UNIVERSITÉ DE LORRAINE | BIBLIOTHÈQUES UNIVERSITAIRES

AVERTISSEMENT
DROIT D'AUTEUR

Ce document est le fruit d'un long travail approuvé par le jury de soutenance et mis à disposition de l'ensemble de la communauté universitaire élargie.

Il est soumis à la propriété intellectuelle de l'auteur. Ceci implique une obligation de citation et de référencement lors de l'utilisation de ce document. Toute contrefaçon, plagiat ou reproduction illicite encourt une poursuite pénale.

Contact bibliothèque : ddoc-theses-contact@univ-lorraine.fr
(Cette adresse ne permet pas de contacter les auteurs)

Code de la Propriété Intellectuelle. articles L 122. 4
Code de la Propriété Intellectuelle. articles L 335.2- L 335.10

UNIVERSITÉ DE LORRAINE
34 COURS LÉOPOLD — CS 25233
54052 NANCY CEDEX
TÉL : 03 54 50 54 00
CONTACT@UNIV-LORRAINE.FR
WWW.UNIV-LORRAINE.FR

Détermination au bord pour le problème inverse de Steklov

THÈSE

présentée et soutenue publiquement le 25 Juin 2026

pour l'obtention du

Doctorat de l'Université de Lorraine

Mention : Mathématiques

par

Benjamin FLORENTIN

Composition du jury

Président	Samuel TAPIE	Université de Lorraine, IECL
Rapporteurs	Thibault LEFEUVRE	Professeur, Université Paris-Saclay, LMO
	Mikko SALO	Professor, University of Jyväskylä
Examinateurs	Nalini ANANTHARAMAN	Professeure, Collège de France
	Yves COLIN DE VERDIÈRE	Professeur émérite, Université de Grenoble-Alpes, Institut Fourier
	Thierry DAUDÉ	Maître de Conférences, Université de Bourgogne Franche-Comté, LmB
	Anna SIFFERT	Professor, University of Münster
Encadrant de thèse	David DOS SANTOS FERREIRA	Professeur, Université de Lorraine, IECL

Au Très-Haut, source de toute vérité,

Remerciements

À l'heure de terminer ce manuscrit, vient le moment, à la fois heureux et délicat, des remerciements. Heureux, car il permet de mesurer combien cette thèse doit aux rencontres, aux échanges, aux encouragements et aux soutiens reçus au fil de ces années. Délicat, car il est toujours difficile de n'oublier personne ; si tel était le cas, j'espère que l'on voudra bien m'en excuser.
Au terme de ce long travail, il m'importe donc de remercier toutes celles et tous ceux qui, de près ou de loin, ont contribué à rendre cette thèse possible, mais aussi plus humaine. Car enfin, que serait la vie — et a fortiori une thèse — sans les rencontres qui l'accompagnent ?

Tout d'abord, ce travail n'aurait pu voir le jour sans David dos Santos Ferreira, qui a toujours répondu à mes questions avec patience et bienveillance, tout en me laissant une très grande liberté et une réelle autonomie dans mon travail. La proposition de sujet de thèse que tu m'avais faite alors que j'étais encore étudiant en Master 2 m'avait immédiatement parlé. Tu m'as énormément appris sur le plan mathématique, mais j'apprécie aussi profondément la personne que tu es. Nous avons d'ailleurs, je crois, quelques points communs et, d'une certaine manière, nous nous sommes bien trouvés. Tu m'avais dit, à l'époque, de ne pas commencer trop vite par les ouvrages de Hörmander ; je ne t'avais pas vraiment écouté, mais j'ai rapidement compris que tu avais raison. Cette thèse aura aussi été faite de ces apprentissages-là, où j'ai été parfois un peu têtu, mais qui traduisent la confiance et la liberté que tu m'as accordée. Je me réjouis à l'idée de pouvoir poursuivre notre collaboration dans les années à venir.

Merci également à Rémi et Samuel d'avoir toujours été à l'écoute, de m'avoir encouragé, et de m'avoir convaincu, dans les moments où cela était nécessaire, de la qualité des résultats obtenus. Vous êtes à la fois d'excellents chercheurs et de très belles personnes. J'espère que nous aurons l'occasion de rester en contact. Rémi, peut-être irons-nous voir un match de rugby ensemble un jour !

Dans la même veine, merci à Tristan d'avoir été présent, aussi bien pour partager le trajet des Journées EDP à Aussois que pour m'avoir entraîné et conseillé avant le premier séminaire en anglais que je devais donner. Tes connaissances mathématiques sont aussi impressionnantes que ta personnalité est attachante. Merci pour ta bonne humeur et ton humour.

Plus largement, je souhaite remercier l'ensemble des membres du laboratoire que j'ai eu la chance de croiser, de près ou de loin, au cours de ces années. Je pense en particulier à Anne Gégout-Petit, notre chère directrice de laboratoire, ainsi qu'à Anne de Roton — merci, entre autres, pour les commandes de pain. Merci également à Matei pour les dis-

cussions toujours animées en salle café, et à Laurence et Élodie pour votre bonne humeur, qui force d'autant plus l'admiration au regard du travail colossal que vous accomplissez chaque jour pour le laboratoire. J'ai toujours été accueilli dans vos bureaux respectifs avec bienveillance, humour et gentillesse. À bientôt. Merci à toute l'équipe EDP ainsi qu'à l'équipe Inria Sphinx, et tout particulièrement à Ingrid, Rémi, Takéo et Alessandro. Je ne doute pas que nos chemins auront bientôt l'occasion de se croiser à nouveau. Plus largement, je tiens à remercier toutes celles et ceux que j'ai eu le plaisir de rencontrer et de côtoyer à l'Inria au fil de ces quelques années. Les personnes que j'aimerais citer sont nombreuses, mais je me contenterai ici d'une pensée particulière pour l'incontournable Isabelle, à la cafétéria, dont les desserts ravissent les palais autant que sa bonne humeur et son sourire illuminent nos journées. A très vite. Enfin, à l'IECL, une mention toute particulière va à Valérie pour son aide précieuse autant que pour son humour, que l'on apprécie tous beaucoup. Je ne compte plus les fois où tu m'as aidé administrativement, voire sauvé — notamment lorsque je me suis retrouvé bloqué en Finlande à la suite de l'annulation de mon trajet retour. Merci.

Je souhaite à présent remercier tout particulièrement les rapporteurs de cette thèse, Thibault Lefeuvre et Mikko Salo, pour leur relecture attentive, leurs corrections précieuses et leurs remarques toujours pertinentes, qui ont largement contribué à améliorer la qualité de ce manuscrit.

Tout d'abord, merci à Thibault Lefeuvre, dont j'admire beaucoup le parcours, et qui s'est toujours montré accessible et bienveillant à mon égard. J'ai eu le plaisir de le rencontrer une première fois à Zurich en 2024, avec Mihajlo Cekić, lors du Workshop "Rigidity phenomena in Dynamics and Spectral theory", puis à Paris en novembre dernier, dans le cadre de son séminaire ADG à l'Institut Henri Poincaré, coorganisé avec Mihajlo. Merci encore à vous pour cette invitation.

Merci ensuite à Mikko Salo de m'avoir accueilli pendant deux mois dans la belle ville de Jyväskylä. Ce séjour en Finlande m'a beaucoup apporté, tant du point de vue de l'expérience de recherche que sur le plan personnel. Merci également pour la bienveillance constante qu'il a manifestée à l'égard de mon travail ; elle a joué un rôle indéniable dans la confiance que j'ai pu prendre au cours de cette thèse. A bientôt !

Merci à Yves Colin de Verdière pour les discussions mathématiques — et parfois historiques — toujours bienveillantes et chaleureuses, ainsi que pour avoir accepté de faire partie du jury de soutenance. Votre parcours mathématique est une véritable source d'inspiration pour les jeunes mathématiciens.

Enfin, je remercie chaleureusement l'ensemble des membres du jury pour avoir accepté d'examiner ce travail et pour l'intérêt qu'ils ont porté à ce sujet. C'est pour moi un grand honneur.

*

* *

Ces années de thèse n'auraient pas eu la même saveur sans les amitiés qui les ont accompagnées. Je voudrais donc me tourner vers celles et ceux qui, par leur amitié, leur présence et leur affection, ont accompagné ces années d'une manière tout aussi essentielle.

Je pense d'abord à mes amis de longue date : Micka, Pierre et Lulu. Je resterai sobre, non parce qu'il y aurait peu à dire, mais parce que certaines amitiés anciennes n'ont pas besoin de grands discours pour dire ce qu'elles représentent. Nous en avons vécu des choses ensemble et j'espère que nous en vivrons encore beaucoup d'autres. Il y a d'ailleurs toujours ce petit voyage à organiser — je ne l'oublie pas.

Merci ensuite aux amis du groupe de Master Agreg, en particulier Blaise, Nathan, Yann, Oriane, Pierrick et Étienne. Ces années auraient été bien différentes sans vous. Étienne, j'espère que nous aurons bientôt l'occasion de jouer une partie d'échecs. Nathan, merci pour ta bonne humeur, ta simplicité, et pour le soutien que tu m'as apporté dans certains moments ; je t'apprécie beaucoup.
De même, Blaise, merci également pour ton soutien, pour nos discussions mathématiques, et plus largement pour tous ces échanges toujours stimulants. Tes connaissances mathématiques, comme ta culture générale, m'impressionnent énormément, et j'ai pour toi une très grande estime.

Je voudrais aussi m'arrêter sur ces doctorants devenus, au fil du temps, de précieux amis : Rodolphe, Mabrouk, Pierrick — alias Patrick, le docteur du groupe. Je n'ai aucun doute que nos aventures ne font que commencer. Il nous faudra bientôt organiser un nouveau tournoi FIFA : j'ai un titre à reconquérir, autour d'une bonne truite fumée, et sans oublier quelques matchs de basket à aller voir. Je suis très heureux de vous avoir rencontrés. Rodolphe, habibi, merci de m'avoir accueilli à Rome en Octobre dernier ; nous en avons fait des choses, en quelques jours. Merci également à Amine d'avoir récemment élargi le groupe : on t'attend pour un prochain tournoi — siuuu ! Vous serez bien sûr tous les bienvenus en Finlande ces deux prochaines années, lorsque je me serai installé.

Merci plus largement à tous les doctorants du laboratoire que j'ai pu croiser au fil de l'eau. Je pense en particulier à Victor (Hector Dubois) — félicitations encore pour ton poste de MCF —, Julia, Mathilde, Bastien et Élie, pour leur bonne humeur et les moments partagés. Merci aussi à Saïd pour ces repas du midi en tête-à-tête à l'Inria, qui resteront de bons souvenirs. Nathan T., félicitations encore pour ta soutenance récente, qui fait qu'avec l'autre Nathan, la promotion que nous avons formée est désormais entièrement docteure ! Merci enfin à Virgile pour sa gentillesse et pour avoir partagé son template !

Je souhaite également remercier mes premières colocataires, Célia, Claire et Nawelle. Je suis heureux d'avoir croisé votre route, et j'espère que nous partagerons encore de nombreux moments ensemble, autour d'une partie de bowling, d'un jeu de société, ou simplement d'une soirée comme nous savons les faire. Nawelle, merci pour toutes ces discussions au cours desquelles nous nous sommes confiés l'un à l'autre. Tu es devenue pour moi comme une petite sœur. Tamara, toi qui es arrivée ensuite, tu sais toute l'affection que j'ai également pour toi, ainsi que pour tes grands-parents d'ailleurs. Je ne doute pas que nous aurons encore de nombreuses occasions de partager des moments chaleureux tous ensemble, malgré la distance. En attendant, prends soin de toi et bien sûr à très vite. Dans le même registre, je pense aussi à toi, Karine. Garde confiance et n'oublie pas de te ménager. Depuis notre rencontre lors de la retraite d'écriture, nous avons noué une relation simple, belle et sincère, à laquelle je tiens beaucoup. Tu es une personne douce et profondément honnête. À quand la prochaine course avec Arco ?

En parlant de course, il est temps de remercier chaleureusement mon groupe d'entraînement de la section Vandœuvre athlétisme. Un grand merci à notre cher coach Manu, dont certaines séances nous auront souvent fait souffrir, mais toujours dans la bonne humeur et avec de nombreux fous rires à la clé. Merci à toute l'équipe, et tout particulièrement à Pierric, Maël, Lauriane, Lucie, Emma, Louis et Noah. À très bientôt ! Merci enfin au tout aussi attachant groupe demi-fond, et en premier lieu à FR, qui m'a accueilli avec beaucoup de bienveillance dès mon arrivée, alors même que j'étais encore licencié à Strasbourg. Tu es un super coach et une personne que j'apprécie beaucoup, et je ne doute pas que nous aurons encore l'occasion de partager à l'avenir quelques-unes de ces séances intenses dont tu as le secret.

Parmi les rencontres qui ont aussi largement marqué cette fin de thèse, je ne peux pas ne pas évoquer la paroisse de Notre-Dame-de-Lourdes. J'y ai trouvé bien plus qu'un lieu : une véritable famille, accueillante, fraternelle, et portée par un amour qui a le pouvoir de changer le monde.

C'est dans ce cadre que nos chemins se sont croisés, Daniel, et que notre lien s'est tissé avec une évidence qui, j'en suis sûr, ne doit rien au hasard. Nos nombreuses discussions m'ont beaucoup apporté ; elles m'ont aidé à grandir, à réfléchir et à avancer. Tu es aujourd'hui comme un grand frère pour moi — en plus d'être mon parrain —, et je t'en remercie énormément.

Merci plus largement à tous les membres de la paroisse pour leur accueil et leur affection. Merci en particulier à Denis, ainsi qu'aux pères Vallin et Martin, avec qui l'enthousiasme est toujours de mise. Je vous aime beaucoup, et vous suis très reconnaissant. Père Vallin, merci d'avoir partagé avec moi un peu de votre expérience de docteur, ainsi que pour vos paroles toujours si bienveillantes et encourageantes.

*

* *

On dit souvent que toutes les bonnes choses ont une fin ; je préfère dire qu'elles se transforment. Cette thèse marque la fin d'une aventure, mais aussi le commencement d'une autre.

Je voudrais d'abord remercier ma famille : Maman, Papa, Élise. Car il est essentiel d'honorer ceux qui nous ont accompagnés depuis toujours. Vous avez toujours été là pour moi, même lorsque mon parcours s'éloignait de ce que vous connaissiez. Vous m'avez soutenu, encouragé, aidé, chacun à votre manière, et je vous en remercie du fond du cœur. Je vous aimerai toujours.

Depuis quelque temps maintenant, Dylan est venu agrandir un peu la famille. Je t'apprécie beaucoup, et j'espère que nous aurons encore l'occasion de partager de nombreux petits moments ensemble — autour du rugby, notamment.

Pour finir, je voulais réserver quelques lignes à celles et ceux que je considère aujourd'hui comme une seconde famille : des rencontres plus récentes, que j'ai eu la grâce de faire et qui comptent profondément pour moi.

Je pense d'abord à Maggie, ma marraine et grand-mère spirituelle — même si tu n'es pas vieille. J'ai été guidé vers toi avec une clarté particulière comme tu le sais... Tu as ce don rare de m'apprendre beaucoup tout en me faisant sourire. Je t'aime énormément !
Je pense aussi à J.-M., que j'ai eu la joie de rencontrer par ton intermédiaire. Je lui porte une grande affection, et j'admire tout particulièrement sa curiosité intellectuelle, si vive, si généreuse, et, pour être honnête, assez contagieuse. Enfin, Katia, ces derniers mots sont pour toi. Tu es un petit rayon de soleil, et l'amour pur qui nous a touchés au même instant est un cadeau du Ciel. Je ne sais pas ce que l'avenir nous réserve exactement — même si nous commençons peut-être à en entrevoir les grandes lignes —, mais je l'accueille avec confiance, émerveillement et gratitude.

*

* *

Résumé

Ce travail s'inscrit dans la vaste tradition des problèmes inverses spectraux, dont l'idée générale peut être résumée par la célèbre question de Kac dans les années 1960 au sujet du spectre du Laplacien: "Peut-on entendre la forme d'un tambour ?". Nous abordons ici le problème analogue pour l'opérateur de Dirichlet-à-Neumann (DN map) ou opérateur de Steklov: Dans quelle mesure la géométrie d'une variété Riemannienne compacte à bord, ou un coefficient d'un problème avec conditions au bord (par exemple un potentiel électrique ou magnétique), peuvent-ils être determinés de manière unique à partir du spectre de cet opérateur ? Ce spectre, appelé *spectre de Steklov*, consiste en une suite croissante de valeurs propres réelles, et, tout comme dans le cas du spectre du Laplacien, il est en fait bien connu qu'il ne détermine en général ni la structure Riemannienne ni un potentiel. Cette thèse s'attache donc à l'inverse, à donner des classes d'exemples pour lesquels l'unique détermination est vraie, autrement dit à apporter des réponses positives au *problème inverse de Steklov*.

Dans l'esprit de Guillemin-Kazhdan, on étudie d'abord le problème linéarisé ou rigidité infinitésimale: une *déformation Steklov isospectrale* est une famille lisse à un paramètre de métriques Riemanniennes préservant le spectre de Steklov. Un résultat est obtenu pour les métriques conformes: si le bord est Anosov et que son *spectre des longueurs* est simple, pour une métrique initiale donnée, alors ses jets au bord (i.e. la donnée de ses séries de Taylor en chaque point du bord) sont préservés le long de toute déformation conforme Steklov isospectrale. Concernant le problème non linéaire, des résultats positifs sont également établis en dimension $n \geq 3$, sous les mêmes hypothèses dynamiques sur le flot géodésique du bord. On démontre que la connaissance du spectre de Steklov détermine les jets au bord de la métrique Riemannienne sous-jacente au sein de sa classe conforme, pourvu que les métriques coïncident au bord. Comme corollaire simple, en supposant que le facteur conforme soit analytique réel, de telles métriques Steklov isospectrales doivent être égales.

L'idée principale consiste à exploiter des formules de trace d'onde pour montrer que le spectre de Steklov détermine les intégrales de données au bord appropriées le long des géodésiques fermées, puis à appliquer les résultats d'injectivité de la transformée à rayons X géodésiques sur les variétés fermées Anosov. Les techniques mises en places sont étendues aux opérateurs de type Schrödinger et aux opérateurs magnétiques, permettant ainsi de récupérer les jets au bord des potentiels électriques et des champs magnétiques à partir du spectre.

Mots-clés : Géométrie spectrale, problèmes inverses, spectre de Steklov, opérateur de Dirichlet-à-Neumann, invariants des ondes, détermination au bord.

Abstract

This work is part of the long tradition of spectral inverse problems, which can be summed up in Kac's celebrated question from the 1960s, regarding the spectrum of the Laplacian: "Can one hear the shape of a drum?". Here, we address the analogous problem for the Dirichlet-to-Neumann map (DN map) or Steklov operator: to what extent can the geometry of a compact Riemannian manifold with boundary, or a coefficient of a problem with boundary conditions (such as an electric or a magnetic potential), be uniquely determined from the spectrum of this operator ? This spectrum, known as the *Steklov spectrum*, consists of a non-decreasing sequence of real eigenvalues, and, just as in the case of the laplacian spectrum, it is actually well known that it generally determines neither the Riemannian structure nor a potential. This thesis therefore focuses on the converse: providing classes of examples for which the unique determination holds, in other words, providing positive answers to the *Steklov inverse problem*.

In the spirit of Guillemin-Kazhdan, one first study the linearized problem or infinitesimal rigidity: a *Steklov isospectral deformation* is a smooth one-parameter family of Riemannian metrics that preserves the Steklov spectrum. A result is obtained for conformal metrics: if the boundary is Anosov and its length spectrum is simple, with respect to a given initial metric, then its boundary jet (i.e. the data of its Taylor expansions at each point of the boundary) is preserved along any Steklov isospectral conformal deformation. Regarding the non-linear problem, positive results are also established in dimension $n \geq 3$ under the same assumptions on the dynamic of the boundary geodesic flow. It is shown that knowledge of the Steklov spectrum determines the boundary jets of the underlying Riemannian metric within its conformal class, provided that the metrics coincide at the boundary. As a simple corollary, assuming real-analyticity of the conformal factor, such Steklov isospectral metrics must be equal.

The main ideas is to exploit wave trace formulas to show that the Steklov spectrum determines integrals of appropriate boundary data along closed geodesics, and then to apply the injectivity results of the geodesic X-ray transform on closed Anosov manifolds. The techniques implemented are extended to Schrödinger-type operators and magnetic operators, thereby making it possible to recover the boundary jets of electric potentials and magnetic fields from the spectrum.

Keywords : Spectral geometry, inverse problems, Steklov spectrum, Dirichlet-to-Neumann map, wave invariants, boundary determination.

Table des matières

Résumé **ix**

Abstract **x**

Introduction en français **1**

— La géométrie spectrale 1

— Les problèmes inverses en géométrie spectrale 3

◊ Spectre du Laplacien: le tambour de Kac 4

◊ Spectre de l'opérateur de Steklov: le "tambour de Steklov" 7

— Organisation de la thèse 9

I Introduction : Inverse problems for the Steklov spectrum **13**

I.1 Recovering geometry from the Steklov spectrum 21

I.1.1 The Steklov inverse problem 24

I.1.2 The linearized problem: Steklov isospectral deformations 26

I.1.3 Main results 29

I.2 Recovering physical data from the Steklov spectrum 31

I.2.1 The magnetic Steklov inverse problem 32

I.2.2 Main results 34

I.3 An additional rigidity result for the magnetic Schrödinger operator . . . 36

I.4 A new DG–type trace formula: the key to boundary determination . . . 39

I.4.1 Brief description of the induction procedure 40

I.5 A roadmap for the reader and a few notes 41

I.6 Some definitions and notation 44

II Looking for spectral invariants **51**

II.1 Spectral functions and invariants 51

II.2 The approach of trace formulas 56

II.2.1 Towards the Duistermaat-Guillemin trace formula 57

II.3 The Duistermaat-Guillemin wave trace formula 60

II.4 Trace asymptotics for perturbed wave propagators 67

III From wave invariants to geometry: X-ray transform and Livšic theory **75**
III.0.1 Anosov manifolds . . . 76
III.1 Elements of Livšic theory . . . 77
III.1.1 The geodesic X-ray transform . . . 78
III.1.2 The smooth Livšic theorem for cocycles . . . 81
III.1.3 A transport equation with real-valued potential . . . 83
III.2 Infinitesimal spectral rigidity . . . 85
III.2.1 The marked length spectrum and its linearization . . . 85
III.2.2 Marked length spectrum and isospectral deformations . . . 87
Appendices to § III.1.2 . . . 90
Appendix III.A. Another proof of Proposition III.1.10 . . . 90
Appendices to § III.1.3 . . . 92
Appendix III.B. Cohomology of a sphere bundle . . . 92
Appendix III.C. Another proof of Theorem III.1.11 . . . 94
Appendices to § III.2.1 . . . 100
Appendix III.D. Higher-order variation of the marked length spectrum 101

IV Positive results for the Steklov inverse problem **105**
IV.1 Boundary determination of Riemannian metrics . . . 107
IV.1.1 Steklov isospectral deformations of conformal metrics . . . 107
IV.1.2 Non-conformal deformations: an approach and perspectives . . . 110
IV.1.3 The non-linear problem for conformal metrics . . . 113
IV.1.4 The non-linear problem for general metrics: Perspectives . . . 125
IV.2 Boundary determination of potentials . . . 127
IV.2.1 Recovery of electric potentials with no magnetic influence . . . 127
IV.2.2 Recovery of electric and magnetic potentials . . . 129
IV.2.3 Towards the framework of connections . . . 142
Appendices to § IV.1.3 . . . 144
Appendix IV.A. Dirichlet-to-Neumann map of a conformal metric . . . 145
Appendix IV.B. On the recovery of the conformal factor . . . 147

Conclusion and outlook **151**

Introduction en français

Table des matières pour le chapitre

— **La géométrie spectrale** 1
— **Les problèmes inverses en géométrie spectrale** 3
◇ Spectre du Laplacien: le tambour de Kac 4
◇ Spectre de l'opérateur de Steklov: le "tambour de Steklov" 7
— **Organisation de la thèse** 9

— La géométrie spectrale

La *géométrie spectrale*[1] étudie les liens entre la géométrie d'un espace et le spectre d'opérateurs (linéaires) naturels canoniquement associés à cette espace. L'exemple le plus classique, qui sera discuté plus en détail ci-dessous dans la section sur le problème de Kac, est donné par l'opérateur laplacien. Considérons, sur un domaine borné $\Omega \subset \mathbb{R}^n$ à bord lisse $\partial\Omega$, le *problème aux valeurs propres de Dirichlet*, modélisant les vibrations d'une membrane fixée le long de son bord :

$$\begin{cases} -\Delta_\Omega u = \lambda u \\ u_{|\partial\Omega} = 0 \end{cases} . \tag{0.0.1}$$

Ce problème possède un spectre discret, formé d'une suite de valeurs propres réelles de multiplicité finie

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \to +\infty \,,$$

qui constituent le *spectre de Dirichlet* du laplacien sur Ω (voir, parmi beaucoup d'autres, [Dav95], [Hen06, Chapter 1] ou le plus récent [LMP24]). Il s'agit alors de comprendre dans quelle mesure cette suite encode la géométrie du domaine Ω. L'étude com-

[1] Pour une introduction générale et accessible le lecteur pourra consulter [Ana23].

mence par plusieurs questions naturelles : Peut-on calculer chaque valeur propre individuellement ? Quel est le comportement asymptotique des valeurs propres $\lambda_j(\Omega)$ lorsque $j \to +\infty$? Les réponses à ces questions permettent-elles de trouver des quantités géométriques qui peuvent être reconstruites à partir du spectre, i.e des quantités géométriques associées à Ω (volume, dimension, nombres de composantes connexes, etc.) qui soient entièrement déterminées par cette suite de valeurs propres ? On parle alors d'*invariants spectraux*, en d'autres termes, ce sont des quantités qui peuvent être "entendues", cela fait écho au problème du tambour de Kac discuté ci-dessous et qui est modélisé exactement par ce spectre de Dirichlet en dimension 2.

La réponse à la première question est généralement négative excepté sur des domaines simples (rectangles, disques du plan, boules en dimensions supérieures, etc.); voir par exemple [Hen06, §1.2.5]. Ainsi pour des domaines généraux, l'étude du spectre repose plutôt sur des méthodes variationnelles, des développements asymptotiques et des calculs numériques. Le lecteur pourra consulter l'excellent [LMP24] en guise d'introduction à ces sujets et plus généralement à la géométrie spectrale.

Un exemple fondamental, répondant d'ailleurs à la deuxième question, est donné par la fonction de comptage

$$N_\Omega(\lambda) = \#\{\, j \in \{1, ..., n\} \;\mid\; \lambda_j(\Omega) \leq \lambda \,\}\,,$$

qui satisfait la *loi de Weyl* [2]

$$N_\Omega(\lambda) \sim \frac{\mathrm{Vol}(\mathbb{B}^n)}{(2\pi)^n} \mathrm{Vol}(\Omega)\, \lambda^{n/2}\,, \quad \lambda \to +\infty\,,$$

si bien que que le volume de Ω constitue un invariant spectral. On retrouve cette information avec la trace de l'opérateur de la chaleur (voir la discussion à ce sujet au début du Chapitre II)

$$\mathrm{Tr}(e^{-t\Delta_\Omega}) := \sum_{j\geq 1} e^{-t\lambda_j(\Omega)}\,,$$

qui admet, lorsque $t \to 0^+$, un développement asymptotique dont le premier coefficient fait intervenir également le volume du domaine Ω et dont les deux coefficients suivants ont été explicitement calculés notamment en dimension 2 (cf. (0.0.2)).
Ces exemples, considérés dans le cas simple du spectre du laplacien euclidien, s'étendent naturellement au cadre riemannien: on remplace le domaine euclidien Ω par une variété Riemannienne (M, g), et le laplacien usuel Δ_Ω par l'opérateur de Laplace–

[2] Voir par exemple [Ivr16, §1.1] pour un bref historique.

Beltrami[3] Δ_g. Le même type de questions se posent alors: dans quelle mesure le volume, la courbure, ou même plus largement la dynamique du flot
géodésique sont-ils contenus dans le spectre ? quels invariants géométriques apparaissent dans les asymptotiques spectrales ? C'est dans ce cadre plus général que la géométrie spectrale s'est développée ces dernières décennies, nous référons à [Bér86], [Cha84] ou encore une fois au plus récent [LMP24]. Notons que, pour ne pas nous éloigner de notre sujet, nous n'avons volontairement pas mentionné un autre pan important (et naturellement étroitement lié aux valeurs propres) de la géométrie spectrale concernant l'étude des fonctions propres associées. Dans le cas de l'opérateur de Laplace–Beltrami, le lecteur pourra consulter par exemple [Zel17].

— Les problèmes inverses en géométrie spectrale

Dans ce travail, nous nous intéressons à un type particulier de problème de géom étrie spectrale: les *problèmes inverses*. Comme la dénomination l'indique, ces questions s'opposent aux *problèmes directs*, pour lesquels la géométrie (domaine ou variété Riemannienne) est connue et l'on cherche à expliciter son spectre, i.e le spectre d'un opérateur linéaire associé. Au contraire ici, il s'agit de se demander si la connaissance de cette suite de valeurs propres permet de connaître les caractéristiques de la géométrie sous-jacente. Ceci revient donc naturellement à la question discutée dans la partie précédente, à savoir: quelles informations géométriques cette suite de valeurs propres encode t-elle ? En d'autres termes, il s'agit de trouver des invariants spectraux, i.e. des quantités géométriques qui restent invariantes sous n'importe quelle transformation de l'opérateur, préservant son spectre.

Par exemple, comme on l'a déjà mentionné, la loi de Weyl montre que le volume d'un domaine, et en fait même d'une variété Riemannienne compacte (voir la discussion au début du Chapitre II), est un invariant spectral pour le laplacien, et les asymptotiques de la trace de la chaleur font apparaître des quantités géométriques supplémentaires, précisément des intégrales de termes de courbures.
Une fois que des invariants spectraux ont été révélés l'étape suivante consiste à comprendre leur rigidité, i.e. dans quelle mesure ils contraignent la géométrie sous-jacente: si deux géométries ont le même spectre, donc les mêmes invariants, sont-ils nécessairement isométriques, ou bien peut-il exister des objets distincts mais *isospectraux* ? On parle de *rigidité spectrale*.

Dans le cas simple du laplacien sur un domaine borné $\Omega \subset \mathbb{R}^n$, il s'agit, informelle-

[3] Voir Sous-section I.6 pour un rappel.

ment, d'étudier l'injectivité de l'application

$$\Omega \mapsto \big(\lambda_j(\Omega)\big)_{j\geq 1} \ .$$

Il est dès lors important de mentionner que, de manière générale, il n'existe en fait que très peu de classes de géométries spectralement déterminées. Nous référons par exemple à [Zel04], [Gui21] et renvoyons aux deux prochaines sous-sections, dans lesquelles sont discuté les cas du spectre du laplacien et du spectre de Steklov. Le premier cas a donné lieu à la formulation historique probablement la plus célèbre d'un problème inverse spectral, tandis que le second en constitue aujourd'hui un analogue naturel.

◊ Spectre du Laplacien: le tambour de Kac

Considérons un domaine borné $\Omega \subset \mathbb{R}^2$, que l'on interprète comme une membrane de tambour, tendue et fixée le long de son bord. La fonction

$$\varphi \ : \ \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$$

qui modélisent les vibrations de la membrane i.e son déplacement vertical, vérifie l'équation des ondes

$$\partial_t^2 \varphi - c^2 \Delta_\Omega \varphi = 0 \quad \text{dans } \mathbb{R}^+ \times \Omega \, ,$$

où c est la célérité des ondes dans la membrane, avec la condition au bord

$$\varphi(t,x) = 0 \quad \text{pour } x \in \partial\Omega \, .$$

Cette condition de Dirichlet exprime le fait que le bord de la membrane est maintenue immobile. Cherchant les solutions découplées en temps et en espace de la forme

$$\varphi(t,x) = T(t)u(x) \, ,$$

en substituant dans l'équation des ondes, on obtient

$$T''(t)u(x) - c^2 T(t)\Delta u(x) = 0 \, ,$$

c'est-à-dire, informellement,

$$\frac{T''(t)}{c^2 T(t)} = \frac{\Delta u(x)}{u(x)} \, .$$

Le membre de gauche ne dépend que de t, tandis que le membre de droite ne dépend que de x ; ils sont donc tous deux égaux à une constante, que l'on écrit $-\lambda$ et on obtient alors, en tenant compte aussi de la condition au bord sur φ, le problème aux valeurs propres de Dirichlet donné précédemment en (0.0.1):

$$\begin{cases} -\Delta_\Omega u = \lambda u \\ u_{|\partial\Omega} = 0 \end{cases} ,$$

ainsi que

$$T''(t) + c^2 \lambda T(t) = 0 \, ,$$

qui est l'équation d'un oscillateur harmonique de pulsation propre $\omega = c\sqrt{\lambda}$. Les valeurs propres de Dirichlet $\lambda_k(\Omega)$ déterminent donc les *pulsations propres* de la membrane, données par

$$\omega_k = c\sqrt{\lambda_k(\Omega)} \, ,$$

et les fonctions u_k associées désignent ses *modes propres*. Il est bien connu qu'une vibration générale φ s'obtient alors par superposition de ces modes.

C'est en ce sens qu'en 1966, le mathématicien Kac pose la question suivante : *peut-on entendre la forme d'un tambour ?* Autrement dit, si deux domaines plans Ω_1 et Ω_2 ont même spectre de Dirichlet, sont-ils nécessairement congruents, i.e. image l'un de l'autre par une isométrie euclidienne ? (voir l'article original de Kac [Kac66]). Comme l'explique bien [Gui21], les premiers résultats positifs viennent de la trace de la chaleur que l'on a déjà mentionnée plus haut: Pour un domaine plan lisse,

$$\mathrm{Tr}(e^{-t\Delta_\Omega}) = \frac{\mathrm{Aire}(\Omega)}{4\pi t} - \frac{\ell(\partial\Omega)}{8\sqrt{\pi t}} + \frac{2-r}{6} + o(1) \, , \quad t \to 0^+ \, , \tag{0.0.2}$$

où r désigne le nombre de composantes connexes de Ω. Autrement dit l'aire, le périmètre et le nombre de composantes connexes sont des quantités *audibles* (voir par exemple [LMP24, Example 6.1.14]). Cela suffit déjà à montrer qu'un disque est déterminé par son spectre parmi les domaines plans à bord lisse: même spectre implique même aire et même périmètre, puis l'inégalité isopérimétrique force le domaine à être un disque (cf. [Kac66], [Gui21]).

Il serait pourtant faux d'en conclure que le spectre détermine toujours la géométrie: d'abord dans le cas riemannien Milnor construit en 1964 des tores plats de dimension 16 qui sont isospectraux mais non isométriques [Mil64], Vignéras donne ensuite des surfaces hyperboliques isospectrales non isométriques [Vig80], et Sunada [Sun85]

fournit une méthode de construction générale et robuste. En ce qui concerne les domaines plans euclidiens, la réponse négative à la question de Kac est obtenue par Gordon, Webb et Wolpert en 1992, avec des domaines plans non congruents mais isospectraux [GWW92].

Il s'agit dès lors de savoir s'il existe des classes de géométries spectralement déterminées et, si oui, est-ce un phénomène fréquent ? Par exemple Osgood, Phillips et Sarnak donnent une réponse en montrant que les métriques riemanniennes isospectrales d'une surface sont compactes dans la topologie C^∞ [OPS88b]. De nombreux résultats positifs ensuite sont obtenus dans certaines classes analytiques sous l'impulsion notamment de Zelditch et Hezari (see, for example, [Zel00, HZ10, HZ12, HZ22]). On réfère plus généralement au surveys [Zel04, Zel15], [DH13] ou encore [Gui21] pour une vision globale.

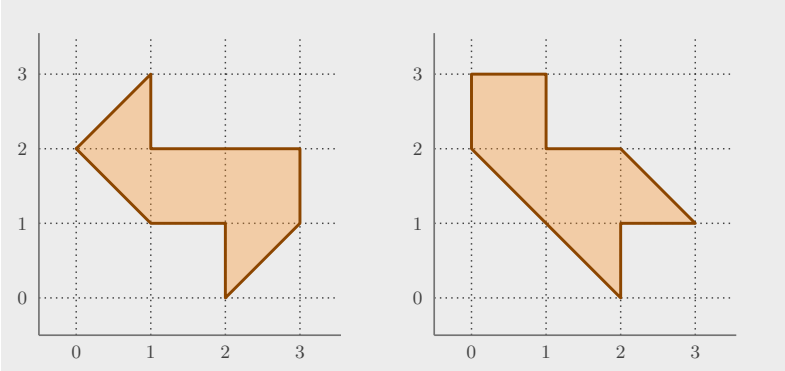


Figure 1: Domaines plans non isométriques mais isospectraux pour le Laplacien de Dirichlet [GWW92].

Au-delà du modèle du tambour, la motivation physique générale des problèmes inverses spectraux vient du fait qu'un système physique inaccessible est souvent observé à travers les fréquences qu'il émet, absorbe ou diffuse: en mécanique quantique, les valeurs propres d'un opérateur de Schrödinger décrivent des niveaux d'énergie ; en spectroscopie, les raies d'absorption observées renseignent sur la composition et la structure d'objets lointains ; en geophysique les ondes sismiques permettent d'inférer des propriétés internes de la Terre. Dans chacun de ces cadres, le problème inverse correspondant consiste à déterminer les informations géométriques ou physiques encodées dans les données spectrales: potentiel, métrique, structure interne, forme d'un obstacle (voir, par exemple, les références données dans [DH13]).

♢ Spectre de l'opérateur de Steklov: le "tambour de Steklov"

Le spectre de Steklov, qui est au centre de ce travail, admet une interprétation particulièrement intéressante d'un point de vue de l'analogie avec le problème de Kac sur le spectre du tambour. Pour un domaine lisse et compact $\Omega \subset \mathbb{R}^2$, le problème de Steklov s'écrit

$$\begin{cases} -\Delta_\Omega u = 0 & \text{dans } \Omega \\ \partial_\nu u = \sigma u & \text{sur } \partial\Omega \end{cases},$$

et les scalaires σ constituent le *spectre de Steklov* de Ω. Ce spectre consiste en une suite de valeurs propres réelles de multiplicité finie

$$0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \to +\infty\,.$$

De manière équivalente, ces valeurs propres sont celles de l'opérateur linéaire

$$\Lambda_\Omega \,:\, f \longmapsto \partial_\nu(Hf)\,,$$

où Hf désigne le prolongement harmonique de f à Ω. Cet opérateur est appelé l'*opérateur de Dirichlet-à-Neumann* de Ω.

En termes physiques, contrairement aux valeurs propres du laplacien de Dirichlet i.e aux modes propres de vibration d'un tambour modelisé par une membrane fixée le long de son bord et de masse uniformément répartie, ici avec le spectre de Steklov, on écoute plutôt une membrane à bord libre dont la masse est concentrée sur ce bord: c'est le "tambour de Steklov". Cette image peut être rendue mathématiquement rigoureuse, c'est ce qui est expliqué par exemple par Dalla Riva et Provenzano [DRP18]. Ils considèrent une membrane libre occupant un domaine plan Ω, de masse totale fixée, mais dont la densité se concentre dans une fine couche

$$\omega_\varepsilon = \{\, x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \varepsilon \,\}$$

au voisinage du bord (voir la figure 2 ci-dessous). Lorsque $\varepsilon \to 0$, les valeurs propres du problème de vibration correspondant convergent vers celles d'un problème de Steklov. En ce sens, le spectre de Steklov peut être compris comme un spectre limite d'une membrane libre dont la masse s'accumule asymptotiquement sur le bord. Nous renvoyons le lecteur à [DRP18, §1] et à toutes les références qui y sont données.

Dans ce cadre, le problème inverse associé consiste donc là encore à se demander quelles sont les informations géométriques sur Ω, et surtout sur $\partial\Omega$, qui peuvent être

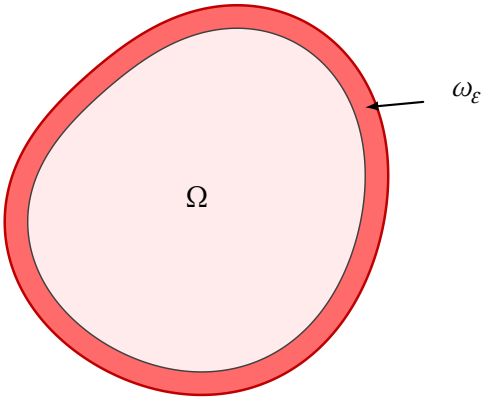


Figure 2: Schéma inspiré de la figure 1 de Dalla Riva–Provenzano [DRP18] : la masse est concentrée dans une fine couche ω_ε au voisinage du bord.

entendues du tambour de Steklov, qui sont *audibles* à partir de ce spectre ?
La loi de Weyl pour l'opérateur de Dirichlet-à-Neumann montre que le volume du bord est déterminé par le spectre (voir **Proposition II.1.1** et la discussion autour). Dans le contexte du tambour de Steklov, donc en dimension 2, Girouard, Parnovski, Polterovich et Sher montrent que le nombre de composantes connexes du bord ainsi que leurs longueurs sont des invariants spectraux [GPPS14]; ce résultat est retrouvé plus tard dans un cadre plus général dans [LSA21]. En ce qui concerne plus généralement la détermination d'une métrique Riemannienne, le cadre de la dimension 2 comporte une *invariance de jauge* naturelle: Si une métrique $g_2 = cg_1$ est une métrique conforme à une métrique g_1 et si $c|_{\partial M} = 1$, alors les deux métriques ont le même spectre de Steklov (voir par exemple [GPPS14, §1.4] et [JS14] pour une discussion sur ce point). Dans le cas d'un domaine simplement connexe $\Omega \subset \mathbf{R}^2$, le problème inverse revient à déterminer une fonction $a \in C^\infty(\partial\mathbb{D})$ sur le cercle à partir du spectre de l'opérateur $a^{-1}\Lambda_{\mathbb{D}}$ (cf. [JS14, JS18]).
Concernant les résultats négatifs, il existe évidemment bien des variétés à bord Steklov isospectrales mais non isométriques [GHW21], les constructions reposent sur des adaptations de méthodes générales élaborées initialement pour le cas du laplacien; en première place se trouve la méthode de Sunada [Sun85] (voir aussi [Gor00, Gor09]). En revanche, dans le cadre euclidien, l'existence de domaines plans non isométriques mais isospectraux reste, à l'heure actuelle, une question ouverte (cf. [CGGS24, §9.2]). On voit là une différence importante avec le tambour de Kac !
Du côté des résultats positifs, le disque est déterminé par son spectre de Steklov parmi les domaines euclidiens plans lisses [GP17, Proposition 5.2.1]. En dimension 3, Polterovich et Sher montrent qu'une boule euclidienne à bord lisse connexe est déterminée par son spectre de Steklov [PS15] en utilisant les invariants de la chaleur.

Dans le même esprit, Jollivet et Sharafutdinov obtiennent un théorème de compacité pour les familles isospectrales de domaines plans [JS18]. Plus récemment, dans la même veine, Dryden, Gordon, Moreno, Rowlett et Villegas-Blas montrent que, pour presque tout polygone convexe, l'ensemble des polygones non congruents ayant le même spectre de Steklov est fini [DGM$^+$25].

Nous n'avons ici pas évoqué de résultats positifs dans le cas riemannien, mais nous soulignons tout de même que, de manière générale, la littérature positive dans ce cadre demeure encore très limitée (voir Chapitre I). C'est dans ce contexte, que s'inscrivent les résultats de cette thèse. On établit notamment un résultat de *détermination au bord* pour des métriques conformes sur toute variété Riemannienne compacte (M, g) de dimension $n \geq 3$. Plus précisément, sous l'hypothèse d'uniforme hyperbolicité de la dynamique au bord (en particulier si le bord est de courbure négative) et de simplicité de son spectre des longueurs (i.e. que toutes ses courbes géodésiques périodiques sont de longueurs distinctes), on montre que le spectre de Steklov détermine le jet au bord de la métrique dans une classe conforme (cf. **Theorem I.1.1**), entraînant sous l'hypothèse d'analyticité du facteur conforme, la détermination de la métrique sur toute la variété, autrement dit si $g_2 = cg_1$ et c est un facteur conforme analytique, alors $g_2 = g_1$. Un résultat de *rigidité spectrale infinitésimale* est aussi obtenu (cf. **Theorem I.1.2**) dans ce même contexte géométrique, informellement: il n'existe pas de famille $((M, c_s g))_s$ à un paramètre, non-triviale, de "tambours de Steklov" conformes ayant même spectre. Nous référons au Chapitre I pour les détails. Les autres résultats concernent la détermination de potentiels électriques et magnétiques, à partir du spectre de Steklov, ce qui sort du cadre de cette partie. Nous renvoyons donc aux sections concernées.

— Organisation de la thèse

Ce travail, est consacré à l'étude du problème inverse spectral de Steklov sur une variété Riemannienne compacte à bord (M, g), que l'on peut voir comme l'analogue riemannien du "tambour de Steklov". Il s'agit de comprendre dans quelle mesure la donnée de la suite des valeurs propres de Steklov

$$0 < \sigma_1 \leq \sigma_2 \leq \cdots \to +\infty\,,$$

c'est à-dire ce que l'on peut "entendre" du tambour de Steklov, permet de reconstituer la géométrie ou la physique sous-jacente: métrique Riemannienne, potentiel

électrique (fonction lisse), potentiel magnétique (forme différentielle de degré 1)...

1. Le premier chapitre a pour but d'introduire tout d'abord le domaine de la géométrie spectrale puis d'y situer les problèmes relatifs au spectre de Steklov. Après avoir introduit les différents contextes étudiés dans cette thèse, on présente les résultats majeurs qui ont été obtenus. Ces derniers sont des résultats de *détermination au bord*, car ils donnent des réponses positifs à la question posée ci-dessus mais les données ne sont déterminées que dans un voisinage du bord. En d'autres termes, on "entend" la géométrie ou les potentiels physiques qu'au voisinage du bord, mais cela suffit pour les entendre partout sous hypothèse d'analyticité.

2. Le deuxième chapitre est consacré à la recherche d'invariants spectraux, on se focalise ici sur des types d'invariants bien particuliers: les *invariants d'onde*. Si Λ désigne un opérateur de Dirichlet-à-Neumann sur une variété Riemannienne compacte à bord, on étudie la trace de son propagateur des ondes:

$$\mathrm{Tr}\left(e^{-it\Lambda}\right), \quad t \in \mathbb{R}\,.$$

Cette trace est à comprendre au sens des distributions et sa richesse réside en ses singularités, i.e en les temps t en lesquels cette distribution n'est plus lisse. Il s'avère que ces temps sont tous des périodes de trajectoires fermées du flot géodésique, autrement dit de courbes géodésiques périodiques. Sous certaines hypothèses dynamiques sur le flot géodésique, ils deviennent des quantités complètement déterminées par le spectre. Cette trace des ondes est en effet la transformée de Fourier de la *distribution spectrale*

$$\mathrm{d}N_\Lambda = \mu_\Lambda := \sum_{j=1}^{+\infty} \delta(\lambda - \lambda_j)\,,$$

si bien que si deux opérateurs ont même spectre ils ont en particulier même trace des ondes et donc même *spectre des longueurs*. Dans ce travail, ces invariants d'onde sont exploités de manière plus précise via les *formules de trace* et donne un ensemble complet d'invariants spectraux appelés *invariants principaux des ondes*. On est réduit typiquement à des quantités de la forme

$$\mathcal{I}f(\gamma) := \int_\gamma f,$$

où γ est une géodésique périodique et f est une fonction, une 1-forme différentielle ou un tenseur symétrique d'ordre 2 (typiquement une métrique Riemannienne), représentant de l'information partielle sur la quantité que l'on souhaite récupérer. Il ne s'agit dès lors plus de comparer deux spectres, mais de comprendre si une fonction, un tenseur ou un coefficient est déterminé par ses intégrales le long des géodésiques fermées. C'est un problème inverse de *tomographie*.

Cette formulation est particulièrement riche lorsque le flot géodésique est Anosov. Dans ce cadre, la dynamique est hyperbolique et chaotique : les orbites périodiques sont nombreuses et cette abondance suggère que cette donnée contient suffisamment d'information pour reconstruire l'objet f inconnu, à condition de disposer de résultats d'injectivité adaptés.

3. Le troisième chapitre introduit justement les outils qui permettent de faire ce passage des invariants d'onde à la géométrie. Il est consacré à la *théorie de Livšic*, aux *transformées à rayons X géodésiques* et aux propriétés d'injectivité qui leur sont associées. Dans sa forme la plus simple, le théorème de Livšic affirme que si une fonction f vérifie

$$\int_\gamma f = 0$$

pour toute orbite périodique γ, alors f est un cobord i.e s'écrit comme une dérivée dans la direction du flot géodésique:

$$f = Xu\,,$$

où X désigne le champ de vecteur générateur du flot.

Autrement dit, ce théorème caractérise le noyau de l'opérateur linéaire $f \mapsto \mathcal{I}f$ appelé transformée à rayons X géodésique, associant donc à une fonction ou à un tenseur la collection de ses intégrales de long des géodésiques périodiques.

