text{for} \quad i \in [1,k]. \end{cases}$$

The generating tree growing on the left for inversion sequences lacks a property that is typically desired when working with objects that avoid patterns: unlike the trees studied in Sections 2.2 and 2.3, it does not systematically restrict to a generating tree for families of pattern-avoiding inversion sequences. Specifically, restricting this tree to keep only sequences which avoid some patterns may disconnect it. This occurs because an inversion sequence which contains a pattern ρ may have children which avoid ρ. For instance, the sequence $(0, 0, 2, 1)$ avoids the pattern 010, but its parent sequence is $(0, 1, 0)$. We now characterize the patterns for which this restriction can be used safely, in Propositions 2.4.1 and 2.4.2. First, we show that this restriction is always possible when avoiding a pattern which contains only one zero, or two zeros that are both at the beginning of the pattern.

Proposition 2.4.1. *Let ρ be a pattern of length $k \geq 1$ which either contains a single zero, or contains exactly two zeros and begins with 00. Let σ be an inversion sequence which contains an occurrence $(\sigma_{q_i})_{i\in[1,k]}$ of ρ, and let τ be a child of σ in the generating tree growing on the left. Then τ contains the pattern ρ.*

Proof. If ρ contains a single zero, then $(\sigma_{q_i})_{i\in[1,k]}$ contains at most one zero. It follows that $(\sigma_{q_i})_{i\in[1,k]}$ is order-isomorphic to $(\tau_{q_i+1})_{i\in[1,k]}$ (every nonzero value is increased by 1, and the hypothetical zero can either increase by 1 or remain a zero), hence τ contains the pattern ρ.

If ρ contains exactly two zeros and begins with 00, we distinguish some cases depending on the values of $(\sigma_{q_i})_{i\in[1,k]}$.

- If $(\sigma_{q_i})_{i\in[1,k]}$ does not contain any zero, then $(\tau_{q_i+1})_{i\in[1,k]} = (\sigma_{q_i} + 1)_{i\in[1,k]}$ is still an occurrence of ρ, in τ.

- If $(\sigma_{q_i})_{i\in[1,k]}$ contains some zeros, i.e. $\sigma_{q_1} = \sigma_{q_2} = 0$, then $\tau_{q_1+1}, \tau_{q_2+1} \in \{0, 1\}$.
  - If $(\tau_{q_1+1}, \tau_{q_2+1}) \in \{(0, 0), (1, 1)\}$, then $(\tau_{q_i+1})_{i\in[1,k]}$ is still an occurrence of ρ.
  - If $(\tau_{q_1+1}, \tau_{q_2+1}) = (0, 1)$, then $(\tau_1) \cdot (\tau_{q_i+1})_{i\in\{1\}\cup[3,k]}$ is an occurrence of ρ.
  - If $(\tau_{q_1+1}, \tau_{q_2+1}) = (1, 0)$, then $(\tau_1) \cdot (\tau_{q_i+1})_{i\in[2,k]}$ is an occurrence of ρ. □

Proposition 2.4.1 provides sufficient conditions on a pattern ρ to ensure that the generating tree growing on the left for $\mathcal{I}$ can be restricted to a generating tree for the class $\mathcal{I}(\rho)$. We now show that these conditions are also necessary.

Proposition 2.4.2. *Let ρ be a pattern of length $k \geq 2$ which contains at least two zeros, and either begins with 000, or has a zero to the right of a nonzero value. Then there exist two inversion sequences σ and τ such that*

- *τ is a child of σ in the generating tree growing on the left,*
- *σ contains ρ,*
- *τ avoids ρ.*

Proof. If ρ begins with 000, let $\rho' = (\rho_i + k \cdot \chi(\rho_i > 0))_{i\in[3,k]}$ be the sequence obtained by deleting the first two zeros of ρ and adding k to every nonzero value. Let $\sigma = (0, 0, 1, 1, \ldots, k-1, k-1) \cdot \rho'$. It is easily checked that σ is an inversion sequence which contains the pattern ρ. Let $\tau = \mathbf{child}(\sigma, \{1, 2\})$. By construction, τ has one fewer zero than ρ, hence no occurrence of ρ in τ can contain the value 0. Similarly, every value in $[1, k]$ appears exactly twice in τ, while the value 0 appears at least three times in ρ, so any occurrence of τ in ρ must only contain values greater than k. Only the rightmost $k-2$ entries of τ may have a value greater than k; this is not enough to form an occurrence of ρ.

If ρ has a zero to the right of a nonzero value, let $\sigma = (0)^k \cdot \rho$. It is easy to see that σ is an inversion sequence which contains ρ. Let $\tau = \mathbf{child}(\sigma, \{\mathbf{max}(\mathbf{Zeros}(\sigma))\})$. By hypothesis, the rightmost zero of σ (or equivalently, ρ) is necessarily an inversion bottom, i.e. there exists an entry of greater value to its left. This implies that the number of zeros that are inversion bottoms in τ is one fewer than in ρ, hence no occurrence of ρ in τ can contain the value 0. Additionally, the value 1 appears only once in τ, while the value 0 appears at least twice in ρ, so any occurrence of ρ in τ must only contain values greater than 1. There are at most $k - 2$ entries of τ having value at least 2; this is not enough to form an occurrence of ρ. □

By Proposition 2.4.1, the generating tree growing on the left is well-defined for inversion sequences avoiding the patterns 201 and 210 studied in Section 2.4.2, since both patterns contain a single zero. Even for problematic patterns like 010, the problem can be worked around by modifying our construction, in some cases. For instance, we describe a generating tree for the class $\mathcal{I}(010, 102)$ in Section 2.4.3. Some other workarounds for the classes $\mathcal{I}(000)$, $\mathcal{I}(010)$, and $\mathcal{I}(100)$ can be found in Section 2.4.4.

2.4.2 Avoiding {201, 210}

For all $n \geq 0$ and $\sigma \in \mathcal{I}_n(201, 210)$, let $L(\sigma) = \{i \in [1, n] : \forall j \leq i,\ \sigma_j = 0\}$ be the set of leading zeros of σ, and let $R(\sigma) = \mathbf{Zeros}(\sigma) \backslash L(\sigma)$. Let $\ell(\sigma) = |L(\sigma)|$ be the number of leading zeros of σ. If $\ell(\sigma) < n$ (i.e. if σ is not constant), then $\sigma_{\ell(\sigma)+1}$ is the value of the first nonzero entry of σ. Notice that $L(\sigma)$ (resp. $R(\sigma)$) is the set of zeros of σ to the left (resp. right) of position $\ell(\sigma)+1$.

Proposition 2.4.3. *Let $\sigma \in \mathcal{I}_n(201, 210)$, $Z \subseteq \mathbf{Zeros}(\sigma)$, and let $\tau = \mathbf{child}(\sigma, Z) \in \mathcal{I}_{n+1}$. Then τ avoids 201 and 210 if and only if $Z \cap R(\sigma) \in \{\emptyset, R(\sigma)\}$.*

Proof. Since the patterns 201 and 210 both begin with the value 2, the first value of an occurrence of either pattern in τ must be at least 2. For all $i \in [1, \ell(\sigma) + 1]$, we have $\tau_i \leq 1$, therefore any occurrence of 201 or 210 in τ must be in $(\tau_i)_{i \in [\ell(\sigma)+2, n+1]}$. If $Z \cap R(\sigma) \in \{\emptyset, R(\sigma)\}$, then $(\tau_i)_{i \in [\ell(\sigma)+2, n+1]}$ is order-isomorphic to $(\sigma_i)_{i \in [\ell(\sigma)+1, n]}$, which avoids both patterns.

Assume $Z \cap R(\sigma) \notin \{\emptyset, R(\sigma)\}$. Let $i, j \in R(\sigma)$ be such that $i \in Z$ and $j \notin Z$. In particular, $\tau_{\ell(\sigma)+2} \geq 2$, $\tau_{i+1} = 1$, and $\tau_{j+1} = 0$. If $i < j$, then $(\tau_{\ell(\sigma)+2}, \tau_{i+1}, \tau_{j+1})$ is an occurrence of 210. If $j < i$, then $(\tau_{\ell(\sigma)+2}, \tau_{j+1}, \tau_{i+1})$ is an occurrence of 201. □

Proposition 2.4.3 establishes that, in the generating tree growing on the left for $\mathcal{I}(201, 210)$, the children of σ are the sequences such that the subset Z of zeros to which the value 1 is added must contain either all or none[7] of $R(\sigma)$. Any subset of $L(\sigma)$ may intersect with Z without creating an occurrence of 201 or 210. It follows that the number of children of σ is $2^{\ell(\sigma)+\chi(R(\sigma)\neq\emptyset)}$.

For all $\sigma \in \mathcal{I}(201, 210)$, let $s(\sigma) = \ell(\sigma) + \chi(R(\sigma) \neq \emptyset)$.

Lemma 2.4.4. *The generating tree growing on the left for $\mathcal{I}(201, 210)$ is described by the succession rule*

$$\Omega_{\{201,210\}} = \begin{cases} (0) & & & \\ (s) & \rightsquigarrow & (i) & \text{for} \quad i \in [1, s+1] \\ & & (i+1)^{2^{s-i}-1} & \text{for} \quad i \in [1, s-1]. \end{cases}$$

Proof. We show that the succession rule $\Omega_{\{201,210\}}$ is obtained by replacing each sequence σ by the label $s(\sigma)$ in the generating tree growing on the left for $\mathcal{I}(201, 210)$. First, the sequence ε corresponding to the root has label $s(\varepsilon) = 0$.

Let $\sigma \in \mathcal{I}(201, 210)$. We know from Proposition 2.4.3 that the set of children of σ in the generating tree growing on the left for $\mathcal{I}(201, 210)$ is

$$\{\mathbf{child}(\sigma, Z) \ : \ Z \subseteq L(\sigma)\} \cup \{\mathbf{child}(\sigma, Z' \sqcup R(\sigma)) \ : \ Z' \subseteq L(\sigma)\}.$$

Note that the above union is disjoint if $R(\sigma)$ is nonempty, and both sets in the union are equal if $R(\sigma)$ is empty. Let Z be such that $\tau = \mathbf{child}(\sigma, Z) \in \mathcal{I}(201, 210)$ (i.e. $Z \subseteq L(\sigma)$, or $Z = Z' \sqcup R(\sigma)$ for some $Z' \subseteq L(\sigma)$). We distinguish several cases:

- If $R(\sigma) = \emptyset$, then $s(\sigma) = \ell(\sigma)$.
  - If Z is a (possibly empty) interval of positions at the end of $L(\sigma)$, that is if $Z = [i, \ell(\sigma)]$ for some $i \in [1, \ell(\sigma) + 1]$ (the case $i = \ell(\sigma) + 1$ corresponding to $Z = \emptyset$), then $L(\tau) = [1, i]$ and $R(\tau) = \emptyset$, therefore $s(\tau) = i$.
  - Otherwise, let $i = \mathbf{min}(Z)$. It holds that $i = \ell(\tau) \in [1, \ell(\sigma) - 1]$, and there exists $j \in \mathbf{Zeros}(\sigma) \backslash Z$ such that $j > i$. In particular, $R(\tau) \neq \emptyset$, and $s(\tau) = i + 1$. There are $2^{s(\sigma)-i} - 1$ possible sets Z satisfying the present assumptions (that is all sets of the form $Z = \{i\} \sqcup Z'$ for some $Z' \subsetneq [i + 1, \ell(\sigma)]$).
- If $R(\sigma) \neq \emptyset$, then $s(\sigma) = \ell(\sigma) + 1$.

[7] When $R(\sigma)$ is not empty, one can think of it as containing only a single zero, since all zeros in this set are modified in the same way.

– If $Z = \emptyset$, then $L(\tau) = [1, \ell(\sigma) + 1]$ and $R(\tau) = \{i + 1 \mid i \in R(\sigma)\}$, therefore $s(\tau) = s(\sigma) + 1$.
– If $Z = \mathbf{Zeros}(\sigma) \cap [i, n]$ for some $i \in [1, \ell(\sigma) + 1]$, then $L(\tau) = [1, i]$ and $R(\tau) = \emptyset$, therefore $s(\tau) = i$.
– Otherwise, Z is not empty nor consists of a set of final zeros of σ. This implies as before that $R(\tau) \neq \emptyset$, and that $s(\tau) = i + 1$, where $i = \ell(\tau) = \mathbf{min}(Z) \in [1, \ell(\sigma)]$. There are $2^{s(\sigma)-i} - 1$ possible sets Z satisfying the present assumptions: either $Z = \{i\} \sqcup Z'$ for some $Z' \subseteq [i + 1, \ell(\sigma)]$, or $Z = \{i\} \sqcup Z' \sqcup R(\sigma)$ for some $Z' \subsetneq [i + 1, \ell(\sigma)]$). □

2.4.2.1 The algebraic generating function of $\mathcal{I}$(201,210)

The succession rule $\Omega_{\{201,210\}}$ of Theorem 2.4.4 provides a new and simple way of computing the generating function of $\mathcal{I}(201, 210)$. For all $n, k \geq 0$, let $\mathfrak{a}_{n,k} = |\{\sigma \in \mathcal{I}_n(201, 210) \ : \ k = s(\sigma)\}|$. Let also $A(t, u) = \sum_{n,k\geq 0} \mathfrak{a}_{n,k} t^n u^k$.

Theorem 2.4.5. *The generating function of* $\mathcal{I}(201, 210)$ *is* $A(t, 1) = \frac{2-t-t\sqrt{1-8t}}{4t^2-4t+2}$.

Proof. The succession rule $\Omega_{\{201,210\}}$ corresponds to the following equation:

$$\begin{aligned}
A(t, u) &= 1 + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} t^{n+1} \left(\sum_{i=1}^{k+1} u^i + \sum_{i=1}^{k-1} (2^{k-i} - 1) u^{i+1} \right) \\
&= 1 + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} t^{n+1} \left(u + \sum_{i=1}^{k} 2^{k-i} u^{i+1} \right) \\
&= 1 + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} t^{n+1} \left(u + 2^k u \sum_{i=1}^{k} \left(\frac{u}{2} \right)^i \right) \\
&= 1 + \sum_{n,k\geq 0} \mathfrak{a}_{n,k} t^{n+1} \left(u + u^2 \frac{2^k - u^k}{2 - u} \right) \\
&= 1 + tuA(t, 1) + \frac{tu^2}{2 - u} (A(t, 2) - A(t, u)).
\end{aligned}$$

This can be rewritten as

$$\left(1 + \frac{tu^2}{2 - u} \right) A(t, u) = \frac{tu^2}{2 - u} A(t, 2) + tuA(t, 1) + 1. \tag{2.7}$$

We apply the kernel method. The kernel in (2.7) is $1 + \frac{tu^2}{2-u}$. To cancel the kernel, we substitute u for $U(t) = \frac{1-\sqrt{1-8t}}{2t}$. This yields the equation

$$A(t, 2) = tU(t)A(t, 1) + 1. \tag{2.8}$$

Evaluating (2.7) for $u = 1$ gives

$$A(t, 1) = tA(t, 2) + 1.$$

Replacing $A(t, 2)$ by its expression from (2.8), we obtain

$$A(t, 1) = t^2 U(t)A(t, 1) + t + 1.$$

This gives an expression for $A(t, 1)$:

$$A(t, 1) = \frac{t + 1}{1 - t^2 U(t)} = \frac{2 - t - t\sqrt{1 - 8t}}{4t^2 - 4t + 2}.$$ □

2.4.3 Avoiding {010, 102}

We begin with two simple but useful properties satisfied by inversion sequences avoiding the pattern 010.

Proposition 2.4.6. *Let $\sigma \in \mathcal{I}(010)$. Then*

1. *all entries of value 0 in σ must be consecutive and at the beginning of σ,*
2. *all entries of value 1 in σ must be consecutive.*

Proof. 1. Assume for the sake of contradiction that σ contains two non-consecutive entries of value 0. In other words, there exists a subsequence $(\sigma_i, \sigma_j, \sigma_k)$ such that $i < j < k$, $\sigma_i = \sigma_k = 0$, and $\sigma_j \neq 0$. In particular, $\sigma_j \geq 1$, therefore $(\sigma_i, \sigma_j, \sigma_k)$ is an occurrence of the pattern 010.

We showed that all entries of value 0 in σ must be consecutive. Since any nonempty inversion sequence σ satisfies $\sigma_1 = 0$, the consecutive entries of value 0 must be at the beginning of σ.

2. Assume for the sake of contradiction that σ contains two non-consecutive entries of value 1. In other words, there exists a subsequence $(\sigma_i, \sigma_j, \sigma_k)$ such that $i < j < k$, $\sigma_i = \sigma_k = 1$, and $\sigma_j \neq 1$. We showed that all entries of value 0 are consecutive and at the beginning of σ, hence $\sigma_j \neq 0$. It follows that $\sigma_j \geq 2$, therefore $(\sigma_i, \sigma_j, \sigma_k)$ is an occurrence of the pattern 010. □

2.4.3.1 Modifying the construction

As we showed in Proposition 2.4.2, the generating tree growing on the left is not always well-defined for inversion sequences avoiding a pattern with several zeros. Specifically, the children of sequences of $\mathcal{I}_n(010, 102)$ in this tree are precisely the sequences of $\mathcal{I}_{n+1}(010, 102)$ in which the value 1 does not appear to the right of a greater value (otherwise their parent sequence would contain a 0 to the right of a greater value, and therefore contain the pattern 010). To complete this construction, we shall introduce some new steps so that the value 1 may appear at any position (while avoiding the patterns 010 and 102).

A possible solution could be to modify the way that entries of value 1 are placed, allowing them to be a newly inserted element in the sequence rather than a replacement of a zero entry. However, we remark that there are some unwanted restrictions on inversion sequences that can be generated by inserting a new 1 in this manner. Indeed, if a sequence τ were obtained by inserting a new entry of value 1 at position i in an inversion sequence $\sigma \in \mathcal{I}_n$, then for all $j \in [i+1, n+1]$, we would have $\tau_j = \sigma_{j-1} < j - 1$. In particular, a sequence τ which contains a saturated entry $\tau_j = j - 1$ for some $j > i$ cannot be generated this way. Nevertheless, for the class $\mathcal{I}(010, 102)$, we specifically want to generate a sequence τ which avoids 102, and in which the value 1 inserted at position i is to the right of a value $k > 1$. In particular, for all $j > i$, τ must satisfy $\tau_j \leq k < i - 1 < j - 1$ to avoid 102, so the aforementioned problem disappears. Do note that inserting an entry of value 1 to the left of every greater value would still cause this problem, so we cannot place every single entry of value 1 by insertion either. This leads us to consider a hybrid approach in which entries of value 1 are only placed by insertion if they are to the right of a greater value, and still placed by replacing a zero otherwise. This describes a new generating tree for $\mathcal{I}(010, 102)$, in which the parent of each nonempty sequence is obtained by

- removing an entry of value 1, if such an entry appears to the right of a greater value (it does not matter which entry of value 1 is removed since they are all consecutive by Proposition 2.4.6);
- removing the first zero and subtracting 1 from all nonzero entries (as in Section 2.4.1), otherwise.

We introduce a second small change in this construction. It is difficult to explain precisely the reason for this choice, but it allows for a simpler presentation of the computation of the generating function of $\mathcal{I}(010, 102)$ in Theorem 2.4.11, by reducing the number of cases to consider. If a sequence contains an entry of value 1 with no greater value to its left (i.e. the sequence begins with a prefix of zeros, followed by a 1), then its parent is now obtained by subtracting 1 from the leftmost entry of value 1 only[8]. Otherwise, its parent is determined as described just above. The corresponding tree is pictured on Figure 2.7. It should be noted that this tree does not quite conform to the usual definition of a generating tree, as some nodes are on the same level as their parent.

The article [51] introduces the notion of "jumping" generating trees, in which each edge has a (bounded) positive integer length. If a node at level n has a child through an edge of length k, then this child is at level $n + k$ (in a classical generating tree, all edges have length 1). The tree in Figure 2.7 could be seen as a jumping generating tree having edges of length 0 or 1.

We claim that the following succession rule describes a generating tree isomorphic to the one described above (and pictured on Figure 2.7); the arrow $\overset{k}{\rightsquigarrow}$ marks an edge of length k.

$$\Omega_{\{010,102\}} = \begin{cases} (\mathtt{C}, 0) & & & & \\ (\mathtt{C}, z) & \overset{1}{\rightsquigarrow} & (\mathtt{C}, z+1) & & \\ & \overset{0}{\rightsquigarrow} & (\mathtt{B}, z-1, 2) & \text{if} & z \geq 2 \\ (\mathtt{B}, z, s) & \overset{1}{\rightsquigarrow} & (\mathtt{P}, z+1, s) & & \\ & \overset{0}{\rightsquigarrow} & (\mathtt{B}, z-1, s+1) & \text{if} & z \geq 2 \\ (\mathtt{P}, z, s) & \overset{1}{\rightsquigarrow} & (\mathtt{P}, z+1, s) & & \\ & \overset{1}{\rightsquigarrow} & (\mathtt{R}, z, s+1) & & \\ & \overset{1}{\rightsquigarrow} & (\mathtt{M}, z, i) & \text{for} & i \in [3, s] \\ & \overset{0}{\rightsquigarrow} & (\mathtt{L}, z-1) & & \\ (\mathtt{L}, z) & \overset{1}{\rightsquigarrow} & (\mathtt{P}, z+1, 2) & & \\ & \overset{0}{\rightsquigarrow} & (\mathtt{L}, z-1) & \text{if} & z \geq 2 \\ (\mathtt{R}, z, s) & \overset{1}{\rightsquigarrow} & (\mathtt{P}, z+1, s) & & \\ & \overset{1}{\rightsquigarrow} & (\mathtt{R}, z, s+1) & & \\ (\mathtt{M}, z, s) & \overset{1}{\rightsquigarrow} & (\mathtt{P}, z+1, s) & & \\ & \overset{1}{\rightsquigarrow} & (\mathtt{M}, z, s). & & \end{cases}$$

The generating function of $\mathcal{I}(010, 102)$ can be computed from the succession rule $\Omega_{\{010,102\}}$, with the kernel method. However, this result is easier to achieve by studying only the subset of sequences of $\mathcal{I}(010, 102)$ which do not contain the value 1, and this is how we will proceed

[8] Starting from a sequence of this form and going up towards the root of our tree by following its ancestors, we would eventually reach the sequence which was its parent in our original definition (once all entries of value 1 have been turned into zeros). We are simply placing some intermediate sequences on that path.

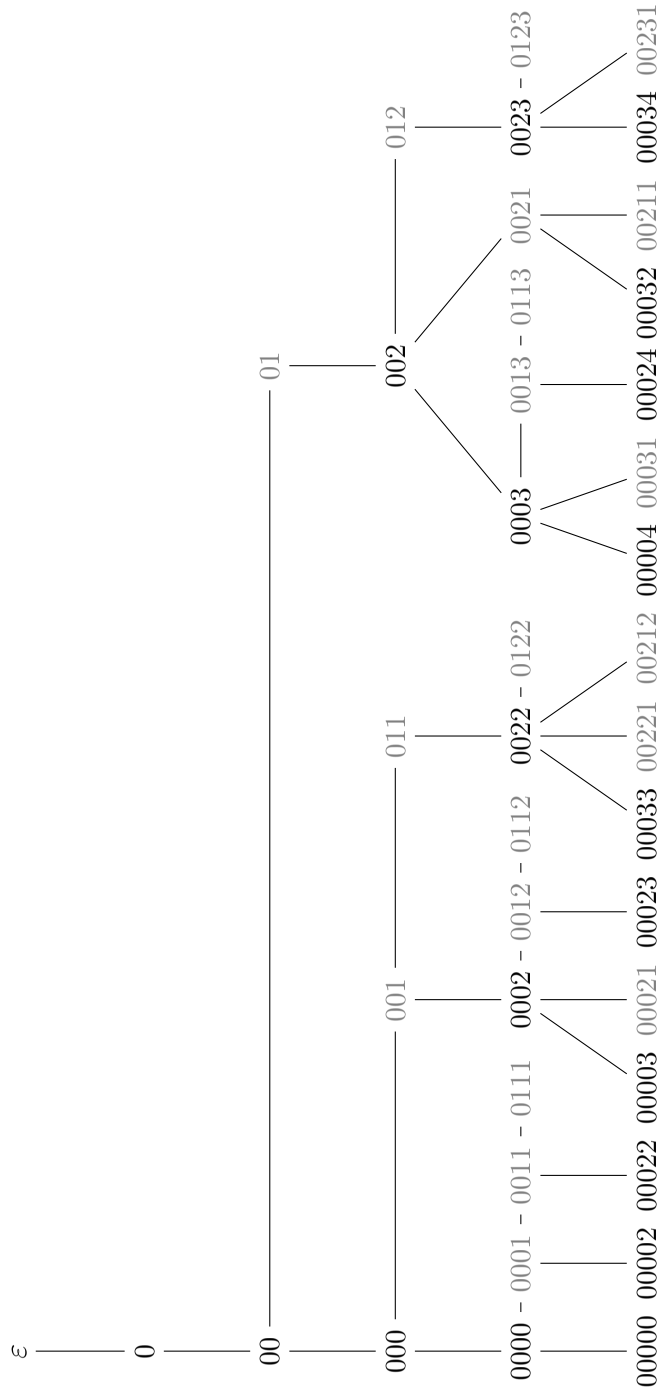

Figure 2.7: Part of a generating tree for $\mathscr{Z}(010, 102)$. Sequences containing the value 1 are colored in gray.

in Section 2.4.3.2. Due to this restriction, the construction used in Section 2.4.3.2 "jumps" over sequences containing the value 1, so they are grayed out in Figure 2.7.

For the proof of Theorem 2.4.11 to come in Section 2.4.3.2, it is not necessary to prove our claim that the rule $\Omega_{\{010,102\}}$ describes a generating tree isomorphic to that of Figure 2.7. Nevertheless, to help convince the reader of the validity of our claim, we now briefly explain the correspondence between inversion sequences and the labels appearing in this rule. In the following descriptions, the letters μ and ν denote nonempty sequences of values greater than or equal to 2.

- The letter `C` marks *constant* sequences of the form $(0)^a$ with $a \geq 0$.
- The letter `B` marks *binary* sequences of the form $(0)^a \cdot (1)^b$ with $a, b > 0$.
- The letter `P` marks *pushed* sequences of the form $(0, \ldots, 0) \cdot \mu$. These sequences are named after the **push** function of Section 2.4.3.2, and they correspond to the set $\mathfrak{B}$ in Section 2.4.3.2.
- The letter `L` marks sequences having 1-entries on the *left*, of the form $(0, \ldots, 0, 1 \ldots, 1) \cdot \mu$.
- The letter `R` marks sequences having 1-entries on the *right*, of the form $(0, \ldots, 0) \cdot \mu \cdot (1, \ldots, 1)$.
- The letter `M` marks sequences having 1-entries in the *middle*, of the form $(0, \ldots, 0) \cdot \mu \cdot (1, \ldots, 1) \cdot \nu$.
- The parameter z counts the *zeros* of an inversion sequence.
- The parameter s counts the *active sites* of an inversion sequence, for some definition of active sites that is similar, albeit slightly different, from that of Section 2.4.3.2.

2.4.3.2 Establishing a recurrence relation

For all $n \geq 0$, let $\mathfrak{A}_n = \{\sigma \in \mathcal{I}_n(010, 102) \ : \ 1 \notin \mathbf{Vals}(\sigma)\}$ be the set of inversion sequences of length n which avoid 010 and 102, and do not contain the value 1. Let $\mathfrak{A} = \coprod_{n \geq 0} \mathfrak{A}_n$. For all $\sigma \in \mathcal{I}$, let $\mathbf{push}(\sigma) = 0 \cdot (\sigma_i + \chi(\sigma_i > 0))_{i \in [1,n]}$ be the sequence obtained by inserting a zero at the beginning of σ, and adding 1 to the value of all nonzero entries.

Proposition 2.4.7. *The image* $\mathbf{push}(\mathcal{I}(010, 102))$ *is the set* $\mathfrak{A} \backslash \{\varepsilon\}$.

Proof. Let $\sigma \in \mathcal{I}(010, 102)$, and let $\tau = \mathbf{push}(\sigma)$. By construction, τ is an inversion sequence, τ does not contain the value 1, and $|\tau| > |\sigma|$ so τ cannot be empty. Let us now show that τ avoids 010 and 102. The sequence $(\tau_i)_{i \in [2,n+1]}$ is order-isomorphic to σ, which avoids 010 and 102. This means that any occurrence of either pattern in τ must involve the first entry of τ. Since $\tau_1 = 0$, it cannot be part of an occurrence of 102. If τ contained an occurrence (τ_1, τ_i, τ_j) of 010 for some $i < j$, then σ would contain a zero σ_{j-1} after a nonzero value σ_{i-1}, which is impossible by Proposition 2.4.6.

