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The Weil representation for a finite field of characteristic two

La représentation de Weil pour un corps fini de caractéristique deux

THÈSE

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par

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Résumé

L'objectif principal de cette thèse est de fournir trois constructions explicites de la représentation de Weil pour un corps fini de caractéristique deux, et de présenter une ouverture vers la théorie de l'information quantique. Ce manuscrit est organisé en quatre chapitres. Nous esquissons ici un aperçu du contenu de chacun d'entre eux et nous renvoyons à l'introduction pour un résumé détaillé.

Dans le prolongement des travaux de Weil, nous donnons dans le premier chapitre une construction d'une représentation de Weil unitaire, fidèle et irréductible d'un revêtement à deux feuillets du groupe pseudosymplectique associé à un espace vectoriel symplectique défini sur un corps fini de caractéristique deux arbitraire. Cette représentation est le relèvement d'une représentation projective du groupe pseudosymplectique, provenant d'opérateurs d'entrelacement entre des représentations irréductibles spécifiques d'un groupe de Heisenberg. Nous exposons des formules explicites pour les opérateurs issus de cette représentation ainsi que le cocycle et le caractère associés.

Le groupe pseudosymplectique admettant une projection sur le groupe orthogonal, et ce dernier étant en général strictement contenu dans le groupe symplectique, nous analysons dans le deuxième chapitre la manière de définir une représentation de Weil associée au groupe symplectique tout entier pour le corps à deux éléments. Cette construction nous invite à élargir le cadre des espaces vectoriels considérés précédemment à celui des modules sur l'anneau $\mathbb{Z}/4\mathbb{Z}$. D'une part, suivant une démarche analogue à celle du premier chapitre, nous construisons une représentation de Weil unitaire, fidèle et irréductible d'un revêtement du groupe affine symplectique, en donnant des formules explicites pour les opérateurs, le cocycle et le caractère associés. Nous étudions différentes possibilités pour la taille du revêtement à choisir (une extension par deux, quatre ou huit éléments) selon la dimension de l'espace considéré et les liens que nous pouvons établir avec la construction du premier chapitre. D'autre part, nous considérons un groupe de Heisenberg permettant d'aboutir de manière indépendante à la définition d'une représentation de Weil unitaire et irréductible d'un recouvrement à deux feuillets du groupe symplectique associé à $\mathbb{Z}/4\mathbb{Z}$, offrant ainsi une structure plus étroitement liée au groupe symplectique lui-même. Nous examinons enfin les connexions entre cette dernière construction et les précédentes.

Le troisième chapitre est consacré à l'exposition explicite des résultats obtenus précédemment dans le cas d'un espace vectoriel de dimension deux sur le corps à deux éléments. Ainsi, nous présentons tous les groupes mis en jeu et les opérateurs des représentations de Weil associés à ceux-ci. En particulier, nous identifions chacun de ces groupes grâce à une présentation par générateurs et relations. Nous obtenons également des résultats supplémentaires valides uniquement dans cette dimension.

Enfin, le quatrième chapitre fournit un dictionnaire entre les opérateurs des représentations de Weil que nous avons construites et des portes quantiques de Clifford. Plus précisément, nous identifions en premier lieu des sous-groupes explicites du groupe de Pauli à des sous-groupes des revêtements construits dans les deux premiers chapitres. Ceci nous conduit à exprimer les générateurs de stabilisateurs au sein de codes correcteurs d'erreurs quantiques classiques en termes d'opérateurs de représentations de Weil. Ensuite, nous identifions des éléments des revêtements mentionnés ci-dessus correspondant à la transformation

de Hadamard(-Rademacher-Walsh). Pour finir, nous dressons une liste de sous-groupes finis de portes de Clifford à un qubit mis en évidence par la description explicite des représentations de Weil du troisième chapitre.

Mots-clés: représentation de Weil, caractéristique deux, groupe symplectique, codes correcteurs d'erreurs quantiques.

Abstract

The main objective of this thesis is to provide three explicit constructions of the Weil representation for a finite field of characteristic two, and to present an opening towards Quantum Information Theory. This manuscript is organized into four chapters. Here we outline the content of each chapter, and we refer to the introduction for a detailed summary.

Following Weil's work, we provide in the first chapter a construction of a unitary, faithful and irreducible Weil representation of a two-fold covering of the pseudosymplectic group associated with a symplectic vector space defined over an arbitrary finite field of characteristic two. This representation is the lift of a projective representation of the pseudosymplectic group, arising from intertwining operators between specific irreducible representations of a Heisenberg group. We present explicit formulas for the operators arising from this representation, as well as the associated cocycle and character.

Since the pseudosymplectic group projects onto the orthogonal group, and the latter is generally strictly contained in the symplectic group, in the second chapter we analyze how to define a Weil representation associated with the entire symplectic group for the field of two elements. This construction invites us to extend the framework of vector spaces considered previously to that of modules over the ring $\mathbb{Z}/4\mathbb{Z}$. On one hand, following an approach similar to that of the first chapter, we construct a unitary, faithful and irreducible Weil representation of a covering of the affine symplectic group, giving explicit formulas for the associated operators, cocycle and character. We study different possibilities for the size of the covering to be chosen (an extension by two, four, or eight elements) depending on the dimension of the space considered and the links we can establish with the construction of the first chapter. On the other hand, we consider a Heisenberg group that allows us to independently lead to the definition of a unitary and irreducible Weil representation of a two-fold covering of the symplectic group associated with $\mathbb{Z}/4\mathbb{Z}$, thus offering a structure more closely related to the symplectic group itself. Finally, we examine the connections between this last construction and the previous ones.

The third chapter is devoted to an explicit exposition of the results previously obtained in the case of a two-dimensional vector space over the field of two elements. Thus, we present all the groups involved and the operators of the Weil representations associated with them. In particular, we identify each of these groups using a presentation by generators and relations. We also obtain additional results that are valid only in this dimension.

Finally, the fourth chapter provides a dictionary between the operators of the Weil representations we have constructed and Clifford quantum gates. More specifically, we first identify explicit subgroups of the Pauli group with subgroups of the coverings constructed in the first two chapters. This leads us to express the stabilizer generators within classical quantum error-correcting codes in terms of operators of the Weil representations. Then, we identify elements of the aforementioned coverings corresponding to the Hadamard(-Rademacher-Walsh) transform. Lastly, we list finite subgroups of one-qubit Clifford gates highlighted by the explicit description of the Weil representations in the third chapter.

Keywords: Weil representation, characteristic two, symplectic group, quantum error-correcting codes.

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"[Mathematics] is a science which requires a great amount of imagination, and one of the leading mathematicians of our century states the case quite correctly when he says that it is impossible to be a mathematician without being a poet in soul."

– *Sofia Kovalevskaya, in a letter to Madame Schabelskoy* [Kov95]

"To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature. […] If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in."

– *Richard Feynman* [Fey65]

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Introduction (version française)

Afin de retracer l'origine de la représentation de Weil, nous devons d'abord présenter le groupe qui en est le fondement, à savoir le groupe de Heisenberg. Au début du vingtième siècle, les physiciens ont fait une série de découvertes expérimentales qui ne pouvaient être expliquées par la physique classique, telles que le rayonnement du corps noir, l'effet photoélectrique, la quantification de la matière et la dualité onde-particule. Toutes les tentatives visant à expliquer ces phénomènes ont conduit à un pan entièrement nouveau de la physique, aujourd'hui connu sous le nom de mécanique quantique. Nous n'aborderons pas en détail la fascinante histoire de la naissance de la mécanique quantique; nous mentionnons seulement que l'une de ses toutes premières avancées majeures a été l'observation faite par Planck en 1900 que l'énergie n'est pas continue. En effet, l'énergie E d'un photon isolé de lumière est proportionnelle à sa fréquence f:

$$E = hf, \quad \text{où } h \text{ est aujourd'hui appelée constante de Planck.} \tag{1}$$

Suite à quelques tentatives initiales visant à expliquer ces phénomènes quantiques, c'est vers 1925 que deux théories formelles différentes ont émergé : la "mécanique matricielle" et la "mécanique ondulatoire". La première théorie a été développée par Heisenberg alors qu'il travaillait sur les raies spectrales de l'hydrogène. Dans son célèbre article [Hei25], il a donné une formulation de la mécanique quantique en termes de tableaux qui représentent des observables physiques. Il a notamment remarqué que les tableaux représentant la position P et la quantité de mouvement Q ne satisfaisaient pas aux règles de multiplication habituelles : ils ne commutaient pas. La description des observables en termes d'opérateurs non commutatifs est en réalité une caractéristique primordiale de la mécanique quantique. Cependant, Heisenberg n'a pas directement utilisé de matrices ou d'opérateurs dans ses travaux, et c'est grâce à l'aide de Born et Jordan que la formulation complète de la théorie a été fournie dans [BHJ25], en tirant parti du cadre mathématique développé par Weyl. Selon cette approche, un système quantique est donné par un vecteur d'état et son évolution est décrite par des multiplications de matrices. Ceci a conduit à la formulation des relations de commutation canoniques formelles (CCR) :

$$[P, Q] = PQ - QP = i\frac{h}{2\pi}I. \tag{2}$$

En parallèle, Schrödinger a développé la deuxième formulation précédemment mentionnée de la mécanique quantique, dans laquelle un système quantique est décrit par une fonction d'onde ψ et son évolution est régie par une équation aux dérivées partielles, désormais connue sous le nom d'équation de Schrödinger. Dans ce cadre, la position P et la quantité de mouvement

Q peuvent être définies comme les opérateurs suivants :

$$\hat{p}(\psi)(x,t) = x\psi(x,t) \quad \text{et} \quad \hat{q}(\psi)(x,t) = -i\frac{h}{2\pi}\frac{\partial\psi}{\partial x}(x,t) \quad \text{pour tous } x \in \mathbb{R} \text{ et } t \in \mathbb{R}. \tag{3}$$

De cette manière, on retrouve la relation

$$[\hat{p},\hat{q}]\psi(x,t) = (\hat{p}\hat{q} - \hat{q}\hat{p})\psi(x,t) = i\frac{h}{2\pi}\psi(x,t). \tag{4}$$

Cela a conduit à la formulation du principe d'incertitude, qui stipule qu'il existe des paires d'observables physiques (comme la position et la quantité de mouvement) que nous ne pouvons pas mesurer simultanément avec une précision arbitraire. Même si les formalismes de la mécanique matricielle et de la mécanique ondulatoire étaient complètement différents, les travaux ultérieurs de Schrödinger, Dirac, von Neumann et Stone (entre 1926 et 1932) ont prouvé qu'ils étaient en fait équivalents. Le théorème de Stone-von Neumann formalise l'idée que toutes les formulations des relations de commutation canoniques sont équivalentes. Cette unification a mis en évidence l'importance de l'utilisation d'un cadre mathématique et a ouvert la voie au développement de la physique quantique. Pour éviter des considérations techniques, nous ne donnerons pas plus de détails sur le théorème de Stone-von Neumann dans ce contexte, mais nous l'énoncerons précisément et en donnerons une preuve dans cette thèse dans le langage de la théorie des représentations, dans le cadre des corps finis.

L'idée de non-commutativité découlant des relations de commutation canoniques peut être formalisée dans un cadre mathématique, et plus précisément à travers la théorie des groupes. Dans sa description standard, le groupe de Heisenberg est l'ensemble $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $n \geqslant 1$, muni du produit suivant :

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z' + \frac{1}{2}(x\cdot y' - y\cdot x')), \tag{5}$$

où $\cdot$ désigne le produit scalaire canonique sur $\mathbb{R}^n$. Le groupe de Heisenberg est un groupe de Lie, et son algèbre de Lie a une base composée d'éléments X_i, Y_i et Z qui satisfont

$$[X_i, Y_j] = \delta_{i,j}Z \quad \text{et} \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0 \quad \text{pour tous } 1 \leqslant i,j \leqslant n. \tag{6}$$

En identifiant respectivement X_i et Y_i aux opérateurs de position et de quantité de mouvement, ces relations permettent de retrouver formellement un analogue des relations de commutation canoniques. Des variations de la définition ci-dessus conduisent à des groupes de Heisenberg plus généraux : nous pouvons par exemple remplacer $\mathbb{R}$ par d'autres corps, ou encore définir un groupe de Heisenberg à partir d'un espace vectoriel symplectique, ce qui est la situation sur laquelle nous nous concentrerons dans cette thèse.

Si W est un espace vectoriel non trivial de dimension finie sur un corps $\mathbb{F}$ de caractéristique différente de deux, et si $\langle\cdot,\cdot\rangle$ est une forme bilinéaire non dégénérée et alternée sur W, alors on définit le groupe de Heisenberg $\mathrm{H}(W)$ comme l'ensemble $W \times \mathbb{F}$, muni du produit suivant:

$$(w,t)(w',t') = (w+w', t+t' + \frac{1}{2}\langle w, w'\rangle). \tag{7}$$

Le groupe de Heisenberg joue un rôle prépondérant dans un nombre conséquent de domaines différents des mathématiques, et en dresser une liste exhaustive dépasserait le cadre de cette introduction. Nous nous concentrerons donc uniquement sur l'aspect le plus pertinent pour nos travaux, à savoir l'influence de ce groupe en théorie des représentations ; voir par exemple [How80]. Plus précisément, le théorème de Stone-von Neumann stipule que le groupe de Heisenberg possède, à isomorphisme près, une unique représentation unitaire et irréductible qui agit sur son centre comme un caractère non trivial fixé. Ce résultat est la pierre angulaire de la construction classique de la représentation de Weil.

Sous sa forme la plus générale, la représentation de Weil (également appelée représentation de Segal-Shale-Weil, représentation oscillatoire ou représentation métaplectique) a été étudiée pour la première fois par Weil en 1964 [Wei64], et celui-ci se réfère aux travaux antérieurs de Segal et Shale pour le cas réel. Les travaux de Weil sont basés sur un groupe de Heisenberg qui n'est pas exactement celui défini par (7). En effet, on remarque que cette définition exclut complètement les corps de caractéristique deux. Comme Weil traitait le cas des corps locaux et des anneaux adéliques dont la caractéristique sous-jacente est arbitraire, la définition du groupe de Heisenberg a dû être généralisée.

Soit $W \neq \{0\}$ un espace vectoriel symplectique de dimension finie sur un corps fini $\mathbb{F}$, et soit $W = X \oplus Y$ une polarisation complète de W (ce qui signifie que X et Y sont deux sous-espaces vectoriels supplémentaires dans W de dimension maximale tels que $\langle \cdot, \cdot \rangle$ s'annule sur $X \times X$ et $Y \times Y$, c'est-à-dire deux sous-espaces lagrangiens). On pose

$$B_{\text{Weil}}(w, w') = \langle x, y' \rangle \quad \text{pour tous } w = x + y \text{ et } w' = x' + y' \text{ dans } W = X \oplus Y. \tag{8}$$

Alors B_{Weil} est une forme bilinéaire B telle que

$$\langle w, w' \rangle = B(w, w') - B(w', w) \quad \text{pour tous } w, w' \in W. \tag{9}$$

On définit alors le groupe de Heisenberg $\mathrm{H}(B)$ comme l'ensemble $W \times \mathbb{F}$ muni du produit

$$(w, t)(w', t') = (w + w', t + t' + B(w, w')). \tag{10}$$

(Notons que ce groupe est isomorphe au groupe $\mathrm{H}(W)$ si la caractéristique de $\mathbb{F}$ est différente de deux. Il s'agit donc véritablement d'une extension de la première définition.) Ce groupe coïncide avec le groupe de Heisenberg étudié par Weil lorsque $B = B_{\text{Weil}}$. Weil construit le groupe pseudosymplectique $\mathrm{Ps}(B_{\text{Weil}})$, qui est un sous-groupe spécifique du groupe des automorphismes de $\mathrm{H}(B_{\text{Weil}})$ agissant trivialement sur son centre. Sans le mentionner explicitement, Weil utilise un analogue du théorème de Stone-von Neumann dans son cadre afin de construire une représentation unitaire projective de $\mathrm{Ps}(B_{\text{Weil}})$. Cette représentation projective peut être relevée en une représentation linéaire d'une extension centrale Mp de $\mathrm{Ps}(B_{\text{Weil}})$ par le cercle unité. Pour un corps local $\mathbb{F}$ dont la caractéristique est différente de deux, le groupe pseudosymplectique $\mathrm{Ps}(B_{\text{Weil}})$ est isomorphe au groupe symplectique $\mathrm{Sp}(W)$ des transformations linéaires de W qui préservent la forme symplectique $\langle \cdot, \cdot \rangle$. Si $\mathbb{F} = \mathbb{C}$, alors l'extension Mp est scindée, et si $\mathbb{F} \neq \mathbb{C}$, alors l'extension Mp peut être réduite à un revêtement à deux feuillets de $\mathrm{Sp}(W)$. Dans le cas de la caractéristique deux, la situation est complètement différente. Par exemple, $\mathrm{Ps}(B_{\text{Weil}})$ définit une projection uniquement sur le groupe orthogonal O, défini comme le groupe des transformations linéaires σ de W satisfaisant $B_{\text{Weil}}(\sigma(w), \sigma(w)) = B_{\text{Weil}}(w, w)$ pour tout $w \in W$. Ainsi, l'extension Mp n'a pas de rapport direct avec le groupe symplectique $\mathrm{Sp}(W)$ tout entier.

Les travaux pionniers de Weil ont d'abord été orientés vers des applications dans la théorie des fonctions thêta et des formes automorphes. Néanmoins, ils ont profondément influencé d'autres domaines des mathématiques et de la physique. Nous ne mentionnerons ici que la correspondance de Howe (ou thêta), qui est certainement l'une des principales motivations de notre travail. Considérons un corps local $\mathbb{F}$ de caractéristique différente de deux. On dit que deux sous-groupes réductifs G et G' de $\mathrm{Sp}(W)$ qui sont centralisateurs l'un de l'autre dans $\mathrm{Sp}(W)$ forment une paire duale réductive (G, G'). Soient $\widetilde{G}$ et $\widetilde{G}'$ les images réciproques respectives de G et G' dans le groupe métaplectique. L'idée centrale développée par Howe est que la restriction de la représentation de Weil lisse à $\widetilde{G}\widetilde{G}'$ fournit une correspondance bijective entre certaines représentations admissibles irréductibles de $\widetilde{G}$ et certaines représentations admissibles irréductibles de $\widetilde{G}'$. Si $\mathbb{F} = \mathbb{R}$, le groupe métaplectique est l'unique revêtement à deux feuillets connexe de $\mathrm{Sp}(W)$, et la représentation de Weil ne peut pas être linéarisée en une représentation de $\mathrm{Sp}(W)$.

Pour en venir à présent à l'objet de cette thèse, nous citons Weil lui-même lorsqu'il évoque l'intérêt potentiel d'étendre sa construction au cadre des groupes finis :

> "[…] qu'il me soit permis, en passant, de signaler l'intérêt qu'il y aurait peut-être à examiner de plus près, du point de vue de la présente théorie, le cas des groupes finis."
>
> – André Weil [Wei64]

L'axe de recherche suggéré par la citation ci-dessus s'est avéré fructueux dans le cas des corps finis de caractéristique impaire. Nous mentionnons par exemple les travaux de Howe [Howa], [How73], Gérardin [Ger77], Duflo [Duf], Ranga Rao [Ran93], Neuhauser [Neu02] et Thomas [Tho08], [Tho13]. Une construction géométrique de la représentation de Weil a été fournie par Gurevich-Hadani [GH08]. L'article plus récent [AP14] d'Aubert et Przebinda présente une construction uniforme (avec des formules explicites) dans le cas où $\mathbb{F} = \mathbb{R}$, $\mathbb{F}$ est un corps p-adique avec $p \neq 2$, ou $\mathbb{F}$ est un corps fini de caractéristique impaire.

Les recherches menées dans le cadre des corps finis de caractéristique deux n'ont pas atteint le même niveau de satisfaction. En effet, rappelons qu'en caractéristique deux, le groupe pseudosymplectique de Weil est une extension du groupe orthogonal d'une forme bilinéaire particulière satisfaisant (9), ce groupe orthogonal étant un sous-groupe propre du groupe symplectique (à l'exception d'un cas, voir la proposition 1.13 dans la présente thèse). Une autre différence par rapport au cas de caractéristique impaire réside dans la définition (10) du groupe de Heisenberg lui-même. En effet, (7) fournit un groupe dont la loi produit est entièrement déterminée par la forme symplectique de W, tandis que (10) dépend du choix d'une forme bilinéaire B satisfaisant (9). Comme B_{Weil} n'est pas la seule forme bilinéaire de ce type, cela soulève la question du nombre de groupes de Heisenberg non isomorphes, et donc du nombre de représentations de Weil non équivalentes que nous obtenons. Dans la littérature, il existe deux approches principales pour la construction de la représentation de Weil en caractéristique deux. Nous les décrivons brièvement ici. Plus de précisions seront données plus loin dans cette introduction, où nous clarifierons le lien entre ces travaux et nos résultats. La première approche est due à Blasco [Bla93]. Elle suit les travaux de Weil afin de construire une

représentation de Weil d'un revêtement à deux feuillets du groupe pseudosymplectique, et établit la classification des paires duales dans le groupe pseudosymplectique. Son point de départ est le groupe de Heisenberg (10) pour n'importe quelle forme bilinéaire B satisfaisant (9). La seconde voie, empruntée par Gurevich-Hadani [GH12] et Genestier-Lysenko [GL12], vise à fournir un analogue géométrique de la représentation de Weil en caractéristique deux et à remédier à l'absence de lien entre la représentation de Weil et le groupe symplectique. L'idée de retrouver le lien avec le groupe symplectique apparaît déjà dans les travaux de Griess [Gri73], Glasby [Gla95] et Schmid [Sch00] dans leur étude des extensions des groupes 2-extraspéciaux. Nous décrirons ce cadre plus précisément lors de la présentation des résultats du deuxième chapitre. Nous ajoutons seulement ici que la théorie des groupes p-extraspéciaux est un élément clé pour résoudre la question du nombre de groupes de Heisenberg non isomorphes définis par (10) pour un corps premier $\mathbb{F}$ de caractéristique p. Cette thèse fait le lien entre les deux approches précédentes. Notre objectif est double : fournir des formules explicites dans la construction de la représentation de Weil pour un corps fini de caractéristique deux, et rétablir simultanément un lien avec le groupe symplectique.