4. Le dernier chapitre rassemble ces différents ingrédients dans les preuves des résultats principaux. La stratégie générale peut être résumée de la façon suivante :

$$\boxed{\text{spec}(\Lambda) \to \text{invariants d'onde} \to \left(\int_\gamma f\right)_{\gamma \in \mathcal{G}_{per}} \to \text{détermination de } f}$$

Chapter I

Introduction : Inverse problems for the Steklov spectrum

Table of Contents for the chapter

I.1 Recovering geometry from the Steklov spectrum 21
I.1.1 The Steklov inverse problem 24
I.1.2 The linearized problem: Steklov isospectral deformations 26
I.1.3 Main results 29
I.2 Recovering physical data from the Steklov spectrum 31
I.2.1 The magnetic Steklov inverse problem 32
I.2.2 Main results 34
I.3 An additional rigidity result for the magnetic Schrödinger operator 36
I.4 A new DG–type trace formula: the key to boundary determination 39
I.4.1 Brief description of the induction procedure 40
I.5 A roadmap for the reader and a few notes 41
I.6 Some definitions and notation 44

The subject of this thesis falls within the field of *spectral geometry*, at the intersection of differential geometry and spectral theory. Spectral geometry studies the extent to which the spectral data of geometric linear operators reflect the geometry of the underlying manifold or domain. To explain the idea, let us consider a family of linear operators $(P_\theta)_{\theta\in\Theta}$ acting on functions of a domain with or without boundary, where Θ is a parameter space and $\theta \in \Theta$ can generally represent a Riemannian metric, a scalar function, a 1-form (or a connection), or even the domain itself. If these operators are self-adjoint with compact resolvent and bounded below, say positive,

they admit a discrete real spectrum consisting in a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \, .$$

Ultimately, everything comes down to analyzing the spectral map

$$\mathrm{spec} \, : \, \left| \begin{array}{ccc} \Theta & \longrightarrow & [0, +\infty)^{\mathbb{N}} \\ \theta & \longmapsto & \left(\lambda_j\right)_{j \geq 1} \end{array} \right. .$$

For example, a central question is whether the spectral map is injective, that is, whether the spectrum determines the datum θ. This is the object of an important class of problems called *spectral inverse problems*. In general, one cannot expect injectivity in the usual sense, since there are often natural transformations of the operator that leave its spectrum unchanged. In other words, the inverse problem must be formulated modulo a group G of *gauge transformations*, that is, injectivity has to be considered on the quotient Θ/G, as well as other closely related issues, such as the structure of *isospectral* sets, and, more generally, the extent to which spectral rigidity can be expected. We refer, for example, to the surveys [Zel04, Zel15] and [DH13].

The other aspect of spectral geometry concerns *direct problems*, that is, the study of spectral data associated with a given geometry. Basically, the geometric setting is given, and the goal is to understand the associated spectrum itself: eigenvalues and individual eigenfunctions, asymptotic laws, variational characterizations, geometric bounds, and inequalities between eigenvalues. For a good general overview, we refer, for example, to the recent book [LMP24].

The present work focuses instead on inverse problems where the spectral data consist of eigenvalues and their multiplicities, but an important principle is shared: in practice, one rarely has direct access to individual eigenvalues, except in a few specific geometries; therefore, one tends to consider spectral generating functions or distributions constructed from the eigenvalues, notably the *spectral counting function*, the *heat trace*, the *spectral zeta function*, informally defined by

$$N(\lambda) := \#\{\lambda_j \leq \lambda\} \quad , \quad Z(t) := \sum_{j \geq 1} e^{-t\lambda_j} \quad , \quad \zeta(s) = \sum_{j \geq 1} \lambda_j^{-s} \quad ,$$

and, most importantly for us, the *wave trace*

$$W(t) := \sum_{j \geq 1} e^{-it\lambda_j} \, .$$

These functions/distributions therefore organize raw spectral data into analytical objects from which geometric information can be extracted. This principle is discussed much longer at the beginning of Chapter II. Our main object of interest will be the wave trace, as it provides the natural bridge between the spectral data and the underlying dynamics. Indeed, the first quantities that emerges from the wave trace are dynamical in nature and concerns in our case, the *length spectrum* :

Definition I.0.1. *The length spectrum of a closed* [1] *Riemannian manifold* (N, h) *is the set of lengths, counted with multiplicities, of its periodic geodesics :*

$$\mathrm{Lsp}(N,h) := \{\, \ell_h(\gamma) \mid \gamma \in \mathcal{G}_{per}(N,h) \} \ .$$

One says that h *has simple length spectrum if distinct closed geodesics have distinct lengths.*

It should be noted that richer information is obtained by marking this length spectrum via homotopy, that is, by assigning to each free homotopy class the length of its representative periodic geodesic, when such a geodesic is unique. This defines the famous *marked length spectrum* of the Riemannian manifold; see Subsection III.2.1 and references therein for details. Regarding the simple length spectrum assumption, it will appear in the wave trace arguments because it isolates the contribution of every periodic geodesic in the asymptotics given by **Theorem II.3.4** and **Theorem II.4.2**, therefore preventing cancellations between different orbits. It is well-known that this assumption is satisfied by generic Riemannian metrics (cf. [Abr], [Ano83]). This latter point is briefly discussed in Subsection III.2.1.

More precisely, as we will explain in Chapter II, the first link between the wave trace and the geometry is provided by the *Poisson relation* (cf. **Proposition II.3.2**): the length spectrum contains the singular support[2] of $W(t)$ as a distribution in time. This relation finds its deepest and most general expression in wave trace formulas: in the case of closed manifolds, the canonical result is the Duistermaat-Guillemin trace formula [DG75, Theorem 4.5], which precisely describe the nature of the singularities of the trace of the wave group associated with any positive, elliptic and self-adjoint classical pseudodifferential operator, to the set of periods of its *bicharacteristic flow* (see **Definition II.3.1**). This work appeared shortly after the papers by Colin de Verdière [CdV73a, CdV73b] and Chazarain [Cha74b] on the Laplacian, we

[1] compact without boundary.
[2] see the end of Subsection I.6 for this notion.

refer to Subsection II.2.1 for a more exhaustive discussion of this topic. On manifolds or domains with boundary, an analogous result is given by the Poisson summation formula of Guillemin and Melrose [GM79], where the underlying dynamics is that of the billiard flow.
From a historical perspective, this connection between Laplacian eigenvalues and length spectrum had already emerged even before the general theory of wave trace formulas was developed. Using Selberg's trace formula [Sel56], Huber had in particular proved (see also [Bus92, Theorem 9.2.9]), for compact hyperbolic surfaces, a remarkably strong equivalence :

Theorem I.0.1 ([Hub59]). *Two compact hyperbolic surfaces of genus $g \geq 2$ are Laplace isospectral if and only if they have the same length spectrum.*

Of course, in general, outside of these rigid geometries, the spectrum and the length spectrum do not provide equivalent information. Penrose–Lifshits mushrooms are a classic example: building on an idea attributed to Penrose (see [Rau78]), Lifshits constructed non-isometric flat domains with the same length spectrum, and Fulling and Kuchment [FK05] showed that, generically, these domains are not isospectral with respect to the Dirichlet Laplacian. This illustrates in particular that, although the wave trace contains highly non-trivial dynamical information, the transition from this information to determination of the underlying geometry is not guaranteed and naturally depends on the geometric framework. We refer once again to the excellent surveys of Zelditch [Zel04, Zel15] and Datchev-Hezari [DH13]. As already mentioned, the marked length spectrum, on the other hand, provides significantly richer information, as shown by some strong rigidity theorems which solve positively the famous *marked length spectrum conjecture*, due to Burns and Katok [BK85] (see also [GL19, CONJECTURE 1]). The first positive results date back to Otal [Ota90] and Croke [Cro90] and have been extensively pursued since then, in particular by Guillarmou and Lefeuvre [GL19] recently. The marked length spectrum plays an important role for us, notably through its linearization; we refer to Section III.2 for more details.

One might therefore ask whether there is a chance that the singularities of the wave trace determine the marked length spectrum itself, but it is easy to see that this is absolutely not the case in general (see **Remark** III.2.2). This constitutes a significant obstacle to exploiting the rigidity of the marked length spectrum to address spectral inverse problems whose underlying dynamics is, of course, a geodesic flow. However, as mentioned above, one can consider a linearization of the problem in

which this difficulty disappears or can be controlled; this is the *infinitesimal* problem of *isospectral deformations* :

Definition I.0.2. *Let (N, h) a closed Riemannian manifold. A deformation of h is a smooth one-parameter family of (smooth) Riemannian metrics $(h_s)_{0\leq s\leq \varepsilon}$ on N such that $h_0 = h$. It is called (Laplace) isospectral if the spectrum of $-\Delta_{h_s}$ is preserved :*

$$\forall s \in [0, \varepsilon] \ , \ \operatorname{spec}\left(-\Delta_{h_s}\right) = \operatorname{spec}\left(-\Delta_h\right) \ .$$

If (h_s) is an isospectral deformation, the singularities of the wave trace $W_{\Delta_h}(t)$ are preserved along the deformation. Thus, if you consider a singularity $\ell_h(\gamma_0)$ of $W_{\Delta_h}(t)$, which is the length of a periodic geodesic γ_0 for $h = h_0$, and if you assume that it is isolated in the length spectrum, then this length is preserved along the deformation. On negatively curved, or more generally Anosov, manifolds (see Subsection III.0.1), any periodic geodesic γ_0 for h deforms smoothly as a periodic geodesic γ_s for h_s and the length map $s \mapsto \ell_{h_s}(\gamma_s)$ is smooth. Thus, in this case, an isospectral deformation preserves the marked length spectrum. The infinitesimal spectral rigidity problem is therefore reduced to the *linearized marked length spectrum rigidity problem*, or equivalently to the injectivity of a linear operator, called the geodesic X-ray transform (see **Definition** III.1.2). Indeed, by differentiating this equality of the marked length spectrum with respect to the parameter s, and evaluating at $s = 0$, we obtain (see, for example, [GM, Lemma 15.4]) for any periodic geodesic γ_0 of h,

$$\int_\gamma \dot{h}\, ds = 0 \ , \tag{I.0.1}$$

where

$$\dot{h} := \frac{\partial}{\partial s}\Big|_{s=0} h_s$$

denotes the first variation of the metric. This idea was successfully used in the pioneering work of Guillemin and Kazhdan [GK80a] on negatively curved closed surfaces, and extended by the same authors [GK80b] to higher dimensions under a pointwise curvature pinching assumption. This assumption was later removed by Croke and Sharafutdinov in [CS98], who proved (infinitesimal) spectral rigidity of closed negatively curved manifolds in any dimension. In other words, there cannot exist non-trivial isospectral deformation of Riemannian metric with negative curvature. This result has been extended to arbitrary Anosov surfaces by Paternain, Salo, and Uhlmann [PSU14] but the case of higher dimension remains open and difficult, as it

is, of course, closely related to the injectivity of the geodesic X-ray transform which is not known on closed Anosov manifolds of higher dimension (see Subsection III.1.1). As noted above, infinitesimal spectral rigidity has been addressed in this work, and a result has been established for the Steklov spectrum (cf. Theorem I.1.2).

In addition to the length spectrum, the wave trace provides a complete set of spectral invariants, known as *wave invariants*. These arise from a deeper analysis of the singularities that led to the wave trace formulas mentioned above. The latter show that each period of the geodesic flow (or, more generally, of the bicharacteristic flow) gives rise to a singularity of the wave trace whose structure is strongly constrained. More precisely, near an isolated period, the wave trace is described by a distribution supported at that point and admitting a complete asymptotic expansion. The associated *principal wave invariants* are precisely the leading terms of this singular expansion and are the ones that will be of interest to us here. The computation of wave invariants has, of course, been investigated: Guillemin proposed an approach based on normal forms [Gui96], and Zelditch expanded on this through explicit calculations for elliptic geodesics, and more generally for non-degenerate closed geodesics [Zel97, Zel98]. We also refer to the surveys [Zel04, Zel15].

The use of these spectral invariants has proven effective in the context of spectral inverse problems related to the Laplacian: whether in the case of closed manifolds, particularly with the [GK80a, GK80b] and [CS98] mentioned above, or in the case with boundary, with the work of Marvizi and Melrose for convex planar domains [MM82] using the billiard wave trace invariants, and most explicitly in Zelditch's work [Zel00] on analytic symmetric plane domains, where the proof proceeds through a detailed analysis of the wave trace invariants of bouncing-ball trajectories. Hezari and Zelditch's results [HZ10] on symmetric analytics in higher dimensions fall within the same framework [HZ10], as does the infinitesimal spectral rigidity of the ellipse proved by the same authors in [HZ12]. We refer once again to the surveys [Zel04, Zel15] and [DH13] for a much longer discussion on this topic.

It should also be noted that wave trace techniques are not the only source of spectral invariants leading to interesting spectral rigidity results. The aforementioned heat trace, spectral zeta functions, and zeta-regularized determinants constitute a complementary family of tools (see the introduction of Chapter II). For example, in dimension two, the coefficients of the heat expansion recover, area, boundary length, and curvature integrals (see [LMP24, Ch. 6]) and naturally lead to the spectral rigidity of the planar disk [Kac66] among domains with smooth boundary. Osgood, Phillips,

and Sarnak used the determinant

$$\det_\zeta \Delta_h := e^{-\zeta'_{\Delta_h}(0)},$$

to show compactness results for isospectral sets of surfaces and plane domains in [OPS88a, OPS88b]. All of these examples are useful for putting things into perspective and, of course, by no means constitute an exhaustive list; however, we will stop here and refer, for example, to [Zel04, Zel15], [DH13], [CdV96, CdV07], [Gui21], [LMP24, Ch. 6] or even [Gor00] for a good general overview regarding the Laplacian spectrum. As we mentioned earlier, this work does indeed focus primarily on the wave trace. The initial idea to address spectral rigidity for the Steklov spectrum was obviously to adapt to the Steklov setting the techniques that proved most effective for Laplace-type spectra. The main observation that lends weight to this naive approach is the fact that the Steklov spectrum of a compact Riemannian manifold (M, g) with boundary is equivalently the spectrum of the Dirichlet-to-Neumann map (DN map), which can actually be viewed as a perturbation of the square root of the boundary Laplacian:

$$\Lambda_g = \sqrt{-\Delta_{g_{|\partial M}}} + B, \quad B \in \Psi^0_{\mathrm{cl}}(\partial M).$$

If ∂M is smooth, it is actually a first-order classical self-adjoint positive elliptic pseudodifferential operator with the same principal symbol as that of $\sqrt{\Delta_{g_{|\partial M}}}$. This point of view has been actively investigated in the last decades, the reader may refer in particular to [GKLP22] and more generally to the survey [GP17].

At the same time, the Steklov case is not a mere formal copy of the Laplacian case. A key difference for us is that the DN map generally admits a non-trivial subprincipal symbol and in particular, the corresponding wave trace (cf. **Theorem II.3.4**) reveals this subprincipal contribution in the principal wave invariants. This observation was one of the initial motivations for our approach, and it will be made more precise later on. Coming back to our initial introduction, in this work we will therefore consider the general spectral map

$$\theta \mapsto \mathrm{spec}(P_\theta),$$

with P_θ being a DN-type map associated either to a Riemannian metric or to potentials (cf. Section I.1 below), and will ask to what extent the wave trace approaches that work in the Laplacian case can be transplanted to the Steklov spectral inverse problem. Unless otherwise specified, the manifolds discussed in this work will be smooth.

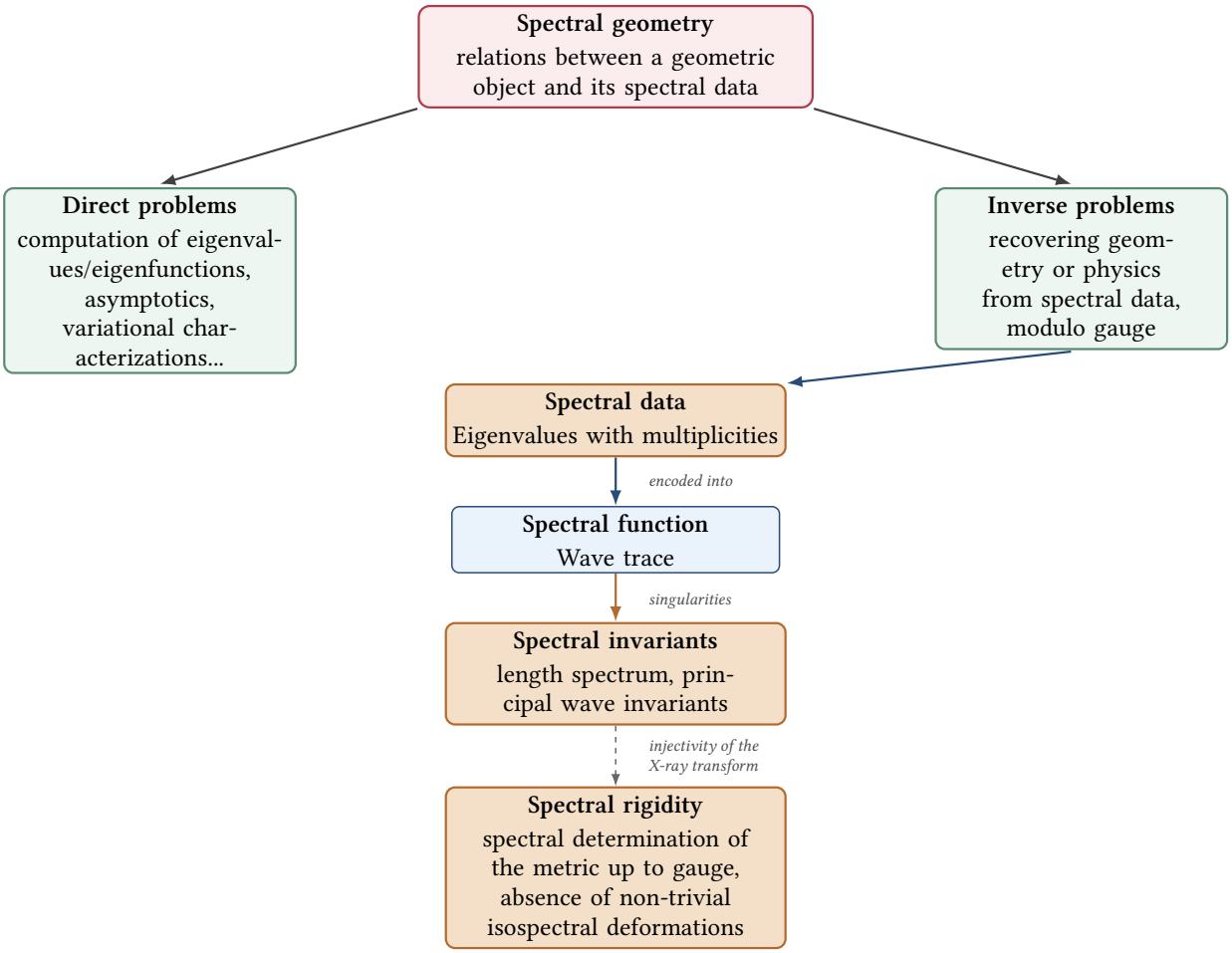


Figure I.1: An overview of spectral inverse problems, using the example where the spectral data consist of eigenvalues and are processed via the wave trace.

I.1 Recovering geometry from the Steklov spectrum

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M.
Recall that the *Steklov eigenvalue problem* consists in finding scalars σ for which the system

$$\begin{cases} -\Delta_g u = 0 & \text{in } M \\ \partial_\nu u = \sigma u & \text{on } \partial M \end{cases}$$

admits a non-trivial solution in $H^1(M)$; where Δ_g is the Laplace-Beltrami operator on (M, g) and ∂_ν denotes the Riemannian outward normal derivative along the boundary. This famous problem, introduced at the beginning of the 20th century by the mathematician V.A. Steklov (see [KKK+14] for background), lies at the interface between spectral geometry and mathematical physics. We refer the reader to the excellent surveys [GP17] and [CGGS24] for details and directly to the next paragraph for the main physical interpretations of these spectral data.
Equivalently, one can consider the eigenvalues of the Dirichlet-to-Neumann map (DN map) Λ_g defined by

$$\Lambda_g : \left| \begin{array}{ccc} H^{1/2}(\partial M) & \longrightarrow & H^{-1/2}(\partial M) \\ f & \longmapsto & \partial_\nu u_{|\partial M} \end{array} \right. ,$$

where u is the unique solution in $H^1(M)$ of the Dirichlet problem

$$\begin{cases} -\Delta_g u = 0 \\ u_{|\partial M} = f \end{cases} .$$

The DN map is a positive self-adjoint elliptic and classical (in the sense of (I.6.4)) pseudodifferential operator of order 1 on the closed manifold ∂M.
In particular, its spectrum is discrete and is given by a sequence of eigenvalues

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to +\infty .$$

The multiset

$$\operatorname{spec}(\Lambda_g) := (\sigma_k)_{k \geqslant 0}$$

defines the so-called Steklov spectrum of the Riemannian manifold (M, g).

The microlocal nature of the DN map gives it a rich spectral theory, closely related to that of the boundary laplacian as mentioned above: first, defining $g^\circ := g_{|\partial M}$, we have the following identity of principal symbols

$$\sigma_{\Lambda_g}(x,\xi) = |\xi|_{g^\circ} = \sigma_{\sqrt{\Delta_{g^\circ}}}(x,\xi)\,, \quad (x,\xi) \in T^*\partial M \setminus \{0\}\,,$$

showing that Λ_g is, to principal order, the square root of the boundary Laplacian. This gives first, by application of Hörmander's standard results, the famous Weyl's law for Steklov eigenvalues (see **Proposition II.1.1**), revealing the volume of the boundary as a first spectral invariant. More generally, as already mentioned, at the level of operators we have

$$\Lambda_g - \sqrt{\Delta_{g^\circ}} \in \Psi^0(\partial M)\,.$$

Thus, if (λ_k) denotes the eigenvalues of the Laplace-Beltrami operator $-\Delta_{g^\circ}$ on the boundary $(\partial M, g^\circ)$, one can see that the Steklov and boundary laplacian spectra have the same first-order asymptotics:

$$\sigma_k \underset{k\to\infty}{\sim} \sqrt{\lambda_k}\,.$$

We refer the reader to [GKLP22, §1.2]. In fact, one has quantitative estimates of the form

$$|\sigma_k - \sqrt{\lambda_k}| \leq C\,, \quad C > 0$$

uniformly in k, and with C depending on the geometry near the boundary (cf. [CGH20], or [GKLP22] and references therein). We show that the Steklov spectrum then encode the boundary geometry, even though the eigenfunctions arise from harmonic functions defined on the whole manifold.

Beyond these asymptotic and comparison results, which we haven't even cited in full, there is a vast literature related to *direct problems* for the Steklov spectrum, that we will not mention here as that would take us off topic. We will simply refer to well-known surveys, notably [GP17], [CGGS24], and [LMP24, Chapter 7].

Physical interpretations of the Steklov spectrum

A natural physical interpretation starts from the weighted Steklov problem

$$\begin{cases} -\Delta_g u = 0 & \text{in } M \\ \partial_\nu u = \sigma \rho\, u & \text{on } \partial M \end{cases},$$

where $\rho \geq 0$ is a boundary mass density. In dimension 2, this models the free transverse vibration of a membrane whose mass is concentrated on the boundary; more generally, Steklov eigenvalues arise as limiting Neumann eigenvalues when the mass density ρ concentrates in a thin strip near ∂M (see, for example [DRP18], or the survey [GP17] and references therein). See also Figure 2 for the case where M is a planar domain.

Another classical physical interpretation of the Steklov problem, which is actually a special case of the previous one, is provided by the *sloshing problem*. Consider an ideal incompressible and irrotational fluid occupying a container at rest. The velocity potential u is harmonic in the fluid domain Ω, satisfies a Neumann condition on the rigid walls B, and a Steklov condition on the free surface F:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } B \\ \partial_\nu u = \dfrac{\omega^2}{g} u & \text{on } F \end{cases},$$

where g denotes the gravitational constant and ω is the oscillation frequency. Thus the sloshing eigenvalues are precisely the quantities ω^2/g, and they encode the natural frequencies of small free-surface oscillations. In other words, the sloshing problem is a mixed Steklov-Neumann eigenvalue problem, and may be viewed as one of the most classic physical realizations of the Steklov spectrum. (see, among many others, [FK83], or the book [KMV02]).

We refer the reader to [GP17] and references therein for more examples of physical interpretations.

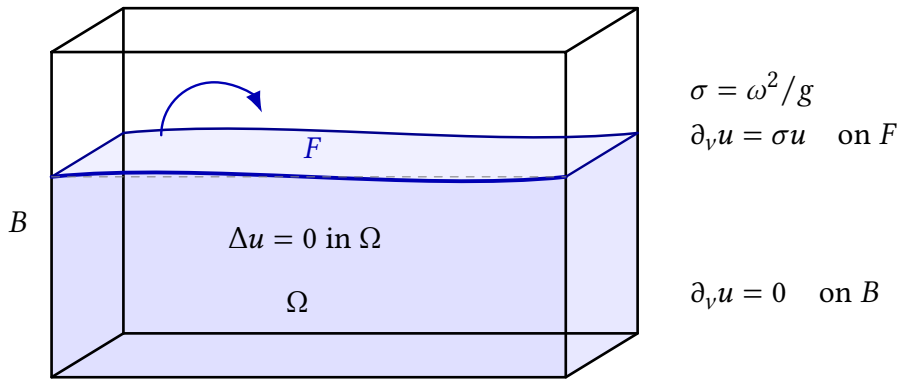


Figure I.2: A three-dimensional figure illustrating the sloshing problem. The fluid potential is harmonic in the domain Ω; it satisfies a Neumann condition on the rigid wall B and a Steklov condition on the free surface F.

I.1.1 The Steklov inverse problem

As already mentioned, following Kac's celebrated question *Can one hear the shape of a drum?* [Kac66], one may ask for an analogous inverse problem for the Steklov spectrum: does the Steklov spectrum determine the underlying Riemannian metric up to the natural gauge ? Indeed, if (M, g) is a compact Riemannian manifold with boundary and $\varphi : M \to M$ is a diffeomorphism, then the DN map Λ_g transforms by conjugacy

$$\left(\varphi_{|\partial M}\right)^* \Lambda \left(\varphi_{|\partial M}\right)_*$$

under the boundary action of φ, and therefore

$$\operatorname{spec}(\Lambda_{\varphi^* g}) = \operatorname{spec}(\Lambda_g) \, .$$

In other words, do we have

$$\operatorname{spec}(\Lambda_{\tilde{g}}) = \operatorname{spec}(\Lambda_g) \implies \exists \varphi \in \operatorname{Diff}(M) \, ; \, \tilde{g} = \varphi^* g \, ?$$

This inverse problem is known as the *Steklov inverse problem.* A first elementary observation that can be made is, of course, that one cannot expect to obtain positive results without possibly very strong assumptions, since, by comparison, we start with even less information than in the famous anisotropic Calderón inverse problem. This problem consists in recovering the metric g from knowledge of the DN map Λ_g and is still far from being fully understood despite a vast literature on the subject and striking positive results for real-analytic metrics [LU89, LU01]. The reader may wish to consult, for example, the survey [DKN18] and all references therein. Regarding this problem, a standard step is to show a boundary determination result, namely that the DN map determines, in boundary normal coordinates, all the normal derivatives of the metric tensor g at the boundary at any order $j \geqslant 0$ (and thus the full Taylor series of g at the boundary in these coordinates). We would like to mention in particular the important papers [LU89], [LU01] mentioned above, and [LTU03], [DSFKSU09] or even the more recent [Cek20], [LLS22].

In this work, we establish similar boundary determination results for the Steklov spectrum. In particular, combining the boundary determination mentioned above, the use of wave trace invariants and the injectivity of the X-ray transform, we show that, under natural assumptions on the geodesic flow of $g_{|\partial M}$, the Steklov spectrum determines all normal covariant derivatives at the boundary of the metric g within

a fixed conformal class. This is to our knowledge, the first positive result on the Steklov inverse problem in such a general setting.

Prior to this work, however, several partial results were already known, notably the following: on smooth compact surfaces, sharp asymptotics imply that one can hear the number of boundary components and their lengths [GPPS14] which gives interesting consequences, for example that Steklov isospectrality automatically imposes Laplace isospectrality of the boundaries. In special geometric classes, Gendron and Daudé obtained uniqueness and stability results for warped-product manifolds from Steklov spectral data [Gen20, DKN21]. In another setting, different approach, specific to the simply connected planar case, was developed by Jollivet and Sharafutdinov. As mentioned above, they reformulated the Steklov inverse problem for a smooth simply connected surface in terms of the spectrum of the operator $a\Lambda$ on the circle [JS14]. Then they introduced Steklov zeta invariants and proved, among other results, a compactness theorem for isospectral families of smooth, simply connected planar domains [JS18, JS22].

For a longer discussion and further references, we refer to the surveys [GP17, CGGS24].

On the negative side, non-uniqueness issues arise even on simple examples. The explicit formula for the Steklov spectrum of a cylinder $[0, L]\times\Sigma$, obtained by separation of variables, shows that it depends only on L and the Laplace spectrum of the closed basis Σ. In particular, the following consequence follows from this.

Proposition I.1.1 ([CESG11]). *Let (Σ_1, h_1) and (Σ_2, h_2) be closed Laplace isospectral Riemannian manifolds. Then for every $L > 0$, the product cylinders*

$$\Omega_1 = [0, L] \times \Sigma_1 \,, \quad \Omega_2 = [0, L] \times \Sigma_2 \,,$$

equipped with the product metrics $dt^2 + h_1$ and $dt^2 + h_2$, are Steklov isospectral.

More sophisticated counterexamples were later constructed by Gordon, Herbrich, and Webb [GHW21], who produced broad families of Steklov isospectral non-isometric manifolds by adapting Sunada-type and torus-action methods to the Steklov setting.

The problem in a conformal class

As mentioned above, the main results obtained on the Steklov inverse problem restrict to the class of conformal metrics, that is, metrics of the form $h = cg$ where $c : M \to \mathbb{R}$ is a smooth positive function in M. In that case, the Steklov spectral

inverse problem boils down to showing that the map

$$h \mapsto \mathrm{spec}(\Lambda_h)$$

is injective on the space of metrics conformal to g on M. Of course the problem is already difficult since this map is highly non-linear and it is also possible to find many counterexamples, particularly when considering the aforementioned warped product manifolds (cf. [Gen20] and references therein). It should be noted that the adaptation of Sunada's method mentioned above (and more specifically of [BPY89, Theorem 1]) to obtain counterexamples is not known, but that could certainly be possible in dimension greater than 2, using [GHW21].

In the present work, we are working in dimension $n \geqslant 3$, the case of dimension 2 is specific due to the well-known conformal invariance of the DN map, namely

$$\Lambda_{cg} = \Lambda_g \,,$$

if

$$c_{|\partial M} \equiv 1 \,.$$

Consequently, the Steklov inverse problem in a conformal class then reduces, in dimension 2, to showing that two conformal and isospectral metrics coincide at the boundary. In the non-linear case, it is a priori as difficult (and not true in general) as in higher dimensions. This point is actually the most troublesome one in the boundary determination, since wave invariants do not provide this information. We show that the latter can be recovered, under much more restrictive geometric assumptions, by using more global spectral invariants, specifically the volume of the boundary (see Appendix IV.B).

We refer the reader to **Theorem I.1.1** below for the exact statement of the result proved. A linearized version of this Steklov inverse problem in a conformal class has also been considered, this is the object of the following.

I.1.2 The linearized problem: Steklov isospectral deformations

As already mentioned, a natural approach to address the Steklov inverse problem, and more generally spectral inverse problems, is to consider *wave invariants*. On a closed Riemannian manifold, the spectral invariants arising from the trace of the wave operator associated with any self-adjoint positive elliptic pseudodifferential operator are given by the trace formula of Duistermaat and Guillemin [DG75, Theo-

rem 4.5], providing a careful analysis of its singularities as a distribution in time. In the case of the Laplace-Beltrami operator, this formula was first established independently by Y.Colin de Verdière [CdV73a], [CdV73b] and then by Chazarain [Cha74b]. Note that the exploitation of these trace formulas requires important assumptions about the dynamics of the Hamiltonian flow associated with the principal symbol of the operator under consideration, e.g. geodesic flows in the case of the Laplacian and the DN map. For geodesic flows, a particularly appropriate framework is provided by the negative curvature assumption or, more generally, by the Anosov property (see Subsection III.0.1).

One of the most important examples of application is given by Guillemin and Kazhdan [GK80a], who were able to show the absence of non-trivial Laplace isospectral deformations (i.e one-parameter families (g_s) of smooth metrics preserving the Laplacian spectrum) on surfaces of negative curvature, and then in any dimension in [GK80b] (see also [CS98]).
Recently, Paternain, Salo and Uhlmann [PSU14] were able to extend these results to the case of Anosov surfaces. The restriction of this spectral inverse problem to isospectral deformations is actually a linearization and leads more precisely to a linear tomography problem as discussed above (cf. (I.0.1)). The latter asks whether it is possible to recover a tensor from its integrals along closed geodesics, which corresponds to an injectivity property of the so-called X-ray transform (see Section III.1.1). Inspired by this approach, this work also considers isospectral deformations for the Steklov spectrum :

Definition I.1.1. *A deformation of g is a smooth one-parameter family of (smooth) Riemannian metrics $(g_s)_{0\leqslant s\leqslant \varepsilon}$ on M such that $g_0 = g$. It is called Steklov isospectral if the spectrum of Λ_{g_s} is preserved :*

$$\forall s \in [0,\varepsilon] \ , \ \operatorname{spec}\left(\Lambda_{g_s}\right) = \operatorname{spec}\left(\Lambda_g\right) \ .$$

It is actually possible to establish, under natural assumptions, that isospectral deformations of conformal metrics are trivial in the sense that each metric automatically coincides at the boundary (see **Proposition IV.1.1**). But, on the other hand, the trace formula techniques do not allow us to prove such a result for the non-linear problem (i.e. considering two conformal metrics). However, as mentioned above, we point out that this can be shown under more restrictive assumptions on the geometry, exploiting the volume of the boundary as a spectral invariant, even if this is not very satisfactory in practice because very little spectral information is used, and

consequently the geometric assumptions required are relatively strong. We refer the reader to Appendix IV.B.

What lies behind this linearization phenomenon can be formulated more generally at the Hamiltonian level: the following lemma is the relevant first-variation statement.

Lemma I.1.2 (Hamiltonian first variation of the period). *Let (N, h) be a closed Riemannian manifold and let $(p_s)_{0\leq s\leq 1}$ be a smooth one-parameter family of positive Hamiltonian functions on $T^*N \setminus \{0\}$, homogeneous of degree 1 in the cotangent variable.*
For any s, let H_{p_s} be the Hamiltonian vector field defined by

$$\omega(H_{p_s}, \cdot) = dp_s \,,$$

where $\omega = d\alpha$ is the canonical symplectic form on T^*N. Let (γ_s) be a smooth family of periodic orbits of H_{p_s} [3], such that for any s, γ_s lies on the unit-energy hypersurface $\{p_s = 1\} \subset T^*N \setminus \{0\}$; also denote by T_s the period of γ_s. Then

$$\frac{\partial}{\partial t}\Big|_{t=s} T_t = \int_0^{T_s} \dot{p}_s(\gamma_s(t))\, dt.$$

In particular, if the periods are preserved along the deformation, then for any s,

$$\int_{\gamma_s} \dot{p}_s = 0 \,.$$

We refer to [GK80a, Theorem 2.1] and [DlLMM86, Proposition 1.7] (and the related discussions). See also [MR23].

In the Riemannian case $p_s(x, \xi) = |\xi|_{g_s}$, **Lemma I.1.2** essentially reduces to the usual first-variation formula for the marked length spectrum (cf. **Proposition III.2.3**). Thus, if the lengths are preserved, the infinitesimal deformation lies in the kernel of the geodesic X-ray transform (cf. Subsection III.2). This is the mechanism behind infinitesimal marked length rigidity and spectral rigidity.

The Steklov problem actually fits into the same framework. Let (M, g) be a smooth compact Riemannian manifold with boundary, and let Λ_g denote its Dirichlet-to-Neumann map. Then, as mentioned earlier, Λ_g is a positive self-adjoint elliptic pseudodifferential operator of order 1 on the closed manifold ∂M and its principal symbol

[3] $t \mapsto \gamma_s(t)$ is an integral curve of H_{p_s}.

coincides with that of $\sqrt{\Delta_{g^\circ}}$, namely

$$\sigma_1(\Lambda_g)(x,\xi) = |\xi|_{g^\circ} ,$$

Moreover, one has the symbolic expansion

$$\Lambda_g = \sqrt{\Delta_{g^\circ}} + B , \quad B \in \Psi^0(\partial M)$$

for smooth boundary. Thus, at principal order, the Hamiltonian dynamics underlying the Steklov problem are exactly the geodesic dynamics of the boundary metric g°. In particular, by the trace formula of Duistermaat–Guillemin applied to Λ_g, the singularities of the Steklov wave trace are located at the lengths of the closed geodesics of $(\partial M, g^\circ)$. This explains why Steklov isospectral deformations should be viewed as a boundary analogue of the Guillemin–Kazhdan approach.

There is, however, a crucial difference with the Laplace case: the subprincipal symbol of the Dirichlet-to-Neumann map is in general non-zero, whereas the corresponding subprincipal term for $\sqrt{\Delta_{g^\circ}}$ vanishes. This has two important consequences. On the one hand, unlike in the Laplace case, one have to make sure that there are no cancellations in the principal wave invariants, which leads naturally to the assumption that the boundary metric has simple length spectrum. On the other hand, this same non-zero subprincipal symbol contains genuinely new geometric information: in the problem for conformal metrics, it is precisely this term that makes it possible to recover the normal derivatives of the conformal factor at the boundary (cf. **Proposition IV.1.2**). We refer to Subsection II.3 for more details on this discussion.

I.1.3 Main results

This section gathers all the main spectral rigidity results obtained on the Steklov inverse problem for metrics. As noted above, these apply to metrics in the same conformal class. Note that for the non-linear problem, we need to require equality of the metrics at the boundary, since it is information that we are unable to obtain directly via the wave trace invariants and that is actually not verified in general.
The following result is established in Subsection IV.1.3 and more particularly in **Theorem IV.1.10**.

Theorem I.1.1 ([Flo25], Theorem I.2). *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with boundary. Assume that $(\partial M, g_{|\partial M})$ is Anosov with simple length spectrum.*

Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying $c_{|\partial M} \equiv 1$. If the Steklov spectra agree

$$\mathrm{spec}(\Lambda_{cg}) = \mathrm{spec}(\Lambda_g)\,,$$

then for all $j \geq 1$, $\partial_\nu^j c = 0$ on ∂M and the two operators coincide modulo a smoothing operator :

$$\Lambda_{cg} \equiv \Lambda_g\,.$$

In particular, if c is real-analytic and M is connected, then $c \equiv 1$ on M.

The proof relies on a recursive procedure (describe in Subsection I.4.1) using an adapted version of the aforementioned Duistermaat-Guillemin trace formula and the injectivity of the X-ray transform. Initialization, which requires the use of (non-Abelian) Livšic theory (cf. Subsection III.1.2), is dealt with in **Theorem IV.1.8**.
As already mentioned, note that it is possible to remove this condition of equality of metrics at the boundary by adding some relatively strong geometric assumptions, due to the lack of usable spectral invariants. This point is covered in Appendix IV.B.

For the linear problem, i.e., considering isospectral deformations, the major difference is that it is possible to establish that for all $s \in [0, \varepsilon]$,

$$\begin{cases} c_{s\,|\partial M} \equiv 1 \\ \partial_\nu c_{s\,|\partial M} \equiv 0 \end{cases} \tag{I.1.2}$$

by differentiating with respect to the parameter s (see **Proposition IV.1.1** and **Proposition IV.1.2**). Then, from there, everything else works the same way at higher order just as in the non-linear problem, so we actually get the following infinitesimal spectral rigidity result:

Theorem I.1.2 ([Flo25], Theorem I.3). *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with boundary. Assume that $(\partial M, g_{|\partial M})$ is Anosov with simple length spectrum. Let $(g_s)_{0\leq s\leq \varepsilon}$ be a Steklov isospectral deformation of the metric g s.t there exists a family of smooth functions $\left(c_s : M \to \mathbb{R}_*^+\right)_{0\leq s\leq \varepsilon}$ satisfying for any $s \in [0, \varepsilon]$,*

$$g_s = c_s g\,.$$

If the deformation map $s \mapsto c_s$ is real-analytic, then,

$$\forall s \in [0, \varepsilon]\ ,\ c_{s\,|\partial M} \equiv 1\,. \tag{I.1.3}$$

In addition, if (I.1.3) *is verified, then for all* $s \in [0, \varepsilon]$, *for all* $j \geq 1$, $\partial_\nu^j c_s = 0$ *on* ∂M *and the two operators coincide modulo a smoothing operator :*

$$\Lambda_{g_s} \equiv \Lambda_g \, .$$

In particular, if the conformal factors c_s *are real-analytic and* M *is connected, then* $c_s \equiv 1$ *on* M, *for all* $s \in [0, \varepsilon]$.

I.2 Recovering physical data from the Steklov spectrum

In this section, we are interested in spectral inverse problems for other Steklov operators that are, this time, parametrized by a smooth 1-form, while the metric is fixed.

The magnetic Schrödinger operator

Let (N, h) be a smooth closed[4] Riemannian manifold of dimension $n \geq 2$. A magnetic potential is a real-valued one form $a \in \Omega^1(N; \mathbb{R})$ and the corresponding magnetic field is the exterior differential $b = \mathrm{d}a \in \Omega^2(N; \mathbb{R})$. If $q \in C^\infty(N; \mathbb{R})$ is a smooth electric potential, we consider the magnetic Schrödinger operator

$$P_{a,q} = (\mathrm{d} + ia)^*(\mathrm{d} + ia) + q \, ,$$

where the adjonction is taken with respect to the scalar product on forms given by

$$(\chi | \omega) = \int_N \langle \chi, \bar{\omega} \rangle \, \mathrm{d}V_g = \int_N \chi \wedge * \bar{\omega} \, .$$

In local coordinates, with the classical notation $D_j := \frac{1}{i} \frac{\partial}{\partial x_j}$ and denoting $a = \sum_{i=1}^n a_j dx_j$, the magnetic Schrödinger operator reads

$$P_{a,q} = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^n \big(D_j + a_j\big)\Big(\sqrt{\det g}\, g^{jk}\big(D_k + a_k\big)\Big) + q \, .$$

If we define the domain of $P := P_{a,q}$ to be

$$D(P) = \big\{ u \in H^1(N) \, : \, Pu \in L^2(N) \big\} \, ,$$

[4] or even with boundary.

then $(P, D(P))$ is a self-adjoint elliptic operator, thus it has a compact resolvent and, therefore, only a pure point spectrum. Besides, since

$$(Pu|u) + c\|u\|^2 = \|\mathrm{d}u + ua\|^2 + ((q+c)u|u) \geq 0$$

with $c = \min q$, the eigenvalues are contained in the interval $[-c, \infty)$. Assuming that q is non-negative or, if necessary, translating, the spectrum consists in a sequence of eigenvalues

$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \ldots \to \infty\,.$$

I.2.1 The magnetic Steklov inverse problem

The magnetic Steklov spectrum

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 2$ with boundary and $a \in \Omega^1(M;\mathbb{R})$, $q \in C^\infty(M;\mathbb{R})$ respectively (smooth) magnetic and electric potentials on M. The magnetic Dirichlet-to-Neumann map (DN map) $\Lambda_{a,q}$ associated with A and q is the DN map associated with the magnetic Schrödinger operator :

$$\Lambda_{A,q} : \left| \begin{array}{ccc} H^{1/2}(\partial M) & \longrightarrow & H^{-1/2}(\partial M) \\ f & \longmapsto & (\mathrm{d}u + iau)\,|_{\partial M}\,(v) = (\partial_\nu u + i\langle a, \nu\rangle u)\,|_{\partial M} \end{array} \right. ,$$

where u is the unique solution in $H^1(M)$ of the Dirichlet problem

$$\begin{cases} P_{a,q}u = 0 \\ u_{|\partial M} = f \end{cases} ,$$

provided that 0 is not a Dirichlet eigenvalue of $P_{a,q}$. The magnetic DN map is a classical, elliptic and formally self-adjoint pseudodifferential operator of order 1 on the closed manifold ∂M (cf. [DSFKSU09] or [Cek20]).

Since the unbounded operator $\left(\Lambda_{a,q}\,,\, H^1(\partial M)\right)$ is self-adjoint elliptic, the spectrum of $\Lambda := \Lambda_{a,q}$ is discrete and is given by a sequence of eigenvalues

$$\sigma_1(\Lambda) \leq \sigma_2(\Lambda) \leq \cdots \to +\infty\,.$$

This sequence is called the magnetic Steklov spectrum, as it is equivalently the se-

quence of eigenvalues of the *magnetic Steklov eigenvalue problem* :

$$\begin{cases} P_{a,q}u = 0 \ \text{ in } M \\ \Lambda_{a,q}u = \sigma u \ \text{ on } \partial M \end{cases}.$$

Unlike the case of the magnetic Laplacian mentioned above, the literature on the Steklov spectrum in the presence of a magnetic field seems rather limited at the moment ; let us mention the very recent [CGHP24] — which is the analogue of Shigekawa's result [Shi87] for the magnetic Steklov lowest eigenvalue (see Subsection I.3 below) —, [PS25] on the direct problem, [LT23], [HN25] on the Steklov heat trace asymptotics, and [CS25] providing important spectral invariants in dimension 2.

Magnetic gauge invariance

As in the case of the Laplacian (cf. Subsection I.3 below), there is a natural gauge invariance on the spectral problem that is completely analogue, namely if $\vartheta \in C^\infty(M; \mathbb{S}^1)$ is a smooth map with values in the unit circle $\mathbb{S}^1 \subset \mathbb{C}$, then

$$\tilde{\Lambda} := \bar{\vartheta}|_{\partial M} \Lambda_{a,q} \vartheta|_{\partial M}$$

is a magnetic DN map with electric potential q and magnetic potential

$$\tilde{a} = a - i\bar{\vartheta}\mathrm{d}\vartheta$$

and obviously with the same spectrum as $\Lambda_{a,q}$. Indeed, a direct calculation shows that for any $f \in H^{1/2}(\partial M)$,

$$\begin{aligned} \widetilde{\Lambda} f &= \left(\mathrm{d}_a u + \bar{\vartheta}(\mathrm{d}\vartheta) u\right)|_{\partial M}(\nu) \\ &= \left(\mathrm{d}u + i(a - i\bar{\vartheta}\mathrm{d}\vartheta) u\right)|_{\partial M}(\nu) \\ &= \mathrm{d}_{\tilde{a}} u|_{\partial M}(\nu) \end{aligned}$$

where u denotes the unique solution in $H^1(M)$ of the Dirichlet problem

$$\begin{cases} P_{\tilde{a},q} u = 0 \\ u_{|\partial M} = f \end{cases}.$$

Therefore, from the magnetic Steklov spectrum one can only hope, at best, to recover the magnetic potential up to gauge, i.e. up to a term of the form $i\bar{\vartheta}\mathrm{d}\vartheta$ where $\vartheta \in$

$C^\infty(M;\mathbb{S}^1)$. In this work, we are only able to do it on the boundary provided the latter is Anosov with simple length spectrum, this is the object of the next part. Since the gauge is closed, it is therefore not an obstruction to the unique identifiability of the magnetic field at the boundary, as already mentioned.

Note that the only source of non-uniqueness in magnetic spectral inverse problems such as this one or the one discussed in Section I.3 arises from the natural gauge invariance on magnetic potentials. In particular, on manifolds with non-trivial topology, gauge-equivalent magnetic potentials may produce the same magnetic field while remaining spectrally indistinguishable. This phenomenon is closely related to the *Aharonov–Bohm effect* (see [AB59], [Esk15]), which represents the fact that a zero magnetic field does not necessarily imply the invisibility of the magnetic potential. Indeed, the spectrum only detects the magnetic fluxes modulo 2π (cf. **Corollary II.3.8** and **Lemma I.3.2**). On this subject, the reader may also consult [Hel88, Proposition 1.1] and the discussion therein.

I.2.2 Main results

Our most important result gives a boundary determination statement similar to that obtained in the metric case in **Theorem I.1.1**, namely the recovery of the jet at the boundary of an electric potential and a magnetic field from the spectrum of its magnetic Dirichlet-to-Neumann map, provided that the boundary is Anosov with simple length spectrum (see **Theorem I.2.2** below). As a simple corollary, an analytic magnetic field is entirely determined by its Steklov spectrum. To our knowledge, these are the first positive results for the magnetic Steklov inverse problem in a general framework. Once again, an important observation from our work is that, in contrast to the magnetic Schrödinger operator which is just a differential operator of order 2, the magnetic DN map is a classical pseudodifferential operator of order 1 whose full symbol, considered in suitable local coordinates, contains much more information about the potentials. More precisely, in boundary normal coordinates, each of its homogeneous terms of order $-j \leq -1$ involves the normal derivative at the boundary of both magnetic and electric potentials, respectively of order j and $j-1$ (see **Lemma IV.2.9**). Consequently, using the approach as for the metric problem, we are able to recover, modulo gauge, the full jet at the boundary of these potentials by induction on the order j. However, in this work, the procedure is much more delicate because of the gauge on the magnetic potential that must be managed at each step (cf. Subsection IV.2.2).