Let $\tau \in \mathfrak{A} \backslash \{\epsilon\}$, and let σ be the sequence obtained by removing the first zero of τ and subtracting 1 from the value of all nonzero entries. Then $\sigma \in \mathcal{I}(010, 102)$, and $\mathbf{push}(\sigma) = \tau$. □

Corollary 2.4.8. *For all* $n \geq 0$, $|\mathcal{I}_n(010, 102)| = |\mathfrak{A}_{n+1}|$.

Proof. Let $n \geq 0$. The function **push** is injective, so it describes a bijection between $\mathcal{I}_n(010, 102)$ and $\mathbf{push}(\mathcal{I}_n(010, 102))$. By construction, the image under **push** of a sequence of length n is a sequence of length $n+1$, so it follows from Proposition 2.4.7 that $\mathbf{push}(\mathcal{I}_n(010, 102)) = \mathfrak{A}_{n+1}$. □

For all $n \geq 0$ and $\sigma \in \mathcal{I}_n(010, 102)$, let

$$\mathbf{Sites}(\sigma) = \{i \in [2, n+1] \ : \ (\sigma_j)_{j\in[1,i-1]} \cdot 1 \cdot (\sigma_j)_{j\in[i,n]} \in \mathcal{I}_{n+1}(010, 102)\}$$

be the set of positions, called *active sites*, where inserting the value 1 in σ does not create an occurrence of 010 or 102. Note that we do not allow the value 1 to be inserted at the beginning of σ (i.e. at position 1), since the result would not be an inversion sequence.

If σ already contains the value 1, then a new entry of value 1 can only be inserted next to the current entries of value 1, by Proposition 2.4.6, and we have little interest in the active sites of σ in that case. We next characterize the active sites of σ when σ does not contain the value 1. Figure 2.8 represents the position of such an active site, as described in Proposition 2.4.9 below.

Proposition 2.4.9. *Let $n \geq 1$, $\sigma \in \mathfrak{A}_n$, and $z = |\mathbf{Zeros}(\sigma)|$. Then*

$$\mathbf{Sites}(\sigma) = \{i \in [z+1, n+1] \ : \ \mathbf{min}((\sigma_j)_{j\in[z+1,i-1]}) \geq \mathbf{max}((\sigma_j)_{j\in[i,n]})\}.$$

Proof. The value 1 cannot be inserted at positions $[2, z]$, since it would create a subsequence $(0, 1, 0)$. Inserting the value 1 at any other position (i.e. in $[z+1, n+1]$) cannot create an occurrence of 010, since there are no zeros to the right and no other entry of value 1 in σ. It follows that $\mathbf{Sites}(\sigma)$ is the subset of $[z+1, n+1]$ of positions where inserting the value 1 does not create an occurrence of 102. Note that all values of $(\sigma_i)_{i\in[z+1,n]}$ are greater than 1, so the new entry of value 1 could only take the role of the minimal value in an occurrence of 102. In other words, $i \in [z+1, n+1]$ is an active site of σ if and only if no value of $(\sigma_j)_{j\in[z+1,i-1]}$ is less than a value of $(\sigma_j)_{j\in[i,n]}$. □

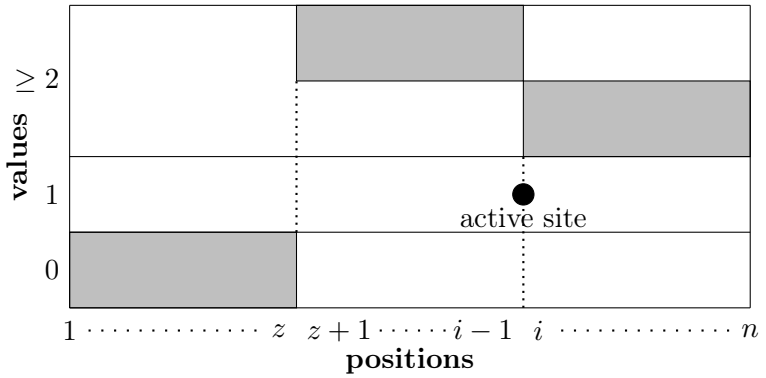

Figure 2.8: Illustration of Proposition 2.4.9.

Let $\mathfrak{B} = \mathfrak{A} \backslash \{(0)^n \ : \ n \geq 0\}$ be the subset of non-constant sequences in $\mathfrak{A}$. Equivalently, $\mathfrak{B}$ is the set of inversion sequences which avoid 010 and 102, do not contain the value 1, and contain a value greater than 1. For all $n, z, s \geq 0$, let $\mathfrak{B}_n = \{\sigma \in \mathfrak{B} \ : \ |\sigma| = n\}$, let $\mathfrak{B}_{n,z} = \{\sigma \in \mathfrak{B}_n \ : \ |\mathbf{Zeros}(\sigma)| = z\}$, and let $\mathfrak{B}_{n,z,s} = \{\sigma \in \mathfrak{B}_{n,z} \ : \ |\mathbf{Sites}(\sigma)| = s\}$. Let also $\mathfrak{b}_n = |\mathfrak{B}_n|$, $\mathfrak{b}_{n,z} = |\mathfrak{B}_{n,z}|$, and $\mathfrak{b}_{n,z,s} = |\mathfrak{B}_{n,z,s}|$.

Lemma 2.4.10. *Let $n, z, s \geq 0$. If $n \geq 3$, $z \in [2, n-1]$, and $s \in [2, n-z+1]$, then*

$$\mathfrak{b}_{n,z,s} = \mathfrak{b}_{n-1,z-1,s} + \chi(z+s = n+1) + \chi(s=2) \cdot \left(\sum_{z' \geq z} \mathfrak{b}_{n-1,z'} \right) + \left(\sum_{k \geq 1} \mathfrak{b}_{n-k-1,z-1,s-k} \right) + \chi(s \geq 3) \cdot \left(\sum_{n' \leq n-2} \sum_{s' \geq s} \mathfrak{b}_{n',z-1,s'} \right).$$

Otherwise, $\mathfrak{b}_{n,z,s} = 0$.

Proof. Let $\sigma \in \mathfrak{B}_{n,z,s}$. By definition, σ is not constant, so we have $z < n$. By definition, σ cannot contain the value 1, hence σ begins by $(0,0)$, and $z \geq 2$. This implies that $n \geq 3$. By Proposition 2.4.9, $\mathbf{Sites}(\sigma) \subseteq [z+1, n+1]$, and $z+1$ and $n+1$ are always active sites of σ, so $s \in [2, n-z+1]$.

Let $\sigma' = \mathbf{push}^{-1}(\sigma)$ be the sequence obtained by removing the first zero from σ, and subtracting 1 from every nonzero entry. In particular, $\sigma' \in \mathcal{I}_{n-1}(010, 102)$.

1 If σ' does not contain the value 1, then $\sigma' \in \mathfrak{B}_{n-1,z-1,s}$.

2 Otherwise, σ' has entries of value 1, and they are all consecutive by Proposition 2.4.6.

2.1 If $\sigma'_z = 1$, then σ' is of the form $(0)^{z-1} \cdot (1, \dots, 1) \cdot \omega$, where ω is a sequence of values strictly greater than 1, by Proposition 2.4.6.

2.1.1 If ω is empty, then $\sigma' = (0)^{z-1} \cdot (1)^{n-z}$, so $\sigma = (0)^z \cdot (2)^{n-z}$. In that case, $\mathbf{Sites}(\sigma) = [z+1, n+1]$, so $z + s = n + 1$.

2.1.2 If ω is not empty, we have $\sigma_{z+1} = 2$ and $\sigma_n > 2$. This implies that inserting the value 1 in σ at any position in $[z+2, n]$ would create an occurrence of 102. It follows that $\mathbf{Sites}(\sigma) = \{z+1, n+1\}$, so $s = 2$.
Since the factor of ones of σ' is consecutive to its factor of zeros, replacing all the ones by zeros yields a sequence $\tau = (0, \dots, 0) \cdot \omega$, which still avoids 010 and 102. Notice that τ does not contain the value 1, but contains a value greater than 1, so $\tau \in \mathfrak{B}_{n-1}$. By construction, the number z' of zeros of τ is at least z.

2.2 If $\sigma'_z > 1$ and $\sigma'_{n-1} = 1$, let k be the number of entries of value 1 in σ', and let $\tau = (\sigma'_i)_{i \in [1, n-k-1]}$ be the sequence obtained by removing all entries of value 1 from σ'. In other words, $\sigma' = \tau \cdot (1)^k$, and $\tau = (0)^{z-1} \cdot \omega$, where ω is a nonempty sequence of values strictly greater than 1. Notice that σ ends with k entries of value 2, and that inserting the value 1 after any of these entries would not create an occurrence of 102. These k entries (or, equivalently, the rightmost k active sites of σ) are no longer present τ. This means that τ has $s-k$ actives sites, and so $\tau \in \mathfrak{B}_{n-k-1,z-1,s-k}$.

2.3 If $\sigma'_z > 1$ and $\sigma'_{n-1} > 1$, then σ' is of the form $(0)^{z-1} \cdot \mu \cdot (1, \dots, 1) \cdot \nu$, where μ and ν are two nonempty sequences of values strictly greater than 1, which satisfy $\mathbf{min}(\mu) \geq \mathbf{max}(\nu)$ since σ' avoids 102. Let $p = z + 1 + |\mu| \in [z+2, n-1]$ be the position of the leftmost entry of value 2 in σ. By Proposition 2.4.9, we have $\{z+1, p, n+1\} \subseteq \mathbf{Sites}(\sigma)$, so $s \geq 3$.
Let $\tau = (0)^{z-1} \cdot \mu \cdot \nu$ be the sequence obtained by removing all entries of value 1 from σ'. Let $n' = |\tau|$, and $s' = |\mathbf{Sites}(\tau)|$, so that $\tau \in \mathfrak{B}_{n',z-1,s'}$. By construction, $n' \leq n-2$. Notice that $\mathbf{Sites}(\sigma) \subseteq [z+1, p] \sqcup \{n+1\}$, since inserting the value 1 at any position in $[p+1, n]$ would create an occurrence $(\sigma_p, 1, \sigma_n)$ of the pattern 102.

This means that all active sites of σ are still present in τ, more precisely $\{i-1 \,:\, i \in \mathbf{Sites}(\sigma)\backslash\{n+1\}\} \sqcup \{n'+1\} \subseteq \mathbf{Sites}(\tau)$, so $s' \geq s$.

To complete the proof, it remains to show that each of the five cases above induces a bijection between the set of sequences σ and the set of sequences σ' or τ. We now describe the inverse bijection in each case.

1 Let $n \geq 3, z, s \geq 2$, $\sigma' \in \mathfrak{B}_{n-1,z-1,s}$, and $\sigma = \mathbf{push}(\sigma')$. Then $\sigma \in \mathfrak{B}_{n,z,s}$ and does not contain the value 1.

2.1.1 Let $n \geq 3, z \geq 2$, $\sigma' = (0)^{z-1} \cdot (1)^{n-z}$, and $\sigma = \mathbf{push}(\sigma')$. Then $\sigma = (0)^z \cdot (2)^{n-z}$.

2.1.2 Let $n \geq 3, z \geq 2, z' \geq z$, and $\tau \in \mathfrak{B}_{n-1,z'}$. Let σ' be the sequence obtained by replacing all but the first $z-1$ zeros of τ by ones, and let $\sigma = \mathbf{push}(\sigma')$. Note that the transformation from τ to σ' cannot create any occurrence of 010 or 102, since there are no zeros to the right of the ones in σ'. It follows from Proposition 2.4.7 that σ still avoids 010 and 102. By construction, $\sigma_{z+1} = \sigma'_z + 1 = 2$, and $\sigma_n = \tau_{n-1} + 1 > 2$, hence inserting the value 1 in σ at any position in $[z+2, n]$ would create an occurrence of 102. It follows that $z+1$ and $n+1$ are the only two active sites of σ, so $\sigma \in \mathfrak{B}_{n,z,2}$.

2.2 Let $n \geq 3, z, s \geq 2, k \geq 1$, and $\tau \in \mathfrak{B}_{n-k-1,z-1,s-k}$. Let $\sigma' = \tau \cdot (1)^k$, and let $\sigma = \mathbf{push}(\sigma')$. By Proposition 2.4.9, inserting the value 1 at the end of a sequence cannot create an occurrence of 010 or 102, so σ still avoids both patterns. It can be seen that σ ends with k entries of value 2, and inserting the value 1 to the right of such an entry cannot create an occurrence of 102, hence all positions in $[n-k+2, n+1]$ are active sites of σ. Adding to that the $s-k$ active sites from τ, σ has a total of s actives sites, so $\sigma \in \mathfrak{B}_{n,z,s}$.

2.3 Let $n, s \geq 3$, $z \geq 2$, $n' \leq n-2$, $s' \geq s$, and $\tau \in \mathfrak{B}_{n',z-1,s'}$. Let $p_1 < p_2 < \cdots < p_{s'}$ be the active sites of τ. Let σ' be the sequence obtained by inserting $n - n' - 1$ entries of value 1 at position p_{s-1} in τ, and let $\sigma = \mathbf{push}(\sigma')$. Inserting the value 1 in σ at any position in $[p_{s-1}+2, n]$ would create an occurrence of 102, since $\sigma_{p_{s-1}+1} = 2$ and $\sigma_n > 2$. It follows that the active sites of σ are $\{p_i + 1 \,:\, i \in [1, s-1]\} \sqcup \{n+1\}$, so $\sigma \in \mathfrak{B}_{n,z,s}$. □

2.4.3.3 The algebraic generating function of $\mathcal{I}$(010,102)

Let $F(t) = \sum_{n\geq 0} |\mathcal{I}_n(010, 102)| t^n$ be the ordinary generating function of $\mathcal{I}(010, 102)$. Let also $B(t, u, v) = \sum_{n,z,s\geq 0} \mathfrak{b}_{n,z,s} t^n u^z v^s$.

Theorem 2.4.11. *The function $F(t)$ is algebraic with minimal polynomial*

$$(t^5 - 3t^4 + 4t^3 - 3t^2 + t)F(t)^3 + (4t^4 - 8t^3 + 6t^2 - 2t)F(t)^2 + (-t^4 + 8t^3 - 11t^2 + 6t - 1)F(t) - 2t^3 + 5t^2 - 4t + 1.$$

Proof. By the definition of $\mathfrak{B}$, we have $|\mathfrak{A}_n| = \mathfrak{b}_n + 1$ for all $n \geq 0$, so it follows from Corollary 2.4.8 that

$$F(t) = \frac{1}{t}\left(B(t, 1, 1) + \frac{1}{1-t} - 1\right),$$

and we proceed by computing $B(t, 1, 1)$.

By Lemma 2.4.10, we have

$$B(t,u,v) = \sum_{\substack{n\geq 3\\ z,s\geq 2}} \left[\mathfrak{b}_{n-1,z-1,s} + \chi(z+s=n+1) + \chi(s=2)\cdot\left(\sum_{z'\geq z} \mathfrak{b}_{n-1,z'}\right) \right.$$
$$\left. + \left(\sum_{k\geq 1} \mathfrak{b}_{n-k-1,z-1,s-k}\right) + \chi(s\geq 3)\cdot\left(\sum_{n'\leq n-2}\sum_{s'\geq s} \mathfrak{b}_{n',z-1,s'}\right)\right] t^n u^z v^s.$$

We now simplify each of the five terms in this expression to obtain a functional equation characterizing $B(t,u,v)$.

$$\sum_{\substack{n\geq 3\\ z,s\geq 2}} \mathfrak{b}_{n-1,z-1,s} t^n u^z v^s = tuB(t,u,v)$$

$$\sum_{\substack{n\geq 3\\ z,s\geq 2}} \chi(z+s=n+1) t^n u^z v^s = \sum_{z,s\geq 2} t^{-1}(tu)^z (tv)^s = \frac{t^3u^2v^2}{(1-tu)(1-tv)}$$

$$\begin{aligned}
\sum_{\substack{n\geq 3\\ z,s\geq 2}} \chi(s=2)\cdot\left(\sum_{z'\geq z} \mathfrak{b}_{n-1,z'}\right) t^n u^z v^s &= \sum_{\substack{n\geq 2\\ z\geq 2}}\sum_{i\geq 0} \mathfrak{b}_{n,z+i} t^{n+1} u^z v^2 \\
&= \sum_{\substack{n\geq 2\\ z\geq 2}}\sum_{i\in[0,z-2]} \mathfrak{b}_{n,z} t^{n+1} u^{z-i} v^2 \\
&= tv^2 \sum_{\substack{n\geq 2\\ z\geq 2}} \mathfrak{b}_{n,z} t^n \cdot \frac{u^2-u^{z+1}}{1-u} \\
&= tv^2 \left(\frac{u^2 B(t,1,1) - uB(t,u,1)}{1-u}\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{\substack{n\geq 3\\ z,s\geq 2}} \left(\sum_{k\geq 1} \mathfrak{b}_{n-k-1,z-1,s-k}\right) t^n u^z v^s &= \sum_{\substack{z\geq 1\\ n,s\geq 2}}\sum_{k\geq 1} \mathfrak{b}_{n,z,s} t^{n+k+1} u^{z+1} v^{s+k} \\
&= \sum_{\substack{z\geq 1\\ n,s\geq 2}} \mathfrak{b}_{n,z,s} t^{n+1} u^{z+1} v^s \sum_{k\geq 1} (tv)^k \\
&= \sum_{\substack{z\geq 1\\ n,s\geq 2}} \mathfrak{b}_{n,z,s} t^{n+1} u^{z+1} v^s \cdot \frac{tv}{1-tv} \\
&= t^2 uv \left(\frac{B(t,u,v)}{1-tv}\right)
\end{aligned}$$

$$\sum_{\substack{n\geq 3\\ z,s\geq 2}} \chi(s\geq 3)\cdot\left(\sum_{n'\leq n-2}\sum_{s'\geq s}\mathfrak{b}_{n',z-1,s'}\right)t^n u^z v^s = \sum_{\substack{n,s\geq 3\\ z\geq 1}}\sum_{\substack{i\geq 2\\ j\geq 0}}\mathfrak{b}_{n-i,z,s+j}t^n u^{z+1}v^s$$
$$= \sum_{\substack{n,s\geq 3\\ z\geq 1}}\mathfrak{b}_{n,z,s}\sum_{\substack{i\geq 2\\ j\in[0,s-3]}} t^{n+i}u^{z+1}v^{s-j}$$
$$= \sum_{\substack{n,s\geq 3\\ z\geq 1}}\mathfrak{b}_{n,z,s}t^n u^{z+1}\cdot\frac{t^2}{1-t}\cdot\frac{v^3-v^{s+1}}{1-v}$$
$$= t^2u\left(\frac{v^3B(t,u,1)-vB(t,u,v)}{(1-t)(1-v)}\right)$$

In summary, the formula of Lemma 2.4.10 is equivalent to the functional equation

$$B(t,u,v) = tuB(t,u,v) + \frac{t^3u^2v^2}{(1-tu)(1-tv)} + tv^2\left(\frac{u^2B(t,1,1)-uB(t,u,1)}{1-u}\right)$$
$$+ t^2uv\left(\frac{B(t,u,v)}{1-tv}\right) + t^2u\left(\frac{v^3B(t,u,1)-vB(t,u,v)}{(1-t)(1-v)}\right).$$

This can be rewritten as

$$K_1(t,u,v)B(t,u,v) = K_2(t,u,v)B(t,u,1) + \frac{tu^2v^2}{1-u}B(t,1,1) + \frac{t^3u^2v^2}{(1-tu)(1-tv)}, \tag{2.9}$$

where K_1 and K_2 are rational functions of (t,u,v):

$$K_1(t,u,v) = \frac{-t^3uv^2-t^2v^2+t^2u+t^2v+tuv+tv^2-tu-t-v+1}{(1-t)(1-v)(1-tv)},$$

$$K_2(t,u,v) = \frac{tuv^2(-tuv+t+v-1)}{(1-t)(1-u)(1-v)}.$$

We apply the kernel method. The kernel in (2.9) is $K_1(t,u,v)$. This kernel can be cancelled by substituting v for a formal power series $V(t,u)$ which is algebraic with minimal polynomial

$$(t^3u+t^2-t)V(t,u)^2 + (-t^2-tu+1)V(t,u) - t^2u + tu + t - 1.$$

An expression for $V(t,u)$ in terms of radicals is

$$V(t,u) = \frac{t^2+tu-1+\sqrt{4t^5u^2-4t^4u^2+t^4-2t^3u+t^2u^2-4t^3+4t^2u+6t^2-2tu-4t+1}}{2t(t^2u+t-1)}.$$

Substituting v for $V(t,u)$ in (2.9) yields the equation

$$K_2(t,u,V(t,u))B(t,u,1) = -\frac{tu^2V(t,u)^2}{1-u}B(t,1,1) - \frac{t^3u^2V(t,u)^2}{(1-tu)(1-tV(t,u))}. \tag{2.10}$$

Once again, we use the kernel method. The kernel in (2.10) is $K_2(t,u,V(t,u))$, and it can be cancelled by substituting u for a formal power series $U(t)$ which is algebraic with minimal polynomial

$$t^2U(t)^3 - 2tU(t)^2 + (-t^3+2t^2+1)U(t) - t^2 + t - 1.$$

Substituting u for $U(t)$ in (2.10), we obtain

$$B(t,1,1) = -\frac{(1-U(t))t^3U(t)^2V(t,U(t))^2}{tU(t)^2V(t,U(t))^2(1-tU(t))(1-tV(t,U(t)))}.$$

From this equation, and the minimal polynomials of U and V, we can compute the minimal polynomial of $B(t,1,1)$ using algebraic elimination[9]:

$$(t^3-2t^2+2t-1)B(t,1,1)^3+(t^3-t^2-t)B(t,1,1)^2+(-t^4+2t^3-4t^2+t)B(t,1,1)-t^4.$$

Now that we have the minimal polynomial of $B(t,1,1)$, we use algebraic elimination again to compute that of $F(t) = \frac{1}{t}\left(B(t,1,1)+\frac{1}{1-t}-1\right)$:

$$\begin{aligned}&(t^5-3t^4+4t^3-3t^2+t)F(t)^3+(4t^4-8t^3+6t^2-2t)F(t)^2\\&+(-t^4+8t^3-11t^2+6t-1)F(t)-2t^3+5t^2-4t+1.\end{aligned}$$ □

2.4.4 A broader perspective on the generating tree growing on the left

In this section, we do a quick survey of the succession rules describing the generating tree growing on the left for inversion sequences avoiding a pattern of length 3. Our goal is mainly to show that the generating tree growing on the left can be used in most of these cases, rather than to provide new enumerative results as before. Consequently, we do not prove the correctness of the succession rules presented in this section. None of these rules are "better" than previously known constructions, but many are similar (in terms of how many distinct labels they require).

We know from Propositions 2.4.1 and 2.4.2 that the generating tree growing on the left can be applied to 10 of the 13 classes of inversion sequences avoiding one pattern of length 3 (that is, for every pattern except 000, 010, and 100). On page 83, we present succession rules describing the generating tree growing on the left for 9 of these 10 classes, the exception being the class $\mathcal{I}(120)$. The letters z, p, and s used in these succession rules correspond to the following statistics on inversion sequences:

- z is the number of zeros,
- p is the number of consecutive zeros appearing as a prefix, i.e. at the beginning of an inversion sequence,
- s is the number of consecutive zeros appearing as a suffix, i.e. at the end of an inversion sequence.

Additionally, some cases require distinguishing whether a sequence is constant. In that case, we mark labels corresponding to constant sequences with the letter `A`, and non-constant sequences with the letter `B`.

For inversion sequences avoiding the patterns 101, 102, or 110, the generating tree growing on the left can be described with fewer distinct labels than the more naive generating tree growing on the right. For the other 7 patterns (including 120), they require a similar number of labels at each level.

The class $\mathcal{I}(120)$ is not included in the table of page 83, because the generating tree growing on the left for $\mathcal{I}(120)$ cannot be described by a succession rule involving only one or two parameters

[9] We use the *eliminate* command in Maple.

Pattern	Succession rule	Comment	OEIS [96]
001	$(\mathtt{A}, 0)$ $(\mathtt{A}, z) \rightsquigarrow (\mathtt{A}, z+1)$ $(\mathtt{B}, i)$ for $i \in [1, z]$ $(\mathtt{B}, z) \rightsquigarrow (\mathtt{B}, i)$ for $i \in [1, z]$	Isomorphic to the generating tree growing on the right. It is easy to find simpler constructions for this class.	A011782 (powers of 2)
011	(0) $(z) \rightsquigarrow (z)^z, (z+1)$	Isomorphic to the generating tree growing on the right used in [46].	A000110 (Bell numbers)
012	$(\mathtt{A}, 0)$ $(\mathtt{A}, z) \rightsquigarrow (\mathtt{A}, z+1)$ $(\mathtt{B}, i)^{2^{z-1-i}}$ for $i \in [0, z-1]$ $(\mathtt{B}, s) \rightsquigarrow (\mathtt{B}, s)$ $(\mathtt{B}, i)^{2^{s-1-i}}$ for $i \in [0, s-1]$	The construction of [46] is much simpler (it can be seen as a generating tree with only two distinct labels).	A001519 (odd-index Fibonacci numbers)
021	(0) $(p) \rightsquigarrow (p+1)$ $(i)^{2^{p-i}}$ for $i \in [1, p]$	See [46] for two bijections between $\mathcal{I}(021)$ and Schröder paths, and a bijection with some binary trees. See also [73] for a bijection with two-colored Dyck paths.	A155069 (large Schröder numbers)
101 110	(0) $(z) \rightsquigarrow (z+1)$ $(i)^i$ for $i \in [1, z]$	This construction is used in [46] and [19].	A113227
102	$(0, 0)$ $(z, z) \rightsquigarrow (z+1, z+1)$ $(i, j)^{\binom{z-i-1}{z-j}}$ for $i \in [0, z-1]$, $j \in [i+1, z]$ if $s \neq z$: $(s, z) \rightsquigarrow (s, j)$ for $j \in [s+1, z+1]$ $(i, j)^{\sum_{k=s-j}^{z-j} \binom{s-i-1}{k}}$ for $i \in [0, s-1]$, $j \in [i+1, z]$	A different construction with two parameters is used in [85] to compute the generating function of $\mathcal{I}(102)$.	A200753
201 210	$(0, 0)$ $(p, z) \rightsquigarrow (p+1, j)$ for $j \in [p+1, z+1]$ $(i, j)^{\sum_{k=p-j}^{z-j} \binom{p-i}{k}}$ for $i \in [1, p]$, $j \in [i, z]$	Two other constructions for this class, involving two parameters each, can be found in [46] and [85].	A263777

as in the other cases. We can label each inversion sequence by the sequence of lengths of its (maximal) factors of zeros, and obtain the unpalatable succession rule below.

$$\Omega_{120} = \begin{cases} (0) & \\ (\ell_0, \ldots, \ell_k) & \rightsquigarrow \quad (q_0, q_1, \ldots, q_r)^{\binom{\ell_0+1-\sum_{j=0}^{r} q_j}{r}} \\ & \qquad \text{for} \quad r \geq 0,\ q_0, \ldots, q_r \in [1, \ell_0 + 1] \\ & \qquad (\ell_0 + 1, \ell_1, \ldots, \ell_{i-1}, q_0, \ldots, q_r)^{\binom{\ell_i+1-\sum_{j=0}^{r} q_j}{r+1}} \\ & \qquad \text{for} \quad i \in [1, k],\ r \geq 0,\ q_0, \ldots, q_r \in [1, \ell_i] \end{cases}$$

The number of distinct labels appearing at level n of the tree described by the rule Ω_{120} is exponential in n. In this case, the approach of [85] is much better since it can be used to compute the counting sequence of $\mathcal{I}(120)$ in polynomial time (this is sequence A263778 in [96]). We present some more serviceable succession rules for 120-avoiding inversion sequences (Ω'_{120} and Ω''_{120}) towards the end of this section.

We now show that even for the classes $\mathcal{I}(000)$, $\mathcal{I}(010)$, and $\mathcal{I}(100)$, it is possible to modify the generating tree growing on the left to obtain some succession rules, in which some of the labels correspond to objects that are not counted ("phantom objects" as in Section 2.2.3). The counting sequences of these three classes can be found in [96] as sequences A000111, A263780, and A263779.