Nous décrivons maintenant les résultats obtenus dans cette thèse, qui est divisée en quatre chapitres. Les deux premiers chapitres fournissent trois constructions explicites différentes de la représentation de Weil. Le premier chapitre concerne une représentation de Weil liée au groupe pseudosymplectique. Cette construction est valable pour un corps fini arbitraire de caractéristique deux. Les deux constructions suivantes, qui nous permettent de retrouver le lien avec le groupe symplectique pour $\mathbb{F} = \mathbb{F}_2$, sont étudiées dans le deuxième chapitre. Dans le troisième chapitre, nous présentons les résultats des deux premiers chapitres dans le cas de dimension minimale d'un $\mathbb{F}_2$-espace vectoriel symplectique W. Travailler en basse dimension nous permet d'exhiber explicitement tous les groupes, extensions centrales et représentations obtenus de cette manière. Enfin, le dernier chapitre vise à traduire certains de nos résultats dans le langage de la théorie de l'information quantique.

Dans le **premier chapitre,** nous considérons un $\mathbb{F}$-espace vectoriel symplectique $(W, \langle \cdot, \cdot \rangle)$ de dimension $2n$, où $\mathbb{F}$ désigne un corps fini arbitraire de caractéristique deux et $n \geqslant 1$. Étant donnée une forme bilinéaire B satisfaisant (9), nous définissons le groupe de Heisenberg $\mathrm{H}(B)$ selon (10). En nous appuyant sur la théorie des groupes extraspéciaux exposée dans l'annexe A, nous montrons dans la proposition 1.6 que pour $\mathbb{F} = \mathbb{F}_2$, il existe à isomorphisme près exactement deux groupes de Heisenberg de ce type. Nous fixons un caractère non trivial χ de $\mathbb{F}$ et l'étendons à un caractère $\widetilde{\chi}$ d'un sous-groupe abélien maximal de $\mathrm{H}(B)$. En nous basant sur les travaux de Weil, nous définissons le groupe pseudosymplectique $\mathrm{Ps}(B)$ comme un sous-groupe spécifique du groupe d'automorphismes de $\mathrm{H}(B)$ qui agit trivialement sur son centre, et nous décrivons la structure de $\mathrm{Ps}(B)$ dans la proposition 1.12. En particulier, $\mathrm{Ps}(B)$ définit une projection sur le groupe orthogonal $\mathrm{O}(Q)$, où Q est la forme quadratique associée à B, c'est-à-dire $Q(w) = B(w, w)$ pour tout $w \in W$. Ensuite, nous énonçons et donnons une preuve courte du théorème de Stone-von Neumann dans ce cadre (Théorème 1.23), même si nous n'en faisons pas usage pour construire la représentation de Weil. En effet, puisque nous calculons le caractère du modèle de Schrödinger $\rho_{\widetilde{\chi}}$ de la représentation de Heisenberg dans le lemme 1.17, et puisque cette représentation est unitaire et irréductible, le lemme de Schur nous permet de déduire directement l'existence et l'unicité (à un facteur unitaire près)

des opérateurs d'entrelacement dont nous avons besoin pour construire la représentation de Weil projective de $\mathrm{Ps}(B)$ que nous cherchons. La prochaine étape consiste à fournir dans la proposition 1.28 et le corollaire 1.31 des formules explicites pour cette représentation projective de Weil $s \mapsto \omega_{\tilde{\chi}}(s, w_s)$, où $s = (\sigma, f) \in \mathrm{Ps}(B)$ et $w_s \in W$ satisfait (37), et pour le cocycle c_χ associé. Remarquons que nous travaillons avec une famille de sections de cette représentation projective paramétrée par des constantes $\gamma(\sigma)$; voir (38). Cela autorise une certaine flexibilité dans la définition des opérateurs unitaires $\omega_{\tilde{\chi}}(s, w_s)$. Comme l'un de nos objectifs est de relever cette représentation projective en une représentation linéaire de la plus petite extension centrale possible de $\mathrm{Ps}(B)$, nous montrons d'abord que le cocycle c_χ n'est pas un cobord (corollaire 1.33). Pour les corps finis de caractéristique impaire, la représentation de Weil est définie comme une représentation linéaire du groupe symplectique lui-même. En revanche, en caractéristique deux, il n'existe aucune section de la représentation projective de Weil de $\mathrm{Ps}(B)$ qui définisse une représentation linéaire du même groupe. Par conséquent, puisque nous démontrons dans le corollaire 1.35 que pour tous choix de w_s et $\gamma(\sigma)$, l'ensemble image minimal du cocycle c_χ est $\mu_2(\mathbb{C}) = \{\pm 1\}$, le mieux que nous puissions espérer est de linéariser la représentation projective de Weil à un revêtement à deux feuillets de $\mathrm{Ps}(B)$. Nous procédons à la construction d'un tel revêtement $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ dans la proposition 1.36. Même si l'extension centrale $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ de $\mathrm{Ps}(B)$ par $\mu_2(\mathbb{C})$ dépend a priori des choix $\tilde{\chi}$, w_s et $\gamma(\sigma)$, nous donnons une condition nécessaire et suffisante (50) sur $\gamma(\sigma)$ pour garantir que différents choix définissent en réalité des extensions isomorphes, et nous exhibons un isomorphisme correspondant. Cela nous permet de relever $s \mapsto \omega_{\tilde{\chi}}(s, w_s)$ en une représentation linéaire $\widetilde{\omega}_{\tilde{\chi}}$ unitaire et fidèle du groupe $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ dans la proposition 1.39. Cette construction peut être illustrée à l'aide du diagramme suivant :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \widetilde{\omega}_{\tilde{\chi}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Ps}}} & & \\ & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{O}(Q) & & \end{array} \qquad (D_1)$$

Soulignons ici que, contrairement au cas de caractéristique impaire, la représentation de Weil $\widetilde{\omega}_{\tilde{\chi}}$ de $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ est irréductible. Son image $\widetilde{\omega}_{\tilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}})$ contient une copie du groupe de Heisenberg $\mathrm{H}(B)$ si et seulement si $\mathbb{F} = \mathbb{F}_2$. Le caractère de $\widetilde{\omega}_{\tilde{\chi}}$ est ensuite calculé dans le corollaire 1.41. Enfin, si le paramètre $\gamma(\sigma)$ satisfait une certaine condition, nous montrons dans la proposition 1.46 que l'extension $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ de $\mathrm{Ps}(B)$ ne dépend pas du choix du caractère χ de $\mathbb{F}$, et nous montrons dans les corollaires 1.45 et 1.47 que la représentation de Weil de $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ ne dépend pas des choix effectués pour χ, $\tilde{\chi}$, w_s et $\gamma(\sigma)$.

Notre travail est inspiré de ceux de Blasco [Bla93] et nous utilisons les méthodes de Glasby [Gla95]. Néanmoins, plusieurs différences importantes sont à noter. Premièrement, tous ces travaux considèrent une forme bilinéaire arbitraire B pour définir le groupe de Heisenberg $\mathrm{H}(B)$. Cependant, Glasby ne traite que le cas $\mathbb{F} = \mathbb{F}_2$. En revanche, notre construction, tout comme celle de Blasco, est valable pour un corps fini arbitraire de caractéristique deux. Deuxièmement, Blasco et Glasby utilisent tous deux le théorème de Stone-von Neumann pour fournir une formule pour $\omega_{\tilde{\chi}}(s, w_s)$ qui est définie à un signe près, en fonction du choix de w_s.

Nous n'utilisons pas ce théorème et nous clarifions la dépendance de $\omega_{\widetilde{\chi}}(s, w_s)$ par rapport au choix de w_s. Blasco donne une formule générale non explicite pour $\omega_{\widetilde{\chi}}(s, w_s)$ impliquant le choix d'une polarisation complète de W et l'action de s sur cette polarisation. Elle fournit également une formule explicite pour $\omega_{\widetilde{\chi}}(s, w_s)$, mais uniquement sur le sous-groupe strict de $\mathrm{Ps}(B)$ formé des éléments $s = (\sigma, f)$ tels que $1 - \sigma$ est inversible. En revanche, notre formule est valable sur l'ensemble du groupe $\mathrm{Ps}(B)$ et pour tout corps fini de caractéristique deux. De plus, Blasco fixe une section explicite de la représentation projective de Weil et Glasby en fixe deux. En comparaison, le paramètre $\gamma(\sigma)$ dans la définition (38) nous permet de travailler sans fixer aucune section explicite de ce type. De cette manière, nous pouvons examiner toutes les possibilités pour l'ensemble image du cocycle c_χ et les revêtements qu'il fournit. Plus précisément, nous trouvons des conditions nécessaires et suffisantes sur $\gamma(\sigma)$ pour définir un revêtement à deux feuillets de $\mathrm{Ps}(B)$. Enfin, Blasco définit d'abord une extension de $\mathrm{Ps}(B)$ par $\mathbb{C}^\times$, puis obtient théoriquement l'existence d'un revêtement à deux feuillets de $\mathrm{Ps}(B)$ en plongeant l'espace vectoriel W dans un espace vectoriel symplectique de dimension deux fois supérieure à celle de W, tandis que nous utilisons l'expression explicite du cocycle c_χ afin de prouver que son ensemble image minimal est $\{\pm 1\}$. L'avantage d'avoir un cocycle explicite est de pouvoir calculer le produit de deux éléments quelconques dans le revêtement $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, puisque le produit dans ce groupe dépend entièrement de c_χ.

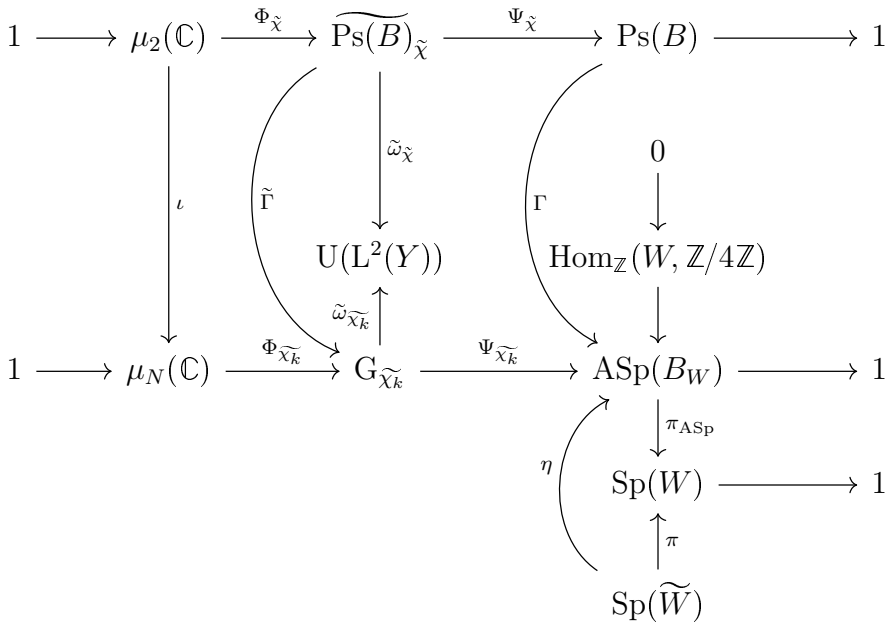
Le premier objectif du **deuxième chapitre** est de construire une représentation de Weil d'un revêtement d'une extension du groupe symplectique $\mathrm{Sp}(W)$. Dans cette mesure, nous supposons que $\mathbb{F} = \mathbb{F}_2$ et nous considérons un $\mathbb{Z}/4\mathbb{Z}$-module libre $\widetilde{W}$ de rang $2n$, où $n \geqslant 1$. On retrouve le cadre du premier chapitre en considérant $\widetilde{W}/2\widetilde{W}$, qui est un $\mathbb{F}_2$-espace vectoriel symplectique de dimension $2n$ (cf. lemme 2.1). Dans les sections 2.1 à 2.3, l'idée est de fournir des constructions parallèles à celles du premier chapitre. Ainsi, la forme bilinéaire B est remplacée par B_W définie juste avant (77), et nous l'utilisons pour définir le groupe de Heisenberg $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ par (79). Il s'agit du groupe de Heisenberg de Glasby [Gla95]. Gurevich et Hadani utilisent cette définition uniquement pour $B = B_{\mathrm{Weil}}$ dans [GH12], mais nous pouvons montrer, grâce à la théorie des groupes presque extraspéciaux exposée dans l'annexe A, qu'il n'existe qu'un seul groupe $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ à isomorphisme près. Le groupe pseudosymplectique $\mathrm{Ps}(B)$ est remplacé par le groupe affine symplectique $\mathrm{ASp}(B_W)$, qui est le groupe des automorphismes de $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ agissant trivialement sur son centre. Le groupe $\mathrm{ASp}(B_W)$ contient $\mathrm{Ps}(B)$, comme le montre la proposition 2.6, et il définit une projection sur le groupe symplectique $\mathrm{Sp}(W)$. Nous étudions sa structure dans la proposition 2.7. Ensuite, nous fixons un caractère fidèle χ_k de $\mathbb{Z}/4\mathbb{Z}$ et l'étendons à un sous-groupe abélien maximal de $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, nous donnons une courte preuve du théorème de Stone-von Neumann (théorème 2.16) dans ce cadre, et nous décrivons le modèle de Schrödinger $\rho_{\widetilde{\chi_k}}$ de la représentation de Heisenberg. Nous utilisons la même méthode que dans le premier chapitre pour trouver dans la proposition 2.21 et le corollaire 2.24 des formules explicites pour la représentation projective de Weil $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ de $\mathrm{ASp}(B_W)$, où w_s satisfait (100), et pour le cocycle c_{χ_k} associé. En particulier, nous autorisons également une certaine liberté dans le choix des paramètres $\gamma(\sigma)$ dans (103), afin d'éviter de fixer une section spécifique. Ensuite, nous montrons dans le corollaire 2.26 que le cocycle c_{χ_k} n'est pas un cobord. Comme pour $\mathrm{Ps}(B)$, cela signifie qu'il n'existe aucune section de la représentation projective de Weil de $\mathrm{ASp}(B_W)$ qui nous permette de définir une

représentation linéaire du même groupe. Nous étudions ensuite l'ensemble image du cocycle c_{χ_k} dans le corollaire 2.28. Nous montrons que, quels que soient les choix effectués pour w_s et $\gamma(\sigma)$, le plus petit ensemble image possible de c_{χ_k} est $\mu_2(\mathbb{C}) = \{\pm 1\}$ si l'indice de Witt de Q est nul (ou, de manière équivalente, si l'invariant d'Arf de Q vaut 1 et $\dim_{\mathbb{F}_2}(W) = 2$), et $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$ sinon. Nous donnons des conditions nécessaires et suffisantes sur $\gamma(\sigma)$ afin d'obtenir ces ensembles images. Pour être tout à fait rigoureux, nous devons mentionner que le paramètre $\gamma(\sigma)$ n'est pas le paramètre le plus général que nous pourrions considérer. En particulier, nous n'avons pas prouvé que le corollaire 2.28 fournit le meilleur résultat possible en général. Par exemple, nous obtenons un ensemble image plus petit pour c_{χ_k} lorsque $\dim_{\mathbb{F}_2}(W) = 2$ et que l'invariant d'Arf de Q est nul (cf.remarques 2.22, 2.36 et 3.8). Ce dernier résultat concernant l'ensemble image de c_{χ_k} nous permet de construire dans la proposition 2.29 un revêtement à quatre feuillets $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ de $\mathrm{ASp}(B_W)$, ainsi qu'un revêtement à deux feuillets $\mathrm{G}^0_{\widetilde{\chi_k}}$ si l'indice de Witt de Q est nul. Nous donnons une condition nécessaire et suffisante (143) sur $\gamma(\sigma)$ pour garantir que différents choix pour $\widetilde{\chi_k}$, w_s et $\gamma(\sigma)$ définissent des extensions de $\mathrm{ASp}(B_W)$ isomorphes, et nous explicitons un isomorphisme correspondant. Nous relevons ensuite l'application $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ en une représentation linéaire unitaire et fidèle $\widetilde{\omega}_{\widetilde{\chi_k}}$ du revêtement correspondant dans la proposition 2.32. Cette construction est illustrée par le diagramme suivant

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_N(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{G}_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & \overset{\eta}{\curvearrowleft} & \mathrm{Sp}(W) & & \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) & &
\end{array}
\qquad (D_2)
$$

où $\mathrm{G}_{\widetilde{\chi_k}}$ désigne le revêtement $\mathrm{G}^0_{\widetilde{\chi_k}}$ si $N = 2$ (auquel cas l'indice de Witt de Q est nul), et $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ si $N = 4$. Ici encore, la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}}$ est irréductible et son image $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi_k}})$ contient une copie du groupe de Heisenberg $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Le corollaire 2.33 fournit le caractère de $\widetilde{\omega}_{\widetilde{\chi_k}}$. Pour terminer l'analogie avec le premier chapitre, nous donnons dans le corollaire 2.34 et la proposition 2.35 des conditions sur le paramètre $\gamma(\sigma)$ afin de garantir que les revêtements de $\mathrm{ASp}(B_W)$ et les représentations de Weil que nous construisons ne dépendent pas des choix effectués pour χ_k, $\widetilde{\chi_k}$, w_s et $\gamma(\sigma)$. Bien qu'elle aboutisse à une extension plus grande que dans le cas de $\mathrm{Ps}(B)$, cette construction a l'avantage d'être reliée au groupe symplectique $\mathrm{Sp}(W)$ par l'intermédiaire de $\mathrm{ASp}(B_W)$. Dans cette partie, nous utilisons les méthodes de Glasby [Gla95], mais nous obtenons des résultats plus généraux car nous ne fixons aucune section particulière de la représentation projective $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$. Nous retrouvons également un revêtement à quatre feuillets de $\mathrm{ASp}(B_W)$, comme proposé par Gurevich et Hadani [GH12], sous une forme très explicite. De plus, notons que ni [Gla95] ni [GH12] n'abordent les questions de la taille minimale de l'extension de $\mathrm{ASp}(B_W)$, ou de l'existence de revêtements non isomorphes de $\mathrm{ASp}(B_W)$. Nous considérons ces questions et donnons des réponses précises dans certains cas.

Une autre question à laquelle nous répondons dans la section 2.4 concerne la compatibilité des deux constructions des représentations de Weil mentionnées plus haut, au sens de l'existence d'un morphisme de groupes $\widetilde{\Gamma}$ tel que le diagramme suivant commute :

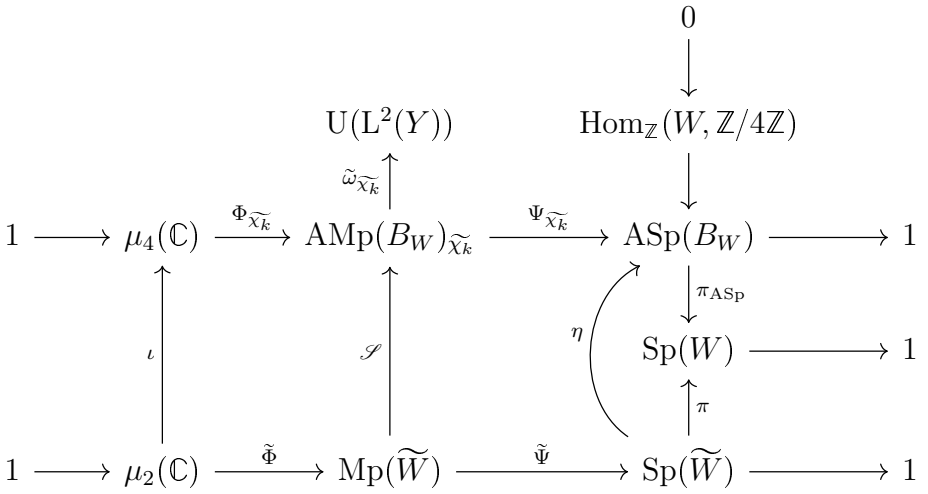


(D_3)

En d'autres termes, nous examinons si les constructions faites au début du deuxième chapitre prolongent réellement les travaux de Weil. Si l'indice de Witt de Q est nul, nous donnons dans la proposition 2.39 une condition nécessaire et suffisante sur les choix effectués pour $\gamma(\sigma)$ dans les constructions de $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ et $\mathrm{G}^0_{\widetilde{\chi_k}}$ afin que le diagramme (D_3) commute. Nous donnons une condition similaire pour la commutativité de ce diagramme lorsque $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ et que W est un espace vectoriel de dimension deux. Cependant, si l'indice de Witt de Q est supérieur ou égal à deux, quels que soient les choix effectués dans les deux premières constructions, nous montrons que $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ n'est pas compatible avec $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$: le diagramme (D_3) ne commute pas. Enfin, nous montrons qu'obtenir cette compatibilité entre les deux constructions des représentations de Weil nécessite de définir un revêtement à huit feuillets de $\mathrm{ASp}(B_W)$ plutôt qu'un revêtement à quatre feuillets. Dans la proposition 2.29, nous construisons le revêtement à huit feuillets approprié de $\mathrm{ASp}(B_W)$. Ceci fait entrevoir une nouvelle perspective dans la théorie de la représentation de Weil : en fonction de notre objectif, nous devons décider soit de définir la représentation de Weil d'une extension la plus petite possible de $\mathrm{ASp}(B_W)$, soit de définir une représentation de Weil qui soit compatible avec celle de $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$.