Provided that suitable gauge transformations have been fixed for each magnetic potentials (we will refer to these potentials as *normalized*, see **Remark** IV.2.4 and the discussion above it), we can first show :

Theorem I.2.1 ([dSFF26] , Theorem 1.1). *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with boundary and (a, q), $(\tilde{a}, \tilde{q})$ be two pairs of potentials in M such that a and $\tilde{a}$ are normalized. Assume that $(\partial M, g_{|\partial M})$ is Anosov with simple length spectrum.*
If the spectra of the magnetic Steklov operators $\Lambda_{\tilde{a},\tilde{q}}$ and $\Lambda_{a,q}$ coincide

$$\mathrm{spec}(\Lambda_{\tilde{a},\tilde{q}}) = \mathrm{spec}(\Lambda_{a,q}) ,$$

then the electric potentials at the boundary are equal,

$$\tilde{q}|_{\partial M} = q_{|\partial M}$$

and the magnetic potentials at the boundary are gauge-equivalent, namely there exists a smooth function $\vartheta \in C^\infty(\partial M, \mathbb{S}^1)$ with values in the unit circle such that

$$\tilde{a}|_{\partial M} = a_{|\partial M} - i\bar{\vartheta}\mathrm{d}\vartheta .$$

In particular, the magnetic fields coincide at the boundary

$$\mathrm{d}\tilde{a}|_{\partial M} = \mathrm{d}a|_{\partial M} .$$

As mentioned at the end of the introduction, it is actually possible to recover the full jet at the boundary of the magnetic field and of the electric potential from the Steklov spectrum, thereby obtaining identifiability in the real-analytic setting, this is the object of the next result. In the following, we will use the notation $\tilde{f} \simeq f$ to denote that $\tilde{f}$ and f have the same Taylor series at a point p in some local coordinates. Note that in **Theorem** I.2.2 below, we do not need to assume that magnetic potentials are normalized contrary to **Theorem** I.2.1 (cf. **Remark** IV.2.4).

Theorem I.2.2 ([dSFF26] , Theorem 1.2). *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with boundary and (a, q), $(\tilde{a}, \tilde{q})$ be two pairs of potentials in M. Assume that $(\partial M, g_{|\partial M})$ is Anosov with simple length spectrum.*
If the spectra of the magnetic Steklov operators $\Lambda_{\tilde{a},\tilde{q}}$ and $\Lambda_{a,q}$ coincide

$$\mathrm{spec}(\Lambda_{\tilde{a},\tilde{q}}) = \mathrm{spec}(\Lambda_{a,q}) ,$$

then in boundary normal coordinates at any point $p \in \partial M$,

$$\mathrm{d}\tilde{a} \simeq \mathrm{d}a \quad \textit{and} \quad \tilde{q} \simeq q\,.$$

In particular, if a, $\tilde{a}$ and q, $\tilde{q}$ are real-analytic and M is connected, then

$$\mathrm{d}\tilde{a} = \mathrm{d}a \quad \textit{and} \quad \tilde{q} = q\,.$$

I.3 An additional rigidity result for the magnetic Schrödinger operator

A result similar to **Theorem I.2.1** on the magnetic Steklov problem was also obtained in [dSFF26] for magnetic Schrödinger operators on closed Anosov manifolds. There, the Duistermaat–Guillemin trace formula is used exactly in the same way to extract magnetic information from the singularities of the wave trace. The magnetic Schrödinger operator indeed carries a non-zero subprincipal symbol essentially given by the magnetic potential (cf. IV.2.2).

Before stating the result, let us recall the magnetic gauge invariance for this problem.

A natural gauge invariance

Let (N, h) be a closed Riemannian manifold. There is a well-known and natural gauge invariance in the spectral inverse problem for the magnetic Schrödinger operator $P := P_{a,q}$, namely if $\vartheta \in C^\infty(N; \mathbb{S}^1)$ is a smooth map with values in the unit circle $\mathbb{S}^1 \subset \mathbb{C}$, then

$$\widetilde{P} = \bar{\vartheta} P \vartheta$$

is a magnetic Schrödinger operator with electric potential q and magnetic potential

$$\tilde{a} = a - i\bar{\vartheta}\mathrm{d}\vartheta$$

and obviously with the same spectral data as P. Indeed, writing $\mathrm{d}_a := \mathrm{d} + ia$ the magnetic exterior derivative, we have $P = \mathrm{d}_a^* \mathrm{d}_a + q$ and

$$\bar{\vartheta} P \vartheta = \bar{\vartheta} \mathrm{d}_a^* \vartheta \bar{\vartheta} \mathrm{d}_a \vartheta = (\bar{\vartheta} \mathrm{d}_a \vartheta)^* \bar{\vartheta} \mathrm{d}_a \vartheta\,,$$

where the conjugated factor is given by

$$\bar{\vartheta} \mathrm{d}_a \vartheta = \mathrm{d} + i(a - i\bar{\vartheta}\mathrm{d}\vartheta) .$$

Note that the one-form $i\bar{\vartheta}\mathrm{d}\vartheta$ is real-valued since

$$0 = \mathrm{d}|\vartheta|^2 = 2\,\mathrm{Re}(\bar{\vartheta}\mathrm{d}\vartheta) .$$

Remark I.3.1. *Since $p : t \mapsto e^{it}$ is a covering map, if N is simply connected, $\vartheta_*(\pi_1(N)) = \{0\} = p_*(\pi_1(\mathbb{R}))$. Consequently, one can lift the function ϑ, in other words, there exists $\psi \in C^\infty(N;\mathbb{R})$ a smooth real-valued function such that $\vartheta = e^{i\psi}$.*
However, an Anosov manifold is never simply connected (see [Kli74]) and the function ϑ cannot be lifted in general. In that case, as already mentioned at the end of the introduction of Subsection I.2.1, the gauge is related to the so-called Aharonov-Bohm effect, where one tries to recover the subprincipal terms $a - \tilde{a}$ from the spectral data on domains with nontrivial topology.

In other words, this is an obstruction to identifiability of the magnetic potential from the spectrum, and a gauge invariance that has to be taken into account in the spectral inverse problems. Nevertheless, the gauge is closed

$$\mathrm{d}(i\bar{\vartheta}\,\mathrm{d}\vartheta) = i\mathrm{d}\left(\frac{\mathrm{d}\vartheta}{\vartheta}\right) = 0$$

and is therefore not an obstruction to identifiability of the magnetic field $b = \mathrm{d}a$. Conversely, one can show that two magnetic potentials with the same magnetic fields are gauge-equivalent under some additional cohomological assumption. The following lemma recalls a suitable version of [Shi87, Proposition 3.1] :

Lemma I.3.2. *Let (N,h) be a closed Riemannian manifold, and $a, \tilde{a} \in \Omega^1(N)$ be two magnetic potentials. The following statements are equivalent :*

1. *the two magnetic potentials are gauge-equivalent, i.e. there exists $\vartheta \in C^\infty(N;\mathbb{S}^1)$ such that*
$$\tilde{a} = a - i\bar{\vartheta}\mathrm{d}\vartheta .$$
2. *the two magnetic fields are equal $\mathrm{d}a = \mathrm{d}\tilde{a}$ and*
$$\int_\gamma \pi_1^*(a - \tilde{a}) \in 2\pi\mathbb{Z}$$

for any closed orbit γ of the geodesic flow of N.

This result is a direct consequence of [Shi87, Proposition 3.1] and the fact that there is always at least one geodesic in each free homotopy class of a closed Riemannian manifold.

The boundary determination result

One of the epitome of spectral inverse results on magnetic operators is [Shi87] where it is shown that if the first eigenvalue of a magnetic Schrödinger operator is zero, that is, is equal to the first eigenvalue of a Schrödinger operator with trivial magnetic field, then the magnetic potential is zero modulo the natural gauge invariance. This can be seen as a spectral inverse problem with spectral datum the lowest eigenvalue. Recently, Cekić and Lefeuvre [CL25c] obtained that the spectrum of the Bochner Laplacian on closed manifolds with negative curvature and simple length spectrum determines a connection, up to gauge, under a low rank assumption. This is closely related to the framework of the magnetic Laplacian (cf. Section IV.2.3). Let us also mention the earlier results of [Gui90], [ER13] on the 2-torus. We are not aware of other positive results on spectral inverse problems for the magnetic Schrödinger operator in the case of closed Riemannian manifolds. Spectral inverse problems in the presence of a magnetic field have been studied quite extensively but rather in the case of manifolds with boundary, where the spectral data include information about the (Neumann) traces of the (Dirichlet) eigenfunctions in addition to the sequence of eigenvalues. This type of spectral inverse problems are usually referred to as Borg-Levinson theorems [BCDSF+21, Ser10] and are also related to boundary inverse problems [NSU95, DSFKSU09] (cf. survey [Sal06] on the subject).

The following result concerns the identifiability, up to a natural gauge, of the magnetic and electric potentials from the spectrum of their magnetic Schrödinger operator on a closed Anosov manifold, provided the length spectrum is simple. The latter assumption is known to be necessary, as there are counterexamples of isospectral electric potentials when the length spectrum is not simple (cf. [Kuw90], [Bro87]). In particular, our result extends a special instance of that obtained by Cekić and Lefeuvre on the magnetic Laplacian in the aforementioned [CL25c]:

Theorem I.3.3 ([dSFF26] , Theorem 1.3). *Let (N, h) be a closed Riemannian manifold of dimension $n \geq 2$ and (a, q), $(\tilde{a}, \tilde{q})$ be two pairs of potentials in N. Assume that the cogeodesic flow ϕ_t is Anosov with simple length spectrum.*

If the spectra of the magnetic Schrödinger operators $P_{a,q}$ and $P_{\tilde{a},\tilde{q}}$ coincide

$$\big(\lambda_k(P)\big)_{k\in\mathbb{N}} = \big(\lambda_k(\tilde{P})\big)_{k\in\mathbb{N}} ,$$

then the electric potentials are equal,

$$\tilde{q} = q$$

and the magnetic potentials are gauge-equivalent, namely there is a smooth function $\vartheta \in C^\infty(N, \mathbb{S}^1)$ with values in the unit circle such that

$$\tilde{a} = a - i\bar{\vartheta}\,\mathrm{d}\vartheta .$$

In particular, the magnetic fields coincide

$$\mathrm{d}\tilde{a} = \mathrm{d}a .$$

Note that this result also extends [Gui78b, Theorem 3(a)] to Schrödinger operators with non-zero magnetic potential.

I.4 A new DG–type trace formula: the key to boundary determination

In a paper of Guillemin [Gui78b] from 1978, a trace formula for a Schrödinger laplacian is established and led soon after to the unique determination of an electric potential q from the laplacian spectrum [GK80a]. Inspired by this work, we were able to derive a similar trace formula in a more general setting where the perturbation is no longer of order 0 and the perturbed operator can be any self-adjoint elliptic and classical pseudodifferential operator of order 1. In the following, we state a version of this trace formula which is both sufficient for our inverse problems and faithful to the precise contribution made here: the underlying asymptotic expansion is a consequence of the general trace formula of Duistermaat and Guillemin [DG75], so that our contribution is not the proof of the full expansion (which is a hard point), but rather the computation of its leading term:

Theorem I.4.1 ([Flo25, Theorem IV.2]). *Let (N, h) be a closed Riemannian manifold $k \in \mathbb{N}^*$, $A \in \Psi^1_{cl}(N)$ and $B \in \Psi^{-k}_{cl}(N)$ such that A and $A + B$ are self-adjoint operators.*

Assume A is elliptic. If A only admits a finite number of closed bicharacteristic[5] *curves $\gamma_1, ..., \gamma_r$ of period T and each of these curves are non-degenerate, then*

$$\mathrm{Tr}\left(e^{it(A+B)} - e^{itA}\right) \underset{t\to T}{\sim} \left(\sum_{j=1}^{r} \frac{|T_j^\sharp| e^{im_j\frac{\pi}{2}} e^{iT\overline{\gamma}_j}}{|\det\left(Id - \mathcal{P}_j\right)|^{1/2}}\, i\int_{\gamma_j} \sigma_B\right) C_k\,(t-T)^{k-1}\log\left(t-T+i0\right)\,,$$

where

$$\overline{\gamma_j} := \frac{1}{T}\int_{\gamma_j} \mathrm{sub}(A)\,,$$

and $\mathcal{P}_j$, $T_j^\sharp$, m_j, C_k are respectively the linearized Poincaré map, the primitive period, the Maslov index[6] *of γ_j, and C_k a non-zero universal constant depending on k.*

In particular, we will use the following consequence in practice:

Corollary I.4.2. *Under the assumptions of the theorem above, If A and $A+B$ have the same spectrum*

$$\mathrm{spec}(A+B) = \mathrm{spec}(A)\,,$$

then

$$\int_\gamma \sigma_B = 0\,.$$

for any closed orbit γ of the bicharacteristic flow of A.

Remark I.4.3. *Compare with the aforementioned [GK80a, Theorem 2].*

This is proved respectively in **Theorem II.4.2** and **Corollary II.4.4**, and will be the key ingredient in all the boundary determination results obtained in this work on the Steklov spectrum. From a heuristic perspective, by analogy with the aforementioned result of Guillemin and Kazhdan [GK80a] on the spectral determination of a 0- th order perturbation of the Laplacian, we wish here to study perturbations of the Dirichlet-to-Neumann map by an increasingly smooth pseudodifferential operator. This is in fact the general principle of the induction argument that allows one to recover the full boundary jet at the boundary of a metric or a potential. The following part is devoted to precise this idea a little bit more.

I.4.1 Brief description of the induction procedure

Consider two DN maps Λ and $\widetilde{\Lambda}$ that coincide at principal and subprincipal order (these points are treated separately in the linear problem through a deformation pa-

[5] cf. Subsection II.3. In our problems, these curves will essentially be geodesics.
[6] These indexes will be zero in our framework (cf. **Remark II.3.3** and **Remark II.3.5**).

rameter, and in the non-linear case through Livčic theory) and set $A := \Lambda \in \Psi^1_{cl}(N)$ and $B := \widetilde{\Lambda} - \Lambda \in \Psi^{-1}_{cl}(N)$. If A and $A + B$ are isospectral, one can then apply the previous corollary to obtain that

$$\int_\gamma \sigma_B = 0 \,.$$

From there, we use injectivity results on the geodesic X-ray transform to recover σ_B, and, in the best scenario, to show that σ_B is zero. In that case, we have shown that the order of B is now -2 and we continue by induction. At the end, we have then shown that $\widetilde{\Lambda}$ and Λ have the same full symbol in boundary normal coordinates and that

$$B := \widetilde{\Lambda} - \Lambda \in \Psi^{-\infty}_{cl}(N) \,.$$

The last step, if we manage to reach this point, is to use the boundary determination results of the Calderon problem ([LU89] in the metric case and [DSFKSU09], [Cek20] for the magnetic case) stating that the full symbol of the DN map, in boundary normal coordinates, completely determine the full jet of the object under consideration (metric, potentials).

I.5 A roadmap for the reader and a few notes

Roadmap

This manuscript is divided into four chapters, including this one. Let's start by outlining the structure of the next three chapters.

Outline of Chapter II: This chapter is intended to introduce and explain the concept of spectral invariants and to introduce the spectral invariants on which we rely in this work: the *wave invariants*. Subsection II.1 first provides an overview based on the *spectral distribution*, and then Subsection II.2 focuses more specifically on understanding the mechanisms behind spectral functions constructed by taking the trace of certain carefully chosen operators, notably the heat and wave operators. Important examples of trace formulas, predating those of Duistermaat-Guillemin, are discussed in subsection II.2.1. Finally, the last two subsections are devoted to the trace formulas used in this work. In Section II.3, we present the Duistermaat–Guillemin trace formula, and in Section II.4, we prove the new trace formula described in the previous section.

Outline of Chapter III: Here we provide the theoretical tools for converting wave invariants and their nature as dynamic integrals along the geodesic flow into information about the integrands. We begin with a brief review of the notion of Anosov flow, which is central for applying not only wave invariants but also the aforementioned theory. In Section III.1, we review the definitions and important results of Livšic theory, beginning in Subsection III.1.1 with the X-ray transform, which is the operator whose kernel this theory describes. In Subsection III.1.2, we recall Livšic's theorem for cocycles, which is useful for describing the subprincipal contribution given by the Duistermaat–Guillemin trace formula. Finally, Subsection III.1.3 deals with the analysis of the transport-type equation given by the previous theorem. Section III.2 focuses on the linearization of the metric problem via a deformation parameter. The central tool here is the marked length spectrum, whose definition is recalled in Subsection III.2.1, along with its linearization. We conclude, in Subsection III.2.2, by applying this linearization to the case of isospectral Steklov deformations, which allows us to highlight a recursive procedure useful for determining the conformal factor at the boundary in the linearized problem.

Outline of Chapter IV: Chapter IV essentially provides the proofs of all the main results concerning the inverse problems considered in this work. Section IV.1 focuses on metric inverse problems, beginning with the case of isospectral deformations of a conformal metric in Subsection IV.1.1, and then the possible approaches for dealing with the case of non-conformal metric deformations are discussed in Subsection IV.1.2. The following two subsections, IV.1.3 and IV.1.4, deal respectively with the non-linear problem in a conformal class and with perspectives for the non-conformal case. Finally, Section IV.2 focuses on the magnetic side: Subsection IV.2.1 first addresses the toy-model problem, in which we seek to recover an electric potential without a magnetic potential, and then a magnetic term is added in Subsection IV.2.2. We thus prove the boundary determination results obtained for the Schrödinger operator on a closed manifold and for the magnetic DN map. Finally, we discuss a possible generalization to the case of connections in Subsection IV.2.3.

The work is mainly based on the following two scientific articles :

1. [Flo25] *Steklov isospectrality of conformal metrics*, preprint, `arXiv:2501.15535`. Accepted for publication in **Annales de l'Institut Fourier**.

2. [dSFF26] *Magnetic spectral inverse problems on compact Anosov manifolds*, preprint, `arXiv:2601.12058`. Joint work with D. Dos Santos Ferreira.

An important remark

Remark I.5.1. *The following points are worth mentioning here :*

- *The Anosov assumption is made only on the boundary ; the entire manifold does not need to be Anosov, it can be anything. Moreover, the simple length spectrum assumption is satisfied by generic Riemannian metrics (cf. [Ano83], [Kli82, Lemma 4.4.3] and [Abr]); in other words, the set of Riemannian metrics with simple length spectrum is residual (countable intersection of dense open subsets) in the sets of metrics.*
  In particular, since the Anosov property is C^2-open with respect to the metric, one can always perturb an Anosov metric to a metric with simple length spectrum while staying Anosov, and the construction is generic.

- *The class of manifolds considered here seems rather special, although natural for the method, so it is important to note that one can easily construct such manifolds and there are plenty of them. As a simple example, take any closed Anosov manifold (N, h) of dimension $n \geq 2$ and consider a cylinder (see an example below with Figure I.3)*
  $$M := (N, h) \times \left([-1, 1], dt^2\right)$$
  equipped with the product metric. Once such a compact manifold with Anosov boundary has been constructed, one can perturb the ambient metric in M near the boundary so as to make the length spectrum simple while preserving the Anosov property.

- *In all the main results of this thesis, there is no need to assume boundary connectedness. First, we rely on wave trace formula techniques on closed manifolds, and these techniques do not require connectedness of the underlying manifold. It enters only at the dynamical stage of the argument, through the use of Livšic theory but this does not result in any real loss of generality for the main results: if the boundary consists of several connected components, the argument can simply be applied separately to each of them.*

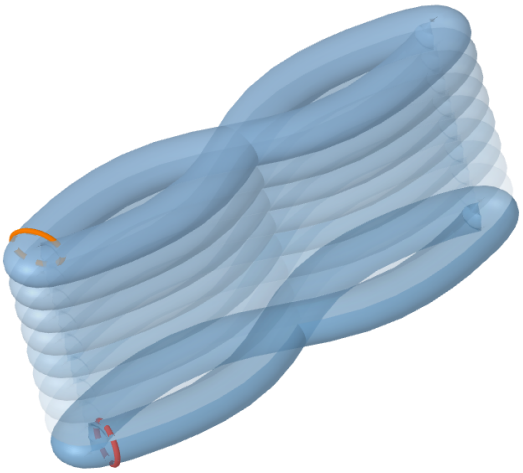


Figure I.3: Representation of a compact cylinder manifold N based on a hyperbolic surface of genus 2 with two periodic geodesics on the boundary.

I.6 Some definitions and notation

We collect here the main notation and definitions used throughout the text. For certain notions, we will provide references when we use them in the presentation.

Manifolds and boundary notation. A *closed manifold* will always mean a smooth compact manifold without boundary. We will denote by (N, h) a closed Riemannian manifold and by (M, g) a smooth compact Riemannian manifold with boundary. If $\iota : \partial M \hookrightarrow M$ is the canonical inclusion, we write

$$g^\circ := \iota^* g$$

for the metric induced on the boundary, and then $(\partial M, g^\circ)$ is a closed Riemannian manifold of dimension $n-1$, if n is the dimension of M. More generally, the symbol $^\circ$ will always denote restriction to ∂M. For instance, if $\Phi : M \to M$ is a diffeomorphism

preserving the boundary, we write for convenience

$$\mathring{\Phi} := \Phi|_{\partial M} .$$

If φ is a diffeomorphism, φ^* denotes the pull-back and φ_* the push-forward.

Some elements of Riemannian geometry. The Riemannian volume form associated with a metric h is denoted by $\mathrm{d}V_h$. We use the convention

$$\Delta_h u := \mathrm{div}_h(\nabla^h u) ,$$

so that the positive Laplace-Beltrami operator is $-\Delta_h$. In local coordinates,

$$\Delta_h u = \frac{1}{\sqrt{|h|}} \partial_i \Big(\sqrt{|h|}\, h^{ij} \partial_j u \Big) ,$$

where $|h| = \det(h_{ij})$, and we use Einstein's summation convention. For an Euclidean metric, it reduces to

$$-\Delta_h = -\sum_i \partial_i^2 ,$$

which is of course the usual positive laplacian.

When M has boundary, ν denotes the outward pointing unit normal vector field along ∂M. For $u \in C^\infty(M)$, the normal derivative is

$$\partial_\nu u := \mathrm{d}u(\nu) = \langle \nabla u, \nu \rangle_g \quad \text{on } \partial M .$$

Near the boundary, we shall use boundary normal coordinates defined as follows. If x' are local coordinates on ∂M, then x_n denotes the distance to the boundary measured along the inward normal geodesics. Thus $x_n \geq 0$ in M, $x_n = 0$ on ∂M, and locally

$$\partial M = \{x_n = 0\} .$$

In these coordinates, the metric has the form

$$g = \mathrm{d}x_n^2 + g_{x_n} ,$$

where g_{x_n} is a smooth family of Riemannian metrics on the hypersurfaces $\{x_n = \text{constant}\}$, with $g_0 = g^\circ$. Since x_n increases towards the interior of M, the outward unit normal vector field is $\nu = -\partial_{x_n}$ on ∂M.

Functional spaces. The space $L^2(N)$ is defined with respect to the Riemannian volume measure $\mathrm{d}V_h$. For $s \in \mathbb{R}$, we denote by $H^s(N)$ the Sobolev space on the closed manifold N. Using the positive operator $I - \Delta_h$, we may write

$$\|u\|_{H^s(N)} := \left\|(I - \Delta_h)^{s/2} u\right\|_{L^2(N)} .$$

More generally, if $E \to N$ is a smooth vector bundle, $H^s(N; E)$ denotes the Sobolev space of sections of E.

Hamiltonian and geodesic flows. We denote by

$$SN := \{(x, v) \in TN \ : \ |v|_h = 1\} , \quad S^*N := \{(x, \xi) \in T^*N \ : \ |\xi|_h = 1\}$$

the unit tangent and unit cotangent bundles. The canonical projection will be denoted by π. We use the musical isomorphisms

$$v^\flat := h(v, \cdot) , \quad \xi^\sharp := h^{-1}(\xi, \cdot) ,$$

to identify tangent and cotangent variables when convenient.

The geodesic flow on SN, or equivalently the cogeodesic flow on S^*N, is denoted by $(\varphi_t)_{t\in\mathbb{R}}$, and its infinitesimal generator by X or by H (on the cotangent). Thus, for a smooth function f,

$$Xf = \mathcal{L}_X f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \varphi_t .$$

More generally, if $p \in C^\infty(T^*N \setminus \{0\})$, we denote by H_p its Hamiltonian vector field with respect to the canonical symplectic structure on T^*N, and by

$$\Phi_t^P = \exp(tH_p)$$

the associated Hamiltonian flow. If P is a pseudodifferential operator, σ_P denotes its principal symbol.

Closed geodesics and length spectrum. We denote $\mathcal{G}_{\mathrm{per}}(N, h)$ for the set of non-constant closed geodesics of (N, h), including iterates. If $\gamma \in \mathcal{G}_{\mathrm{per}}(N, h)$ is parametrized on $[0, T]$, its length is

$$\ell_h(\gamma) := \int_0^T |\dot\gamma(t)|_h \, \mathrm{d}t .$$

If $\gamma : \mathbb{S}^1 \to N$ is a closed geodesic and $k \geq 2$, we define its k-fold iterate as the closed geodesic

$$\gamma^k(e^{it}) := \gamma(e^{ikt}) .$$

A closed geodesic is called *primitive* if it is not a k-fold iterate of another closed geodesic. The length spectrum of (N, h) is denoted by

$$\mathrm{Lsp}(N, h) := \{\ell_h(\gamma) : \gamma \in \mathcal{G}_{\mathrm{per}}(N, h)\} .$$

We may also use the set of periods

$$\mathcal{T}(N, h) := \pm \mathrm{Lsp}(N, h) = \{\pm\ell_h(\gamma) : \gamma \in \mathcal{G}_{\mathrm{per}}(N, h)\} .$$

Symmetric tensors and natural lifts. For $m \geq 0$, we denote by $S^m T^*N$ the bundle of symmetric covariant m-tensors. If

$$f \in C^\infty(N; S^m T^*N) ,$$

we denote by $\pi_m^* f$ its natural lift to the unit tangent bundle:

$$\pi_m^* f(x, v) := f_x(v, \dots, v) , \quad (x, v) \in SN.$$

Using the musical isomorphism, the same notation will also be used on the unit cotangent bundle:

$$\pi_m^* f(x, \xi) := f_x(\xi^\sharp, \dots, \xi^\sharp) , \quad (x, \xi) \in S^*N.$$

For $m = 0$, this reduces to the usual pull-back $\pi^* f = f \circ \pi$.

Classical pseudodifferential operators. We briefly recall some basics on pseudodifferential operators. We work, when convenient, with operators acting on half-densities. Thus, if $u, v \in C^\infty(N; \Omega_N^{1/2})$, the product $u\overline{v}$ is a density and the pairing

$$(u, v)_{L^2(N;\Omega^{1/2})} := \int_N u\overline{v}$$

is intrinsically defined. We denote by $\Psi^m(N; \Omega_N^{1/2})$ the space of pseudodifferential operators of order m acting on half-densities: A pseudodifferential operator of order m acting on half-densities on a closed manifold N is an operator

$$P : C^\infty(N; \Omega_N^{1/2}) \longrightarrow C^\infty(N; \Omega_N^{1/2})$$

which extends continuously to

$$P \,:\, \mathcal{D}'(N;\Omega_N^{1/2}) \longrightarrow \mathcal{D}'(N;\Omega_N^{1/2})\,.$$

and is given, in local coordinates and modulo a *smoothing*[7] operator, by an oscillatory integral

$$Pu(x) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}p(x,\xi)u(y)\,dy\,d\xi\ ,$$

where $p \in S^m$ is a symbol of order m. We denote the space of such operators by

$$\Psi^m(N;\Omega_N^{1/2})\,.$$

The class

$$\Psi^{-\infty}(N;\Omega_N^{1/2})$$

consists of smoothing pseudodifferential operators, equivalently operators with smooth Schwartz kernel.[8]

An operator $P \in \Psi^m(N;\Omega_N^{1/2})$ is called *classical*, and we write

$$P \in \Psi^m_{\mathrm{cl}}(N;\Omega_N^{1/2})\,,$$

if, in any local coordinate chart, its full symbol admits a polyhomogeneous asymptotic expansion

$$\sigma_P^{full}(x,\xi) \sim \sum_{j=0}^{+\infty} p_{m-j}\,, \quad |\xi| \to +\infty\,, \tag{I.6.4}$$

where p_{m-j} is positively homogeneous of degree $m-j$ in ξ for $|\xi| \gg 1$.
The principal symbol is given by

$$\sigma_P \,:=\, p_m\,.$$

For operators acting on half-densities, the subprincipal symbol is also invariantly defined; in local coordinates (x,ξ) it is given by

$$\mathrm{sub}(P) \,:=\, p_{m-1} - \frac{1}{2i}\sum_{j=1}^{n}\frac{\partial^2 p_m}{\partial x_j \partial \xi_j}\,.$$

Finally, note that if two classical pseudodifferential operators P and $\widetilde{P}$ of order m have

[7] see below.
[8] They map distributions of half-densities to smooth half-densities.

same symbolic asymptotic expansion then $\widetilde{P} - P$ is *smoothing*

$$\forall j \in \mathbb{N}\,,\ \tilde{p}_{m-j} = p_{m-j} \implies \widetilde{P} - P \in \Psi^{-\infty}(N)\,.$$

Wavefront set and singular support. The wavefront set of a distribution u in N is defined as follows:

$$(x_0, \xi_0) \notin \mathrm{WF}(u) \subset T^*N \setminus \{0\}$$

iff there exist a cutoff $\chi \in C_c^\infty(N)$, with $\chi \equiv 1$ near x_0, and a conic neighbourhood Γ of ξ_0 such that

$$|\widehat{\chi u}(\xi)| \leq C_N \langle \xi \rangle^{-N}$$

for every $N \geq 0$, uniformly for $\xi \in \Gamma$ ([Hör90, Ch. VIII]).

The singular support is the projection of the wave front set:

$$\mathrm{Suppsing}(u) = \pi(\mathrm{WF}(u))\,, \tag{I.6.5}$$

where $\pi : T^*N \setminus \{0\} \to N$ denotes the projection. The singular support $\mathrm{Suppsing}(u)$ records where the distribution u is singular, i.e. fail to be smooth, in the space variable, while $\mathrm{WF}(u)$ records both the space variable and the cotangent directions of the singularities.

Distributions in the trace formula. We refer specifically to [Hör90, Chapter III] for the following notions.

We shall use the standard notation $(t - T + i0)^\alpha$ for the boundary value, in the sense of distributions, of $(t - T + i\varepsilon)^\alpha$ as $\varepsilon \downarrow 0$, with the logarithm chosen in the upper half-plane. These distributions are smooth on $\mathbb{R} \setminus \{T\}$, but may be singular at $t = T$. The basic example, which is the main singular distribution in the wave trace asymptotics, is

$$(t - T + i0)^{-1} = \mathrm{p.\,v.}\,\frac{1}{t - T} - i\pi\delta_T\,.$$

Similarly for $j \geq 0$, $(t - T + i0)^j \log(t - T + i0)$ is defined by the same boundary value procedure. These logarithmic terms are more regular than $(t - T + i0)^{-1}$, but they are still singular at $t = T$. They appear as local models for the singularities of the wave trace at periods T of the Hamiltonian flow.

Chapter II

Looking for spectral invariants

Table of Contents for the chapter

II.1 Spectral functions and invariants **51**
II.2 The approach of trace formulas **56**
II.2.1 Towards the Duistermaat-Guillemin trace formula 57
II.3 The Duistermaat-Guillemin wave trace formula **60**
II.4 Trace asymptotics for perturbed wave propagators **67**

II.1 Spectral functions and invariants

As already mentioned, the study of spectral inverse problems relies on the identification of spectral invariants, that is, quantities that are entirely determined by the spectrum of the linear operator under consideration (Laplacian-type operator, DN map). In other words, they are quantities that remain unchanged under any transformation of the operator preserving its spectrum. Ideally, a set of spectral invariants have to encode as much information as possible about geometry of the underlying domain or Riemannian manifold. Such quantities are usually provided by packaging the eigenvalues into suitable "spectral generating functions", which collect the spectral data into a single analytic object. A natural way to formalize this idea is through the *spectral distribution* :

Let (N, h) be a closed Riemannian manifold of dimension $n \in \mathbb{N}^*$ and let $P \in \Psi^m_{\mathrm{cl}}(N)$ be a positive self-adjoint elliptic and classical pseudodifferential operator of order $m > 0$ in N. Its spectrum $\mathrm{spec}(P)$ consists in a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty \, .$$

This sequence can be naturally encoded by the *spectral distribution*

$$\mathrm{d}N_P = \mu_P := \sum_{j=1}^{+\infty} \delta(\lambda - \lambda_j)\,,$$

which defines a tempered distribution on $\mathbb{R}$ by

$$\langle \mu_P, \varphi \rangle = \sum_{j=1}^{+\infty} \varphi(\lambda_j)\,.$$

A first interesting function built from this model, known as the *spectral counting function*, is defined by

$$N_P(\lambda) := \#\{\lambda_j \leq \lambda\}\,,$$

and can be formally[1] written as

$$N_P(\lambda) = \int_{-\infty}^{\lambda} \mathrm{d}N_P(\mu) = \langle \mu_P, \mathbf{1}_{(-\infty,\lambda]} \rangle\,.$$

In this setting, the celebrated Weyl's law describes the high-energy asymptotics of the spectral distribution. Precisely, if $p_m(x,\xi)$ denotes the principal symbol of P, then

$$N_P(\lambda) = C_P\,\lambda^{n/m} + \underset{\lambda\to+\infty}{\mathcal{O}}\left(\lambda^{\frac{n-1}{m}}\right)\,,$$

where

$$C_P := (2\pi)^{-n}\mathrm{Vol}\{\,(x,\xi) \in T^*N \;:\; p_m(x,\xi) < 1\,\}$$

and the volume is taken with respect to the canonical Liouville measure on the cotangent bundle T^*N. This general form of Weyl's law follows from Hörmander's analysis of the spectral function for elliptic operators [Hör68]. The author may also refer to [GS94, Ch. 12], [Hör85b, Ch. 29], [Shu01, §13.3] and [Ivr98]. Applying this to the DN map $\Lambda_g \in \Psi^1_{cl}(N)$, with $(N,h) = (\partial M, g^\circ)$, we obtain the following distribution of Steklov eigenvalues :

Proposition II.1.1 (Weyl's law for Steklov eigenvalues).

$$\#(\sigma_k \leq \sigma) = \frac{\mathrm{Vol}(\mathbb{B}^{n-1})}{(2\pi)^{n-1}}\mathrm{Vol}_{g^\circ}(\partial M)\sigma^{n-1} + \underset{\sigma\to+\infty}{\mathcal{O}}(\sigma^{n-2})\,,$$

[1]The function $\mathbf{1}_{(-\infty,\lambda]}$ is not a test function, so the pairing (in the sense of distributions) is only formal in this case.

where $\mathbb{B}^{n-1}$ denotes the unit ball in $\mathbb{R}^{n-1}$.

Consequently, since two isospectral metrics in M have the same spectral counting function, they have same volume :

Corollary II.1.2. *Given any smooth compact Riemannian manifold with boundary, the volume of its boundary is a spectral invariant for the Steklov spectrum.*

Going back to the spectral distribution, it is standard that for appropriate functions φ one can define, via the functional calculus, the operator

$$\varphi(P) = \int_{\mathrm{spec}(P)} \varphi(\lambda)\, dE_P(\lambda)\,,$$

where E_P is the spectral resolution of P (see [RS72, §VIII.3]), and if φ decays sufficiently fast at infinity (for instance if $\varphi \in \mathcal{S}(\mathbb{R})$ or $\varphi \in C_c^\infty(\mathbb{R})$), then $\varphi(P)$ is trace class and its trace is given by

$$\mathrm{Tr}\,(\varphi(P)) = \sum_{j=1}^{+\infty} \varphi(\lambda_j) = \langle \mu_P, \varphi \rangle\,.$$

This identity provides a convenient way to construct "spectral functions" by suitable choices of φ. The most classical examples are summarized in Table II.1 below. The "special values" of such spectral functions then define a complete set of spectral invariants : Taking first $\varphi(\lambda) = e^{-t\lambda}$, we obtain the *heat trace*

$$Z_P(t) := \sum_{j=1}^{+\infty} e^{-t\lambda_j}\,, \quad t > 0\,,$$

which defines a smooth function and admits a short-time asymptotic expansion of the form

$$Z_P(t) \sim \sum_{k=0}^{+\infty} a_k(P)\, t^{(k-n)/m} + \sum_{l=1}^{+\infty} b_l(P)\, t^l \log(t)\,, \quad t \to 0^+\,.$$

The coefficients $a_k(P)$ and $b_l(P)$ are then spectral invariants of P known as its *heat invariants*. We refer to [GG98] and references therein for details. Note that for differential operators, the heat invariants simply reduce to the coefficients $a_k(P)$; see, for example, the discussion in [DG75, §2]. In the case of Laplacian operators, these coefficients are local invariants: there are given by integrals of universal polynomials in the jets of the metric, that is, of local geometric invariants. For instance, the

first two coefficients gives the volume and the total scalar curvature of the manifold (see [Gil04, Ch. 3], [LMP24, Ch. 6]). For more general Laplacian-type operators, the reader may wish to consult [Vas03]. The geometric quantities that appear in the heat trace expansion provide, in some cases, sufficient geometric constraints to ensure spectral rigidity. For example, in the case of the Laplacian, it can be shown that planar disks are spectrally determined among all domain with smooth boundary (cf. [Kac66]). We refer to Zelditch's survey [Zel04] and references therein for other examples and details.

In the case of first-order elliptic pseudodifferential operators such as the Dirichlet-to-Neumann map, the coefficients $a_k(P)$ are boundary spectral invariants that may be computed directly from the symbol and are local for $k = 0, ..., n-1$; this structure has been described in detail in [PS15], where the first coefficients were also calculated explicitly. They involve the boundary volume, the mean curvature, and more generally higher-order boundary curvature terms. Similar investigations were conducted in the presence of a magnetic potential (cf. [LT23] and [HN25]) .

Another interesting spectral function that can be considered is obtained by taking $\varphi(\lambda) = \lambda^{-s}$, which gives the *spectral zeta function*

$$\zeta_P(s) := \mathrm{Tr}(P^{-s}) = \sum_{j=1}^{+\infty} \lambda_j^{-s}\,, \quad \Re s > \frac{n}{m}\,.$$

This function is closely related to the heat trace by the Mellin transform

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Z_P(t)\, dt\,,$$

providing a meromorphic extension[2] of ζ_P to the complex plane $\mathbb{C}$ (see, for example, [Gil95, §1.10] and [DG75, §2]). If P is a differential operator, this Mellin formula, combined with the heat trace asymptotic, shows that ζ_P has simple poles at $s = \frac{n-j}{m}$ and

$$\operatorname*{Res}_{s=(n-j)/m} \Gamma(s)\zeta_P(s) = a_j(P)\,.$$

Hence the meromorphic part of ζ_P encodes exactly the same local spectral information as that contained in the heat trace asymptotic. We refer once again to [Gil95, §1.10] and [DG75, §2] for details. By contrast, the holomorphic part of ζ_P yield *global* (non-local) invariants, that is, invariants that cannot be expressed as the integral of

[2] See also Seeley's construction of complex powers [See67].

a local geometric invariant and that generally depend on the global geometry[3]. In particular, when $s = 0$ is not a pole, one defines the zeta-regularized determinant of P by

$$\det_\zeta P := e^{-\zeta_P'(0)},$$

which may be viewed as a regularization of the divergent product of the positive eigenvalues (see [RS71], [KV94]). The zeta-regularized determinant of the Laplacian has proven particularly useful for obtaining compactness results concerning classes of surfaces and Laplace-isospectral domains [OPS88a, OPS88b].
Regarding spectral zeta invariants for the Dirichlet-to-Neumann map, we refer once again to [PS15], which identify the Steklov heat invariants and compute the first coefficients of $Z_\Lambda(t)$, thereby describing the local invariants of ζ_Λ as (local) boundary geometric quantities. In [GG07], the zeta-regularized determinant of the DN map on compact surfaces with boundary is computed and related to dynamical zeta functions; to our knowledge, this remains one of the main reference for $\det_\zeta \Lambda$. In the case of planar domains, we refer to [Edw93], [MS15], [JS18], and [JS22], which has yielded interesting results regarding compactness. The author may consult, for example, [CGGS24] for a more complete and comprehensive overview of the literature.

Heat invariants, and thus local zeta invariants, are essentially computable local spectral invariants derived from the short-time heat expansion; however, they are often too restrictive to allow the determination of smooth metrics or domains, even though they have led to several remarkable rigidity results, as mentioned above (see the introduction of Chapter I). The main issue is that these invariants do not capture information about the underlying classical dynamics.
On the other hand, as we shall see in detail in Subsection II.3 below, this is not the case for wave invariants, which are therefore global in the sense mentioned above and are of particular interest for tackling spectral inverse problems, provided that it is possible to compute them. These invariants arise from the singularities of the wave trace

$$W_P(t) := \mathrm{Tr}(e^{-itP^{1/m}}) = \sum_j e^{-it\lambda_j^{1/m}},$$

as a distribution in t, which occur at the periods of the bicharacteristic flow of the principal symbol $p_m^{1/m}$ (cf. Subsection II.3 and the references therein). The reader will have noticed that the heat trace and the wave trace can be regarded, respectively, as the Laplace and Fourier transforms of the spectral measure $\mathrm{d}N_P$; the former in the classical sense, the latter in the sense of distributions (see Subsection II.3).

[3] In the sense that they are sensitive to small variations of the metric.

Function	$\varphi(\lambda)$	Expression	Invariants
$N_P(\lambda)$	$\mathbf{1}_{(-\infty,\lambda]}(\lambda)$	$\int_{(-\infty,\lambda]} \mathrm{d}N_P(\mu)$	geometric, local
$Z_P(t)$	$e^{-t\lambda}$	$\mathrm{Tr}\left(e^{-tP}\right)\ ,\ t>0$	geometric, local
$\zeta_P(s)$	λ^{-s}	$\mathrm{Tr}\left(P^{-s}\right)\ ,\ \Re s > \frac{n}{m}$	geometric, local /spectral, global
$W_P(t)$	$e^{-it\lambda^{1/m}}$	$\mathrm{Tr}\left(e^{-itP^{1/m}}\right)$	dynamical, global

Table II.1: Spectral functions obtained from the spectral distribution.

As explained at the beginning of Chapter I, wave invariants have been successfully used in various contexts to obtain a range of positive results for spectral inverse problems on the spectrum of Laplacian-type operators. We refer once again to the excellent surveys [CdV96],[Zel04], [Zel15], [DH13] for a more exhaustive presentation.

II.2 The approach of trace formulas

The spectral functions presented above are obtained by taking the trace of suitable operators, in particular the heat and wave operators, so as to encode the entire spectral data into a single generating function. The logical next step, and the fundamental idea behind the construction of spectral invariants, is to relate this generating function to geometric quantities. Ideally, one would like to recover enough geometric and dynamical information to reconstruct the domain or the manifold itself. This is precisely the role of trace formulas. The natural idea is to write, for a suitable function φ,

$$\mathrm{Tr}\left(\varphi(P)\right) = \int_N \mathrm{K}_{\varphi(P)}(x,x)\,\mathrm{d}V_h(x)\,, \tag{II.2.1}$$

where $K_{\varphi(P)}$ denotes the Schwartz kernel of the operator $\varphi(P)$. The main challenge is then to compute this integral exactly, or at least to obtain asymptotic expansions in which the geometry and dynamics of the underlying space appear explicitly. The most simple example of trace formula is probably given by the Poisson summation formula on the unit disk $\mathbb{S}^1$ (see Subsection II.2.1). What at first glance appears to be merely a nice duality becomes, with the Selberg trace formula [Sel56] on hyperbolic surfaces and later with the work of Colin de Verdière [CdV73a, CdV73b], a more profound relationship between eigenvalues and lengths of closed geodesics. Shortly after, with the Duistermaat–Guillemin formula [DG75, Theorem 4.5], it took the form of a much more general principle arising from the singularities of the wave trace.

The following aims to explain this progression, in Subsection II.2.1, before presenting, in detail in Subsection II.3, the Duistermaat-Guillemin trace formula, which lies at the heart of the approach developed in this work. Finally, a new version of this trace formula will be proved in Subsection II.4; this is one of our main results, **Theorem I.4.1**. As already mentioned in Chapter I, this result will prove decisive in establishing the framework necessary for recovering Taylor series at the boundary in the Steklov inverse problems studied in this thesis.

II.2.1 Towards the Duistermaat-Guillemin trace formula

The extraction of geometric information from spectral traces already appears in the study of the heat kernel of the Laplacian on a closed Riemannian manifold N. In their work [MP49], Minakshisundaram and Pleijel obtained the short-time asymptotic expansion of the heat kernel $K(t,x,y)$ near the diagonal of $\mathbb{R}^+ \times N \times N$. Then restricting to the diagonal and integrating over N immediately gave the aforementioned short-time asymptotic expansion of the heat trace

$$\mathrm{Tr}(e^{-t\Delta}) = \int_N K(t,x,x)\, dV_h(x)\,.$$

We also refer to McKean–Singer [MS67] on the subject. This shows a first mechanism by which geometric information may be revealed through the Schwartz kernel of an operator of the form (II.2.1). The mechanism we are interested in here also relies on asymptotic expansions, but near the singularities of the wave trace distribution, which naturally link them to periodic dynamics, as mentioned earlier.

The unit circle provides the simplest instance: Since the eigenvalues of the Laplacian $-\Delta_{\mathbb{S}^1} = -\mathrm{d}/\mathrm{d}x^2$ are given by

$$\lambda_k = k^2\,,\ k \in \mathbb{N}\,,$$

the distribution $\sum_{k\in\mathbb{Z}} e^{ikt}$ is identified [4] with the wave trace of $Q := \sqrt{-\Delta_{\mathbb{S}^1}}$

$$W_Q(t) = \sum_{k\in\mathbb{Z}} e^{\pm i\sqrt{\lambda_k}\, t}\,.$$

[4] up to a factor $\frac{1}{2}$

By the Poisson summation formula,

$$\sum_{k\in\mathbb{Z}} e^{ikt} = 2\pi \sum_{k\in\mathbb{Z}} \delta(t - 2\pi k)\,,$$

so the wave trace is singular precisely at the times $t = 2\pi k$, which are exactly the periods of closed geodesics in $\mathbb{S}^1$ i.e of circles. Of course, up to orientation, there is only one primitive closed geodesic (see the definition in Subsection I.6); it has a period of 2π, and the other singular times correspond to the periods of its iterates. In other words, the right hand side of the Poisson summation implies that the singular support of the wave trace is supported exactly at periods of closed geodesics. Inspired by the work of Colin de Verdière [CdV73a, CdV73b], this observation on the singular support was extended by Chazarain [Cha74b] to arbitrary compact Riemannian manifolds. Note, however, that unlike the Poisson formula on the unit disk, which gives an exact identity of distributions, the trace formulas of Colin de Verdière and Chazarain are local in nature: they provide an asymptotic expansion near periods of the geodesic flow, rather than providing an exact formula valid for all times t. We refer once again to Subsection II.3, and more precisely to the Duistermaat-Guillemin formula **Theorem II.3**. Before the results of Colin de Verdière [CdV73a, CdV73b] and Chazarain [Cha74b], Selberg had already obtained in [Sel56], an exact summation formula on locally symmetric spaces, which was interpreted by Huber [Hub59] as a trace formula relating the spectrum of the Laplacian of closed hyperbolic surfaces to the length of their closed geodesics. This interpretation by Huber is mentioned in [CdV07].

More precisely, let

$$N = \Gamma\backslash\mathbb{H}$$

be a closed hyperbolic surface, and let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$$

be the eigenvalues of the nonnegative Laplace-Beltrami operator on N. It is standard to write for any $j \in \mathbb{N}$,

$$\lambda_j = \frac{1}{4} + r_j^2\,,$$

and, of course, if

$$\lambda_j \geq \frac{1}{4}\,,$$

then $r_j = \sqrt{\lambda_j - \frac{1}{4}}$ is real, whereas if

$$0 \leq \lambda_j < \frac{1}{4} ,$$

then r_j is purely imaginary. The latter correspond to the so-called *small eigenvalues*, we refer to [Bus92, Chapter 8] and the references therein.