For inversion sequences avoiding the pattern 000, we can use the generating tree growing on the left by allowing sequences to contain more than two zeros. In other words, only occurrences of the pattern 000 involving nonzero values of an inversion sequence are forbidden. This yields a simple succession rule in which sequences are labelled by their number of zeros z.

$$\Omega_{000} = \begin{cases} (0) & \\ (z) & \rightsquigarrow \quad (z-1)^{\binom{z}{2}}, (z)^z, (z+1) \end{cases}$$

This rule can be used to enumerate 000-avoiding inversion sequences, by only counting nodes of the tree labelled (0), (1), or (2). Note that nodes labelled (z) for $z \geq 3$ are still a necessary part of this construction, because some of their descendants have a label whose value is less than 3. This approach is easy to generalize to inversion sequences avoiding a constant pattern of any length $k \geq 2$. We obtain the following succession rule, where only nodes labelled (z) for $z < k$ are to be counted.

$$\Omega_{0^k} = \begin{cases} (0) & \\ (z) & \rightsquigarrow \quad (z+1-i)^{\binom{z}{i}} \quad \text{for} \quad i \in [0, k-1] \end{cases}$$

Inversion sequences avoiding a constant pattern of any length have been studied before, and their exponential generating function is given in [55]. We have not tried to compute generating functions from the rule Ω_{0^k}.

Similarly to the 000 case, we can define the generating tree growing on the left for inversion sequences avoiding occurrences of the pattern 100 which use only nonzero values. This tree can be labelled by pairs of integers (p, z), where p is the length of the prefix of zeros, and z is the number of zeros of the corresponding inversion sequence. It is described by the succession rule

below.

$$\Omega_{100} = \begin{cases} (0,0) & & & \\ (p,z) & \rightsquigarrow & (p+1,z+1) & \\ & & (p+1,z)^{z-p} & \\ & & (i,j)^{\binom{p-i}{z-j}} & \text{for} \quad i \in [1,p],\ j \in [i,z] \\ & & (i,j)^{\binom{p-i}{z-j-1}(z-p)} & \text{for} \quad i \in [1,p],\ j \in [i,z-1] \end{cases}$$

In this tree, inversion sequences which avoid 100 are those that have at most one zero outside of their prefix, and they correspond to nodes labelled (p,z) for $z-p \in \{0,1\}$.

More generally, if ρ is a pattern such that the generating tree growing on the left does not restrict to a tree for $\mathcal{I}(\rho)$ (such patterns are characterized in Proposition 2.4.2), it is still possible to define the generating tree growing on the left for inversion sequences avoiding occurrences of ρ which use only nonzero values. However, in order to enumerate $\mathcal{I}(\rho)$ from such a tree, one must be able to determine which nodes correspond to inversion sequences containing ρ, as they must not be counted. Consequently, the labels may need to encode more information than the strict minimum necessary to describe the shape of the tree.

By applying this idea to the pattern 010, we can obtain a generating tree having similar labels to those of the rule Ω_{120} (to be precise, each inversion sequence is labelled by the number of zeros in its prefix, and the multiset of lengths of the other factors of zeros). The number of distinct labels at level n of this tree is exponential in $\sqrt{n}$, so this does not provide an efficient way of computing $|\mathcal{I}_n(010)|$. Nevertheless, it is possible to improve this construction to obtain a tree which only has a quadratic number of distinct labels at level n, by using the idea of *commitments* from [99]. That tree is described by the succession rule below, where the terms in brackets are unsigned Stirling numbers of the first kind, and only nodes labelled $(p,0)$ must be counted to enumerate $\mathcal{I}(010)$.

$$\Omega_{010} = \begin{cases} (0,0) & & & \\ (p,c) & \rightsquigarrow & (p+1,c) & \\ & & (p+1,c-1) & \text{if} \quad c > 0 \\ & & (p+1-\ell,c+k)^{\binom{c+k}{k}\left[\begin{smallmatrix}\ell\\ \ell-k\end{smallmatrix}\right]} & \text{for} \quad 0 \le k < \ell \le p \end{cases}$$

With this rule, the counting sequence of $\mathcal{I}(010)$ can be computed with a similar efficiency to the method previously used in Section 1.1.10.

Lastly, we remark that the same approach using commitments also yields a simpler succession rule Ω'_{120} for inversion sequences avoiding 120.

$$\Omega'_{120} = \begin{cases} (0,0) & & & \\ (p,c) & \rightsquigarrow & (p+1,c) & \\ & & (p+1,c-1) & \text{if} \quad c > 0 \\ & & (p+1-\ell,k)^{a_{\ell,k}} & \text{if} \quad c = 0, \quad \text{for} \quad 0 \le k < \ell \le p \end{cases}$$

The numbers $(a_{\ell,k})_{\ell,k\ge 0}$ appearing in the rule Ω'_{120} count some constrained 120-avoiding words, and their generating function is

$$\sum_{\ell,k\ge 0} a_{\ell,k} t^\ell u^k = \frac{2t-1+\sqrt{4t^2u+4t^2-4tu-4t+1}}{2u(t-1)}.$$

The rule Ω'_{120} has a nicer form than Ω_{010}, making it possible to skip the commitments. This yields the rule Ω''_{120} below, which only has one parameter, although it requires allowing arbitrarily long jumps. As a consequence, some nodes of the corresponding "tree" have an infinite number of children. This construction of $\mathcal{I}(120)$ is actually quite similar[10] to the one employed in [85].

$$\Omega''_{120} = \begin{cases} (0) & \\ (p) & \overset{1}{\rightsquigarrow} \quad (p+1) \\ & \overset{k}{\rightsquigarrow} \quad (p+k-\ell)^{b_{\ell,k}} \quad \text{for} \quad 1 \leq \ell \leq p, \quad k \geq 1 \end{cases}$$

The numbers $(b_{\ell,k})_{\ell,k\geq 0}$ appearing in the rule Ω''_{120} count another family of constrained 120-avoiding words, and their generating function is

$$\sum_{\ell,k\geq 0} b_{\ell,k} t^\ell u^k = \frac{-2tu + 2t + u - 1 + \sqrt{-4t^2u + 4t^2 + 4tu + u^2 - 4t - 2u + 1}}{2(t-1)}.$$

In conclusion, up to allowing some small modifications, a succession rule involving at most two integer parameters can be found to describe a kind of generating tree growing on the left for every class of inversion sequences avoiding a pattern of length 3. This shows that the construction is quite robust, and it could be further explored in a larger setting; for instance, to study longer patterns, sets of patterns, or different definitions of patterns. We believe this construction could also be applied to *restricted growth functions* [94], since they form a subtree of the generating tree growing on the left for inversion sequences: the parent of a (nonempty) restricted growth function is always a restricted growth function. However, this does not hold for *ascent sequences* [29], another interesting subset of inversion sequences.

[10] The construction of $\mathcal{I}(120)$ from [85] works by splitting an inversion sequence into factors starting at each left-to-right maximum, and building each factor from left to right. Here, we essentially build these same factors from right to left.

Asymptotics

Looking back at the first few terms of the counting sequences in Table 2, it could appear that among the pairs of patterns we have studied, the one avoided by the most inversion sequences is {101, 210}, and the one avoided by the fewest inversion sequences is {000, 010}. Interestingly, it can be shown that the counting sequence of $\mathcal{I}(101, 210)$ is bounded above by an exponential function, and that of $\mathcal{I}(000, 010)$ is super-exponential (see Propositions 2.0.12 and 2.0.13 below). In fact, computation indicates there are more inversion sequences of length n avoiding {000, 010} than {101, 210} starting at $n = 41$.

Marcus and Tardos [92, Theorem 9] prove that the number of $n \times n$ 0-1 matrices avoiding a permutation matrix P is bounded above by c_P^n for some constant c_P. The subset of $n \times n$ 0-1 matrices which contain exactly one 1-entry in each column is clearly in bijection with the set $\mathcal{W}_{n,n}$ of words of length n over the alphabet $[0, n-1]$. For any permutation π, this restricts to a bijection between matrices which avoid the permutation matrix of π and words avoiding the pattern π. Observing that $\mathcal{W}_{n,n}$ includes the set $\mathcal{I}_n$ of inversion sequences of length n, we obtain the following proposition.

Proposition 2.0.12. *For any permutation π, for all $n \geq 0$, we have $|\mathcal{I}_n(\pi)| \leq c_\pi^n$ for some constant c_π.*

In particular, the growth of $\mathcal{I}(101, 210)$ is at most exponential since 210 is a permutation pattern (i.e. a pattern without any repeated values).

Proposition 2.0.13 below implies that, among the classes of inversion sequences we studied, those avoiding only patterns with repeated values all have a super-exponential growth: $\mathcal{I}(010)$, $\mathcal{I}(000, 010)$, $\mathcal{I}(000, 100)$, $\mathcal{I}(100, 110)$, $\mathcal{I}(100, 101)$, and $\mathcal{I}(010, 110)$.

Proposition 2.0.13. *The counting sequence of the class $\mathcal{I}(000, 010, 100, 101, 110)$ grows super-exponentially.*

Proof. Let $\mathfrak{A}_{n,k} = \{\sigma \in \mathcal{I}_n : \sigma = \alpha \cdot (\beta + k), \alpha = (0, 0, 1, 1, \ldots, k-1, k-1), \beta \in \mathcal{I}^k_{n-2k}(00)\}$ be the set of inversion sequences of length n which begin by $(0, 0, 1, 1, \ldots, k-1, k-1)$, followed by $n-2k$ distinct values greater than or equal to k. Let $\mathfrak{A}_n = \coprod_{k \geq 0} \mathfrak{A}_{n,k}$. For all $k \geq 0$ and $n \geq 2k$, we have $|\mathfrak{A}_{n,k}| = (k+1)^{n-2k}$, hence for any $k \geq 0$, $|\mathfrak{A}_n|$ is asymptotically larger than k^n. Since this holds for an arbitrarily large value of k, $|\mathfrak{A}_n|$ has a super-exponential asymptotic behavior. By construction, every sequence in $\mathfrak{A}_n$ avoids the patterns 000, 010, 100, 101, and 110. □

Constructions similar to the above proof can be used to show that many classes of inversion sequences avoiding patterns with repeated letters have a super-exponential growth. However this is not always the case: for instance, [46, Theorem 10] proves that $|\mathcal{I}_n(001)| = 2^{n-1}$ for all $n \geq 1$. Remarkably, 001 is the only pattern ρ of length 3 with repeated values such that the growth of $\mathcal{I}(\rho)$ is at most exponential.

Using his software [97], Pantone provided us with precise conjectures about the asymptotic behavior of some of the counting sequences, based on the initial terms we computed. We present these conjectures in Table 1. In particular, it seems that the classes $\mathcal{I}(010, 201)$ and $\mathcal{I}(101, 210)$ have the same growth rate, and the classes $\mathcal{I}(100, 120)$ and $\mathcal{I}(110, 201)$ as well.

Class	μ	α	β
$\mathcal{I}(000, 201)$	~ 10.9282032	~ -7.3906	?
$\mathcal{I}(010, 201)$	~ 10.1566572	~ -7.6168360	?
$\mathcal{I}(100, 102)$	$\sim 5.06613049471619559669960 0$	$-3/2$	0
$\mathcal{I}(100, 120)$	~ 7.72334814688	$-3/2$	0
$\mathcal{I}(101, 210)$	~ 10.156657	~ -6.273831	0
$\mathcal{I}(110, 201)$	$\sim 7.7233481468847308370$	$-3/2$	0
$\mathcal{I}(120, 201)$	$\sim 7.4563913226671221339$	$-3/2$	0

Table 1: Conjectured asymptotic behavior of the form $C\mu^n n^\alpha \log(n)^\beta$ for the counting sequences of some classes of pattern-avoiding inversion sequences.

Chapter 3

Algebraicity and rationality for 021-avoiding sequences

3.1 Introduction

Inversion sequences avoiding the pattern 021 are known to be counted by the Schröder numbers [85, 46], whose generating function is algebraic of degree 2. Additionally, it was shown in [113, 69, 87, 86] that inversion sequences avoiding both 021 and any other pattern of length at most 5 always have an algebraic generating function.

Motivated by these results, we show in this chapter that for any finite set of patterns P such that P contains 021, the generating function of the class $\mathcal{I}(P)$ is algebraic. We then show that the same result holds for ascent sequences and restricted growth functions, with rational generating functions for the latter. Additionally, we show that when the set P contains both 021 and a strictly increasing pattern, the generating functions of P-avoiding inversion sequences and P-avoiding ascent sequences are rational. Our proofs rely on some encodings of 021-avoiding inversion sequences as words recognized by counter automata or finite automata. We show that, under those encodings, pattern avoidance in inversion sequences corresponds to nice (regular) constraints on words. The algebraic or rational nature of the associated generating function is then easily obtained using classical results of formal language theory.

3.1.1 Notation

For all $n \geq 1$ and $\sigma \in \mathbb{N}^n$, let $\mathbf{asc}(\sigma) = |\{i \in [1, n-1] \ : \ \sigma_i < \sigma_{i+1}\}|$ be the number of *ascents* of σ, with the convention $\mathbf{asc}(\varepsilon) = -1$. If $\sigma_i < \sigma_{i+1}$, then σ_i is called an *ascent bottom* and σ_{i+1} is called an *ascent top*.

In addition to inversion sequences, this chapter considers two other types of constrained integer sequences. For all $n \in \mathbb{N}$, let

- $\mathcal{A}_n = \{\sigma \in \mathbb{N}^n \ : \ \sigma_i \leq 1 + \mathbf{asc}((\sigma_j)_{1 \leq j < i}) \quad \forall i \in [1, n]\}$ be the set of *ascent sequences* [29] of length n,
- $\mathcal{R}_n = \{\sigma \in \mathbb{N}^n \ : \ \sigma_i \leq 1 + \mathbf{max}((\sigma_j)_{1 \leq j < i}) \quad \forall i \in [1, n]\}$ be the set of *restricted growth functions* [94] (RGFs for short) of length n.

We denote by $\mathcal{A}$ and $\mathcal{R}$ the sets of all ascent sequences and restricted growth functions. In particular, we have $\mathcal{R} \subset \mathcal{A} \subset \mathcal{I}$. Since $\mathcal{R}, \mathcal{A} \subset \mathcal{I}$, we also have $\mathcal{R}(P) = \mathcal{I}(P) \cap \mathcal{R}$, and $\mathcal{A}(P) = \mathcal{I}(P) \cap \mathcal{A}$ for any set of patterns P.

We will not only consider sequences of integers in this chapter, but also words that are sequences of letters taken from a finite alphabet. Given a word ω and some set of letters L, we denote by $L[\omega] = \{i \in [1, |\omega|] \ : \ \omega_i \in L\}$ the set of positions in ω of letters in L. For instance, if $\omega = \texttt{abcaccba}$, we have $\{\texttt{a}\}[\omega] = \{1, 4, 8\}$, and $\{\texttt{b}, \texttt{c}\}[\omega] = \{2, 3, 5, 6, 7\}$.

3.1.2 Automata and formal languages

We now introduce some basic concepts of automata and formal language theory, which will be useful later. A classical reference for these concepts is the "Cinderella book" [56].

Definition 3.1.1. A *finite automaton* is a 5-tuple $(Q, q_0, Q_f, \Sigma, \delta)$, where

- Q is a finite set of *states*,
- $q_0 \in Q$ is the *initial* state,
- $Q_f \subseteq Q$ is the set of *final* states,
- Σ is a finite set of letters called the *input alphabet*,
- δ is a finite subset of $Q \times \Sigma \times Q$ called the *transition relation*.

A finite automaton can be seen as a rooted directed multigraph M on the set of vertices Q, whose edges are labelled by letters in Σ. For any $n \geq 0$, a word $\omega \in \Sigma^n$ is *recognized* by the automaton M if there exists a walk of length n on M, starting at q_0 and ending in a final state $q_n \in Q_f$, such that ω is the sequence of labels of traversed edges $(s_i)_{i\in[1,n]} \in \Sigma^n$. The language recognized by an automaton is the set of all words over its input alphabet which it recognizes.

An automaton is *deterministic* if for every state $q \in Q$, there is at most one transition labelled by each letter $\ell \in \Sigma$. An automaton is *ambiguous* if it contains two distinct walks recognizing the same word. A deterministic automaton is always non-ambiguous. For instance, the automaton pictured on Figure 3.1 below is deterministic.

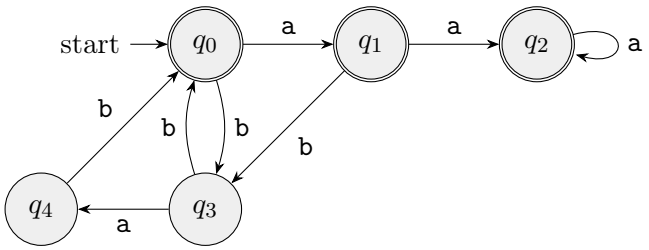

Figure 3.1: Finite automaton recognizing all words over the alphabet $\{\texttt{a}, \texttt{b}\}$ which contain an even number of letters $\texttt{b}$ and do not contain the factor $\texttt{aab}$. The final states q_0, q_1 and q_2 are represented by a double circle.

A language (i.e. a set of words) is said to be *regular* if it is recognized by a finite automaton. Every regular language can be recognized by a *deterministic* finite automaton. In other words, deterministic and non-deterministic finite automata have the same expressive power (however, this does not hold for the counter automata we consider later). It is well known that a regular language always has a rational generating function. Indeed, counting words in a regular language comes down to counting walks on a graph; this can be done by computing the powers of its adjacency matrix, yielding a linear recurrence relation with constant coefficients. Regular languages

satisfy some nice properties, in particular the union and the intersection of two regular languages is always regular (this can be seen by considering the Cartesian product of the corresponding automata). There are many other ways to define regular languages; another definition we will use is that a language is regular if it can be expressed in monadic second-order logic, see e.g. [108].

A typical example of a language that is not regular is the *Dyck language*. An underdiagonal Dyck path of semilength $k \in \mathbb{N}$ is a walk in the plane which

- starts at the point $(0, 0)$,
- ends at the point (k, k),
- consists of a sequence of *east steps* $(1, 0)$ and *north steps* $(0, 1)$,
- does not go above the diagonal (i, i) for $i \in \mathbb{N}$.

A Dyck path can also be written as a word by representing each north step by a letter `n`, and each east step by a letter `e`, see Figure 3.2. Equivalently, a Dyck word over the alphabet $\{\texttt{e}, \texttt{n}\}$ is a word which contains as many letters `e` as letters `n`, and whose every prefix contains at least as many letters `e` as letters `n`. These words will have an important role in the present chapter, and we now define another family of automata that can recognize them (more generally, these automata are designed to recognize a type of lattice paths called *excursions* [11]).

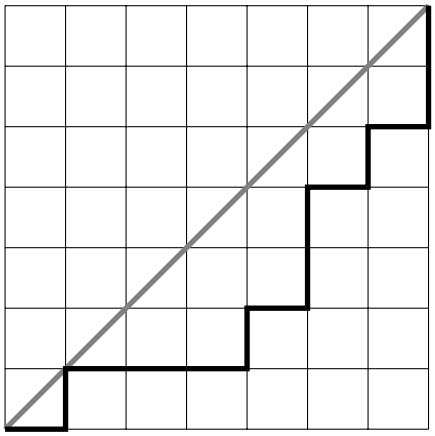

Figure 3.2: Dyck path corresponding to the word `eneeeenennenenn`.

Definition 3.1.2. We define a *counter automaton* as a 5-tuple $(Q, q_0, v_0, \Sigma, \delta)$, where

- Q, q_0 and Σ are the set of states, the initial state, and the input alphabet, as in Definition 3.1.1,
- $v_0 \in \mathbb{N}$ is the *initial value* of the counter (we will always take $v_0 = 0$, so we do not specify the initial value in the future),
- the transition relation δ is now a finite subset of $Q \times \Sigma \times \mathbb{Z} \times Q$,
- unlike finite automata, we do not distinguish a subset of final states.

As before, a counter automaton can be seen as a rooted directed multigraph M on the set of vertices Q, whose edges are labelled by elements of $\Sigma \times \mathbb{Z}$. For any $n \geq 0$, a word $\omega \in \Sigma^n$ is *recognized* by the automaton M if there exists a walk of length n on M starting at q_0 such that

- the sequence of Σ-labels of traversed edges $(s_i)_{i\in[1,n]} \in \Sigma^n$ is the word ω,
- the sequence of $\mathbb{Z}$-labels of traversed edges $(k_i)_{i\in[1,n]} \in \mathbb{Z}^n$ satisfies
  - $v_0 + \sum_{i=1}^{j} k_i \geq 0$ for all $j \in [1, n]$,
  - $v_0 + \sum_{i=1}^{n} k_i = 0$.

These conditions can be understood as follows: the process of reading the word ω begins in state q_0, and goes through edges labelled by the letters of ω, while keeping track of the sum of integer labels of the traversed edges in a *counter* initialized at v_0 and updated along transitions. The counter is not allowed to take negative values, and the word ω is recognized only if the counter has value 0 once the process of reading ω is complete.

An example of a counter automaton for constrained Dyck words is pictured on Figure 3.3 below.

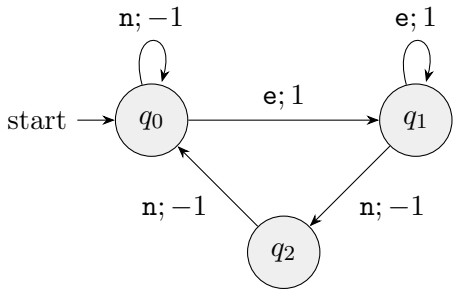

Figure 3.3: Counter automaton recognizing Dyck words over the alphabet $\{\mathtt{e}, \mathtt{n}\}$ which do not contain the factor `ene`.

Our definition of a counter automaton is very similar to the one-counter machine described in [56], which is a particular case of *pushdown automaton*. Our counter automaton model is also easy to simulate using a pushdown automaton. A language recognized by a pushdown automaton is called *context-free*. The earlier definition of deterministic and ambiguous finite automata also applies to counter automata and pushdown automata. Deterministic pushdown automata and non-ambiguous pushdown automata only recognize strict subsets of the context-free languages. We call these sets *deterministic context-free* and *non-ambiguous context-free* languages. It is well known that a non-ambiguous context-free language always has an algebraic generating function [43]. Therefore, any language recognized by a deterministic counter automaton has an algebraic generating function (this will be our main tool for proving algebraicity).

Note that any finite automaton can be turned into a counter automaton with integer labels in $\{-1, 0, 1\}$, by ensuring that the counter always has value 0 in final states and 1 in non-final states. Conversely, if a counter automaton is such that the counter's value is always less than a constant $k \in \mathbb{N}$, then it can be turned into a finite automaton by making at most k copies of each state (one for each value the counter could take in that state). Additionally, the intersection of a regular language and a deterministic context-free language is still a deterministic context-free language (as before, this can be seen by considering some type of Cartesian product of deterministic automata).

3.2 Inversion sequences avoiding 021

We begin with a simple and well known [46, 73] characterization of 021-avoiding inversion sequences.

Proposition 3.2.1. *An inversion sequence σ avoids the pattern 021 if and only if its nonzero entries are weakly increasing, i.e. $\forall i < j \in [1, |\sigma|]$, $\sigma_i > \sigma_j \implies \sigma_j = 0$.*

Proof. We show that σ contains 021 if and only if $\sigma_i > \sigma_j > 0$ for some $i < j$.

If $\sigma \in \mathcal{I}$ is such that $\sigma_i > \sigma_j > 0$ for some $i < j$, then $(\sigma_1, \sigma_i, \sigma_j)$ is an occurrence of 021, since $\sigma_1 = 0$.

If $\sigma \in \mathcal{I}$ contains 021, let $k < i < j$ be such that $(\sigma_k, \sigma_i, \sigma_j)$ is an occurrence of 021. Then $\sigma_i > \sigma_j > \sigma_k \geq 0$. □

Proposition 3.2.2. *Let $n, k \geq 2$, let $\sigma \in \mathcal{I}_n(021)$, and let ρ be a pattern of length k which contains an inversion, i.e. $\exists i < j \in [1, k] : \rho_i > \rho_j$. If ρ' is an occurrence of ρ in σ, then for all $i \in [1, k]$, $\rho'_i = 0$ if and only if $\rho_i = 0$.*

Proof. By definition of ρ and ρ', we have $0 \leq \rho_i \leq \rho'_i$ for all $i \in [1, k]$, hence $\rho_i = 0$ if $\rho'_i = 0$.

Let $i < j \in [1, k]$ be such that $\rho_i > \rho_j$. In particular $\rho'_i > \rho'_j$ since ρ and ρ' are order-isomorphic, hence $\rho'_j = 0$ by Proposition 3.2.1 since ρ' is a subsequence of $\sigma \in \mathcal{I}(021)$. Because ρ contains an inversion, and therefore an index j satisfying the previous assumption exists, we showed that $\rho_j = \rho'_j = 0$ for some $j \in [1, k]$. Since ρ and ρ' are order-isomorphic, this is enough to conclude that every zero of ρ corresponds to a zero of ρ'. □

Proposition 3.2.3. *Let ρ be a pattern.*

1. *If ρ contains 021, then $\mathcal{I}(021, \rho) = \mathcal{I}(021)$.*
2. *If $\rho_1 > 0$, then $\mathcal{I}(021, \rho) = \mathcal{I}(021, 0 \cdot \rho)$.*
3. *If ρ avoids 021 and $\rho_1 = 0$, then ρ is a RGF (a restricted growth function).*

Proof. Let $k = |\rho|$.

1. If ρ contains 021, then $\mathcal{I}(021, \rho) = \mathcal{I}(021) \cap \mathcal{I}(\rho) = \mathcal{I}(021)$ since $\mathcal{I}(021) \subseteq \mathcal{I}(\rho)$.

2. Assume that $\rho_1 > 0$. Let $\sigma \in \mathcal{I}(021)$ be an inversion sequence. Clearly, if σ avoids ρ, then σ also avoids $0 \cdot \rho$. Since any pattern must contain a zero, the assumption $\rho_1 > 0$ implies that ρ contains an inversion. If σ contains an occurrence ρ' of ρ, then by Proposition 3.2.2, ρ' contains the value 0. Moreover, σ_1 is not part of ρ', since $\rho'_1 \geq \rho_1 > \sigma_1$. This means that $\sigma_1 \cdot \rho'$ is an occurrence of $0 \cdot \rho$ in σ.

3. We proceed by contrapositive: we prove that if $\rho_1 = 0$ and ρ is not a RGF, then ρ contains 021. Assume that ρ is not a RGF. In particular, there exists some $i \in [2, k]$ such that $\rho_i > m_i$, where $m_i = 1 + \mathbf{max}((\rho_j)_{1 \leq j < i})$. Since ρ is a pattern and $0 < m_i < \rho_i$, we have $m_i \in \mathbf{Vals}(\rho)$. Let $q \in [1, k]$ be such that $\rho_q = m_i$. Since $m_i \neq \rho_j$ for all $j \in [1, i]$, we must have $q \in [i + 1, k]$. It follows that $(\rho_1, \rho_i, \rho_q) = (0, \rho_i, m_i)$ is an occurrence of 021. □

Corollary 3.2.4. *For any finite set of patterns P, there exists a finite set of 021-avoiding RGFs R such that $\mathcal{I}(\{021\} \cup P) = \mathcal{I}(\{021\} \cup R)$ and $|R| \leq |P|$.*

Proof. Let P be a set of patterns. Let $P_0 = \{\rho \in P \ : \ \rho_1 = 0\}$ be the subset of patterns of P which begin by 0, let $P_1 = P \backslash P_0$, and let $P_1' = \{0 \cdot \rho \ : \ \rho \in P_1\}$. By item 2 of Proposition 3.2.3,

$$\mathcal{I}(\{021\} \cup P_1) = \mathcal{I}(\{021\} \cup P_1').$$

Let $R = \{\rho \in P_0 \cup P_1' \ : \ \rho \text{ avoids } 021\}$. By item 1 of Proposition 3.2.3,

$$\mathcal{I}(\{021\} \cup P_0 \cup P_1') = \mathcal{I}(\{021\} \cup R).$$

By item 3 of Proposition 3.2.3, R is a set of RGFs, concluding the proof. □

Corollary 3.2.4 has an important consequence for our work: since we only study 021-avoiding inversion sequences in this chapter, we can now consider that all other patterns are 021-avoiding RGFs, without loss of generality. Note that Corollary 3.2.4 and its proof would still hold for infinite sets P and R, although we will not use this result.

3.2.1 The word encoding of $\mathcal{I}(021)$

In this section, we describe a way to encode sequences of $\mathcal{I}(021)$ as words over an alphabet of three letters $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$. These words can also be seen as coloured Dyck paths, and we will show that they are recognized by a counter automaton.