En lien avec les travaux de Gurevich-Hadani [GH12], une question supplémentaire abordée dans la section 2.5 consiste à fournir des constructions explicites du revêtement métaplectique à deux feuillets $\mathrm{Mp}(\widetilde{W})$ de $\mathrm{Sp}(\widetilde{W})$, et du "splitting homomorphism" $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ qui permettrait aux auteurs d'étendre la représentation de Weil au groupe $\mathrm{Mp}(\widetilde{W})$. Notre première remarque est que pour obtenir une extension appropriée de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}}$ de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ à un tel revêtement $\mathrm{Mp}(\widetilde{W})$ à deux feuillets, on s'attendrait à ce

que la représentation étendue $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ hérite des propriétés de $\widetilde{\omega}_{\widetilde{\chi_k}}$, en particulier de son injectivité. Cependant, un argument combinatoire énoncé dans le lemme 2.42 montre que cette injectivité est impossible pour $n \geqslant 2$. Par conséquent, nous ne pouvons qu'espérer construire un revêtement $\mathrm{Mp}(\widetilde{W})$ à deux feuillets et un morphisme d'extensions centrales $\mathscr{S}$ tels que le diagramme suivant (D_4) commute.



En outre, nous avons appris par Lysenko l'existence d'une lacune dans [GH12], qui a été confirmée par Gurevich lors d'une conversation avec Przebinda. Comme nous fournissons un contre-exemple à l'argument utilisé dans [GH12] (cf. lemme 2.44), la question de l'existence de $\mathrm{Mp}(\widetilde{W})$ et $\mathscr{S}$ restait ouverte, et cette thèse y répond. La voie que nous suivons nous conduit à une construction nouvelle et indépendante de la représentation de Weil. En effet, nous commençons par définir un nouveau groupe de Heisenberg $\mathrm{H}(\widetilde{W})$ par (196). La motivation derrière cette définition est de retrouver le cadre de la caractéristique différente de deux, en ce sens que $\mathrm{H}(\widetilde{W})$ est un groupe sur lequel $\mathrm{Sp}(\widetilde{W})$ agit par automorphismes. Après avoir fixé un caractère fidèle de $\mathbb{Z}/4\mathbb{Z}$, nous utilisons une méthode similaire aux deux premières constructions. De cette manière, nous définissons une représentation de Weil projective $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ de $\mathrm{Sp}(\widetilde{W})$, et déduisons des formules explicites pour cette représentation et pour le cocycle associé $\widetilde{c}_{\chi_k}$ dans la proposition 2.56 et le corollaire 2.58. Nous soulignons le fait que le paramètre $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ utilisé dans (215) englobe toutes les sections possibles de $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$. Une fois encore, nous montrons que cette représentation projective ne peut être relevée en une représentation linéaire de $\mathrm{Sp}(\widetilde{W})$ lui-même, puisque $\widetilde{c}_{\chi_k}$ n'est pas un cobord (cf. corollaire 2.59). Cependant, nous montrons dans le corollaire 2.61 que la restriction du cocycle à un sous-groupe parabolique maximal de $\mathrm{Sp}(\widetilde{W})$ est un cobord. Par la suite, le corollaire 2.62 aborde la question de l'ensemble image de $\widetilde{c}_{\chi_k}$: nous donnons une condition nécessaire et suffisante sur le paramètre $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ pour que $\widetilde{c}_{\chi_k}$ prenne ses valeurs dans $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$. Améliorer ce résultat en passant de $\mu_4(\mathbb{C})$ à $\mu_2(\mathbb{C})$ à partir de l'expression explicite de $\widetilde{c}_{\chi_k}$ semble bien plus difficile. Néanmoins, nous sommes en mesure de prouver que le plus petit ensemble image possible de $\widetilde{c}_{\chi_k}$ est en fait $\mu_2(\mathbb{C})$. Si le rang $2n$ du $\mathbb{Z}/4\mathbb{Z}$-module libre $\widetilde{W}$ vaut deux, nous trouvons explicitement les deux choix possibles de cocycles $\widetilde{c}_{\chi_k}$ non cohomologues dont l'ensemble image est $\mu_2(\mathbb{C})$ (voir section 3.3). Si $n \geqslant 2$, nous utilisons le second groupe de cohomologie

$\mathrm{H}_2(\mathrm{Sp}(\widetilde{W}),\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ explicité par Funar et Pitsch [FP20] pour déduire l'existence d'un choix de $\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})$ tel que l'ensemble image de $\widetilde{c}_{\chi_k}$ soit $\mu_2(\mathbb{C})$. Cela nous permet de construire dans la proposition 3.20 les revêtements métaplectiques à deux feuillets $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ de $\mathrm{Sp}(\widetilde{W})$ correspondants. Plus précisément, il existe exactement deux possibilités pour $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ si $n = 1$, et un seul revêtement à deux feuillets pour $n \geqslant 3$ car $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}),\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Il s'agit d'un résultat crucial puisqu'il fournit un véritable analogue du groupe métaplectique dans le cas réel (et cela montre également l'unicité à un cobord près du choix de $\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})$ garantissant que l'ensemble image de $\widetilde{c}_{\chi_k}$ est $\mu_2(\mathbb{C})$). Dans la proposition 2.65, nous relevons ensuite la représentation projective $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})$ de $\mathrm{Sp}(\widetilde{W})$ en une représentation linéaire unitaire irréductible $\widetilde{\omega}_{\chi_k}$ du groupe métaplectique $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. À l'exception d'un cas particulier de basse dimension, cette représentation n'est pas fidèle, mais nous l'appelons tout de même la représentation de Weil de $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ car sa construction est similaire à celle des deux précédentes. Nous calculons son caractère dans le corollaire 2.67 et nous examinons les liens entre les revêtements $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ et les représentations de Weil pour les deux choix de caractères χ_k dans la proposition 2.68. À ce stade, nous avons déjà répondu à la question de la construction explicite d'une représentation de Weil d'un revêtement à deux feuillets du groupe symplectique $\mathrm{Sp}(\widetilde{W})$, par une approche totalement indépendante des deux premières constructions décrites ci-dessus. La fin de la section 2.5 élucide la question de l'existence du morphisme d'extensions centrales $\mathscr{S}$ pour $B = B_{\mathrm{Weil}}$. Plus précisément, la proposition 2.73 fournit une condition nécessaire et suffisante sur les choix des paramètres $\gamma(\sigma)$ pour $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$ et $\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})$ pour $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ garantissant l'existence (et l'unicité) du morphisme d'extensions centrales $\mathscr{S}$ rendant le diagramme suivant commutatif :

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathrm{Hom}_{\mathbb{Z}}(W,\mathbb{Z}/4\mathbb{Z}) & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & \overset{\mathscr{S}}{\curvearrowleft} & \mathrm{U}(\mathrm{L}^2(Y)) & \overset{\eta}{\curvearrowleft} & \mathrm{Sp}(W) & \longrightarrow & 1 \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\chi_k}} & & \uparrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{Mp}(\widetilde{W})_{\chi_k} & \xrightarrow{\widetilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) & \longrightarrow & 1
\end{array}
\qquad (D_5)
$$

Cela apporte un éclairage nouveau sur les affirmations de Gurevich et Hadani.

Le **troisième chapitre** vise à présenter les résultats obtenus dans les deux premiers, dans le cas de dimension minimale où $\widetilde{W}$ est un $\mathbb{Z}/4\mathbb{Z}$-module libre de rang deux et W est un $\mathbb{F}_2$-espace vectoriel de dimension deux. Étant donnée la petite taille de ces espaces, nous sommes en mesure de décrire explicitement les éléments de tous les groupes impliqués et les opérateurs des représentations de Weil que nous avons construites. Ainsi, nous identifions ces groupes

dans la base de données [GN] (dans chaque cas avec un isomorphisme explicite) et les diagrammes commutatifs qui apparaissent dans ce cadre. Une liste exhaustive de ces groupes est fournie dans l'annexe B.2, avec une présentation et leur identification dans la bibliothèque SmallGroup. Nous fixons une polarisation complète $W = X \oplus Y$, où $X = \mathrm{Vect}\{(1,0)\}$ et $Y = \mathrm{Vect}\{(0,1)\}$. Comme déjà mentionné dans le résumé du premier chapitre, il existe exactement deux groupes de Heisenberg $\mathrm{H}(B)$ à isomorphisme près. Plus précisément, $\mathrm{H}(B)$ est d'ordre huit et est isomorphe au groupe diédral D_4 ou au groupe quaternionique Q_8. Les deux premières sections 3.1 et 3.2 étudient ces deux possibilités séparément. Avant d'aborder spécifiquement chacun de ces deux cas, nous donnons dans la proposition 3.1 les listes des éléments de $\mathrm{Sp}(W) = \mathrm{SL}_2(\mathbb{F}_2)$ et $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$, qui sont respectivement isomorphes au groupe symétrique S_3, d'ordre 6, et au groupe $\mathrm{A}_4 \rtimes \mathrm{C}_4$, d'ordre 48.

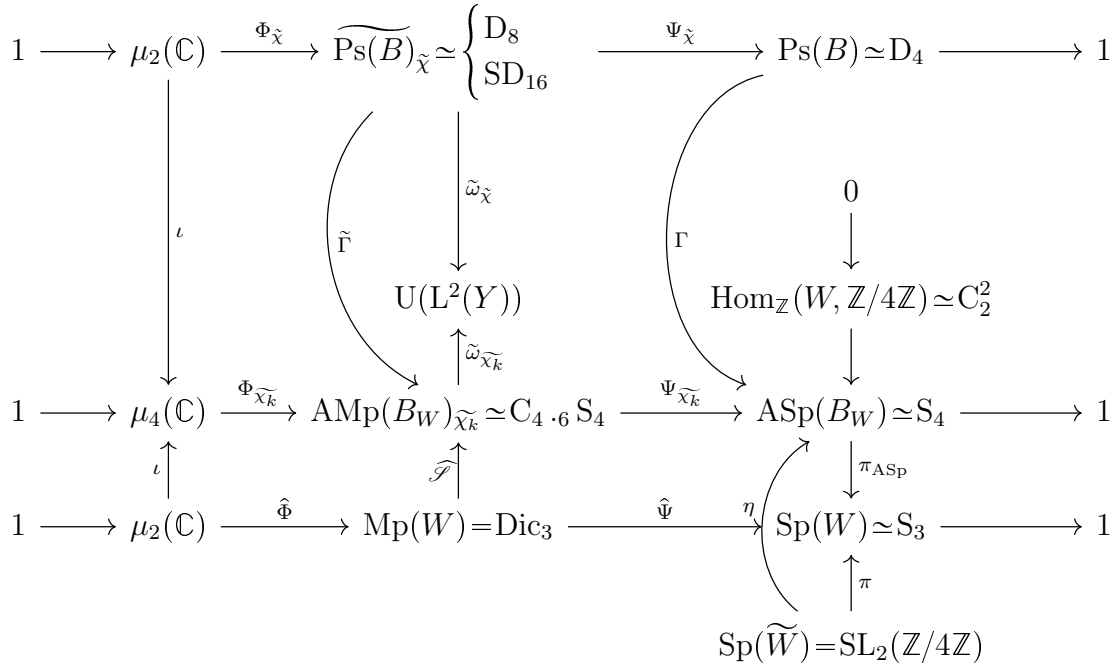
Si $\mathbf{H}(B)$ est isomorphe à $\mathbf{D_4}$, alors nous choisissons $B = B_{\mathrm{Weil}} : ((x,y),(x',y')) \mapsto xy'$ pour effectuer tous les calculs. Dans ce cas, la forme quadratique associée à B est $Q : (x,y) \mapsto xy$ et le groupe orthogonal $\mathrm{O}(Q)$ comporte exactement deux éléments. Dans la proposition 3.2, nous exhibons les éléments du groupe pseudosymplectique $\mathrm{Ps}(B)$ et les différents revêtements à deux feuillets $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ possibles. Cela fournit également la liste des opérateurs qui appartiennent à l'image $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi}}$. Le groupe pseudosymplectique $\mathrm{Ps}(B)$ est isomorphe au groupe diédral D_4, d'ordre huit. En considérant tous les choix mentionnés précédemment pour w_s, $s \in \mathrm{Ps}(B)$, et $\gamma(\sigma)$, $\sigma \in \mathrm{O}(Q)$, nous montrons qu'il y a exactement deux possibilités pour le revêtement à deux feuillets : $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ est soit isomorphe au groupe diédral D_8, soit au groupe semi-diédral SD_{16} d'ordre 16. Le calcul du second groupe de cohomologie $\mathrm{H}^2(\mathrm{Ps}(B), \mathbb{Z}/2\mathbb{Z})$ et [GN] nous permettent de remarquer que ce sont les seuls revêtements à deux feuillets de $\mathrm{Ps}(B)$ qui contiennent une copie du groupe de Heisenberg $\mathrm{H}(B)$. Nous poursuivons par l'étude du groupe affine symplectique. Nous avons déjà mentionné qu'il n'existe qu'un seul groupe $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ à isomorphisme près : dans le cas de dimension minimale, celui-ci est isomorphe au groupe de Pauli $\mathrm{C}_4 \circ \mathrm{D}_4$, d'ordre 16. Dans la proposition 3.5, on montre que $\mathrm{ASp}(B_W)$ est isomorphe au groupe symétrique S_4, d'ordre 24, et on donne les listes des 96 éléments des revêtements affines métaplectiques $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, où $k = 1$ ou 3. Cela fournit également les opérateurs qui appartiennent à l'image $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi_k}})$ de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}}$. Selon les choix de $\gamma(\sigma)$, $\sigma \in \mathrm{Sp}(W)$, il y a exactement deux possibilités : $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est soit isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, soit au groupe unitaire $\mathrm{U}_2(\mathbb{F}_3)$. En calculant $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z})$ et en utilisant [GN], nous remarquons comme pour $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ que ce sont les seuls revêtements à quatre feuillets de $\mathrm{ASp}(B_W)$ qui contiennent une copie du groupe de Heisenberg $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. La description de $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z})$ implique que la classe d'isomorphisme de l'extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ de $\mathrm{ASp}(B_W)$ ne dépend pas dans cet exemple du caractère fidèle χ_k de $\mathbb{Z}/4\mathbb{Z}$ choisi. Cela nous permet également de montrer qu'un revêtement à quatre feuillets n'est pas l'extension centrale non scindée minimale de $\mathrm{ASp}(B_W)$ qu'on peut obtenir (cf. la discussion ci-dessus sur la non-optimalité du paramètre $\gamma(\sigma)$ dans le deuxième chapitre). Plus précisément, nous exhibons dans la remarque 3.8 deux choix différents de w_s, $s \in \mathrm{ASp}(B_W)$, et $\gamma(\sigma)$, $\sigma \in \mathrm{Sp}(W)$, qui conduisent aux deux revêtements à deux feuillets possibles de $\mathrm{ASp}(B_W)$, à savoir le groupe conforme spécial unitaire (ou groupe octaédral binaire) $\mathrm{CSU}_2(\mathbb{F}_3)$ et le groupe linéaire général $\mathrm{GL}_2(\mathbb{F}_3)$, d'ordre 48. On remarque que ces groupes ne contiennent pas $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, contrairement à $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. La question

de la compatibilité des constructions de $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ et $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est traitée dans la proposition 3.9 : le diagramme (D_3) commute si et seulement si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$.

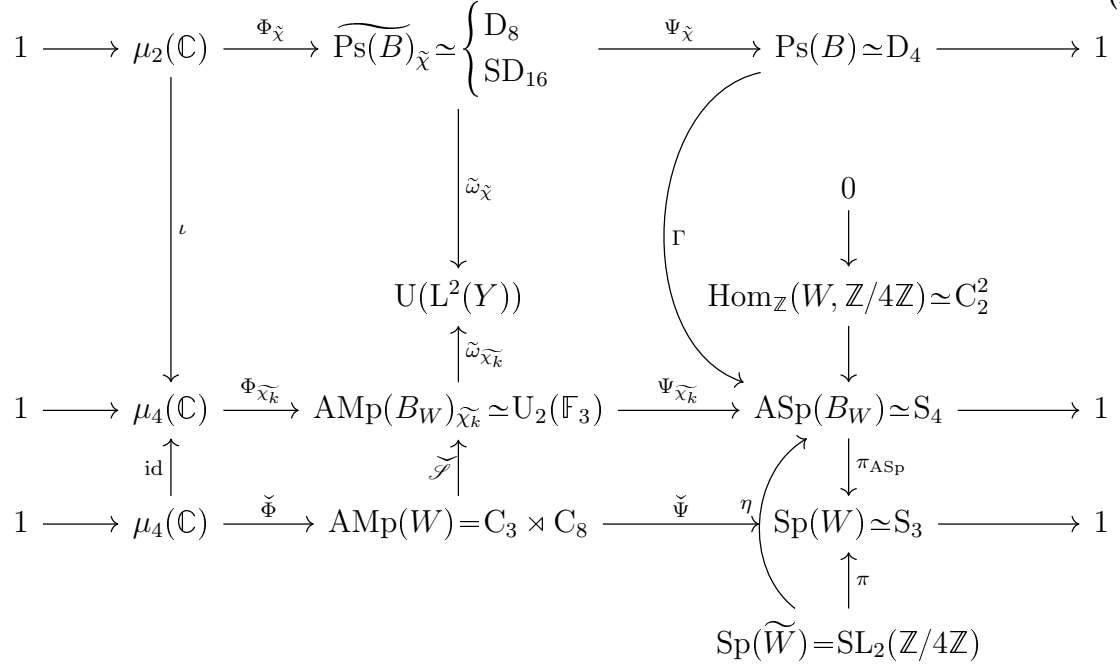
À la suite des constructions de $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ et $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, la première voie que nous empruntons dans la section 3.1.4 est la recherche d'un revêtement métaplectique à deux feuillets $\mathrm{Mp}(\widetilde{W})$ de $\mathrm{Sp}(\widetilde{W})$ afin d'obtenir un diagramme commutatif (D_4). Les résultats de cette section ont été obtenus avant que nous ne réalisions la troisième construction indépendante de la représentation de Weil de $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ dans la section 2.5 (qui est présentée dans la section 3.3). Nous les incluons néanmoins ici pour deux raisons principales. Premièrement, ils montrent que dans cet exemple, nous sommes en mesure de construire un revêtement à deux feuillets $\mathrm{Mp}(\widetilde{W})$ de $\mathrm{Sp}(\widetilde{W})$ et sa représentation de Weil, de telle sorte que le diagramme (D_4) commute, en utilisant uniquement les deux premières constructions et des considérations sur les présentations abstraites des groupes impliqués. Deuxièmement, notons que les diagrammes (D_4) et (D_5) sont légèrement différents l'un de l'autre, et que nous avons fourni le morphisme d'extensions centrales $\mathscr{S}$ dans la section 2.5 uniquement pour $B = B_{\mathrm{Weil}}$. Par ailleurs, les méthodes utilisées dans la section 3.1.4 peuvent être réemployées dans la section 3.2. Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$, alors nous montrons dans la proposition 3.12 qu'obtenir le meilleur résultat possible nécessite que $\mathrm{Mp}(\widetilde{W})$ soit isomorphe au groupe $\mathrm{U}_2(\mathbb{F}_3)$. Dans ce cas, on peut choisir l'application identité pour le morphisme $\mathscr{S}$ qui rend le diagramme (D_4) commutatif, et l'extension $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ de la représentation de Weil de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ à $\mathrm{Mp}(\widetilde{W})$ que nous obtenons reste unitaire, irréductible et fidèle. Rappelons que cette situation est impossible en dimension supérieure (cf. lemme 2.42). D'autre part, si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, alors la commutativité du diagramme (D_4) impose à $\mathrm{Mp}(\widetilde{W})$ d'être isomorphe au groupe $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. Dans la proposition 3.16, nous étudions les différentes possibilités pour le morphisme $\mathscr{S}$ rendant le diagramme (D_4) commutatif. En particulier, l'extension $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ de la représentation de Weil de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ à $\mathrm{Mp}(\widetilde{W})$ obtenue dans le corollaire 3.17 est unitaire et irréductible, mais pas fidèle, car son image $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ est d'indice deux dans $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ et est isomorphe à $\mathrm{CSU}_2(\mathbb{F}_3)$ ou $\mathrm{GL}_2(\mathbb{F}_3)$. Par conséquent, la construction au sein de laquelle $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$ préserve l'injectivité de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, mais perd le lien avec les travaux de Weil, au sens de l'existence de $\widetilde{\Gamma}$. En revanche, la construction dans laquelle $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$ préserve le lien avec les travaux de Weil, c'est-à-dire que le morphisme $\widetilde{\Gamma}$ existe, mais perd l'injectivité de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$.