Selberg's trace formula states that for a suitable even test function h, with Fourier transform $\hat{h}$, one has

$$\sum_{j=0}^{+\infty} h(r_j) = \frac{\mathrm{Area}(N)}{4\pi} \int_{\mathbb{R}} r h(r) \tanh(\pi r)\, dr + \sum_{\substack{[\gamma] \in \mathcal{C}(\Gamma) \\ [\gamma] \neq [\mathrm{id}]}} \frac{\ell(\gamma_0)}{2\sinh(\ell(\gamma)/2)} \hat{h}\left(\ell(\gamma)\right) .$$

Here the second sum runs over the nontrivial conjugacy classes $[\gamma]$ in $\Gamma \simeq \pi_1(N)$. For a compact hyperbolic surface, every non-trivial conjugacy class determines a unique closed geodesic (see the discussion at the beginning of Chapter III). Such a closed geodesic γ is written as a k-fold iterate $\gamma = \gamma_0^k$ of the associated primitive closed geodesic γ_0, so that the sum at the right-hand side can also be written as

$$\sum_{\gamma \in \mathcal{P}} \sum_{k=1}^{+\infty} \frac{\ell(\gamma)}{2\sinh(k\ell(\gamma)/2)} \hat{h}\left(k\ell(\gamma)\right) ,$$

where $\mathcal{P}$ denotes the set of primitive closed geodesics. The integral term corresponds to the contribution of the identity class, while the sum gives the contributions of the non-trivial classes. For more details on Selberg's trace formula [Sel56], the reader may consult, for example, [Bus92, Chapter 9,§5], [Hej76b, Vol. 2], [Hej76a, §5], or [Mar12, §10].

We see that the Selberg trace formula relates the spectral parameters r_j, and so the eigenvalues of the Laplacian, to the lengths of closed geodesics in N. In other words, the Laplacian spectrum on a closed hyperbolic surface determines[5] the *length spectrum*. As mentioned above, this result is attributed to Huber [Hub59] and precedes the ones of Colin de Verdière [CdV73a, CdV73b] and of Chazarain [Cha74b]. The following section is devoted to a presentation of the Duistermaat-Guillemin formula [DG75, Theorem 4.5], which sets out the general principle underlying all the results mentioned above.

[5] In fact, a much stronger claim can be proven, see **Theorem I.0.1**.

II.3 The Duistermaat-Guillemin wave trace formula

The main idea we began to outline above is that the singularities of the wave trace are determined by the periodic trajectories of the underlying classical dynamics. The purpose of this section is to explain in more detail the reasoning behind this idea, formalized in its most general form by Duistermaat and Guillemin [DG75], and inspired by the work of Colin de Verdière [CdV73a, CdV73b] and Chazarain [Cha74b] as mentioned above.

Let (N, h) be a closed Riemannian manifold of dimension $n \in \mathbb{N}^*$ and let $P \in \Psi^1_{\mathrm{cl}}(N; \Omega_N^{1/2})$ be a positive elliptic self-adjoint classical pseudodifferential operator of order 1, acting on half-densities (see Subsection I.6). We denote once again by

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to +\infty$$

the spectrum of P, counted with multiplicity, and set

$$U(t) := e^{-itP}, \quad t \in \mathbb{R},$$

the unitary wave group generated by P (see, for example, [RS72, §VIII.3] or [Shu01, §20]), so that the wave trace is written

$$W_P(t) := \mathrm{Tr}\,(U(t)) .$$

As already mentioned, it is a tempered distribution in $\mathbb{R}$ (cf. [Hör68]) and the Fourier transform of the spectral measure

$$\langle \mu_P, \varphi \rangle = \sum_{j=1}^{+\infty} \varphi(\lambda_j) ,$$

so that two operators with the same sequence of eigenvalues have the same wave trace.

Since P is classical of order 1, its principal symbol $p := \sigma_P$ is a smooth function on $T^*N \setminus \{0\}$, which is positively homogeneous of degree 1 in the fiber variable:

$$p(x, \lambda\xi) = \lambda\, p(x, \xi) , \quad \lambda > 0 .$$

If ω denotes the canonical symplectic form on T^*N, the Hamiltonian vector field of

p is the vector field H_p on $T^*N \setminus \{0\}$, which is uniquely determined by

$$\omega\left(H_p, \cdot\right) = dp\,.$$

In local coordinates (x, ξ), this reads

$$H_p = \sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \sum_{j=1}^{n} \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\,.$$

The associated Hamiltonian flow is the one-parameter family of diffeomorphisms $\left(\Phi_t^P\right)$ defined by

$$\Phi_t^P := \exp(tH_p)\,, \quad t \in \mathbb{R}\,.$$

Definition II.3.1. *The Hamiltonian flow $\left(\Phi_t^P\right)$ defines the bicharacteristic flow of P and the integral curves of the Hamiltonian vector field H_p are called bicharacteristics*[6].

Since

$$\omega\left(H_p, H_p\right) = 0\,,$$

we see that the Hamiltonian function p is constant along every bicharacteristic; thus, in particular, the unit-energy hypersurface

$$\Sigma_P := \{(x,\xi) \in T^*N \setminus \{0\} \mid \ p(x,\xi) = 1\}$$

is invariant under the bicharacteristic flow. In our case, we have $p(x,\xi) = |\xi|_h$; therefore, throughout this work, we will always consider $\left(\Phi_t^P\right)$ as a flow acting on the unit cotangent bundle

$$\Sigma_P := S^*N$$

and consequently, that the bicharacteristics lie on this hypersurface.

In the following, we refer the reader to the standard general references regarding pseudodifferential/Fourier integral operators theory, for example: [Hör85a, Hör85b], [Dui73], [GS94], [Zwo12]; or even [Shu01], [RS72].

A first important result is the aforementioned Poisson relation, which identifies the times at which the wave trace may fail to be smooth :

Proposition II.3.2 (Poisson relation). *One has*

$$\text{Suppsing}\, W_P \subset \{T \in \mathbb{R} \mid \ \exists (x,\xi) \in \Sigma_P\,;\ \Phi_T^P(x,\xi) = (x,\xi)\}\,.$$

[6] or bicharacteristic curves.

In particular, for $P = \sqrt{\Delta_h}$,

$$\text{Suppsing}\, W_P \subset \{0\} \cup \{\pm L \mid L \in \text{Lsp}(N,h)\}\,.$$

In other words, the singularities of the wave trace are contained in the set of period of periodic orbits.

For the Laplacian on a closed manifold, this result can be attributed to Chazarain [Cha74b], who was inspired by the work of Colin de Verdière [CdV73a, CdV73b], as we mentioned earlier. In the more general setting of classical pseudodifferential operators, this is a direct consequence of the description of the wave propagator as a Fourier integral operator, developed by Hörmander and Duistermaat [Hör71, Dui73] and described in particular in [DG75, Cha74a]. In fact, it was this very tool that Chazarain relied on in the specific case of the Laplacian.

More precisely, for each $t \in \mathbb{R}$, [DG75, Theorem 1.1] shows that the Schwartz kernel $K_U(t,x,y)$ of U, viewed as a distribution on $\mathbb{R}_t \times N \times N$, is a *Lagrangian distribution* of class $I^{-\frac{1}{4}}(\mathbb{R}\times N\times N\,; C)$ with *canonical relation*

$$\begin{aligned} C \;:= \{\,&(t,\tau),(x,\xi),(y,\eta) \mid (x,\xi),(y,\eta) \in T^*N \setminus \{0\},\\ &(t,\tau) \in T^*\mathbb{R}\setminus\{0\},\, \tau + p(x,\xi) = 0\,,\, (x,\xi) = \Phi_t^P(y,\eta)\,\}\,. \end{aligned}$$

In particular, the wavefront set of $K_U(t,x,y)$ is contained in the twisted canonical relation :

$$\text{WF}(K_U) \subset C',$$

where

$$\begin{aligned} C' \;:= \{\,&(t,\tau),(x,\xi),(y,-\eta) \mid (x,\xi),(y,-\eta) \in T^*N \setminus \{0\},\\ &(t,\tau) \in T^*\mathbb{R}\setminus\{0\},\, \tau + p(x,\xi) = 0\,,\, (x,\xi) = \Phi_t^P(y,\eta)\,\}\,. \end{aligned}$$

In addition, it is easy to show that

$$W_P(t) = \pi_*\Delta^*\nu\,,$$

where Δ^* is the pull-back by the diagonal map $\Delta : \mathbb{R}\times N \to \mathbb{R}\times N\times N$ and π_* is the push-forward by the projection map $\pi : \mathbb{R}\times N \to \mathbb{R}$ (see for example [DG75, Lemma 6.4]).
Microlocally, the pull-back Δ^* identifies the two space variables, so that only those points in $\text{WF}(K_U)$ with $x = y$ contribute: After restriction to the diagonal, a covector $(t,\tau;x,\xi;y,-\eta)$ becomes $(t,\tau;x,\xi-\eta)$. Then the action of the push-forward by π on

the wavefront set implies that a singularity of W_P at time t can occur only if $\xi - \eta = 0$. In other words, by putting everything together, we obtain

$$\mathrm{WF}(W_P) \subset \{\, (t,\tau) \in T^*\mathbb{R} \setminus \{0\} \mid \exists (x,\xi) \in T^*N \setminus \{0\}\,;\, (x,\xi) = \Phi_t^P(x,\xi) \text{ and } \tau < 0 \,\}\,.$$

Thus, by projecting onto the first variable (see (I.6.5)), we obtain the desired inclusion (**Proposition II.3.2**) of the singular support of W_P. In other words, the singular support of the wave trace is contained in the set of periods of the bicharacteristic flow on Σ_P.[7]

The natural next question is whether the other inclusion holds: when does a period of the bicharacteristic flow actually produce a singularity of the wave trace ? Equivalently, when do we have equality in the Poisson relation ? This question is much more complex and an answer does not follow directly from Hörmander's theory of propagations of singularities for FIOs, as was the case for the Poisson relation. In general, although each singularity in the wave trace must occur at a period of the bicharacteristic flow, the converse is not necessarily true, since cancellations can occur between the contributions of distinct periodic orbits having the same period (see Zelditch's surveys [Zel15] and [Zel04]). This statement will become clear once we recall the Duistermaat-Guillemin wave trace formula. The version that we will give in the following is in the case where all periodic bicharacteristics γ of period T_γ are non-degenerate:

$$\det\left(Id - \mathcal{P}_\gamma\right) \neq 0\,,$$

where $\mathcal{P}_\gamma$ denotes the linearized Poincaré return map of the closed orbit γ, i.e. the linear application induced by $d\phi^P_{T_\gamma}$ in $T\left(S^*N\right)/\mathbb{R}H_p$.
In the Riemannian case, it is easy to see that this property holds if the underlying manifold N is Anosov (see Subsection III.0.1), since in this case $\mathcal{P}_\gamma$ may be identified with

$$d\phi^P_{T_\gamma}\Big|_{E_s \oplus E_u}\,.$$

In what follows, we consider the case where all periodic bicharacteristics are non-degenerate.

Using the global symbolic calculus of Fourier integral operators Duistermaat and Guillemin proved [DG75] that any such periodic bicharacteristic of period $T_\gamma \in \mathcal{T}$ contributes to a singularity of $W_P(t)$ through a one-dimensional Lagrangian distribu-

[7] Given that p is of homogeneous order 1, it is indeed equivalent to consider the dynamics on Σ_P.

tion $e_T(t)$ that admits an asymptotic expansion of the form

$$e_T(t) \underset{t\to T}{\sim} c_{T,-1}(t-T+i0)^{-1} + \sum_{j=0}^{+\infty} c_{T,j}(t-T+i0)^j \log(t-T+i0)\,.$$

The leading coefficient is called the *principal wave invariant* at $t = T$ and expressed as

$$c_{T,-1} = \sum_{j=1}^{r} \frac{|T_j^\sharp| e^{im_j\frac{\pi}{2}} e^{-iT\overline{\gamma_j}}}{|\det\left(Id - \mathcal{P}_j\right)|^{1/2}}\,,$$

where

$$\overline{\gamma_j} := \frac{1}{T}\int_{\gamma_j} \mathrm{sub}(P)\,,$$

and $\mathcal{P}_j$, $T_j^\sharp$, m_j are respectively the linearized Poincaré map, the primitive period and the Maslov index of γ_j.

Remark II.3.3. *In the cases considered in this work, the Maslov index m_j appearing in this result will be the Morse index of the periodic geodesic γ_j (or more precisely here the projection of γ_j). Recall that these indices are zero in the case where the underlying manifold N is Anosov (see* ***Remark II.3.5****).*

Application to the DN map

Exactly the same formalism applies to DN maps on a smooth compact Riemannian manifold (M, g) with boundary:

Let $P := \Lambda \in \Psi^1_{\mathrm{cl}}(\partial M)$ be either the usual DN map Λ_g or the magnetic one $\Lambda_{a,q} = \Lambda_{g,a,q}$. It is self-adjoint elliptic and its principal symbol is given by

$$\sigma_\Lambda(x,\xi) = |\xi|_{g^\circ}\,.$$

Therefore the bicharacteristic flow of Λ lives in the usual unit cotangent bundle $S^*\partial M$ and is precisely the geodesic flow of the boundary metric. Applying the above trace formula to Λ yields:

Theorem II.3.4 (Duistermaat-Guillemin, [DG75]). *Suppose that $(\partial M, g^\circ)$ only admits a finite number of closed geodesics $\gamma_1, ..., \gamma_r$ of period T and that each of these geodesics*

is nondegenerate[8].
Then

$$W_\Lambda(t) := \mathrm{Tr}\,(U(t))$$

is a tempered distribution on $\mathbb{R}$ *admitting an isolated singularity in* $t = T$ *given by a Lagrangian distribution* e_T *with*

$$\mathrm{Suppsing}\,(e_T) \subseteq \{T\}$$ [9]

and complete asymptotic expansion

$$e_T(t) \underset{t\to T}{\sim} c_{T,-1}(t - T + i0)^{-1} + \sum_{j=0}^{+\infty} c_{T,j}(t - T + i0)^j \log(t - T + i0)\,.$$

The leading coefficient is expressed as

$$c_{T,-1} = \sum_{j=1}^{r} \frac{|T_j^\sharp| e^{im_j \frac{\pi}{2}} e^{-iT\overline{\gamma_j}}}{|\det\left(Id - \mathcal{P}_j\right)|^{1/2}}\,,$$

where

$$\overline{\gamma_j} := \frac{1}{T}\int_{\gamma_j} \mathrm{sub}(\Lambda)\,.$$

Remark II.3.5. *The Morse index* m_γ *of a closed geodesic* γ *is the maximal dimension of a vector subspace on which the second-order differential of the energy functional at* γ *is negative definite. We refer to [Dui76] for details. It is zero for any non-degenerate closed orbit with no conjugate point, which is the case in particular for all closed geodesics of an Anosov manifold. We refer again to [Dui76] and, for a proof of this result, the reader can consult the aforementioned [GM]. We also refer to [Kli82] and more generally to [PP94].*

The leading term $c_{T,-1}$ in the previous theorem, called the principal wave trace invariant at $t = T$, is the leading singular contribution. In particular, if this term is canceled, then the corresponding period T may or may not be a singularity, we don't know.
Consequently, to take full advantage of this theorem, we systematically consider the case where $(\partial M, g^\circ)$ is Anosov with the generic assumption that the length spectrum

[8] This is the case if the manifold is Anosov (see, for example, the recent [GM] and references therein).

[9] Equality holds if at least one coefficient in the following singular expansion is non-zero.

is simple (i.e. $r = 1$ in the theorem), so that this contribution is not canceled out.
It is then possible to formulate a result analogous to [GK80a, Theorem 3] in our context, highlighting two spectral invariants :

Corollary II.3.6. *In the class of Riemannian metrics g such that g° is Anosov with simple length spectrum, the Steklov spectrum determines :*

(a) the lengths of the closed geodesics on the boundary.

(b) for any period $T \in \mathcal{T}$, the quantity

$$c_{T,-1} = \frac{|T^\sharp| exp\left(-i \int_{\gamma_{g^\circ}} \mathrm{sub}(\Lambda)\right)}{|\det\left(Id - \mathcal{P}_{\gamma_{g^\circ}}\right)|^{1/2}} .$$

Remark II.3.7. *As already mentioned, in the standard case of the Laplacian on a closed manifold, the subprincipal symbol is zero and consequently the Anosov property (or the negative curvature in the case of [GK80a, GK80b]) is sufficient to ensure, in this context, the point (a) of the previous corollary. Indeed in this case, given the expression of the principal wave trace invariant, no cancellation takes place in the wave trace formula i.e. every period contributes to a singularity.*
In our case however, the subprincipal symbol is non-zero and contributes the additional phase

$$exp\left(-i \int_\gamma \mathrm{sub}(\Lambda)\right)$$

along the periodic orbit γ. This forces us to ensure that there are no such cancellations by adding the simple length spectrum assumption. On the other hand, it also provides additional information, namely point (b).

The important formulation to keep in mind is that, under suitable dynamical assumption on the geodesic flow, it leads to exploitable spectral invariants at subprincipal level:

Corollary II.3.8. *Let Λ and $\widetilde{\Lambda}$ be DN maps (resp. magnetic DN maps) on (M, g) and $(M, \tilde{g})$, with the same spectrum. If $\tilde{g}^\circ = g^\circ$ (resp. $\tilde{g} = g$), then under the above assumptions on the metric g°,*

$$\int_\gamma \left(\mathrm{sub}(\Lambda) - \mathrm{sub}(\widetilde{\Lambda})\right) \in 2\pi\mathbb{Z} ,$$

for any closed orbit γ of the geodesic flow of ∂M.

Remark II.3.9. *The same statement, of course, holds for Schrödinger operators (magnetic or not) on closed Anosov manifolds with simple length spectrum*

II.4 Trace asymptotics for perturbed wave propagators

This section is based on the first part of [Flo25, §IV] and is devoted to prove **Theorem I.4.1**. Let us consider a classical self-adjoint elliptic pseudodifferential operator

$$A \in \Psi^1_{\mathrm{cl}}(N)$$

and a self-adjoint lower-order perturbation

$$B \in \Psi^{-k}_{\mathrm{cl}}(N)\,, \quad k \geq 1\,.$$

By [DG75, Theorem 1.1], the Schwartz kernels of the associated propagators e^{itA} and $e^{it(A+B)}$, viewed as distributions on $\mathbb{R} \times N \times N$, are Lagrangian distributions of class $I^{-\frac{1}{4}}(\mathbb{R} \times N \times N\,; C)$ defined by the same canonical relation

$$\begin{aligned} C := \{&(t,\tau),(x,\xi),(y,\eta) \mid (x,\xi),(y,\eta) \in T^*N \setminus \{0\}, \\ &(t,\tau) \in T^*\mathbb{R} \setminus \{0\},\, \tau + \sigma_A(x,\xi) = 0\,,\, (x,\xi) = \Phi^A_t(y,\eta)\}\,. \end{aligned}$$

In other words,

$$U_0(t) := e^{itA} \quad , \quad U(t) := e^{it(A+B)}$$

are Fourier integral operators with the same canonical transformation, namely the bicharacteristic flow Φ^A_t of A. The natural object for comparing the two evolutions is therefore the so-called *return operator*

$$R(t) := U(t)U_0(-t)\,.$$

Heuristically, one first evolves according to the perturbed dynamics, then we comes back with the unperturbed one and since the two propagators have the same canonical relation, by composition, $R(t)$ defines a Fourier integral operator with the identity as canonical transformation and with polyhomogeneous symbol of order 0, in other words :

$$\forall t \in \mathbb{R}\,, \quad R(t) \in \Psi^0_{cl}(N)\,.$$

Basically, $R(t)$ is the "microlocal defect" between the two evolutions after eliminating the common principal transport term. This is the point of view adopted by Guillemin [Gui78b] in the particular case where A is the square root of the laplacian and B is a zero-order perturbation, that is, a potential.

The following result is therefore a generalization of [Gui78b, Theorem 3] :

Theorem II.4.1. *Let $k \in \mathbb{N}^*$, $A \in \Psi^1_{cl}(N)$ and $B \in \Psi^{-k}_{cl}(N)$ self-adjoint operators. Then*

$$W(t) := R(t) - Id$$

is a pseudodifferential operator of order $-k$ on N and its principal symbol is given by :

$$\sigma_{W(t)}(x,\xi) = i \int_{\gamma^t(x,\xi)} \sigma_B \quad ,$$

where $\gamma^t(x,\xi) : [0,t] \ni s \mapsto \Phi^A_s(x,\xi)$ is the bicharacteristic arc of length t passing through (x,ξ).

Proof. Differentiating in t the expression of $W(t)$, we get

$$\begin{aligned} -i\dot{W}(t) &= (A+B)e^{it(A+B)}e^{-itA} - e^{it(A+B)}Ae^{-itA} \\ &= (A+B)R(t) - R(t)A \\ &= (A+B)\left(W(t)+Id\right) - \left(W(t)+Id\right)A \\ &= AW(t) - W(t)A + B + BW(t)\,, \end{aligned}$$

where the second equality comes from the fact that A and e^{-itA} commute.
In other words,

$$\begin{cases} -i\dot{W}(t) = [A, W(t)] + B + BW(t) \\ W(0) = 0 \end{cases} \tag{II.4.2}$$

Then let us note $w(t)$ the full symbol of $W(t) \in \Psi^0_{cl}(N)$ in any local coordinates and

$$w(t) \sim \sum_{j=0}^{+\infty} w_{-j}(t)\,. \tag{II.4.3}$$

If b denotes the full symbol of $B \in \Psi^{-k}_{cl}(N)$ in the same local coordinates, we also note

$$b \sim \sum_{j=0}^{+\infty} b_{-k-j}\,. \tag{II.4.4}$$

Inserting (II.4.3) and (II.4.4) into (II.4.2) and collecting homogeneous terms with same order $-j$, one can access the leading term in the symbol of $W(t)$:

- For $j \in \{0, ..., k-1\}$, we show by induction that $w_{-j}(t) = 0$: First, $w_0(t)$ satisfies

$$-i\dot{w}_0(t) = i^{-1} H_{\sigma_A} w_0(t) \,,$$

so that

$$\begin{cases} \dot{w}_0(t) = H_{\sigma_A} w_0(t) \\ w_0(0) = 0 \end{cases}$$

and then we can write

$$w_0(t) = \exp\left(tH_{\sigma_A}\right) w_0(0) = 0 \,.$$

Next for any $j \in \{0, ..., k-2\}$, if we assume that $w_{-j}(t) = 0$, we obtain in the same way

$$w_{-j-1}(t) = \exp\left(tH_{\sigma_A}\right) w_{-j-1}(0) = 0 \,.$$

- For $j = k$, we obtain :

$$-i\dot{w}_{-k}(t) = i^{-1} H_{\sigma_A} w_{-k}(t) + b_{-k} + b_{-k} w_0(t) \,,$$

and since $w_0(t) = 0$,

$$\begin{cases} \dot{w}_{-k}(t) = H_{\sigma_A} w_{-k}(t) + ib_{-k} \\ w_{-k}(0) = 0 \end{cases}$$

Then we can apply Duhamel's principle to get

$$\begin{aligned} w_{-k}(t) &= \exp\left(tH_{\sigma_A}\right) w_{-k}(0) + i \int_0^t \exp\left((t-s)H_{\sigma_A}\right) b_{-k} \, ds \\ &= i \int_0^t \sigma_B \circ \Phi_s^A \, ds \end{aligned}$$

□

In the same spirit as [Gui78b, Theorem 4], one can then formulate a proof to **Theorem I.4.1**. Going back to the heuristic point of view mentioned above, this result shows that the microlocal defect $R(t)$ contributes to the singularities of the "perturbed" wave trace

$$S(t) := \mathrm{Tr}\left(U(t) - U_0(t)\right)$$

essentially through linear terms, given by the integrals of the principal symbol σ_B

over periodic bicharacteristics of A. We will note that the complete asymptotic result relies on the arguments of Duistermaat-Guillemin. As mentioned in Subsection I.4, our main contribution consists solely in calculating the leading term in this framework:

Theorem II.4.2. *Let (N,h) be a closed Riemannian manifold, $k \in \mathbb{N}^*$, $A \in \Psi^1_{cl}(N)$ and $B \in \Psi^{-k}_{cl}(N)$ such that A and $A+B$ are self-adjoint operators. Assume A is elliptic. If A only admits a finite number of closed bicharacteristic curves $\gamma_1, ..., \gamma_r$ of period T and each of these curves are non-degenerate, then*

$$S(t) := \mathrm{Tr}\left(U(t) - U_0(t)\right)$$

is a tempered distribution on $\mathbb{R}$ admitting an isolated singularity in $t = T$ given by a Lagrangian distribution e_T with

$$\mathrm{Suppsing}\left(e_T\right) \subseteq \{T\}$$

and complete asymptotic expansion

$$e_T(t) \underset{t\to T}{\sim} \sum_{j=0}^{+\infty} c_{T,k-1+j}\,(t-T)^{k-1+j}\log(t-T+i0)\,.$$

The leading coefficient is expressed [10] *as*

$$c_{T,k-1} = \sum_{j=1}^{r} \frac{|T_j^\sharp| e^{im_j\frac{\pi}{2}} e^{iT\overline{\gamma_j}}}{|\det\left(Id - \mathcal{P}_j\right)|^{1/2}}\, i\int_{\gamma_j} \sigma_B\,,$$

where

$$\overline{\gamma_j} := \frac{1}{T}\int_{\gamma_j} \mathrm{sub}(A)\,,$$

and $\mathcal{P}_j$, $T_j^\sharp$, m_j are respectively the linearized Poincaré map, the primitive period and the Maslov index of γ_j.

Proof. The proof is similar to that of [Gui78b, Theorem 4], it uses the same arguments as [DG75, Theorem 4.5] on the symbolic calculus for Fourier integral operators.
For the sake of convenience, let us briefly review the main points in our setting : As we pointed out in the introduction of this section, if we denote $\mu_0(t,x,y)$ the Schwartz

[10] up to a non-zero universal constant depending on k.

kernel of $U_0(t)$,

$$\mu_0 \in I^{-\frac{1}{4}}(\mathbb{R}\times N\times N, C)\ .$$

Moreover, by **Theorem II.4.1**, $W(t)\in\Psi^{-k}(N)$ and consequently, the distribution

$$U(t)-U_0(t)=W(t)U_0(t)$$

defines a Fourier integral operator ; more precisely its Schwartz kernel v is a Lagrangian distribution of order $-k-\frac{1}{4}$ on $\mathbb{R}\times N\times N$:

$$v\in I^{-k-\frac{1}{4}}(\mathbb{R}\times N\times N, C)\ .$$

In addition, it is easy to show that $S(t)=\pi_*\Delta^* v$, where Δ^* is the pull-back by the diagonal map $\Delta : \mathbb{R}\times N\to\mathbb{R}\times N\times N$ and π_* is the push-forward by the projection map $\pi : \mathbb{R}\times N\to\mathbb{R}$ (see for example [DG75, Lemma 6.4]).
Then the theorem follows in the same way as [DG75, Section 6] from *clean intersection theory* developed in [DG75, Section 5], which can be applied here by assumption on the periodic orbits $\gamma_1,...,\gamma_r$.
At this point, it should be noted that, compared to the standard Duistermaat-Guillemin formula, the order of the amplitude is shifted by k in the sense that the corresponding one-dimensional Lagrangian distribution has, near $t=T$, an oscillatory integral representation of the form

$$e_T(t)=\int_{-\infty}^{+\infty} e^{i(t-T)\tau}a_T(\tau)\,d\tau \quad \mod C^\infty ; \quad a_T(\tau)\sim\sum_{j=0}^{+\infty} a_{T,k-1+j}\,\tau^{-k-j}\ ,\ \tau>0$$

and this homogeneous expansion can be equivalently rewritten as an expansion in the time variable :

$$e_T(t)\underset{t\to T}{\sim}\sum_{j=0}^{+\infty} c_{T,k-1+j}\,(t-T)^{k-1+j}\log(t-T+i0)\ . \tag{II.4.5}$$

Let us briefly explain this point: denoting for $\mathrm{Re}\,\beta>-1$,

$$\tau_+^\beta=\begin{cases}\tau^\beta & \text{if} \quad \tau>0\\ 0 & \text{if} \quad \tau\le 0\end{cases},$$

the passage from the symbolic expansion to the time expansion is given by applying the one-dimensional inverse Fourier transform of homogeneous distributions in the

time variable $s = t - T$, given by [Hör90, Example 7.1.17]

$$\mathcal{F}^{-1}(\tau_+^\beta)(s) = \int_0^{+\infty} e^{is\tau}\tau^\beta \,\mathrm{d}\tau = \Gamma(\beta+1)e^{\frac{i\pi}{2}(\beta+1)}(s+i0)^{-1-\beta} \quad , \quad \Re(\beta) > -1 \,.$$

The distribution-valued function $\beta \mapsto \tau_+^\beta$, extends meromorphically to $\beta = -m$, $m \in \mathbb{N}^*$, and near such points, we write

$$\tau_+^\beta = \frac{\alpha_m}{\beta+m} + \mathrm{F.\,p.}\,\tau_+^{-m} + \underset{\beta\to -m}{O}(\beta+m)\,,$$

where F. p. τ_+^{-m} denotes the finite part, i.e. the constant term of the Laurent expansion at $\beta = -m$. This identity has to be understood in the sense of distributions in $\mathcal{D}'(\mathbb{R}_\tau)$ i.e. for every $\varphi \in C_c^\infty(\mathbb{R}_\tau)$,

$$\left\langle \tau_+^\beta - \frac{\alpha_m}{\beta+m} - \mathrm{F.\,p.}\,\tau_+^{-m}, \varphi \right\rangle = \underset{\beta\to -m}{\mathcal{O}}(\beta+m)\,.$$

By writing $\beta = -m + z$, it is standard to see that, as $z \to 0$,

$$\Gamma(1-m+z) = \frac{(-1)^{m-1}}{(m-1)!}\frac{1}{z} + \mathcal{O}(1)$$

and

$$(s+i0)^{m-1-z} = s^{m-1}\left(1 - z\log(s+i0) + O(z^2)\right)\,,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R}_s)$, i.e. for every $\varphi \in C_c^\infty(\mathbb{R}_s)$,

$$\left\langle (s+i0)^{m-1-z} - s^{m-1} + zs^{m-1}\log(s+i0), \varphi \right\rangle = \underset{z\to 0}{\mathcal{O}}(z^2)\,.$$

Thus by multiplying these two expansions, we see that the finite part of $\mathcal{F}^{-1}(\tau_+^{-m})$ is of the form

$$C_m\, s^{m-1}\log(s+i0) \quad , \quad C_m \neq 0\,,$$

where C_m depends only on m and on the Fourier convention.[11]
The pole term is proportional to s^{m-1}, hence gives a smooth function that does not contribute to the singular part of e_T.
In other words, this shows that the inverse Fourier transform of the distribution

[11] Here

$$C_m = -\frac{(-1)^{m-1}}{(m-1)!}e^{\frac{i\pi}{2}(1-m)}$$

and the second factor depends on the Fourier transform convention.

F. p. τ_+^{-m} is a non-zero multiple of $s^{m-1}\log(s+i0)$:

$$\mathcal{F}^{-1}(\text{F. p. } \tau_+^{-m})(s) = C_m\, s^{m-1}\log(s+i0) \quad \mod C^\infty \quad , \quad C_m \neq 0\,.$$

Applying this with $m = k+j$, we precisely obtain (II.4.5) and that

$$c_{T,k-1+j} = C_{k+j} a_{T,k-1+j}\,.$$

In particular, the leading term $c_{T,k-1}$ in the asymptotic expansion is given, up to a non-zero universal constant (coming from the Fourier transform), by the principal symbol of $\pi_*\Delta^*\nu$ and using notations from [Gui78b, Section 4], it can be expressed in terms of the principal symbol $\sigma(\nu)$ of ν as follows :

$$\sigma_{S(t)} = \pi_*\Delta^*\sigma(\nu) = \sum_{j=1}^{r} \sigma(\nu)|_{C_j}\,,$$

where

$$C_j := \{(t,\tau),(x,\xi),(y,\eta) \mid (x,\xi) = (y,\eta) \in \gamma_j\,,\, t = T\} \subset C\,.$$

Moreover, we note that the principal symbol of ν is given by

$$\sigma(\nu)((t,\tau),(x,\xi),(y,\eta)) = \sigma_{W(t)}(x,\xi)\cdot\sigma(\mu_0)((t,\tau),(x,\xi),(y,\eta))\,,$$

where $\sigma(\mu_0)$ denotes the principal symbol of μ_0 . By [DG75, Theorem 4.5], we know that

$$\sigma(\mu_0)|_{C_j} = \frac{|T_j^\sharp| e^{im_j\frac{\pi}{2}} e^{iT\overline{\gamma}_j}}{|\det\left(Id - \mathcal{P}_j\right)|^{1/2}}\,.$$

Finally, by **Theorem II.4.1**,

$$\sigma_{W(t)}|_{C_j} = i\int_{\gamma_j}\sigma_B\,,$$

which conclude the sketch of proof. □

Remark II.4.3. *In our case, once again, the Maslov index m_j appearing in this result will be the Morse index of the periodic geodesic γ_j (or more precisely here the projection of γ_j) just as in **Theorem II.3.4**. Recall that these indices are zero in the case where the underlying manifold N is Anosov (see **Remark II.3.5**).*

The following corollary, formulated in a slightly more general context in **Corollary I.4.2**, will be the key for the induction procedures used in the proofs of our main

boundary determination results (cf. Chapter IV).

Corollary II.4.4. *If A is either the square root of a magnetic Schrödinger operator on a closed manifold, a DN map, or a magnetic DN map, and has the same spectrum as $A + B$, then under the assumption that the underlying metric is Anosov with simple length spectrum,*

$$\int_\gamma \sigma_B = 0$$

for any closed orbit γ of its geodesic flow.

Proof. Since A and $A + B$ have the same spectrum their wave traces are identical and then, in particular, the leading terms given by **Theorem II.4.2** are zero. □

Chapter III

From wave invariants to geometry: X-ray transform and Livšic theory

Table of Contents for the chapter

III.0.1 Anosov manifolds 76

III.1 Elements of Livšic theory 77

III.1.1 The geodesic X-ray transform 78

III.1.2 The smooth Livšic theorem for cocycles 81

III.1.3 A transport equation with real-valued potential 83

III.2 Infinitesimal spectral rigidity 85

III.2.1 The marked length spectrum and its linearization 85

III.2.2 Marked length spectrum and isospectral deformations 87

Appendices to § III.1.2 90

Appendix III.A. Another proof of Proposition III.1.10 90

Appendices to § III.1.3 92

Appendix III.B. Cohomology of a sphere bundle 92

Appendix III.C. Another proof of Theorem III.1.11 94

Appendices to § III.2.1 100

Appendix III.D. Higher-order variation of the marked length spectrum 101

The purpose of this chapter is to present the tools for translating the spectral invariants derived above into the geometric information they encode. The invariants we obtain are integrals over the closed geodesics of the boundary and such data are naturally encoded by a linear operator called the *geodesic X-ray transform*.

In the following, we recall the definition of this transform and discuss the injectivity results needed in our setting. These results are closely related to the standard *Livšic*

theory, which aim to identify the kernel of the X-ray transform (cf. Section III.1). We refer to the introduction of [Lef25, §11.1] for a nice motivation of the Livšic theory through sheaf-theoretic considerations.

In this chapter, we will assume that the closed Riemannian manifold (N, h) is connected.

III.0.1 Anosov manifolds

Recall that a flow (ϕ_t) generated by a smooth vector field X on a closed manifold $\mathcal{M}$ is Anosov if there is a continuous splitting

$$T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u \ ,$$

of the tangent bundle $T\mathcal{M}$ into three flow-invariant subbundles, respectively, tangential to the flow direction, exponentially contracting and expanding. Note that two Anosov flows that are C^1-close enough are topologically conjugate : This is known as the *structural stability*. In particular, the Anosov property is robust to C^1-perturbations, it is a C^1-open condition with respect to the vector field. We refer, for example, to [FH19, Section 5.1] and to the fundamental [DlLMM86] for more details.

A particularly important special case, which is considered exclusively in this work, is that where $\mathcal{M}$ is the unit tangent bundle (resp. the unit cotangent bundle)

$$SN := \{(x, v) \in TN \ ; \ h_x(v, v) = 1\}$$

$$(\text{resp. } S^*N := \{(x, \xi) \in T^*N \ ; \ h_x^{-1}(\xi, \xi) = 1\})$$

of the closed Riemannian manifold (N, h) and (ϕ_t) the associated geodesic flow. We say that (N, h), or simply h, is Anosov if its geodesic flow is Anosov.
As already mentioned, any closed Riemannian manifold with negative curvature is Anosov (see [FH19, Theorem 5.2.4], [KH97, Section 17.6] or [Kni02] for a proof).

If the manifold is connected, such flows are transitive[1], their periodic orbits are dense, and knowledge of a function along these periodic integral curves allows us to reconstruct it up to natural obstructions. This is the purpose of the standard Livšic theory, and in particular of **Theorem III.1.1** below. More recent elements of the so-called non-Abelian Livšic theory are used via **Proposition III.1.10**. Finally, as explained in Subsection II.3, the Anosov property provides a general framework par-

[1] They admit a dense orbit.

ticularly suited to obtaining spectral rigidity results by using trace formulas. Anosov geodesic flows indeed form a strongly chaotic, or non-integrable, class of flows and, in contrast with completely integrable systems, where invariant tori may organize the dynamics, periodic geodesics of an Anosov flow are isolated and hyperbolic (in particular non-degenerate).

III.1 Elements of Livšic theory

The following theorem, originally due[2] to Livšic [Liv71, Theorem 1] in Hölder regularity C^α, $\alpha \in (0,1)$, is undoubtedly the most standard and representative result of the theory. Here we give the smooth regularity version proved in [DlLMM86, Theorem 2.1] :

Theorem III.1.1 (Livšic [Liv71], De La Llave-Marco-Moriyon [DlLMM86]). *Let $\mathcal{M}$ be a closed Riemannian manifold, $f : \mathcal{M} \to \mathbb{R}$ a smooth function and (ϕ_s) a smooth transitive Anosov flow on $\mathcal{M}$ generated by the vector field X.*

The following propositions are equivalent :

1. *For every periodic orbit γ of the flow,*
$$\int_\gamma f = 0 \, .$$
2. *There is $u \in C^\infty(\mathcal{M}, \mathbb{R})$ such that $f = Xu$.*

Note that only 1. $\Rightarrow$ 2. is non-trivial, the idea is to define u as the integral of f over a dense orbit (such an orbit exists by transitivity of the flow, see the discussion above), it is straightforward to see that such a function is a solution to the transport equation

$$Xu = f \, ,$$

called *cohomological equation*. Then a standard calculation using the Anosov closing lemma (see [FH19, Theorem 5.3.11]) ensures Hölder regularity and the main difficulty lies in showing smooth regularity. A recent proof of this point, based on a microlocal approach and under the additional assumption that the flow preserves a smooth measure, can be found in [Gui17]. Precisely, the proof relies on the Guillarmou's operator Π initially introduced in [Gui17, Theorem 1.1].

[2] see also [GK80a, Appendix] and [KH97, Theorem 19.2.4].

We will also use it in the same way in the alternative proof of **Proposition III.1.10** given in Appendix to show the regularity of a solution to a certain transport equation with attenuation.

As mentioned above, one sees that **Theorem III.1.1** characterizes the kernel of a linear integration operator along periodic orbits. In the following, we recall the definition of this operator for geodesic flows.

III.1.1 The geodesic X-ray transform

Let us denote by $\mathcal{C}$ the set of free homotopy classes of N. The latter is in 1-1 correspondence with the set of conjugacy classes of the fundamental group $\pi_1(N)$ of N and consequently $\mathcal{C}$ is countable (see, for example, [Hat02, Chapter 1]). In addition, if N is Anosov, it is well-known that there exists a unique closed geodesic for h in each free homotopy class (see [Kli74], [Kni02] or, more generally, [CIPP00, Theorem 29]).
Thus in this case, the set $\mathcal{G}$ of periodic orbits of the geodesic flow (ϕ_t) can be identified with $\mathcal{C}$. The *X-ray transform* on SN is the operator defined by :

$$\mathcal{I}^h \; : \; \left| \begin{array}{ccc} C^0(SN) & \longrightarrow & \ell^\infty(\mathcal{C}) \\ f & \longmapsto & \left(\mathcal{I}^h f([\gamma])\right)_{[\gamma]\in\mathcal{C}} \end{array} \right.$$

and

$$\mathcal{I}^h f([\gamma]) := \frac{1}{\ell_h(\gamma_h)} \int_0^{\ell_h(\gamma_h)} f\left(\phi_t(x,v)\right) dt \, ,$$

where $\gamma_h : [0, \ell_h(\gamma_h)] \to N$ denotes any parametrization by arclength of the unique closed geodesic in the free homotopy class $[\gamma] \in \mathcal{C}$, ℓ_h the length functional for h and $(x,v) \in SN$ is any point on the closed curve $t \mapsto (\gamma_h(t), \dot{\gamma}_h(t))$.

We usually consider this operator acting on symmetric tensor fields on N :
Let S^mT^*N be the bundle of symmetric tensors of order $m \in \mathbb{N}$ on N, we refer the reader to the recent books [Lef25], or [GM] for definitions in this formalism. These references are the main we use to recall the following basics on integral geometry.
Denoting by $C^\infty(N, S^mT^*N)$ the set of smooth sections of $S^mT^*N \to N$, we can see any symmetric tensor field f of order m on N as a function on SN by considering the lift $\pi_m^* : C^\infty(N, S^mT^*N) \to C^\infty(SN)$ defined by

$$\pi_m^* f \; : \; (x,v) \mapsto f_x(v, ..., v) \; ,$$

with the convention that $S^0T^*N := N \times \mathbb{R}$ is the trivial bundle, so that π_0^* is simply the pullback by the projection $\pi : SN \to N$.

Then with the same notations as above:

Definition III.1.2 (Geodesic X-ray transform on symmetric tensors). *The X-ray transform on symmetric tensors of order m is the operator*

$$\mathcal{I}_m^h := \mathcal{I}^h \circ \pi_m^* \left| \begin{array}{rcl} C^\infty(N, S^mT^*N) & \longrightarrow & \ell^\infty(\mathcal{C}) \\ f & \longmapsto & \left(\mathcal{I}^h \pi_m^* f([\gamma])\right)_{[\gamma]\in\mathcal{C}} \end{array} \right. ,$$

where

$$\mathcal{I}^h \pi_m^* f([\gamma]) := \frac{1}{\ell_h(\gamma_h)} \int_0^{\ell_h(\gamma_h)} f_{\gamma_h(t)}\left(\dot\gamma_h(t), ..., \dot\gamma_h(t)\right) dt \, .$$

Remark III.1.3. *Of course, the definition remains the same if we instead consider π_m as the lift to smooth functions of the unit cotangent bundle.*

The problems considered in this work will mainly involve $\mathcal{I}_m^h$ for $m = 0, 1, 2$, respectively the X-ray transform on functions, on 1-forms and on symmetric 2-tensors. The simplest injectivity result applies to functions:

Theorem III.1.4 ([DS03]). *Suppose h is Anosov. Then for any $f_0 \in C^\infty(N)$,*

$$\mathcal{I}_0^h f_0 = 0 \quad \Rightarrow \quad f_0 = 0 \, .$$

The proof is actually an easy consequence of the Livšic theorem (**Theorem III.1.1**) combined with the following well-known L^2 energy identity :

Proposition III.1.5 (Pestov identity). *For any $u \in C^\infty(SN)$,*

$$\|\overset{v}{\nabla} Xu\|^2_{L^2(SN)} - \|X\overset{v}{\nabla} u\|^2_{L^2(SN)} = (n-1)\|Xu\|^2_{L^2(SN)} - \langle R\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2(SN)} \, .$$

We refer the reader to [PSU15, Proposition 2.2] for a proof of this result.

Recall that R denotes the operator induced by the Riemann curvature tensor and $\overset{v}{\nabla}$ is the *vertical gradient* with respect to the Sasaki metric (see for example [PSU15, Section 2] or the recent [PSU23, Section 4.7]).

To exploit the Pestov identity, we usually use [PSU15, Lemma 4.2] which implies that if h is Anosov,

$$\|X\overset{v}{\nabla} u\|^2_{L^2(SN)} - \langle R\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2(SN)} \geqslant 0$$

for any $u \in C^\infty(SN)$.

proof of ***Theorem III.1.4****.* Assume $f \in C^\infty(N)$ satisfies $\mathcal{I}_0^h f = 0$. Then by the Livšic theorem, there exists $u \in C^\infty(SN)$ such that $Xu = \pi_0^* f$. Applying the Pestov identity and using that $\overset{v}{\nabla}\pi_0^* f = 0$, we get

$$0 = \|\overset{v}{\nabla} Xu\|^2_{L^2(SN)} = \underbrace{\|X\overset{v}{\nabla} u\|^2_{L^2(SN)} - \langle R\overset{v}{\nabla} u, \overset{v}{\nabla} u\rangle_{L^2(SN)}}_{\geqslant 0} + (n-1)\|Xu\|^2_{L^2(SN)} .$$

Thus, $Xu = 0$ and consequently $f = 0$. □

The conclusion of **Theorem III.1.4** is actually a simple special case of a more general notion of injectivity on symmetric tensors of order m, called *s-injectivity*. Indeed, the X-ray transform on symmetric tensors of order $m \geq 1$ has a non-trivial kernel: If p is a symmetric $(m-1)$-tensor and

$$Dp := \mathrm{Sym}(\nabla p)$$

is its symmetric (covariant) derivative, then

$$\pi_m^*(Dp) = X(\pi_{m-1}^* p) ,$$

where X denotes the geodesic vector field. Hence the integral of $\pi_m^*(Dp)$ over any closed orbit vanishes. Consequently, one cannot hope to recover a tensor f completely from its X-ray data; at best, one can recover it modulo potential tensors of the form Dp. This leads to the notion of solenoidal injectivity, or s-injectivity:

Definition III.1.6 (s-injectivity). *The transform I_m^h is said to be* solenoidal injective*, or* s-injective*, if*

$$\ker \mathcal{I}_m^h = D\big(C^\infty(N, S^{m-1}T^*N)\big) .$$

The literature on this subject is quite large; we refer, for example, to [DS03], [CS98], or the more recent [PSU14] and [Gui17]. Its study in the context of general Anosov manifolds is still largely open, at least in dimension $n \geqslant 3$, even if major progress has recently been made by Cekić and Lefeuvre in [CL25a].

Note that for $m = 1$, potential tensors are given by exact forms, and a proof similar to the one used on functions shows s-injectivity of $\mathcal{I}_1$ on closed Anosov manifolds. The proof is also originally due to Dairbekov and Sharafutdinov [DS03]. In the case of $m = 0$ and $m = 1$, however, injectivity statements can be strengthened to a joint one:

Theorem III.1.7 (Joint injectivity in degrees 0 and 1). *Assume h is Anosov and denote by X the geodesic vector field. Let $f_0 \in C^\infty(N)$ and $f_1 \in C^\infty(N, T^*N)$ and $f := f_0 + f_1$ such that for any $\gamma \in \mathcal{G}_{per}(N,h)$,*

$$\int_\gamma f = 0 \,.$$

Then

$$f_0 = 0 \quad \text{and} \quad f_1 \text{ is exact} \,.$$

Equivalently, if there exists $u \in C^\infty(SN)$ such that $\pi_0^* f_0 + \pi_1^* f_1 = Xu$, then the same conclusion holds.

This theorem is formulated (and proven) in a more general setting in [DP08, Theorem B]; we refer the reader to that article and the comments contained therein. The proof of this is,of course, also a consequence of the use of Pestov identity—a more general form of the Pestov identity in the case of Dairbekov and Paternain.

Remark III.1.8. *The reader may compare **Theorem III.1.7** above with our **Theorem III.1.11**.*

III.1.2 The smooth Livšic theorem for cocycles

To deal with the information provided by the principal wave invariants in the Duistermaat-Guillemin trace formula **Theorem II.3.4**, we need an appropriate version of the Livšic theorem, since this information is, a priori, nonlinear. The purpose of this section is therefore to present this theorem.

For convenience, let us first recall [Pat09, Definition 2.1]. Let G be a compact Lie group–which we will take to be $\mathbb{S}^1$–,$\mathcal{M}$ a closed Riemannian manifold, and (ϕ_s) a smooth transitive Anosov flow on $\mathcal{M}$.

Definition III.1.9 (cocycles). *A G-valued cocycle over the flow (ϕ_s) is a map $C : \mathcal{M} \times \mathbb{R} \to G$ such that for all $x \in \mathcal{M}$, $s,t \in \mathbb{R}$,*

$$C(x, t+s) = C(\phi_t x, s) C(x,t) \,.$$

The following is basically just a possible formulation of the Livšic theorem for cocycles (see [Pat09, Theorem 2.2] based on [Liv74] and [NT98]):

Proposition III.1.10. *Let $\mathcal{M}$ be a closed Riemannian manifold, $f : \mathcal{M} \to \mathbb{R}$ be a smooth function, and (ϕ_s) a smooth transitive Anosov flow on $\mathcal{M}$ generated by the vector field*

X.