We define Φ as the function which associates to each inversion sequence $\sigma \in \mathcal{I}(021)$ a word over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$ as follows:

- each zero of σ is turned into a letter $\mathtt{z}$,
- each positive entry σ_i is replaced by $\mathtt{d}^{\sigma_i - \mathbf{max}(\sigma_j)_{j<i}} \cdot \mathtt{p}$.

The function Φ is an encoding of inversion sequences in which the letter $\mathtt{z}$ marks a zero entry, the letter $\mathtt{p}$ marks a positive entry, and the letters $\mathtt{d}$ mark the difference between a positive entry and the previous one (where the "0th" positive entry is 0 by convention). For instance, $\Phi(0, 0, 2, 0, 2, 3) = \mathtt{zzddpzpdp}$.

It can be seen that Φ is injective, and Φ^{-1} turns a word $\omega \in \Phi(\mathcal{I}(021))$ into an inversion sequence as follows:

- each letter $\mathtt{z}$ of ω is turned into a 0,
- each letter $\mathtt{d}$ of ω is removed,
- each letter $\omega_i = \mathtt{p}$ is turned into the value $|\{j < i \ : \ \omega_j = \mathtt{d}\}|$.

For all $n \geq 1$ and $\sigma \in \mathcal{I}_n(021)$, if σ contains k zeros, then $\Phi(\sigma)$ contains k letters $\mathtt{z}$, $n - k$ letters $\mathtt{p}$, and $\mathbf{max}(\sigma)$ letters $\mathtt{d}$. Let $\Phi_{\mathrm{Inv}}(\sigma) = \Phi(\sigma) \cdot \mathtt{d}^{|\sigma| - \mathbf{max}(\sigma)}$ be the word obtained by placing $|\sigma| - \mathbf{max}(\sigma)$ occurrences of the letter $\mathtt{d}$ at the end of $\Phi(\sigma)$, so that $|\Phi_{\mathrm{Inv}}(\sigma)| = 2|\sigma|$, with the convention $\Phi_{\mathrm{Inv}}(\varepsilon) = \varepsilon$. It is easy to see that Φ_{Inv} is also a bijection and, although the sets $\Phi(\mathcal{I}(021))$ and $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$ are different, the image of a word under Φ_{Inv}^{-1} is obtained in the exact same way as described above for Φ^{-1} (since extra letters d at the end of the word are simply removed).

Proposition 3.2.5. *The set $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$ is the set of words ω over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$ which satisfy the following:*

1. *no letter $\mathtt{p}$ appears before the first letter $\mathtt{d}$ in ω,*

2. *ω does not contain the factor* $\mathtt{dz}$,

3. $|\{\mathtt{d}\}[\omega]| = |\{\mathtt{p},\mathtt{z}\}[\omega]|$,

4. *for every prefix α of ω, we have* $|\{\mathtt{d}\}[\alpha]| \leq |\{\mathtt{p},\mathtt{z}\}[\alpha]|$.

In particular, if ω is a nonempty word in $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$, conditions 1 and 4 imply that ω starts with the letter $\mathtt{z}$*, and conditions 3 and 4 imply that ω ends with the letter* $\mathtt{d}$.

Proof. We show that for any $\sigma \in \mathcal{I}(021)$, the word $\omega = \Phi_{\mathrm{Inv}}(\sigma)$ satisfies all four conditions.

1. By definition of Φ, the first letter $\mathtt{p}$ of ω corresponds to the first positive entry σ_i of σ, hence it is preceded by $\sigma_i > 0$ letters $\mathtt{d}$.

2. Each letter $\mathtt{d}$ of ω is either part of a factor of letters $\mathtt{d}$ created by Φ, which is always followed by a $\mathtt{p}$, or part of the suffix of letters $\mathtt{d}$ added in Φ_{Inv}.

3. By definition of Φ and Φ_{Inv}, ω contains exactly $|\sigma|$ letters $\mathtt{p}$ or $\mathtt{z}$, as well as $|\sigma|$ letters $\mathtt{d}$.

4. Let $a \in [1, |\omega|]$, and let $\alpha = (\omega_i)_{i\in[1,a]}$ be a prefix of ω.

   - If $\omega_i = \mathtt{d}$ for all $i \geq a$, then $|\{\mathtt{d}\}[\alpha]| \leq |\{\mathtt{p},\mathtt{z}\}[\alpha]|$ as a consequence of item 3.
   - Otherwise, let $b = \mathbf{min}(i \in [a, |\omega|] \ : \ \omega_i \neq \mathtt{d})$, and let $\beta = (\omega_i)_{i\in[1,b]}$. Notice that β is a prefix of ω which ends with a letter $\mathtt{p}$ or $\mathtt{z}$, so by definition of Φ, the sequence σ admits a prefix γ such that $\beta = \Phi(\gamma)$. Since γ is a nonempty inversion sequence, we have
   $$|\{\mathtt{d}\}[\beta]| = \mathbf{max}(\gamma) < |\gamma| = |\{\mathtt{p},\mathtt{z}\}[\beta]|.$$
   If $b = a$, then $\alpha = \beta$ and the proof is finished. Otherwise $b > a$, and $\beta = \alpha \cdot \mathtt{d}^{b-a-1} \cdot \ell$ for some $\ell \in \{\mathtt{p},\mathtt{z}\}$, so
   $$|\{\mathtt{d}\}[\alpha]| \leq |\{\mathtt{d}\}[\beta]| \leq |\{\mathtt{p},\mathtt{z}\}[\beta]| - 1 = |\{\mathtt{p},\mathtt{z}\}[\alpha]|.$$

Conversely, it can be seen that any word satisfying all four conditions can be turned into a 021-avoiding inversion sequence by turning each $\mathtt{z}$ into a 0, each $\mathtt{p}$ into the number of letters $\mathtt{d}$ to its left, and removing all letters $\mathtt{d}$. □

It follows from items 3 and 4 of Proposition 3.2.5 that the words $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$ can be represented as coloured Dyck paths under the diagonal, with east steps $\mathtt{p}$ and $\mathtt{z}$, and north steps $\mathtt{d}$, see Figure 3.4. This representation of 021-avoiding inversion sequences as coloured Dyck paths was already used by Lin and Kim [73, Definition 2.1] to study statistics of 021-avoiding inversion sequences.

Proposition 3.2.6. *The set $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$ is a deterministic context-free language.*

Proof. The deterministic counter automaton of Figure 3.5 recognizes the language $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$. Indeed, it can be seen that the language recognized by this automaton satisfies all four conditions of Proposition 3.2.5.

- The transition from q_0 to q_1 reading a letter $\mathtt{d}$ must be traversed before any letter $\mathtt{p}$ can be read, hence condition 1 is satisfied.

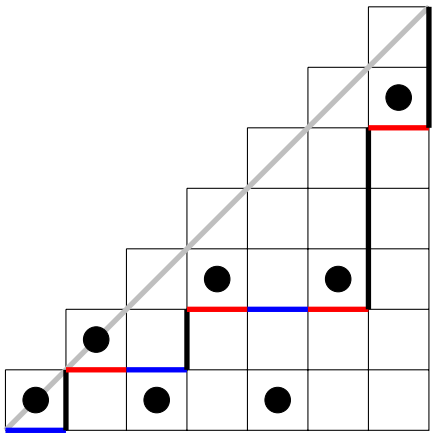

Figure 3.4: The diagram of the inversion sequence $\sigma = (0, 1, 0, 2, 0, 2, 5)$, and the coloured Dyck path from $(0, 0)$ to $(7, 7)$ representing $\Phi_{\mathrm{Inv}}(\sigma) = \mathtt{zdpzdpzpdddpdd}$. Each blue east step represents a letter $\mathtt{z}$, each red east step represents a letter $\mathtt{p}$, and each north step represents a letter $\mathtt{d}$.

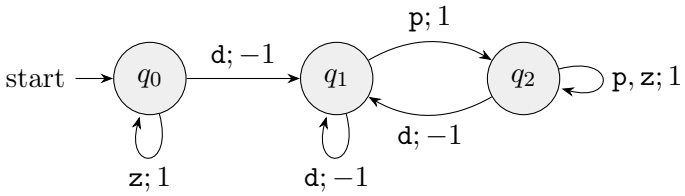

Figure 3.5: Counter automaton recognizing the language $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$.

- All transitions labelled by the letter $\mathtt{d}$ arrive in state q_1, and the letter $\mathtt{z}$ cannot be read from state q_1, hence condition 2 is satisfied.
- The counter's value increases by 1 whenever a letter $\mathtt{z}$ or $\mathtt{p}$ is read, and decreases by 1 whenever a letter $\mathtt{d}$ is read. Condition 3 is satisfied since words are accepted only if the counter is at 0. Condition 4 is satisfied since the counter cannot take negative values.

Conversely, any word satisfying all four conditions of Proposition 3.2.5 is accepted by this automaton. Note that its counter tracks the (Manhattan) distance between the diagonal and the lattice path representation of the word being read. □

3.2.2 Encoding patterns as ternary words

The aim of this section is to transfer the definition of pattern avoidance from 021-avoiding inversion sequences to their word encoding through Φ_{Inv}. Recall that we can restrict ourselves to patterns that are 021-avoiding RGFs, by Corollary 3.2.4.

Let τ be a nonempty word over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$. We say that τ is a *ternary pattern* if $\tau \in \Phi(\mathcal{R}(021))$. It is not difficult to see that $\Phi(\mathcal{R}(021))$ is the subset of $\Phi(\mathcal{I}(021))$ of words which do not contain consecutive letters d. It follows that ternary patterns can be equivalently described as words over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$ which begin with $\mathtt{z}$, have no $\mathtt{p}$ before their first $\mathtt{d}$, and whose every $\mathtt{d}$ is part of a factor $\mathtt{dp}$. We denote by $\mathcal{T}$ the set of all ternary patterns, and

by $\mathcal{T}_k$ the set of ternary patterns of length k for any $k \geq 1$. For instance, $\mathcal{T}_1 = \{\mathtt{z}\}$, $\mathcal{T}_2 = \{\mathtt{zz}\}$, $\mathcal{T}_3 = \{\mathtt{zzz}, \mathtt{zdp}\}$, and $\mathcal{T}_4 = \{\mathtt{zzzz}, \mathtt{zzdp}, \mathtt{zdpz}, \mathtt{zdpp}\}$.

Definition 3.2.7. Let Σ be a finite set of letters such that $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\} \subseteq \Sigma$. Let $n \geq k \geq 1$, let $\omega \in \Sigma^n$ be a word of length n over the alphabet Σ, and let $\tau \in \mathcal{T}_k$ be a ternary pattern of length k. We say that ω *contains* the ternary pattern τ if there exists a strictly increasing function $\varphi : \{\mathtt{p}, \mathtt{z}\}[\tau] \to [1, n]$ such that

- $\tau_i = \omega_{\varphi(i)}$ for all $i \in \{\mathtt{p}, \mathtt{z}\}[\tau]$ (i.e. the letters $\mathtt{p}$ and $\mathtt{z}$ of τ form a subword of ω), and
- for all $a < b \in \{\mathtt{p}, \mathtt{z}\}[\tau]$ such that $\tau_a = \tau_b = \mathtt{p}$, the factor $(\omega_i)_{i \in [\varphi(a), \varphi(b)]}$ contains the letter $\mathtt{d}$ if and only if the factor $(\tau_i)_{i \in [a,b]}$ contains the letter $\mathtt{d}$, regardless of the number of occurrences of the letter $\mathtt{d}$ in these factors.

We say that ω *avoids* the ternary pattern τ if ω does not contain τ.

Note that a ternary pattern τ contained by a word ω is not always a subword of ω. For instance, the word $\Phi(0, 1, 0, 1) = \mathtt{zdpzp}$ contains the ternary pattern $\Phi(0, 0, 1) = \mathtt{zzdp}$. From here on, we sometimes call classical patterns in inversion sequences "integer patterns" in order to clearly distinguish them from ternary patterns in words.

Proposition 3.2.8. *Let Σ be a finite set such that $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\} \subseteq \Sigma$, and let $\tau \in \mathcal{T}$ be a ternary pattern. The language of τ-avoiding words over the alphabet Σ is regular.*

Proof. It is easy to translate Definition 3.2.7 into a first-order logic formula. Let τ' be the subword of letters $\mathtt{p}$ and $\mathtt{z}$ of τ, let $q = |\tau'|$, and let f be the increasing bijection from $[1, q]$ to $\{\mathtt{p}, \mathtt{z}\}[\tau]$. A word ω over Σ contains the ternary pattern τ if and only if ω satisfies the formula

$$\exists x_1 \dots \exists x_q \left[\bigwedge_{a<b} x_a < x_b \wedge \bigwedge_{i \in [1,q]} P_{\tau'_i}(x_i) \wedge \bigwedge_{a<b \in \{\mathtt{p}\}[\tau']} \Big(\big(\exists y \, (P_{\mathtt{d}}(y)) \wedge x_a \leq y \leq x_b\big) \leftrightarrow D_{a,b} \Big) \right]$$

where

- each variable x_i corresponds to $\varphi(i)$ in Definition 3.2.7, so the the variables $x_1 \dots x_q$ represent the indices of a subword τ' in ω,
- $P_\ell(x)$ is the predicate indicating $\omega_x = \ell$,
- $D_{a,b}$ is the boolean constant $\mathtt{d} \in \{\tau_i\}_{i \in [f(a), f(b)]}$.

By the Büchi–Elgot theorem [30, 49], a language is regular if and only if it can be defined in monadic second-order logic. Since monadic second-order logic is a generalization of first-order logic, this concludes the proof. □

More directly, a finite automaton recognizing the language of τ-avoiding words over Σ can be obtained by following a process similar to the Knuth–Morris–Pratt algorithm [67]. We show an example on Figure 3.6, which is easy to generalize to any pattern.

For all $\sigma \in \mathcal{I}(021)$ and $i \in [1, |\sigma|]$, let $\psi_\sigma(i) = i + \mathbf{max}(\sigma_j : j \leq i)$. The function ψ_σ maps each $i \in [1, [\sigma|]$ to the position of the letter $\mathtt{p}$ or $\mathtt{z}$ of $\Phi(\sigma)$ corresponding to σ_i. In other words, ψ_σ is the increasing bijection from $[1, |\sigma|]$ to $\{\mathtt{p}, \mathtt{z}\}[\Phi(\sigma)]$. The next results establish a connection between integer patterns and ternary patterns.

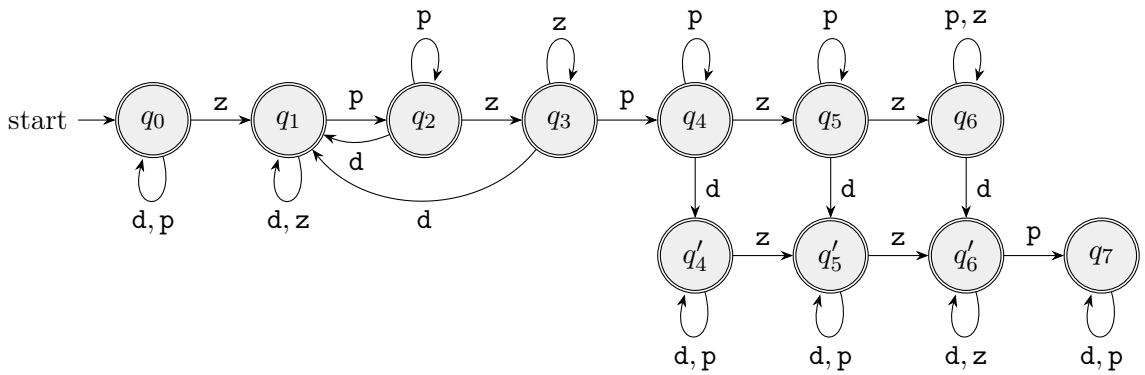

Figure 3.6: Finite automaton recognizing all words over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$ which avoid the ternary pattern $\Phi(01010020) = \mathtt{zdpzpzzdpz}$.

Lemma 3.2.9. *Let ρ be a 021-avoiding RGF. An inversion sequence $\sigma \in \mathcal{I}(021)$ contains an occurrence μ of the integer pattern ρ such that $0 \in \mathbf{Vals}(\mu)$ if and only if $\Phi(\sigma)$ contains the ternary pattern $\Phi(\rho)$.*

Proof. Let $\sigma \in \mathcal{I}(021)$, $\rho \in \mathcal{R}(021)$, $k = |\rho|$, $\omega = \Phi(\sigma)$, and $\tau = \Phi(\rho)$.

Assume that $f : [1, k] \to [1, |\sigma|]$ is an increasing function such that $\mu = (\sigma_{f(i)})_{i \in [1,k]}$ is an occurrence of ρ and $0 \in \mathbf{Vals}(\mu)$. In particular, for all $i \in [1, k]$, $\mu_i = 0$ if and only if $\rho_i = 0$. For all $i \in [1, k]$, we have $\omega_{\psi_\sigma \circ f(i)} = \tau_{\psi_\rho(i)}$ since

- $\omega_{\psi_\sigma \circ f(i)} = \mathtt{z} \iff \sigma_{f(i)} = 0 \iff \rho_i = 0 \iff \tau_{\psi_\rho(i)} = \mathtt{z}$, and
- $\omega_{\psi_\sigma \circ f(i)} = \mathtt{p} \iff \sigma_{f(i)} > 0 \iff \rho_i > 0 \iff \tau_{\psi_\rho(i)} = \mathtt{p}$.

The first and third equivalences hold by definition of ψ, while the second one holds by definition of f. For all $a < b \in [1, k]$ such that $\tau_{\psi_\rho(a)} = \tau_{\psi_\rho(b)} = \mathtt{p}$, we have

$$\mathtt{d} \in \{\tau_i\}_{i \in [\psi_\rho(a), \psi_\rho(b)]} \iff \rho_a < \rho_b \iff \sigma_{f(a)} < \sigma_{f(b)} \iff \mathtt{d} \in \{\omega_i\}_{i \in [\psi_\sigma \circ f(a), \psi_\sigma \circ f(b)]}.$$

This shows that ω contains the ternary pattern τ (notice that $\psi_\sigma \circ f \circ \psi_\rho^{-1}$ corresponds to the function denoted φ in Definition 3.2.7).

Conversely, assume that ω contains the ternary pattern τ. Let $\varphi : \psi_\rho([1, k]) \to [1, |\omega|]$ be a function satisfying the conditions of Definition 3.2.7, and let $f = \psi_\sigma^{-1} \circ \varphi \circ \psi_\rho$. For all $i \in [1, k]$, we have

$$\rho_i = 0 \iff \tau_{\psi_\rho(i)} = \mathtt{z} \iff \omega_{\varphi \circ \psi_\rho(i)} = \mathtt{z} \iff \sigma_{f(i)} = 0.$$

For all $a < b \in [1, k]$ such that $\rho_a, \rho_b > 0$, we have

$$\rho_a < \rho_b \iff \mathtt{d} \in \{\tau_i\}_{i \in [\psi_\rho(a), \psi_\rho(b)]} \iff \mathtt{d} \in \{\omega_i\}_{i \in [\varphi \circ \psi_\rho(a), \varphi \circ \psi_\rho(b)]} \iff \sigma_{f(a)} < \sigma_{f(b)}.$$

This shows that $(\sigma_{f(i)})_{i \in [1,k]}$ is an occurrence of ρ which contains the value 0. □

Corollary 3.2.10. *Let $\sigma \in \mathcal{I}(021)$, and let $\rho \in \mathcal{R}(021)$ be a pattern which contains a single 0, or contains an inversion. Then σ contains the integer pattern ρ if and only if $\Phi(\sigma)$ contains the ternary pattern $\Phi(\rho)$.*

Proof. If ρ has a single zero (it must be ρ_1), then for any occurrence μ of ρ in σ, the sequence $\sigma_1 \cdot (\mu_i)_{i\in[2,|\rho|]}$ is also an occurrence of ρ and uses the value 0.

If ρ contains an inversion, then by Proposition 3.2.2, any occurrence of ρ in σ must use the value 0.

In either case, σ contains ρ if and only if there exists an occurrence of ρ in σ which uses the value 0. Applying Lemma 3.2.9 concludes the proof. □

Corollary 3.2.11. *Let $\sigma \in \mathcal{I}(021)$, let $\rho \in \mathcal{R}(021)$, and let $\rho' = 0 \cdot (\rho_i + 1)_{i\in[1,|\rho|]}$. Then σ contains the integer pattern ρ if and only if $\Phi(\sigma)$ contains the ternary pattern $\Phi(\rho)$ or the ternary pattern $\Phi(\rho')$.*

Proof. Only three cases can occur.

- If σ contains an occurrence of ρ which uses the value 0, then $\Phi(\sigma)$ contains the ternary pattern $\Phi(\rho)$ by Lemma 3.2.9.
- If σ contains an occurrence μ of ρ such that $0 \notin \mathbf{Vals}(\mu)$, then $\sigma_1 \cdot \mu$ is an occurrence of ρ' in σ which uses the value 0, hence $\Phi(\sigma)$ contains the ternary pattern $\Phi(\rho')$ by Lemma 3.2.9.
- If σ avoids ρ, then σ also avoids ρ', therefore $\Phi(\sigma)$ avoids both ternary patterns $\Phi(\rho)$ and $\Phi(\rho')$ by Lemma 3.2.9. □

We are now ready to prove the main theorem of this chapter, by putting together all of the previous results.

Theorem 3.2.12. *For any finite set of patterns P, the generating function of $\mathcal{I}(\{021\} \cup P)$ is algebraic.*

Proof. Let P be a finite set of patterns.

- By Corollary 3.2.4, there is a finite set $R \subset \mathcal{R}(021)$ such that $\mathcal{I}(\{021\}\cup P) = \mathcal{I}(\{021\}\cup R)$.
- By Corollary 3.2.11, an inversion sequence $\sigma \in \mathcal{I}(021)$ avoids R if and only if $\Phi(\sigma)$ avoids the set of ternary patterns $R' = \bigcup_{\rho\in R}\{\Phi(\rho), \Phi(0 \cdot (\rho_i + 1)_{i\in[1,|\rho|]}\}$.
- By Proposition 3.2.8, for all $\tau \in R'$, the language of τ-avoiding words over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$ is regular.
- Regular languages are closed under finite intersection, so the language $L_{R'}$ of R'-avoiding words over the alphabet $\{\mathtt{d}, \mathtt{p}, \mathtt{z}\}$ is regular.
- By Proposition 3.2.6, $\Phi_{\mathrm{Inv}}(\mathcal{I}(021))$ is recognized by a deterministic counter automaton, it is therefore a deterministic context-free language.
- For any $\sigma \in \mathcal{I}(021)$, the word $\Phi_{\mathrm{Inv}}(\sigma)$ contains the same ternary patterns as $\Phi(\sigma)$, since adding or removing some letters $\mathtt{d}$ at the end of a word does not affect its ternary patterns. Consequently, $\Phi_{\mathrm{Inv}}(\mathcal{I}(\{021\} \cup R)) = \Phi_{\mathrm{Inv}}(\mathcal{I}(021)) \cap L_{R'}$.
- The intersection of a deterministic context-free language and a regular language is a deterministic context-free language. Deterministic context-free languages have algebraic generating functions, hence the (ordinary) generating function $F(x)$ of $\Phi_{\mathrm{Inv}}(\mathcal{I}(\{021\} \cup R))$ is algebraic.
- The function Φ_{Inv} is a bijection satisfying $|\Phi_{\mathrm{Inv}}(\sigma)| = 2|\sigma|$ for any $\sigma \in \mathcal{I}(021)$, hence the generating function of $\mathcal{I}(\{021\} \cup P) = \mathcal{I}(\{021\} \cup R)$ is $F(\sqrt{x})$, which is algebraic. □

3.3 Ascent sequences avoiding 021

Recall that $\mathcal{A}$ denotes the set of ascent sequences, that is, the subset of inversion sequences σ whose each entry σ_i is at most the number of ascents (i.e. the number of pairs (σ_k, σ_{k+1}) such that $\sigma_k < \sigma_{k+1}$) of the prefix $(\sigma_j)_{j \leq i}$. Similarly to inversion sequences, it was shown in [90] that ascent sequences avoiding 021 and a pattern of length 4 always have an algebraic generating function. In the present section, we adapt our previous encoding of inversion sequences to the ascent sequence case, and prove that Theorem 3.2.12 still holds on ascent sequences. Our new encoding introduces a fourth letter $\mathtt{f}$ that is placed each time an entry has a value less than or equal to the preceding one; this indicates entries that are not ascent tops.

For all $\sigma \in \mathcal{A}(021)$, let $\Phi_{\mathrm{Asc}}(\sigma)$ be the word over the alphabet $\{\mathtt{d}, \mathtt{f}, \mathtt{p}, \mathtt{z}\}$ obtained by applying the following steps to σ:

- the first zero of σ is turned into a letter $\mathtt{z}$,
- all other zeros of σ are turned into factors $\mathtt{zf}$,
- each positive entry σ_i is replaced by $\mathtt{d}^{\sigma_i - \mathbf{max}(\sigma_j)_{j<i}} \cdot \mathtt{p}$ if $\sigma_i > \sigma_{i-1}$ (i.e. if σ_i is an ascent top), or by $\mathtt{pf}$ otherwise,
- if $\sigma \neq \varepsilon$, then $1 + \mathbf{asc}(\sigma) - \mathbf{max}(\sigma)$ letters $\mathtt{d}$ are placed at the end of the resulting word.

Equivalently, $\Phi_{\mathrm{Asc}}(\sigma)$ can be defined by making some small changes to $\Phi(\sigma)$. If $\sigma \in \mathcal{A}(021)$ and $\omega = \Phi(\sigma)$, then $\Phi_{\mathrm{Asc}}(\sigma)$ can be obtained from ω as follows:

- each letter $\mathtt{z}$ of ω except for the first one is turned into a factor $\mathtt{zf}$,
- a letter $\mathtt{p} = \omega_i$ is turned into a factor $\mathtt{pf}$ if $\omega_{i-1} = \mathtt{p}$,
- if $\omega \neq \varepsilon$, the factor $\mathtt{d}^{1+\mathbf{asc}(\sigma)-\mathbf{max}(\sigma)}$ is inserted at the end of the resulting word.

It is easy to see that Φ_{Asc} is injective, since Φ is injective and $\Phi(\sigma)$ can be recovered from $\Phi_{\mathrm{Asc}}(\sigma)$ by removing every letter $\mathtt{f}$, as well as any letters $\mathtt{d}$ appearing in a suffix of $\Phi_{\mathrm{Asc}}(\sigma)$.

Once again, the word encoding of an ascent sequence can be represented as a coloured Dyck path, as detailed in the two propositions below. See Figure 3.7 for an example.

Proposition 3.3.1. *For all $\sigma \in \mathcal{A}(021)$, we have $|\{\mathtt{p}, \mathtt{z}\}[\Phi_{\mathrm{Asc}}(\sigma)]| = |\{\mathtt{d}, \mathtt{f}\}[\Phi_{\mathrm{Asc}}(\sigma)]| = |\sigma|$. In particular, $|\Phi_{\mathrm{Asc}}(\sigma)| = 2|\sigma|$*

Proof. Let $\sigma \in \mathcal{A}(021)$, and let $\omega = \Phi_{\mathrm{Asc}}(\sigma)$. If $\sigma = \varepsilon$, then $\omega = \varepsilon$. If σ is not empty, then

- $|\{\mathtt{z}\}[\omega]|$ is the number of zeros of σ, and $|\{\mathtt{p}\}[\omega]|$ is the number of positive entries of σ, so $|\{\mathtt{p}, \mathtt{z}\}[\omega]| = |\sigma|$,
- $|\{\mathtt{d}\}[\omega]| = 1 + \mathbf{asc}(\sigma)$, since the number of letters $\mathtt{d}$ in $\Phi(\sigma)$ is $\mathbf{max}(\sigma)$, and some additional $1 + \mathbf{asc}(\sigma) - \mathbf{max}(\sigma)$ letters $\mathtt{d}$ are present at the end of ω,
- $|\{\mathtt{f}\}[\omega]| = |\sigma| - \mathbf{asc}(\sigma) - 1$, since a letter $\mathtt{f}$ is placed after each $\mathtt{z}$ except for the initial $\mathtt{z}$ of $\Phi(\sigma)$, and each $\mathtt{p}$ of $\Phi(\sigma)$ whose corresponding entry in σ is not an ascent top.