Étant donné que tous les groupes sont explicites et relativement petits dans cet exemple, nous explorons dans la section 3.1.5 une autre manière d'étendre la représentation de Weil de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, à un groupe plus proche de $\mathrm{Sp}(W)$ que de $\mathrm{Mp}(\widetilde{W})$, c'est-à-dire à une extension centrale de $\mathrm{Sp}(W)$ lui-même. Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, nous montrons dans la proposition 3.20 que le meilleur résultat que nous pouvons obtenir (c'est-à-dire la plus petite extension possible de $\mathrm{Sp}(W)$ qui est compatible avec $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ au sens du diagramme commutatif (D_6) ci-dessous) est de définir une représentation de Weil unitaire, irréductible et fidèle $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ du revêtement à deux feuillets $\mathrm{Mp}(W) = \mathrm{Dic}_3$ de $\mathrm{Sp}(W)$. Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$, nous obtenons un résultat similaire pour un revêtement à quatre

feuillets $\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8$ de $\mathrm{Sp}(W)$ et sa représentation de Weil associée $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}}$. Nous pouvons visualiser ces résultats à l'aide des deux diagrammes suivants, en fonction de la classe d'isomorphisme de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.



(D_6)



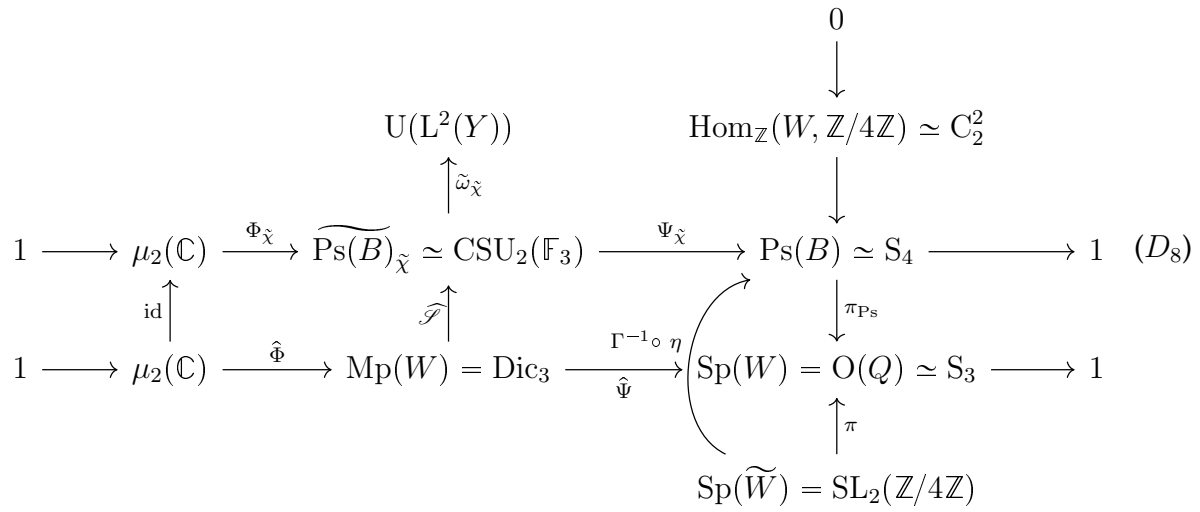
(D_7)

Si $\mathbf{H}(B)$ est isomorphe à $\mathbf{Q_8}$, alors nous choisissons $B : ((x, y), (x', y')) \mapsto xy' + xx' + yy'$ pour effectuer tous les calculs. La forme quadratique associée à B est alors $Q : (x, y) \mapsto xy + x^2 + y^2$. Il s'agit là du cas très exceptionnel dans lequel le groupe orthogonal et le groupe

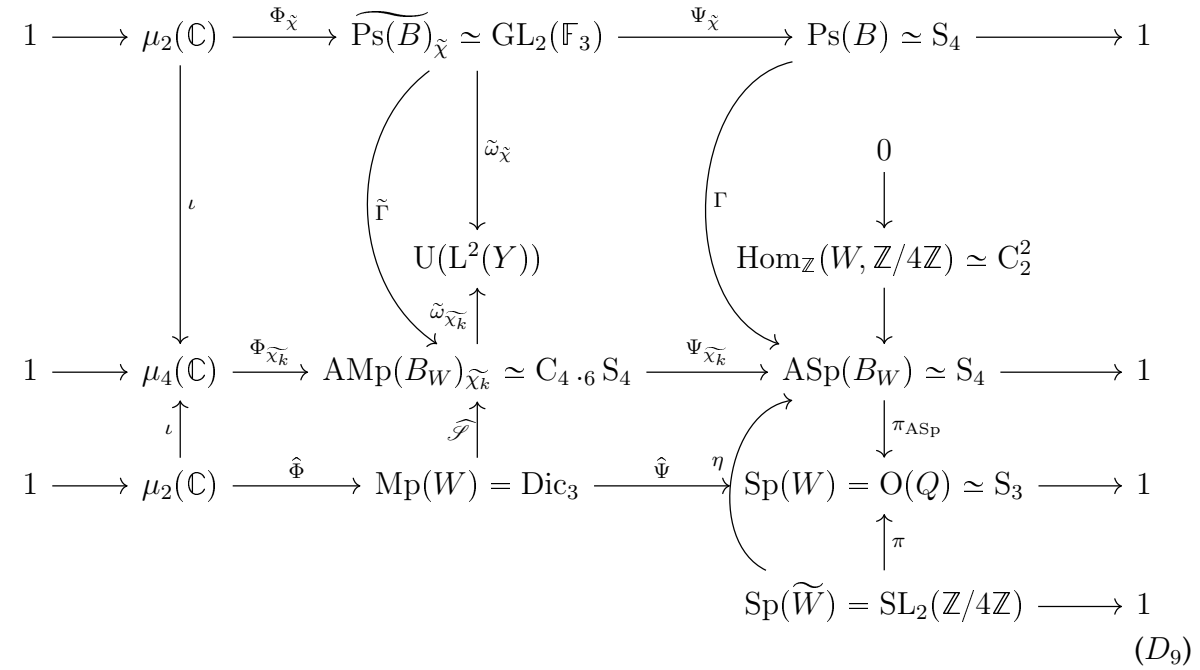
symplectique coïncident (cf. proposition 1.13). Comme le raisonnement effectué dans la section 3.2 est le même que celui de la section 3.1, nous ne soulignerons ici que les différences dans les résultats obtenus. Comme $\mathrm{O}(Q) = \mathrm{Sp}(W)$ dans ce cas, $\mathrm{O}(Q)$ est isomorphe à S_3. Alors le groupe pseudosymplectique $\mathrm{Ps}(B)$ et le groupe symplectique affine $\mathrm{ASp}(B_W)$ sont tous deux isomorphes au groupe symétrique S_4, d'ordre 24. Il y a deux possibilités pour le revêtement à deux feuillets $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, à savoir le groupe conforme spécial unitaire (ou groupe octaédral binaire) $\mathrm{CSU}_2(\mathbb{F}_3)$, ou le groupe linéaire général $\mathrm{GL}_2(\mathbb{F}_3)$ (cf. proposition 3.28). Rappelons qu'au deuxième chapitre, nous avons affirmé que cet exemple est le seul dans lequel nous pouvons définir à partir du cocycle c_{χ_k} un revêtement à deux feuillets $\mathrm{G}^0_{\widetilde{\chi_k}}$ de $\mathrm{ASp}(B_W)$, en plus du revêtement à quatre feuillets $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Dans la proposition 3.30, nous retrouvons pour $\mathrm{G}^0_{\widetilde{\chi_k}}$ les deux possibilités mentionnées précédemment pour $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Cela est cohérent avec le fait que $\mathrm{Ps}(B)$ et $\mathrm{ASp}(B_W)$ sont isomorphes. Dans cette perspective, la construction de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ perd son intérêt puisque $\mathrm{Ps}(B)$ définit déjà une projection sur $\mathrm{Sp}(W)$. Cependant, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ sera utile dans la suite. Nous trouvons deux possibilités pour ce revêtement $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$: il est soit isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, soit à $\mathrm{U}_2(\mathbb{F}_3)$. Remarquons que ce sont précisément les mêmes possibilités que pour B_{Weil}, ce qui est cohérent puisqu'il n'y a qu'un seul groupe de Heisenberg $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ à isomorphisme près. Les différences résident dans les expressions explicites des éléments de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ pour $B = B_{\mathrm{Weil}}$ ou $B \neq B_{\mathrm{Weil}}$. Ensuite, nous montrons dans la section 3.2.3 que les constructions de $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ et $\mathrm{G}^0_{\widetilde{\chi_k}}$ sont compatibles, au sens de la commutativité de (D_3), si et seulement si les groupes $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ et $\mathrm{G}^0_{\widetilde{\chi_k}}$ sont isomorphes. Par ailleurs, les constructions de $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ et $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ sont compatibles si et seulement si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$.

Concernant le revêtement métaplectique à deux feuillets $\mathrm{Mp}(\widetilde{W})$ de $\mathrm{Sp}(\widetilde{W})$ et sa représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, satisfaisant à la commutativité du diagramme (D_4), nous obtenons dans la section 3.2.4 des résultats similaires à ceux de la section 3.1.4. Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$, alors $\mathrm{Mp}(\widetilde{W})$ l'est également. Nous conservons l'injectivité de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, mais nous perdons le lien avec les travaux de Weil, dans le sens que $\widetilde{\Gamma}$ n'existe pas. Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, alors $\mathrm{Mp}(\widetilde{W})$ est isomorphe à $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. Nous conservons ainsi le lien avec les travaux de Weil, dans le sens que le morphisme $\widetilde{\Gamma}$ existe, mais nous perdons l'injectivité de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, puisque son image $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ est soit isomorphe à $\mathrm{CSU}_2(\mathbb{F}_3)$, soit à $\mathrm{GL}_2(\mathbb{F}_3)$.

La section 3.2.5 vise à fournir une représentation de Weil unitaire, irréductible et fidèle de la plus petite extension centrale possible de $\mathrm{Sp}(W)$ (au lieu de $\mathrm{Sp}(\widetilde{W})$) dont la construction est compatible avec celle de $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ si possible, ou $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ sinon, au sens des diagrammes commutatifs (D_8), (D_9) et (D_{10}) ci-dessous. Si $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ est isomorphe à $\mathrm{CSU}_2(\mathbb{F}_3)$, alors nous n'avons pas besoin de considérer $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, et le meilleur résultat consiste en une représentation de Weil $\widetilde{\omega}_{\widetilde{\chi}} \circ \widehat{\mathscr{S}}$ unitaire, irréductible et fidèle du revêtement à deux feuillets $\mathrm{Mp}(W) = \mathrm{Dic}_3$ telle que le diagramme (D_8) commute.

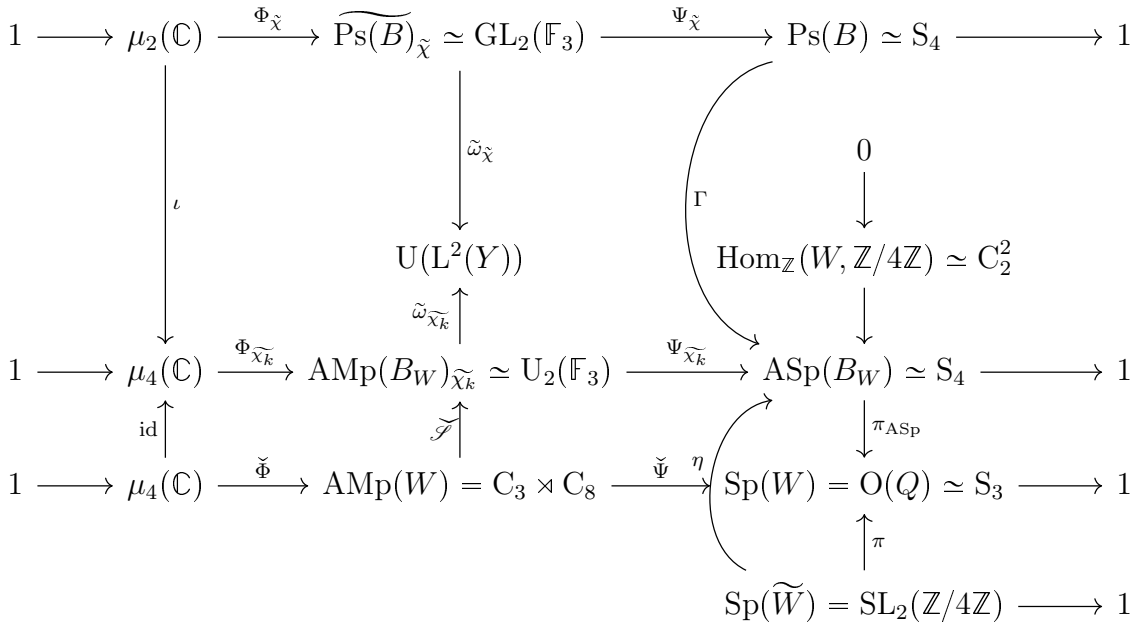


Si $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ est isomorphe à $\mathrm{GL}_2(\mathbb{F}_3)$, alors nous devons considérer l'extension de taille supérieure $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ de $\mathrm{ASp}(B_W)$ afin de relever la représentation de Weil à une extension centrale de $\mathrm{Sp}(W)$. Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, alors le meilleur résultat est une représentation de Weil $\widetilde{\omega}_{\tilde{\chi}} \circ \widetilde{\mathscr{S}}$ unitaire, irréductible et fidèle du revêtement à deux feuillets $\mathrm{Mp}(W) = \mathrm{Dic}_3$ de $\mathrm{Sp}(W)$ telle que le diagramme suivant commute :



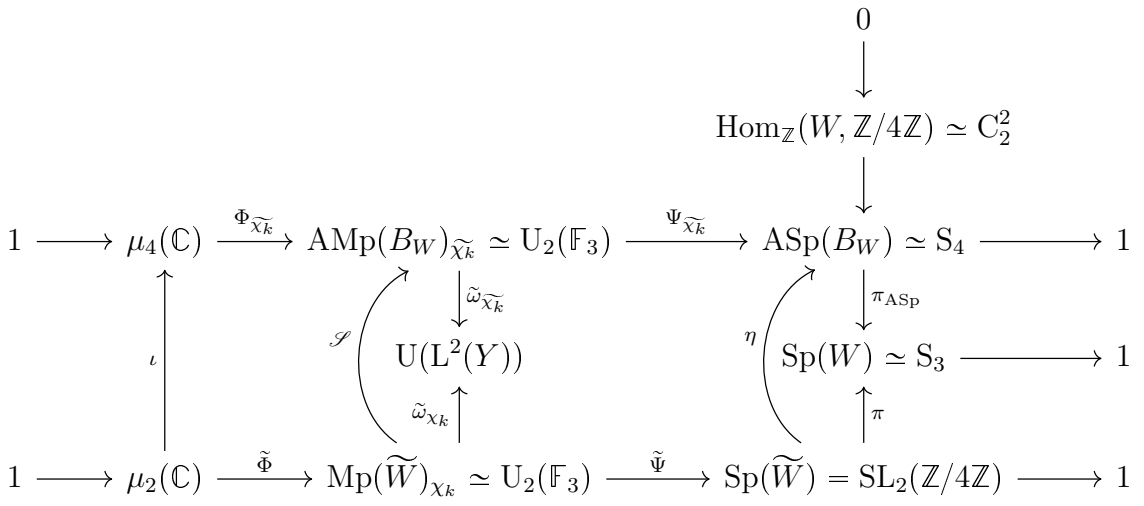
(D_9)

Si $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$, alors le meilleur résultat consiste en une représentation de Weil $\widetilde{\omega}_{\tilde{\chi}} \circ \widetilde{\mathscr{S}}$ unitaire, irréductible et fidèle du revêtement à quatre feuillets $\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8$ de $\mathrm{Sp}(W)$ telle que le diagramme suivant commute :



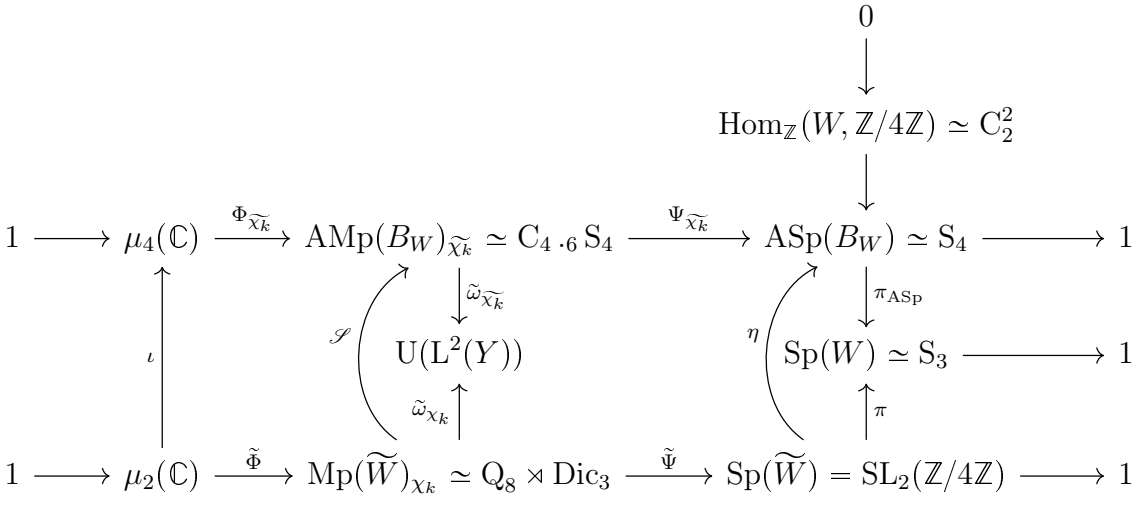
(D_{10})

La section 3.3 est consacrée à la présentation de la construction du revêtement à deux feuillets $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ de $\mathrm{Sp}(\widetilde{W})$ de la section 2.5, à partir du groupe de Heisenberg $\mathrm{H}(\widetilde{W})$ qui dans cet exemple est isomorphe au groupe $\mathrm{C}_4^2 \rtimes_4 \mathrm{C}_4$, d'ordre 64. La proposition (3.49) présente deux choix valables différents pour les ensembles $\{\widetilde{w}_{\widetilde{\sigma}}, \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ et $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ qui conduisent à deux possibilités pour $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. En dressant la liste explicite des éléments qui appartiennent à $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, nous montrons que ce groupe est soit isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$, soit à $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ (ce qui est cohérent avec les résultats des sections 3.1.4 et 3.2.4). En utilisant [GN] et le calcul de $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/2\mathbb{Z})$, nous montrons également que ces deux choix particuliers définissent en fait les deux seules possibilités pour le revêtement à deux feuillets $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ de $\mathrm{Sp}(\widetilde{W})$ (voir la remarque 3.50). Dans chacun de ces cas, nous relevons la représentation de Weil projective ω_{χ_k} de $\mathrm{Sp}(\widetilde{W})$ en la représentation de Weil linéaire $\widetilde{\omega}_{\chi_k}$ de $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. Cette représentation est unitaire et irréductible. De plus, elle est fidèle si $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$. En revanche, si $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ est isomorphe à $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$, alors cette représentation possède un noyau à deux éléments et son image $\widetilde{\omega}_{\chi_k}(\mathrm{Mp}(\widetilde{W})_{\chi_k})$ est isomorphe soit à $\mathrm{CSU}_2(\mathbb{F}_3)$, soit à $\mathrm{GL}_2(\mathbb{F}_3)$. Dans le cas où $B = B_{\mathrm{Weil}}$, nous concluons cette section par l'étude du morphisme d'extensions centrales $\mathscr{S}$ qui rend le diagramme (D_5) commutatif. Si $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ est isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$, alors la commutativité de (D_5) nécessite que $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ soit isomorphe à $\mathrm{U}_2(\mathbb{F}_3)$. Dans ce cas, $\mathscr{S}$ est un isomorphisme :



(D_{11})

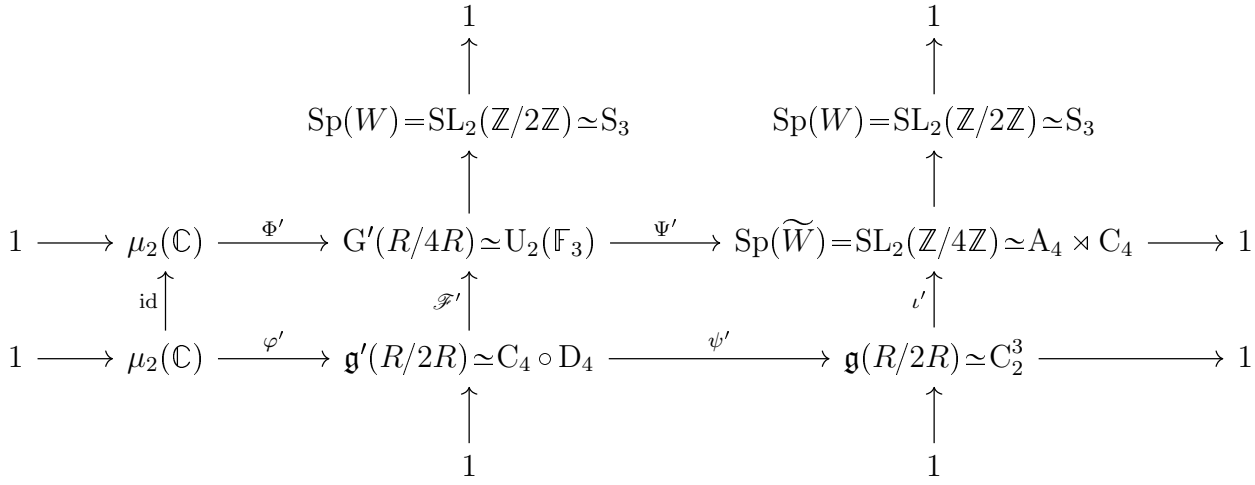
Si $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ est isomorphe à $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$, alors la commutativité de (D_5) impose que $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ soit isomorphe à $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$. Dans ce cas, l'image de $\mathscr{S}$ est d'indice deux dans $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$:



(D_{12})

Dans son article [Sav12], Savin établit une correspondance bijective entre certaines représentations d'un revêtement à deux feuillets $\mathrm{G}'(R/4R)$ de $\mathrm{G}(R/4R)$ et les représentations de $\mathrm{G}(R/2R)$, où $\mathrm{G}(A)$ désigne le groupe de Chevalley sur un anneau commutatif unitaire A correspondant à un système de racines simplement lacé Φ fixé (c'est-à-dire de type A, D ou E). Si R est l'anneau des entiers $\mathbb{Z}_2$ de $\mathbb{Q}_2$ et Φ est le système de racines A_1, alors $\mathrm{G}(R/2R)$ et $\mathrm{G}(R/4R)$ sont respectivement les groupes symplectiques $\mathrm{Sp}(W)$ et $\mathrm{Sp}(\widetilde{W})$ que nous manipulons dans notre exemple de dimension minimale. La section 3.4 établit un lien entre les résultats de Savin et ceux que nous obtenons dans cette thèse. Grâce au logiciel de calcul formel **GAP**, nous pouvons identifier dans la base de données [GN] les groupes qui apparais-

sent dans l'article de Savin et nous donnons une réalisation explicite du diagramme commutatif qu'il présente. Les codes **GAP** sont fournis dans l'annexe B.1.