The following propositions are equivalent :

1. *For every periodic orbit γ of the flow,*

$$\int_\gamma f \in 2\pi\mathbb{Z}\,.$$

2. *There is $u \in C^\infty(\mathcal{M}, \mathbb{S}^1)$ such that*

$$f = -i\frac{Xu}{u}\,.$$

Proof. $(2) \Rightarrow (1)$: Suppose there exists $u \in C^\infty(\mathcal{M}, \mathbb{S}^1)$ such that

$$f = -i\frac{Xu}{u} = -i\bar{u}Xu\,.$$

Define for $x \in \mathcal{M}$,

$$\Gamma\ :\ \left|\begin{array}{ccc} [0,\ell] & \longrightarrow & \mathbb{S}^1 \\ t & \longmapsto & \phi_t^* u(x) \end{array}\right.$$

and denotes by $\mathrm{Ind}_\Gamma(0)$ the index of 0 with respect to Γ, seen as a closed curve in the complex plane. Then for each periodic orbit γ of length ℓ starting at x,

$$\begin{aligned}\int_\gamma f\ :=\ \int_0^\ell \phi_t^* f(x)dt &= -i\int_0^\ell \frac{\frac{\partial}{\partial s}\big|_{s=t}(\phi_s^* u)(x)}{\phi_t^* u(x)}dt \\ &= -i\int_\Gamma \frac{dz}{z} \\ &= -i\,(2i\pi\,\mathrm{Ind}_\Gamma(0)) \\ &= 2\pi\,\mathrm{Ind}_\Gamma(0) \in 2\pi\mathbb{Z}\end{aligned}$$

$(1) \Rightarrow (2)$ Assume that for every periodic orbit γ of the flow,

$$\int_\gamma f \in 2\pi\mathbb{Z}\,.$$

Consider the smooth cocycle $C\ :\ \mathcal{M}\times\mathbb{R} \to \mathbb{S}^1$ over (ϕ_s) defined by

$$C(x,t)\ :=\ exp\left(i\int_0^t f(\phi_s x)\,ds\right)\,.$$

The above assumption ensures that C satisfies

$$C(x,T) = 1$$

whenever $\phi_T x = x$.

Consequently, by the smooth Livšic theorem for cocycles [Pat09, Theorem 2.2], there exists $u \in C^\infty(\mathcal{M}, \mathbb{S}^1)$ such that

$$C(x,t) = u(\phi_t x)u(x)^{-1} ,$$

for all $(x,t) \in \mathcal{M} \times \mathbb{R}$. Differentiating with respect to t and evaluating at $t = 0$, we get :

$$f = -i\frac{Xu}{u} .$$

□

Of course, it is possible to prove this result in a way that is perhaps more explicit, by following the same initial proof as that of Livšic theorem (**Theorem III.1.1**), i.e. by first showing the statement in the Hölder C^α regularity and then extending it to smooth regularity. The latter can also be shown using the more recent microlocal approach of [Gui17]. This point of view is presented in Appendix III.A.

III.1.3 A transport equation with real-valued potential

By **Proposition III.1.10**, we are reduced to studying a transport-type equation with real-valued potential, namely

$$Xu + ifu = 0 .$$

In fact, our problem is similar to what appears when considering two transparent connections (see [Pat09, Section 3]) on a Hermitian line bundle over an Anosov manifold as in the proof of [Pat09, Theorem 3.2]. The latter shows that two transparent connections are gauge-equivalent. It is based on the use of a standard algebraic tool called Gysin sequences (cf. Appendix III.B) and on the solenoidal injectivity of $\mathcal{I}_1$ on Anosov manifolds (see [CS98, Theorem 1.3]).
We reason in a very similar way to show the next result below,

Theorem III.1.11. *Let (N,h) be a closed Anosov Riemannian manifold. Let $f \in C^\infty(SN;\mathbb{R})$ be a smooth real-valued function, if there exists a solution $u \in H^2(SN;\mathbb{S}^1)$ of the trans-*

port equation with potential f

$$Xu + ifu = 0$$

and if $f = f_0 + f_1$ is the sum [3] *of a function $f_0 \in C^\infty(N)$ and a one-form $f_1 \in \Omega^1(N)$ then $f_0 = 0$, f_1 is closed and u is a function on N.*

Proof. Let us introduce the closed 1-form

$$\varphi := -\frac{du}{iu} \in C^\infty(SN, T^*SN) .$$

By **Lemma III.B.1**, $\pi^* : H^1_{\mathrm{dR}}(N) \to H^1_{\mathrm{dR}}(SN)$ is an isomorphism since N cannot be the 2-torus (cf. **Remark III.B.2**).
As a consequence, there exists a closed 1-form ω in N and a smooth function $\psi : SN \to \mathbb{R}$ such that

$$\varphi = \pi^*\omega + d\psi .$$

Applying the vector field X and using that

$$Xu + ifu = 0 ,$$

we get

$$\pi_0^* f_0 + \pi_1^* f_1 = \pi_1^*\omega + X\psi .$$

Then we are essentially reduced to the injectivity result of **Theorem III.1.7**, and we immediately obtain that $f_0 = 0$ and that $f_1 - \omega$ is an exact 1-form. Consequently, f_1 is in particular closed. □

This result is actually related to a much more general theory (tensor tomography for connections) developed over the last ten years following the work of Guillarmou-Paternain-Salo-Uhlmann (see, for example, [Pat13], [GPSU16a], [PSU12]). Note that the proof works more generally if the manifold has no conjugate points but only if it is not the 2-torus.
In Appendix III.C, we present a new proof, rather in the spirit of the famous Pestov identity in relation to transport equations (see, for example, [PSU23]), which allows us to formulate the result more naturally on manifolds without conjugate points. Recall that Anosov manifolds do not have conjugate points (cf. [Kli74]). We believe

[3] More precisely, as functions on the unit tangent bundle: $\pi^* f = \pi_0^* f_0 + \pi_1^* f_1$.

this alternative proof to be quite natural (cf. **Remark III.1.8**) and instructive and hope that it may be useful in different contexts, involving gauge invariances.

III.2 Infinitesimal spectral rigidity

The aim of this section is to explain how the spectral invariants obtained from the Steklov wave trace can lead to infinitesimal rigidity statements. As we have already briefly explained in Chapter I, in the presence of a deformation parameter s one is naturally led to study the linearization of marked length spectrum, as it has been successfully done for the laplacian spectrum [GK80a, GK80b], [CS98].

In the following, we shall recall standard material on marked length spectrum rigidity and tensor tomography, following in particular the work of Guillarmou-Lefeuvre and the expositions in Lefeuvre's book [Lef25] and in that of Guillarmou and Mazuchelli [GM].

III.2.1 The marked length spectrum and its linearization

Let (N, h) be a closed Riemannian manifold with Anosov geodesic flow. Then, as mentionned at the beginning of Subsection III.1.1, for each $[\gamma] \in \mathcal{C}$, there is a unique closed geodesic γ_h in the class $[\gamma]$, that we assume to be parametrized by the arclength.

Definition III.2.1 (Marked length spectrum). *The* marked length spectrum *of* (N, h) *is the map*

$$\mathcal{L}_h : \left| \begin{array}{ccc} \mathcal{C} & \longrightarrow & (0, \infty) \\ [\gamma] & \longmapsto & \ell_h(\gamma_h) \end{array} \right.$$

Remark III.2.2. *The spectrum of the operator under consideration (laplacian or DN map) may determined the unmarked length spectrum via the wave trace formula and under suitable dynamical assumptions but it does not determine the marked length spectrum in general: If* $\varphi : N \to N$ *is a diffeomorphism and* $h_2 = \varphi^* h_1$*, then* (N, h_1) *and* (N, h_2) *are isometric, hence have the same eigenvalues and the same unmarked length spectrum. However,*

$$\mathcal{L}_{h_2}([\gamma]) = \mathcal{L}_{h_1}(\varphi_* [\gamma]), \quad [\gamma] \in \mathcal{C}.$$

Thus the marked length spectra may differ by a non-trivial permutation of $\mathcal{C}$*. For deformations of a fixed Anosov metric, this difficulty is partly removed by structural stability: closed orbits can be continued along the deformation [DlLMM86] (cf. below).*

Note that assuming the metric h has simple length spectrum is equivalent to assuming that no two distinct free homotopy classes have the same length; that is, the marked length spectrum $\mathcal{L}_h$ is injective on the free homotopy classes. This assumption is known to be generic even in the non-Anosov case ([Abr], [Kli82, Lemma 4.4.3] or [Ano83]).

The key point is the first variation formula. Let (h_s) be a smooth family of Anosov metrics with $h_0 = h$, and write

$$\dot{h} = \frac{\partial}{\partial s}\Big|_{s=0} h_s \, .$$

Then, we have

Proposition III.2.3. *For any free homotopy class $[\gamma] \in \mathcal{C}$,*

$$\frac{\partial}{\partial s}\Big|_{s=0} \mathcal{L}_{h_s}([\gamma]) = \frac{1}{2}\mathcal{L}_h([\gamma])\mathcal{I}_2^h(\dot{h})([\gamma]) \, .$$

Infinitesimal marked length spectrum rigidity is therefore a tensor tomography problem. A clean proof of this interesting relation between the first variation of the marked length spectrum and the X-ray transform can be found in [GM, Lemma 15.4]. It will actually serve as the initialization step for establishing **Proposition IV.1.1** by induction on the order of derivation of the metric tensor. Indeed, denoting for all $k \in \mathbb{N}_{\geq 2}$,

$$(h)^{(k)} = \frac{\partial^k}{\partial s^k}\Big|_{s=0} h_s$$

the k-th variation of the metric tensor and differentiating k times the equality of **Lemma III.2.4** with respect to s, we observe more generally that for all $[\gamma] \in \mathcal{C}$,

$$\frac{\partial^k}{\partial s^k}\Big|_{s=0} \mathcal{L}_{h_s}([\gamma]) = \frac{1}{2}\mathcal{L}_h([\gamma])\mathcal{I}_2^h\,(h)^{(k)}([\gamma]) \; + \; \mathcal{R}_k([\gamma])\left[\dot{h}^\circ, (h)^{(2)}, ..., (h)^{(k-1)}\right] \, , \tag{III.2.1}$$

where the second term at the right is a finite sum of integrals over the closed h-geodesic γ of (tensorial) polynomials in the lower-order variations $\dot{h}$, $h^{(2)}$,..., $h^{(k-1)}$ and their covariant derivatives of order at most $k-2$ along $\gamma := \gamma_0$. This identity (III.2.1) follows from a technical but standard calculation of the k-th derivative of $s \mapsto \mathcal{L}_{h_s}([\gamma])$ with respect to s (cf. Appendix III.D). Thus, we see that at each step of a possible induction, the main term is the X-ray transform of the next metric variation. This recursive structure is precisely what allows us to obtain **Theorem I.1.2** in the conformal case, as we will explain in the subsection below and in the proof of this

result, which is essentially done in **Proposition IV.1.1**.

III.2.2 Marked length spectrum and isospectral deformations

We now want to specify what we have just seen above to the Steklov spectrum on a compact Riemannian manifold (M, g) with boundary, and to study more precisely the relation between Steklov isospectral deformations and linearized marked length spectrum. This turned out to be much more delicate than in the case of the Laplace spectrum ([GK80a, GK80b], [CS98]), because in reality, in this case, we do not really reduce to linear marked length rigidity, but to a weaker version, as we will see below.

If $(g_s)_{0\leqslant s\leqslant\varepsilon}$ is any deformation of g (see **Definition I.1.1**) s.t g_s° is an Anosov metric for all s, any closed geodesic γ for g° deforms smoothly as a closed geodesic γ_s for g_s° and its length is a smooth function in s (cf. [DlLMM86], or [GM, Lemma 15.3] for a precise statement in this specific context). If, in addition, $(g_s)_{0\leqslant s\leqslant\varepsilon}$ is Steklov isospectral and g° has simple length spectrum, one can show that the marked length of a closed geodesic γ is locally preserved along the deformation :

Lemma III.2.4. *For any free homotopy class $[\gamma] \in \mathcal{C}$, there exists $\eta_{[\gamma]} \in]0, \varepsilon]$ such that*

$$\forall s \leq \eta_{[\gamma]}\ ,\ \mathcal{L}_{g_s^\circ}([\gamma]) = \mathcal{L}_{g^\circ}([\gamma])\ .$$

The key point in the proof of this result, is to see that the length of any closed geodesic remains uniformly isolated under small perturbations of the metric. This is a consequence of the work of Guillarmou and Lefeuvre [GL19, Proposition 2.1] :

Lemma III.2.5. *For any free homotopy class $[\gamma] \in \mathcal{C}$, there exists $s_0 \in]0, \varepsilon[$ and $\delta > 0$ such that for all $s < s_0$,*

$$I_s \cap |\mathcal{T}_{g_s^\circ}| = \{\mathcal{L}_{g_s^\circ}([\gamma])\}\ ,$$

where

$$I_s := \left]\mathcal{L}_{g_s^\circ}([\gamma]) - \delta\ ,\ \mathcal{L}_{g_s^\circ}([\gamma]) + \delta\right[$$

and $|\mathcal{T}_{g_s^\circ}|$ denotes the set of lengths of closed geodesics for g_s°.

Proof. Let $[\gamma] \in \mathcal{C}$. Since g° is Anosov, its length spectrum is discrete and consequently, there exists $\eta > 0$ s.t

$$\left]\mathcal{L}_{g^\circ}([\gamma]) - \eta\ ,\ \mathcal{L}_{g^\circ}([\gamma]) + \eta\right[\cap\, |\mathcal{T}_{g^\circ}| = \{\mathcal{L}_{g^\circ}([\gamma])\}\ . \tag{III.2.2}$$

By [GL19, Proposition 2.1], the g°-*normalized marked length spectrum* converges in $\ell^\infty(\mathcal{C})$ as s tends to 0

$$\left\| \left(\frac{\mathcal{L}_{g_s^\circ}([\omega])}{\mathcal{L}_{g^\circ}([\omega])} \right)_{[\omega]\in\mathcal{C}} - \mathbf{1} \right\|_{\ell^\infty(\mathcal{C})} \xrightarrow[s\to 0]{} 0 \,.$$

Consequently, for all $\epsilon > 0$ and s small enough,

$$(1-\epsilon)\mathcal{L}_{g^\circ} \leq \mathcal{L}_{g_s^\circ} \leq (1+\epsilon)\mathcal{L}_{g^\circ} \,. \tag{III.2.3}$$

Taking $\epsilon > 0$ sufficiently small, shrinking η if necessary, and using the continuity of $s \mapsto \mathcal{L}_{g_s^\circ}([\gamma])$ at 0, we can ensure that there exists $s_0 > 0$ such that for any $s < s_0$,

$$\left]\frac{\mathcal{L}_{g_s^\circ}([\gamma]) - \delta}{1+\epsilon} \,,\, \frac{\mathcal{L}_{g_s^\circ}([\gamma]) + \delta}{1-\epsilon}\right[\subset \left]\mathcal{L}_{g^\circ}([\gamma]) - \eta \,,\, \mathcal{L}_{g^\circ}([\gamma]) + \eta\right[\,,$$

with $\delta := \eta/2$. Assuming then that there exists $[\omega] \neq [\gamma]$ such that

$$\mathcal{L}_{g_s^\circ}([\omega]) \in \left]\mathcal{L}_{g_s^\circ}([\gamma]) - \delta \,,\, \mathcal{L}_{g_s^\circ}([\gamma]) + \delta\right[\,,$$

(III.2.3) would lead to

$$\mathcal{L}_{g^\circ}([\omega]) \in \left]\mathcal{L}_{g_s^\circ}([\gamma]) - \eta \,,\, \mathcal{L}_{g_s^\circ}([\gamma]) + \eta\right[\,,$$

which would contradict (III.2.2).

□

Let us point out that the proof of **Lemma III.2.5** does not use the simple length spectrum assumption on g° at all.

Proof of ***Lemma III.2.4.*** Let $[\gamma] \in \mathcal{C}$ be any free homotopy class. Using **Lemma III.2.5** and its notations, there exists $s_0 \in]0, \varepsilon[$ and $\delta > 0$ such that for all $s < s_0$,

$$I_s \cap |\mathcal{T}_{g_s^\circ}| = \{\mathcal{L}_{g_s^\circ}([\gamma])\} \,.$$

In addition, denoting also by $|\mathcal{T}_{g^\circ}|$ the length spectrum of g°, we have

$$|\mathcal{T}_{g^\circ}| \subseteq |\mathcal{T}_{g_s^\circ}| \,. \tag{III.2.4}$$

This fact is a direct consequence of isospectrality, which gives the inclusion of the

singular supports of the wave traces associated with Λ_{g_s} and Λ_g. Note that (III.2.4) is a priori just an inclusion because the length spectrum of g_s° is not necessarily simple.

To conclude, we use that the map $s \mapsto \mathcal{L}_{g_s^\circ}([\gamma])$ is continuous (even smooth as mentioned above). In particular, there is $\eta_{[\gamma]} \in]0, s_0[$ s.t for any $s \leq \eta_{[\gamma]}$,

$$|\mathcal{L}_{g_s^\circ}([\gamma]) - \mathcal{L}_{g^\circ}([\gamma])| \leq \delta$$

and since $\mathcal{L}_{g^\circ}([\gamma]) \in |\mathcal{T}_{g_s^\circ}|$ (by (III.2.4)), we get that for any $s \leq \eta_{[\gamma]}$,

$$\mathcal{L}_{g_s^\circ}([\gamma]) = \mathcal{L}_{g^\circ}([\gamma]) .$$

□

Note that this result is less direct and much weaker than in the case of Laplace isospectral deformations, since in the latter, inclusion (III.2.4) is an equality for any s and the length spectrum does not need to be simple. In relation to this last point, we would like to point out to the reader that **Lemma III.2.4** does not imply that the simple length spectrum assumption is preserved along the deformation even for s sufficiently small. For this to be the case, η would have to be uniform (independent of any free homotopy class $[\gamma]$) which is absolutely not true in general even in the class of Steklov isospectral metrics. This is a significant obstacle to linearizing the problem, but we can nevertheless consider a weaker version, which is the subject of the following discussion.

Differentiating for each free homotopy class the equality of marked length spectra (given by **Lemma III.2.4**) with respect to the parameter, we obtain[4] that the first variation

$$\dot{g}^\circ = \frac{\partial}{\partial s}\Big|_{s=0} g_s^\circ$$

has to be in the kernel of the X-ray transform $\mathcal{I}_2^{g^\circ}$:

Proposition III.2.6. *For any free homotopy class* $[\gamma] \in \mathcal{C}$,

$$\mathcal{I}_2^{g^\circ} \dot{g}^\circ([\gamma]) = 0 .$$

Denoting for all $k \in \mathbb{N}_{\geq 2}$,

$$(g^\circ)^{(k)} = \frac{\partial^k}{\partial s^k}\Big|_{s=0} g_s^\circ$$

[4]Direct by **Proposition III.2.3**.

the k-th variation of the metric tensor and using (III.2.1), we observe more generally that for all $[\gamma] \in \mathcal{C}$,

$$\mathcal{I}_2^{g^\circ}\,(g^\circ)^{(k)}([\gamma]) \;+\; \widetilde{\mathcal{R}}_k([\gamma])\left[\dot{g}^\circ, (g^\circ)^{(2)}, ..., (g^\circ)^{(k-1)}\right] = 0\,, \tag{III.2.5}$$

where the second term is a finite sum of integrals over the closed g°-geodesic $\gamma := \gamma_0$ of (tensorial) polynomials in the lower-order variations $\dot{g}^\circ$, $(g^\circ)^{(2)}$,..., $(g^\circ)^{(k-1)}$ and their covariant derivatives of order at most $k-2$ along γ. The fact that the left-hand side of (III.2.1) is zero follows from differentiation at $s=0$ of the equality from **Lemma III.2.4**.

Finally, it should be noted that in all this, it is not essential to assume that the deformation preserves the Anosov property at the boundary, due to the structural stability of an Anosov flow and the fact that we differentiate at $s=0$.

Appendix III.A Another proof of Proposition III.1.10

We give here an alternative proof to **Proposition III.1.10**, even though an essentially equivalent argument appears in an exercise of [Lef25, §11.1]. The proof given here was written independently, before the author became aware of the formulation in terms of Livšic theory for cocycles and of Paternain's argument. We have kept it because it is quite explicit: it shows directly which objects are being transported along the flow and how the cohomological equation arises. It is also more in the spirit of the initial proof of the standard Livšic theorem **Theorem III.1.1** and uses Guillarmou's operator Π (see [Gui17, Corollary 1.2]) to show smooth regularity. Note that this argument requires the additional assumption that $\mathcal{M}$ is connected and that the flow preserves a smooth measure.
To avoid going into unnecessary detail, we simply give precise references to the important notions used, without repeating them. In general, readers may refer to [Lef25, Chapters 8,11].

Alternative proof of $(1) \Rightarrow (2)$. We first deal with Hölder regularity, that is, we assume $f \in C^\alpha(\mathcal{M})$, $\alpha \in (0,1)$. We denote by d the distance induced by any Riemannian metric on $\mathcal{M}$ and follow the proof of [FH19, Claim 6.3.3]. Let us consider $x_0 \in \mathcal{M}$ a point whose orbit $\mathcal{O}(x_0)$ is dense in $\mathcal{M}$ (exists by transitivity of the flow) and define along the orbit

$$u(\phi_t x_0) = exp\left(i\int_0^t f(\phi_s x_0)ds\right).$$

Let $t_1 < t_2$ such that $\varepsilon := d(\phi_{t_1}(x_0), \phi_{t_2}(x_0))$ is small enough for the Anosov closing lemma to hold (see [FH19, Theorem 5.3.11] and [FH19, Theorem 6.2.4]).
Then there exists a T-periodic orbit with $|T - t_2 + t_1| < \varepsilon$, which shadows (we refer to [FH19, Section 5.3]) the orbit segment of extremities $\phi_{t_1}(x_0)$ and $\phi_{t_2}(x_0)$. Moreover, this periodic orbit contains a point $p \in \mathcal{M}$ of period $T + \tau$, with $|\tau| \leq C\varepsilon$ such that for all $s \in [0, T]$,

$$d(\phi_{t_1+s}(x_0), \phi_s(p)) \leq C\varepsilon\eta^{\min(s,T-s)},$$

where $C, \eta > 0$. By assumption,

$$\int_0^{T+\tau} f(\phi_s p)ds = 2k\pi \in 2\pi\mathbb{Z}.$$

Writing $x = \phi_{t_1}(x_0)$, $y = \phi_{t_2}(x_0)$ and $\tilde{x} = \phi_{t_1+T}(x_0)$ we get :

$$\begin{aligned}
|u(x) - u(\tilde{x})| &= \left|exp\left(i\int_0^{t_1} f(\phi_s x_0)ds\right) - exp\left(i\int_0^{t_1+T} f(\phi_s x_0)ds\right)\right| \\
&= \left|exp\left(i\int_0^{t_1} f(\phi_s x_0)ds + 2ik\pi\right) - exp\left(i\int_0^{t_1+T} f(\phi_s x_0)ds\right)\right| \\
&\leq \left|\int_0^T f(\phi_s x)ds - 2k\pi\right| \\
&\leq \left|\int_0^T [f(\phi_s x) - f(\phi_s p)]\,ds + 2k\pi - \int_T^{T+\tau} f(\phi_s p)ds - 2k\pi\right| \\
&\leq \left|\int_0^T [f(\phi_s x) - f(\phi_s p)]\,ds - \int_T^{T+\tau} f(\phi_s p)ds\right| \\
&\leq \|f\|_{C^\alpha}\int_0^T d\left(\phi_s x, \phi_s p\right)^\alpha ds + \|f\|_\infty |\tau| \\
&\leq C^\alpha\varepsilon^\alpha\|f\|_{C^\alpha}\int_0^T \eta^{\alpha\min(s,T-s)}ds + C\varepsilon\|f\|_\infty
\end{aligned}$$

On the other hand,

$$|u(y) - u(\tilde{x})| \leq \left| \int_0^{t_2} f(\phi_s x_0) ds - \int_0^{t_1+T} f(\phi_s x_0) ds \right| \leq \varepsilon \|f\|_\infty .$$

Consequently, u is C^α on $\mathcal{O}^\alpha(x_0)$ and since $\mathcal{O}^\alpha(x_0)$ is dense in $\mathcal{M}$, u admits a unique C^α-extension in $\mathcal{M}$ and it satisfies $Xu = ifu$.

Now we assume that $f \in C^\infty(\mathcal{M})$ and we want to show that $u \in C^\infty(\mathcal{M})$.
As already mentioned, we use Guillarmou's Π operator introduced in [Gui17].
We refer the reader to [Gui17, Section 2.3]. As $C^\alpha(N) \hookrightarrow H^{\alpha-\epsilon}(N)$ [5] (continuous embedding) for any $\epsilon \in]0, \alpha[$, we can apply Guillarmou's Π operator to ifu .
By [Gui17, Theorem 2.6], we obtain

$$\Pi(ifu) = \Pi(Xu) = 0$$

and then there exists a smooth function v and a constant w such that

$$Xv + w = ifu = Xu .$$

Note that $w = 0$ as ifu integrates to 0 along every closed geodesics. Then

$$X(u - v) = 0$$

and by ergodicity of the flow (Anosov flow preserving a smooth measure), $u - v$ is constant. In particular, u is smooth. □

Appendix III.B Cohomology of a sphere bundle

In this appendix, we show how to use the algebraic tool called Gysin sequences (see [BT82, Chapter 14, p.177-179]), mentioned in Section III.1.3 and describing the cohomology of a sphere bundle, to prove an important argument used in the proof of **Theorem III.1.11**. As explained, the argument is actually known and used, for example, by G. Paternain in [Pat09] but, to the author's knowledge, the proof is never really written explicitly in the literature. For convenience, we write this elementary

[5] This is an easy consequence of the following characterization of Sobolev spaces :

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x+y) - u(x)|^2}{|y|^{n+2s}} dxdy < \infty\} .$$

proof here.
In the following, given a smooth manifold N, we denote $H^k_{\mathrm{dR}}(N)$ its de Rham cohomology group of degree k.

Lemma III.B.1. *Assume that the closed Riemannian manifold (N, h) of dimension n satisfies one of the following conditions :*

- *$n = 2$ and the Euler characteristic satisfies $\chi(N) \neq 0$.*
- *$n \geq 3$.*

If $\pi : SN \to N$ denotes the projection, then

$$\pi^* : H^1_{\mathrm{dR}}(N) \to H^1_{\mathrm{dR}}(SN)$$

is an isomorphism.

Proof. If $n = 2$, the Gysin exact sequence (see [BT82, Proposition 14.33]) for the unit circle bundle $SN \to N$ gives :

$$0 \to H^1_{\mathrm{dR}}(N) \xrightarrow{\pi^*} H^1_{\mathrm{dR}}(SN) \xrightarrow{\pi_*} H^0_{\mathrm{dR}}(N) = \mathbb{R} \xrightarrow{\wedge e} H^2_{\mathrm{dR}}(N) = \mathbb{R} \to \cdots$$

where π_*, $\wedge e$, π^* are respectively integration along the fiber, multiplication by the Euler class and pullback. Therefore we have

$$\ker \pi^* = \{0\}$$

as well as

$$\operatorname{ran} \pi^* = \ker \pi_* \, .$$

Besides,

$$\dim H^1_{\mathrm{dR}}(SN) = \dim \ker \pi_* + \dim \operatorname{ran} \pi_* = \dim \ker \pi_* + \dim \ker(\wedge e)$$

and since the multiplication by the characteristic class

$$e = \frac{1}{2\pi}[K_h dV_h]$$

has trivial kernel unless e is exact, i.e. unless the Euler characteristic of the surface satisfies

$$\chi(N) = \frac{1}{2\pi}\int_N K_h dV_h = 0\,,$$

this implies

$$\operatorname{ran}\pi^* = \ker\pi_* = H^1_{\mathrm{dR}}(SN)$$

and proves that π^* is an isomorphism.

The case $n \geq 3$ is straightforward, as the Gysin exact sequence takes a simpler form, giving in particular :

$$0 \to H^1_{\mathrm{dR}}(N) \xrightarrow{\pi^*} H^1_{\mathrm{dR}}(SN) \to 0\,.$$

□

Remark III.B.2. *The first condition in the previous lemma is equivalent to saying that N is not the 2-torus, this point is verified if N is, for example, an Anosov surface. This follows, for example, from [Ghy84, Théorème A], which characterizes Anosov flows on 3-manifolds that are circle bundles. Applied to the unit fiber bundle of a Riemannian surface, it shows, in particular, that the 2-torus cannot carry any Anosov metric.*

Appendix III.C Another proof of Theorem III.1.11

This alternative proof is taken from [dSFF26, §3], where the natural preference was to perform calculations in the cotangent space rather than the tangent space. To keep this Appendix concise, we present here only the two-dimensional case, simply to illustrate the idea, since the geometry of the unitary cotangent bundle is much nicer there. We refer to [dSFF26, §3.2] for the case of higher dimensions.

The two-dimensional case

In dimension two, one can choose local isothermal coordinates in which the metric reads

$$g = e^{2\varphi}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right)$$

and the symbol p of the Laplacian

$$p(x,\xi) = |\xi|_x^2 = e^{-2\varphi}\left(\xi_1^2 + \xi_2^2\right).$$

We denote by

$$K = -\Delta_g \varphi$$

the Gauss curvature[6] of the manifold (N, h). The Hamiltonian vector field of the symbol p takes the simple form

$$\begin{aligned} H_p &= \frac{\partial p}{\partial \xi} \cdot \frac{\partial}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial}{\partial \xi} \\ &= 2p(x,\xi)\left(\frac{\xi_1}{\xi_1^2 + \xi_2^2} \frac{\partial}{\partial x_1} + \frac{\xi_2}{\xi_1^2 + \xi_2^2} \frac{\partial}{\partial x_2} + \partial_{x_1}\varphi \frac{\partial}{\partial \xi_1} + \partial_{x_2}\varphi \frac{\partial}{\partial \xi_2} \right). \end{aligned}$$

The group $(\mathbb{R}_+^*, \times)$ acts on the cotangent bundle by multiplication in the fibre variables

$$M_\lambda(x,\xi) = (x, \lambda\xi)$$

and the Euler (or radial, or canonical) vector field is the infinitesimal generator of the one parameter group $(M_{e^t}^*)_{t\in\mathbb{R}}$

$$\varrho u = \frac{\partial}{\partial t} M_{e^t}^* u \big|_{t=0} .$$

It reads in local coordinates

$$\varrho = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} .$$

Similarly rotations act on the cosphere bundle

$$S^*N = \left\{ (x,\xi) \in T^*N \,:\, \xi_1^2 + \xi_2^2 = e^{2\varphi(x)} \right\}$$

in the fiber variables

$$(x,\xi) \mapsto (x, R_\theta \xi), \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and we denote its infinitesimal generator by V

$$Vu := \frac{\partial}{\partial \theta} u(x, R_\theta \xi)|_{\theta=0} .$$

The unit cotangent bundle can be parametrized by the angle variable θ

$$S^*N \ni \xi = (\xi_1, \xi_2) = e^{\varphi}(\cos\theta, \sin\theta)$$

[6] That is one half of the scalar curvature.

and the Euler vector field ϱ is normal to the tangent space of the cosphere bundle

$$T_{(x,\theta)}(T^*N) = \mathbb{R}\varrho \oplus T_{(x,\theta)}(S^*N) .$$

We consider the following vector field belonging to the tangent space of the cosphere bundle

$$H = e^{-\varphi}\Big(\cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + a_\varphi \frac{\partial}{\partial \theta}\Big)$$

where the coefficient a_φ is given by

$$a_\varphi = -\sin\theta\, \partial_{x_1}\varphi + \cos\theta\, \partial_{x_2}\varphi .$$

The Hamiltonian vector field decomposes as

$$\frac{1}{2}H_p = pe^{\varphi} b_\varphi \varrho + \sqrt{p} H$$

where the coefficient b_φ is given by

$$b_\varphi = \cos\theta\, \partial_{x_1}\varphi + \sin\theta\, \partial_{x_2}\varphi .$$

Now we set ourselves in the cosphere bundle and consider H, V acting on functions on S^*M. We introduce

$$H_\perp = [H, V]$$

so that in coordinates

$$H_\perp = e^{-\varphi}\Big(\sin\theta \frac{\partial}{\partial x_1} - \cos\theta \frac{\partial}{\partial x_2} + b_\varphi \frac{\partial}{\partial \theta}\Big) .$$

The vector fields satisfy the following bracket algebra

$$[V, H_\perp] = H, \quad [H_\perp, H] = KV .$$

We now start with a solution $u \in H^2(S^*N; \mathbb{S}^1)$ of the transport equation with real-valued potential $f \in C^\infty(S^*N)$

$$(H + if)u = 0 . \tag{III.3.6}$$

Consider the following functions in $H^1(S^*N)$

$$\alpha = i\bar{u}H_\perp u, \quad \beta = i\bar{u}Vu\,.$$

From the fact that u has values in the circle

$$|u|^2 = 1\,,$$

we get that α and β are real-valued functions :

$$\operatorname{Im}\alpha = \operatorname{Re}(\bar{u}H_\perp u) = \frac{1}{2i}H_\perp(|u|^2) = 0, \quad \operatorname{Im}\beta = \operatorname{Re}(\bar{u}Vu) = \frac{1}{2}V(|u|^2) = 0\,.$$

Besides, from the transport equation (III.3.6), we have

$$\begin{aligned} H\alpha &= iH\bar{u}\,H_\perp u + i\bar{u}\,HH_\perp u = if\alpha - K\beta + \bar{u}H_\perp(fu) \\ &= if\alpha - K\beta + H_\perp f - if\alpha = -K\beta + H_\perp f \end{aligned}$$

and similarly

$$\begin{aligned} H\beta &= iH\bar{u}\,Vu + i\bar{u}\,HVU = if\beta + i\bar{u}\,VHu + \alpha \\ &= if\beta + Vf - if\beta + \alpha = \alpha + Vf\,. \end{aligned}$$

Therefore we get the following system of equations on (α, β)

$$\begin{cases} H\alpha = -K\beta + H_\perp f \\ H\beta = \alpha + Vf \end{cases}$$

which we can rewrite under the matrix form

$$\begin{pmatrix} 1 & -H \\ H & K \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -Vf \\ H_\perp f \end{pmatrix}\,.$$

Having endowed the product space $L^2(S^*N) \times L^2(S^*N)$ with the scalar product

$$\left(\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}\middle|\begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}\right) = (\phi_1|\phi_2) + (\psi_1|\psi_2)\,,$$

where $(\cdot\,|\,\cdot)$ denotes the scalar product on $L^2(S^*N)$, the matrix operator

$$\mathcal{Q} = \begin{pmatrix} 1 & -H \\ H & K \end{pmatrix}$$

with domain $\mathrm{Dom}(H)\times\mathrm{Dom}(H) \subset L^2(S^*N)\times L^2(S^*N)$ defines a quadratic form which can be extended to $\mathrm{Dom}(H) \times L^2(S^*N)$

$$\begin{aligned}\left(\mathcal{Q}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\middle|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right) &= \|\phi\|^2 - 2\,\mathrm{Re}(\phi|H\psi) + (K\psi|\psi) \\ &= -\|H\psi\|^2 + (K\psi|\psi) + \|\phi - H\psi\|^2. \end{aligned} \tag{III.3.7}$$

Lemma III.1. *The operator $\mathcal{Q}$ is self-adjoint*

$$\mathcal{Q}^* = \mathcal{Q}\,.$$

and its kernel is trivial

$$\ker \mathcal{Q} = \{0\}\,.$$

Therefore the polarisation of (III.3.7) *is a non-degenerate sesquilinear form. Its restriction to the graph of H is definite negative.*

Proof. The formal adjoint of $\mathcal{Q}$ reads

$$\begin{pmatrix} 1 & -H \\ H & K \end{pmatrix}^* = \begin{pmatrix} 1 & H^* \\ -H^* & K \end{pmatrix} = \begin{pmatrix} 1 & -H \\ H & K \end{pmatrix},$$

hence the operator $\mathcal{Q}$ is symmetric and since $(\phi_2, \psi_2) \in D(\mathcal{Q}^*)$ if and only if

$$\begin{aligned}\left|\left(\mathcal{Q}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}\middle|\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\right)\right| &= |(\phi_1|\phi_2) - (H\phi_1|\psi_2) - (H\psi_1,|\phi_2) + (K\psi_1|\psi_2)| \\ &\leq C\sqrt{\|\phi_1\|^2 + \|\psi_1\|^2}\end{aligned}$$

for all $(\phi_1, \psi_1) \in \mathrm{Dom}(H) \times \mathrm{Dom}(H)$, choosing $\psi_1 = 0$ we deduce $\psi_2 \in \mathrm{Dom}(H)$ and choosing $\phi_1 = 0$ we deduce $\phi_2 \in \mathrm{Dom}(H)$. Hence $\mathcal{Q}$ is self-adjoint. Besides, if $(\phi, \psi) \in \ker \mathcal{Q}$, we get from (III.3.7)

$$\phi = H\psi, \qquad \text{and} \qquad \|H\psi\|^2 - (K\psi|\psi) = 0\,.$$

If (N,h) has no conjugate points this implies $\psi = 0$ and therefore $\phi = 0$. The negativity of the restriction of the quadratic form (III.3.7) to the graph of H comes from the fact that the absence of conjugate points implies $\|H\psi\|^2 - (K\psi|\psi \geq 0$ for all $\psi \in \mathrm{Dom}(H)$ (see [Hop48]). □

Now we have

$$\left(\begin{pmatrix} -Vf \\ H_\perp f \end{pmatrix} \middle| \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = (f|V\alpha - H_\perp \beta)$$

and it remains to calculate the right-hand side term.

Lemma III.2. $H_\perp \beta = V\alpha - f$.

Proof. Using the commutation rules of the vector fields V and $H_\perp$, we get

$$\begin{aligned} H_\perp \beta &= i(H_\perp \bar{u})\, Vu + i\bar{u} H_\perp Vu = i\bar{\alpha}\,\beta + i\bar{u} V H_\perp u - i\bar{u} H u \\ &= i\bar{\alpha}\,\beta + V\alpha - i\alpha\,\beta - f \end{aligned}$$

and this gives the expression since α and β are real-valued. □

Proof of ***Theorem III.1.11*** *in the two-dimensional case.* Taking the scalar product of

$$\mathcal{Q}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -Vf \\ H_\perp f \end{pmatrix}$$

with (α, β) and using **Lemma III.2**, we get

$$\mathcal{Q}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\begin{pmatrix} H_\perp f \\ -Vf \end{pmatrix} \middle| \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = \|f\|^2.$$

This gives the identity

$$-\|H\beta\|^2 + (K\beta|\beta) + \|\alpha - H\beta\|^2 = \|f\|^2$$

and since $\alpha - H\beta = -Vf$,

$$-\|H\beta\|^2 + (K\beta|\beta) + \|Vf\|^2 = \|f\|^2. \tag{III.3.8}$$

Until now we have not used the special form of the function f

$$f(x,\theta) = f_0(x) + f_1(x)\cos\theta + f_2(x)\sin\theta$$

which implies that f solves the equation

$$V^2 f + f = f_0 \,.$$

This implies in turn

$$\|f\|^2 - \|Vf\|^2 = (V^2 f + f|f) = (f_0|f) = \|f_0\|^2$$

and therefore (III.3.8) reads

$$\underbrace{-\|H\beta\|^2 + (K\beta|\beta)}_{\leq 0} = \underbrace{\|f_0\|^2}_{\leq 0} \,.$$

This finally gives $f_0 = 0$ and since the left-hand side is negative definite in the absence of conjugate points by the aforementioned classic result of Hopf [Hop48],

$$\beta = 0 \,.$$

Hence $Vu = 0$, which means that $\vartheta := u$ does not depend on the fiber variable θ. The transport equation now takes the form

$$f = i\bar{\vartheta}\,\mathrm{d}\vartheta = i\frac{\mathrm{d}\vartheta}{\vartheta}$$

and we conclude that f is closed :

$$\mathrm{d}f = 0 \,.$$

Note that we have shown the existence of a function $\vartheta : N \to \mathbb{S}^1$ such that $f = i\bar{\vartheta}\,\mathrm{d}\vartheta$. □

Appendix III.D Higher-order variation of the marked length spectrum

This appendix is devoted to discuss in more details the differentiation formula (III.2.1).

Let $(h_s)_{0\leq s\leq \varepsilon}$ be a smooth family of Anosov Riemannian metrics on a closed manifold N, and for $k \geq 1$, let

$$h := h_0 \quad , \quad h^{(k)} := \left.\frac{\partial^k}{\partial s^k}\right|_{s=0} h_s \, .$$

For each $[\gamma] \in \mathcal{C}$, the unique h-geodesic γ in $\mathcal{C}$ deforms smoothly as a closed geodesic γ_s for h_s in $[\gamma]$, so that $s \mapsto \mathcal{L}_{h_s}([\gamma])$ is smooth and

$$\mathcal{L}_{h_s}([\gamma]) = \ell_{h_s}(\gamma_s) \, .$$

Proposition III.D.1. *For every $k \geq 1$,*

$$\left.\frac{\partial^k}{\partial s^k}\right|_{s=0} \mathcal{L}_{h_s}([\gamma]) = \frac{1}{2}\,\mathcal{L}_h([\gamma])\,\mathcal{I}_2^h\big(h^{(k)}\big)([\gamma]) + \mathcal{R}_k([\gamma])\big[\dot{h}, h^{(2)}, \dots, h^{(k-1)}\big] \, , \qquad \text{(III.4.9)}$$

where the remainder term is a finite sum of integrals over the closed h-geodesic $\gamma := \gamma_0$ of tensorial polynomials in the lower-order variations

$$\dot{h},\ h^{(2)},\ \dots,\ h^{(k-1)}$$

and their covariant derivatives along γ, of order at most $k-2$.

Sketch of proof. To avoid making this unnecessarily complicated, we will focus here on describing the main ideas of the calculation without going into too much technical details.

Fix $[\gamma] \in \mathcal{C}$, and let

$$L(s) := \mathcal{L}_{h_s}([\gamma]) \, .$$

Choose γ_s parametrized by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with constant speed and consider the energy functional

$$E_s \,:\, \alpha \mapsto \int_0^1 h_s(\dot{\alpha}(t), \dot{\alpha}(t))\, dt \, .$$

Since γ_s is a closed h_s-geodesic[7] with constant speed,

$$E_s(\gamma_s) = L(s)^2 \quad , \quad \mathrm{d}E_s(\gamma_s) = 0 \, .$$

By differentiating $s \mapsto E_s(\gamma_s)$ exactly k times at $s = 0$ via the chain rule, we see that the only term involving the k-th variation $h^{(k)}$ and no derivative of the curve $s \mapsto \gamma_s$ is

$$\left.\frac{\partial^k}{\partial s^k}\right|_{s=0} E_s(\gamma) = \int_0^1 h^{(k)}(\dot\gamma(t), \dot\gamma(t))\, dt = \mathcal{L}_h([\gamma])^2 \, \mathcal{I}_2^h(h^{(k)})([\gamma]) \, .$$

Indeed, all the other terms contain at least one derivative

$$\Gamma_j := \left.\frac{\partial^j}{\partial s^j}\right|_{s=0} \gamma_s \, , \quad j \geq 1$$

and these derivatives are obtained recursively by differentiating the critical point equation

$$\mathrm{d}E_s(\gamma_s) = 0 \quad :$$

At order j, we obtain for all vector fields Ξ along γ an equation of the form

$$\mathrm{d}^2 E_0(\gamma)\,[\Gamma_j, \Xi] = \mathcal{S}_j(\Xi)$$

where $\mathcal{S}_j$ is a linear form in Ξ depending only on the metric terms $\dot h,\ h^{(2)}, \dots, h^{(j)}$ and on the lower-order variations $\Gamma_1, \dots, \Gamma_{j-1}$. Hence, by induction[8] on j, each Γ_j depends only on $\dot h,\ h^{(2)},\ \dots,\ h^{(j)}$ and consequently, every term in the chain rule formula other than

$$\left.\frac{\partial^k}{\partial s^k}\right|_{s=0} E_s(\gamma)$$

depends only on the lower-order metric terms $\dot h,\ h^{(2)},\ \dots,\ h^{(k-1)}$.

On the other hand, since $E_s(\gamma_s) = L(s)^2$,

$$\left.\frac{\partial^k}{\partial s^k}\right|_{s=0} E_s(\gamma_s) = 2\,\mathcal{L}_h([\gamma])\, L^{(k)}(0) + Q_k[L'(0), \dots, L^{(k-1)}(0)] \, ,$$

where Q_k is a universal polynomial expression. By induction on k, each lower derivative $L^{(j)}(0)$, $j < k$, is already of the form claimed in the proposition, hence depends

[7] it is in particular a critical point of the energy functional E_s.

[8] provided the Hessian is invertible, but this is the case here, it follows from the fact that γ is hyperbolic since h is Anosov.

only on $h^{(1)}, \dots, h^{(j)}$.

Putting everything together exactly gives (III.4.9). □

Remark III.D.2 (The case $k = 2$). *Let*

$$\Gamma_1 := \frac{\partial}{\partial s}\Big|_{s=0} \gamma_s \,.$$

Differentiating once the critical point equation $\mathrm{d}E_s(\gamma_s) = 0$ *at* $s = 0$*, one gets for all* Ξ

$$\mathrm{d}^2 E_0(\gamma)[\Gamma_1, \Xi] + \mathrm{d}\Big(\frac{\partial}{\partial s}\Big|_{s=0} E_s\Big)(\gamma)[\Xi] = 0 \,.$$

Evaluating at $\Xi = \Gamma_1$*, this becomes*

$$\mathrm{d}\Big(\frac{\partial}{\partial s}\Big|_{s=0} E_s\Big)(\gamma)[\Gamma_1] = -\mathrm{d}^2 E_0(\gamma)[\Gamma_1, \Gamma_1] \,.$$

Differentiating twice the identity $E_s(\gamma_s) = L(s)^2$*, one finds*

$$\mathcal{L}_h([\gamma])^2\, \mathcal{J}_2^h\big(h^{(2)}\big)([\gamma]) - \mathrm{d}^2 E_0(\gamma)[\Gamma_1, \Gamma_1] = 2\,\mathcal{L}_h([\gamma])\, L''(0) + 2\big(L'(0)\big)^2 \,,$$

hence

$$L''(0) = \frac{1}{2}\,\mathcal{L}_h([\gamma])\, \mathcal{J}_2^h\big(h^{(2)}\big)([\gamma]) - \frac{1}{2\,\mathcal{L}_h([\gamma])}\, \mathrm{d}^2 E_0(\gamma)[\Gamma_1, \Gamma_1] - \frac{(L'(0))^2}{\mathcal{L}_h([\gamma])} \,.$$

In particular, if the marked length is preserved up to order 2*, so that* $L'(0) = L''(0) = 0$*, then*

$$\mathcal{J}_2^h\big(h^{(2)}\big)([\gamma]) = \frac{1}{\mathcal{L}_h([\gamma])^2}\, \mathrm{d}^2 E_0(\gamma)[\Gamma_1, \Gamma_1] \,,$$

with Γ_1 *depending only on* $\dot{h}$*.*

Chapter IV

Positive results for the Steklov inverse problem

Table of Contents for the chapter

IV.1 Boundary determination of Riemannian metrics 107
IV.1.1 Steklov isospectral deformations of conformal metrics 107
IV.1.2 Non-conformal deformations: an approach and perspectives . . . 110
IV.1.3 The non-linear problem for conformal metrics 113
IV.1.4 The non-linear problem for general metrics: Perspectives 125
IV.2 Boundary determination of potentials 127
IV.2.1 Recovery of electric potentials with no magnetic influence 127
IV.2.2 Recovery of electric and magnetic potentials 129
IV.2.3 Towards the framework of connections 142
Appendices to § IV.1.3 144
Appendix IV.A. Dirichlet-to-Neumann map of a conformal metric . . 145
Appendix IV.B. On the recovery of the conformal factor 147

Armed with the wave invariants and the tools to interpret them, we are now ready to prove the main theorems on the inverse problems for the Steklov spectrum. As explained in Chapter I, the information recovered is the boundary jets of either metrics, electric potentials, or magnetic fields depending on the problem under consideration. This boundary determination is enough for global recovery in the usual case where the data to recover are real-analytic and the manifold is connected.

Regarding the magnetic case, let us mention that the proof of the additional result on the magnetic Schrödinger operator $P_{a,q}$, formulated in **Theorem I.3.3**, is given at the beginning of Subsection IV.2.2 below. As already mentioned, unlike the mag-

netic Schrödinger operator, which is a differential operator of order 2, the magnetic Dirichlet-to-Neumann map is a classical pseudodifferential operator of order 1. Its full symbol is considerably richer: in addition to the principal symbol, it contains lower-order contributions carrying geometric and magnetic information. The subprincipal symbol is the first of these, appearing through the principal wave invariants in the Duistermaat-Guillemin formula.

Remark (Subprincipal symbols and half-densities). *In this work, the subprincipal symbols are defined for operators acting on half-densities, although they are used below for operators acting on functions. Let us briefly explain how to switch between the two frameworks and why this makes no difference to our work.*

Let $P \in \Psi^m(N)$ be an operator acting on functions, and let μ be a positive smooth half-density on N. In local coordinates $x = (x_1, \dots, x_n)$, write

$$\mu = \mu_x(x)|dx|^{1/2} .$$

In these coordinates, the operator P^μ on half-densities associated with P (obtained by the identification $u \mapsto u\mu$) is written locally as

$$P^\mu_x := \mu_x P \mu_x^{-1} .$$

Set

$$\mathrm{sub}_\mu(P) := \mathrm{sub}(P^\mu)$$

and assume that P is classical. Writing (for the usual left quantization), in the local coordinates (x, ξ),

$$\sigma_P^{\mathrm{full}}(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots ,$$

we have,

$$\mathrm{sub}_\mu(P) = p_{m-1} - \frac{1}{2i}\sum_{j=1}^{n} \partial_{x_j}\partial_{\xi_j} p_m + iH_{p_m}(\log \mu_x) . \tag{IV.0.1}$$

We refer for example to [DH72, §5.2]. In particular, if an operator $Q \in \Psi^m_{cl}(N)$ acts on functions and have the same principal symbol as that of P, then, for any two positive half-densities μ and $\tilde{\mu}$ on N,

$$\mathrm{sub}_{\tilde{\mu}}(P) - \mathrm{sub}_{\tilde{\mu}}(Q) = \mathrm{sub}_\mu(P) - \mathrm{sub}_\mu(Q)$$

and in particular, there is no additional density term that appears in the expression of such quantities. This is, of course, a direct consequence of (IV.0.1) applied for P and for

Q . In other words, if the two operators have the same principal symbol, the computation of this difference at subprincipal order is independent of the choice of a half-density.