We conclude by summing the number of occurrences of each letter. □

Proposition 3.3.2. *For all $\omega \in \Phi_{\mathrm{Asc}}(\mathcal{A}(021))$, every prefix α of ω satisfies*

$$|\{\mathtt{p}, \mathtt{z}\}[\alpha]| \geq |\{\mathtt{d}, \mathtt{f}\}[\alpha]|.$$

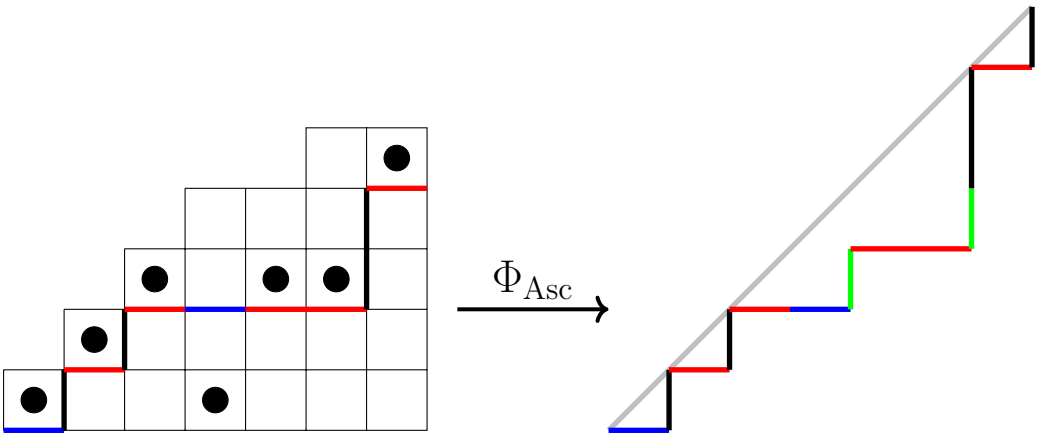

Figure 3.7: On the left, the diagram of the ascent sequence $\sigma = (0, 1, 2, 0, 2, 2, 4)$, and the coloured lattice path representation of $\Phi(\sigma) = \mathtt{zdpdpzppddp}$. On the right, the coloured Dyck path from $(0, 0)$ to $(7, 7)$ representing $\Phi_{\mathrm{Asc}}(\sigma) = \mathtt{zdpdpzfppfddpd}$. Each blue east step is a letter $\mathtt{z}$, each red east step is a letter $\mathtt{p}$, each black north step is a letter $\mathtt{d}$, and each green north step is a letter $\mathtt{f}$.

Proof. Let $\sigma \in \mathcal{A}(021)$, $\omega = \Phi_{\mathrm{Asc}}(\sigma)$, $a \in [1, |\omega|]$, and let $\alpha = (\omega_i)_{i\in[1,a]}$ be a prefix of ω.

- If $\omega_i \in \{\mathtt{d}\}$ for all $i \geq a$, then $|\{\mathtt{d}, \mathtt{f}\}[\alpha]| \leq |\{\mathtt{p}, \mathtt{z}\}[\alpha]|$ as a consequence of Proposition 3.3.1.
- Otherwise, let $b = \mathbf{min}(i \in [a, |\omega|] \ : \ \omega_i \neq \mathtt{d})$, and let $\beta = (\omega_i)_{i\in[1,b]}$. Let $k = |\{\mathtt{p}, \mathtt{z}\}[\beta]|$, and let $\gamma = (\sigma_i)_{i\in[1,k]}$ be the prefix of σ which satisfies $\Phi_{\mathrm{Asc}}(\gamma) = \beta \cdot \mathtt{d}^{1+\mathbf{asc}(\gamma)-\mathbf{max}(\gamma)}$. In particular, we have
  - $|\{\mathtt{d}\}[\beta]| = \mathbf{max}(\gamma)$,
  - $|\{\mathtt{f}\}[\beta]| = |\gamma| - \mathbf{asc}(\gamma) - 1$,
  - $|\{\mathtt{p}, \mathtt{z}\}[\beta]| = |\gamma|$.

  Since γ is an ascent sequence, we have $\mathbf{max}(\gamma) \leq \mathbf{asc}(\gamma)$, hence $|\{\mathtt{d}, \mathtt{f}\}[\beta]| < |\{\mathtt{p}, \mathtt{z}\}[\beta]|$.

  If $\alpha = \beta$ then the proof is finished. Otherwise $a < b$, and $\beta = \alpha \cdot \mathtt{d}^{b-a-1} \cdot \ell$ for some $\ell \in \{\mathtt{f}, \mathtt{p}, \mathtt{z}\}$, so

$$|\{\mathtt{d}, \mathtt{f}\}[\alpha]| \leq |\{\mathtt{d}, \mathtt{f}\}[\beta]| \leq |\{\mathtt{p}, \mathtt{z}\}[\beta]| - 1 \leq |\{\mathtt{p}, \mathtt{z}\}[\alpha]|. \qquad \square$$

The following Proposition provides the ascent sequence equivalent of Proposition 3.2.5, and can be proved in much the same way (the nontrivial parts are already covered by Propositions 3.3.1 and 3.3.2).

Proposition 3.3.3. *The set $\Phi_{\mathrm{Asc}}(\mathcal{A}(021))$ can be described as the set of words ω over the alphabet $\{\mathtt{d}, \mathtt{f}, \mathtt{p}, \mathtt{z}\}$ which satisfy the following:*

1. $|\{\mathtt{p}, \mathtt{z}\}[\omega]| = |\{\mathtt{d}, \mathtt{f}\}[\omega]|$,
2. *for every prefix α of ω, we have $|\{\mathtt{p}, \mathtt{z}\}[\alpha]| \geq |\{\mathtt{d}, \mathtt{f}\}[\alpha]|$,*
3. *no letter $\mathtt{p}$ appears before the first letter $\mathtt{d}$ of ω (if there is no $\mathtt{d}$, then $\mathtt{p}$ cannot appear),*

4. *ω does not contain any factor* `dz`, `df`, *or* `ff`,
5. *every letter* `z` *except the first one is immediately followed by the letter* `f`,
6. *a letter* `p` *is immediately followed by the letter* `f` *if and only if it is immediately preceded by* `p` *or* `pf`.

Clearly, words satisfying only conditions 1 and 2 of Proposition 3.3.3 are recognized by a deterministic counter automaton (with a single state), since they are bicolored Dyck paths. It is a simple task to describe finite automata which recognize words satisfying conditions 3, 4, 5, or 6. It follows that the intersection $\Phi_{\mathrm{Asc}}(\mathcal{A}(021))$ of these languages is recognized by a deterministic counter automaton; such an automaton is pictured on Figure 3.8.

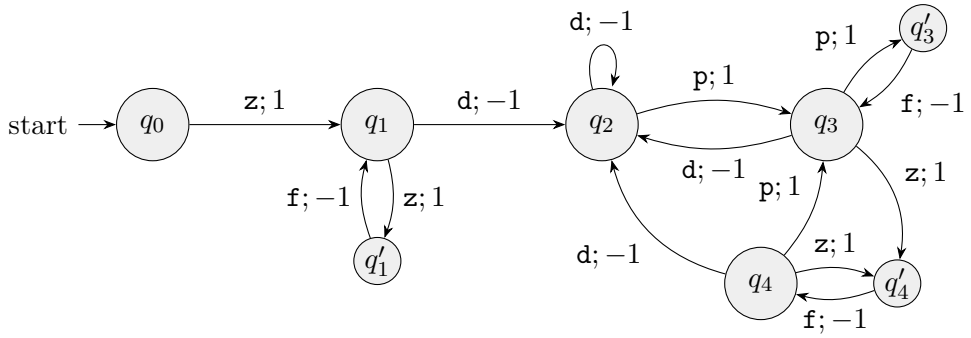

Figure 3.8: Counter automaton recognizing the language $\Phi_{\mathrm{Asc}}(\mathcal{A}(021))$.

This shows that $\Phi_{\mathrm{Asc}}(\mathcal{A}(021))$ is a deterministic context-free language. We now have all the tools necessary to prove the ascent sequence equivalent of Theorem 3.2.12.

Theorem 3.3.4. *For any finite set of patterns P, the generating function of $\mathcal{A}(\{021\} \cup P)$ is algebraic.*

Proof. For all $\sigma \in \mathcal{A}(021)$, the word $\Phi_{\mathrm{Asc}}(\sigma)$ contains the exact same ternary patterns as $\Phi(\sigma)$, since the insertion of some letters `f` and a suffix of letters `d` has no effect on ternary patterns. Proposition 3.3.1 implies that $|\Phi_{\mathrm{Asc}}(\sigma)| = 2|\sigma|$, so the proof of Theorem 3.2.12 easily extends to ascent sequences, replacing Φ_{Inv} by Φ_{Asc}. □

3.4 Restricted growth functions avoiding 021

Recall that $\mathcal{R}$ is the set of restricted growth functions (or RGFs), that is, the set of inversion sequences in which the first occurrence of each value is a strict left-to-right maximum. We now adapt the encoding Φ to RGFs. This case is simpler than the previous ones: the languages we obtain are regular, so their generating functions are rational.

Let $\sigma \in \mathcal{R}(021)$, and $\omega = \Phi(\sigma)$. We define $\Phi_{\mathrm{RGF}}(\sigma)$ as the word over the alphabet $\{\mathtt{d}, \mathtt{f}, \mathtt{p}, \mathtt{z}\}$ obtained from ω by

- replacing every letter `z` of ω, except the first one, by a factor `zf`,
- replacing every letter `p` of ω, except those preceded by a letter `d`, by a factor `pf`,
- appending a single letter `d`, if $\omega \neq \varepsilon$.

The first two bullet points can be expressed more intuitively as follows: every letter p or z of $\Phi_{\mathrm{RGF}}(\sigma)$ must be followed by an f unless the corresponding entry of σ is a strict left-to-right maximum. In particular, we have

$$|\{\mathtt{f}\}[\Phi_{\mathrm{RGF}}(\sigma)]| = |\sigma| - \mathbf{max}(\sigma) - 1,$$

since every value in $[0, \mathbf{max}(\sigma)]$ occurs once as a strict left-to-right maximum (since σ is a RGF). Moreover, we have $|\{\mathtt{d}\}[\Phi_{\mathrm{RGF}}(\sigma)]| = \mathbf{max}(\sigma) + 1$, so we observe once again that

$$|\{\mathtt{d}, \mathtt{f}\}[\Phi_{\mathrm{RGF}}(\sigma)]| = |\{\mathtt{p}, \mathtt{z}\}[\Phi_{\mathrm{RGF}}(\sigma)]| = |\sigma|.$$

Earlier, we encoded inversion sequences and ascent sequences as words recognized by counter automata. The value of the counter was used to track how great the next value of a sequence could be (while satisfying the constraints characterizing inversion or ascent sequences) compared to its current maximum. Equivalently, the counter tracked how many consecutive letters d could be placed in the word encoding of these sequences. However, in the case of RGFs, a value cannot exceed the greatest value to its left by more than 1; equivalently, there cannot be several consecutive letters d in their word encoding. Consequently, a counter is no longer necessary in this case: a finite automaton is sufficient to recognize $\Phi_{\mathrm{RGF}}(\mathcal{R}(021))$, see Figure 3.9.

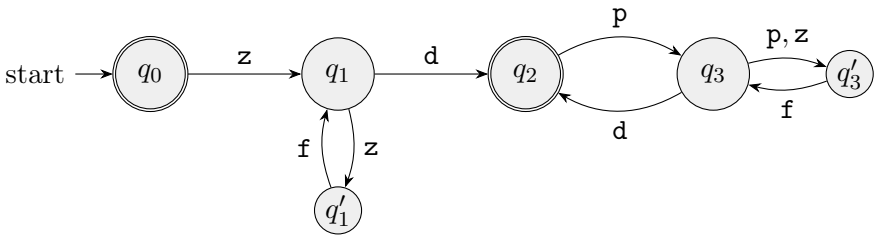

Figure 3.9: Finite automaton recognizing the language $\Phi_{\mathrm{RGF}}(\mathcal{R}(021))$.

Theorem 3.4.1. *For any finite set of patterns P, the generating function of $\mathcal{R}(\{021\} \cup P)$ is rational.*

Proof. The proof follows the same line as the proof of Theorem 3.2.12, since

- for all $\sigma \in \mathcal{R}(021)$, we have $|\Phi_{\mathrm{RGF}}(\sigma)| = 2|\sigma|$,
- $\Phi_{\mathrm{RGF}}(\mathcal{R}(021))$ is a regular language since it is recognized by a finite automaton,
- the intersection of two regular languages is a regular language,
- a regular language always has a rational generating function. □

3.5 Avoiding 021 and an increasing pattern

In this section, we show that both inversion sequences and ascent sequences avoiding 021, an increasing pattern, and any finite set of patterns always have a rational generating function. To achieve this, we use another encoding of inversion sequences Ψ, which is similar to our earlier encoding Φ in some ways, but also uses the idea of *commitments* from [99].

3.5.1 Inversion sequences

For all $\sigma \in \mathcal{I}(021)$, let $\Psi(\sigma)$ be the word over the alphabet $\{\mathtt{d}, \mathtt{f}, \mathtt{p}, \mathtt{z}\}$ obtained by turning each entry σ_i into a factor abc defined as follows.

- If $\sigma_i > 0$ and $\sigma_i > \sigma_j$ for all $j < i$, then $a = \mathtt{d}$. Otherwise, $a = \varepsilon$.
- If $\sigma_i = 0$, then $b = \mathtt{z}$. If $\sigma_i > 0$, then $b = \mathtt{p}$.
- If $i \in \mathbf{Vals}(\sigma)$, then $c = \varepsilon$. Otherwise, $c = \mathtt{f}$.

We now provide a more intuitive explanation of the encoding Ψ. Let $\sigma \in \mathcal{I}(021)$, and assume that $\sigma_i = v > 0$ is a strict left-to-right maximum of σ. The function Φ we used earlier was a simple encoding reading σ from left to right: the value v of the term σ_i was determined only after encoding the prefix $(\sigma_j)_{j<i}$, and this value was represented by a number of letters $\mathtt{d}$ placed in $\Phi(\sigma)$. With our new encoding Ψ, the decision of whether the value v appears is taken earlier, when encoding the term σ_v. If the v-th letter $\mathtt{z}$ or $\mathtt{p}$ of $\Psi(\sigma)$ is followed by an $\mathtt{f}$, then the value v is forbidden and will not appear in the sequence. Otherwise, we commit to placing the value v later in σ. This commitment is not fulfilled until position i is reached; the value of i may be much greater than v, and several unfulfilled commitments may exist at a given time. The letter $\mathtt{d}$ marks the fulfillment of a commitment, and it must always be followed by a letter $\mathtt{p}$ which corresponds to the smallest committed value that has not yet appeared. In particular, each letter $\mathtt{d}$ is paired with a "missing" letter $\mathtt{f}$ to its left, so the number of letters $\mathtt{d}$ or $\mathtt{f}$ in a prefix of $\Psi(\sigma)$ can never exceed its number of letters $\mathtt{p}$ or $\mathtt{z}$. Moreover, a sequence is complete only when all commitments have been fulfilled, this means there are as many letters $\mathtt{d}$ as "missing" letters $\mathtt{f}$, i.e. $|\{\mathtt{d}, \mathtt{f}\}[\Psi(\sigma)]| = |\sigma|$. Figure 3.10 depicts the function Ψ on an example.

Proposition 3.5.1. *Let $\sigma \in \mathcal{I}(021)\backslash\{\varepsilon\}$, and let $\omega = \Psi(\sigma)$. Then*

- $|\{\mathtt{d}\}[\omega]| = \mathbf{dist}(\sigma) - 1$,
- $|\{\mathtt{p}, \mathtt{z}\}[\omega]| = |\sigma|$,
- $|\{\mathtt{f}\}[\omega]| = |\sigma| - \mathbf{dist}(\sigma) + 1$.

It follows that $|\{\mathtt{p}, \mathtt{z}\}[\omega]| = |\{\mathtt{d}, \mathtt{f}\}[\omega]|$, and $|\omega| = 2|\sigma|$.

Proof. From the definition of Ψ, it can be seen that

- each letter $\mathtt{d}$ corresponds to the first occurrence of a positive value of σ,
- each letter $\mathtt{p}$ or $\mathtt{z}$ corresponds to an entry of σ,
- a letter $\mathtt{f}$ appears for each $i \in [1, n]\backslash(\mathbf{Vals}(\sigma)\backslash\{0\})$. □

Proposition 3.5.2. *For all $\sigma \in \mathcal{I}(021)$, the words $\Phi(\sigma)$ and $\Psi(\sigma)$ contain the same ternary patterns.*

Proof. Let $\sigma \in \mathcal{I}(021)$.

- Clearly, $\Phi(\sigma)$ and $\Psi(\sigma)$ have the same subword of letters $\mathtt{p}$ and $\mathtt{z}$, since each letter $\mathtt{p}$ corresponds to a positive entry of σ, and each letter $\mathtt{z}$ corresponds to a zero of σ, in both $\Phi(\sigma)$ and $\Psi(\sigma)$.

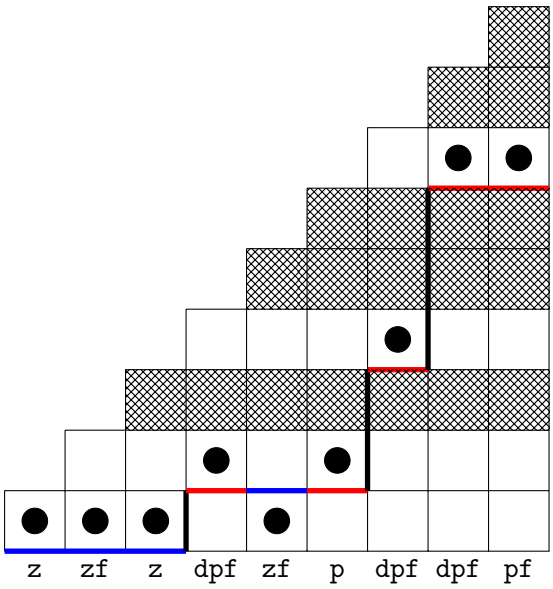

Figure 3.10: The diagram of the inversion sequence $\sigma = (0, 0, 0, 1, 0, 1, 3, 6, 6)$, with decorations representing $\Psi(\sigma) = \texttt{zzfzdpfzfpdpfdpfpf}$. Each letter $\texttt{f}$ of $\Psi(\sigma)$ (except the last one) is represented by a crosshatched line marking a forbidden value. The letters $\texttt{d}$ $\texttt{p}$, and $\texttt{z}$ of $\Psi(\sigma)$ are represented by a lattice path: each letter $\texttt{z}$ is a blue east step, each letter $\texttt{p}$ is a red east step, and each letter $\texttt{d}$ is a vertical line (of variable length).

- The letters $\texttt{p}$ of $\Phi(\sigma)$ and $\Psi(\sigma)$ are separated by letters $\texttt{d}$ in the same manner. More precisely, let $i < j \in [1, |\sigma|]$ be such that $\sigma_i > 0$ and $\sigma_j > 0$, let $i_\Phi < j_\Phi$ be the indices of the corresponding letters $\texttt{p}$ of $\Phi(\sigma)$ (which can be defined as the i-th and j-th smallest elements of $\{\texttt{p}, \texttt{z}\}[\Phi(\sigma)]$), and let $i_\Psi < j_\Psi$ be the indices of the corresponding letters $\texttt{p}$ of $\Psi(\sigma)$ (defined similary). Then

$$\texttt{d} \in \{\Phi(\sigma)_k\}_{k\in[i_\Phi, j_\Phi]} \iff \sigma_i < \sigma_j \iff \texttt{d} \in \{\Psi(\sigma)_k\}_{k\in[i_\Psi, j_\Psi]}.$$

  Indeed, the number of such letters $\texttt{d}$ in $\Phi(\sigma)$ is $\sigma_j - \sigma_i$, while the number of such letters $\texttt{d}$ in $\Psi(\sigma)$ is $|\mathbf{Vals}(\sigma) \cap [\sigma_i + 1, \sigma_j]|$ (i.e. the number of strict left-to-right maxima of σ in $(\sigma_k)_{k\in[i+1,j]}$).

Since the definition of ternary pattern depends only on the relative positions of letters $\texttt{p}$ and $\texttt{z}$ and on the existence of a letter $\texttt{d}$ in-between two letters $\texttt{p}$, this concludes the proof. □

As in the previous sections, we provide an explicit characterization of the language $\Psi(\mathcal{I}(021))$ in the following proposition. We omit its proof, since it is similar to that of Proposition 3.2.5.

Proposition 3.5.3. *The set $\Psi(\mathcal{I}(021))$ is the set of words ω over the alphabet $\{\texttt{d}, \texttt{f}, \texttt{p}, \texttt{z}\}$ which satisfy the following:*

1. $|\{\texttt{d}, \texttt{f}\}[\omega]| = |\{\texttt{p}, \texttt{z}\}[\omega]|$,
2. *for every prefix α of ω, we have $|\{\texttt{d}, \texttt{f}\}[\alpha]| \leq |\{\texttt{p}, \texttt{z}\}[\alpha]|$.*
3. *no letter $\texttt{p}$ appears before the first letter $\texttt{d}$ in ω (if there is no $\texttt{d}$, then $\texttt{p}$ cannot appear),*

4. *every letter* `d` *is immediately followed by a letter* `p`,

5. *there is no factor* `ff`.

From the characterization of Proposition 3.5.3, we could describe a counter automaton recognizing $\Psi(\mathcal{I}(021))$, and prove Theorem 3.2.12 using Ψ instead of Φ. Indeed, it is clear that conditions 1 and 2 describe bicolored Dyck paths as before (so they are recognized by a counter automaton having a single state), while conditions 3, 4 and 5 describe a regular language. Instead, we now consider inversion sequences which avoid both 021 and an increasing pattern, and show that the function Ψ turns them into a regular language.

Lemma 3.5.4. *Let $k \geq 1$, and let $\rho = 01 \dots k$ be the strictly increasing pattern of length $k+1$. The language $\Psi(\mathcal{I}(021, \rho))$ is regular.*

Proof. We know that in a 021-avoiding inversion sequence σ, the first occurrence of each value is a strict left-to-right maximum; it follows that the longest increasing subsequence of σ has length $\mathbf{dist}(\sigma)$. In other words, $\sigma \in \mathcal{I}(021)$ avoids the pattern ρ if and only if $\mathbf{dist}(\sigma) \leq k$.

Let $\sigma \in \mathcal{I}(021, \rho)$, and $\omega = \Psi(\sigma)$. Then σ avoids ρ if and only if ω contains at most $k-1$ letters `d`, which also implies that at most $k-1$ letters `p` or `z` of ω are not followed by an `f`. Since k is a constant, this means that a counter is no longer necessary to ensure that conditions 1 and 2 of Proposition 3.5.3 are satisfied. Indeed, a counter tracking the value $|\{\mathtt{p}, \mathtt{z}\}[\alpha]| - |\{\mathtt{d}, \mathtt{f}\}[\alpha]|$ for each prefix α of ω would only take values between 0 and k, so a finite number of states is enough to simulate the counter. Moreover, the new constraint that ω must contain less than k letters `d` clearly corresponds to a finite automaton with k states. Since conditions 3, 4, and 5 of Proposition 3.5.3 all describe regular languages as well, it follows that $\Psi(\mathcal{I}(021, \rho))$ is regular, as regular languages are closed under finite intersection. □

We now explicitly describe a non-deterministic and non-ambiguous finite automaton recognizing $\Psi(\mathcal{I}(021, \rho))$, where $\rho = 01 \dots k$ is the strictly increasing pattern of length $k+1$. To simplify its presentation, we allow transitions to be labelled by a nonempty factor rather than one letter. This does not affect which languages are recognizable by such a machine, since these factor transitions can be split into individual letters by adding some non-final states. For instance, given two states x and y, a transition $x \xrightarrow{\mathtt{ab}} y$ can be rewritten as $x \xrightarrow{\mathtt{a}} z$ and $z \xrightarrow{\mathtt{b}} y$, where z is a non-final state. Our automaton is described below.

- The set of states is $\{(c, d) \in \mathbb{N}^2 \ : \ c + d < k\}$. The first component c indicates the number of unfulfilled commitments, while the second component d tracks the number of distinct positive values of the corresponding inversion sequence (or equivalently, the number of letters `d` read so far).

- The initial state is $(0, 0)$.

- The final states are $(0, d)$ for all $d \in [0, k-1]$.

- The transitions are

$$\begin{cases} (c,d) \xrightarrow{\texttt{zf}} (c,d) & \\ (c,d) \xrightarrow{\texttt{z}} (c+1,d) & \text{if } c+d < k-1 \\ (c,d) \xrightarrow{\texttt{pf}} (c,d) & \text{if } d > 0 \\ (c,d) \xrightarrow{\texttt{p}} (c+1,d) & \text{if } d > 0 \quad \text{and } c+d < k-1 \\ (c,d) \xrightarrow{\texttt{dpf}} (c-1,d+1) & \text{if } c > 0 \\ (c,d) \xrightarrow{\texttt{dp}} (c,d+1) & \text{if } c > 0 \quad \text{and } c+d < k-1. \end{cases}$$

The automaton for $k = 4$ is pictured on Figure 3.11.

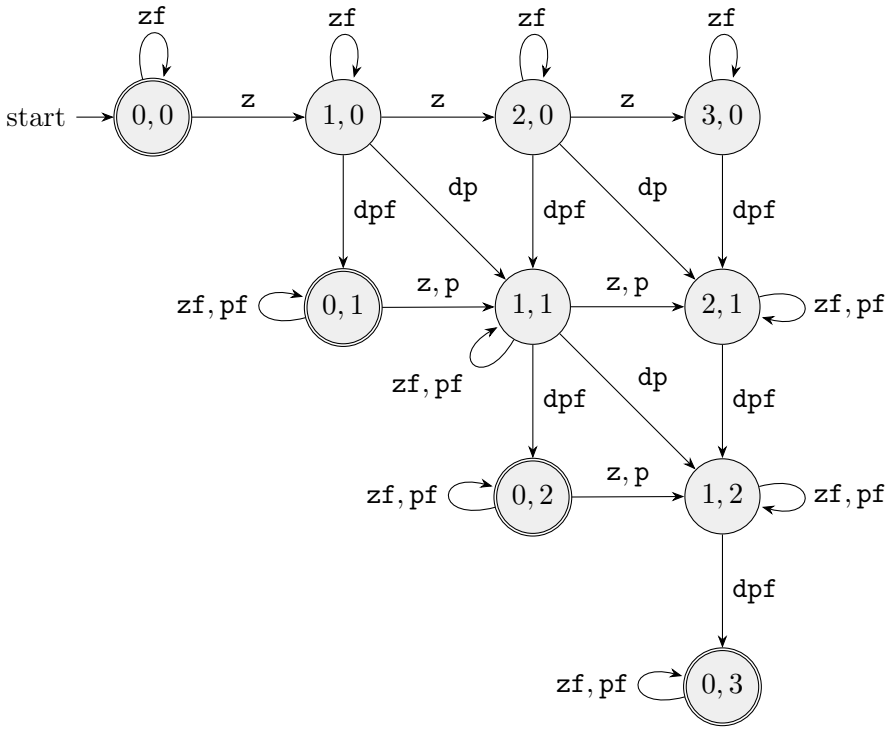

Figure 3.11: Finite automaton recognizing the language $\Psi(\mathcal{I}(021, 01234))$.

Theorem 3.5.5. *For any strictly increasing pattern ρ and any finite set of patterns P, the generating function of $\mathcal{I}(\{021, \rho\} \cup P)$ is rational.*

Proof. If $|\rho| = 1$, then the generating function of $\mathcal{I}(\{021, \rho\} \cup P)$ is 1. Otherwise, Lemma 3.5.4 states that the language $\Psi(\mathcal{I}(021, \rho))$ is regular. The proof can then follow the same line as for Theorem 3.2.12, since Ψ has the same ternary pattern properties as Φ by Proposition 3.5.2, and Ψ also turns inversion sequences into words of double length. □

3.5.2 Ascent sequences

In this section, we extend Theorem 3.5.5 to ascent sequences, using an encoding Ψ_{Asc} which follows the same approach as Ψ. In the previous section, we used a construction which could

commit a value for future use every time an entry was placed. Now, we only allow a commitment to be made when placing an ascent top (or the initial zero of a sequence), since that is when a new value becomes available in an ascent sequence.

Given an ascent sequence $\sigma \in \mathcal{A}$ and a value $v \geq 1$, we denote $\mathbf{com}_\sigma(v)$ the *commitment position* of v, that is the position of the v-th ascent top of σ, with the convention that σ_1 is an ascent top, and that $\mathbf{com}_\sigma(v) = +\infty$ if $v > \mathbf{asc}(\sigma)$. If $p = \mathbf{com}_\sigma(v)$, then the value v can only appear at position at least $p+1$ (by definition of an ascent sequence). Our encoding Ψ_{Asc} will follow a construction which decides whether the value v appears once the p-th entry is placed.