(D_{13})

Remarquons que $\mathfrak{g}'(R/2R)$ est notre groupe de Heisenberg $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, et que $\mathrm{G}'(R/4R) \simeq \mathrm{U}_2(\mathbb{F}_3)$ correspond à l'une des deux possibilités pour le revêtement métaplectique à deux feuillets $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ de $\mathrm{Sp}(\widetilde{W})$. Etant donné un caractère fidèle χ du centre $\mathbb{Z}/4\mathbb{Z}$ de $\mathfrak{g}'(R/2R)$, Savin définit une représentation irréductible ρ_χ de $\mathfrak{g}'(R/2R)$ agissant comme χ sur son centre. D'après le théorème de Stone-von Neumann, nous savons que ρ_χ est isomorphe au modèle de Schrödinger $\rho_{\widetilde{\chi_k}}$ de la représentation de Heisenberg de $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, à condition que $\chi_k = \chi$. Savin étend ensuite ρ_χ en une représentation ρ'_χ du groupe $\mathrm{G}'(R/4R)$ et il souligne que cette extension n'est pas unique dans ce cas spécifique, comme nous pouvons le vérifier directement. Notre représentation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ est l'une des possibilités pour ρ'_χ. Enfin, en comptant le nombre de représentations irréductibles des groupes $\mathrm{G}(R/2R) = \mathrm{Sp}(W)$ et $\mathrm{G}'(R/4R) = \mathrm{Mp}(\widetilde{W})_{\chi_k}$, et en utilisant [GN], nous vérifions qu'il n'existe pas de correspondance bijective $\sigma \mapsto \sigma \otimes \rho'_\chi$ entre les classes d'isomorphisme de représentations irréductibles de $\mathrm{G}(R/2R) = \mathrm{Sp}(W)$ et les classes d'isomorphismes de représentations irréductibles de $\mathrm{G}'(R/4R) = \mathrm{Mp}(\widetilde{W})_{\chi_k}$ telles que le centre de $\mathfrak{g}'(R/2R)$ agit par le caractère χ. Il s'agit là du cas exceptionnel auquel le théorème 4.1 de Savin ne s'applique pas.

Le but du **quatrième chapitre** est d'établir un dictionnaire entre nos constructions de la représentation de Weil et le cadre de la théorie de l'information quantique. Seul un résumé des résultats obtenus est donné ici, et nous renvoyons le lecteur au début de ce chapitre pour une introduction plus précise et plus complète. L'information quantique circule à travers des circuits quantiques, et les qubits (bits quantiques) sont représentés par des vecteurs dans un espace de Hilbert complexe de dimension deux. Plus précisément, pour tout entier $n \geqslant 1$, un système composé de n qubits est représenté par un vecteur d'état dans un espace de Hilbert complexe H de dimension 2^n. Les manipulations sur un système de n qubits sont effectuées par des portes quantiques, qui sont des matrices unitaires complexes de taille 2^n. Le lien principal entre ce cadre et nos résultats est que les opérateurs définis par les représentations de

Weil sont des portes quantiques spécifiques, appelées portes de Clifford. La pertinence de ces portes est particulièrement évidente dans le contexte de la correction d'erreurs quantiques, et plus précisément dans le formalisme des stabilisateurs. Nous mentionnons que Heinrich [Hei21] étudie dans sa thèse les relations entre le formalisme des stabilisateurs et le cadre des représentations de Weil. Néanmoins, il soulève des difficultés techniques pour un corps fini de caractéristique deux et renvoie à l'article de Gurevich et Hadani [GH12] comme approche possible pour traiter ce cas. À la lumière de nos constructions explicites, nous clarifions certains aspects de ces relations pour $\mathbb{F} = \mathbb{F}_2$. Il existe un groupe de portes quantiques, appelé groupe de Pauli $\mathscr{P}_n$, dont les éléments forment une base de l'espace des matrices complexes unitaires de taille 2^n. Étant donné un code correcteur d'erreurs quantiques dont l'espace du code est un sous-espace vectoriel C de H de dimension 2^k, la capacité à corriger toute erreur qui se produit par l'action d'une porte de Pauli sur k qubits dans C nous permet de corriger une erreur arbitraire sur k qubits dans C. Après quelques préliminaires nécessaires sur le produit tensoriel des représentations dans la section 4.1, notre premier résultat dans la proposition 4.9 consiste à identifier le groupe de Pauli $\mathscr{P}_n$ et son sous-groupe $\mathscr{P}_n^{\mathbb{R}}$ (appelé groupe de Pauli réel) avec des sous-groupes des revêtements $\mathrm{AMp}(B_W)_{\widehat{\chi_k}}$ de $\mathrm{ASp}(B_W)$ et $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ de $\mathrm{Ps}(B)$, respectivement, lorsque W est un $\mathbb{F}_2$-espace vectoriel de dimension $2n$. Plus précisément, si $B = B_{\mathrm{Weil}}$, alors la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi}}$ fournit un isomorphisme explicite entre l'image réciproque dans $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ du sous-groupe de $\mathrm{Ps}(B)$ qui agit trivialement sur $\mathrm{O}(Q)$ et le groupe $\mathscr{P}_n^{\mathbb{R}}$. De manière similaire, nous montrons que si $B = B_{\mathrm{Weil}}$ ou si $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ (dans une base symplectique adaptée à la polarisation complète $W = X \oplus Y$), alors la représentation de Weil $\widetilde{\omega}_{\widehat{\chi_k}}$ fournit un isomorphisme explicite entre l'image réciproque dans $\mathrm{AMp}(B_W)_{\widehat{\chi_k}}$ du sous-groupe de $\mathrm{ASp}(B_W)$ qui agit trivialement sur $\mathrm{Sp}(W)$ et le groupe $\mathscr{P}_n$. En prenant appui sur cette correspondance, nous traduisons les générateurs des stabilisateurs de codes stabilisateurs bien connus en termes d'opérateurs de la représentation de Weil.

Le groupe de Clifford (infini) $\mathscr{C}_n$ est le groupe des opérations quantiques qui préservent le groupe de Pauli par conjugaison. Une porte de Clifford essentielle est connue sous le nom de transformation de Hadamard(-Rademacher-Walsh). Du point de vue de la théorie des représentations, cette transformation est la transformation de Fourier sur le groupe abélien obtenu comme produit tensoriel de n copies de $\mathbb{F}_2$. En information quantique, cette porte quantique est par exemple utilisée pour créer une superposition d'états quantiques ou pour effectuer des changements de bases afin de mieux détecter certaines erreurs. Ces propriétés en font un ingrédient fondamental de nombreux algorithmes quantiques, tels que ceux de Deutsch-Josza, Grover, Shor et de la transformation de Fourier quantique. Notre deuxième résultat est la proposition 4.11 : à l'intérieur des groupes $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, $\mathrm{AMp}(B_W)_{\widehat{\chi_k}}$ (à nouveau pour les deux possibilités mentionnées plus haut pour la forme bilinéaire B) et $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, nous déterminons les éléments dont l'opérateur correspondant par les représentations de Weil $\widetilde{\omega}_{\widetilde{\chi}}$, $\widetilde{\omega}_{\widehat{\chi_k}}$ et $\widetilde{\omega}_{\chi_k}$, respectivement, est exactement la porte de Hadamard.

Enfin, nous utilisons les calculs explicites et les identifications effectués dans le troisième chapitre lorsque $\widetilde{W}$ est un $\mathbb{Z}/4\mathbb{Z}$-module libre de rang deux et W est un $\mathbb{F}_2$-espace vectoriel de dimension deux, afin de sélectionner certains sous-groupes finis de portes de Clifford à un

qubit. Plus précisément, dans la section 4.2.3, nous écrivons en termes de portes quantiques les générateurs de tous les sous-groupes finis de portes de Clifford que nous obtenons grâce à notre construction des représentations de Weil. Ceci peut être utile pour les calculs quantiques, car les groupes finis nous permettent d'effectuer des calculs en nous appuyant sur des structures et des propriétés algébriques.

Pour conclure cette introduction, nous mentionnons quatre questions ouvertes qui découlent naturellement des résultats obtenus dans cette thèse :

- La construction de la représentation de Weil $\widetilde{\omega}_{\widetilde{\chi}}$ de $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ dans le premier chapitre est valable pour tout corps fini $\mathbb{F}$ de caractéristique deux. En revanche, les représentations de Weil $\widetilde{\omega}_{\widetilde{\chi_k}}$ de $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ et $\widetilde{\omega}_{\chi_k}$ de $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, qui nous permettent de retrouver le lien avec le groupe symplectique $\mathrm{Sp}(W)$ tout entier, ne sont présentées que pour $\mathbb{F} = \mathbb{F}_2$. Nous pouvons alors nous demander comment étendre les deux dernières constructions à un corps fini arbitraire de caractéristique deux. Une piste prometteuse mentionnée par Heinrich dans [Hei21] pourrait être d'utiliser, au lieu de l'anneau $\mathbb{Z}/4\mathbb{Z}$ pour $\mathbb{F}_2$, l'extension de Galois $\mathrm{G}(4, m)$ de degré m de l'anneau $\mathbb{Z}/4\mathbb{Z}$ pour un corps général $\mathbb{F} = \mathbb{F}_{2^m}$.

- La construction de la représentation de Weil n'est que la première étape nécessaire à des développements ultérieurs dans différents domaines des mathématiques. Nous avons déjà mentionné la correspondance de Howe (ou thêta), qui prend appui sur la représentation de Weil pour fournir une bijection entre certaines représentations admissibles irréductibles des images réciproques dans le groupe métaplectique $\mathrm{Mp}(W)$ de deux groupes qui forment une paire duale réductive dans le groupe symplectique $\mathrm{Sp}(W)$. Pour les corps finis, cette correspondance a diverses réincarnations. L'une d'elles est appelée correspondance eta, mais cette théorie est loin d'être complète. En caractéristique impaire, la représentation de Weil est une représentation linéaire du groupe $\mathrm{Sp}(W)$ lui-même, de sorte que les paires duales réductives sont à chercher dans le groupe symplectique. Pour développer cette théorie dans la caractéristique deux, nous devons d'abord rechercher les paires duales. À l'intérieur du groupe pseudosymplectique $\mathrm{Ps}(B)$, les paires duales ont été classifiées par Blasco dans [Bla93]. En revanche, à notre connaissance, la question de la détermination de telles paires dans le groupe affine symplectique $\mathrm{ASp}(B_W)$ et le groupe symplectique $\mathrm{Sp}(\widetilde{W})$ n'a pas encore été résolue. D'autre part, nous avons montré que des éléments qui commutent dans ces groupes ont des images réciproques qui ne commutent pas nécessairement dans les revêtements correspondants. Nous devons donc clarifier comment l'image réciproque d'une telle paire duale se comporte dans les revêtements $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ et $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, respectivement, et ce pour tous les revêtements possibles, car à l'heure actuelle nous ne savons pas combien il existe de revêtements non isomorphes en général pour les deux premiers.

- Pour le moment, nous avons seulement traduit une partie de nos résultats dans le langage de la théorie de l'information quantique afin de fournir une structure de groupe fini impliquant certaines portes quantiques pertinentes. Il serait intéressant d'approfondir cet axe de recherche à l'avenir et de déterminer si ces constructions explicites des représentations de Weil pourraient déboucher sur de nouvelles applications, par exemple dans le

cadre de la correction d'erreurs quantiques, comme le suggère notamment [MG21] pour le cas de caractéristique impaire.

- La récente prépublication de Fintzen et Schwein [FS25] souligne la pertinence de la représentation de Weil pour la construction des représentations supercuspidales des groupes p-adiques. Pour $p \neq 2$, des constructions générales sont connues. Parmi celles-ci, la construction de Yu [Yu01] s'appuie sur les représentations de Heisenberg-Weil via une induction compacte. Cette construction ne s'applique pas au cas $p = 2$ en raison de la division par deux dans la définition du groupe de Heisenberg donnée par Yu et parce que la représentation projective ne peut être linéarisée en une représentation du groupe pseudosymplectique. Dans [FS25], Fintzen et Schwein généralisent la construction de Yu, y compris dans le cas où $\mathbb{F} = \mathbb{F}_2$, en se limitant à des sous-groupes spécifiques pour lesquels cette linéarisation est possible. Trouver le lien entre leurs travaux et notre linéarisation explicite de la représentation projective de Weil pour le groupe pseudosymplectique, et éventuellement étendre leurs constructions au cas symplectique que nous étudions dans les chapitres 2 et 3, nous semble tout à fait naturel et constitue une direction de recherche intéressante à explorer.

Introduction (English version)

Introduction

In order to trace back the origin of the Weil representation, we first need to introduce the group which lies at its root, namely the Heisenberg group. In the early twentieth century, physicists made a series of experimental discoveries that could not be explained by classical physics, such as the black-body radiation, the photoelectric effect, the quantization of matter and the wave-particle duality. All the attempts made to explain these phenomena led to an entirely new facet of physics, today known as quantum mechanics. We will not dive into the fascinating story of the birth of quantum mechanics: we only mention that one of its very first breakthroughs was Planck's observation in 1900 that energy is not continuous. In fact, the energy E of a single photon of light is proportional to its frequency f:

$$E = hf, \quad \text{where } h \text{ is now known as Planck's constant.} \tag{1}$$

After some initial attempts to explain these quantum phenomena, it is around 1925 that two different formal theories emerged: the "matrix mechanics" and the "wave mechanics". The first theory was developed by Heisenberg while working on the spectral lines of hydrogen. In his famous paper [Hei25], he gave a formulation of quantum mechanics in terms of arrays that represent physical observables. In particular, he noticed that the arrays representing the position P and momentum Q did not satisfy the usual multiplication rules: they did not commute. The description of observables in terms of non-commuting operators is in fact a key feature of quantum mechanics. However, Heisenberg did not directly use matrices or operators in his work, and it is with the help of Born and Jordan that the complete formulation of the theory was provided in [BHJ25], taking into advantage the mathematical framework developed by Weyl. In this vision, a quantum system is given by a state vector and its evolution is described by matrix multiplications. This led to the formal canonical commutation relations (CCR):

$$[P, Q] = PQ - QP = i\frac{h}{2\pi}I. \tag{2}$$

In the meantime, Schrödinger developed the second aforementioned formulation of quantum mechanics, in which a quantum system is described by a wavefunction ψ and its evolution is governed by a partial differential equation, now known as Schrödinger's equation. In this setting, the position P and momentum Q can be defined as the following operators:

$$\hat{p}(\psi)(x,t) = x\psi(x,t) \quad \text{and} \quad \hat{q}(\psi)(x,t) = -i\frac{h}{2\pi}\frac{\partial\psi}{\partial x}(x,t) \quad \text{for all } x \in \mathbb{R} \text{ and } t \in \mathbb{R}. \tag{3}$$

In this way, we recover the relation

$$[\hat{p}, \hat{q}]\psi(x,t) = (\hat{p}\hat{q} - \hat{q}\hat{p})\psi(x,t) = i\frac{h}{2\pi}\psi(x,t). \tag{4}$$

This led to the formulation of the uncertainty principle, which states that there are pairs of physical observables (like position and momentum) that we cannot measure simultaneously with an arbitrary precision. Even if the formalisms of matrix mechanics and wave mechanics were completely different, later works of Schrödinger, Dirac, von Neumann and Stone (between 1926 and 1932) proved that they were in fact equivalent. The Stone-von Neumann theorem formalizes the idea that all the formulations of the canonical commutation relations are equivalent. This unification highlighted the importance of the use of a mathematical framework and paved the way for the development of quantum physics. To avoid technical considerations, we will not give more details on the Stone-von Neumann theorem in this context, but we shall precisely state it and prove it in this thesis in the language of representation theory, in the setting of finite fields.

The idea of non-commutativity arising from the canonical commutations relations can be formalized in a mathematical framework, and more precisely through group theory. In its standard description, the Heisenberg group is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $n \geqslant 1$, endowed with the following product:

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z' + \frac{1}{2}(x\cdot y' - y\cdot x')), \tag{5}$$

where $\cdot$ denotes the usual scalar product on $\mathbb{R}^n$. The Heisenberg group is a Lie group, and its Lie algebra has a basis consisting of elements X_i, Y_i and Z which satisfy

$$[X_i, Y_j] = \delta_{i,j}Z \quad \text{and} \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0 \quad \text{for all } 1 \leqslant i,j \leqslant n. \tag{6}$$

If we identify X_i and Y_i with position and momentum operators, respectively, then these relations formally recover an analog of the canonical commutation relations. Variations on the above definition lead to more general Heisenberg groups: we can for example replace $\mathbb{R}$ by some other fields, or we can define a Heisenberg group from a symplectic vector space, which is the situation we will focus on in this thesis.

If W is a nontrivial finite-dimensional vector space over a field $\mathbb{F}$ of characteristic different from two, and $\langle \cdot, \cdot \rangle$ is a nondegenerate and alternating bilinear form on W, then we define the Heisenberg group $\mathrm{H}(W)$ as the set $W \times \mathbb{F}$, endowed with the following product:

$$(w,t)(w',t') = (w+w', t+t' + \frac{1}{2}\langle w, w' \rangle). \tag{7}$$

The Heisenberg group plays a prominent role in so many different areas of mathematics that a comprehensive list is beyond the scope of this introduction. Therefore, we only focus on the aspect most relevant to our work, namely the influence of this group in representation theory; see e.g. [How80]. More precisely, the Stone-von Neumann theorem states that

the Heisenberg group has, up to isomorphism, a unique unitary irreducible representation which acts on its center as a fixed nontrivial character. This is the cornerstone for the usual construction of the Weil representation.

In its most general form, the Weil representation (which is also called the Segal-Shale-Weil, oscillator, or metaplectic representation) was first studied by Weil in 1964 [Wei64]. He refers to the earlier works of Segal and Shale for the real case. Weil's work starts from a Heisenberg group which is not exactly the one defined by (7). In fact, observe that this definition completely excludes the fields of characteristic two. Since Weil treated the case of local fields and adelic rings with arbitrary underlying characteristic, the definition of the Heisenberg group had to be generalized.

Let $W \neq \{0\}$ denote a finite-dimensional symplectic vector space over a finite field $\mathbb{F}$, and let $W = X \oplus Y$ be a complete polarization of W (which means that X and Y are two supplementary subspaces of W of maximal dimension such that $\langle \cdot, \cdot \rangle$ vanishes on $X \times X$ and $Y \times Y$, i.e. two Lagrangian subspaces). We set

$$B_{\text{Weil}}(w, w') = \langle x, y' \rangle \quad \text{for all } w = x + y \text{ and } w' = x' + y' \text{ in } W = X \oplus Y. \tag{8}$$

Then B_{Weil} is a bilinear form B such that

$$\langle w, w' \rangle = B(w, w') - B(w', w) \quad \text{for all } w, w' \in W. \tag{9}$$

We define the Heisenberg group $\mathrm{H}(B)$ as the set $W \times \mathbb{F}$ endowed with the product

$$(w, t)(w', t') = (w + w', t + t' + B(w, w')). \tag{10}$$

(Notice that this group is isomorphic to the group $\mathrm{H}(W)$ if the characteristic of $\mathbb{F}$ is different from two. So this is really an extension of the first definition.) This is Weil's Heisenberg group when $B = B_{\text{Weil}}$. Weil constructs the pseudosymplectic group $\mathrm{Ps}(B_{\text{Weil}})$, which is a specific subgroup of the group of automorphisms of $\mathrm{H}(B_{\text{Weil}})$ acting trivially on its center. Without explicitly mentioning it by name, Weil uses an analog of the Stone-von Neuman theorem in his setting in order to construct a projective unitary representation of $\mathrm{Ps}(B_{\text{Weil}})$. This projective representation can be lifted to a linear representation of a central extension Mp of $\mathrm{Ps}(B_{\text{Weil}})$ by the unit circle. For a local field $\mathbb{F}$ whose characteristic is different from two, the pseudosymplectic group $\mathrm{Ps}(B_{\text{Weil}})$ is isomorphic to the symplectic group $\mathrm{Sp}(W)$ of linear transformations of W which preserve the symplectic form $\langle \cdot, \cdot \rangle$. If $\mathbb{F} = \mathbb{C}$ then the extension Mp splits, and if $\mathbb{F} \neq \mathbb{C}$ then the extension Mp can be reduced to a two-fold covering of $\mathrm{Sp}(W)$. In the case of characteristic two, the situation is completely different. For example, $\mathrm{Ps}(B_{\text{Weil}})$ projects only onto the orthogonal group O, defined as the group of linear transformations σ of W satisfying $B_{\text{Weil}}(\sigma(w), \sigma(w)) = B_{\text{Weil}}(w, w)$ for all $w \in W$. Thus the extension Mp has no direct relation with the entire symplectic group $\mathrm{Sp}(W)$.