For the DN maps considered below we use the Riemannian half-density

$$\mu_{g^\circ} := |dV_{g^\circ}|^{1/2} .$$

In the conformal case, if $c|_{\partial M} = 1$, then $\sigma_{\Lambda_{cg}} = \sigma_{\Lambda_g}$ and the quantity

$$\mathrm{sub}_{\mu_{g^\circ}}(\Lambda_{cg}) - \mathrm{sub}_{\mu_{g^\circ}}(\Lambda_g)$$

does not depend on μ_{g° . In the magnetic case, the metric is fixed and $\sigma_{\Lambda_{a,q}}(x,\xi) = |\xi|_{g^\circ}$ is independent of a and q. Therefore the same statement holds for

$$\mathrm{sub}_{\mu_{g^\circ}}(\Lambda_{\tilde{a},\tilde{q}}) - \mathrm{sub}_{\mu_{g^\circ}}(\Lambda_{a,q}) .$$

IV.1 Boundary determination of Riemannian metrics

This section is devoted to the boundary determination results for the metric problem. The first section deals with the case of isospectral deformations within a conformal class, while the following section presents the case of general metric deformations, which is currently only an overview. Finally, we turn to the proof of the non-linear problem in a conformal class, followed once again by a section devoted to perspectives on the non-linear problem for general metrics.

IV.1.1 Steklov isospectral deformations of conformal metrics

The aim of this part is to prove **Theorem I.1.2** which establish an infinitesimal spectral rigidity result for conformal metric deformations.

The first thing to note is that, if we consider a deformation of isospectral conformal metrics $(g_s) := (c_s g)$, any variation

$$(g^\circ)^{(m)} := \frac{\partial^m}{\partial s^m}\Big|_{s=0} g_s^\circ$$

is reduced to $\left(\partial_s^m c_s|_{s=0}\right) g^\circ$, so if each of these derivatives is zero at the boundary up to order k, using (III.2.5) we get:

$$\forall\, [\gamma] \in \mathcal{C} \ , \ \mathcal{I}_0^{g^\circ} \partial_s^k c_s|_{s=0}([\gamma]) = 0 . \tag{IV.1.2}$$

Consequently, the first part of the statement in **Theorem** I.1.2 follows:

Proposition IV.1.1. *Let (M, g) be a smooth compact Riemannian manifold with boundary. Assume that g° is Anosov with simple length spectrum. Let $(g_s)_{0\leq s\leq \varepsilon}$ be a Steklov isospectral deformation of the metric g s.t there exists a family of smooth functions $\left(c_s : M \mapsto \mathbb{R}_*^+\right)_{0\leq s\leq\varepsilon}$ satisfying for any $s \in [0, \varepsilon]$,*

$$g_s = c_s g \, .$$

Then the boundary deformation is flat at $s = 0$:

$$\forall k \in \mathbb{N}^* \, , \; \partial_s^k c_s|_{s=0} = 0 \;\; in \; \partial M.$$

In particular, if $s \mapsto c_s$ is real-analytic, then for all $s \in [0, \varepsilon]$,

$$g_s^\circ = g^\circ \, . \tag{IV.1.3}$$

Proof. First, **Proposition** III.2.6 combined with the injectivity of the X-ray transform on functions for closed Anosov manifolds (**Theorem** III.1.4) gives $\partial_s c_s|_{s=0} = 0$. Then, as already mentioned, the proof is done by induction on the order of derivation k of the length maps $s \mapsto \mathcal{L}_{g_s^\circ}([\gamma])$, $[\gamma] \in \mathcal{C}$. At each step, we combine (IV.1.2) with the injectivity of the X-ray transform on functions. □

Provided that (IV.1.3) holds, combining it with **Theorem** II.3.4, we are able to prove the second part of the infinitesimal spectral rigidity result (*I*.1.2) :

Proposition IV.1.2. *Let (M, g) be a smooth compact Riemannian manifold with boundary such that $(\partial M, g^\circ)$ is Anosov with simple length spectrum.*
Let $(g_s)_{0\leq s\leq \varepsilon}$ be a Steklov isospectral deformation of the metric g s.t there exists a family of smooth functions $\left(c_s : M \mapsto \mathbb{R}_*^+\right)_{0\leq s\leq\varepsilon}$ satisfying for any $s \in [0, \varepsilon]$,

$$g_s = c_s g$$

and

$$c_{s\,|\partial M} \equiv 1 \, .$$

Then for all $s \in [0, \varepsilon]$,

$$\partial_\nu c_{s\,|\partial M} \equiv 0 \, .$$

Proof. For all $s \in [0, \varepsilon]$, $g_s^\circ = g^\circ$, so the closed geodesics for g_s° are exactly the closed geodesics for g° and since g_s° and g° are Anosov with simple length spectrum, then

by isospectrality, the principal wave trace invariant given by **Theorem II.3.4** remains constant :

$$\frac{|T_\gamma^\sharp| e^{-i\int_\gamma \mathrm{sub}(\Lambda_{g_s})}}{|\det\left(Id-\mathcal{P}_\gamma\right)|^{1/2}} = \frac{|T_\gamma^\sharp| e^{-i\int_\gamma \mathrm{sub}(\Lambda_{g})}}{|\det\left(Id-\mathcal{P}_\gamma\right)|^{1/2}} ,$$

for any $\gamma := \gamma_0$ closed orbit of the (co)geodesic flow for g° with period T_γ (recall that $T_\gamma^\sharp$ is its primitive period).
Thus for all $s \in [0,\varepsilon]$ and any γ closed orbit,

$$exp\left(-i\int_\gamma \left[\mathrm{sub}(\Lambda_{g_s}) - \mathrm{sub}(\Lambda_g)\right]\right) = 1$$

and then by **Lemma IV.A.5** we obtain

$$exp\left(-i\alpha_n \int_\gamma (\partial_\nu c_s)\mid_{\partial M}\right) = 1 .$$

Differentiating with respect to s, we finally get for any $[\gamma] \in \mathcal{C}$,

$$\mathcal{I}_0^{g^\circ}\partial_\nu \dot{c}_s\,([\gamma]) = \int_\gamma (\partial_\nu \dot{c}_s)\mid_{\partial M} = 0 .$$

From there we can conclude that for all $s \in [0,\varepsilon]$,

$$(\partial_\nu \dot{c}_s)\mid_{\partial M} = 0$$

by the injectivity of the X-ray transform on functions for Anosov manifolds (**Theorem III.1.4**). Hence for all $s \in [0,\varepsilon]$, $\partial_\nu c_s = \partial_\nu c_0 = 0$ on ∂M. □

As we mentioned in Subsection I.1.3, once we know that such Steklov isospectral deformations of conformal metrics automatically coincide at the boundary and their normal derivative at the boundary is zero, we easily obtain **Theorem I.1.2**: it suffices to apply the same procedure as in the non-linear problem to each fixed conformal factor c_s. The non-linear problem effectively becomes linear at this stage by the trace formula of **Theorem II.4.2**.

A natural sequel is to look at isospectral deformations of metrics that are no longer in the same conformal class. The main difficulty is that in this case, the linearization provided by (IV.1.2), described in Subsection III.2.2, concern the X-ray transform $\mathcal{I}_2$ on symmetric 2-tensors but contrary to $\mathcal{I}_0$, it has a non-trivial kernel (cf Subsection

III.1.1). The next subsection is devoted to explain how we could deal with this obstruction, and could maybe offer a better understanding to adress the corresponding non-linear problem presented in Subsection IV.1.4.

IV.1.2 Deformations of non-conformally equivalent metrics : an approach and perspectives

This subsection presents ideas and formal statements that could be proven. At this stage, these are merely promising avenues for a solution. They stem from an ongoing project conducted in collaboration with Sebastián Muñoz-Thon (Université Paris-Saclay).

The expected main statement is as follows:

Theorem IV.1.3. *Let (M, g) be a compact Riemannian manifold with boundary, such that $(\partial M, g^\circ)$ is Anosov with simple length spectrum. Assume that the geodesic X-ray transform $\mathcal{I}_2^{g^\circ}$ is s-injective on symmetric 2-tensors of $(\partial M, g^\circ)$.*
Let $(g_s)_{0\leq s\leq \varepsilon}$ be a Steklov isospectral deformation of $g = g_0$. Then, there exists a smooth one-parameter family of diffeomorphism $(\Phi_s)_{0\leq s\leq\varepsilon}$ in M such that for any $s \in [0, \varepsilon]$, $\Phi_s(\partial M) = \partial M$ and for any order $k \in \mathbb{N}^*$,

$$\partial_s^k|_{s=0} \left(\mathring{\Phi}_s^* g_s^\circ\right) = 0 \, .$$

In particular, if the deformation map $s \mapsto g_s$ is real-analytic, then for s sufficiently small,

$$g_s^\circ = \mathring{\Psi}_s^* g^\circ \, ,$$

where $\Psi_s := \Phi_0 \circ \Phi_s^{-1}$.

As mentioned at the end of the previous subsection, the main issue is that the geodesic X-ray transform on 2-tensors has a natural kernel, given by the potential tensors. Our idea is then to put each metric of the deformation in the same solenoidal gauge, in other words to make them divergence-free. This point of view likely goes back to Ebin's work on the action of the group of diffeomorphisms on the space of Riemannian metrics [Ebi68, Ebi70], and is in fact closely related to the decomposition of symmetric tensor fields into a sum of a solenoidal and a potential part [Sha94]. The approach has recently played a major role, for example, in the work of Guillarmou-Lefeuvre on marked length spectrum rigidity for Anosov manifolds [GL19] and even more recently for entropy rigidity problems [Hum26].

This idea is stated in a lemma, known as the *slice lemma*, and is essentially proven in [Ebi68], but we also refer to [CDS00, Theorem 2.1] and to the more recent [GL19, Lemma 4.1] and [Hum26, Lemma 2.3].

In our problem, one needs a boundary version of this construction for one-parameter families of metrics. Here is the statement that we believe could be proven :[1]

Lemma IV.1.4 (Boundary solenoidal reduction for one-parameter deformations). *Let (M, g) be a compact Riemannian manifold with boundary, such that $(\partial M, g^\circ)$ is Anosov[2] and $(g_s)_{0\leq s\leq \varepsilon}$ be a smooth one-parameter family of smooth Riemannian metrics with $g_0 = g$. Then there exists $\eta \in]0, \varepsilon]$ and a smooth one-parameter family of diffeomorphisms $(\Phi_s)_{0\leq s<\eta}$ in M such that for any $s \in [0, \eta[$, $\Phi_s(\partial M) = \partial M$, $\Phi_0 = \mathrm{Id}_M$, and*

$$\delta_{g^\circ}\left(\mathring{\Phi}_s^* g_s^\circ\right) = 0\,.$$

Sketch of proof. We follow the same ideas as in the proof of the usual slice lemma; basically we want to solve locally

$$F(\Phi, s) = 0\,,$$

where F is defined below : Let $k \in \mathbb{N}_{\geq 2}$ and $0 < \alpha < 1$. Consider the Banach manifold $\mathrm{Diff}^{k+1,\alpha}(\partial M)$ of α-Hölder diffeomorphisms of class C^{k+1} and define

$$F \,:\, \mathrm{Diff}^{k+1,\alpha}(\partial M) \times [0, \varepsilon[\longrightarrow C^{k-1,\alpha}(\partial M; T^*\partial M)$$

by

$$F(\phi, s) \,:=\, \delta_{g^\circ}\left(\phi^* g_s^\circ\right)\,.$$

This map is well-defined, $F(\mathrm{Id}, 0) = 0$ and is smooth in a neighborhood of $(\mathrm{Id}, 0)$. For $X \in C^{k+1,\alpha}(T\partial M)$, its derivative with respect to ϕ is given by

$$D_\phi F(\mathrm{Id}, 0)(X) = \delta_{g^\circ}(\mathcal{L}_X g^\circ) = 2\mathrm{D}^*_{g^\circ}\mathrm{D}_{g^\circ} X^\sharp\,,$$

where we recall that D_{g° is the symmetric derivative with respect to g° and $\mathrm{D}^*_{g^\circ} = \delta_{g^\circ}$ is its formal adjoint. Therefore, $\mathrm{D}^*_{g^\circ}\mathrm{D}_{g^\circ}$ is an elliptic differential operator of order 2 and its kernel is the space of Killing fields of $(\partial M, g^\circ)$ which is trivial by ergodicity of the geodesic flow of g°. As a consequence, we get an isomorphism and then one

[1] Once again, let us emphasize that the sketch of proof presented at this stage is only a preliminary draft, not yet a properly proven result.

[2] Ergodicity of the geodesic flow is actually enough.

can apply the (parameter version) implicit function theorem in Banach spaces (see, for exemple, [Lan93, Chapter XIV]), which gives $\eta > 0$ and a unique smooth one-parameter family $(\phi_s)_{0\leq s<\eta}$ such that for any s,

$$\phi_s \in \mathrm{Diff}^{k+1,\alpha}(\partial M)\,,$$

$\phi_0 = \mathrm{Id}_{\partial M}$ and

$$\delta_{g^\circ}(\phi_s^* g_s^\circ) = F(\phi_s, s) = 0\,.$$

Repeating the argument for every regularity should then give smoothness in the C^∞ topology, i.e smoothness of $s \mapsto \phi_s$ as a map into $C^\infty(\partial M)$ and all that would remain is to expand each ϕ_s to the whole manifold by choosing a collar neighborhood of the boundary. □

From this point, the proof of **Theorem IV.1.3** would essentially follow from (III.2.1) by the same induction procedure as in **Proposition IV.1.1**: Choose a boundary solenoidal reduction $(\Phi_s^* g_s)$ of the deformation (g_s) :

By construction,

$$\delta_{g^\circ}\left(\mathring{\Phi}_s^* g_s^\circ - g^\circ\right) = 0\,.$$

Since the linear differential operator δ_{g° does not depend on the parameter, all derivatives of the solenoidal reduction are also solenoidal:

$$\delta_{g^\circ}\left(\partial_s^k \mathring{\Phi}_s^* g_s^\circ\big|_{s=0}\right) = 0\,, \quad k \geq 1\,.$$

In addition, note that this solenoidal reduction also preserves the Steklov spectrum since the initial deformation is Steklov isospectral and the Steklov spectrum is preserved by isometry. For $k = 1$, we can then apply **Proposition III.2.3** to get that

$$\mathcal{I}_2^{g^\circ}\left(\partial_s \mathring{\Phi}_s^* g_s^\circ\big|_{s=0}\right) = 0\,,$$

and by s-injectivity of $\mathcal{I}_2^{g^\circ}$, the symmetric 2-tensor $\partial_s \mathring{\Phi}_s^* g_s^\circ|_{s=0}$ is potential, which implies that it is zero as it is also solenoidal. By induction for a fixed order $k \geq 2$, assume that the jets vanish up to order $k-1$, so that the reminder in (III.2.1) is zero, then

$$\mathcal{I}_2^{g^\circ}\left(\partial_s^k \mathring{\Phi}_s^* g_s^\circ\big|_{s=0}\right) = 0\,,$$

and we conclude as in the case $k = 1$, by injectivity of $\mathcal{I}_2^{g^\circ}$.

Remark IV.1.5 (Analytic dependence on the deformation parameter). *The real-analyticity mentioned in* ***Theorem*** *IV.1.3 is only on the parameter s, not on the metric variable. In this setting,* ***Lemma*** *IV.1.4 above should probably generalize well via the analytic version of the implicit function theorem.*

Note that real-analytic deformations of a metric is not difficult to construct in itself. One can perturb the initial metric g by a convergent[3] *power series of the form*

$$\sum_{j=1}^{+\infty} h_j s^j \,,$$

where (h_j) are symmetric two-tensors, and s is small enough.

IV.1.3 The non-linear problem for conformal metrics

In the following, we will use suitable coordinates on the manifold (M, g), called *boundary normal coordinates*.
If $x' := (x_1, ..., x_{n-1})$ denotes any local coordinates on the boundary near $p \in \partial M$, we can define the associated boundary normal coordinates (x', x_n) at p by considering x_n to be the distance to the boundary along unit speed geodesics normal to ∂M. See for example [LU89, Section 1] for more details.
It follows that in these coordinates, $x_n > 0$ in M and ∂M is locally characterized by $x_n = 0$. The metric in these coordinates has the form

$$g = \sum_{1 \leq \alpha, \beta \leq n-1} g_{\alpha\beta}(x)\, dx^\alpha \otimes dx^\beta + dx^n \otimes dx^n \,.$$

Finally, this local coordinate system naturally defines a diffeomorphism

$$\phi : \mathcal{U}_p \to \mathcal{V}_p \,,$$

where $\mathcal{U}_p$ is some open set in the half-space

$$\mathbb{R}^n_+ := \{(x_1, ..., x_n) \,;\, x_n \geq 0\}$$

[3] In a suitable Banach space.

and $\mathcal{V}_p := \phi\left(\mathcal{U}_p\right)$ is a neighborhood of p in M. For two metrics g_1 and g_2 on M, let ϕ_1 and ϕ_2 be such diffeomorphisms, then

$$\phi := \phi_1 \circ \phi_2^{-1}$$

defines a local diffeomorphism sending the unit speed geodesics for g_2 normal to ∂M onto the unit speed geodesics for g_1 normal to ∂M. Recall that this diffeomorphism can be extended globally into a diffeomorphism that is again denoted ϕ and that if (x', x_n) are boundary normal coordinates near a fixed boundary point for the metric g_1, they are also boundary normal coordinates for the metric $\phi^* g_2$.
This leads to an important remark that we will use in the following to exploit the formulas in [LU89] :

Remark IV.1.6. *Since ϕ is a diffeomorphism of M fixing the boundary we have the well-known gauge invariance*

$$\Lambda_{\phi^* g_2} = \Lambda_{g_2}$$

and then in particular, assuming that g_2-boundary normal coordinates are also g_1-boundary normal coordinates does not change the DN map Λ_{g_2}.
*For our problem in the conformal case and more precisely in **Lemma IV.1.9** below, we will therefore consider the normalized metric*

$$\tilde{g} := \phi^*(cg) .$$

Although $\tilde{g}$ is no longer exactly conformal to g, the fact that $c_{|\partial M} \equiv 1$ and the inductive assumption at order $j \geq 1$

$$\partial_n^m c|_{x_n=0} = 0 \quad , \quad \forall\ 1 \leq m \leq j$$

in g-boundary normal coordinates implies

$$c = 1 + O(x_n^{j+1}) , \tag{IV.1.4}$$

where the notation $O(x_n^k)$ denotes an expression of the form $x_n^k F_k(x', x_n)$ near the boundary with F_k smooth up to the boundary. In particular, in g-boundary normal coordinates one has

$$(cg)_{n\alpha} = 0 \quad , \quad (cg)_{nn} = 1 + O(x_n^{j+1}) .$$

Thus, g-boundary normal coordinates fail to be exact cg-boundary normal coordinates

only at order $j+1$ in the normal component and the diffeomorphism ϕ actually changes the tangential part of the metric only by $O(x_n^{j+2})$:

$$\tilde{g}_{\alpha\beta} = c g_{\alpha\beta} + O(x_n^{j+2})$$

and therefore for the inverse metric

$$\tilde{g}^{\alpha\beta} = c^{-1} g^{\alpha\beta} + O(x_n^{j+2}) \, .$$

This point is justified in the following lemma below.

Lemma IV.1.7. *Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying $c_{|\partial M} \equiv 1$ and for any $j \geq 1$,*

$$\partial_n^m c|_{x_n=0} = 0 \ , \ \ \forall \ 1 \leq m \leq j$$

in g-boundary normal coordinates (x', x_n). Then the tangential components of

$$\tilde{g} \ := \phi^*(cg)$$

in these coordinates satisfy

$$\tilde{g}_{\alpha\beta} = c g_{\alpha\beta} + O(x_n^{j+2}) \, .$$

Proof. Write

$$\phi(x', x_n) = \left(\phi^1(x', x_n), \dots, \phi^{n-1}(x', x_n), \phi^n(x', x_n)\right)$$

in g-boundary normal coordinates and note that, for tangential indices α, β,

$$\tilde{g}_{\alpha\beta} = c\left(\phi(\cdot)\right) \left(\sum_{\gamma,\delta=1}^{n-1} g_{\gamma\delta}(\phi(\cdot)) \partial_\alpha \phi^\gamma \partial_\beta \phi^\delta + \partial_\alpha \phi^n \partial_\beta \phi^n \right) . \tag{IV.1.5}$$

First, recall that

$$\forall \alpha \in \{1, ..., n-1\} \, , \ \phi^\alpha(x', 0) = x^\alpha \quad , \quad \phi^n(x', 0) = 0 \tag{IV.1.6}$$

and since $c|_{\partial M} = 1$, the unit normals for g and cg agree on ∂M, so

$$\forall \alpha \in \{1, ..., n-1\} \, , \ \partial_n \phi^\alpha(x', 0) = 0 \quad \text{and} \quad \partial_n \phi^n(x', 0) = 1 \, . \tag{IV.1.7}$$

In particular, we already have the estimate

$$\phi^n(x', x_n) = x_n + O(x_n^2) = O(x_n) \tag{IV.1.8}$$

near the boundary.
In addition, the geodesic equation for cg in g-boundary normal coordinates is written

$$\partial_n^2 \phi^k + \Gamma(cg)_{ij}^k (\phi(\cdot)) \, \partial_n \phi^i \partial_n \phi^j = 0 \quad , \quad k = 1, ..., n \, ,$$

and the Christoffel symbols of cg and g satisfy

$$\Gamma(cg)_{ij}^k - \Gamma(g)_{ij}^k = \frac{1}{2c} \left(\delta_{ik} \partial_j c + \delta_{jk} \partial_i c - g_{ij} g^{\ell k} \partial_\ell c \right) \, .$$

Since $\Gamma(g)_{nn}^k = 0$, by using (IV.1.4) and noting that tangential derivatives do not consume powers of x_n, we get

$$\Gamma(cg)_{nn}^n = \frac{1}{2c} \partial_n c = O(x_n^j) \, , \tag{IV.1.9}$$

and

$$\forall \alpha \in \{1, ..., n-1\} \, , \; \Gamma(cg)_{nn}^\alpha = -\frac{1}{2c} g^{\alpha\beta} \partial_\beta c = O(x_n^{j+1}) \, . \tag{IV.1.10}$$

Now the above geodesic equation for tangential indices $k = \alpha$ gives

$$\begin{aligned} \partial_n^2 \phi^\alpha = & -\Gamma(cg)_{nn}^\alpha (\phi(\cdot)) \left(\partial_n \phi^n \right)^2 \\ & - 2\Gamma(cg)_{n\beta}^\alpha (\phi(\cdot)) \, \partial_n \phi^n \partial_n \phi^\beta \\ & - \Gamma(cg)_{\beta\delta}^\alpha (\phi(\cdot)) \, \partial_n \phi^\beta \partial_n \phi^\delta \end{aligned}$$

and combining (IV.1.8) with (IV.1.10), we also have

$$\Gamma(cg)_{nn}^\alpha (\phi(\cdot)) = O(x_n^{j+1}) \, . \tag{IV.1.11}$$

The remaining Christoffel symbols appearing in the geodesic equation are smooth, so in particular they are bounded in a sufficiently small collar neighborhood of the boundary. Also by (IV.1.7), $\partial_n \phi^n$ remains bounded in this neighborhood, even if this means reducing it.

Therefore, defining near the boundary

$$M(t) := \max_{1\le\beta\le n-1} |\partial_n \phi^\beta(x',t)|$$

and bounding each term of the geodesic equation above, we get for any tangential index α and sufficiently small parameter t,

$$|\partial_n^2 \phi^\alpha(x',t)| \lesssim t^{j+1} + M(t) + M(t)^2 ,$$

where $\lesssim$ of course denotes inequality up to a constant. Since $M(0) = 0$ (cf. (IV.1.7)), after shrinking the neighborhood if necessary we may assume $M(t) \le 1$ and then

$$|\partial_n^2 \phi^\alpha(x',t)| \lesssim t^{j+1} + M(t) .$$

Using (IV.1.7) and integrating, we get near the boundary for any α

$$\begin{aligned} |\partial_n \phi^\alpha(x',x_n)| &\le \int_0^{x_n} |\partial_n^2 \phi^\alpha(x',t)|\, dt \\ &\lesssim x_n^{j+2} + \int_0^{x_n} M(t)\, dt \end{aligned}$$

and therefore, taking the maximum over each α,

$$M(x_n) \lesssim x_n^{j+2} + \int_0^{x_n} M(t)\, dt .$$

To conclude, a simple use of Gronwall's inequality gives for x_n small enough

$$M(x_n) \lesssim x_n^{j+2}$$

and then[4] for any tangential index α,

$$\partial_n \phi^\alpha(x',x_n) = O(x_n^{j+2}) . \tag{IV.1.12}$$

[4]Since the functions involved are smooth up to the boundary, an estimate of the form $|F| \lesssim x_n^k$ imply $F = O(x_n^k)$. Indeed, the bound forces all normal derivatives $\partial_n^m F|_{\partial M}$ to vanish for $0 \le m < k$ and Taylor's formula with integral remainder then yields $F = x_n^k F_k$, with F_k smooth.

Finally, integrating once more and using (IV.1.6), we obtain :

$$\begin{aligned}|\phi^\alpha(x', x_n) - x^\alpha| &\leq \int_0^{x_n} |\partial_n \phi^\alpha(x', t)|\, dt \\ &\lesssim x_n^{j+3}\end{aligned}$$

and then

$$\phi^\alpha(x', x_n) = x^\alpha + O(x_n^{j+3})\,. \tag{IV.1.13}$$

We now estimate the normal component in the same way: Taking $k = n$ in the geodesic equation gives

$$\begin{aligned}\partial_n^2 \phi^n = &-\Gamma(cg)^n_{nn}(\phi\,(\cdot))\,(\partial_n \phi^n)^2 \\ &- 2\Gamma(cg)^n_{n\beta}(\phi\,(\cdot))\partial_n \phi^n \partial_n \phi^\beta \\ &- \Gamma(cg)^n_{\beta\delta}(\phi\,(\cdot))\partial_n \phi^\beta \partial_n \phi^\delta\,.\end{aligned}$$

Now combining (IV.1.11) with (IV.1.12) and using once again that $\partial_n \phi^n$ remains bounded in a sufficiently small neighborhood of the boundary, we obtain

$$\partial_n^2 \phi^n(x', x_n) = O(x_n^j)\,,$$

and since $\partial_n \phi^n(x', 0) = 1$, integration gives

$$\partial_n \phi^n(x', x_n) = 1 + O(x_n^{j+1})\,.$$

Finally, using $\phi^n(x', 0) = 0$, we integrate once more and obtain

$$\phi^n(x', x_n) = x_n + O(x_n^{j+2})\,. \tag{IV.1.14}$$

To conclude, substituting (IV.1.13) and (IV.1.14) in the pull-back metric formula (IV.1.5) gives the desired identity. □

Recovery of the normal derivative at the boundary

The theorem below provides boundary determination at the subprincipal level of the symbol of the DN map and is a consequence of the Livšic theorem for cocycles (cf. **Proposition III.1.10**):

Theorem IV.1.8. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary. Assume that $(\partial M, g^\circ)$ is Anosov with simple length*

spectrum. Let $c : M \to \mathbb{R}_*^+$ *be a smooth function satisfying* $c_{|\partial M} \equiv 1$.
If

$$\mathrm{spec}(\Lambda_{cg}) = \mathrm{spec}(\Lambda_g) ,$$

then

$$\partial_\nu c_{|\partial M} \equiv 0 .$$

Proof. Since g and cg are two isospectral metrics that coincide at the boundary and g° is Anosov with simple length spectrum, **Theorem II.3.4** and **Lemma IV.A.5** implies that for any closed geodesic γ for g°,

$$exp\left(-i\alpha_n \int_\gamma (\partial_\nu c)\,_{|\partial M}\right) = 1$$

in the same way as in the proof of **Proposition IV.1.2**.

Equivalently, if we put $f := \alpha_n(\partial_\nu c)\,_{|\partial M}$, then for any closed geodesic γ,

$$\int_\gamma f \in 2\pi\mathbb{Z} .$$

Combining **Proposition III.1.10** and **Theorem III.1.11**, we obtain $f \equiv 0$. □

We have thus proved the initialization step of the induction procedure used below to recover the boundary jets and then **Theorem I.1.1**.

Boundary determination of the conformal factor: Proof of Theorem I.1.1

At this point, we know that the total symbols of Λ_{cg} and Λ_g coincide at principal and subprincipal order, so these two operators differ only by a pseudodifferential operator of order -1. Therefore, we are now able to combine **Corollary II.4.4** with the injectivity of the X-ray transform on functions (**Theorem III.1.4**) to access the homogeneous terms of orders $j \leq -1$ contained in the symbol of $\Lambda_{cg} - \Lambda_g \in \Psi^{-1}(\partial M)$ and, more precisely, to show recursively that they are all zero if the metrics g and cg are Steklov isospectral. Indeed, following the work of Lee and Uhlmann [LU89, Proposition 1.3], we show that these homogeneous terms of order $j \leq -1$ essentially consist in the normal derivatives at the boundary of the conformal factor:

Lemma IV.1.9. *Let* (M, g) *be a smooth compact Riemannian manifold of dimension* $n \geq 2$ *with smooth boundary and* $c : M \to \mathbb{R}_*^+$ *be a smooth function satisfying* $c_{|\partial M} \equiv 1$.

Let

$$\sigma^{full}_{\Lambda_{cg}} \sim \sum_{j=-1}^{+\infty} \tilde{a}_{-j}$$

and

$$\sigma^{full}_{\Lambda_g} \sim \sum_{j=-1}^{+\infty} a_{-j}$$

be respectively the full symbols of Λ_{cg} and Λ_g considered in any local coordinate system $x' := (x_1, ...x_{n-1})$ on the boundary.

In the g-boundary normal coordinates (x', x_n) :

For any $j \geq 1$, if

$$\partial_n^m c|_{x_n=0} = 0 \ , \ \ \forall \ 1 \leq m \leq j \ ,$$

then

$$\tilde{a}_{-j} - a_{-j} = \frac{(n-2)}{4} \left(-\frac{1}{2|\xi'|_{g^\circ}} \right)^j \partial_n^{j+1} c|_{x_n=0} \ .$$

Proof. Let $x' := (x_1, ..., x_{n-1})$ be any local coordinate system on the boundary and (x', x_n) the associated g-boundary normal coordinates. By **Remark IV.1.6**, we have to replace cg by

$$\tilde{g} := \phi^*(cg) \quad , \quad \phi|_{\partial M} = \mathrm{Id}_{\partial M} \ ,$$

so that g-boundary normal are also $\tilde{g}$-boundary normal coordinates. Recall that

$$\Lambda_{\tilde{g}} = \Lambda_{cg} \ ,$$

so the full symbol of Λ_{cg} may be computed from that of $\Lambda_{\tilde{g}}$ in these coordinates.

Using [LU89, Proposition 1.3], we denote by $(g^{\alpha\beta})$ the inverse matrix of $(g_{\alpha\beta})$ and introduce for any $1 \leq \alpha, \beta \leq n-1$,

$$h^{\alpha\beta} := \partial_n g^{\alpha\beta} \ , \ h = \sum_{1 \leq a,b \leq n-1} g_{ab} h^{ab} \ ,$$

$$\tilde{h}^{\alpha\beta} := \partial_n \tilde{g}^{\alpha\beta} \ , \ \tilde{h} = \sum_{1 \leq a,b \leq n-1} \tilde{g}_{ab} \tilde{h}^{ab}$$

and the quadratic forms

$$k^{\alpha\beta} := h^{\alpha\beta} - h g^{\alpha\beta} \ , \ \tilde{k}^{\alpha\beta} := \tilde{h}^{\alpha\beta} - \tilde{h} \tilde{g}^{\alpha\beta}$$

respectively associated with the metrics g and $\tilde{g}$.

Now suppose that for any $j \geq 1$,

$$\partial_n^m c|_{x_n=0} = 0 \ , \quad \forall\ 1 \leq m \leq j \ .$$

By **Remark IV.1.6**,

$$\tilde{g}^{\alpha\beta} = c^{-1} g^{\alpha\beta} + O(x_n^{j+2})$$

and then

$$\tilde{h}^{\alpha\beta} = \partial_n(c^{-1} g^{\alpha\beta}) + O(x_n^{j+1}) \, ,$$

so that

$$\tilde{h} = \tilde{g}_{ab}\tilde{h}^{ab} = c\, g_{ab}\, \partial_n(c^{-1} g^{ab}) + O(x_n^{j+1}) \, .$$

Consequently, we have

$$\tilde{k}^{\alpha\beta} = \partial_n(c^{-1} g^{\alpha\beta}) - \Big(c\, g_{ab} \partial_n(c^{-1} g^{ab})\Big)\, c^{-1} g^{\alpha\beta} + O(x_n^{j+1}) \, ,$$

with

$$\partial_n(c^{-1} g^{\alpha\beta}) = c^{-1} h^{\alpha\beta} - \frac{\partial_n c}{c^2} g^{\alpha\beta} \, .$$

Furthermore,

$$\begin{aligned} c\, g_{ab} \partial_n(c^{-1} g^{ab}) &= c\, g_{ab} \left(c^{-1} h^{ab} - \frac{\partial_n c}{c^2} g^{ab}\right) \\ &= h - (n-1)\frac{\partial_n c}{c} \end{aligned}$$

and then

$$\begin{aligned} \tilde{k}^{\alpha\beta} &= c^{-1} h^{\alpha\beta} - \frac{\partial_n c}{c^2} g^{\alpha\beta} \\ &\quad - \left(h - (n-1)\frac{\partial_n c}{c}\right) c^{-1} g^{\alpha\beta} + O(x_n^{j+1}) \\ &= c^{-1} k^{\alpha\beta} + (n-2)\frac{\partial_n c}{c^2} g^{\alpha\beta} + O(x_n^{j+1}) \, . \end{aligned}$$

Since

$$c|_{x_n=0} = 1 \quad , \quad \partial_n^m c|_{x_n=0} = 0 \, , \quad 1 \leq m \leq j \, ,$$

we get

$$\partial_n^j \left(c^{-1} k^{\alpha\beta}\right)\Big|_{x_n=0} = \partial_n^j k^{\alpha\beta}\Big|_{x_n=0} \, ,$$

and

$$\partial_n^j \left(\frac{\partial_n c}{c^2} g^{\alpha\beta}\right)\Big|_{x_n=0} = \partial_n^{j+1} c\Big|_{x_n=0}\, g^{\alpha\beta}\Big|_{x_n=0} \, .$$

It should be noted that when we apply ∂_n^j and restrict ourselves to $x_n = 0$, the remain-

der $O(x_n^{j+1})$ makes no contribution, which finally gives us

$$\partial_n^j \tilde{k}^{\alpha\beta}\Big|_{x_n=0} = \partial_n^j k^{\alpha\beta}\Big|_{x_n=0} + (n-2)\partial_n^{j+1} c\Big|_{x_n=0} g^{\alpha\beta}\Big|_{x_n=0} .$$

By [LU89, Proposition 1.3], for $j \geq 1$,

$$\tilde{a}_{-j} = \left(-\frac{1}{2|\xi'|_{g^\circ}}\right)^{j+2} \sum_{1\leq\alpha,\beta\leq n-1} \left(\partial_n^j \tilde{k}^{\alpha\beta}\right) \xi_\alpha \xi_\beta + T_{-j}(\tilde{g}_{\alpha\beta}) ,$$

where $T_{-j}(\tilde{g}_{\alpha\beta})$ only involves the boundary values of $\tilde{g}_{\alpha\beta}$, $\tilde{g}^{\alpha\beta}$, and their normal derivatives of order at most j.
Obviously we also have the same formula for a_{-j}, just replacing $\tilde{k}^{\alpha\beta}$ by $k^{\alpha\beta}$ and $\tilde{g}$ by g :

$$a_{-j} = \left(-\frac{1}{2|\xi'|_{g^\circ}}\right)^{j+2} \sum_{1\leq\alpha,\beta\leq n-1} \left(\partial_n^j k^{\alpha\beta}\right) \xi_\alpha \xi_\beta + T_{-j}(g_{\alpha\beta}) .$$

In addition, since

$$\partial_n^m \tilde{g}_{\alpha\beta}\big|_{x_n=0} = \partial_n^m (c g_{\alpha\beta})\big|_{x_n=0} \quad , \quad 0 \leq m \leq j+1 ,$$

and

$$\partial_n^m c|_{x_n=0} = 0 , \quad 1 \leq m \leq j ,$$

we have equality of the remainders

$$T_{-j}(\tilde{g}_{\alpha\beta}) = T_{-j}(g_{\alpha\beta}) .$$

Thus, noticing that

$$\sum_{1\leq\alpha,\beta\leq n-1} g^{\alpha\beta}\xi_\alpha\xi_\beta = |\xi'|^2_{g^\circ}$$

and noting

$$T_{-j} := \sum_{1\leq\alpha,\beta\leq n-1} T_{-j}(g_{\alpha\beta}) ,$$

we get for $j \geq 1$,

$$\begin{aligned}\tilde{a}_{-j} &= \left(-\frac{1}{2|\xi'|}\right)^{j+2} \sum_{1\leq\alpha,\beta\leq n-1} \left[(n-2)\left(\partial_n^{j+1} c|_{x_n=0}\right) g^{\alpha\beta} + \partial_n^j k^{\alpha\beta}\right] \xi_\alpha \xi_\beta + T_{-j} \\ &= \frac{(n-2)}{4|\xi'|^2_{g^\circ}} \left(-\frac{1}{2|\xi'|_{g^\circ}}\right)^{j} \partial_n^{j+1} c|_{x_n=0} \sum_{1\leq\alpha,\beta\leq n-1} g^{\alpha\beta}\xi_\alpha\xi_\beta \ + \ a_{-j} \\ &= \frac{(n-2)}{4} \left(-\frac{1}{2|\xi'|_{g^\circ}}\right)^{j} \partial_n^{j+1} c|_{x_n=0} \ + \ a_{-j} \, .\end{aligned}$$

This proves the lemma. □

We deduce the boundary determination of the jet of the conformal factor c at any point $p \in \partial M$ as stated:

Theorem IV.1.10. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary. Assume that $(\partial M, g^\circ)$ is Anosov with simple length spectrum. Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying $c_{|\partial M} \equiv 1$.*
If

$$\mathrm{spec}(\Lambda_{cg}) = \mathrm{spec}(\Lambda_g) \, ,$$

then the boundary normal derivatives of the conformal factor are zero :

$$\partial_\nu^j c|_{\partial M} = 0 \, , \ \forall j \geq 1 \, .$$

Consequently, the Taylor series of c at any point $p \in \partial M$ in any local coordinates are reduced to the constant function identically equal to 1.

Proof. Let us consider $x' := (x_1, ...x_{n-1})$ any local coordinate system on the boundary and (x', x_n) the associated g-boundary normal coordinates near ∂M.
We proceed by induction on the order of derivation $j \geq 1$:

- $j = 1$: By the above **Theorem IV.1.8**,

$$\partial_n c|_{x_n=0} = 0$$

since $(\partial M, g^\circ)$ is Anosov with simple length spectrum.

- Let $j \geq 1$ s.t for all $1 \leq m \leq j$,

$$\partial_n^m c|_{x_n=0} = 0 \, .$$

Then using **Lemma IV.1.9**,

$$\Lambda_{cg} - \Lambda_g \in \Psi^{-j}(\partial M) \, .$$

Now we can apply **Corollary II.4.4** to $(N, h) = (\partial M, g^\circ)$,

$$A = \Lambda_g \in \Psi^1(\partial M) \text{ and } B = \Lambda_{cg} - \Lambda_g \in \Psi^{-j}(\partial M) \quad :$$

Recalling that

- $\operatorname{spec}(\Lambda_{cg}) = \operatorname{spec}(\Lambda_g)$.
- $c_{|\partial M} \equiv 1$.
- $(\partial M, g^\circ)$ is Anosov with simple length spectrum.

and then using **Corollary II.4.4**, we get that for each periodic bicharacteristic $s \mapsto \Phi_s^A(x, \xi)$,

$$0 = \int_0^\ell \sigma_B\left(\Phi_s^A(x, \xi)\right) ds \, ,$$

where ℓ obviously denotes the length of the bicharacteristic. Given the expression for the principal symbol of $B \in \Psi^{-j}(\partial M)$ in **Lemma IV.1.9**, we conclude as usual by injectivity of the geodesic X-ray transform on functions, that

$$\partial_n^{j+1} c|_{x_n=0} = 0 \, .$$

It is then easy to see that the Taylor series of the metrics g and cg at any $p \in \partial M$ are identical in g-boundary normal coordinates (x', x_n) near this point :

By simply using **Remark IV.1.6** and [LU89, Proposition 1.3], we know that the knowledge of the full symbols of Λ_{cg} and Λ_g determine, in g-boundary normal coordinates the Taylor series of cg and g respectively.
But with the same notations as in **Lemma IV.1.9**, we have just shown that in these coordinates,

$$\tilde{a}_{-j} - a_{-j} = 0 \quad , \quad \forall j \geq -1 \quad . \tag{IV.1.15}$$

Taylor series of c at p are then reduced to the constant function identically equal to 1 in any local coordinates since the nullity of derivatives is obviously an invariant property under coordinate changes. □

Remark IV.1.11. *Note that* (IV.1.15) *is equivalent to saying that* Λ_{cg} *and* Λ_g *coincide modulo a smoothing operator.*

By a simple argument of analytic continuation on the function c, we then obtain that if c is real-analytic and M is connected, then $c \equiv 1$ on M.

IV.1.4 The non-linear problem for general metrics: Perspectives

As already mentioned, the discussion of Subsection IV.1.2 on the linearized problem suggests a possible strategy for trying to extend the results proved in Subsection IV.1.1, on conformal metrics, to the case where the metrics are no longer in the same conformal class. Below, we offer a brief analysis of this topic and present some preliminary ideas that still require further verification at this stage.

This problem obviously poses significant technical issues, since we are no longer reasoning about a function (the conformal factor) but directly about the metric tensor; in other words, the natural gauge invariance of the problem (cf. Subsection III.1.1) has to be addressed explicitly in this case.

More precisely, consider two metrics g_1 and g_2 and assume that $(\partial M, g_i^\circ)$, $i \in \{1, 2\}$ is Anosov with simple length spectrum. Then if the Steklov spectrum is preserved

$$\mathrm{spec}(\Lambda_{g_1}) = \mathrm{spec}(\Lambda_{g_2}) \,,$$

can we show that the Taylor series of g_1 and g_2 in boundary normal coordinates are equal at each point on the boundary, at least modulo gauge ?
Of course, the answer is a priori no, at least without further assumptions. Indeed, as in the case of conformal metrics, information at principal order poses a significant problem, and therefore it seems necessary here to assume at least that the two metrics are isometric at the boundary, i.e. there is a diffeomorphism $\varphi : \partial M \to \partial M$ such that

$$g_2^\circ = \varphi^* g_1^\circ \,.$$

This is a natural analogue for non-conformal metrics of fixing the conformal factor at the boundary.

At subprincipal order, the relevant quantity that appears is the second fundamental form of the boundary, since for any metric g,

$$\mathrm{sub}(\Lambda_g)(x,\xi) = \frac{1}{2}\left(\frac{\mathrm{II}_{\partial M}(\xi^\sharp,\xi^\sharp)}{|\xi|^2} - H_g\right),$$

where $\mathrm{II}_{\partial M}(\xi^\sharp,\xi^\sharp)$ is the second fundamental form of ∂M in (M,g) and H_g its trace, also called *mean curvature* of M. We refer to [Tay23, Chapter 12, Proposition C.1] for this standard result. Since the second fundamental form is, in boundary normal coordinates, essentially given by the normal derivative of the metric tensor at the boundary, we can expect, at this stage, that the non-linear problem will be, in terms of the information to be recovered, quite analogous to the conformal case, which is of course not surprising. Indeed, more generally, the work of Lee and Uhlmann [LU89, Proposition 1.3] — already used in the problem of conformal metrics (see **Lemma IV.1.9**) — shows that, in these coordinates, the full symbol of the DN map determines the jets of the metric at the boundary.

Based on these observations, we can try to mimick the conformal case:

Before trying to compare the full Taylor expansions of g_1 and g_2 in boundary normal coordinates, one should first place the boundary metrics in a suitable gauge. Suppose, for instance, that g_2 is sufficiently close to g_1 so that one can use a boundary solenoidal reduction argument just as in the linear case (see **Lemma IV.1.4**). More precisely, we would like to find a diffeomorphism $\phi : \partial M \to \partial M$, close to the identity, such that

$$\delta_{g_1^\circ}\left(\phi^* g_2^\circ\right) = 0\,,$$

and extend it to a diffeomorphism Φ in M i.e. $\Phi(\partial M) = \partial M$ and $\Phi|_{\partial M} = \phi$. This would be an appropriate version of the slice lemma [CDS00, Theorem 2.1]; see the discussion preceding **Lemma IV.1.4** and the references therein.

In the case of conformal metrics, the wave trace invariants produce integrals of a scalar function over closed geodesics of the boundary, here however, once the gauge Φ has been constructed, one should instead expect integrals over closed geodesics of $(\partial M, g_1^\circ)$ of the solenoidal 2-tensor

$$\phi^* g_2^\circ - g_1^\circ\,,$$

and of its normal derivatives at the boundary.
Thus, the problem would no longer require the injectivity of the X-ray transform on functions, but rather an injectivity property for the X-ray transform on symmetric

tensors of order 2 which, for closed Anosov manifolds, is only known in dimension 2 (see [PSU14, Theorem 1.1] or more generally [Gui17, Theorem 1.4] for tensors of any order), or in any dimension but with non-positive sectional curvature (see [CS98, Theorem 1.3]). However, we should mention here that the X-ray transform of symmetric tensors of arbitrary degree is generically injective on closed Anosov manifolds of dimension $n \geq 3$, as shown very recently by Cekić and Lefeuvre in [CL25a].
We refer to Subsection III.1.1 for a more explicit discussion on injectivity of the X-ray transform.

In summary, two important additional concepts that would naturally be useful in addressing this problem are:

- a boundary solenoidal reduction lemma that allows one to compare g_1° and g_2° modulo boundary diffeomorphisms.
- s-injectivity of the geodesic X-ray transform on symmetric 2-tensors.

IV.2 Boundary determination of potentials

IV.2.1 Recovery of electric potentials with no magnetic influence

This section recall the main result obtained in [Flo25, Section V] which is merely a special case of **Theorem I.2.2** but the latter was obtained afterwards.

More specifically, we show—in exactly the same way as for the conformal metric problem—how to determine the boundary jets of an electric potential from the spectrum of the magnetic DN map $\Lambda_{g,q}$ in the absence of a magnetic influence. This can be seen as an interesting comparison with the case of conformal metrics as well as a warm-up before introducing a magnetic influence, which substantially complicates the problem (cf. Subsection IV.2.2).

Theorem IV.2.1. *Assume that $(\partial M, g^\circ)$ is Anosov with simple length spectrum. Let q_1 and q_2 two smooth potentials on M.*
If the Steklov spectra agree

$$\operatorname{spec}(\Lambda_{g,q_1}) = \operatorname{spec}(\Lambda_{g,q_2}) \,,$$

then the boundary normal derivatives at any order coincide :

$$\partial_\nu^{j-1} (q_2 - q_1) \,|_{\partial M} = 0 \,, \ \forall j \geq 1 \,.$$

Consequently, the Taylor series of q_1 and q_2 at each point $p \in \partial M$ in any local coordinates are equal and

$$\Lambda_{g,q_1} = \Lambda_{g,q_2}$$

modulo a smoothing operator.

Proof. The proof follows exactly the same ideas as **Theorem IV.1.10** above : We proceed by induction on the order of derivation $j \geq 0$ and apply at each **Corollary II.4.4** to

$$(N,h) = (\partial M, g^\circ) \ , \ \ A = \Lambda_{g,q_1} \in \Psi^1(\partial M) \ , \ B = \Lambda_{g,q_2} - \Lambda_{g,q_1} \in \Psi^{-j-1}(\partial M)$$

which, combined with the isospectrality assumption, gives us for any periodic bicharacteristic $s \mapsto \Phi_s^A(x,\xi)$,

$$0 = \int_0^\ell \sigma_B\left(\Phi_s^A(x,\xi)\right) ds \, ,$$

where ℓ denotes the length of the bicharacteristic.