For all $\sigma \in \mathcal{A}(021)$, let $\Psi_{\mathrm{Asc}}(\sigma)$ be the word over the alphabet $\{\mathtt{d}, \mathtt{f}, \mathtt{p}, \mathtt{z}\}$ obtained by turning each entry σ_i into a factor abc defined as follows (for a and b, the definition is identical to Ψ).

- If $\sigma_i > 0$ and $\sigma_i > \sigma_j$ for all $j < i$, then $a = \mathtt{d}$. Otherwise, $a = \varepsilon$.
- If $\sigma_i = 0$, then $b = \mathtt{z}$. If $\sigma_i > 0$, then $b = \mathtt{p}$.
- If there exists a value $v \in \mathbf{Vals}(\sigma)$ such that $i = \mathbf{com}_\sigma(v)$, then $c = \varepsilon$. Otherwise, $c = \mathtt{f}$.

For instance, $\sigma = (0,1,0,1,0,2,4,4) \in \mathcal{A}_8(021)$ satisfies $\{\mathbf{com}_\sigma(v)\}_{v\in\{1,2,3,4\}} = \{1,2,4,6\}$, and $\Psi_{\mathrm{Asc}}(\sigma) = \mathtt{z\,dp\,zf\,pf\,zf\,dp\,dpf\,pf}$.

This encoding yields a language $\Psi_{\mathrm{Asc}}(\mathcal{A}(021))$ which is very similar to $\Psi(\mathcal{I}(021)$. This language is described in the proposition below, where conditions 1–5 are identical to Proposition 3.5.3, while conditions 6 and 7 ensure that a commitment can only be made upon placing an ascent top.

Proposition 3.5.6. *The set $\Psi_{\mathrm{Asc}}(\mathcal{A}(021))$ is the set of words ω over the alphabet $\{\mathtt{d}, \mathtt{f}, \mathtt{p}, \mathtt{z}\}$ which satisfy the following:*

1. $|\{\mathtt{d}, \mathtt{f}\}[\omega]| = |\{\mathtt{p}, \mathtt{z}\}[\omega]|$,
2. *for every prefix α of ω, we have $|\{\mathtt{d}, \mathtt{f}\}[\alpha]| \leq |\{\mathtt{p}, \mathtt{z}\}[\alpha]|$.*
3. *no letter* `p` *appears before the first letter* `d` *in ω (if there is no* `d`*, then* `p` *cannot appear),*
4. *every letter* `d` *is immediately followed by a letter* `p`*,*
5. *there is no factor* `ff`*,*
6. *every letter* `z` *except possibly the first one is immediately followed by a letter* `f`*,*
7. *any letter* `p` *which is immediately preceded by* `p` *or* `pf` *is immediately followed by* `f`*.*

Lemma 3.5.7. *Let $k \geq 1$, and let $\rho = 01\ldots k$ be the strictly increasing pattern of length $k+1$. The language $\Psi_{\mathrm{Asc}}(\mathcal{A}(021, \rho))$ is regular.*

Proof. Words satisfying conditions 1–5 of Proposition 3.5.6 form the language $\Psi(\mathcal{I}(021))$, which is rational by Lemma 3.5.4. Conditions 6 and 7 are local constraints, so they also describe regular languages. Since regular languages are closed under finite intersection, this concludes the proof. □

Similarly to the language $\Psi(\mathcal{I}(021, \rho))$, we can also describe an explicit finite automaton recognizing $\Psi_{\mathrm{Asc}}(\mathcal{A}(021, \rho))$, where $\rho = 01\ldots k$ is a strictly increasing pattern.

- The set of states is $q_0 \sqcup \{(c, d, \ell) \in \mathbb{N} \times \mathbb{N} \times \{\mathtt{p}, \mathtt{z}\} \ : \ c + d < k\}$. The components c and d are defined in the same way as in our previous automaton for $\Psi(\mathcal{I}(021, \rho))$, the component ℓ indicates which letter among $\mathtt{p}$ or $\mathtt{z}$ was last read, and q_0 is the initial state which is only visited by the empty word.
- The final states are q_0 and $(0, d, \ell)$ for all $d \in [0, k-1]$ and $\ell \in \{\mathtt{p}, \mathtt{z}\}$.
- The transitions are

$$\begin{cases} q_0 \xrightarrow{\mathtt{zf}} (0, 0, \mathtt{z}) & \\ q_0 \xrightarrow{\mathtt{z}} (1, 0, \mathtt{z}) & \text{if } \ k > 1 \\ (c, d, \cdot) \xrightarrow{\mathtt{zf}} (c, d, \mathtt{z}) & \\ (c, d, \cdot) \xrightarrow{\mathtt{pf}} (c, d, \mathtt{p}) & \text{if } \ d > 0 \\ (c, d, \mathtt{z}) \xrightarrow{\mathtt{p}} (c+1, d, \mathtt{p}) & \text{if } \ d > 0 \quad \text{and } \ c + d < k - 1 \\ (c, d, \cdot) \xrightarrow{\mathtt{dpf}} (c-1, d+1, \mathtt{p}) & \text{if } \ c > 0 \\ (c, d, \cdot) \xrightarrow{\mathtt{dp}} (c, d+1, \mathtt{p}) & \text{if } \ c > 0 \quad \text{and } \ c + d < k - 1. \end{cases}$$

Theorem 3.5.8. *For any strictly increasing pattern ρ and any finite set of patterns P, the generating function of $\mathcal{A}(\{021, \rho\} \cup P)$ is rational.*

Proof. Identical to the proof of Theorem 3.5.5. □

3.6 Future work

In this chapter, we showed general algebraicity and rationality results for sequences avoiding 021 and other patterns. Since 021 contains the pattern 10, avoiding a set of patterns P which contains 10 is equivalent to avoiding $P \cup \{021\}$. Consequently, our results still hold by replacing the pattern 021 with 10. An interesting question for future work is whether there are other patterns, or (more likely) sets of patterns, satisfying this property. If so, what characterizes these patterns?

A second important question is the effectivity aspect. While the theorems we used directly yield an effective process for computing generating functions, some parts of this process are unpractical and computationally heavy. We believe that in favorable cases (for instance, inversion sequences avoiding 021 and one other arbitrary pattern), we could provide a more efficient process (based on a decomposition of inversion sequences and patterns into blocks) yielding equations in one catalytic variable solvable by the kernel method. It may also be possible to obtain similar results in the general case, with the *vectorial kernel method* developed in [4].

Lastly, we showed that our approach is robust enough to also apply to ascent sequences. Another direction for future work could be to further generalize these results to other families of restricted integer sequences. We also believe our approach could be generalized to other types of patterns (for instance, inversion sequences avoiding 021 and a set of vincular patterns, or barred patterns) by modifying the definition of ternary patterns.

Chapter 4

Minimal inversion sequences containing a pattern

4.1 Motivation and outline

We begin with a short introduction to some classical concepts of the permutation patterns literature, which may be found in [23]. The set of patterns that can appear in a permutation is (up to reduction) itself the set of permutations, yielding a nice correspondence between permutations and their patterns. More precisely, the pattern containment relation is a partial order on the set of all permutations. A *permutation class* is a down-set in this partial order: if σ and τ are two permutations such that σ contains τ as a pattern, then any permutation class which contains σ also contains τ. Every permutation class can be characterized by the minimal set of permutations that it avoids; this set is called the *basis* of the class. The basis of a permutation class is an antichain (a set of pairwise incomparable elements) under the pattern containment order, and may be infinite.

In the context of inversion sequences, we can observe patterns that are not inversion sequences, even after reduction (for instance, the inversion sequence 0021401 contains the pattern 120). From this observation, it is natural to wonder whether the set $\mathcal{I}(\rho)$ of inversion sequences avoiding a pattern ρ can be described as the set of inversion sequences $\mathcal{I}(S_\rho)$ avoiding some set of *inversion sequences* $S_\rho \subseteq \mathcal{I}$. If we allow infinite sets, the obvious answer is yes: a possible choice for S_ρ is the set of all inversion sequences which contain the pattern ρ. A more interesting refinement of this question is to ask whether there is a *finite* set (or equivalently, a finite antichain) S_ρ satisfying the above, and how small can such a set be? This question already surfaced in [69, page 3], although it is incorrectly answered. We recall their definition and approach, before introducing our own in the next section.

Definition 4.1.1 (From [69])**.** Given a pattern $\tau = \tau_1 \dots \tau_k$, let L_τ be the set of all inversion sequences of the form $\theta^{(1)}\tau_1\theta^{(2)}\tau_2 \dots \theta^{(k)}\tau_k$ such that the length of the inversion sequence $\theta^{(1)}\tau_1\theta^{(2)}\tau_2 \dots \theta^{(j)}\tau_j$ is minimal for each $j \in \{1, \dots, k\}$. Note that some words $\theta^{(j)}$ might be empty.

Claim 4.1.2 (From [69])**.** For any set of patterns B, an inversion sequence σ avoids B if and only if σ avoids every pattern in $L_B = \cup_{\{\tau \in B\}} L_\tau$.

Unfortunately, Claim 4.1.2 is false. Indeed, taking $B = \{10\}$, we have $L_B = \{010\}$, and Claim 4.1.2 then states that an inversion sequence avoids the pattern 10 if and only if it avoids 010. Clearly this is not true: the sequence $0021 \in \mathcal{I}_4$ contains the pattern 10, yet avoids 010.

In [69], Claim 4.1.2 is then used to argue that inversion sequences containing a pattern τ of length $k \geq 1$ must always contain a pattern τ' that is an inversion sequence of length at most $2k-1$ (since sequences in L_τ have length at most $2k-1$). We give in Corollary 4.3.4 the correct upper bound of $3k-2$, and we show in Section 4.4 that there are patterns τ of any length $k \geq 1$ for which this bound is reached. This change of bound does not invalidate the algorithmic approach of [69], since its purpose is to *guess* succession rules, not to prove them; the enumerating formulas given in [69] are unaffected, as they do not rely on the validity of the $2k-1$ bound.

A summary of our work is as follows.

- In Section 4.2, we introduce some definitions, most notably the notion of minimal inversion sequences for a pattern, and a useful statistic we call **mdd**.
- In Section 4.3, we state properties on the general structure of minimal inversion sequences. In particular, we provide a characterization of ρ-minimal inversion sequences based on conditions satisfied by the occurrences of the pattern ρ they contain. We also find lower and upper bounds on the lengths of ρ-minimal inversion sequences, which depend on the length and **mdd** of ρ.
- In Section 4.4, we show that the aforementioned bounds are almost optimal.
- In Section 4.5, we enumerate ρ-minimal inversion sequences for any pattern ρ such that $\rho_1 = \mathbf{mdd}(\rho)$ through a bijection with some increasing trees, and for any pattern ρ of length at most 5 using computer programs.
- In Section 4.6, we enumerate the class of reduced inversion sequences (once again using a bijection with some increasing trees), and find new combinatorial interpretations of the poly-Bernoulli numbers of type C.
- In Section 4.7, we present open problems and make some closing remarks.

4.2 Definitions, notation, and simple observations

For all $n \geq 0$, we denote by $\mathcal{P}_n$ the set of *Cayley permutations* [95] of length n, defined as integer sequences $\rho_1 \dots \rho_n$ whose set of values is exactly $\{0, \dots, \mathbf{max}(\rho)\}$, and we denote $\mathcal{P} = \cup_{n \geq 0} \mathcal{P}_n$. Note that the set of Cayley permutations is exactly the set of patterns with the addition of the empty sequence. We will use the term "Cayley permutation" when the emphasis is on their own combinatorial structure, and "pattern" when the emphasis is on their role as a substructure in another sequence.

Given two integer sequences σ and τ, we denote $\tau \preceq \sigma$ if σ contains τ as a pattern. Note that $\preceq$ is a partial order on the set $\mathcal{P}$, although it is only a preorder on the set of all integer sequences (it is also a preorder on the set of inversion sequences). This relation gives some information on the relative lengths of sequences:

- if $\tau \preceq \sigma$, then $|\tau| \leq |\sigma|$,
- if $\tau \preceq \sigma$ and $\sigma \not\preceq \tau$, then $|\tau| < |\sigma|$.

Let σ be a nonnegative integer sequence of length $n \geq 1$. We define some functions on σ.

- Let $\mathbf{mdd}(\sigma) = \mathbf{max}(\sigma_i - i + 1)_{i \in [1,n]}$ be the *maximum diagonal difference* of σ, measuring how far σ goes above the "diagonal" sequence $012 \dots (n-1)$. In particular, we have

- $\mathbf{mdd}(\sigma) \geq 0$ (since $\sigma_1 \geq 0$),
- $\mathbf{mdd}(\sigma) = 0$ if and only if σ is an inversion sequence,
- $\mathbf{mdd}(\sigma \cdot \tau) = \mathbf{max}(\mathbf{mdd}(\sigma), \mathbf{mdd}(\tau) - n)$ for any nonempty sequence of nonnegative integers τ.

- Recall that $\mathbf{red}(\sigma)$ is the *reduction* of σ, that is, the sequence obtained by replacing the minimum of σ by 0, the second smallest value of σ by 1, and so forth. Let $\tau = \mathbf{red}(\sigma)$. The following holds:
  - $\tau_i \leq \sigma_i$ for all $i \in [1, n]$,
  - $\mathbf{mdd}(\tau) \leq \mathbf{mdd}(\sigma)$,
  - if σ is an inversion sequence, then τ is an inversion sequence,
  - $\mathbf{dist}(\tau) = \mathbf{dist}(\sigma)$,
  - τ is the only Cayley permutation of length n such that $\tau \preceq \sigma$,
  - τ contains exactly the same patterns as σ.

  Note that a pattern $\rho \in \mathcal{P}\backslash\{\varepsilon\}$ is contained in σ if and only if ρ is the reduction of a nonempty subsequence of σ.

- Let $\mathbf{Sat}(\sigma) = \{i \in [1, n] \ : \ \sigma_i - i + 1 = \mathbf{mdd}(\sigma)\}$ be the set of positions of *saturated* entries of σ, that is, entries achieving the maximum diagonal difference. In particular,
  - every saturated entry of σ is a strict left-to-right maximum (i.e. it has a strictly greater value than every entry to its left),
  - the maximum of σ is not always saturated (e.g. $\sigma = 001$),
  - if $\sigma \in \mathcal{I}_n$, $i \in [1, n]$ and $\sigma' = (\sigma_j)_{j \in [1,n]\backslash\{i\}}$ is the sequence obtained by deleting the entry σ_i from σ, then σ' is an inversion sequence if and only if $\mathbf{Sat}(\sigma) \subseteq [1, i]$ (since deleting σ_i increases the diagonal difference of every entry to its right by 1).

Figure 4.1 gives a visual representation of the functions **mdd** and **red**.

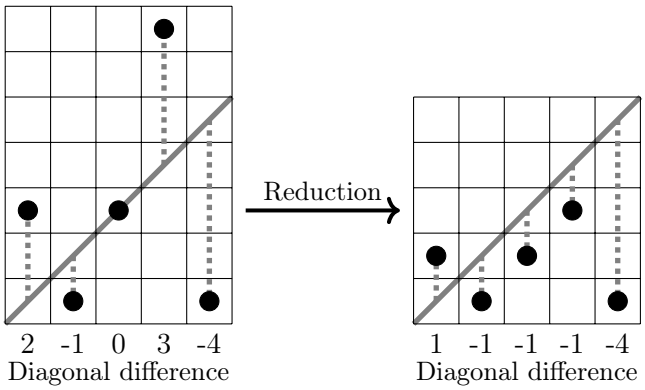

Figure 4.1: On the left, the sequence 20260 of maximum diagonal difference 3. On the right, its reduction 10120 of maximum diagonal difference 1.

Definition 4.2.1. Let $\rho \in \mathcal{P}\backslash\{\varepsilon\}$ be a pattern, and let $\mathcal{IP}[\rho] = \{\sigma \in \mathcal{I} \cap \mathcal{P} \,:\, \rho \preceq \sigma\}$ be the set of inversion sequences which are also Cayley permutations and contain the pattern ρ. We say that σ is a *ρ-minimal inversion sequence* if σ is a minimal element of the partially ordered set $(\mathcal{IP}[\rho], \preceq)$.

Let $\sigma \in \mathcal{I}$ be an inversion sequence, and let $\rho \in \mathcal{P}\backslash\{\varepsilon\}$ be a pattern. Then

- $\rho \preceq \sigma$ if and only if there exists a ρ-minimal inversion sequence τ such that $\tau \preceq \sigma$, since $\mathbf{red}(\sigma) \preceq \sigma$, and $\rho \preceq \sigma$ implies that $\mathbf{red}(\sigma) \in \mathcal{IP}[\rho]$,
- σ is ρ-minimal if and only if $\rho \preceq \sigma$, $\sigma \in \mathcal{P}$, and there is no inversion sequence τ such that $|\tau| < |\sigma|$ and $\rho \preceq \tau \preceq \sigma$.

By the first bullet point above, σ avoids ρ if and only if σ avoids every ρ-minimal inversion sequence. It is worth noting that ρ-minimal inversion sequences form the smallest set of inversion sequences satisfying this property for any $\sigma \in \mathcal{I}$. More precisely, any set $S \subseteq \mathcal{I} \cap \mathcal{P}\backslash\{\varepsilon\}$ such that $\mathcal{I}(S) = \mathcal{I}(\rho)$ must contain every ρ-minimal inversion sequence.

A given pattern ρ may admit several ρ-minimal inversion sequences, and their lengths can be different. For instance, the 10-minimal inversion sequences are $\{010, 0021\}$, and the 0201-minimal inversion sequences are $\{00201, 01201, 001312, 010312, 011302\}$, as we will see. We show in Corollary 4.3.4 that there is only a finite number of ρ-minimal inversion sequences for any pattern ρ, by giving an upper bound on their length. In general, it is no easy task to find every ρ-minimal inversion sequence, although we provide such results for some patterns in Section 4.5.

4.3 Structure of minimal inversion sequences

Given a pattern ρ, a ρ-minimal inversion sequence σ, and an occurrence of ρ in σ, it can be seen that every entry of σ serves one of two purposes: either it is part of the occurrence of ρ, or it "fills up" space in order to allow the placement of larger values to its right (since the value of every entry of an inversion sequence must be less than its position). For instance, in the 10-minimal inversion sequence 0021, the subsequence 21 forms an occurrence of the pattern 10, while the entries 00 are used as "filler". For a non-trivial example, consider the 1306524-minimal inversion sequence 01214027635. This sequence contains three occurrences of the pattern 1306524. For each occurrence, we colour the remaining filler entries in gray: 01214027635, 01214027635, 01214027635.

Proposition 4.3.1 below shows that the ρ-minimality of an inversion sequence can be expressed by conditions on its filler entries for each occurrence of ρ.

Proposition 4.3.1. *Let $n, k \geq 1$, let $\rho \in \mathcal{P}_k$, and let $\sigma \in \mathcal{I}_n \cap \mathcal{P}_n$ be a sequence containing the pattern ρ. Then σ is ρ-minimal if and only if for any set $R \subseteq [1, n]$ such that $\rho' = (\sigma_i)_{i \in R}$ is an occurrence of ρ, both conditions below are satisfied:*

1. *$[\mathbf{max}(\mathbf{Sat}(\sigma)), n] \subseteq R$,*
2. *any value of σ which does not appear in ρ' appears at least twice in σ.*

Proof. Let $s = \mathbf{max}(\mathbf{Sat}(\sigma))$. First, we prove by contrapositive that both conditions are necessary. Let $R \subseteq [1, n]$ be such that $\rho' = (\sigma_i)_{i \in R}$ is an occurrence of ρ.

1. Assume that $u \in [s, n]$ satisfies $u \notin R$. Then $(\sigma_i)_{i \in [1,n]\backslash\{u\}}$ is an inversion sequence which contains the pattern ρ, hence σ is not ρ-minimal.

2. Assume that $u \in [1,n]\backslash R$ is such that the value σ_u appears only once in σ, and let $\sigma' = \mathbf{red}((\sigma_i)_{i\in[1,n]\backslash\{u\}})$. Clearly, σ' contains the pattern ρ. We next show that σ' is an inversion sequence. For each $i \in [1,n]\backslash\{u\}$, the transformation from σ to σ' shifts the diagonal difference of each entry σ_i as follows:

   - if $i > u$, then the removal of σ_u increases the diagonal difference of σ_i by 1,
   - if $\sigma_i > \sigma_u$, then the reduction decreases the diagonal difference of σ_i by 1.

   It follows that the only entries whose diagonal difference is actually increased are those σ_i such that $i > u$ and $\sigma_i < \sigma_u$. Since $\sigma_u < u$, such entries satisfy $\sigma_i < i-2$, so they remain below the diagonal in σ'. We conclude that σ' is an inversion sequence, therefore σ is not ρ-minimal.

Now assume that both conditions are satisfied by every occurrence of ρ in σ. To complete our proof, we show that there is no sequence $\sigma' \in \mathcal{I} \cap \mathcal{P}$ such that $\sigma' \neq \sigma$ and $\rho \preceq \sigma' \preceq \sigma$. Let $J \subsetneq [1,n]$ be such that $\sigma' = \mathbf{red}((\sigma_i)_{i\in J})$ contains the pattern ρ, and let $J^c = [1,n]\backslash J$. By condition 1, we have $J^c \subseteq [1,s-1]$, therefore $\sigma_i \leq s-2$ for all $i \in J^c$. Let $V = \{v \in \{\sigma_i\}_{i\in J^c} : v \notin \{\sigma_i\}_{i\in J}\}$ be the set of values of σ which do not appear in σ'. By condition 2, we have $|J^c| \geq 2\,|V|$. Since $J^c \neq \emptyset$, this implies that $|J^c| > |V|$. Recall that $\sigma_s = s-1$, and for all $i \in J^c$ we have both $i < s$ and $\sigma_i < \sigma_s$. By looking at the value to which σ_s is mapped in σ', we obtain

$$\sigma'_{s-|J^c|} = s - 1 - |V| > s - 1 - |J^c|,$$

therefore σ' is not an inversion sequence. We conclude that σ is ρ-minimal. □

Next, we establish a correspondence between the saturated entries of a pattern ρ and those of ρ-minimal inversion sequences. This will be useful to prove Lemma 4.3.3, which shows a relation between three statistics (namely the length, **dist**, and **mdd**) of a pattern ρ and its minimal inversion sequences. This relation then yields some lower and upper bounds on the lengths of ρ-minimal inversion sequences, in Corollary 4.3.4.

Proposition 4.3.2. *Let $n,k \geq 1$, let $\rho \in \mathcal{P}_k$, let $\sigma \in \mathcal{I}_n$, let $\ell \in [0,k-1]$ be such that $k-\ell \in \mathbf{Sat}(\rho)$, let $r_1 < \cdots < r_k \in [1,n]$ be such that $\rho' = (\sigma_{r_i})_{i\in[1,k]}$ is an occurrence of ρ, and let $R = \{r_i\}_{i\in[1,k]}$. If σ is ρ-minimal, the following holds:*

1. *for all $i \in [1,n]$, if $\sigma_i \geq \sigma_{r_{k-\ell}}$, then $i \in R$,*
2. *for all $i \in [0,\ell]$, we have $r_{k-i} = n-i$,*
3. *$n - \ell \in \mathbf{Sat}(\sigma)$.*

Proof. First, we prove items 1 and 2 by contrapositive. Note that item 2 is equivalent to $[r_{k-\ell}, n] \subseteq R$. Assume that σ does not satisfy item 1 or item 2 (or both), i.e.

$$U = \{i \in [1,n]\backslash R \ : \ \sigma_i \geq \sigma_{r_{k-\ell}} \ \text{ or } \ i \geq r_{k-\ell}\} \neq \emptyset.$$

In particular, we have $\mathbf{min}(U) > \sigma_{r_{k-\ell}}$ since every $i \in U$ satisfies either $i > \sigma_i \geq \sigma_{r_{k-\ell}}$ or $i \geq r_{k-\ell} > \sigma_{r_{k-\ell}}$ because σ is an inversion sequence. Let $\sigma' = \mathbf{red}(\{\sigma_i\}_{i\in[1,n]\backslash U})$, let $n' = |\sigma'|$, let $r'_i = r_i - |\{j \in U \ : \ j < r_i\}|$ for all $i \in [1,k]$, and let $R' = \{r'_i\}_{i\in[1,k]}$. By construction, $r'_{k-i} = n' - i$ for all $i \in [0,\ell]$, and $\rho'' = (\sigma'_i)_{i\in R'}$ is an occurrence of ρ. Note that every entry σ'_i corresponds to some entry σ_j such that $i \leq j$ and $\sigma'_i \leq \sigma_j$; to be precise, j can be defined as the only element of $[1,n]\backslash U$ such that $i = j - |\{u \in U \ : \ u < j\}|$. Now we show that σ' is an inversion sequence.

- Let $i \in [1, n']$ be such that $\sigma'_i \leq \sigma_{r_{k-\ell}}$, and let σ_j be the entry of σ corresponding to σ'_i. Since $\mathbf{min}(U) > \sigma_{r_{k-\ell}}$, the entry σ_j either stays at position j, or moves to a position at least $\sigma_{r_{k-\ell}} + 1$ in σ'. Hence we have either $i = j > \sigma_j \geq \sigma'_i$, or $i \geq \sigma_{r_{k-\ell}} + 1 > \sigma'_i$. In any case, $\sigma'_i < i$.
- Let $i \in [0, n'-1]$ be such that $\sigma'_{n'-i} > \sigma_{r_{k-\ell}}$. By definition of σ' and R', we have $n' - i \in R'$. Because $k - \ell \in \mathbf{Sat}(\rho)$ and $\sigma'_{n'-i} > \sigma_{r_{k-\ell}} \geq \sigma'_{r'_{k-\ell}}$, we have $n' - i > r'_{k-\ell} = n' - \ell$, hence $n' - i = r'_{k-i}$. By definition of U, every entry of σ' of value at least $\sigma'_{n'-\ell}$ is part of ρ''. Since σ' is a Cayley permutation, this implies that $\sigma'_{n'-i} - \sigma'_{n'-\ell} = \rho_{k-i} - \rho_{k-\ell}$. Moreover, we have $\rho_{k-i} - \rho_{k-\ell} \leq \ell - i$ because $k - \ell \in \mathbf{Sat}(\rho)$. Since $\sigma'_{n'-\ell} = \sigma'_{r'_{k-\ell}} \leq \sigma_{r_{k-\ell}}$, we have $\sigma'_{n'-\ell} < n' - \ell$ by the previous bullet point. We conclude that $\sigma'_{n'-i} \leq \sigma'_{n'-\ell} + \ell - i < n' - i$.

We showed that $\sigma'_i < i$ for every $i \in [1, n']$, hence σ' is an inversion sequence, and therefore σ is not ρ-minimal. This concludes the proof of items 1 and 2.

Now we prove item 3 by contrapositive. Assume that σ satisfies items 1 and 2, that $\sigma \in \mathcal{P}$ (these conditions are safe to assume since they are necessary to be ρ-minimal), and that σ does not satisfy item 3. Since σ is an inversion sequence and $n - \ell \notin \mathbf{Sat}(\sigma)$, we must have $\sigma_{n-\ell} < n - \ell - 1$. By items 1 and 2, every entry of σ of value at least $n - \ell$ is part of ρ'. Since σ is a Cayley permutation, by the same reasoning used in the first part of the proof, for all $i \in [0, \ell - 1]$ such that $\sigma_{n-i} > n - \ell$, we have

$$\sigma_{n-i} = \sigma_{n-\ell} + \rho_{k-i} - \rho_{k-\ell} \leq \sigma_{n-\ell} + \ell - i < n - i - 1,$$

hence the entries of σ at positions in $[n-\ell, n]$ are not saturated. Now, assume that $i \in [1, k-\ell-1]$ is such that $r_i \in \mathbf{Sat}(\sigma)$. In particular, we have $\sigma_{r_i} - r_i > \sigma_{r_{k-\ell}} - r_{k-\ell}$ since $r_{k-\ell} = n - \ell$ by item 2. Because $k - \ell \in \mathbf{Sat}(\rho)$, we have $\rho_i - i \leq \rho_{k-\ell} - (k - \ell)$, and in particular $\rho_i < \rho_{k-\ell}$. Because $(\sigma_{r_i})_{i \in [1,k]}$ is an occurrence of ρ, we have $\sigma_{r_i} < \sigma_{r_{k-\ell}}$ and $\sigma_{r_{k-\ell}} - \sigma_{r_i} \geq \rho_{k-\ell} - \rho_i \geq k - \ell - i$. It follows that $r_{k-\ell} - r_i > k - \ell - i$, in other words there exists some $j \in [r_i, r_{k-\ell}]$ such that $j \notin R$. In conclusion, any saturated entry of σ that is part of ρ' must be to the left of some entry of σ that is not part of ρ'. This contradicts the first item of Proposition 4.3.1, hence σ is not ρ-minimal. □

Lemma 4.3.3. *Let ρ be a pattern, and let σ be a ρ-minimal inversion sequence. Then*

$$|\sigma| - \mathbf{dist}(\sigma) = |\rho| - \mathbf{dist}(\rho) + \mathbf{mdd}(\rho).$$

Proof. Let $n = |\sigma|$, let $k = |\rho|$, let $\ell \in [0, k-1]$ be such that $k - \ell \in \mathbf{Sat}(\rho)$, let $R \subseteq [1, n]$ be such that $\rho' = (\sigma_i)_{i \in R}$ is an occurrence of ρ, and let $\tau = (\sigma_i)_{i \in [1,n] \setminus R}$ be the sequence obtained by removing ρ' from σ.