Weil's pioneering work was first directed towards applications in the theory of theta functions and automorphic forms. Nevertheless, it deeply influenced other areas of mathematics and physics. We mention here only the Howe (or theta) correspondence, which is certainly one of the main motivations behind our work. Consider a local field $\mathbb{F}$ of characteristic different from two. Two reductive subgroups G and G' of $\mathrm{Sp}(W)$ that are mutual centralizers in $\mathrm{Sp}(W)$, are said to form a reductive dual pair (G, G'). Let $\widetilde{G}$ and $\widetilde{G}'$ be the preimages of G and G',

respectively, in the metaplectic group. The core idea developed by Howe is that the restriction of the smooth Weil representation to $\widetilde{G}\widetilde{G}'$ provides a bijective correspondence between some irreducible admissible representations of $\widetilde{G}$ and some irreducible admissible representations of $\widetilde{G}'$. If $\mathbb{F} = \mathbb{R}$, the metaplectic group is the unique connected two-fold covering of $\mathrm{Sp}(W)$, and the Weil representation does not factor to a representation of $\mathrm{Sp}(W)$.

We now turn to the object of this thesis, and quote Weil himself when he refers to the potential interest in extending his construction to the setting of finite groups:

> "[…] qu'il me soit permis, en passant, de signaler l'intérêt qu'il y aurait peut-être à examiner de plus près, du point de vue de la présente théorie, le cas des groupes finis."
>
> – André Weil [Wei64]

The line of research suggested by the above quotation has been fruitful in the case of finite fields of odd characteristics. We mention for example the works of Howe [Howa], [How73], Gérardin [Ger77], Duflo [Duf], Ranga Rao [Ran93], Neuhauser [Neu02] and Thomas [Tho08], [Tho13]. A geometric analog of the Weil representation has been provided by Gurevich-Hadani [GH08]. The more recent work [AP14] of Aubert and Przebinda presents a uniform construction (with explicit formulas) in the case in which $\mathbb{F} = \mathbb{R}$ or $\mathbb{F}$ is a p-adic field with $p \neq 2$ or $\mathbb{F}$ is a finite field of odd characteristic.

The research conducted in the framework of finite fields of characteristic two did not reach the same level of satisfaction. In fact, recall that in characteristic two, Weil's pseudosymplectic group is an extension of the orthogonal group of a particular bilinear form satisfying (9), and this is a proper subgroup of the symplectic group (except for one case, see Proposition 1.13 in this thesis). Another difference from the odd characteristic case lies in the definition (10) of the Heisenberg group itself. In fact, (7) provides a group whose product law is entirely defined by the symplectic form of W, whereas (10) depends on the choice of a bilinear form B satisfying (9). Since B_{Weil} is not the only such bilinear form, this raises the question of the number of non-isomorphic Heisenberg groups, and hence the number of non-equivalent Weil representations we obtain. In the literature, there are two main approaches to the construction of the Weil representation in characteristic two. We briefly describe them here. Further details will be given later in this introduction, where we will clarify how these works relate to our results. The first approach is due to Blasco [Bla93]. She follows the work of Weil in order to construct a Weil representation of a two-fold covering of the pseudosymplectic group, and classifies the dual pairs in the pseudosymplectic group. Her starting point is the Heisenberg group (10) for any bilinear form B satisfying (9). The second path, taken by Gurevich-Hadani [GH12] and Genestier-Lysenko [GL12], aims to provide a geometric analog of the Weil representation in characteristic two and to overcome the missing link between the Weil representation and the symplectic group. The idea for recovering the link with the symplectic group already appears in the works of Griess [Gri73], Glasby [Gla95] and Schmid [Sch00] in their study of extensions of extraspecial 2-groups. We will describe this idea more precisely when presenting the results of the second chapter. We only add here that the theory of extraspecial p-groups is a key ingredient to settle the question of the number of non-isomorphic Heisenberg groups defined

by (10) for a prime field $\mathbb{F}$ of characteristic p. This thesis bridges both approaches. Our goal is twofold: to provide explicit formulas in the construction of the Weil representation for a finite field of characteristic two, and, simultaneously, to recover a connection with the symplectic group.

We now describe the results obtained in this thesis, which is divided into four chapters. The first two chapters provide three different explicit constructions of the Weil representation. The first chapter concerns a Weil representation related to the pseudosymplectic group. This construction is valid for an arbitrary finite field of characteristic two. The other two constructions, which allow us to recover the link with the symplectic group for $\mathbb{F} = \mathbb{F}_2$, are studied in the second chapter. In the third chapter, we present the results of the first two chapters in the lowest-dimensional case of a symplectic $\mathbb{F}_2$-vector space W. Working in low dimension enables us to exhibit explicitly all the groups, central extensions and representations obtained this way. Finally, the last chapter aims to translate some of our results into the language of Quantum Information Theory.

In the **first chapter,** we consider a symplectic $\mathbb{F}$-vector space $(W, \langle \cdot, \cdot \rangle)$ of dimension $2n$, where $\mathbb{F}$ is an arbitrary finite field of characteristic two and $n \geqslant 1$. Given a bilinear form B satisfying (9), we define the Heisenberg group $\mathrm{H}(B)$, according to (10). Building on the theory of extraspecial groups reviewed in appendix A, we state in Proposition 1.6 that for $\mathbb{F} = \mathbb{F}_2$, there are exactly two such Heisenberg groups up to isomorphism. We fix a nontrivial character χ of $\mathbb{F}$ and extend it to a character $\widetilde{\chi}$ of a maximal abelian subgroup of $\mathrm{H}(B)$. Following Weil, we define the pseudosymplectic group $\mathrm{Ps}(B)$ as a specific subgroup of the automorphism group of $\mathrm{H}(B)$ which acts trivially on its center, and we describe the structure of $\mathrm{Ps}(B)$ in Proposition 1.12. In particular, $\mathrm{Ps}(B)$ projects onto the orthogonal group $\mathrm{O}(Q)$, where Q is the quadratic form associated to B, i.e. $Q(w) = B(w, w)$ for all $w \in W$. Then, we state and give a short proof of the Stone-von Neumann theorem in this setting (Theorem 1.23), even if we do not need it to construct the Weil representation. In fact, since we compute the character of the Schrödinger model $\rho_{\widetilde{\chi}}$ of the Heisenberg representation in Lemma 1.17, and since this representation is unitary and irreducible, Schur's lemma allows us to directly deduce the existence and unicity (up to a unitary multiple) of the intertwining operators we need to construct the projective Weil representation of $\mathrm{Ps}(B)$. Our next step is to provide in Proposition 1.28 and Corollary 1.31 explicit formulas for this projective Weil representation $s \mapsto \omega_{\widetilde{\chi}}(s, w_s)$, where $s = (\sigma, f) \in \mathrm{Ps}(B)$ and $w_s \in W$ satisfies (37), and for its associated cocycle c_χ. Observe that we work with a family of sections of this projective representation parametrized by constants $\gamma(\sigma)$; see (38). This allows additional flexibility in the definition of the unitary operators $\omega_{\widetilde{\chi}}(s, w_s)$. Since one of our goals is to lift this projective representation to a linear representation of the smallest possible central extension of $\mathrm{Ps}(B)$, we first prove that the cocycle c_χ is not a coboundary (Corollary 1.33). For finite fields of odd characteristics, the Weil representation is defined as a linear representation of the symplectic group itself. In contrast, in characteristic two, there is no section of the projective Weil representation of $\mathrm{Ps}(B)$ which defines a linear representation of the same group. Therefore, since we prove in Corollary 1.35 that for all choices of w_s and $\gamma(\sigma)$, the smallest possible range of the cocycle c_χ is $\mu_2(\mathbb{C}) = \{\pm 1\}$, the best we can do is to linearize the projective Weil representation on a two-fold covering of $\mathrm{Ps}(B)$. We proceed to the construction of such two-fold covering

$\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ in Proposition 1.36. Even if the central extension $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ by $\mu_2(\mathbb{C})$ depends a priori on the choices $\widetilde{\chi}$, w_s and $\gamma(\sigma)$, we give a necessary and sufficient condition (50) on $\gamma(\sigma)$ to ensure that different such choices define in fact isomorphic extensions, and we exhibit a corresponding isomorphism. This enables us to lift $s \mapsto \omega_{\widetilde{\chi}}(s, w_s)$ to a unitary and faithful linear representation $\widetilde{\omega}_{\widetilde{\chi}}$ of the group $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ in Proposition 1.39. This construction may be visualized through the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} & \xrightarrow{\Psi_{\widetilde{\chi}}} & \mathrm{Ps}(B) & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Ps}}} & & \\ & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{O}(Q) & & \end{array} \qquad (D_1)$$

We highlight that, in constrast to the cases of odd characteristic, the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is irreducible. Its image $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ contains a copy of the Heisenberg group $\mathrm{H}(B)$ if and only if $\mathbb{F} = \mathbb{F}_2$. The character of $\widetilde{\omega}_{\widetilde{\chi}}$ is then computed in Corollary 1.41. Finally, provided one condition on the parameter $\gamma(\sigma)$ is met, we prove in Proposition 1.46 that the extension $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ does not depend on the chosen character χ of $\mathbb{F}$, and we prove in Corollaries 1.45 and 1.47 that the Weil representation of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ does not depend on the choices made for χ, $\widetilde{\chi}$, w_s and $\gamma(\sigma)$.

Our work is inspired by Blasco [Bla93], and we use the methods of Glasby [Gla95]. However, there are several important differences to notice. Firstly, all these works consider an arbitrary bilinear form B to define the Heisenberg group $\mathrm{H}(B)$. However, Glasby only treats the case $\mathbb{F} = \mathbb{F}_2$. On the other hand, our construction, as well as Blasco's, is valid for an arbitrary finite field of characteristic two. Secondly, both Blasco and Glasby use the Stone-von Neumann theorem to provide a formula for $\omega_{\widetilde{\chi}}(s, w_s)$ which is defined up to a sign, depending on the choice of w_s. We do not use this theorem and we clarify the dependence of $\omega_{\widetilde{\chi}}(s, w_s)$ on the choice of w_s. Blasco gives a general inexplicit formula for $\omega_{\widetilde{\chi}}(s, w_s)$ involving a choice of a complete polarization of W and the action of s on this polarization. She also provides an explicit formula for $\omega_{\widetilde{\chi}}(s, w_s)$, but only on the proper subset of the $s = (\sigma, f) \in \mathrm{Ps}(B)$ such that $1 - \sigma$ is invertible. In contrast, our formula is valid on the whole group $\mathrm{Ps}(B)$ and for any finite field of characteristic two. Moreover, Blasco fixes an explicit section of the projective Weil representation and Glasby fixes two of them. In contrast, our parameters $\gamma(\sigma)$ in the definition (38) allow us to work without fixing any such explicit section. This way, we can examine all the possibilities for the range of the cocycle c_χ and the coverings it provides. More precisely, we find necessary and sufficient conditions on $\gamma(\sigma)$ to define a two-fold covering of $\mathrm{Ps}(B)$. Finally, Blasco first defines an extension of $\mathrm{Ps}(B)$ by $\mathbb{C}^\times$, and then obtains theoretically the existence of a two-fold covering of $\mathrm{Ps}(B)$ by embedding the vector space W in a symplectic vector space twice as big as W. In constrast, we use the explicit expression of the cocycle c_χ in order to prove that its minimal range is $\{\pm 1\}$. An advantage of an explicit cocycle is that we can compute the product of any two elements in the covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, since the product in this group entirely depends on c_χ.

The first goal of the **second chapter** is to construct a Weil representation of a covering of an extension of the symplectic group $\mathrm{Sp}(W)$. To this extent, we suppose that $\mathbb{F} = \mathbb{F}_2$ and

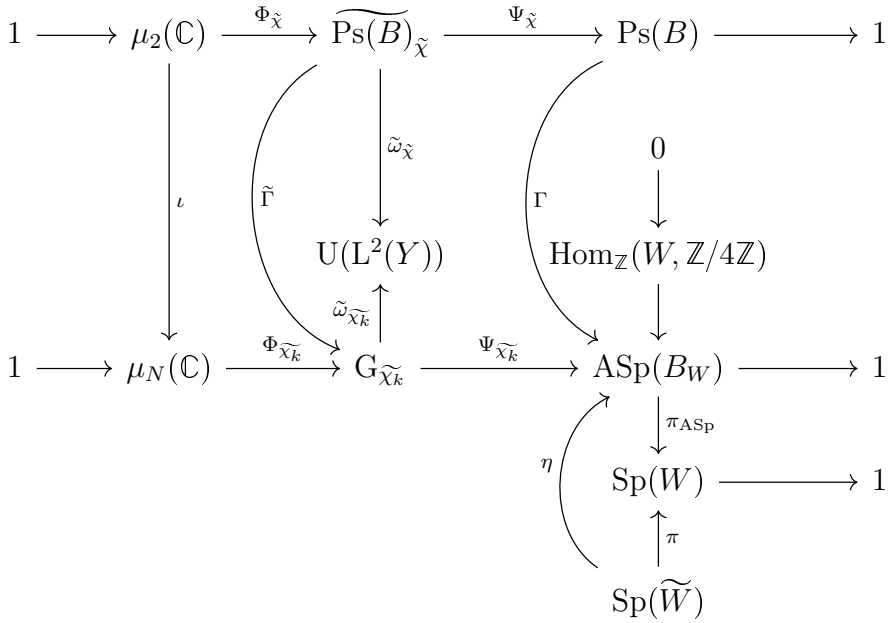
we consider a free $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ of rank $2n$, where $n \geqslant 1$. The setting of the first chapter can be recovered by looking at $\widetilde{W}/2\widetilde{W}$, which is a symplectic $\mathbb{F}_2$-vector space of dimension $2n$ (cf. Lemma 2.1). In sections 2.1 to 2.3, the idea is to provide constructions parallel to those of the first chapter. The bilinear form B is replaced by B_W defined before (77), and we use it to define the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ by (79). This is Glasby's [Gla95] Heisenberg group. Gurevich and Hadani use this definition only for $B = B_{\mathrm{Weil}}$ in [GH12], but we can prove that there is only one group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ up to isomorphism, thanks to the theory of almost extraspecial groups explained in appendix A. The pseudosymplectic group $\mathrm{Ps}(B)$ is replaced by the affine symplectic group $\mathrm{ASp}(B_W)$, which is the group of automorphisms of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ acting trivially on its center. The group $\mathrm{ASp}(B_W)$ contains $\mathrm{Ps}(B)$, see Proposition 2.6, and it projects onto the symplectic group $\mathrm{Sp}(W)$. We study its structure in Proposition 2.7. Then, we fix a faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$, we extend it to a maximal abelian subgroup of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, we give a short proof of the Stone-von Neumann theorem (Theorem 2.16) in this setting, and we describe the Schrödinger model $\rho_{\widetilde{\chi_k}}$ of the Heisenberg representation. We use the same method as in the first chapter to find in Proposition 2.21 and Corollary 2.24 explicit formulas for the projective Weil representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ of $\mathrm{ASp}(B_W)$, where w_s satisfies (100), and for the associated cocycle c_{χ_k}. In particular, we also leave some freedom to choose the parameters $\gamma(\sigma)$ in (103), to avoid fixing any specific section. After that, we prove in Corollary 2.26 that the cocycle c_{χ_k} is not a coboundary. As for $\mathrm{Ps}(B)$, this means that there is no section of the projective Weil representation of $\mathrm{ASp}(B_W)$ which allows us to define a linear representation of the same group. We then study the range of the cocycle c_{χ_k} in Corollary 2.28. We prove that, regardless of the choices made for w_s and $\gamma(\sigma)$, the smallest possible range of c_{χ_k} is $\mu_2(\mathbb{C}) = \{\pm 1\}$ if the Witt index of Q is zero (or equivalently, if Q has Arf invariant 1 and $\dim_{\mathbb{F}_2}(W) = 2$), and $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$ otherwise. We provide necessary and sufficient conditions on $\gamma(\sigma)$ in order to obtain these ranges. To be completely rigorous, we have to mention that the parameter $\gamma(\sigma)$ is not the most general parameter we could use. In particular, we did not prove that Corollary 2.28 provides the best possible result in general. For example, we obtain a smaller range for c_{χ_k} when $\dim_{\mathbb{F}_2}(W) = 2$ and the Arf invariant of Q is zero (cf. Remarks 2.22, 2.36 and 3.8). This last result on the range of c_{χ_k} enables us to construct in Proposition 2.29 a four-fold covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$, and also a two-fold covering $\mathrm{G}^0_{\widetilde{\chi_k}}$ if Q has Witt index zero. We give a necessary and sufficient condition (143) on $\gamma(\sigma)$ to ensure that different choices of $\widetilde{\chi_k}$, w_s and $\gamma(\sigma)$ define isomorphic extensions of $\mathrm{ASp}(B_W)$, and we provide a corresponding isomorphism. We then lift the map $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ to a unitary and faithful linear representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of the corresponding covering in Proposition 2.32. This amounts to the following diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_N(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{G}_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & \overset{\eta}{\curvearrowleft} & \mathrm{Sp}(W) & & \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) & &
\end{array}
\qquad (D_2)
$$

where $\mathrm{G}_{\widetilde{\chi_k}}$ denotes the covering $\mathrm{G}^0_{\widetilde{\chi_k}}$ if $N = 2$ (in which case the Witt index of Q is zero), and

$\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ if $N = 4$. Here again, the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ is irreducible, and its image $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi_k}})$ contains a copy of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Corollary 2.33 provides the character of $\widetilde{\omega}_{\widetilde{\chi_k}}$. To end the analogy with the first chapter, we discuss in Corollary 2.34 and Proposition 2.35 some conditions on the parameter $\gamma(\sigma)$ to ensure that the coverings of $\mathrm{ASp}(B_W)$ and the Weil representations we construct do not depend on the choices made for χ_k, $\widetilde{\chi_k}$, w_s and $\gamma(\sigma)$. Despite resulting in a larger extension than in the case of $\mathrm{Ps}(B)$, this construction has the advantage to be connected to the symplectic group $\mathrm{Sp}(W)$ through $\mathrm{ASp}(B_W)$. In this part, we use the methods of Glasby [Gla95] but we obtain more general results, since we do not fix any particular section of the projective representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$. We also recover, as proposed by Gurevich and Hadani [GH12], a four-fold covering of $\mathrm{ASp}(B_W)$, in a very explicit form. Furthermore, observe that neither [Gla95] nor [GH12] addresses the questions of the minimal extension size, or the existence of non-isomorphic coverings of $\mathrm{ASp}(B_W)$. We address these questions and provide precise answers in some cases.

Another question we answer in section 2.4 is the compatibility of the two aforementioned constructions of the Weil representations, in the sense of the existence of a group homomorphism $\widetilde{\Gamma}$ such that the following diagram commutes:



(D_3)

In other words, we examine whether the constructions made at the beginning of the second chapter really extend Weil's work. If the Witt index of Q is zero, we give in Proposition 2.39 a necessary and sufficient condition on the choices made for $\gamma(\sigma)$ in the constructions of $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ and $\mathrm{G}^0_{\widetilde{\chi_k}}$ so that the diagram (D_3) commutes. We give a similar condition for the commutativity of this diagram when $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and W is a two-dimensional vector space. However, if the Witt index of Q is greater than one, regardless of the choices made in the first two constructions, we prove that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is not compatible with $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$: the diagram (D_3) does not commute. Finally, we show that attaining this compatibility between the two constructions of the Weil representations requires defining an eight-fold covering of $\mathrm{ASp}(B_W)$

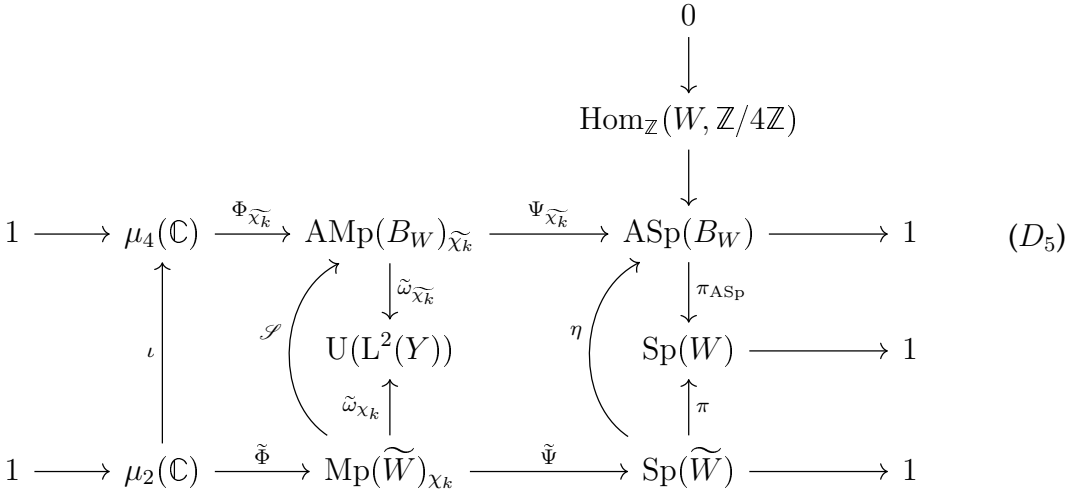
rather than a four-fold covering. In Proposition 2.29, we construct the appropriate eight-fold covering of $\mathrm{ASp}(B_W)$. This opens a new perspective in the theory of the Weil representation: depending on our objective, we have to decide whether to define the Weil representation of a smallest possible extension of $\mathrm{ASp}(B_W)$, or to define a Weil representation which is compatible with that of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$.