Then combining the expression for the principal symbol of $B \in \Psi^{-j-1}(\partial M)$ given by **Lemma IV.2.9** with the injectivity of the X-ray transform on functions (**Theorem III.1.4**), we obtain in the g-boundary normal coordinates (x', x_n),

$$\partial_n^j (q_2 - q_1) \big|_{x_n=0} = 0 \, ,$$

equality that is again completely invariant under coordinate changes, of course.
Using the notations from **Lemma IV.2.9**, we have just shown that,

$$\tilde{a}_{-j} - a_{-j} = 0 \ , \ \ \forall j \geq -1$$

or equivalently that Λ_{g,q_1} and Λ_{g,q_2} coincide modulo a smoothing operator.
To conclude, we can then simply use the calculations of [DSFKSU09, Section 8] or [Cek20, Section 3.2] (see **Lemma IV.2.9**) which show that in g-boundary normal coordinates the Taylor series of q_1 and q_2 at any point $p \in \partial M$ are determined by the knowledge of the full symbols of Λ_{g,q_1} and Λ_{g,q_2} respectively. □

Remark IV.2.2. *Note that initialization here is much simpler, since it is obtained directly by the same argument as at higher orders, unlike the metric case. In the latter, initialization consisted in showing that the normal derivative of the conformal factor at the boundary is zero, but this quantity appears at the subprincipal level (via the trace formula **Theorem II.3.4**) and then is not directly covered by our induction.*

To complete the above remark, let us note that this problem is actually made completely linear by the fact that the operators Λ_{g,q_1} and Λ_{g,q_2} already coincide at principal and subprincipal order and that the information on the potentials is given from order -1, which allows us to apply the **Corollary II.4.4** directly. In this sense, the problem is simpler than the case of conformal metrics, for which obtaining information in the principal order is a non-linear problem and requires the addition of relatively strong geometric assumptions as discussed in Appendix IV.B.
Finally, let us note that just as **Theorem II.4.2** is an analogue of [Gui78b, Theorem 4], its application **Theorem IV.2.1** above can be compared with [GK80a, Theorem 2] on the recovery of a potential q from the spectrum of the associated Schrödinger operator. Moreover, it seems worth mentioning here that **Theorem II.4.2** also gives us in particular, and this is actually equivalent to **Theorem IV.2.1**, an analogue of [GK80a, Theorem 3(a)], namely :

Theorem IV.2.3. *Let (M, g) be a compact smooth Riemannian manifold with boundary and q be a smooth potential on M. Assume that $(\partial M, g^\circ)$ is Anosov with simple length spectrum. Then the spectrum of $\Lambda_{g,q}$ determines the integrals of q and all its normal derivatives at the boundary over the periodic geodesics of g°.*

A natural question raised by **Theorem IV.2.1** is whether we can explicitly construct an example of Steklov isospectral potentials which are not the same, obviously when the simple length spectrum condition is not verified.
In the case of the Schrödinger operator $P_{g,q} := -\Delta_g + q$, such examples are shown in [Bro87] via the general method of Sunada [Sun85], which remains probably the most widely used and cited technique to construct isospectral manifolds.
We refer to [Gor09] for details and more specifically to [GHW21] for the Steklov spectrum. To the author's knowledge, there is no reference for the operator $\Lambda_{g,q}$, but it may be known in folklore that the Sunada construction can be adapted again.

IV.2.2 Recovery of electric and magnetic potentials

This section is reproduced almost in its entirety from [dSFF26, §5].

The inverse problem for magnetic Schrödinger operators

Here we show how to derive **Theorem I.3.3** as a direct consequence of Subsection III.1.2. In order to exploit the principal invariants of the wave trace, we first recall that the subprincipal symbol of the magnetic Schrödinger operator $P := P_{a,q}$ is given exactly by the magnetic potential :

The (positive) Laplace-Beltrami operator also reads in coordinates

$$\delta d = -\Delta = \sum_{j,k=1}^{n} g^{jk} D_j D_k + i \sum_{j,k=1}^{n} g^{jk} \Gamma_{jk}^{\ell} D_\ell \,,$$

therefore, the homogeneous part of degree one of the full symbol in these local coordinates is given by

$$p_1(x,\xi) = 2\langle a(x), \xi \rangle + i \sum_{j,k=1}^{n} g^{jk}(x) \Gamma_{jk}^{\ell}(x) \xi_\ell \,.$$

Of course this is not an invariantly defined function on the cotangent bundle but the principal symbol is

$$\sigma_P(x,\xi) = p(x,\xi) = |\xi|_x^2 = \sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k \,,$$

as well as the subprincipal symbol

$$\begin{aligned} \mathrm{sub}(P)(x,\xi) &= p_1(x,\xi) - \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_j \partial \xi_j} \\ &= 2\langle a(x), \xi \rangle + \underbrace{i \sum_{j,k=1}^{n} g^{jk}(x) \Gamma_{jk}^{\ell}(x) \xi_\ell - i \sum_{j=1}^{n} \frac{\partial g^{jk}}{\partial x_j}(x) \xi_j}_{=0} \,. \end{aligned}$$

Up to the natural gauge (see Section I.3), we are now able to recover the magnetic potential from the spectrum of the magnetic Schrödinger operator $P := P_{a,q}$, in the case where the boundary is Anosov with simple length spectrum :

Proof of ***Theorem I.3.3.*** Since $P_{a,q}$ and $P_{\tilde{a},\tilde{q}}$ have the same spectrum, the principal wave invariants gives that for any periodic geodesic γ of N :

$$\int_\gamma \pi_1^* (\tilde{a} - a) \in 2\pi\mathbb{Z} \,.$$

Then by **Proposition III.1.10**, there exists a solution $u \in C^\infty(S^*N; \mathbb{S}^1)$ of the transport equation with potential $f := \pi_1^* (\tilde{a} - a)$:

$$Hu + ifu = 0 \,.$$

Now using **Theorem III.1.11**, we get that the one-form $\pi_1^*(\tilde{a}-a)$ is closed, $\vartheta := u$ does not depend on the fiber variable, and by **Lemma I.3.2**, satisfies the desired identity :

$$\tilde{a} = a - i\bar{\vartheta}\mathrm{d}\vartheta \,.$$

The operator

$$\bar{\vartheta}P_{a,q}\vartheta = P_{a-i\bar{\vartheta}\mathrm{d}\vartheta,q} = P_{\tilde{a},q}$$

has the same spectrum than $P_{\tilde{a},\tilde{q}}$, therefore a variation of the argument of [Gui78a] reduces the spectral inverse problem to that of [GK80a] and gives

$$\int_\gamma (\tilde{q}-q) = 0$$

for all periodic geodesic γ. The injectivity of the geodesic X-ray transform on functions implies $\tilde{q} = q$. □

The magnetic Steklov inverse problem

In this section, we prove **Theorem I.2.1** and **Theorem I.2.2** on the recovery of a magnetic and electric potential from the spectrum of the associated magnetic DN map. Unlike the case of the magnetic Laplacian treated above, we are able to extract more information from the spectrum of the magnetic DN map, namely that it determines the jet at the boundary of the magnetic field and of the electric potential. This is done using the same approach as the one explained so far, that is, by induction on the order of derivation, using at each step the trace formula of **Theorem II.3.4**. We will therefore need to identify, in suitable coordinates, the homogeneous term at any order in the full symbol of

$$B := \Lambda_{a,q} - \Lambda_{\tilde{a},\tilde{q}} \,.$$

In order to do that, we will consider *boundary normal coordinates* near $N := \partial M$ and exploit [DSFKSU09, Section 8] or, more generally, [Cek20, Section 3] for the expression of the full symbol of the magnetic DN map in these coordinates. Note that the computation of this full symbol is made possible by putting the magnetic potential in a suitable gauge and making a conformal normalization of the metric. By nature, these transformations preserve the DN map, so in this section, unless otherwise stated, we will therefore always consider normalized data and note that in our problem it does not reduce the generality. We refer to [DSFKSU09, Section 8] and to [Cek20, Section 3] for details.

Remark IV.2.4. *More precisely, the conformal normalization, which affects both the metric and the electric potential, is the same for each pair of data* (a, q) *and* $(\tilde{a}, \tilde{q})$ *since the metric is fixed in our problem. Therefore, recovering the jet at the boundary of a normalized electric potential also implies recovering the jet at the boundary of the initial one. However, this is not true for the magnetic potential, because for each pair* a *and* $\tilde{a}$*, we must choose different gauge transformations a priori. On the other hand, since the gauge terms are closed, this does not affect our main results on the recovery of magnetic fields.*

To conclude regarding the subject of gauge transformations, note that the natural gauge (cf. Subsection I.2.1) of this inverse problem transform the subprincipal symbol in a simple way :

Lemma IV.2.5. *Let* $\vartheta \in C^\infty(N, \mathbb{S}^1)$ *and* Λ *be a magnetic DN map on* M*. Then the following identity holds on the unit cotangent bundle* S^*N *:*

$$\mathrm{sub}(\bar{\vartheta}\Lambda\vartheta) = \mathrm{sub}(\Lambda) - i\bar{\vartheta}\pi_1^*\mathrm{d}\vartheta \,.$$

Proof. If P and Q are pseudodifferential operators on the closed manifold N, it is well-known that the subprincipal symbol of PQ is given by

$$\mathrm{sub}(PQ) = \mathrm{sub}(P)\cdot\sigma_Q + \sigma_P\cdot\mathrm{sub}(Q) + \frac{1}{2i}\{\sigma_P, \sigma_Q\}\,,$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Using this formula, it is easy to see that

$$\mathrm{sub}(\bar{\vartheta}\Lambda\vartheta) = \mathrm{sub}(\Lambda) + \frac{1}{2i}\left(\bar{\vartheta}\{\sigma_\Lambda, \vartheta\} + \vartheta\{\bar{\vartheta}, \sigma_\Lambda\}\right)\,.$$

Since ϑ does not depend on the fiber variable and has modulus 1, we show that in any coordinate chart (x, ξ) on the unit cotangent bundle S^*N :

$$\begin{aligned}\frac{1}{2i}\left(\bar{\vartheta}\{\sigma_\Lambda, \vartheta\} + \vartheta\{\bar{\vartheta}, \sigma_\Lambda\}\right) &= -i\bar{\vartheta}\sum_{j=1}^{n-1}\frac{\partial\sigma_\Lambda}{\partial\xi_j}\frac{\partial\vartheta}{\partial x_j}\\ &= -i\bar{\vartheta}\sum_{j,k=1}^{n-1} g^{kj}\frac{\partial\vartheta}{\partial x_j}\mathrm{d}x_k\\ &= -i\bar{\vartheta}\pi_1^*\mathrm{d}\vartheta\,.\end{aligned}$$

This proves the identity on subprincipal symbols. □

Remark IV.2.6. *If $\vartheta \in C^\infty(N, \mathbb{S}^1)$ and Λ is a magnetic DN map on M, then $\bar{\vartheta}\Lambda\vartheta$ is not necessarily a magnetic DN map. For this to be the case, ϑ must be the restriction of a function that lives on the whole manifold M as above.*

This identity will be useful for initiating the induction procedure described below.

Recovery of the magnetic potential at the boundary

In this part, we deal with the information provided by the subprincipal order of B. This will give **Theorem I.2.1** and by the same a first step in the direction of the boundary determination. The following lemma shows that the relevant quantity appearing at the subprincipal order is naturally the magnetic potential :

Lemma IV.2.7. *Let a and $\tilde{a}$ be two (normalized) magnetic potentials on M. For any electric potentials q and $\tilde{q}$ on M, the following identity holds in the unit cotangent bundle S^*N :*

$$\mathrm{sub}(\Lambda_{\tilde{a},\tilde{q}}) - \mathrm{sub}(\Lambda_{a,q}) = \pi_1^* (\tilde{a} - a) \ .$$

Proof. Let us denote in boundary normal coordinates (x, ξ)

$$\sigma^{\mathrm{full}}_{\Lambda_{\tilde{a},\tilde{q}}} \sim \sum_{j=-1}^{+\infty} \tilde{p}_{-j} \quad \text{and} \quad \sigma^{\mathrm{full}}_{\Lambda_{a,q}} \sim \sum_{j=-1}^{+\infty} p_{-j} \ .$$

Then using the expression of $\tilde{p}_1$ and $\tilde{p}_0$ provided by [DSFKSU09, Lemma 8.7], and denoting $a = \sum_{i=1}^n a_\alpha \mathrm{d}x_\alpha$ and $\tilde{a} = \sum_{i=1}^n \tilde{a}_\alpha \mathrm{d}x_\alpha$, we get the following identity :

$$\begin{aligned}
\mathrm{sub}(\Lambda_{\tilde{a},\tilde{q}}) = \tilde{p}_0 - \frac{1}{2i} \sum_{j=1}^n \frac{\partial^2 \tilde{p}_1}{\partial x_j \partial \xi_j} &= \mathrm{sub}(\Lambda_{a,q}) + \tilde{p}_0 - p_0 \\
&= \mathrm{sub}(\Lambda_{a,q}) \ + \sum_{1 \leqslant \alpha, \beta \leqslant n-1} g^{\alpha\beta} (\tilde{a}_\alpha - a_\alpha) \xi_\beta \\
&= \mathrm{sub}(\Lambda_{a,q}) + (\tilde{a} - a)\, \xi^\sharp
\end{aligned}$$

on the unit cotangent bundle S^*N. □

Exploiting the information provided by the wave invariants, we are now able to prove **Theorem I.2.1** in the same way as **Theorem I.3.3** for the Laplacian case :

*Proof of **Theorem I.2.1.*** Since $\Lambda_{a,q}$ and $\Lambda_{\tilde{a},\tilde{q}}$ have the same spectrum, the principal

wave invariants give that for any periodic geodesic γ of N :

$$\int_\gamma \pi_1^* (\tilde{a} - a) \in 2\pi\mathbb{Z} .$$

Then by **Proposition III.1.10**, there exists a solution $u \in C^\infty(S^*N; \mathbb{S}^1)$ of the transport equation with potential $f := \pi_1^* (\tilde{a} - a)$:

$$Hu + ifu = 0 .$$

Now using **Theorem III.1.11**, we get that the one form $(\tilde{a} - a)_{|N}$ is closed, $\vartheta := u$ does not depend on the fiber variable, and by **Lemma I.3.2**, satisfies the desired identity :

$$\tilde{a}_{|N} = a_{|N} - i\bar{\vartheta}\mathrm{d}\vartheta .$$

Regarding the recovery of electric potential at the boundary, we refer to **Lemma IV.2.10** in the next section. □

Remark IV.2.8. *Note that unlike the case of the Laplacian, the subprincipal order does not give information on the electric potential at order* 0*. As we shall see in the following, this information is in fact contained in the homogeneous part of order* -1*.*

Determination of the jet at the boundary

In this part, we proceed to the recovery of the full jet at the boundary of the magnetic field and of the electric potential, from the magnetic Steklov spectrum. First, we want to precisely describe the information contained in the homogeneous terms of order $-j \leq -1$ in the full symbol of $B := \Lambda_{a,q} - \Lambda_{\tilde{a},\tilde{q}}$:

Lemma IV.2.9. *Let a and $\tilde{a}$ be two (normalized) magnetic potentials on M and q and $\tilde{q}$ be two (normalized) electric potentials on M. Let*

$$\sigma^{\mathrm{full}}_{\Lambda_{\tilde{a},\tilde{q}}} \sim \sum_{j=-1}^{+\infty} \tilde{p}_{-j} \quad \textit{and} \quad \sigma^{\mathrm{full}}_{\Lambda_{a,q}} \sim \sum_{j=-1}^{+\infty} p_{-j}$$

be respectively the full symbols of $\Lambda_{\tilde{a},\tilde{q}}$ and $\Lambda_{a,q}$ considered in any local coordinate system $x' := (x_1, \ldots x_{n-1})$ on the boundary.

In boundary normal coordinates (x', x_n), denoting $a = \sum_{\alpha=1}^n a_\alpha dx_\alpha$ and $\tilde{a} = \sum_{\alpha=1}^n \tilde{a}_\alpha dx_\alpha$,

the following identity holds on the unit cotangent bundle S^*N, for any $j \geq 1$:

$$p_{-j} - \tilde{p}_{-j} = -2^{-j} \left(\pi_0^* \partial_n^{j-1} (q - \tilde{q}) \big|_{x_n=0} + \pi_1^* \partial_n^j (a - \tilde{a}) \big|_{x_n=0} \right) + T_j \,,$$

where

$$T_j := T_j \left(g^{\alpha\beta}, (a - \tilde{a})_\alpha \,, (q - \tilde{q}) \right)$$

is an expression involving only the boundary values of $g^{\alpha\beta}, (a - \tilde{a})_\alpha, q - \tilde{q}$ (for $j \geq 2$) and their normal derivatives respectively of order at most j, $j-1$ and $j-2$ at the boundary.

Proof. This result is a direct consequence of the expression in boundary normal coordinates of the full symbol of the magnetic DN map (cf. [DSFKSU09, Lemma 8.7] or [Cek20, Theorem 3.4]). □

As already mentioned, the relevant quantities that appear at order $-j \leq -1$ involve the normal derivative at the boundary of both magnetic and electric potentials, respectively, at order j and $j-1$. However, unlike the problems considered so far in this work, here we have to deal with the reminder terms T_j, $-j \leq -1$ at each step of induction since we are only able to recover the magnetic potential at the boundary up to gauge, i.e. up to a term of the form $-i\bar{\vartheta}\mathrm{d}\vartheta$ for some $\vartheta \in C^\infty(N; \mathbb{S}^1)$.
First, using **Theorem I.2.1**, we know that if two magnetic potentials are isospectral, there exists $\vartheta_0 := \vartheta \in C^\infty(N; \mathbb{S}^1)$ s.t

$$\tilde{a}_{|N} = a_{|N} - i\bar{\vartheta}_0 \mathrm{d}\vartheta_0 \,. \tag{IV.2.16}$$

Next, our goal is to obtain the same type of identity for all normal derivatives at the boundary. To describe the idea, we begin with the order 1, this is the object of **Lemma IV.2.10** below. The main ingredient is, of course, the trace formula of **Theorem II.4.2** but we cannot apply it directly to the pseudodifferential operator B, since it is not necessarily in $\Psi^{-1}(N)$, because of the gauge term $-i\bar{\vartheta}_0\mathrm{d}\vartheta_0$ in (IV.2.16). To deal with this, we need to conjugate the operator $\Lambda_{a,q}$ with ϑ_0, that is, consider the operator

$$B_1 := \bar{\vartheta}_0 \Lambda_{a,q} \vartheta_0 - \Lambda_{\tilde{a},\tilde{q}} \in \Psi^{-1}(N) \,.$$

The fact that B_1 is in $\Psi^{-1}(N)$ is a direct consequence of how the subprincipal symbol changes after conjugacy (see **Lemma IV.2.5**). The other thing to verify is that the principal symbol of B_1 also gives the desired information on the potentials. In other

words, we want to be able to apply **Lemma IV.2.9** to B_1, which requires that $\bar{\vartheta}_0\Lambda_{a,q}\vartheta_0$ be a magnetic map. This is the case if ϑ_0 is a true gauge which is not guaranteed unless ϑ_0 can be extended in a suitable way over the entire manifold M as mentioned in **Remark IV.2.6**. It is of course not possible in general to extend it to a function with values in $\mathbb{S}^1$ due to standard topological obstructions. The approach we use to ensure that we are working with a magnetic DN map is to consider the restrictions of $\bar{\vartheta}_0\Lambda_{a,q}\vartheta_0$ to each open chart U_k of a finite cover $(U_k)_{k=1,...,K}$ of N such that

$$\vartheta_0|_{U_k} = e^{i\psi_k} \ , \ \psi_k \in C^\infty(U_k, \mathbb{R}) \ .$$

Such a cover always exists since $t \mapsto e^{it}$ is a covering map of $\mathbb{S}^1$ and M is compact. Let us also note $(\chi_k)_{k=1,...K}$ a partition of unity subordinate to the cover. Then for any $k = 1, ..., K$, we note $\overline{W}_k := Supp\,(\chi_k) \subset U_k$ and $\tilde{\psi}_k$ a smooth extension of $\psi_k|_{\overline{W}_k}$ to M. As we will see below, for our proof to work, we must choose this extension so that it is flat at the boundary in the normal direction, in summary we choose $\tilde{\psi}_k$ such that for all $k = 1, ..., K$,

- $\tilde{\psi}_k$ is a smooth map on M and

$$\tilde{\psi}_k|_{\overline{W}_k} = \psi_k \ .$$

- All its normal derivatives at the boundary vanish :

$$\forall \ell \geq 1 \ , \ \partial_n^\ell \tilde{\psi}_k|_{x_n=0} = 0$$

  in any boundary coordinates (x', x_n).

It is standard to see that this choice is always possible. We are now able to recover the electric potential at the boundary and, up to gauge, the normal derivative of the magnetic potential at the boundary from the magnetic Steklov spectrum :

Lemma IV.2.10. *Assume that the cogeodesic flow ϕ_t of N is Anosov with simple length spectrum. If the magnetic DN maps $\Lambda_{a,q}$ and $\Lambda_{\tilde{a},\tilde{q}}$ have the same eigenvalues with multiplicities*

$$\big(\sigma_k(\Lambda)\big)_{k\in\mathbb{N}} = \big(\sigma_k(\tilde{\Lambda})\big)_{k\in\mathbb{N}} \ ,$$

then

$$\tilde{q}|_N = q_{|N}$$

and the 1-form $\partial_\nu\,(\tilde{a} - a)\,|_N$ is exact, i.e. there is $\beta_1 \in C^\infty(N)$ such that :

$$\partial_v \tilde{a}|_N = \partial_v a|_N + \mathrm{d}\beta_1 \,.$$

In particular, there exists $\vartheta_1 \in C^\infty(N;\mathbb{S}^1)$ *s.t*

$$\partial_v \tilde{a}|_N = \partial_v a|_N - i\bar{\vartheta}_1 \mathrm{d}\vartheta_1 \,.$$

Proof. Since $\Lambda_{\tilde{a},\tilde{q}}$ and $\bar{\vartheta}_0 \Lambda_{a,q} \vartheta_0$ have the same spectrum and N is Anosov with simple length spectrum, one can apply **Corollary II.4.4** to the operators $A := \Lambda_{\tilde{a},\tilde{q}} \in \Psi^1(N)$ and $B_1 \in \Psi^{-1}(N)$. We get that for any closed orbit γ of the geodesic flow on N :

$$\int_\gamma \sigma_{B_1} = 0 \,.$$

Given the expression of **Lemma IV.2.9**, we observe that on the unit cotangent bundle S^*N, the principal symbol of $B_1 \in \Psi^{-1}(N)$ is the sum of a function $f_0 \in C^\infty(N)$ and a 1-form $f_1 \in \Omega_1(N)$:

$$\sigma_{B_1} = \pi_0^* f_0 + \pi_1^* f_1$$

Then, by (joint) injectivity of the X-ray transform on functions and 1-forms (cf. **Theorem III.1.7**), we obtain that $f_0 = 0$ and f_1 is exact

$$f_1 := \mathrm{d}\beta_1 \,.$$

In addition, since for any $k \in \{1, ..., K\}$, the principal symbols of $\bar{\vartheta}_0 \Lambda_{a,q} \vartheta_0$ and $\Lambda_{a+\mathrm{d}\tilde{\psi}_k,q}$ coincide on $S^*\overline{W}_k$ (it is easy to see by using for example the partition of unity $(\chi_k)_{k=1,...K}$), we get

$$\sigma_{B_1}|_{S^*\overline{W}_k} = \left(\sigma_{\Lambda_{a+\mathrm{d}\tilde{\psi}_k,q}} - \sigma_{\Lambda_{\tilde{a},\tilde{q}}}\right)\Big|_{S^*\overline{W}_k} \,.$$

Next, using **Lemma IV.2.9** we deduce that in boundary normal coordinates (x', x_n), for any $k \in \{1, ..., K\}$,

$$\sigma_{B_1}|_{S^*\overline{W}_k} = -\frac{1}{2}\left(\pi_0^*(q-\tilde{q})\,|_{S^*\overline{W}_k} + \pi_1^* \partial_n \left(a - \tilde{a} + \mathrm{d}\tilde{\psi}_k\right)\Big|_{S^*\overline{W}_k}\right) + T_1 \,,$$

where $T_1 = 0$ since it involves only the values of $g^{\alpha\beta}$, $\partial_n g^{\alpha\beta}$, $\left(a + \mathrm{d}\tilde{\psi}_k - \tilde{a}\right)_\alpha$ in $\overline{W}_k \subset N$ and also by (IV.2.16) :

$$\tilde{a}|_{\overline{W}_k} = \left(a + \mathrm{d}\tilde{\psi}_k\right)\Big|_{\overline{W}_k} \,.$$

Then using the fact that f_1 is exact and that for all k, $\tilde{\psi}_k$ is flat at the boundary in the

normal direction, we obtain the desired identities on each $\overline{W}_k$ and then on the whole manifold M since $(\overline{W}_k)_{k=1,\dots,K}$ covers N. In particular

$$\partial_n (\tilde{a} - a) \,|_{x_n=0} = \mathrm{d}\beta_1$$

is closed and **Lemma** I.3.2 gives ϑ_1 as desired. □

Unlike the information (IV.2.16) obtained on the magnetic potential at the subprincipal order, here we obtain a gauge term that is exact, which is much more convenient for continuing the induction. Consequently, one can proceed in the same way to recover the normal derivatives of higher order and it is even easier as it is possible to construct a true gauge transformation at the end of each step of the induction. More explicitly, assume that for $j \in \mathbb{N}^*$, we have constructed $(\beta_\ell)_{\ell=1,\dots,j}$ a family of smooth functions on M such that for all $1 \leq \ell \leq j$,

$$\partial_n^\ell \tilde{a}|_{x_n=0} = \partial_n^\ell a|_{x_n=0} + \mathrm{d}\beta_\ell(x', 0)\,. \tag{IV.2.17}$$

in boundary normal coordinates near N. Then one can define the smooth function

$$\alpha_j := \sum_{\ell=1}^{j} \chi \cdot \beta_\ell \frac{x_n^\ell}{\ell\,!}$$

on M, where $\chi \in C^\infty(M)$ obviously denotes a smooth bump function.
Unlike ϑ_0, the function $e^{i\alpha_j}$ defines a true gauge in the sense that a magnetic DN map $\Lambda_{b,q}$ conjugated by $e^{i\alpha_j}$ (as a unitary operator of order 0) remains a magnetic DN map (see Section I.2.1) :

$$e^{-i\alpha_j}\Lambda_{b,q}e^{i\alpha_j} = \Lambda_{b+\mathrm{d}\alpha_j,q}\,.$$

It is then natural to consider the pseudodifferential operator

$$B_{j+1} := e^{-i\alpha_j}\tilde{\vartheta}_0\Lambda_{a,q}\vartheta_0 e^{i\alpha_j} - \Lambda_{\tilde{a},\tilde{q}}$$

to apply our wave trace formula in the same way as we apply it to B_1 in the proof of **Lemma** IV.2.10.

Lemma IV.2.11. *Assume that for all* $1 \leq \ell \leq j$,

$$\partial_n^{\ell-1}\left(\tilde{q}-q\right)\big|_{x_n=0} = 0\,. \tag{IV.2.18}$$

Then

$$B_{j+1} \in \Psi^{-j-1}(N)\,.$$

Proof. Let

$$\sigma^{\text{full}}_{\Lambda_{\tilde{a},\tilde{q}}} \sim \sum_{j=-1}^{+\infty} \tilde{p}_{-j} \quad \text{and} \quad \sigma^{\text{full}}_{B_j} \sim \sum_{j=-1}^{+\infty} b_{-j}$$

be respectively the full symbols of $\Lambda_{\tilde{a},\tilde{q}}$ and of

$$B_j := e^{-i\alpha_j}\bar{\vartheta}_0\Lambda_{a,q}\vartheta_0 e^{i\alpha_j}$$

considered in any local coordinate system $x' := (x_1, ...x_{n-1})$ on the boundary. The key point here, and once again the principle of our idea, is to observe that the full symbol of B_j coincide on each $S^*\overline{W}_k$ with the full symbol of $\Lambda_{a+\mathrm{d}(\tilde{\psi}_k+\alpha_j)}$ in any boundary coordinates since locally,

$$B_j = \Lambda_{a+\mathrm{d}(\tilde{\psi}_k+\alpha_j)}$$

as operators in U_k. Then using **Lemma IV.2.9**, we get, for any $k \in \{1, ..., K\}$ and any $1 \leq \ell \leq j$, the following identity on $S^*\overline{W}_k$, in boundary normal coordinates (x', x_n) near $\overline{W}_k$:

$$\begin{aligned} b_{-\ell} - \tilde{p}_{-\ell} &= -2^{-\ell}\left(\pi_0^*\partial_n^{\ell-1}\left(q-\tilde{q}\right) + \pi_1^*\partial_n^{\ell}\left(a-\tilde{a}+\mathrm{d}(\tilde{\psi}_k+\alpha_j)\right)\right) + T_\ell \\ &= -2^{-\ell}\pi_1^*\partial_n^{\ell}\left(a-\tilde{a}+\mathrm{d}(\tilde{\psi}_k+\alpha_j)\right) + T_\ell \end{aligned}$$

where the second equality comes from the use of (IV.2.18). Note that T_ℓ involves only the values of $g^{\alpha\beta}$, $\left(a-\tilde{a}+\mathrm{d}(\tilde{\psi}_k+\alpha_j)\right)_\alpha$, $q-\tilde{q}$ and of their normal derivatives respectively of order at most ℓ, $\ell-1$ and $\ell-2$ in $\overline{W}_k \subset N$. We know that the terms involving metrics are known and those involving electric potentials are zero (by (IV.2.18)). Regarding the magnetic terms, since $\tilde{\psi}_k$ and χ are flat at the boundary in the normal direction (recall that χ is a bump function, so it is constant equal to 1 in a neighborhood of N) i.e.

$$\forall \ell \geq 1\,,\ \partial_n^\ell\tilde{\psi}_k\big|_{x_n=0} = 0 = \partial_n^\ell\chi\big|_{x_n=0}\,,$$

then T_ℓ actually depends only on the values in $\overline{W}_k$ of $\left(a-\tilde{a}+\mathrm{d}\tilde{\psi}_k\right)_\alpha$ and of $\partial_n\left(a-\tilde{a}+\mathrm{d}\beta_1\right)_\alpha, ..., \partial_n^j\left(a-\tilde{a}+\mathrm{d}\beta_j\right)_\alpha$.

By (IV.2.16) and (IV.2.17), these values are zero and we finally conclude that $T_\ell = 0$. In the same way, using (IV.2.16), (IV.2.17) and the fact that $\tilde{\psi}_k$ and χ are flat at the boundary in the normal direction, we see that for any $1 \leq \ell \leq j$,

$$b_{-\ell} - \tilde{p}_{-\ell} = -2^{-\ell} \left(\pi_1^* \partial_n^\ell \left(a - \tilde{a} + \mathrm{d}(\tilde{\psi}_k + \alpha_j) \right) \right) = 0$$

on each $S^*\overline{W}_k$ and then in S^*N since $(\overline{W}_k)_{k=1,\dots,K}$ cover N. □

We can then show the following theorem which proves **Theorem I.2.2** :

Theorem IV.2.12. *Assume that the cogeodesic flow ϕ_t of N is Anosov with simple length spectrum. If the spectra of the magnetic DN maps $\Lambda_{a,q}$ and $\Lambda_{\tilde{a},\tilde{q}}$ coincide*

$$\left(\sigma_k(\Lambda)\right)_{k\in\mathbb{N}} = \left(\sigma_k(\tilde{\Lambda})\right)_{k\in\mathbb{N}} ,$$

then in boundary normal coordinates (x', x_n), for any $j \in \mathbb{N}^*$,

$$\partial_n^{j-1} \left(\tilde{q} - q\right) \big|_{x_n=0} = 0$$

and there exists $\beta_j \in C^\infty(M)$ s.t

$$\partial_n^j \tilde{a}|_{x_n=0} = \partial_n^j a|_{x_n=0} + \mathrm{d}\beta_j(x', 0) .$$

In particular, for any $j \in \mathbb{N}^*$, there exists $\vartheta_j \in C^\infty(N; \mathbb{S}^1)$ s.t

$$\partial_n^j \tilde{a}|_{x_n=0} = \partial_n^j a|_{x_n=0} - i\bar{\vartheta}_j \mathrm{d}\vartheta_j .$$

Proof. Consider $x' := (x_1, \dots x_{n-1})$ any local coordinates on the boundary and (x', x_n) the associated g-boundary normal coordinates near N.
We proceed by induction on the order $j \geq 1$:

- $j = 1$: This is established by **Lemma IV.2.10** above.
- Let $j \geq 1$ s.t for all $1 \leq \ell \leq j$,

$$\partial_n^{\ell-1} \left(\tilde{q} - q\right) \big|_{x_n=0} = 0$$

  and there exists $\beta_\ell \in C^\infty(M)$ s.t

$$\partial_n^\ell \tilde{a}|_{x_n=0} = \partial_n^\ell a|_{x_n=0} + \mathrm{d}\beta_\ell(x',0)\,.$$

Since $\Lambda_{\tilde{a},\tilde{q}}$ and $e^{-i\alpha_j}\bar{\vartheta}_0\Lambda_{a,q}\vartheta_0 e^{i\alpha_j}$ have the same spectrum and N is Anosov with simple length spectrum, one can apply **Corollary II.4.4** to the operators $A := \Lambda_{\tilde{a},\tilde{q}} \in \Psi^1(N)$ and $B_{j+1} \in \Psi^{-j-1}(N)$. We get that for any closed orbit γ of the geodesic flow on N :

$$\int_\gamma \sigma_{B_{j+1}} = 0\,.$$

Then it is the exact same proof as that of **Lemma IV.2.10**, just replacing of course B_1 by B_{j+1}, noting that for any $k \in \{1,...,K\}$,

$$\sigma_{A+B_{j+1}}\big|_{S^*\overline{W}_k} = \sigma_{\Lambda_{a+\mathrm{d}(\tilde{\psi}_k+\alpha_j),q}}\big|_{S^*\overline{W}_k}\,,$$

and finally combining this with **Lemma IV.2.9** to deduce that for any $k \in \{1,...,K\}$, in boundary normal coordinates (x',x_n) near U_k, the principal symbol of B_{j+1} in $S^*\overline{W}_k$ is essentially given by the sum of $\pi_0^*\partial_n^j(q-\tilde{q})\big|_{S^*\overline{W}_k}$, $\pi_1^*\partial_n^{j+1}\left(a-\tilde{a}+\mathrm{d}(\tilde{\psi}_k+\alpha_j)\right)\big|_{S^*\overline{W}_k}$ and of T_{j+1}. Moreover, $T_{j+1} = 0$ since it involves only the values of $g^{\alpha\beta}$, $q-\tilde{q}$, $\left(a-\tilde{a}+\mathrm{d}(\tilde{\psi}_k+\alpha_j)\right)_\alpha$ and of their normal derivatives respectively of order at most $j+1$, $j-1$ and j in $\overline{W}_k \subset N$. Note that we also used crucially, as well as to conclude the proof, the identities (IV.2.16), (IV.2.17), (IV.2.18) and the fact that $\tilde{\psi}_k$ and χ are flat at the boundary in the normal direction, namely for all $\ell \geq 1$:

$$\partial_n^\ell \tilde{\psi}_k|_{x_n=0} = 0 = \partial_n^\ell \chi|_{x_n=0}\,.$$

We refer to the end of the proof of **Lemma IV.2.11** for a similar and more detailed use of these identities.

□

Since the gauge on the magnetic potential is closed, it gives the recovery of the magnetic field at the boundary as expected (cf. **Theorem I.2.2**).

IV.2.3 Towards the framework of connections

Inverse problems for magnetic Schrödinger operators and for magnetic DN maps can be naturally understood as special instances of a more general geometric framework. Regarding first a magnetic Schrödinger operator, consider a Hermitian vector bundle $E \to N$ over a closed Riemannian manifold N, a unitary connection ∇ on E, and a Hermitian matrix-valued potential $Q \in C^\infty(N, \mathrm{End}(E))$ which is assumed to be positive (pointwise as Hermitian endomorphism). The corresponding connection Laplacian acts on sections $u \in C^\infty(N, E)$ by

$$\mathcal{L}_{\nabla,Q} := \nabla^*\nabla + Q \,.$$

It is a formally self-adjoint differential operator on $L^2(N, E)$ with discrete non-negative spectrum :

$$0 \leq \lambda_1(\nabla) \leq \lambda_2(\nabla) \leq \ldots \to \infty \,.$$

The spectral inverse problem consists of determining the pair (∇, Q) from the spectrum of $\mathcal{L}_{\nabla,Q}$. The usual magnetic problem is recovered as the *rank-one* case : a real 1-form a defines a unitary connection $\nabla^a := \mathrm{d} + ia$ on the trivial line bundle (and of course Q is a scalar electric potential q in that case). As already mentioned, in absence of electric potential i.e for $q = 0$, Cekić and Lefeuvre answered the question positively when M is negatively curved (cf. [CL25c]) by using the wave trace formula of [Gui74].

In the same way, one can consider the analogue for the DN map associated with connection Laplacians : Let $E \to M$ be a Hermitian vector bundle over a compact Riemannian manifold (M, g) with boundary, equipped with a unitary connection ∇^A and a smooth matrix-valued potential $Q \in C^\infty(M, \mathrm{End}(E))$. The associated connection Laplacian $\Delta_{A,Q} := \mathcal{L}_{\nabla^A,Q}$ induces a Dirichlet-to-Neumann map

$$\Lambda_{A,Q} f = \nabla^A_\nu u|_{\partial M} \quad , \quad u_{|\partial M} = f \quad , \quad \Delta_{A,Q} u = 0 \,,$$

provided that 0 is not a Dirichlet eigenvalue of $\Delta_{A,Q}$. It is an elliptic pseudodifferential operator of order 1 acting on sections $f \in C^\infty(\partial M, E_{|\partial M})$. Moreover, it is not difficult to see that $\Lambda_{A,Q}$ is self-adjoint if Q is Hermitian matrix-valued. Thus, in this case, as usual, it admits a discrete spectrum of eigenvalues :

$$\sigma_1(\nabla^A) \leq \sigma_2(\nabla^A) \leq \ldots \to \infty \,.$$

What we could call the *connection Steklov inverse problem* consists in determining the gauge equivalence class of (A, Q) from the sequence of eigenvalues of $\Lambda_{A,Q}$. Note that gauge transformations act by

$$A \mapsto U^{-1}AU + U^{-1}\mathrm{d}U \quad , \quad Q \mapsto U^{-1}QU \quad , \quad U \in C^\infty(M, \mathbb{U}(r)) \ .$$

If E is the trivial complex line bundle,

$$A := a \in \Omega^1(M; \mathbb{R}) \quad , \quad Q := q \in C^\infty(M) \quad , \quad \nabla^a := \mathrm{d} + ia \, ,$$

and the operator $\Delta_{A,Q}$ becomes the magnetic Schrödinger operator

$$\Delta_{a,q} = (\mathrm{d} + ia)^*(\mathrm{d} + ia) + q \, .$$

Thus the recovery of (a, q) from the magnetic Steklov spectrum is simply the rank-one case of the spectral inverse problem for (A, Q).

The DN map $\Lambda_{A,Q}$ is a classical matrix-valued pseudodifferential operator with full symbol

$$\sigma^{\mathrm{full}}_{\Lambda_{A,Q}}(x, \xi) \sim |\xi|_{g^\circ} \, \mathrm{Id}_{E_x} \ + \ \sigma_0(A, Q)(x, \xi) \ + \ \sigma_{-1}(A, Q)(x, \xi) \ + \cdots \ .$$

While the principal symbol is a scalar multiple of the identity and depends only on the metric, the lower-order homogeneous terms $\sigma_{-j}(A, Q)$, $j \geq 0$, encodes the full Taylor series at the boundary of A and Q (cf. [Cek20, Theorem 3.4]). More precisely, as already mentioned, we have a true generalization to **Lemma IV.2.9**.

We believe that the same approach used in this article could work to relate these symbols to the Steklov spectrum, namely by using wave trace techniques in the setting of pseudodifferential operators on vector bundles. For example, the information contained at subprincipal order could be recovered from the leading terms in the wave trace formula of [Gui74] which essentially consist (under the assumptions that ∂N is Anosov with simple length spectrum) of the traces of the boundary holonomy $\mathrm{Hol}_{\nabla^A|_{\partial M}}(\gamma)$ along closed geodesics γ of period T_γ :

$$\big(t - T_\gamma\big) \sum_{j=0}^{+\infty} e^{it\sigma_j(\nabla^A)} \underset{t \to T_\gamma}{\longrightarrow} \mathrm{Tr}\Big(\mathrm{Hol}_{\nabla^A|_{\partial M}}(\gamma)\Big) \, \frac{|T^\#_\gamma|}{2\pi} |\det\big(\mathrm{Id} - \mathcal{P}_\gamma\big)|^{-1/2} \, .$$

As mentioned above, the principal symbol of $\Lambda_{A,Q}$ is given by

$$\sigma_{\Lambda_{A,Q}}(x,\xi) = |\xi|_{g^\circ} \operatorname{Id}_{E_x}, \quad (x,\xi) \in T^*\partial M,$$

thus, the underlying Hamiltonian flow is simply the cogeodesic flow of the boundary metric. The key point is that the first non-scalar contribution in the symbol appears at subprincipal order and contains the tangential part of the boundary connection (see, once again, [Cek20, Theorem 3.4]), which should explain why the leading singularities of the wave trace $\operatorname{Tr}\left(e^{it\Lambda_{A,Q}}\right)$ at lengths of closed geodesics mainly involve traces of the holonomy of the boundary connection along those geodesics and, consequently, why the residue formula above is exactly what is expected.
In other words, assuming such an identification, the Steklov spectrum would determine the boundary *geodesic Wilson loop operator* (see [CL25c, §4]). One may then use the results of Cekić and Lefeuvre on the holonomy inverse problem (see, for example, [CL25c] and [CL25b]) and more precisely on the injectivity of the geodesic Wilson loop operator. In cases where it is injective–for instance if we assume that the boundary is negatively curved with simple length spectrum and under a low-rank assumption on the vector bundle $E_{|\partial M}$–this would determine the boundary connection $\nabla^A|_{\partial M}$ up to a unitary gauge.

The natural continuation would be to ask whether it is possible to find an appropriate version of Guillemin's trace formula [Gui74, Theorem II] in the manner of **Theorem II.4.2** to deal with lower-order terms and, hopefully, obtain results analogous to **Theorem I.2.2** in the context of connections and matrix-valued potentials.
Beyond the subprincipal level, the expected lower-order invariants should no longer be simple scalar integrals over closed geodesics. On the contrary, once the boundary connection is recovered, we expect expressions depending of course, on symbols σ of lower order with values in $\operatorname{End}(E)$, and on holonomy. We are deliberately keeping this point fairly vague because, as noted above, to specify it, we would need a perturbative trace formula such as the one obtained in the **Theorem II.4.2**, but suitable for elliptic systems.
Assuming this, we would then have at our disposal a vast body of literature to draw upon when dealing with connection-twisted transport equations–such as the one we address in the scalar case–mainly the work of Guillarmou-Paternain-Salo-Uhlmann from these last decades (for example [Pat13], [GPSU16b], [PS23] or even [PSU12]).

We hope to address these topics in future work.

Appendix IV.A Dirichlet-to-Neumann map of a conformal metric

Here we provide a useful identity for the calculation of the subprincipal symbol of Λ_{cg}. This identity is undoubtedly very well known, but the author has found no clear reference for it, so it is reproduced here. The computation of the subprincipal symbol is performed at the end in **Lemma IV.A.5**.

Lemma IV.A.1. *Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying $c_{|\partial M} \equiv 1$. Then*

$$\Lambda_{cg} = \Lambda_{g,-q_c} - \frac{n-2}{4}\left((\partial_\nu c)_{|\partial M}\right) Id\,,$$

where

$$q_c = c^{\frac{n-2}{4}} \Delta_{cg}(c^{-\frac{n-2}{4}})\,.$$

Proof. It is well-known that the Laplace-Beltrami operator transforms under conformal scaling of the metric by

$$\Delta_{cg} = c^{-\frac{n+2}{4}}\left(\Delta_g + q_c\right) c^{\frac{n-2}{4}}\,.$$

Thus, for any $f \in H^{1/2}(\partial M)$, u is solution in $H^1(M)$ of

$$\begin{cases} -\Delta_{cg} u = 0 \\ u_{|\partial M} = f \end{cases}$$

iff

$$v := c^{\frac{n-2}{4}} u$$

is solution in $H^1(M)$ of the following Dirichlet problem

$$\begin{cases} \mathcal{L}_{g,-q_c} v = 0 \\ v_{|\partial M} = f \end{cases},$$

where $\mathcal{L}_{g,-q_c} := -\Delta_g - q_c$. Therefore,

$$\begin{aligned} \Lambda_{cg} f := \partial_\nu u_{|\partial M} &= \partial_\nu (c^{-\frac{n-2}{4}} v)\big|_{\partial M} \\ &= -\frac{n-2}{4}\left((c^{-\frac{n+2}{4}} \partial_\nu c) v\right)\Big|_{\partial M} + \left(c^{-\frac{n-2}{4}} \partial_\nu v\right)\Big|_{\partial M} \end{aligned}$$

and since $v_{|\partial M} = f$,

$$\begin{aligned}\Lambda_{cg} f &= -\frac{n-2}{4}((\partial_\nu c)v)\,|_{\partial M} + \partial_\nu v|_{\partial M} \\ &= -\frac{n-2}{4}(\partial_\nu c)\,|_{\partial M} f + \Lambda_{g,-q_c} f\ .\end{aligned}$$

□

Remark IV.A.2. *If M is of dimension 2, we easily deduce the well-known conformal invariance of the DN map i.e*

$$\Lambda_{cg} = \Lambda_g\ ,$$

if

$$c_{|\partial M} \equiv 1\ .$$

The above formula can be refined with the :

Proposition IV.A.3. *Let $q \in C^\infty(M)$ such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ so that*

$$\Lambda_{g,q} : H^{1/2}(\partial M) \to H^{-1/2}(\partial M)$$

is well defined.
Then

$$\Lambda_{g,q} = \Lambda_g + R_{-1}\quad ,$$

where $R_{-1} \in \Psi^{-1}(\partial M)$ has for principal symbol

$$\sigma_{R_{-1}}(x,\xi) = \frac{q^\circ(x)}{2|\xi|_{g^\circ}}\quad .$$

Proof. This result is a direct consequence of the expressions in boundary normal coordinates for the full symbols of the pseudodifferential operators $\Lambda_{g,q}$ and Λ_g computed, for example, in [DSFKSU09, Lemma 8.6] and [LU89, Section 1] respectively. □

In particular, we deduce the following :

Corollary IV.A.4. *Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying $c_{|\partial M} \equiv 1$.*
Then

$$\Lambda_{cg} = \Lambda_g + \alpha_n \left((\partial_\nu c)\,|_{\partial M}\right) Id + R_{-1}\quad ,$$

where $\alpha_n := -\frac{n-2}{4}$ and $R_{-1} \in \Psi^{-1}(\partial M)$.

We conclude that the exploitable quantity that appears when calculating the subprincipal symbol of the DN map $\Lambda_{cg} \in \Psi^1(\partial M)$ is the normal derivative of the conformal factor at the boundary :

Lemma IV.A.5. *Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying $c_{|\partial M} \equiv 1$.*

Then we have the following equality on $T^*\partial M$

$$\mathrm{sub}(\Lambda_{cg}) - \mathrm{sub}(\Lambda_g) = \alpha_n(\partial_\nu c)\,|_{\partial M}\,,$$

where $\alpha_n = -\dfrac{n-2}{4}$.

Proof. If we note in any local coordinates

$$\sigma_{\Lambda_g}^{full} \sim \sum_{j=-1}^{+\infty} p_{-j}\ ,$$

in the sense of (I.6.4), then by the result of **Corollary IV.A.4** and the expression of the subprincipal symbol in coordinates (see the Section I.6), we obtain the following identity :

$$\begin{aligned}\mathrm{sub}(\Lambda_{cg}) &= p_0 + \alpha_n(\partial_\nu c)\,|_{\partial M} - \frac{1}{2i}\sum_{j=1}^{n}\frac{\partial^2\sigma_{\Lambda_g}}{\partial x_j \partial \xi_j}\\ &= \mathrm{sub}(\Lambda_g) + \alpha_n(\partial_\nu c)\,|_{\partial M}\end{aligned}$$

in any coordinate chart (x,ξ) on $T^*\partial M$. □

Appendix IV.B On the recovery of the conformal factor

Here we establish some simple results on the spectral inverse problem at principal order ; more precisely we give examples where the Steklov isospectrality of two conformal metrics g and cg implies their equality at the boundary i.e

$$c_{|\partial M} \equiv 1\ .$$

As already mentioned, this highly non-linear problem cannot be treated with the techniques used here and it is of course not true in general just as the analogue for the Laplacian is not. We are actually able to make certain statements but only under much stronger geometrical assumptions due to the lack of exploitable spectral information. Indeed, here we only exploit the boundary volume as a spectral invariant, given by the Weyl's law (see **Proposition II.1.1**).