By Proposition 4.3.2, we know that $\sigma_{n-\ell}$ is $\rho'_{k-\ell}$, and every entry of τ has value less than $\sigma_{n-\ell}$. Since σ is a Cayley permutation, this implies that $\sigma_{n-\ell} = \rho_{k-\ell} + \mathbf{dist}(\sigma) - \mathbf{dist}(\rho)$, where the number $\mathbf{dist}(\sigma) - \mathbf{dist}(\rho)$ counts the values of τ which do not appear in ρ'. We conclude by noticing that

- $\sigma_{n-\ell} = n - \ell - 1$, by item 3 of Proposition 4.3.2, and
- $\rho_{k-\ell} = \mathbf{mdd}(\rho) + k - \ell - 1$, by definition of ℓ,

hence $n = \mathbf{mdd}(\rho) + k + \mathbf{dist}(\sigma) - \mathbf{dist}(\rho)$. □

Corollary 4.3.4. *Let ρ be a pattern, and let σ be a ρ-minimal inversion sequence. Then $|\rho| + \mathbf{mdd}(\rho) \leq |\sigma| \leq |\rho| + 2\,\mathbf{mdd}(\rho)$. In particular, $|\sigma| \leq 3\,|\rho| - 2$.*

Proof. The lower bound is satisfied because $|\mathbf{dist}(\sigma)| \geq |\mathbf{dist}(\rho)|$, since σ contains the pattern ρ.

To prove the upper bound, let ρ' be an occurrence of ρ in σ, and let τ be the sequence obtained by removing ρ' from σ. Let $t = |\tau|$, $m = \mathbf{mdd}(\rho)$, and let v be the number of values of τ which do not appear in ρ'. In particular, $t = |\sigma| - |\rho|$, and $v = \mathbf{dist}(\sigma) - \mathbf{dist}(\rho)$. By item 2 of Proposition 4.3.1, any value of τ which is not in ρ' must appear at least twice in τ, therefore $t \geq 2v$. By Lemma 4.3.3, we have $t = v + m$, hence $2v \leq v + m$, hence $v \leq m$, hence $t \leq 2m$, and finally $|\sigma| = |\rho| + t \leq |\rho| + 2m$.

To see that $|\sigma| \leq 3\,|\rho| - 2$, simply notice that $\mathbf{mdd}(\rho) \leq \mathbf{max}(\rho)$, and $\mathbf{max}(\rho) \leq |\rho| - 1$ since ρ is a Cayley permutation. □

For any pattern ρ, we remark that all inversion sequences of the form $\alpha \cdot \rho$ for some $\alpha \in \mathcal{I}_{\mathbf{mdd}(\rho)}$ are ρ-minimal. Indeed, these are ρ-containing inversion sequences of minimal length, and they are Cayley permutations since $\mathbf{max}(\alpha) < \mathbf{mdd}(\rho) \leq \mathbf{max}(\rho)$. In particular, the total number of ρ-minimal inversion sequences is at least $\mathbf{mdd}(\rho)!$.

We conclude this section with a very simple but important consequence of Corollary 4.3.4 on pattern avoidance in inversion sequences.

Theorem 4.3.5. *Let $P \subset \mathcal{P} \backslash \{\varepsilon\}$ be a finite set of patterns. There exists a finite set $Q \subset \mathcal{I} \cap \mathcal{P}$ such that $\mathcal{I}(P) = \mathcal{I}(Q)$.*

Proof. Let Q be the set obtained by replacing each pattern $\rho \in P$ by the set of ρ-minimal inversion sequences. By definition of ρ-minimal inversion sequences, we have $Q \subset \mathcal{I} \cap \mathcal{P}$ and $\mathcal{I}(Q) = \mathcal{I}(P)$. By Corollary 4.3.4, Q contains only inversion sequences of length at most $\mathbf{max}(3\,|\rho| - 2 \ : \ \rho \in P)$, therefore Q is finite. □

4.4 Optimality

In this section, we examine the optimality of the bounds on the lengths of ρ-minimal inversion sequences obtained in Corollary 4.3.4. First, we remark that the upper bound depending only on $|\rho|$ cannot be improved, since it can be reached for a pattern ρ of any length $k \geq 1$. Indeed, if ρ is a pattern of length k starting with the value $k - 1$, take for example $\rho = (k-1)(k-2)\dots 210$, then

$$001122\dots(k-3)(k-3)(k-2)(k-2) \cdot (\rho_i + k - 1)_{i \in [1,k]}$$

is a ρ-minimal inversion sequence of length $3k - 2$. See Proposition 4.4.1 below for a proof.

Next, we investigate the lower and upper bounds depending on both $|\rho|$ and $\mathbf{mdd}(\rho)$. Clearly, the lower bound $|\rho| + \mathbf{mdd}(\rho)$ is reached for any pattern ρ, since $0^{\mathbf{mdd}(\rho)} \cdot \rho$ is always an inversion sequence. Now we show that the upper bound $|\rho| + 2\,\mathbf{mdd}(\rho)$ is reached for most patterns.

Proposition 4.4.1. *Let $k \geq 2$, let $\rho \in \mathcal{P}_k$, and let $m = \mathbf{mdd}(\rho)$. If $\rho_i = 0$ for some $i \geq 2$, then there exists a ρ-minimal inversion sequence of length $k + 2m$.*

Proof. First assume that $\rho_1 \neq 0$ or $\rho_z = 0$ for some $z \geq 3$, and let

$$\sigma = 0011\dots(m-1)(m-1) \cdot (\rho_i + m)_{i \in [1,k]} \in \mathcal{I}_{2m+k} \cap \mathcal{P}.$$

In that case, $(\sigma_i)_{i \in [2m+1, 2m+k]}$ is the only occurrence of ρ in σ, and it satisfies both conditions of Proposition 4.3.1, therefore σ is ρ-minimal.

Now assume that $\rho_1 = \rho_2 = 0$ and $\rho_i > 0$ for all $i \in [3, k]$. If $k = 2$, then 00 is trivially the only ρ-minimal inversion sequence. Otherwise, let

$$\tau = 01 \dots (m-1)mm(m-1) \dots 10 \cdot (\rho_i + m)_{i \in [3,k]} \in \mathcal{I}_{2m+k} \cap \mathcal{P}.$$

Notice that $\mathbf{red}(\rho_1\rho_2\rho_3) = 001$, and the prefix $(\tau_i)_{i \in [1,2m+2]}$ avoids the pattern 001, hence the last $k-2$ entries of any occurrence of ρ in τ must be $(\tau_i)_{i \in [2m+3,2m+k]}$. It follows that τ contains exactly $m+1$ occurrences of ρ, and they are the sequences $zz \cdot (\rho_i + m)_{i \in [3,k]}$ for $z \in [0, m]$. All of these occurrences satisfy both conditions of Proposition 4.3.1, hence τ is ρ-minimal. □

We observe that there exist patterns ρ whose minimal inversion sequences are all of length strictly less than $|\rho| + 2\,\mathbf{mdd}(\rho)$. The smallest such pattern is 021, and its minimal inversion sequences are $\{0021, 0121\}$. Now we show that even when the upper bound $|\rho| + 2\,\mathbf{mdd}(\rho)$ is not reached, there are still some ρ-minimal inversion sequences whose length comes very close to this bound.

Proposition 4.4.2. *For any pattern ρ, there exists a ρ-minimal inversion sequence of length at least $|\rho| + 2\,\mathbf{mdd}(\rho) - 2$.*

Proof. Let $k = |\rho|$, and let $m = \mathbf{mdd}(\rho)$. If $m \leq 2$, then $0^m \cdot \rho$ is a ρ-minimal inversion sequence of length $k + m \geq k + 2m - 2$. Assume now that $m \geq 3$. If $\rho_i = 0$ for some $i \geq 2$, then there exists a ρ-minimal inversion sequence of length $k + 2m$ by Proposition 4.4.1. Assume now that ρ_1 is the only zero of ρ.

Assume that $\rho_2 \geq 2$. Let $\rho' = (\rho_i + m - 1)_{i \in [2,k]}$, which is a sequence such that $\mathbf{mdd}(\rho') = 2m$, and let $\tau = 001122 \dots (m-1)(m-1) \cdot \rho' \in \mathcal{I}_{k+2m-1}$. Every term of value less than m in τ can only take the role of the 0 in an occurrence of ρ, so all occurrences of ρ in τ are subsequences of the form $v \cdot \rho'$ for some $v \in [0, m-1]$. These occurrences all satisfy the conditions of Proposition 4.3.1, hence τ is ρ-minimal.

Assume that $\rho_2 = 1$, and let $\ell = \mathbf{max}(i \in [1, k] \ : \ (\rho_j)_{j \in [1,i]} \in \mathcal{I})$ be the length of the longest prefix of ρ that is an inversion sequence. In particular, $\ell \geq 2$, $\rho_i < \ell$ for all $i \in [1, \ell]$, $\rho_{\ell+1} > \ell$, and there exists some $q \in [\ell + 2, k]$ such that $\rho_q = \ell$. Let $\alpha = (\rho_i)_{i \in [1,\ell-1]}$, let $\beta = (\ell+1)(\ell+1)(\ell+2)(\ell+2) \dots (\ell+m-2)(\ell+m-2)$, and let γ be the sequence of length $k - \ell + 1$ defined by

$$\forall i \in [\ell, k],\ \gamma_{i-\ell+1} = \begin{cases} \rho_i & \text{if} \quad \rho_i \leq \ell, \\ \rho_i + m - 2 & \text{if} \quad \rho_i > \ell. \end{cases}$$

Let $\sigma = 00 \cdot \alpha \cdot \beta \cdot \gamma$, which is a sequence of length $n = k + 2m - 2$. For example, if $\rho = 01165342$, then we have $m = 3$, $\ell = 3$, $\alpha = 01$, $\beta = 44$, and $\gamma = 176352$, hence $\sigma = 000144176352$. Going back to the general case, note that $\mathbf{mdd}(\alpha) = 0$, $\mathbf{mdd}(\beta) = \ell + 1 = |00 \cdot \alpha|$, and

$$\mathbf{mdd}(\gamma) = m + (\ell - 1) + (m - 2) = 2m + \ell - 3 = |00 \cdot \alpha \cdot \beta|,$$

therefore σ is an inversion sequence. It can be seen that σ is also a Cayley permutation:

- every value in $[0, \ell]$ appears in ρ, and remains unchanged in $\alpha \cdot \gamma$,
- every value in $[\ell + 1, \ell + m - 2]$ appears in β,
- every value in $[\ell + m - 1, \mathbf{max}(\rho) + m - 2]$ appears in γ,
- by construction, $\mathbf{max}(\sigma) = \mathbf{max}(\gamma) = \mathbf{max}(\rho) + m - 2$.

Now, we show that in an occurrence of ρ in σ, the values of β are too large to take the role of the term ρ_i for any $i \leq \ell$. Let $r_1 < \cdots < r_k \in [1, n]$ be such that $(\sigma_{r_i})_{i\in[1,k]}$ is an occurrence of ρ. Notice that $r_2 \geq 4$, since $\sigma_1 = \sigma_2 = \sigma_3 = 0$ and ρ_1 is the only zero of ρ. This implies that $r_i \geq i + 2$ for all $i \in [2, k]$, in particular $r_\ell \geq \ell + 2$. Let $b = |00 \cdot \alpha \cdot \beta| = 2m + \ell - 3$, and let $q \in [\ell + 2, k]$ be such that $\rho_q = \ell$.

- Assume for the sake of contradiction that $r_\ell \leq b$ and $r_{\ell+1} > b$. In particular, $\sigma_{r_\ell} > \ell$ since $r_\ell \in [\ell + 2, b]$. Because $\rho_q > \rho_\ell$, this also implies that $\sigma_{r_q} > \ell$. Now, we have

  - $\{i \,:\, \rho_i \geq \ell\} = \{i \,:\, \sigma_{r_i} \geq \sigma_{r_q}\}$ by definition of $(r_i)_{i\in[1,k]}$ and because $\rho_q = \ell$,
  - $|\{i \,:\, \sigma_{r_i} \geq \sigma_{r_q}\}| \leq |\{i \,:\, \gamma_i \geq \sigma_{r_q}\}|$ since
    - $\sigma_{r_i} \geq \sigma_{r_q}$ is equivalent to $\rho_i \geq \ell$, so it implies that $i > \ell$ by definition of ℓ,
    - $r_{\ell+1} > b$ by hypothesis, so $(\sigma_{r_i})_{i\in[\ell+1,k]}$ is a subsequence of γ,
  - $|\{i \,:\, \gamma_i \geq \sigma_{r_q}\}| \leq |\{i \,:\, \gamma_i > \ell\}|$ since $\ell < \sigma_{r_q}$,
  - $|\{i \,:\, \gamma_i > \ell\}| = |\{i \,:\, \rho_i > \ell\}|$ by definition of γ.

  In summary, we have $|\{i \,:\, \rho_i \geq \ell\}| \leq |\{i \,:\, \rho_i > \ell\}|$; this implies that ρ does not contain the value ℓ, a contradiction.

- Assume for the sake of contradiction that $r_{\ell+1} \leq b$. Recall that $\sigma_{r_\ell} < \sigma_{r_q} < \sigma_{r_{\ell+1}}$, and $\ell + 2 \leq r_\ell < r_{\ell+1} < r_q$. Since $\ell + 2 \leq r_\ell < r_{\ell+1} \leq b$ and β is nondecreasing, we have $\ell + 1 \leq \sigma_{r_\ell} \leq \sigma_{r_{\ell+1}} \leq \ell + m - 2$. By construction of σ, every term σ_i for $i \in [1 + r_{\ell+1}, n]$ falls in one of three cases:

  - $\sigma_i \geq \sigma_{r_{\ell+1}}$ if $i \leq b$ since β is nondecreasing,
  - $\sigma_i \leq \ell < \sigma_{r_\ell}$ if $i > b$ and $\rho_{i+k-n} \leq \ell$,
  - $\sigma_i > \ell + m - 2 \geq \sigma_{r_{\ell+1}}$ if $i > b$ and $\rho_{i+k-n} > \ell$.

  This contradicts the existence of σ_{r_q}.

We showed that $r_\ell > b$. Since $|[b + 1, n]| = |[\ell, k]|$, we have $r_i = i + b - \ell + 1 = i + n - k$ for all $i \in [\ell, k]$. In particular, $\sigma_{r_q} = \sigma_{q+b-\ell+1} = \gamma_{q-\ell+1} = \rho_q = \ell$, hence $\sigma_{r_{\ell-1}} < \ell$ since $\rho_{\ell-1} < \rho_q$. This implies that $r_{\ell-1} \leq \ell + 1$, since $r_{\ell-1} < r_\ell = b + 1$ and $\sigma_i > \ell$ for all $i \in [\ell + 2, b]$. Lastly, recalling that $r_i \geq i + 2$ for all $i \geq 2$, we have $r_j = j + 2$ for all $j \in [2, \ell - 1]$. In summary, there are at most three occurrences of ρ in σ, as we completely determined the value of each r_i except for $r_1 \in \{1, 2, 3\}$. It is easy to verify that those three subsequences are occurrences of ρ: indeed, all three are equal to the sequence $\alpha \cdot \gamma$, which is order-isomorphic to ρ by definition. It is also straightforward to verify that all three occurrences of ρ satisfy the conditions of Proposition 4.3.1, since $\mathbf{max}(\mathbf{Sat}(\sigma)) = b + \mathbf{max}(\mathbf{Sat}(\rho)) \geq b + 2$, and the entries of σ which are not used in each occurrence of ρ are exactly two zeros and the factor β. We conclude that σ is ρ-minimal. □

We remark that Proposition 4.4.2 cannot be improved to a length of $|\rho| + 2\,\mathbf{mdd}(\rho) - 1$. Indeed, the pattern 013542 is of length 6 and **mdd** 2, and all 34 of its minimal inversion sequences are of length 8. The process we used to generate these minimal inversion sequences is explained in Section 4.5.2, and Table 4.2 (on page 126) shows that every pattern ρ of length 5 or less admits some ρ-minimal inversion sequence of length at least $|\rho| + 2\,\mathbf{mdd}(\rho) - 1$.

4.5 Enumeration of minimal inversion sequences

4.5.1 Patterns beginning by a large value

In this section, we enumerate ρ-minimal inversion sequences for any pattern ρ such that $\rho_1 = \mathbf{mdd}(\rho)$, i.e. $\rho_1 > \rho_i - i$ for all i. As we shall see, this assumption gives rise to a very simple structure for ρ-minimal inversion sequences (in particular, the number of such sequences depends only on the **mdd** of ρ). Moreover, this assumption is often satisfied by patterns of small size. To be precise, a computation shows that patterns ρ such that $\rho_1 = \mathbf{mdd}(\rho)$ account for more than half of $\mathcal{P}_k$ for any $k \in [1, 11]$, and more than 10% of $\mathcal{P}_k$ up to $k = 231$.

For all $n \geq m \geq 0$, let $\mathcal{A}_{n,m}$ be the set of coloured inversion sequences of length n which satisfy the following conditions:

- every *value* in $[0, n-1]$ is assigned a colour, either blue or red[11], and all occurrences of this value in the inversion sequence receive its colour,
- there are m distinct blue values and $n - m$ distinct red values,
- each red value occurs at least twice in the sequence.

Note that any value of $[0, n-1]$ which does not occur in such a sequence is implicitly blue. For instance, $\underline{0}0\overline{1}3\underline{2}3\overline{1}6\overline{1} \in \mathcal{A}_{9,7}$ and the corresponding colouring of values is $\{\underline{0}, \overline{1}, \underline{2}, \overline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}\}$, but $\overline{0}\underline{12} \notin \mathcal{A}_{3,2}$ since the red $\overline{0}$ appears only once. Note also that $\mathcal{A}_{n,m}$ is empty when $n > 2m$, since its sequences must have at least $2(n-m)$ red entries. For any sequence $\alpha \in \mathcal{A}_{n,m}$, let $\mathbf{uncolour}(\alpha) \in \mathcal{I}_n$ be the uncoloured underlying inversion sequence.

Proposition 4.5.1. *Let $k \geq 1$, $m \geq 0$, and let $\rho \in \mathcal{P}_k$ be such that $\rho_1 = \mathbf{mdd}(\rho) = m$. For all $n \geq 0$, the number of ρ-minimal inversion sequences of length $n + k$ is $|\mathcal{A}_{n,m}|$.*

Proof. We present a simple bijection between ρ-minimal inversion sequences of length $n+k$ and $\mathcal{A}_{n,m}$, depicted on Figure 4.2.

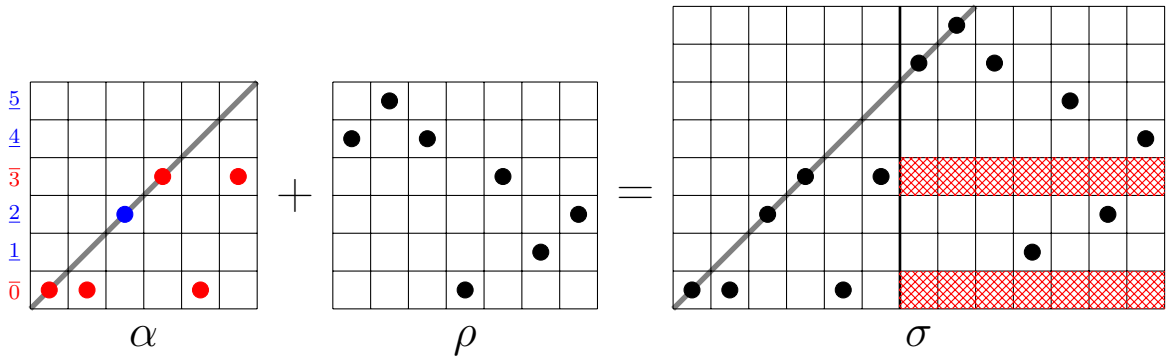

Figure 4.2: The coloured inversion sequence $\alpha = \overline{0}0\underline{2}\overline{3}0\overline{3}$ and the pattern $\rho = 4540312$ together form the ρ-minimal inversion sequence $\sigma = 0023036761524$.

Let σ be a ρ-minimal inversion sequence of length $n+k$. By Proposition 4.3.2, σ has a unique occurrence $\rho' = (\sigma_i)_{i \in [n+1, n+k]}$ of ρ. Let α be the coloured inversion sequence of length n defined

[11] In what follows, we additionally underline blue values and overline red values so that colour is not required to tell them apart.

by

$$\forall i \in [1,n],\ \alpha_i = \begin{cases} \underline{\sigma_i} & \text{if } \sigma_i \in \{\rho'_j\}_{j\in[1,k]}, \\ \overline{\sigma_i} & \text{otherwise.} \end{cases}$$

Note that every red value of α appears at least twice, by item 2 of Proposition 4.3.1. Moreover, the number of distinct red values in α is $\mathbf{dist}(\sigma) - \mathbf{dist}(\rho)$, which equals $n-m$ by Lemma 4.3.3. It follows that $\alpha \in \mathcal{A}_{n,m}$.

To describe the inverse bijection, let $\alpha \in \mathcal{A}_{n,m}$, and let ρ' be the sequence obtained from ρ by mapping the i-th smallest value in $[0, \mathbf{max}(\rho)]$ to the i-th smallest blue value of α, with the convention that any value greater than or equal to n is blue (since it cannot appear in α). More formally, ρ' can be defined as the only sequence which is order-isomorphic to ρ (i.e. such that $\rho \preceq \rho' \preceq \rho$) and whose set of values is

$$[0, \mathbf{max}(\rho) + n - m] \setminus \{v \in [0, n-1] \ :\ \overline{v} \in \{\alpha_i\}_{i\in[1,n]}\}.$$

Now we show that $\sigma = \mathbf{uncolour}(\alpha) \cdot \rho'$ is a ρ-minimal inversion sequence.

- For all $i \in [1,k]$, we have $\rho'_i \leq \rho_i + n - m$, hence $\mathbf{mdd}(\rho') \leq \mathbf{mdd}(\rho) + (n-m) = n$, therefore σ is an inversion sequence.
- There are exactly m values of ρ smaller than ρ_1, and α has exactly m blue values in $[0, n-1]$, so it follows from the construction of ρ' that $\rho'_1 = n$. In other words, σ has a saturated entry at position $n+1$.
- Every value in $\{\sigma_i\}_{i\in[1,n]}$ which does not appear in ρ' occurs at least twice in σ, since it corresponds to a red value of α.
- Lastly, we show that σ contains only one occurrence of the pattern ρ. Let $\ell \in [1, n+k]$ be the position of the leftmost term of an occurrence of ρ in σ. In particular, we have $\ell \geq m+1$ since $\sigma_\ell \geq m$, and $\ell \leq n+1$ since there must be at least $k-1$ entries of σ to its right. If $\ell = n+1$, then clearly the only occurrence of ρ whose first term is σ_ℓ is $(\sigma_i)_{i\in[n+1,n+k]} = \rho'$.

  Assume for the sake of contradiction that $\ell \leq n$. Let $v = \sigma_\ell$, and let

  - $V = \{\sigma_i \ :\ i > \ell \ \text{ and } \ \sigma_i < v\}$ be the set of values of σ which are less than v and appear to the right of position ℓ,
  - $U = \{i \in [0, n-1] \ :\ \overline{i} \in \{\alpha_j\}_{j\leq \ell} \ \text{ and } \ \overline{i} \notin \{\alpha_j\}_{j>\ell}\}$ be the set of values that are red in α and do not appear to the right of position ℓ,
  - $U^- = \{i \in U \ :\ i < v\}$,
  - $U^+ = \{i \in U \ :\ i \geq v\}$.

  Since σ_ℓ takes the role of $\rho_1 = m$ in an occurrence of ρ, we have $|V| \geq m$. Notice that V must contain all values of $[0, v-1]$ that are blue in α, since these values all appear in ρ', and therefore to the right of position ℓ in σ. It follows that $V = [0, v-1] \setminus U^-$. The total number of red values in α is $n-m$, and at most $n-\ell$ of them can fit in the interval of positions $[\ell+1, n]$, so we have $|U| \geq \ell - m$. Moreover, each value of U^+ must appear at least twice in the interval of positions $[v+1, \ell]$, so $|U^+| \leq (\ell - v)/2 \leq \ell - v - 1$. In conclusion, we obtain the following inequality:

  $$|V| = v - |U^-| = v - |U| + |U^+| \leq v - (\ell - m) + (\ell - v - 1) = m - 1,$$

  a contradiction.

It follows from Proposition 4.3.1 that σ is a ρ-minimal inversion sequence. □

In order to count our coloured inversion sequences, we introduce another object. For all $n \geq m \geq 1$, let $\mathcal{B}_{n,m}$ be the set of non-plane coloured labelled rooted trees on n nodes satisfying the following:

1. each node is labelled by a value in $[0, n-1]$ and coloured blue or red,
2. each value in $[0, n-1]$ labels exactly one node,
3. the tree is *increasing*, i.e. each non-root node has a greater label than its parent,
4. there are m blue nodes and $n-m$ red nodes,
5. each red node has at least two children.

Let $\mathcal{A} = \bigcup_{n,m\geq 0} \mathcal{A}_{n,m}$ and $\mathcal{B} = \bigcup_{n,m\geq 1} \mathcal{B}_{n,m}$. We now establish a one-to-one correspondence between these two families. Let $\phi : \mathcal{A} \to \mathcal{B}$ be the function which turns a coloured inversion sequence σ into the tree τ defined as follows:

- the root of τ has label 0,
- the labels of the children of the node labelled v in τ are the positions of the value v in σ,
- the node labelled v in τ has the same colour as the value v in σ; the node labelled $|\sigma|$ is blue by convention.

Figure 4.3 depicts the function ϕ on an example.

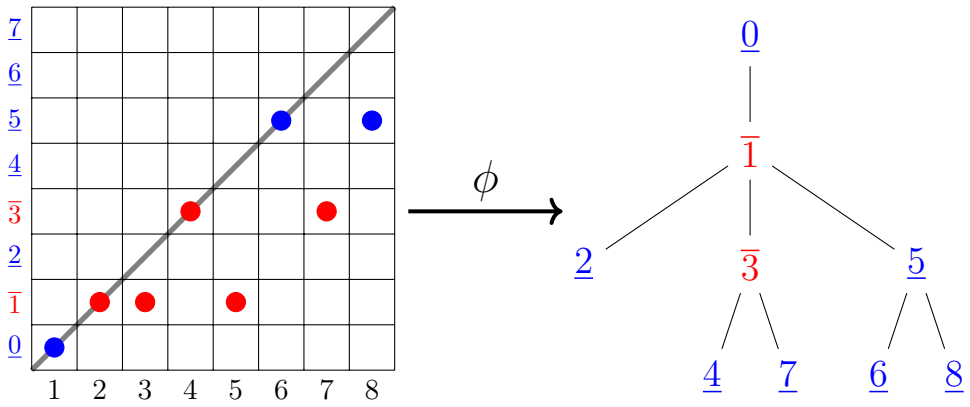

Figure 4.3: Illustration of the function ϕ.

It is easy to check that for any $\sigma \in \mathcal{A}_{n,m}$, its image $\tau = \phi(\sigma)$ satisfies all 5 items in the definition of $\mathcal{B}_{n+1,m+1}$:

- item 1 is satisfied since ϕ assigns a label and a colour to every node of τ,
- item 2 is satisfied since the root of τ has label 0, and every integer in $[1, n]$ is the position of some value of σ,
- item 3 is satisfied since σ is an inversion sequence: a value v can only appear at position at least $v+1$ in σ,

- item 4 is satisfied since σ has m blue values and $n-m$ red values in $[0, n-1]$, and the node labelled n is always blue,
- item 5 is satisfied since red values of σ appear at least twice.

It is also easy to reverse ϕ and construct a coloured inversion sequence $\sigma \in \mathcal{A}$ from a tree $\tau \in \mathcal{B}$, since τ clearly determines the colour and positions of each value of σ. It follows that ϕ is a bijection between $\mathcal{A}_{n,m}$ and $\mathcal{B}_{n+1,m+1}$ for all $n, m \geq 0$.