In connection with Gurevich-Hadani's work [GH12], an additional question addressed in section 2.5 is to provide explicit constructions of the two-fold metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$, and of the "splitting homomorphism" $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ that would allow the authors to extend the Weil representation to the group $\mathrm{Mp}(\widetilde{W})$. Our first remark is that in order to obtain a suitable extension of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to such two-fold covering $\mathrm{Mp}(\widetilde{W})$, we would expect that the extended representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ inherits the properties of $\widetilde{\omega}_{\widetilde{\chi_k}}$, in particular its injectivity. However, a combinatorial argument stated in Lemma 2.42 shows that this injectivity is impossible for $n \geqslant 2$. Therefore, we can only hope to construct a two-fold covering $\mathrm{Mp}(\widetilde{W})$ and a homomorphism of central extensions $\mathscr{S}$ such that the following diagram (D_4) commutes.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) & & \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \mathscr{S}} & \overset{\eta}{\curvearrowleft} & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & & & & \mathrm{Sp}(W) & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W}) & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) & \longrightarrow & 1
\end{array}
\qquad (D_4)
$$

As a matter of fact, we learned from Lysenko that there is a gap in [GH12], which was confirmed by Gurevich in a conversation with Przebinda. Since we provide a counterexample to the argument used in [GH12] (cf. Lemma 2.44), the question of the existence of $\mathrm{Mp}(\widetilde{W})$ and $\mathscr{S}$ was open, and this thesis answers it. The path we follow leads us to a new and independent construction of the Weil representation. In fact, we start by defining a new Heisenberg group $\mathrm{H}(\widetilde{W})$ by (196). The motivation behind this definition is to recover the setting of characteristic different from two, in the sense that $\mathrm{H}(\widetilde{W})$ is a group on which $\mathrm{Sp}(\widetilde{W})$ acts by automorphisms. After fixing a faithful character of $\mathbb{Z}/4\mathbb{Z}$, we use a method which is similar to the first two constructions. This way, we define a projective Weil representation $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ of $\mathrm{Sp}(\widetilde{W})$, and derive explicit formulas for this representation and for the associated cocycle $\widetilde{c}_{\chi_k}$ in Proposition 2.56 and Corollary 2.58. We underline the fact that the parameter $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ used in (215) encompasses all the possible sections of $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$. Again, we prove that this projective representation cannot be lifted to a linear representation of $\mathrm{Sp}(\widetilde{W})$ itself, since $\widetilde{c}_{\chi_k}$ is not

a coboundary (cf. Corollary 2.59). However, we prove in Corollary 2.61 that the restriction of the cocycle to a maximal parabolic subgroup of $\mathrm{Sp}(\widetilde{W})$ is a coboundary. Later, Corollary 2.62 addresses the question of the range of $\widetilde{c}_{\chi_k}$: we are able to find a necessary and sufficient condition on the parameter $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ so that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$. Improving from $\mu_4(\mathbb{C})$ to $\mu_2(\mathbb{C})$ out of the explicit expression of $\widetilde{c}_{\chi_k}$ seems much more difficult. Nevertheless, we are able to prove that the smallest possible range of $\widetilde{c}_{\chi_k}$ is in fact $\mu_2(\mathbb{C})$. If the rank $2n$ of the free $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ equals two, then we explicitly find the two possible non-cohomologous choices for $\widetilde{c}_{\chi_k}$ for which the range is $\mu_2(\mathbb{C})$ (see section 3.3). If $n \geqslant 2$, we use the cohomological result $\mathrm{H}_2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ of Funar and Pitsch [FP20] to derive the existence of a choice of $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ such that $\widetilde{c}_{\chi_k}$'s range is $\mu_2(\mathbb{C})$. This allows us to construct the corresponding metaplectic two-fold coverings $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ in Proposition 3.20. More precisely, there are exactly two possibilities for $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ if $n = 1$, and a unique two-fold covering for $n \geqslant 3$ because $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. This is a crucial result since it yields a true analog of the metaplectic group in the real case. (This also shows that, up to a coboundary, the choice of $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ which ensures that $\widetilde{c}_{\chi_k}$'s range is $\mu_2(\mathbb{C})$, is unique.) Thereafter in Proposition 2.65, we lift the projective representation $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ of $\mathrm{Sp}(\widetilde{W})$ to a linear unitary irreducible representation $\widetilde{\omega}_{\chi_k}$ of the metaplectic group $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. Except for one low-dimensional exception, this representation is not faithful, but we still call it the Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ because its construction arises in a similar way to the previous two. We compute its character in Corollary 2.67, and we examine the links between the coverings $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ and the Weil representations for the two choices of characters χ_k in Proposition 2.68. At this stage, we have already answered the question of the explicit construction of a Weil representation of a two-fold covering of the symplectic group $\mathrm{Sp}(\widetilde{W})$, by an approach which is completely independent of the first two constructions described above. The end of section 2.5 elucidates the question of the existence of the homomorphism $\mathscr{S}$ of central extensions for $B = B_{\mathrm{Weil}}$. More precisely, Proposition 2.73 provides a necessary and sufficient condition on the choices of parameters $\gamma(\sigma)$ for $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ for $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ ensuring the existence (and uniqueness) of the homomorphism of central extensions $\mathscr{S}$ making the following diagram commute:



This sheds some light on the claims of Gurevich and Hadani.

The **third chapter** aims to present the results obtained in the first two, in the lowest-dimensional case in which $\widetilde{W}$ is a free $\mathbb{Z}/4\mathbb{Z}$-module of rank two and W is a two-dimensional $\mathbb{F}_2$-vector space. Given the small size of these spaces, we are able to compute the elements of all the groups involved and the operators of the Weil representations we have constructed. Simultaneously, we identify these groups in the database [GN] (in each case with an explicit isomorphism) and the commutative diagrams which arise in this setting. An exhaustive list of these groups, with a presentation and the SmallGroup identification, is provided in appendix B.2. We fix the complete polarization $W = X \oplus Y$, where $X = \mathrm{Span}\{(1,0)\}$ and $Y = \mathrm{Span}\{(0,1)\}$. As already mentioned in the summary of the first chapter, up to isomorphism there are exactly two Heisenberg groups $\mathrm{H}(B)$. More precisely, $\mathrm{H}(B)$ has order eight and is isomorphic to the dihedral group D_4 or to the quaternionic group Q_8. The first two sections 3.1 and 3.2 study these two possibilities separately. Just before specializing to each of these two specific cases, we exhibit in Proposition 3.1 the lists of elements of $\mathrm{Sp}(W) = \mathrm{SL}_2(\mathbb{F}_2)$ and $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$, which are respectively isomorphic to the symmetric group S_3, of order 6, and the group $\mathrm{A}_4 \rtimes \mathrm{C}_4$, of order 48.

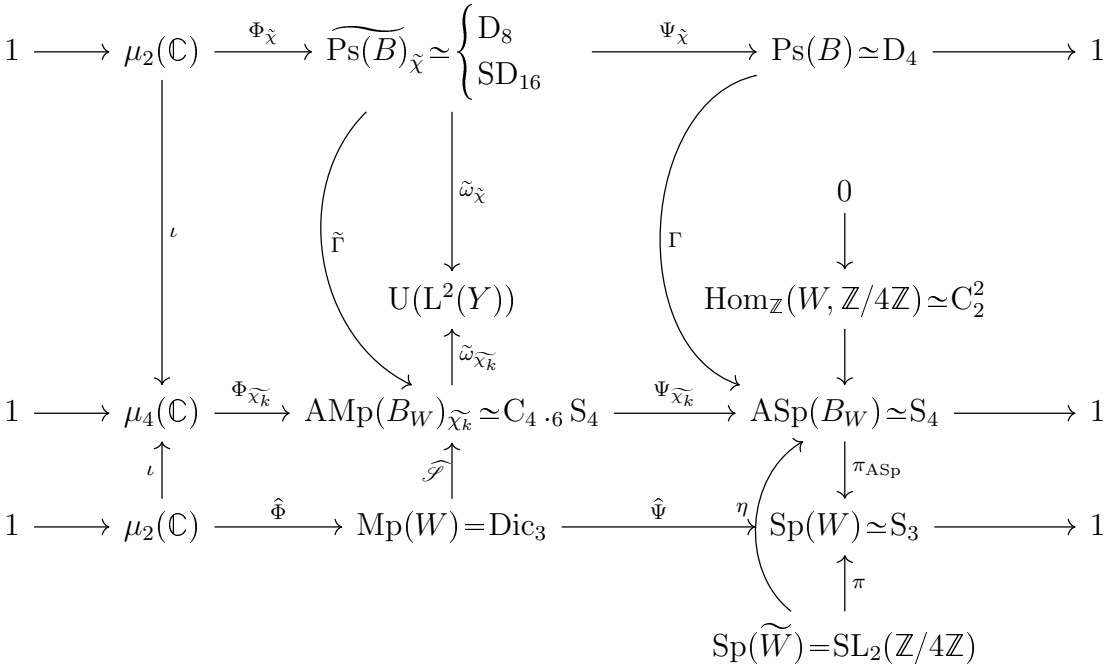
If $\mathrm{H}(B)$ is isomorphic to D_4, then we choose $B = B_{\mathrm{Weil}} : ((x,y),(x',y')) \mapsto xy'$ to carry out all the computations. In this case, the quadratic form associated to B is $Q : (x,y) \mapsto xy$ and the orthogonal group $\mathrm{O}(Q)$ has exactly two elements. In Proposition 3.2, we exhibit the elements of the pseudosymplectic group $\mathrm{Ps}(B)$ and the different possible two-fold coverings $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. This provides also the list of operators which belong to the image $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$. The pseudosymplectic group $\mathrm{Ps}(B)$ is isomorphic to the dihedral group D_4, of order eight. Considering all the aforementioned choices of w_s, $s \in \mathrm{Ps}(B)$, and $\gamma(\sigma)$, $\sigma \in \mathrm{O}(Q)$, we prove that there are exactly two possibilities for the two-fold covering: $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is either isomorphic to the dihedral group D_8 or to the semidihedral group SD_{16} of order 16. The computation of the second cohomology group $\mathrm{H}^2(\mathrm{Ps}(B), \mathbb{Z}/2\mathbb{Z})$ and [GN] help us to observe that these are the only two-fold coverings of $\mathrm{Ps}(B)$ which contain a copy of the Heisenberg group $\mathrm{H}(B)$. After that, we turn to the study of the affine symplectic group. We already mentioned that there is only one group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ up to isomorphism: in the lowest-dimensional case it is isomorphic to the Pauli group $\mathrm{C}_4 \circ \mathrm{D}_4$, of order 16. In Proposition 3.5, we find that $\mathrm{ASp}(B_W)$ is isomorphic to the symmetric group S_4, of order 24, and we exhibit the lists of 96 elements of the affine metaplectic coverings $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, where $k = 1$ or 3. This provides also the operators in the image $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi_k}})$ of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$. Depending on the choices of $\gamma(\sigma)$, $\sigma \in \mathrm{Sp}(W)$, there are exactly two possibilities: $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is either isomorphic to $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$ or to the unitary group $\mathrm{U}_2(\mathbb{F}_3)$. Computing $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z})$ and using [GN], we remark, as for $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, that these are the only four-fold coverings of $\mathrm{ASp}(B_W)$ which contain a copy of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Our description of $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z})$ implies that the isomorphism class of the extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ does not depend on the chosen faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ in this specific example. It also allows us to prove that a four-fold covering is not the minimal non-split central extension of $\mathrm{ASp}(B_W)$ we can obtain (cf. the discussion above on the non-optimality of the parameter $\gamma(\sigma)$ in the second chapter). More precisely, we exhibit in Remark

3.8 two different choices of w_s, $s \in \mathrm{ASp}(B_W)$, and $\gamma(\sigma)$, $\sigma \in \mathrm{Sp}(W)$, that lead to the two possible two-fold coverings of $\mathrm{ASp}(B_W)$, namely the conformal special unitary (or binary octahedral) group $\mathrm{CSU}_2(\mathbb{F}_3)$ and the general linear group $\mathrm{GL}_2(\mathbb{F}_3)$, of order 48. Observe that these groups do not contain $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, unlike $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. The question of the compatibility of the constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is answered in Proposition 3.9: the diagram (D_3) commutes if and only if $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$.

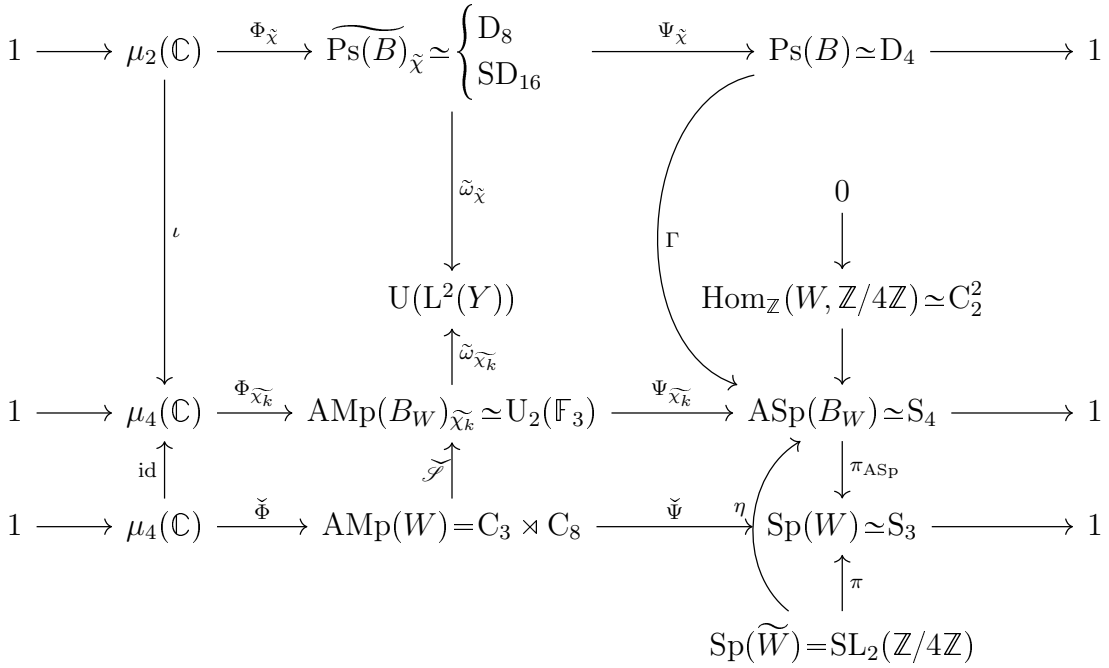
After the constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, the first path we follow in section 3.1.4 is the search for a metaplectic two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ in order to obtain a commutative diagram (D_4). The results of this section were obtained before we made the third independent construction of the Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ in section 2.5 (which is presented in section 3.3). We still include them here for two main reasons. First, they show that in this example, we are able to construct a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and its Weil representation, such that the diagram (D_4) commutes, using only the first two constructions and considerations on abstract presentations of the groups involved. Secondly, observe that the diagrams (D_4) and (D_5) are slightly different from each other, and we provided the homomorphism $\mathscr{S}$ of central extensions in section 2.5 only for $B = B_{\mathrm{Weil}}$. On the other hand, the methods used in section 3.1.4 can be reproduced also in section 3.2. If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$, then we prove in Proposition 3.12 that obtaining the best possible result requires to set $\mathrm{Mp}(\widetilde{W})$ as isomorphic to the group $\mathrm{U}_2(\mathbb{F}_3)$. Then the map $\mathscr{S}$ which makes the diagram (D_4) commute can be chosen as the identity, and the extension $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to $\mathrm{Mp}(\widetilde{W})$ that we obtain stays unitary, irreducible and faithful. Recall that this situation is impossible in higher dimension (cf. Lemma 2.42). On the other hand, if $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$, then the commutativity of (D_4) implies that $\mathrm{Mp}(\widetilde{W})$ is isomorphic to the group $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. In Proposition 3.16 we study the different possibilities for the homomorphism $\mathscr{S}$ making the diagram (D_4) commutative. In particular, the extension $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to $\mathrm{Mp}(\widetilde{W})$ obtained in Corollary 3.17 is unitary and irreducible but not faithful, because its image $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ has index two in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_3)$. Therefore, the construction in which $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$ preserves the injectivity of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, but loses the link with Weil's work, in the sense of the existence of $\widetilde{\Gamma}$. On the contrary, the construction in which $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ keeps the link with Weil's work, i.e. the homomorphism $\widetilde{\Gamma}$ exists, but loses the injectivity of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$.

Since all groups are explicit and relatively small in this example, in section 3.1.5 we explore an alternative way to extend the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to a group which is closer to $\mathrm{Sp}(W)$ than $\mathrm{Mp}(\widetilde{W})$, i.e. a central extension of $\mathrm{Sp}(W)$ itself. If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$, we prove in Proposition 3.20 that the best result we can obtain (in the sense of the smallest possible extension of $\mathrm{Sp}(W)$ which is compatible with $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ by the commutative diagram (D_6) below) is to define a unitary, irreducible and faithful Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ of the two-fold covering $\mathrm{Mp}(W) = \mathrm{Dic}_3$ of $\mathrm{Sp}(W)$. If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$, we obtain a similar result for a four-fold covering $\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8$

of $\mathrm{Sp}(W)$ and its associated Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}}$. We may visualize these results through the following two diagrams, depending of the isomorphism class of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.