More precisely, we adapt arguments[5] from Besson, Gallot and Courtois initially used to establish a similar spectral rigidity result for the Laplacian [BCG91, Proposition B.2]. However, in our case, the statement is much weaker since the total scalar curvature (i.e. the integral of scalar curvature) is not an invariant of the Steklov spectrum :

Theorem IV.B.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$ with boundary such that $(\partial M, g^\circ)$ has a constant negative scalar curvature.*
Let $c : M \to \mathbb{R}_*^+$ be a smooth function satisfying

$$\mathrm{spec}(\Lambda_{cg}) = \mathrm{spec}(\Lambda_g) .$$

If the total scalar curvatures at the boundary coincide

$$S_{(cg)^\circ} = S_{g^\circ} ,$$

then

$$c_{|\partial M} \equiv 1 .$$

Proof. The proof follows exactly that of [BCG91, Proposition B.2].
Let us write $h = cg$ and $c = e^{-2u}$ with u a smooth real-valued function on M.
The well-known expression of the scalar curvature after conformal change gives on ∂M :

$$scal(g^\circ) = e^{-2u} \left(scal(h^\circ) - 2(n-2)\Delta_{h^\circ} u - (n-3)(n-2)|du|^2_{h^\circ} \right) ,$$

then by integrating with respect to the canonical measure of $(\partial M, h^\circ)$:

$$\int_{\partial M} e^{2u} \left(-scal(g^\circ)\right) dV_{h^\circ} = \int_{\partial M} \left(-scal(h^\circ)\right) dV_{h^\circ} + (n-3)(n-2) \int_{\partial M} |du|^2_{h^\circ} dV_{h^\circ} .$$

Indeed, since the integral of the divergence of a vector field on a closed manifold is zero (by the classical Stokes' theorem), we have

$$\int_{\partial M} (\Delta_{h^\circ} u)\, dV_{h^\circ} = \int_{\partial M} \mathrm{div}_{h^\circ}(\nabla_{h^\circ} u) dV_{h^\circ} = 0 .$$

[5] Arguments originally put forward by A.El Soufi and S.Ilias in their work on conformal volume.

Thus, as the scalar curvature of $(\partial M, g^\circ)$ is constant and by the Hölder's inequality applied with exponents $p = \dfrac{n-1}{2}$ and $q = \dfrac{n-1}{n-3}$ we get :

$$\begin{aligned}\int_{\partial M} (-scal(h^\circ))\, dV_{h^\circ} &\leq \int_{\partial M} e^{2u}\, (-scal(g^\circ))\, dV_{h^\circ} \\ &\leq -scal(g^\circ) \left(\underbrace{\int_{\partial M} e^{(n-1)u} dV_{h^\circ}}_{=\mathrm{Vol}_{g^\circ}(\partial M)} \right)^{\frac{1}{p}} \mathrm{Vol}_{h^\circ}(\partial M)^{\frac{1}{q}} .\end{aligned}$$

Since g and h are Steklov isospectral and the volume of the boundary ∂M is an invariant for the Steklov spectrum, we have $\mathrm{Vol}_{g^\circ}(\partial M) = \mathrm{Vol}_{h^\circ}(\partial M)$ and consequently the previous inequality becomes :

$$\begin{aligned}\int_{\partial M} (-scal(h^\circ))\, dV_{h^\circ} &\leq -scal(g^\circ) \left(\mathrm{Vol}_{g^\circ}(\partial M)\right)^{\frac{1}{p}} \mathrm{Vol}_{h^\circ}(\partial M)^{\frac{1}{q}} \\ &\leq -scal(g^\circ)\mathrm{Vol}_{g^\circ}(\partial M) ,\end{aligned}$$

obviously the equality holds iff $u_{|\partial M} = 0$. [6]

By assumption, we do have

$$\int_{\partial M} (-scal(h^\circ))\, dV_{h^\circ} = \int_{\partial M} (-scal(g^\circ))\, dV_{g^\circ} = -scal(g^\circ)\mathrm{Vol}_{g^\circ}(\partial M) ,$$

so the equality in the inequality and therefore $u_{|\partial M} = 0$. □

In dimension 3, the assumption on total scalar curvatures at the boundary is already verified which gives in particular this elementary statement :

Proposition IV.B.2. *Let (M, g) be a smooth compact Riemannian 3-manifold with boundary such that $(\partial M, g^\circ)$ has constant sectional curvature.*

[6] Recall that for f and g two integrable functions and p, q conjugate exponents the equality in Hölder's inequality holds iff

$$|g|^q = \lambda^p |f|^p \text{ almost everywhere,}$$

with $\lambda = \dfrac{\|g\|_q^{q/p}}{\|f\|_p}$. Here we have $f = e^{2u}$ and $g \equiv 1$ on ∂M so we get that the equality holds iff

$$e^{(n-1)u} \equiv \mathrm{Vol}_{g^\circ}(\partial M)/\mathrm{Vol}_{h^\circ}(\partial M) = 1 \text{ on } \partial M,$$

iff

$$u_{|\partial M} = 0 .$$

Let $h := cg$ be a metric conformal to g and assume that h° has constant sectional curvature.[7]
If

$$\mathrm{spec}(\Lambda_{cg}) = \mathrm{spec}(\Lambda_g)\,,$$

then

$$scal(h^\circ) = scal(g^\circ)\,.$$

In addition, if this scalar curvature is negative,

$$c_{|\partial M} \equiv 1\,.$$

Proof. As usual let us note $c = e^{2u}$ with u a smooth real-valued function on M. The expression of the scalar curvature after conformal change in dimension 2 leads to :

$$\int_{\partial M} scal(h^\circ)dV_{h^\circ} = \int_{\partial M} e^{2u} scal(g^\circ)dV_{h^\circ} = scal(g^\circ)\mathrm{Vol}_{g^\circ}(\partial M)\,.$$

By isospectrality, $\mathrm{Vol}_{h^\circ}(\partial M) = \mathrm{Vol}_{g^\circ}(\partial M)$, and since h° has constant curvature we obviously deduce that

$$scal(h^\circ) = scal(g^\circ)\,.$$

Therefore, $\varphi := 2u_{|\partial M}$ satisfies the equation

$$-\Delta_{h^\circ}\varphi + \kappa = \kappa e^{\varphi}\,,$$

where $\kappa := scal(g^\circ)$. But if κ is negative, it is easy to see that $\varphi \equiv 0$ is the unique solution and, equivalently, that there is one and only one metric with scalar curvature κ in the conformal class of g°. We refer, for example, to the proof of [Aub98, Theorem 6.8] and more generally to T.Aubin's work on the prescription of scalar curvature. □

[7] By the uniformization theorem it is always possible to find such metrics.

Conclusion and outlook

Conclusion

The goal of this work was to obtain positive results for the *Steklov inverse problem* in as broad a class of geometries as possible. The first results, obtained in [Flo25], concern the *boundary determination* of Riemannian metrics in their conformal classes (cf. **Theorem I.1.1** and **Theorem I.1.2**) and of electric potentials (cf. [Flo25, Theorem V.2]).

The strategy was to convert the Steklov inverse problem into a *tomography inverse problem* through the singularities of the *wave trace*

$$W_{\Lambda_g}(t) := \mathrm{Tr}\left(e^{-it\Lambda_g}\right)$$

of the unitary wave group associated with the DN map (cf. Chapter II).
As we explain in this work, if the boundary is Anosov with simple length spectrum, the Duistermaat-Guillemin trace formula (cf. **Theorem II.3.4**) indeed shows that every closed geodesic γ on the boundary contributes to the wave trace via a Lagrangian distribution whose asymptotic expansion is essentially controlled by the integral along γ of the subprincipal symbol $\mathrm{sub}(\Lambda_g)$. In addition, a new version of this trace formula (cf. **Theorem II.4.2**), largely inspired by Guillemin's ideas [Gui78a], is also established and generalize the one given in [Gui78a, Theorem 4].
By induction, it allows us to access integrals along any closed geodesic on the boundary of the lower-order symbols that make up the total symbol of Λ_g. This is the key ingredient in each of our results, and more precisely, in recovering the full jet at the boundary of the metric (or potential); see Chapter IV.
One can say that this wave trace approach provides us with exactly three major classes of dynamical information corresponding to three levels within the total symbol of Λ_g :

- The level of the principal symbol σ_{Λ_g} provides the boundary length spectrum

via the underlying dynamic.

- The level of the subprincipal symbol $\mathrm{sub}(\Lambda_g)$ contributes via the phase terms

$$e^{-i\int_\gamma \mathrm{sub}(\Lambda_g)}$$

which control the singularities in the Duistermaat-Guillemin trace formula.

- The level of the homogeneous terms p_{-j} of order $-j \leq -1$ contributes via the integrals

$$\int_\gamma p_{-j}$$

which are essentially the leading terms in the trace formula of **Theorem II.4.2**.

Regarding first the principal order, it is successfully used in a line of ideas going back to Guillemin–Kazhdan [GK80a, GK80b], under the name of *infinitesimal spectral rigidity*. The idea is to consider a smooth one-parameter family of isospectral Riemannian metrics $(g_s)_{0\leq s\leq \varepsilon}$ with $g_0 = g$; such a family is known as *isospectral deformations*. Differentiating the associated one-parameter family of marked length spectra with respect to the parameter s then linearizes the problem, reducing it to the study of a tomography inverse problem, namely the injectivity of the geodesic X-ray transform. This point of view has also been considered in our case and proves to be fruitful (see **Theorem I.1.2** and Subsection IV.1.2), even though trickier to apply than in the case of the laplacian (see Section III.2).

For lower-order symbolic terms, they intervene in the non-linear problem: Since the boundary is Anosov, Livšic theory and injectivity results on the geodesic X-ray transform allow these to be recovered at the boundary from their integrals along closed geodesics.
Boundary determination results for the Calderón problem [LU89], [DSFKSU09] then identify the full symbol of the DN map with the boundary jets of the metric, or of the lower-order coefficients. The entire strategy is summarize below:

$$\mathrm{spec}(\Lambda_g) \xrightarrow{\text{wave trace}} \underset{\text{along closed geodesics}}{\text{integrals of symbols}} \xrightarrow{\substack{\text{injectivity of the}\\ \text{X-ray transform}}} \underset{\text{normal coordinates}}{\sigma^{\mathrm{full}}_{\Lambda_g} \text{ in boundary}}$$

In a more recent work [dSFF26], spectral inverse problems for magnetic Schrödinger operators and the associated magnetic Dirichlet-to-Neumann maps are studied. The latter problem, which may be viewed as a magnetic analogue of the classic Steklov

inverse problem, is the subject of the main results of [dSFF26]. As explained in this manuscript, the strategy described above can be applied to this framework but with an important additional issue, which is a natural gauge invariance on the magnetic part. This gauge invariance significantly complicates the implementation of these techniques, particularly at lower levels and the relevant quantities that can be recovered are therefore the gauge-invariant magnetic field $\mathrm{d}A$, together with the electric potential. We refer to Subsection IV.2.2 for details.

To conclude, an important observation from our work is that, in contrast to the Laplace-type operators which are just differential operators of order 2, DN maps are classical pseudodifferential operator of order 1 whose full symbol, considered in suitable local coordinates, contains the full boundary jets of the metric and the potentials.
Given the techniques used, the dynamical assumptions made at the boundary, are completely natural and likely the most appropriate. In addition, it is fairly easy to construct examples of such geometries (cf. **Remark I.5.1** and Figure I.3).
In the literature on the Steklov inverse problem, where positive results are still rare and counterexamples are abundant, this work provides, to our knowledge, one of the first rigidity results for such a broad geometric class.

Perspectives

Let us finally indicate some interesting directions opened by this work. Below is a list of possible research problems, ordered in our opinion from the most ambitious to the least ambitious. The first two have already been discussed in this manuscript.

- The Steklov inverse problem for connections. This is a logical continuation of the magnetic case study treated in this work.
  We refer to Subsection IV.2.3 for a brief presentation of the problem. This is a joint project with David dos Santos Ferreira (Université de Lorraine).
  At present, there are strong difficulties to overcome, particularly with regard to the generalization of the wave trace formula **Theorem II.4.2** to operators acting on sections of a vector bundle (of any rank).

- The Steklov inverse problem for general metrics (non-necessarily conformal). This is a natural sequel to [Flo25] and a project the author initiated recently with Sebastián Muñoz Thon (Université de Paris-Saclay). It seems very promising at least in the non-linear case of isospectral deformations (cf Subsection

IV.1.2). It appears that we can overcome the obstacles related to gauge invariance via a boundary solenoidal reduction argument.

- Stability estimates for the Steklov spectrum in the case of conformal metrics or for potentials. This is a natural problem that would complement the positive results obtained in this work. There is great hope that this will follow from a combination of the techniques used with the stability estimate for the X-ray transform proved by Guillarmou and Lefeuvre and notably [GL19, Theorem 5].

  The case where the metric is fixed, and only the potentials or the electric or magnetic connections vary, should serve as a first natural try. The wave trace residues should allow to control principal wave invariants, i.e. the integrals over the closed geodesics of the relevant symbols and then the aforementioned stability estimate of Guillarmou and Lefeuvre could be applied to the X-ray transform.

In addition, it would be of interest to investigate the topic of counterexamples to the inverse problems considered here. For example, as mentioned in subsection IV.2.1, it would be natural to combine Brooks's work [Bro87] with the extension of Sunada's method [Sun85] to the Steklov setting [GHW21], in order to construct Steklov isospectral electric or magnetic potentials, and maybe even Steklov isospectral connections.

*

* *

Bibliography

[AB59] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. *Phys. Rev., II. Ser.*, 115:485–491, 1959. doi:10.1103/PhysRev.115.485.

[Abr] Ralph Abraham. Bumpy metrics. Global Analysis, Proc. Sympos. Pure Math. 14, 1-3 (1970).

[Ana23] Nalini Anantharaman. *Histoires de spectres.* Number 314 in Leçons inaugurales du Collège de France. Collège de France, 2023.

[Ano83] Dmitriĭ Viktorovich Anosov. On generic properties of closed geodesics. *Math. USSR, Izv.*, 21:1–29, 1983. doi:10.1070/IM1983v021n01ABEH001637.

[Aub98] Thierry Aubin. *Some nonlinear problems in Riemannian geometry.* Springer Monogr. Math. Berlin: Springer, 1998.

[BCDSF+21] Mourad Bellassoued, Mourad Choulli, David Dos Santos Ferreira, Yavar Kian, and Plamen Stefanov. A Borg-Levinson theorem for magnetic Schrödinger operators on a Riemannian manifold. *Ann. Inst. Fourier*, 71(6):2471–2517, 2021. doi:10.5802/aif.3451.

[BCG91] Gérard Besson, Gilles Courtois, and Sylvestre Gallot. Volume et entropie minimale des espaces localement symétriques. (Minimal volume and entropy of locally symmetric spaces). *Invent. Math.*, 103(2):417–445, 1991. URL: https://eudml.org/doc/143864, doi:10.1007/BF01239520.

[Bér86] Pierre H. Bérard. *Spectral geometry: direct and inverse problems. With an appendix by G. Besson*, volume 1207 of *Lect. Notes Math.* Springer, Cham, 1986. doi:10.1007/BFb0076330.

[BK85] K. Burns and A. Katok. Manifolds with non-positive curvature. *Ergodic Theory Dyn. Syst.*, 5:307–317, 1985. doi:10.1017/S0143385700002935.

[BPY89] Robert Brooks, Peter Perry, and Paul Yang. Isospectral sets of conformally equivalent metrics. *Duke Math. J.*, 58(1):131–150, 1989. doi:10.1215/S0012-7094-89-05808-0.

[Bro87] Robert Brooks. On manifolds of negative curvature with isospectral potentials. *Topology*, 26:63–66, 1987. doi:10.1016/0040-9383(87)90021-8.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Grad. Texts Math.* Springer, Cham, 1982.

[Bus92] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Prog. Math.* Boston, MA: Birkhäuser, 1992.

[CDS00] Christopher B. Croke, Nurlan S. Dairbekov, and Vladimir A. Sharafutdinov. Local boundary rigidity of a compact Riemannian manifold with curvature bounded above. *Trans. Am. Math. Soc.*, 352(9):3937–3956, 2000. doi:10.1090/S0002-9947-00-02532-0.

[CdV73a] Yves Colin de Verdière. Spectre du laplacien et longueurs des géodésiques périodiques. I. *Compos. Math.*, 27:83–106, 1973. URL: https://eudml.org/doc/89181.

[CdV73b] Yves Colin de Verdière. Spectre du Laplacien et longueurs des géodésiques périodiques. II. *Compos. Math.*, 27:159–184, 1973. URL: https://eudml.org/doc/89186.

[CdV96] Yves Colin de Verdière. The spectrum of the Laplacian: Partial survey since Berger-Gauduchon-Mazet and problems. In *Actes de la table ronde de géométrie différentielle en l'honneur de Marcel Berger, Luminy, France, 12–18 juillet, 1992*, pages 233–252. Paris: Société Mathématique de France, 1996.

[CdV07] Yves Colin de Verdière. Spectrum of the Laplace operator and periodic geodesics: thirty years after. *Ann. Inst. Fourier*, 57(7):2429–2463, 2007. URL: https://eudml.org/doc/10303, doi:10.5802/aif.2339.

[Cek20] Mihajlo Cekić. Calderón problem for Yang-Mills connections. *J. Spectr. Theory*, 10(2):463–513, 2020. doi:10.4171/JST/302.

[CESG11] Bruno Colbois, Ahmad El Soufi, and Alexandre Girouard. Isoperimetric control of the Steklov spectrum. *J. Funct. Anal.*, 261(5):1384–1399, 2011. doi:10.1016/j.jfa.2011.05.006.

[CGGS24] Bruno Colbois, Alexandre Girouard, Carolyn Gordon, and David Sher. Some recent developments on the Steklov eigenvalue problem. *Rev. Mat. Complut.*, 37(1):1–161, 2024. doi:10.1007/s13163-023-00480-3.

[CGH20] Bruno Colbois, Alexandre Girouard, and Asma Hassannezhad. The Steklov and Laplacian spectra of Riemannian manifolds with boundary. *J. Funct. Anal.*, 278(6):38, 2020. Id/No 108409. doi:10.1016/j.jfa.2019.108409.

[CGHP24] Tirumala Chakradhar, Katie Gittins, Georges Habib, and Norbert Peyerimhoff. A note on the magnetic Steklov operator on functions. Preprint, arXiv:2410.07462 [math.DG] (2024), 2024. URL: https://arxiv.org/abs/2410.07462.

[Cha74a] J. Chazarain. Flot hamiltonien et spectre d'un opérateur elliptique. Sémin. Goulaouic-Schwartz 1973-1974, Équat. dériv. part. Analyse fonct., Exposé XVI, 12 p. (1974)., 1974. URL: https://eudml.org/doc/111608.

[Cha74b] Jacques Chazarain. Formule de Poisson pour les variétés riemanniennes. *Invent. Math.*, 24:65–82, 1974. URL: https://eudml.org/doc/142270, doi:10.1007/BF01418788.

[Cha84] Isaac Chavel. *Eigenvalues in Riemannian geometry. With a chapter by Burton Randol. With an appendix by Jozef Dodziuk*, volume 115 of *Pure Appl. Math., Academic Press.* Academic Press, New York, NY, 1984.

[CIPP00] Gonzalo Contreras, Renato Iturriaga, Gabriel Pedro Paternain, and Miguel Paternain. The Palais-Smale condition and Mañé's critical values. *Ann. Henri Poincaré*, 1(4):655–684, 2000. doi:10.1007/PL00001011.

[CL25a] Mihajlo Cekić and Thibault Lefeuvre. Generic injectivity of the X-ray transform. *J. Differ. Geom.*, 129(2):261–334, 2025. doi:10.4310/jdg/1738160846.

[CL25b] Mihajlo Cekić and Thibault Lefeuvre. The holonomy inverse problem. *J. Eur. Math. Soc. (JEMS)*, 27(6):2187–2250, 2025. doi:10.4171/JEMS/1409.

[CL25c] Mihajlo Cekić and Thibault Lefeuvre. Isospectral connections, ergodicity of frame flows, and polynomial maps between spheres. *Ann. Sci. Éc. Norm. Supér. (4)*, 58(1):203–229, 2025. doi:10.24033/asens.2604.

[Cro90] Christopher B. Croke. Rigidity for surfaces of non-positive curvature. *Comment. Math. Helv.*, 65(1):150–169, 1990. URL: https://eudml.org/doc/140188, doi:10.1007/BF02566599.

[CS98] Christopher B. Croke and Vladimir A. Sharafutdinov. Spectral rigidity of a compact negatively curved manifold. *Topology*, 37(6):1265–1273, 1998. doi:10.1016/S0040-9383(97)00086-4.

[CS25] Mihajlo Cekić and Anna Siffert. Magnetic Steklov problem on surfaces. *J. Funct. Anal.*, 289(12):47, 2025. Id/No 111159. doi:10.1016/j.jfa.2025.111159.

[Dav95] E. Brian Davies. *Spectral Theory and Differential Operators*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[DG75] Johannes Jisse Duistermaat and Victor W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29:39–79, 1975. doi:10.1007/BF01405172.

[DGM+25] Emily B. Dryden, Carolyn Gordon, Javier Moreno, Julie Rowlett, and Carlos Villegas-Blas. The Steklov spectrum of convex polygonal domains. I: Spectral finiteness. *J. Geom. Anal.*, 35(3):38, 2025. Id/No 91. doi:10.1007/s12220-025-01922-8.

[DH72] J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. *Acta Math.*, 128:183–269, 1972. doi:10.1007/BF02392165.

[DH13] Kiril Datchev and Hamid Hezari. Inverse problems in spectral geometry. In *Inverse problems and applications. Inside out II*, pages 455–485. Cambridge: Cambridge University Press, 2013.

[DKN18] Thierry Daudé, Niky Kamran, and Francois Nicoleau. A survey of non-uniqueness results for the anisotropic Calderón problem with disjoint data. In *Nonlinear analysis in geometry and applied mathematics. Part 2. Based on lectures presented during the program year 2015–2016 on "Nonlinear Equations" at the Harvard Center of Mathematical Sciences and Applications, Cambridge, MA, USA*, pages 77–101. Somerville, MA: International Press, 2018.

[DKN21] Thierry Daudé, Niky Kamran, and François Nicoleau. Stability in the inverse Steklov problem on warped product Riemannian manifolds. *J. Geom. Anal.*, 31(2):1821–1854, 2021. doi:10.1007/s12220-019-00326-9.

[DlLMM86] Rafael De la Llave, José Manuel Marco, and Roberto Moriyón. Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation. *Ann. Math. (2)*, 123:537–611, 1986. doi:10.2307/1971334.

[DP08] Nurlan S. Dairbekov and Gabriel P. Paternain. Rigidity properties of Anosov optical hypersurfaces. *Ergodic Theory Dyn. Syst.*, 28(3):707–737, 2008. doi:10.1017/S0143385707000612.

[DRP18] Matteo Dalla Riva and Luigi Provenzano. On vibrating thin membranes with mass concentrated near the boundary: an asymptotic analysis. *SIAM J. Math. Anal.*, 50(3):2928–2967, 2018. doi:10.1137/17M1118221.

[DS03] Nurlan S. Dairbekov and Vladimir A. Sharafutdinov. Some problems of integral geometry on Anosov manifolds. *Ergodic Theory Dyn. Syst.*, 23(1):59–74, 2003. doi:10.1017/S0143385702000822.

[dSFF26] David dos Santos Ferreira and Benjamin Florentin. Magnetic spectral inverse problems on compact Anosov manifolds. Preprint, arXiv:2601.12058 [math.SP] (2026), 2026. URL: https://arxiv.org/abs/2601.12058.

[DSFKSU09] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann. Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.*, 178(1):119–171, 2009. doi:10.1007/s00222-009-0196-4.

[Dui73] J. J. Duistermaat. Fourier integral operators. New York: Courant Institute of Mathematical Science, New York University. III, 190 p. (1973)., 1973.

[Dui76] Johannes Jisse Duistermaat. On the Morse index in variational calculus. *Adv. Math.*, 21:173–195, 1976. doi:10.1016/0001-8708(76)90074-8.

[Ebi68] D. G. Ebin. On the space of Riemannian metrics. *Bull. Am. Math. Soc.*, 74:1001–1003, 1968. doi:10.1090/S0002-9904-1968-12115-9.

[Ebi70] D. G. Ebin. The manifold of Riemannian metrics. Global Analysis, Proc. Sympos. Pure Math. 15, 11-40 (1970)., 1970.

[Edw93] Julian Edward. An inverse spectral result for the Neumann operator on planar domains. *J. Funct. Anal.*, 111(2):312–322, 1993. doi:10.1006/jfan.1993.1015.

[ER13] G. Eskin and J. Ralston. Gauge equivalence and the inverse spectral problem for the magnetic Schrödinger operator on the torus. *Russ. J. Math. Phys.*, 20(4):413–423, 2013. doi:10.1134/S1061920813040043.

[Esk15] Gregory Eskin. Aharonov-bohm effect revisited. *Rev. Math. Phys.*, 27(2):52, 2015. Id/No 1530001. doi:10.1142/S0129055X15300010.

[FH19] Todd Fisher and Boris Hasselblatt. *Hyperbolic flows*. Zur. Lect. Adv. Math. Berlin: European Mathematical Society (EMS), 2019. doi:10.4171/200.

[FK83] D. W. Fox and J. R. Kuttler. Sloshing frequencies. *Z. Angew. Math. Phys.*, 34:668–696, 1983. doi:10.1007/BF00948809.

[FK05] S. A. Fulling and Peter Kuchment. Coincidence of length spectra does not imply isospectrality. *Inverse Probl.*, 21(4):1391–1395, 2005. doi:10.1088/0266-5611/21/4/013.

[Flo25] Benjamin Florentin. Steklov isospectrality of conformal metrics. Preprint, arXiv:2501.15535 [math.SP] (2025), 2025. URL: https://arxiv.org/abs/2501.15535.

[Gen20] Germain Gendron. Uniqueness results in the inverse spectral Steklov problem. *Inverse Probl. Imaging*, 14(4):631–664, 2020. doi:10.3934/ipi.2020029.

[GG98] Peter B. Gilkey and Gerd Grubb. Logarithmic terms in asymptotic expansions of heat operator traces. *Commun. Partial Differ. Equations*, 23(5-6):777–792, 1998. doi:10.1080/03605309808821365.

[GG07] Colin Guillarmou and Laurent Guillopé. The determinant of the Dirichlet-to-Neumann map for surfaces with boundary. *Int. Math. Res. Not.*, 2007(22):26, 2007. Id/No rnm099. doi:10.1093/imrn/rnm099.

[GHW21] Carolyn Gordon, Peter Herbrich, and David Webb. Steklov and Robin isospectral manifolds. *J. Spectr. Theory*, 11(1):39–61, 2021. doi:10.4171/JST/335.

[Ghy84] Etienne Ghys. Flots d'Anosov sur les 3-variétés fibrées en cercle. *Ergodic Theory Dyn. Syst.*, 4:67–80, 1984. doi:10.1017/S0143385700002273.

[Gil95] Peter B. Gilkey. *Invariance theory, the heat equation and the Atiyah-Singer index theorem.* Boca Raton, FL: CRC Press, 2nd ed. edition, 1995.

[Gil04] Peter B. Gilkey. *Asymptotic formulae in spectral geometry.* Boca Raton, FL: Chapman & Hall/CRC, 2004.

[GK80a] Victor W. Guillemin and David Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology*, 19:301–312, 1980. doi:10.1016/0040-9383(80)90015-4.

[GK80b] Victor W. Guillemin and David Kazhdan. Some inverse spectral results for negatively curved n-manifolds. Geometry of the Laplace operator, Honolulu/Hawaii 1979, Proc. Symp. Pure Math., Vol. 36, 153-180 (1980)., 1980.

[GKLP22] Alexandre Girouard, Mikhail Karpukhin, Michael Levitin, and Iosif Polterovich. The Dirichlet-to-Neumann map, the boundary Laplacian, and Hörmander's rediscovered manuscript. *J. Spectr. Theory*, 12(1):195–225, 2022. doi:10.4171/JST/399.

[GL19] Colin Guillarmou and Thibault Lefeuvre. The marked length spectrum of Anosov manifolds. *Ann. Math. (2)*, 190(1):321–344, 2019. doi:10.4007/annals.2019.190.1.6.

[GM] Colin Guillarmou and Marco Mazzucchelli. An introduction to geometric inverse problems. Book to appear in Graduate Studies in Math, AMS. 600p, to appear.

[GM79] Victor Guillemin and Richard Melrose. The Poisson summation formula for manifolds with boundary. *Adv. Math.*, 32:204–232, 1979. doi:10.1016/0001-8708(79)90042-2.

[Gor00] Carolyn S. Gordon. Survey of isospectral manifolds. In *Handbook of differential geometry. Vol. I*, pages 747–778. Amsterdam: North-Holland, 2000.

[Gor09] Carolyn Gordon. Sunada's isospectrality technique: two decades later. In *Spectral analysis in geometry and number theory. International conference on the occasion of Toshikazu Sunada's 60th birthday, August 6–10, 2007*, pages 45–58. Providence, RI: American Mathematical Society (AMS), 2009.

[GP17] Alexandre Girouard and Iosif Polterovich. Spectral geometry of the Steklov problem (survey article). *J. Spectr. Theory*, 7(2):321–359, 2017. doi:10.4171/JST/164.

[GPPS14] Alexandre Girouard, Leonid Parnovski, Iosif Polterovich, and David A. Sher. The Steklov spectrum of surfaces: asymptotics and invariants. *Math. Proc. Camb. Philos. Soc.*, 157(3):379–389, 2014. doi:10.1017/S030500411400036X.

[GPSU16a] Colin Guillarmou, Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The X-ray transform for connections in negative curvature. *Commun. Math. Phys.*, 343(1):83–127, 2016. doi:10.1007/s00220-015-2510-x.

[GPSU16b] Colin Guillarmou, Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The X-ray transform for connections in negative curvature. *Commun. Math. Phys.*, 343(1):83–127, 2016. doi:10.1007/s00220-015-2510-x.

[GS94] Alain Grigis and Johannes Sjöstrand. *Microlocal analysis for differential operators. An introduction*, volume 196 of *Lond. Math. Soc. Lect. Note Ser.* Cambridge: Cambridge University Press, 1994.

[Gui74] Victor W. Guillemin. Fourier integral operators for systems. Unpublished report, 1974. URL: https://msp.org/extras/Guillemin/V.Guillemin-Fourier_Integral_Operators-1974.pdf.

[Gui78a] V. Guillemin. An Addendum to: Some spectral results on rank one symmetric spaces. *Adv. Math.*, 28:138–147, 1978. doi:10.1016/0001-8708(78)90060-9.

[Gui78b] Victor W. Guillemin. Some spectral results on rank one symmetric spaces. *Adv. Math.*, 28:129–137, 1978. doi:10.1016/0001-8708(78)90059-2.

[Gui90] V. Guillemin. Inverse spectral results on two-dimensional tori. *J. Am. Math. Soc.*, 3(2):375–387, 1990. doi:10.2307/1990958.

[Gui96] Victor Guillemin. Wave-trace invariants. *Duke Math. J.*, 83(2):287–352, 1996. doi:10.1215/S0012-7094-96-08311-8.

[Gui17] Colin Guillarmou. Invariant distributions and X-ray transform for Anosov flows. *J. Differ. Geom.*, 105(2):177–208, 2017. doi:10.4310/jdg/1486522813.

[Gui21] Colin Guillarmou. Spectral rigidities: a brief state of the art. *Gaz. Math., Soc. Math. Fr.*, 168:24–33, 2021.

[GWW92] C. Gordon, D. Webb, and S. Wolpert. Isospectral plane domains and surfaces via Riemannian orbifolds. *Invent. Math.*, 110(1):1–22, 1992. URL: https://eudml.org/doc/144038, doi:10.1007/BF01231320.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[Hej76a] Dennis A. Hejhal. The Selberg trace formula and the Riemann zeta-function. *Duke Math. J.*, 43:441–482, 1976. doi:10.1215/S0012-7094-76-04338-6.

[Hej76b] Dennis A. Hejhal. *The Selberg trace formula for* PSL(2, ℝ). *Vol. I*, volume 548 of *Lect. Notes Math.* Springer, Cham, 1976. doi:10.1007/BFb0079608.

[Hel88] B. Helffer. Effet d'Aharonov-Bohm sur un état borné de l'équation de Schrödinger. (Aharonov-Bohm effect for a bounded state of the Schrödinger equation). *Commun. Math. Phys.*, 119(2):315–329, 1988. doi:10.1007/BF01217743.

[Hen06] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Front. Math. Basel: Birkhäuser, 2006.

[HN25] Bernard Helffer and François Nicoleau. Trace formulas for the magnetic Laplacian and Dirichlet to Neumann operator – explicit expansions. *Asymptotic Anal.*, 141(2):95–118, 2025. doi:10.1177/09217134241308420.

[Hop48] Eberhard Hopf. Closed surfaces without conjugate points. *Proc. Natl. Acad. Sci. USA*, 34:47–51, 1948. URL: europepmc.org/articles/pmc1062913, doi:10.1073/pnas.34.2.47.

[Hör68] Lars Hörmander. The spectral function of an elliptic operator. *Acta Math.*, 121:193–218, 1968. doi:10.1007/BF02391913.

[Hör71] L. Hörmander. Fourier integral operators. I. *Acta Math.*, 127:79–183, 1971. doi:10.1007/BF02392052.

[Hör85a] Lars Hörmander. *The analysis of linear partial differential operators. III: Pseudo-differential operators*, volume 274 of *Grundlehren Math. Wiss.* Springer-Verlag, 1985. doi:10.1007/978-3-540-49938-1.

[Hör85b] Lars Hörmander. *The analysis of linear partial differential operators. IV: Fourier integral operators*, volume 275 of *Grundlehren Math. Wiss.* Springer, Cham, 1985. doi:10.1007/978-3-642-00136-9.

[Hör90] Lars Hörmander. *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.*, volume 256 of *Grundlehren Math. Wiss.* Berlin etc.: Springer-Verlag, 2nd ed. edition, 1990.

[Hub59] Heinz Huber. Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. *Math. Ann.*, 138:1–26, 1959. URL: https://eudml.org/doc/160698, doi:10.1007/BF01369663.

[Hum26] Tristan Humbert. Katok's entropy conjecture near real and complex hyperbolic metrics. *Duke Math. J.*, 175(1):135–184, 2026. doi:10.1215/00127094-2025-0023.

[HZ10] Hamid Hezari and Steve Zelditch. Inverse spectral problem for analytic $(\mathbb{Z}/2\mathbb{Z})^n$-symmetric domains in $\mathbb{R}^n$. *Geom. Funct. Anal.*, 20(1):160–191, 2010. doi:10.1007/s00039-010-0059-6.

[HZ12] Hamid Hezari and Steve Zelditch. C^∞ spectral rigidity of the ellipse. *Anal. PDE*, 5(5):1105–1132, 2012. URL: msp.org/apde/2012/5-5/p06.xhtml, doi:10.2140/apde.2012.5.1105.

[HZ22] Hamid Hezari and Steve Zelditch. One can hear the shape of ellipses of small eccentricity. *Ann. Math. (2)*, 196(3):1083–1134, 2022. doi:10.4007/annals.2022.196.3.4.

[Ivr98] Victor Ivrii. *Microlocal analysis and precise spectral asymptotics*. Springer Monogr. Math. Berlin: Springer, 1998.

[Ivr16] Victor Ivrii. 100 years of Weyl's law. *Bull. Math. Sci.*, 6(3):379–452, 2016. doi:10.1007/s13373-016-0089-y.

[JS14] Alexandre Jollivet and Vladimir A. Sharafutdinov. On an inverse problem for the Steklov spectrum of a Riemannian surface. In *Inverse problems and applications. Proceedings of two conferences in honour of Gunther Uhlmann's 60th birthday, Irvine, CA, USA, June 18–22, 2012 and Hangzhou, China, September 17–21, 2012*, pages 165–191. Providence, RI: American Mathematical Society (AMS), 2014.

[JS18] Alexandre Jollivet and Vladimir Sharafutdinov. Steklov zeta-invariants and a compactness theorem for isospectral families of planar domains. *J. Funct. Anal.*, 275(7):1712–1755, 2018. doi:10.1016/j.jfa.2018.06.019.

[JS22] Alexandre Jollivet and Vladimir Sharafutdinov. An estimate for the Steklov zeta function of a planar domain derived from a first variation formula. *J. Geom. Anal.*, 32(5):53, 2022. Id/No 161. doi:10.1007/s12220-022-00890-7.

[Kac66] Mark Kac. Can one hear the shape of a drum? *Am. Math. Mon.*, 73:1–23, 1966. doi:10.2307/2313748.

[KH97] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems. With a supplement by Anatole Katok and Leonardo Mendoza*, volume 54 of *Encycl. Math. Appl.* Cambridge: Cambridge University Press, 1997.

[KKK+14] Nikolay Kuznetsov, Tadeusz Kulczycki, Mateusz Kwaśnicki, Alexander Nazarov, Sergey Poborchi, Iosif Polterovich, and Bartłomiej Siudeja. The legacy of Vladimir Andreevich Steklov. *Notices Am. Math. Soc.*, 61(1):9–22, 2014. doi:10.1090/noti1073.

[Kli74] Wilhelm Klingenberg. Riemannian manifolds with geodesic flow of Anosov type. *Ann. Math. (2)*, 99:1–13, 1974. doi:10.2307/1971011.

[Kli82] Wilhelm Klingenberg. Lectures on closed geodesics. (Lektsii o zamknutykh geodezicheskikh). Transl. from the Engl. Moskva: Izdatel'stvo "Mir". 414 p. R. 3.40 (1982)., 1982.

[KMV02] N. Kuznetsov, V. Maz'ya, and B. Vainberg. *Linear water waves. A mathematical approach.* Cambridge: Cambridge University Press, 2002. doi:10.1017/CBO9780511546778.

[Kni02] Gerhard Knieper. Hyperbolic dynamics and Riemannian geometry. In *Handbook of dynamical systems. Volume 1A*, pages 453–545. Amsterdam: North-Holland, 2002.

[Kuw90] Ruishi Kuwabara. Isospectral connections on line bundles. *Math. Z.*, 204(4):465–473, 1990. URL: https://eudml.org/doc/183783, doi:10.1007/BF02570886.

[KV94] Maxim Kontsevich and Simeon Vishik. Determinants of elliptic pseudo-differential operators. Preprint, arXiv:hep-th/9404046 [hep-th] (1994), 1994. URL: https://arxiv.org/abs/hep-th/9404046.

[Lan93] Serge Lang. *Real and functional analysis.*, volume 142 of *Grad. Texts Math.* New York: Springer-Verlag, 3. ed. edition, 1993.

[Lef25] Thibault Lefeuvre. *Microlocal analysis in hyperbolic dynamics and geometry. With a contributed chapter by Yann Chaubet*, volume 32 of *Cours Spéc. (Paris).* Paris: Société Mathématique de France (SMF), 2025.

[Liv71] Alexander Nakhimovich Livshits. Homology properties of Y-systems. *Mathematical Notes of the Academy of Sciences of the USSR*, 10:758–763, 1971.

[Liv74] Alexander Nakhimovich Livshits. Cohomology of dynamical systems. *Math. USSR, Izv.*, 6:1278–1301, 1974. doi:10.1070/IM1972v006n06ABEH001919.

[LLS22] Matti Lassas, Tony Liimatainen, and Mikko Salo. The Calderón problem for the conformal Laplacian. *Commun. Anal. Geom.*, 30(5):1121–1184, 2022. doi:10.4310/CAG.2022.v30.n5.a6.

[LMP24] Michael Levitin, Dan Mangoubi, and Iosif Polterovich. *Topics in spectral geometry*, volume 237 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2024. doi:10.1090/gsm/237.

[LSA21] Jean Lagacé and Simon St-Amant. Spectral invariants of Dirichlet-to-Neumann operators on surfaces. *J. Spectr. Theory*, 11(4):1627–1667, 2021. doi:10.4171/JST/382.

[LT23] Genqian Liu and Xiaoming Tan. Spectral invariants of the magnetic Dirichlet-to-Neumann map on Riemannian manifolds. *J. Math. Phys.*, 64(4):47, 2023. Id/No 041501. doi:10.1063/5.0088549.

[LTU03] Matti Lassas, Michael Taylor, and Gunther Uhlmann. The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary. *Commun. Anal. Geom.*, 11(2):207–221, 2003. doi:10.4310/CAG.2003.v11.n2.a2.

[LU89] John M. Lee and Gunther Uhlmann. Determining anisotropic real-analytic conductivities by boundary measurements. *Commun. Pure Appl. Math.*, 42(8):1097–1112, 1989. doi:10.1002/cpa.3160420804.

[LU01] Matti Lassas and Gunther Uhlmann. On determining a Riemannian manifold from the Dirichlet-to-Neumann map. *Ann. Sci. Éc. Norm. Supér. (4)*, 34(5):771–787, 2001. URL: https://eudml.org/doc/82557, doi:10.1016/S0012-9593(01)01076-X.

[Mar12] Jens Marklof. Selberg's trace formula: an introduction. In *Hyperbolic geometry and applications in quantum chaos and cosmology*, pages 83–119. Cambridge: Cambridge University Press, 2012.

[Mil64] John W. Milnor. Eigenvalues of the Laplace operator on certain manifolds. *Proc. Natl. Acad. Sci. USA*, 51:542, 1964. doi:10.1073/pnas.51.4.542.

[MM82] Shahla Marvizi and Richard Melrose. Spectral invariants of convex planar regions. *J. Differ. Geom.*, 17:475–502, 1982. doi:10.4310/jdg/1214437138.

[MP49] S. Minakshisundaram and Åke Pleijel. Some properties of the Eigenfunctions of the Laplace-operator on Riemannian manifolds. *Can. J. Math.*, 1:242–256, 1949. doi:10.4153/CJM-1949-021-5.

[MR23] James Marshall Reber. Deformative magnetic marked length spectrum rigidity. *Bull. Lond. Math. Soc.*, 55(6):3077–3096, 2023. doi:10.1112/blms.12911.

[MS67] H. P. jun. McKean and I. M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differ. Geom.*, 1:43–69, 1967. doi:10.4310/jdg/1214427880.

[MS15] E. G. Mal'kovich and V. A. Sharafutdinov. Zeta-invariants of the Steklov spectrum of a planar domain. *Sib. Math. J.*, 56(4):678–698, 2015. doi:10.1134/S0037446615040114.

[NSU95] Gen Nakamura, Ziqi Sun, and Gunther Uhlmann. Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field. *Math. Ann.*, 303(3):377–388, 1995. URL: https://eudml.org/doc/165373, doi:10.1007/BF01460996.

[NT98] Viorel Niţică and Andrei Török. Regularity of the transfer map for cohomologous cocycles. *Ergodic Theory Dyn. Syst.*, 18(5):1187–1209, 1998. doi:10.1017/S0143385798117480.

[OPS88a] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral of plane domains. *Proc. Natl. Acad. Sci. USA*, 85(15):5359–5361, 1988. doi:10.1073/pnas.85.15.5359.

[OPS88b] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral sets of surfaces. *J. Funct. Anal.*, 80(1):212–234, 1988. doi:10.1016/0022-1236(88)90071-7.

[Ota90] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. (The spectrum marked by lengths of surfaces with negative curvature). *Ann. Math. (2)*, 131(1):151–162, 1990. doi:10.2307/1971511.

[Pat09] Gabriel Pedro Paternain. Transparent connections over negatively curved surfaces. *J. Mod. Dyn.*, 3(2):311–333, 2009. doi:10.3934/jmd.2009.3.311.

[Pat13] Gabriel P. Paternain. Inverse problems for connections. In *Inverse problems and applications. Inside out II*, pages 369–409. Cambridge: Cambridge University Press, 2013.

[PP94] Gabriel P. Paternain and Miguel Paternain. On Anosov energy levels of convex Hamiltonian systems. *Math. Z.*, 217(3):367–376, 1994. URL: https://eudml.org/doc/174695, doi:10.1007/BF02571949.

[PS15] Iosif Polterovich and David A. Sher. Heat invariants of the Steklov problem. *J. Geom. Anal.*, 25(2):924–950, 2015. doi:10.1007/s12220-013-9451-4.

[PS23] Gabriel Pedro Paternain and Mikko Salo. Carleman estimates for geodesic X-ray transforms. *Ann. Sci. Éc. Norm. Supér. (4)*, 56(5):1339–1379, 2023. doi:10.24033/asens.2557.

[PS25] Luigi Provenzano and Alessandro Savo. Geometry of the magnetic Steklov problem on Riemannian annuli. *Commun. Contemp. Math.*, 27(7):49, 2025. Id/No 2550001. doi:10.1142/S0219199725500014.

[PSU12] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The attenuated ray transform for connections and Higgs fields. *Geom. Funct. Anal.*, 22(5):1460–1489, 2012. doi:10.1007/s00039-012-0183-6.

[PSU14] Gabriel Pedro Paternain, Mikko Salo, and Gunther Uhlmann. Spectral rigidity and invariant distributions on Anosov surfaces. *J. Differ. Geom.*, 98(1):147–181, 2014. doi:10.4310/jdg/1406137697.

[PSU15] Gabriel Pedro Paternain, Mikko Salo, and Gunther Uhlmann. Invariant distributions, Beurling transforms and tensor tomography in higher dimensions. *Math. Ann.*, 363(1-2):305–362, 2015. doi:10.1007/s00208-015-1169-0.

[PSU23] Gabriel Pedro Paternain, Mikko Salo, and Gunther Uhlmann. *Geometric inverse problems. With emphasis on two dimensions*, volume 204 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2023. doi:10.1017/9781009039901.

[Rau78] Jeffrey Rauch. Illumination of bounded domains. *Am. Math. Mon.*, 85:359–361, 1978. doi:10.2307/2321342.

[RS71] D. B. Ray and I. M. Singer. R-torsion and the Laplacian on Riemannian manifolds. *Adv. Math.*, 7:145–210, 1971. doi:10.1016/0001-8708(71)90045-4.

[RS72] Michael Reed and Barry Simon. Methods of modern mathematical physics. 1: Functional analysis. New York-London: Academic Press, Inc. xvii, 325 p. $ 12.50 (1972)., 1972.

[Sal06] Mikko Salo. Inverse boundary value problems for the magnetic Schroedinger equation. Preprint, arXiv:math/0611458 [math.AP] (2006), 2006. URL: https://arxiv.org/abs/math/0611458.

[See67] R. T. Seeley. Complex powers of an elliptic operator. Proc. Sympos. Pure Math. 10, 288-307 (1967)., 1967.

[Sel56] Atle Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc., New Ser.*, 20:47–87, 1956.

[Ser10] V. S. Serov. Borg-levinson theorem for magnetic Schrödinger operator. *Bull. Greek Math. Soc.*, 57:321–332, 2010.

[Sha94] Vladimir A. Sharafutdinov. *Integral geometry for tensor fields. Transl. from the Russian.* Inverse Ill-Posed Probl. Ser. Utrecht: VSP, 1994.

[Shi87] Ichiro Shigekawa. Eigenvalue problems for the Schrödinger operator with the magnetic field on a compact Riemann manifold. *J. Funct. Anal.*, 75:92–127, 1987. doi:10.1016/0022-1236(87)90108-X.

[Shu01] M. A. Shubin. *Pseudodifferential operators and spectral theory. Transl. from the Russian by Stig I. Andersson.* Berlin: Springer, 2nd ed. edition, 2001.

[Sun85] Toshikazu Sunada. Riemannian coverings and isospectral manifolds. *Ann. Math. (2)*, 121:169–186, 1985. doi:10.2307/1971195.

[Tay23] Michael E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Appl. Math. Sci.* Cham: Springer, 3rd corrected and expanded edition edition, 2023. doi:10.1007/978-3-031-33700-0.

[Vas03] D. V. Vassilevich. Heat kernel expansion: user's manual. *Phys. Rep.*, 388(5-6):279–360, 2003. doi:10.1016/j.physrep.2003.09.002.

[Vig80] Marie-France Vignéras. Isospectral and non-isometric Riemannian manifolds. *Ann. Math. (2)*, 112:21–32, 1980. doi:10.2307/1971319.

[Zel97] S. Zelditch. Wave invariants at elliptic closed geodesics. *Geom. Funct. Anal.*, 7(1):145–213, 1997. doi:10.1007/PL00001615.

[Zel98] S. Zelditch. Wave invariants for non-degenerate closed geodesics. *Geom. Funct. Anal.*, 8(1):179–217, 1998. doi:10.1007/s000390050052.

[Zel00] S. Zelditch. Spectral determination of analytic bi-axisymmetric plane domains. *Geom. Funct. Anal.*, 10(3):628–677, 2000. doi:10.1007/PL00001633.

[Zel04] Steve Zelditch. The inverse spectral problem. In *Surveys in differential geometry. Vol. IX: Eigenvalues of Laplacians and other geometric operators*, pages 401–467. Somerville, MA: International Press, 2004.

[Zel15] Steve Zelditch. Survey of the inverse spectral problem for the Notices of the ICCM. Preprint, arXiv:1504.02000 [math.SP] (2015), 2015. URL: https://arxiv.org/abs/1504.02000.

[Zel17] Steve Zelditch. *Eigenfunctions of the Laplacian on a Riemannian manifold*, volume 125 of *CBMS Reg. Conf. Ser. Math.* Providence, RI: American Mathematical Society (AMS), published for the Conference Board of the Mathematical Sciences (CBMS), 2017. doi:10.1090/cbms/125.

[Zwo12] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2012.