For all $n < m$, let $\mathcal{A}_{n,m} = \emptyset$ and $\mathcal{B}_{n,m} = \emptyset$. Let $B(x,y) = \sum_{n,m\geq 1} \frac{|\mathcal{B}_{n,m}|}{n!} x^n y^m$ be the mixed generating function of $\mathcal{B}$ that is exponential in x and ordinary in y.

Theorem 4.5.2. *The formal power series $B(x,y)$ satisfies the equations*

$$\frac{\partial B}{\partial x}(x,y) = (y+1)e^{B(x,y)} - B(x,y) - 1, \quad B(0,y) = 0.$$

Proof. We use the symbolic method from [53, Part A]. The class $\mathcal{B}$ is described by the recursive specification below, where $\mathcal{X}$ is an atomic class representing a single labelled node, μ is a neutral object marking the colour blue, and the symbol $^{\square}$ indicates the *boxed product* [53, page 139].

$$\mathcal{B} = \mu\, \mathcal{X}^{\square} \star \textsc{Set}(\mathcal{B}) + \mathcal{X}^{\square} \star \textsc{Set}_{\geq 2}(\mathcal{B})$$

This specification translates to the following functional equation:

$$B(x,y) = \int_0^x \Big(y\, e^{B(t,y)} + e^{B(t,y)} - B(t,y) - 1 \Big)\, dt. \tag{4.1}$$

The proof is easily concluded by taking the derivative of (4.1) with respect to x, and by evaluating (4.1) at $x = 0$. □

Let $k \geq 1$, $m \geq 0$, and let $\rho \in \mathcal{P}_k$ be such that $\rho_1 = \mathbf{mdd}(\rho) = m$. By Proposition 4.5.1, the number of ρ-minimal inversion sequences of length $n+k$ is $|\mathcal{A}_{n,m}|$, or equivalently $|\mathcal{B}_{n+1,m+1}| = (n+1)![x^{n+1}y^{m+1}]B(x,y)$. These numbers are shown for small values of n and m in Table 4.1, and we added them to [96] as entry A393523. The total number of ρ-minimal inversion sequences is $\sum_{n=m}^{2m} |\mathcal{A}_{n,m}|$, now entry A393524 of [96].

We end this subsection with some remarks on the minimal-length and maximal-length cases.

– The ρ-minimal inversion sequences of minimal length are in bijection with $\mathcal{A}_{m,m}$. Inversion sequences in $\mathcal{A}_{m,m}$ only have blue values, and they need not satisfy any additional constraints, so they are trivially in bijection with $\mathcal{I}_m$. Similarly, $\mathcal{B}_{m+1,m+1}$ is trivially in bijection with the set of (non-plane) increasing trees on $m+1$ nodes. Restricting the function ϕ to this particular case yields a simple bijection between inversion sequences and increasing trees. This correspondence appears in [53, page 145], it is used to enumerate inversion sequences avoiding the pattern 000 in [46], and generalized for a constant pattern of any length in [55].

– The ρ-minimal inversion sequences of maximal length are in bijection with $\mathcal{A}_{2m,m}$. Inversion sequences in $\mathcal{A}_{2m,m}$ have m red values and m blue values, each red value appears exactly twice, and blue values do not appear at all. Similarly, $\mathcal{B}_{2m+1,m+1}$ is the set of (non-plane) increasing trees having m red nodes and $m+1$ blue nodes, such that each red node has exactly two children, and all blue nodes are leaves. This family of trees was studied by Poupard [101], who found that their exponential generating function (counting trees with respect to their size $2m+1$) is $\sqrt{2}\tan(x/\sqrt{2})$. Their counting sequence appears in [96] as entry A002105.

$\lvert\mathcal{A}_{n,m}\rvert$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
$n=0$	1	0	0	0	0	0	0	0
$n=1$	0	1	0	0	0	0	0	0
$n=2$	0	1	2	0	0	0	0	0
$n=3$	0	0	5	6	0	0	0	0
$n=4$	0	0	4	27	24	0	0	0
$n=5$	0	0	0	49	168	120	0	0
$n=6$	0	0	0	34	515	1200	720	0
$n=7$	0	0	0	0	790	5471	9720	5040

Table 4.1: The numbers $|\mathcal{A}_{n,m}|$ for $n, m \in [0, 7]$.

4.5.2 Exhaustive generation for small patterns

In this subsection, we use computer programs to generate every ρ-minimal inversion sequence for a pattern ρ of small length. We know from Corollary 4.3.4 that the length of every ρ-minimal inversion sequence is between $|\rho| + \mathbf{mdd}(\rho)$ and $|\rho| + 2\,\mathbf{mdd}(\rho)$, so an easy approach would be to generate each sequence of $\mathcal{I} \cap \mathcal{P}$ whose length lies in this interval, and then test whether it is ρ-minimal using either the definition of ρ-minimal sequences (checking whether there is a smaller element in the poset $(\mathcal{IP}[\rho], \preceq)$), or the characterization of Proposition 4.3.1 (checking whether every occurrence of ρ satisfies some conditions). While this approach works for very small patterns, its limits are already noticeable for some patterns of length 5: their minimal inversion sequences can reach a length of 13, and there are 363 674 407 sequences in $\mathcal{I}_{13} \cap \mathcal{P}_{13}$ to test (see Section 4.6 for the enumeration of $\mathcal{I} \cap \mathcal{P}$).

Instead, we make use of some properties of minimal inversion sequences from Section 4.3 to generate them more efficiently. Assume that we are given a nonempty pattern ρ with $m = \mathbf{mdd}(\rho)$ and $s = \mathbf{min}(\mathbf{Sat}(\rho))$, and that we wish to generate every ρ-minimal inversion sequence. The general idea of our approach is to start from the pattern ρ, and build an inversion sequence σ around it by inserting some new entries, possibly shifting some values of ρ in the process. This way, we are guaranteed to construct an inversion sequence which contains at least one occurrence of ρ (we later refer to it as the *starting occurrence* of ρ). Additionally, Section 4.3 gives several conditions that the newly inserted entries must satisfy for σ to be ρ-minimal. In practice, we generate all such sequences σ which belong to $\mathcal{I} \cap \mathcal{P}$ and satisfy the following conditions.

- By items 1 and 2 of Proposition 4.3.2, each new entry inserted must be to the left and below the leftmost saturated entry of the starting occurrence of ρ (that is ρ_s at the beginning of our construction).
- By Lemma 4.3.3, if $n = |\sigma| - |\rho|$ is the number of new entries inserted in ρ, and $v = \mathbf{dist}(\sigma) - \mathbf{dist}(\rho)$ is the number of new values introduced by these entries, then $n - v = m$.
- By Corollary 4.3.4, the number n of new entries belongs to the interval $[m, 2m]$.
- By item 2 of Proposition 4.3.1, each of the v new values is inserted at least twice.

If $\rho_1 = m$, then we obtain precisely the construction of Proposition 4.5.1 (the colour red marks new values, and the colour blue marks values which also appear in the starting occurrence of ρ). This case is easy because, as we showed, inserting new entries does not create any other occurrences of ρ. In the general case, things are not so simple: while our construction guarantees that the starting occurrence of ρ satisfies the conditions of Proposition 4.3.1, we cannot guarantee this property for *every* occurrence of ρ in σ. Consequently, we must test the ρ-minimality of each inversion sequence generated by this process. Nevertheless, this is much faster than the naive approach presented earlier, since the set of sequences to test is now significantly smaller. With this approach, any modern computer can generate all ρ-minimal inversion sequences for any pattern ρ of length up to 7, taking at most a few minutes. This method also works well for patterns of greater length that have a small **mdd**.

In order to better present some of this data, we introduce a new notation. Given a nonempty pattern ρ, we denote by $\mathbf{ISBT}(\rho)$ the *inversion sequence basis type* of ρ, that is, the counting sequence of ρ-minimal inversion sequences of length ranging from $|\rho|+\mathbf{mdd}(\rho)$ to $|\rho|+2\,\mathbf{mdd}(\rho)$. For instance, if $\rho = 0312$, we have $|\rho| = 4$ and $\mathbf{mdd}(\rho) = 2$, so the length of every ρ-minimal inversion sequence is between 4 and 6; there are 6 of length 4, 5 of length 5, and none of length 6, so $\mathbf{ISBT}(0312) = (6, 5, 0)$. Note that $\mathbf{ISBT}(\rho) = (1)$ if and only if ρ is an inversion sequence.

In Table 4.2, we catalog the inversion sequence basis type of every pattern ρ of length 1 through 5, using a computer program which generates every ρ-minimal inversion sequence. It is remarkable that there are only 27 distinct basis types for those 633 patterns.

4.6 Enumeration of $\mathcal{I} \cap \mathcal{P}$

Minimal inversion sequences for a pattern are, by definition, objects that belong to the combinatorial class $\mathcal{I} \cap \mathcal{P}$. Surprisingly, it appears this class has not yet been enumerated, and we now make a short detour to address this problem.

For all $n \geq 1$ and $k \geq 0$, let $\mathcal{IP}_{n,k} = \{\sigma \in \mathcal{I} \cap \mathcal{P} \ : \ |\sigma| = n \ \text{ and } \ \mathbf{max}(\sigma) = k\}$. Equivalently, we have $\mathcal{IP}_{n,k} = \{\sigma \in \mathcal{I}_n \ : \ \{\sigma_i\}_{i\in[1,n]} = [0, k]\}$. For all $n, k \geq 1$, let $\mathcal{T}_{n,k}$ be the set of non-plane increasing trees having n internal nodes labelled by the integers $[0, n-1]$, and k leaves labelled by the integers $[n, n+k-1]$.

As we remarked at the end of Section 4.5.1, the function ϕ of Section 4.5.1 can be restricted to a bijection between $\mathcal{I}_n$ and (non-plane) increasing trees on $n+1$ nodes. Further restricting its domain to $\mathcal{I} \cap \mathcal{P}$ yields another nice correspondence: ϕ is also a bijection between $\mathcal{IP}_{n,k}$ and $\mathcal{T}_{k+1,n-k}$. This is easy to see since $\mathcal{IP}_{n,k}$ is the subset of inversion sequences of length n whose values are $[0, k]$, and $\mathcal{T}_{k+1,n-k}$ is the subset of increasing trees on $n+1$ nodes whose internal node labels are $[0, k]$.

We now count those trees. Let $\mathcal{T} = \coprod_{n,k\geq 1} \mathcal{T}_{n,k}$, and let $T(x, y) = \sum_{n,k\geq 1} \frac{|\mathcal{T}_{n,k}| x^n y^k}{n!\,k!}$ be their doubly exponential generating function.

Theorem 4.6.1.

$$T(x, y) = \log\left(\frac{1}{e^x + e^y - e^{x+y}}\right)$$

Proof. We once again use the symbolic method from [53, Part A]. In order to provide a specification for $\mathcal{T}$, we consider that objects of a set $\mathcal{T}_{n,k}$ have two different labellings: one for internal nodes, corresponding to the labels $[0, n-1]$, and one for leaves, corresponding to the labels $[n, n+k-1]$. While the book [53] does not explicitly treat the case of objects with two labellings and doubly exponential generating functions, this is a harmless generalization of their approach. The class $\mathcal{T}$ is described by the recursive specification below, where

ISBT	Patterns
(1)	Any ρ such that $\rho_1 = \mathbf{mdd}(\rho) = 0$.
$(1,1)$	Any ρ such that $\rho_1 = \mathbf{mdd}(\rho) = 1$.
$(2,0)$	021, 0211, 0212, 0213, 0221, 0231, 02111, 02112, 02113, 02121, 02122, 02123, 02131, 02132, 02133, 02134, 02143, 02211, 02212, 02213, 02221, 02231, 02311, 02312, 02313, 02314, 02321, 02331, 02341
$(2,3)$	0201, 0210, 02001, 02010, 02011, 02012, 02013, 02021, 02031, 02100, 02101, 02102, 02103, 02110, 02120, 02130, 02201, 02210, 02301, 02310
$(4,0)$	0132, 01322, 01323, 01324, 01332, 01342
$(4,1)$	01312, 01321
$(4,3)$	01302, 01320
$(4,4)$	00312, 00321
$(7,0)$	01243
$(2,5,4)$	Any ρ such that $\rho_1 = \mathbf{mdd}(\rho) = 2$.
$(4,8,5)$	1302, 13024, 13042
$(4,8,6)$	1320, 13002, 13020, 13022, 13023, 13032, 13200, 13202, 13203, 13204, 13220, 13230, 13240, 13302, 13320, 13402, 13420
$(4,11,9)$	13012, 13021, 13102, 13120, 13201, 13210
$(6,5,0)$	0312, 0321, 03112, 03121, 03122, 03123, 03124, 03132, 03142, 03211, 03212, 03213, 03214, 03221, 03231, 03241, 03312, 03321, 03412, 03421
$(6,21,21)$	03012, 03021, 03102, 03120, 03201, 03210
$(7,10,3)$	10423, 10432
$(7,13,9)$	12403, 12430
$(12,6,0)$	02413, 02431
$(18,4,0)$	01423, 01432
$(6,27,49,34)$	Any ρ such that $\rho_1 = \mathbf{mdd}(\rho) = 3$.
$(12,50,86,57)$	24013
$(12,50,87,59)$	24130
$(12,50,88,60)$	24031, 24103, 24301, 24310
$(18,61,95,58)$	14023, 14203
$(18,61,95,60)$	14032, 14230, 14302, 14320
$(24,54,38,0)$	04123, 04132, 04213, 04231, 04312, 04321
$(24,168,515,790,496)$	Any ρ such that $\rho_1 = \mathbf{mdd}(\rho) = 4$.

Table 4.2: The inversion sequence basis type of every pattern of length at most 5.

- $\mathcal{X}$ is an atomic class consisting of a single labelled object (an internal node),
- $\mathcal{Y}$ is an atomic class consisting of a single labelled object (a leaf),
- the boxed product is taken with respect to the labelling of internal nodes only.

$$\mathcal{T} = \mathcal{X}^{\square} \star \textsc{Set}_{\geq 1}(\mathcal{T} + \mathcal{Y})$$

This definition corresponds to the following equation for $T(x, y)$:

$$T(x, y) = \int_0^x (e^{T(t,y)+y} - 1)dt.$$

Equivalently, the formal power series $T(x, y)$ is characterized by the two equations below:

$$\frac{\partial T}{\partial x}(x, y) = e^{T(x,y)+y} - 1, \quad T(0, y) = 0.$$

It is easy to verify that the function $\log\left(1/(e^x + e^y - e^{x+y})\right)$ satisfies both equations, concluding the proof. □

$\lvert\mathcal{T}_{n,k}\rvert$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1	1	1	1	1	1	1
$n=2$	1	3	7	15	31	63	127
$n=3$	1	7	31	115	391	1267	3991
$n=4$	1	15	115	675	3451	16275	72955
$n=5$	1	31	391	3451	25231	164731	999391
$n=6$	1	63	1267	16275	164731	1441923	11467387
$n=7$	1	127	3991	72955	999391	11467387	116914351

Table 4.3: The numbers $|\mathcal{T}_{n,k}|$ for $n, k \in [1, 7]$.

We remark that a consequence of Theorem 4.6.1 is the symmetry $|\mathcal{T}_{n,k}| = |\mathcal{T}_{k,n}|$, which is not obvious from the definition of our increasing trees or inversion sequences. Table 4.3 shows the numbers $|\mathcal{T}_{n,k}|$ for small values of n and k. These numbers appear in [96] as entry A136126. They are called the *poly-Bernoulli numbers*[12] *of type C*, and have several known combinatorial interpretations, see e.g. [20] or [68].

The numbers $|\mathcal{I}_n \cap \mathcal{P}_n|$ are the antidiagonal sums $\sum_{k=0}^{n-1} |\mathcal{T}_{k+1,n-k}|$. For $n \in [1, 7]$, these numbers are $1, 2, 5, 16, 63, 294, 1585$. They appear in [96] as entry A136127, for which no simple formula is known.

4.7 Open questions and perspectives

Our first question concerns the poly-Bernoulli numbers discussed above.

Question 1. Find bijections between the families $\mathcal{IP}_{n,k}$ or $\mathcal{T}_{n,k}$ of Section 4.6 and other families of combinatorial objects known to be counted by the poly-Bernoulli numbers of type C.

[12]The name "poly-Bernoulli numbers" was originally given by Kaneko [61] to the numbers $|\mathcal{T}_{n+1,k}| + |\mathcal{T}_{n,k+1}|$.

One of the main questions that our work gives rise to is to enumerate, or at least better understand the set of minimal inversion sequences for any pattern. While we solved that question for patterns ρ such that $\rho_1 = \mathbf{mdd}(\rho)$ in Section 4.5.1, the general case is much more difficult. Indeed, it is not even easy to determine the exact length of the longest ρ-minimal inversion sequence for any pattern ρ, although Corollary 4.3.4 and Proposition 4.4.2 tell us that it always lies between $|\rho| + 2\,\mathbf{mdd}(\rho) - 2$ and $|\rho| + 2\,\mathbf{mdd}(\rho)$. We have done some preliminary study of this question, and we expect that a complete answer would be difficult to formulate, since it seems to require many different constructions of minimal inversion sequences depending on the pattern considered.

Question 2. Find a way to determine the exact length of the longest ρ-minimal inversion sequence for any pattern ρ.

One area in which we expect progress to be relatively easy is to identify more classes of patterns that have a given **ISBT**. In particular, we believe that the approach of Section 4.5.1 could be generalized to patterns whose prefix to the left of the first saturated entry has a simple form. For instance, we expect this to work well with patterns of the form $0^z \cdot \rho$ where $z \geq 0$ and ρ satisfies $\rho_1 = \mathbf{mdd}(\rho)$; a look at Table 4.2 suggests this should give a way of identifying patterns whose **ISBT** is $(2, 0)$, $(2, 3)$, $(6, 5, 0)$, $(6, 21, 21)$, or $(24, 54, 38, 0)$, among others.

Two patterns ρ and ρ' are said to be *Wilf-equivalent* for inversion sequences if there is the same number of ρ-avoiding as ρ'-avoiding sequences in $\mathcal{I}_n$ for all $n \geq 0$. The complete classification of Wilf-equivalent patterns of length 3 in inversion sequences was done in [46], and an almost complete classification for patterns of length 4 appears in [55]. From this data, we noticed that Wilf-equivalent patterns of length at most 4 always have the same **ISBT**.

Question 3. Does Wilf-equivalence between two patterns always imply they have the same inversion sequence basis type?

A natural extension of our work is to study other *minimal* combinatorial objects defined in a similar way for some notion of pattern containment. Some obvious choices of objects to consider are well-known subsets of inversion sequences in which patterns have also been studied, for the same pattern containment relation. We briefly discuss two such cases below.

- **Restricted growth functions.** A *record* (or strict left-to-right maximum) of an integer sequence σ is an entry σ_i such that $\sigma_i > \sigma_j$ for all $j < i$. Let

$$\mathbf{REC}(\sigma) = \{\sigma_i \,:\, \forall j < i,\ \sigma_i > \sigma_j\}$$

  be the set of record values of σ. Recall that a restricted growth function, or RGF for short, can be defined as an inversion sequence σ such that $\mathbf{REC}(\sigma) = [0, \mathbf{max}(\sigma)]$. In particular, RGFs are always Cayley permutations.

  Given a pattern ρ, it is much easier to determine all ρ-minimal RGFs than ρ-minimal inversion sequences. Let $m = |[0, \mathbf{max}(\rho)] \backslash \mathbf{REC}(\rho)|$ be the number of values of ρ which do not occur as a record. The value m has a role analogous to $\mathbf{mdd}(\rho)$ in the inversion sequence case, since it serves as a measure of how much ρ differs from a RGF. We claim that the ρ-minimal RGFs are precisely the sequences obtained by inserting one occurrence of each value of $[0, \mathbf{max}(\rho)] \backslash \mathbf{REC}(\rho)$ in ρ, in positions such that the resulting sequence is a RGF (each value must be inserted far enough to the left to be a record, and far enough to the right to avoid overtaking other records). Indeed,

- it is clearly necessary to insert at least these entries in order to create a RGF which contains ρ,
- inserting any extra entries (and possibly shifting values in the process) is unnecessary and would create a RGF that is not ρ-minimal, since removing those extra entries (and taking the reduction of the resulting sequence) still yields a RGF which contains ρ.

In particular, ρ-minimal RGFs always have length $|\rho|+m$, they contain a single occurrence of ρ, and they have a much simpler structure than ρ-minimal inversion sequences.

- **Ascent sequences.** An *ascent* (or consecutive pattern $\underline{01}$) of an integer sequence σ is a pair of entries (σ_i, σ_{i+1}) such that $\sigma_i < \sigma_{i+1}$. Let

$$\mathbf{asc}(\sigma) = |\{i \in [1, |\sigma| - 1] \ : \ \sigma_i < \sigma_{i+1}\}|$$

be the number of ascents of σ. Recall that an ascent sequence is a nonnegative integer sequence σ such that $\sigma_i \leq \mathbf{asc}((\sigma_j)_{j<i})$ for all i. In particular, ascent sequences are a subset of inversion sequences, and a superset of restricted growth functions.

Given a pattern ρ, the set of ρ-minimal ascent sequences appears to be more difficult to study than its inversion sequence counterpart. We now make a few remarks about some of their differences.

- Some patterns that are not ascent sequences only have a single minimal ascent sequence. Consequently, some patterns having different lengths are Wilf-equivalent for ascent sequences. For instance, 010 is the only 10-minimal ascent sequence.
- Some patterns have "gaps" in their ascent sequence basis type. For instance, there exist some 210-minimal ascent sequences of length 11 and 13, but none of length 12.
- We believe that a result on the length of ρ-minimal ascent sequences analogous to Corollary 4.3.4 could be given, albeit with an upper bound that is *quadratic* in the length of ρ. More precisely, if ρ is the decreasing pattern of length $k \geq 3$, we can construct a ρ-minimal ascent sequence of length $k^2 + 2k - 2$, and we conjecture this is the greatest achievable length. Such a sequence can be formed by concatenating $k-1$ factors[13] containing two occurrences of each value in the $k-1$ sets $[0,1], [2,4], [5,8], \ldots, [(k^2-k-2)/2, (k^2+k-4)/2]$ (of respective sizes $2, 3 \ldots, k$), then appending an occurrence of ρ using the values $[(k^2+k-2)/2, (k^2+3k-4)/2]$. For instance, with the notation $\mathtt{A} = 10$, $\mathtt{B} = 11$, and $\mathtt{C} = 12$,

$$0101 \cdot 342423 \cdot 78685756 \cdot \mathtt{CBA}9$$

is a 3210-minimal ascent sequence.

We hope to study ρ-minimal ascent sequences more thoroughly in the future.

[13] To be precise, the factor containing the values $[a, b]$ is obtained by interleaving the two nonincreasing sequences $(b-1, b-2, \ldots, a+1) \cdot (a, a)$ and $(b, b) \cdot (b-1, b-2, \ldots, a+1)$.

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Abstract

This thesis is devoted to the study of pattern avoidance in inversion sequences, with a primary focus on their enumeration. Inversion sequences arise naturally as encodings of permutations via the Lehmer code, but they also form a rich combinatorial structure in their own right. While permutation patterns have been extensively studied for several decades, the systematic investigation of patterns in inversion sequences is more recent, and has already revealed numerous connections with well-known integer sequences and other combinatorial objects.

The first two chapters investigate the enumeration of inversion sequences avoiding one or two patterns of length three. Building on previous work that left many cases unresolved, we complete the enumeration of such classes by providing counting formulas for all remaining cases. Our approach relies on two main techniques. The first is based on recursive decompositions of inversion sequences, leading to recurrence relations in several variables that allow us to quickly compute terms of their counting sequences. The second uses generating tree constructions, which represent inversion sequences as nodes in infinite labelled trees whose parent-child relation is determined by succession rules. These methods yield effective enumeration procedures, and provide some structural insight into the combinatorial classes under consideration. In several cases, we derive algebraic generating functions. In the remaining cases, we provide empirical evidence suggesting their generating functions are not D-finite.

In the next chapter, we focus on inversion sequences avoiding the pattern 021 together with additional patterns. Using an encoding in terms of coloured Dyck paths, we show that inversion sequences avoiding any finite set of patterns containing 021 always admit an algebraic generating function. This result extends to some subsets of inversion sequences such as ascent sequences and restricted growth functions. Furthermore, when the forbidden set contains both 021 and a strictly increasing pattern, we show that the corresponding generating functions are rational. These results are obtained through connections with formal language theory, interpreting the relevant classes as languages recognized by counter automata or finite automata.

In the final chapter, we introduce minimal inversion sequences containing a given pattern. These form the smallest set of inversion sequences whose avoidance is equivalent to avoiding the pattern in question. We establish structural properties of these sequences, characterizing them by conditions satisfied by their occurrences of the given pattern. From these properties, we derive upper and lower bounds on their length, proving they are always finite. For some patterns, we obtain functional equations to enumerate minimal inversion sequences via a bijection with increasing trees. We also provide enumerative data for small patterns through exhaustive generation. Lastly, we enumerate reduced inversion sequences, revealing a connection with poly-Bernoulli numbers.

Keywords: Pattern avoidance, inversion sequences, enumeration, generating trees, generating functions, Cayley permutations, counter automata

Résumé

Cette thèse est consacrée à l'étude de l'évitement de motifs dans les séquences d'inversions, et particulièrement au dénombrement de ces objets. Les séquences d'inversions apparaissent naturellement comme des encodages de permutations via le code de Lehmer, mais elles constituent également une structure combinatoire riche à part entière. Alors que les motifs de permutations ont été largement étudiés depuis plusieurs décennies, l'étude systématique des motifs dans les séquences d'inversions est plus récente et a déjà révélé de nombreuses connexions avec des suites d'entiers bien connues et d'autres objets combinatoires.

Les deux premiers chapitres étudient le dénombrement des séquences d'inversions évitant un ou deux motifs de longueur trois. En s'appuyant sur des travaux qui ont laissé de nombreux cas ouverts, nous complétons le dénombrement de ces classes en fournissant des formules de comptage pour toutes les paires de motifs restantes. Notre approche repose sur deux techniques principales. La première utilise des décompositions récursives des séquences d'inversions, qui mènent à des relations de récurrence à plusieurs variables permettant un calcul rapide des termes de leurs suites de comptage. La seconde utilise des constructions par arbres de génération, qui représentent les séquences d'inversions comme des nœuds dans des arbres étiquetés infinis dont la relation parent-enfant est déterminée par des règles de succession. Ces méthodes fournissent des procédures de dénombrement efficaces, et apportent un éclairage sur la structure des classes combinatoires considérées. Dans plusieurs cas, nous obtenons des fonctions génératrices algébriques. Dans les autres cas, nos résultats suggèrent que leurs fonctions génératrices ne sont pas holonomes.

Dans le chapitre suivant, nous nous concentrons sur les séquences d'inversions évitant le motif 021 ainsi que d'autres motifs supplémentaires. En utilisant un encodage par des chemins de Dyck colorés, nous montrons que les séquences d'inversions évitant n'importe quel ensemble fini de motifs contenant 021 ont toujours une fonction génératrice algébrique. Ce résultat s'étend à certains sous-ensembles de séquences d'inversions tels que les "ascent sequences" et les fonctions à croissance restreinte. De plus, lorsque l'ensemble de motifs interdits contient à la fois 021 et un motif strictement croissant, nous montrons que les fonctions génératrices correspondantes sont rationnelles. Ces résultats sont obtenus grâce à des liens avec la théorie des langages formels, en interprétant les classes considérées comme des langages reconnus par des automates à compteur ou des automates finis.

Dans le dernier chapitre, nous introduisons les séquences d'inversions minimales contenant un motif donné. Celles-ci forment le plus petit ensemble de séquences d'inversions dont l'évitement est équivalent à l'évitement du motif en question. Nous établissons des propriétés structurelles de ces séquences, en les caractérisant par des conditions satisfaites par leurs occurrences du motif donné. À partir de ces propriétés, nous déduisons des bornes supérieures et inférieures sur leur longueur, montrant qu'elles sont toujours finies. Pour certains motifs, nous obtenons des équations fonctionnelles pour dénombrer les séquences d'inversions minimales via une bijection avec des arbres croissants. Nous fournissons également des données sur le nombre de séquences d'inversions minimales pour de petits motifs en procédant par génération exhaustive. Enfin, nous étudions le dénombrement des séquences d'inversions sous forme réduite, aboutissant à une connexion avec les nombres de poly-Bernoulli.

Mots-clés: Évitement de motifs, séquences d'inversions, dénombrement, arbres de génération, fonctions génératrices, permutations de Cayley, automates à compteur