(D_6)

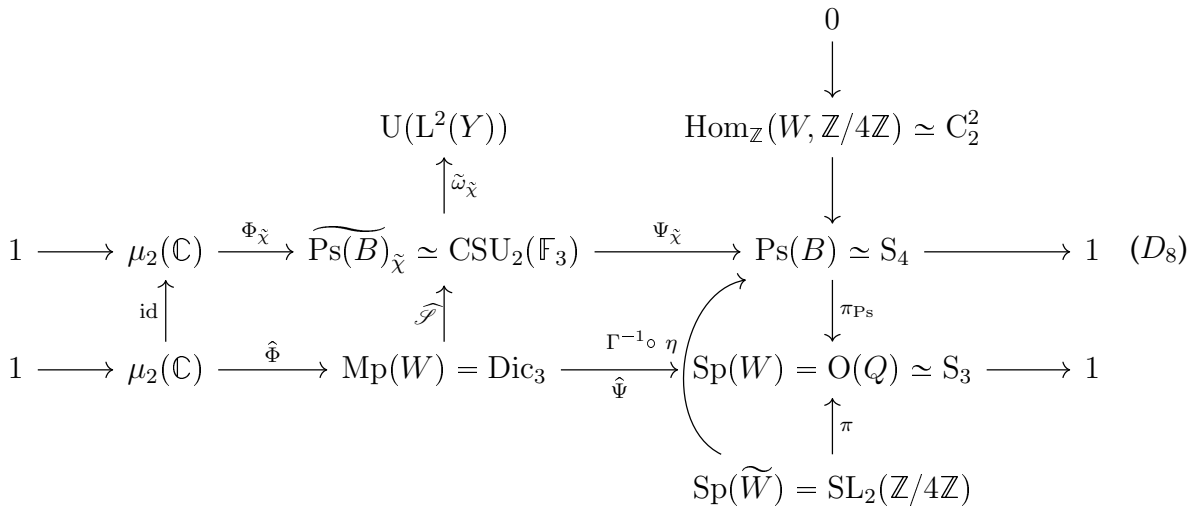


(D_7)

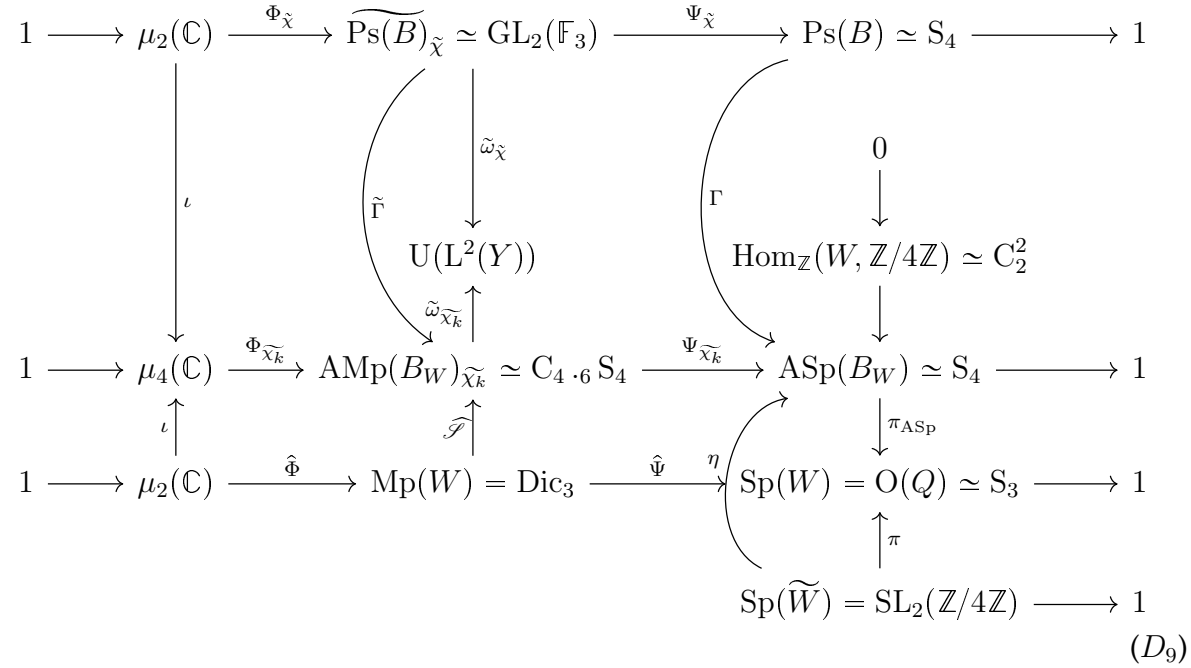
If $\mathbf{H}(\boldsymbol{B})$ is isomorphic to $\mathbf{Q_8}$, then we choose $B : ((x,y),(x',y')) \mapsto xy' + xx' + yy'$ to carry out all the computations. Then the quadratic form associated to B is $Q : (x,y) \mapsto xy + x^2 + y^2$. This is the very exceptional case where the orthogonal and symplectic groups coincide (cf. Proposition 1.13). Since the path followed in section 3.2 is the same as that of section 3.1, we will only highlight here the differences in the results we obtain. Since $\mathrm{O}(Q) = \mathrm{Sp}(W)$ in this case, $\mathrm{O}(Q)$ is isomorphic to S_3. Then the pseudosymplectic group $\mathrm{Ps}(B)$ and the affine symplectic group $\mathrm{ASp}(B_W)$ are both isomorphic to the symmetric group S_4, of order 24. There are two possibilities for the two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, namely the conformal special unitary (or binary octahedral) group $\mathrm{CSU}_2(\mathbb{F}_3)$, or the general linear group $\mathrm{GL}_2(\mathbb{F}_3)$ (cf. Proposition 3.28). Recall from the second chapter that this example is the only one in which we are able to define a two-fold covering $\mathrm{G}^0_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ with the cocycle c_{χ_k}, in addition to the four-fold covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. In Proposition 3.30, we recover that $\mathrm{G}^0_{\widetilde{\chi_k}}$ agrees with the two aforementioned possibilities for $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. This is consistent with the fact that $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$ are isomorphic. In this perspective, the construction of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ loses its interest since $\mathrm{Ps}(B)$ projects already onto $\mathrm{Sp}(W)$. However, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ will be useful later. We find two possibilities for this covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$: it is either isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ or $\mathrm{U}_2(\mathbb{F}_3)$. Observe that these are precisely the same possibilities as for B_{Weil}, which is consistent since there is only one Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ up to isomorphism. The differences lie in the explicit expressions of the elements of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ for $B = B_{\text{Weil}}$ or $B \neq B_{\text{Weil}}$. Then, we prove in section 3.2.3 that the constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}^0_{\widetilde{\chi_k}}$ are compatible, in the sense of the commutativity of (D_3), if and only if the groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}^0_{\widetilde{\chi_k}}$ are isomorphic. Moreover, the constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ are compatible if and only if $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$.

Concerning the metaplectic two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and its Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, satisfying that the diagram (D_4) is commutative, we recover in section 3.2.4 similar results to those from section 3.1.4. If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$, then so is $\mathrm{Mp}(\widetilde{W})$. We preserve the injectivity of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, but we lose the link with Weil's work, in the sense of the non-existence of $\widetilde{\Gamma}$. If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$, then $\mathrm{Mp}(\widetilde{W})$ is isomorphic to $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. We keep this way the link with Weil's work, in the sense of the existence of the homomorphism $\widetilde{\Gamma}$, but we lose the injectivity of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, since its image $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is either isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_3)$.

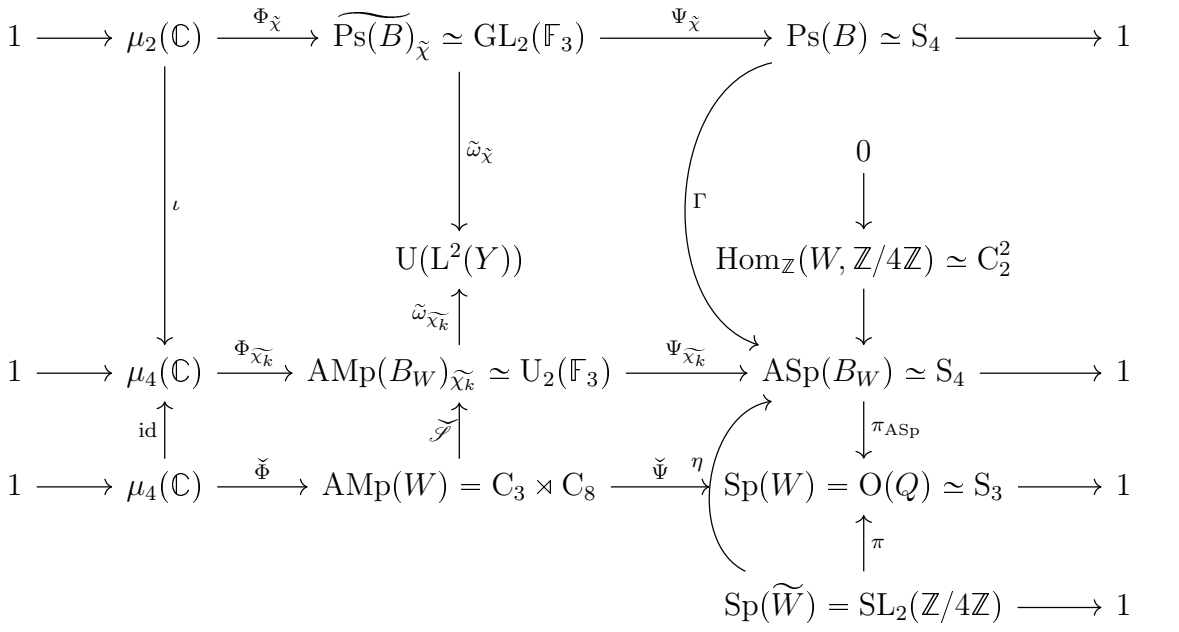
Section 3.2.5 aims to provide a unitary, irreducible and faithful Weil representation of the smallest possible central extension of $\mathrm{Sp}(W)$ (instead of $\mathrm{Sp}(\widetilde{W})$) whose construction is compatible with that of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ if possible, or $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ if not, in the sense of the commutative diagrams (D_8), (D_9) and (D_{10}) below. If $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$, then we do not need to consider $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, and the best result consists of a unitary, irreducible and faithful Weil representation $\widehat{\omega}_{\widetilde{\chi}} \circ \widehat{\mathscr{S}}$ of the two-fold covering $\mathrm{Mp}(W) = \mathrm{Dic}_3$ of $\mathrm{Sp}(W)$ such that the diagram (D_8) commutes.



If $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$, then we need to consider the larger extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ in order to lift the Weil representation to a central extension of $\mathrm{Sp}(W)$. If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$, then the best result is a unitary, irreducible and faithful Weil representation $\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}}$ of the two-fold covering $\mathrm{Mp}(W) = \mathrm{Dic}_3$ of $\mathrm{Sp}(W)$ such that the following diagram commutes:

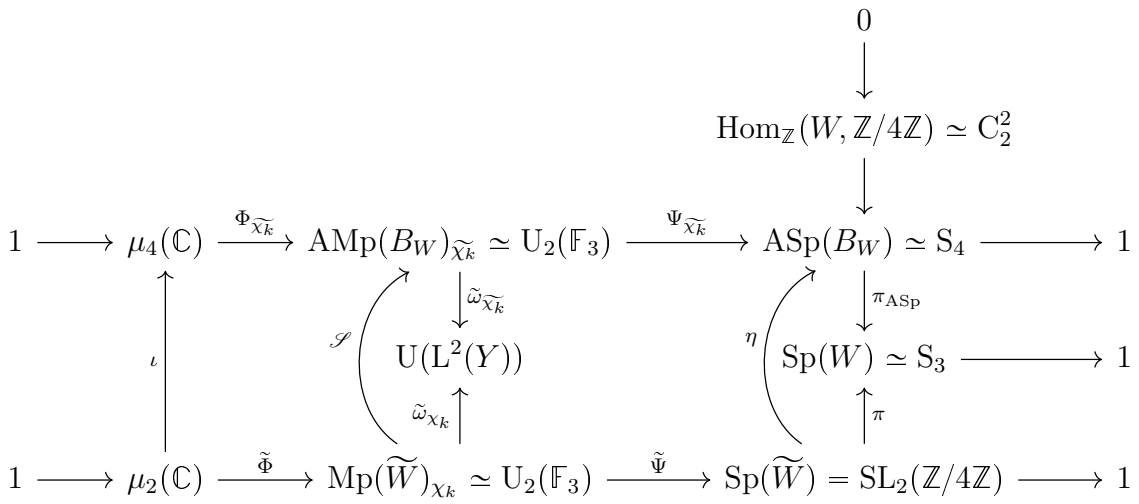


If $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$, then the best result is a unitary, irreducible and faithful Weil representation $\widetilde{\omega}_{\tilde{\chi}} \circ \widetilde{\mathscr{S}}$ of the four-fold covering $\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8$ of $\mathrm{Sp}(W)$ such that the following diagram commutes:



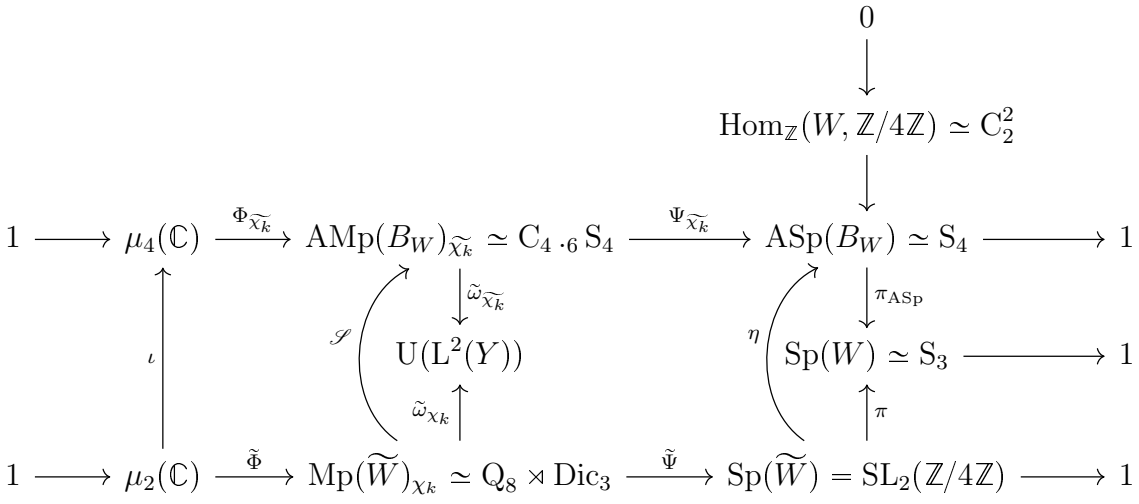
(D_{10})

Section 3.3 is devoted to present the construction of the two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ from section 2.5, starting from the Heisenberg group $\mathrm{H}(\widetilde{W})$, which in this example is isomorphic to the group $\mathrm{C}_4^2 \rtimes_4 \mathrm{C}_4$, of order 64. Proposition (3.49) exhibits two different valid choices for the sets $\{\widetilde{w}_{\widetilde{\sigma}}, \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ which lead to two possibilities for $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. Computing the explicit list of elements which belong to $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, we prove that this group is either isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$ or to $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ (which is consistent with the results of sections 3.1.4 and 3.2.4). Using [GN] and the computation of $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/2\mathbb{Z})$, we prove also that these two particular choices define in fact the only two possibilities for the two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ (see Remark 3.50). In each of these cases, we lift the projective Weil representation ω_{χ_k} of $\mathrm{Sp}(\widetilde{W})$ to the linear Weil representation $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. This representation is unitary and irreducible. Furthermore, it is faithful if $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$. If $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is isomorphic to $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ then it has a two-element kernel and its image $\widetilde{\omega}_{\chi_k}(\mathrm{Mp}(\widetilde{W})_{\chi_k})$ is isomorphic either to $\mathrm{CSU}_2(\mathbb{F}_3)$ or to $\mathrm{GL}_2(\mathbb{F}_3)$. In the case $B = B_{\mathrm{Weil}}$, we conclude this section with a study of the homomorphism $\mathscr{S}$ of central extensions which makes the diagram (D_5) commute. If $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$ then the commutativity of (D_5) requires that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$, and in this case $\mathscr{S}$ is an isomorphism:



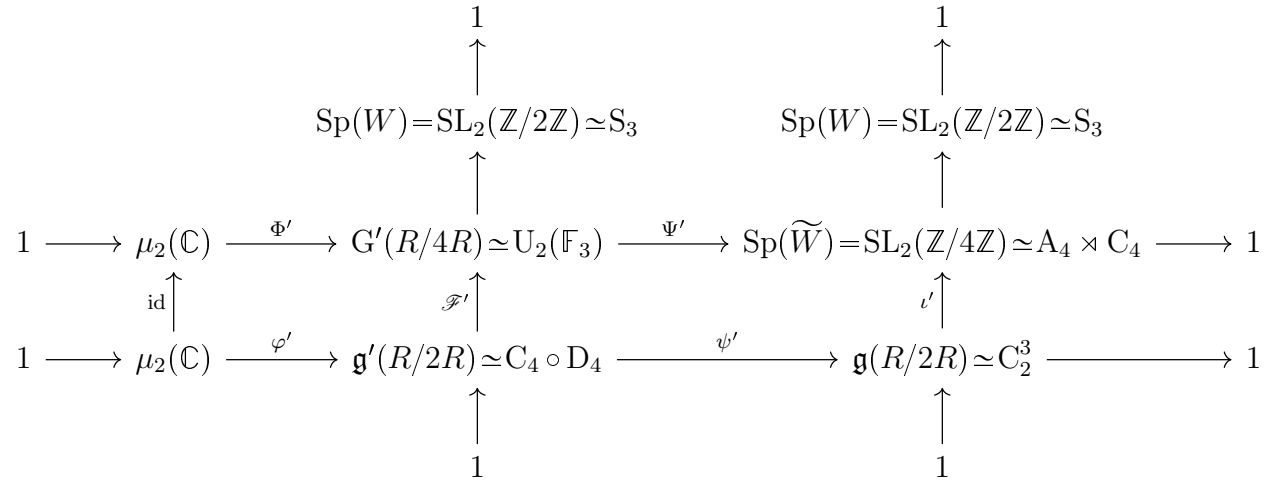
(D_{11})

If $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is isomorphic to $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$, then the commutativity of (D_5) requires that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$, and in this case the image of $\mathscr{S}$ has index two in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$:



(D_{12})

In his article [Sav12], Savin constructs a one-to-one correspondence between genuine representations of a two-fold covering $\mathrm{G}'(R/4R)$ of $\mathrm{G}(R/4R)$ and representations of $\mathrm{G}(R/2R)$, where $\mathrm{G}(A)$ denotes the Chevalley group over a commutative ring with unit A corresponding to a fixed simply laced root system Φ. If R is the ring of integers $\mathbb{Z}_2$ of $\mathbb{Q}_2$ and Φ is the root system A_1, then $\mathrm{G}(R/2R)$ and $\mathrm{G}(R/4R)$ are respectively the symplectic groups $\mathrm{Sp}(W)$ and $\mathrm{Sp}(\widetilde{W})$ that we encounter in our lowest-dimensional example. Section 3.4 provides a link between Savin's results and those we obtain in this thesis. Thanks to the computer algebra system **GAP**, we can identify in the database [GN] the groups which appear in Savin's article and we realize the commutative diagram he presents. **GAP** codes are provided in appendix B.1.



(D_{13})

Observe that $\mathfrak{g}'(R/2R)$ is our Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, and $\mathrm{G}'(R/4R) \simeq \mathrm{U}_2(\mathbb{F}_3)$ corresponds to one of the two possibilities for the metaplectic two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. For a genuine character χ of the center $\mathbb{Z}/4\mathbb{Z}$ of $\mathfrak{g}'(R/2R)$, Savin defines ρ_χ as an irreducible representation of $\mathfrak{g}'(R/2R)$ acting as χ on its center. By the Stone-von Neumann theorem, we know that ρ_χ is isomorphic to the Schrödinger model $\rho_{\widetilde{\chi_k}}$ of the Heisenberg representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, provided $\chi_k = \chi$. Then, Savin extends ρ_χ to a representation ρ'_χ of the group $\mathrm{G}'(R/4R)$, and he points out that this extension is not unique in this specific case, as we verify directly. Our representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is one of the possibilities for ρ'_χ. Finally, counting the numbers of irreducible representations of the groups $\mathrm{G}(R/2R) = \mathrm{Sp}(W)$ and $\mathrm{G}'(R/4R) = \mathrm{Mp}(\widetilde{W})_{\chi_k}$, and using [GN], we verify that there is no bijective correspondence $\sigma \mapsto \sigma \otimes \rho'_\chi$ between isomorphism classes of irreducible representations of $\mathrm{G}(R/2R) = \mathrm{Sp}(W)$ and isomorphism classes of irreducible representations of $\mathrm{G}'(R/4R) = \mathrm{Mp}(\widetilde{W})_{\chi_k}$ such that the center of $\mathfrak{g}'(R/2R)$ acts by the character χ. This is the exceptional case not covered by Savin's Theorem 4.1.

The purpose of the **fourth chapter** is to set up a dictionary between our constructions of the Weil representation and the framework of Quantum Information Theory. Only a summary of the results obtained is given here, and we refer the reader to the beginning of this chapter for a more precise and comprehensive introduction. Quantum information is carried through quantum circuits, and qubits (quantum bits) are represented by vectors in a two-dimensional complex Hilbert space. More precisely, for every integer $n \geqslant 1$, a system consisting of n qubits is represented by a state vector in a 2^n-dimensional complex Hilbert space H. Manipulations on a system of n qubits are performed by quantum gates, which are complex unitary matrices of size 2^n. The main connection between this setting and our results is that the operators defined by the Weil representations are specific quantum gates, known as Clifford gates. The relevance of these gates is particularly evident in the context of quantum error correction, and more precisely in the stabilizer formalism. We mention that Heinrich [Hei21] studies in

his thesis the relations between the stabilizer formalism and the framework of Weil representations. Nevertheless, he raises technical difficulties for a finite field of characteristic two and refers to Gurevich and Hadani's article [GH12] as a possible approach to treat this case. In light of our explicit constructions, we clarify some aspects of these relations for $\mathbb{F} = \mathbb{F}_2$. There is a group of quantum gates, called the Pauli group $\mathscr{P}_n$, whose elements form a basis for all the unitary complex matrices of size 2^n. Given a quantum error-correcting code whose codespace is a vector subspace C of H of dimension 2^k, the ability to correct any error which occurs by the action of some Pauli gate on k qubits in C allows us to correct an arbitrary error on k qubits in C. After some necessary preliminaries on tensor product of representations in section 4.1, our first result in Proposition 4.9 is to identify the Pauli group $\mathscr{P}_n$ and its subgroup $\mathscr{P}_n^{\mathbb{R}}$ (called the real Pauli group) with subgroups of the coverings $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$, respectively, when W is a $2n$-dimensional $\mathbb{F}_2$-vector space. More precisely, if $B = B_{\mathrm{Weil}}$ then the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ provides an explicit isomorphism between the preimage in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of the subgroup of $\mathrm{Ps}(B)$ which acts trivially on $\mathrm{O}(Q)$, and the group $\mathscr{P}_n^{\mathbb{R}}$. In a similar way, we prove that if $B = B_{\mathrm{Weil}}$ or if $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ (in a symplectic basis adapted to the complete polarization $W = X \oplus Y$), then the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ provides an explicit isomorphism between the preimage in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of the subgroup of $\mathrm{ASp}(B_W)$ which acts trivially on $\mathrm{Sp}(W)$, and the group $\mathscr{P}_n$. Using this correspondence, we translate the generators of the stabilizers of well-known stabilizer codes in terms of operators of the Weil representation.

The (infinite) Clifford group $\mathscr{C}_n$ is the group of quantum operations that preserve the Pauli group by conjugation. An essential Clifford gate is known as the Hadamard(-Rademacher-Walsh) transform. In representation-theoretic terms, this transform is the Fourier transform on the abelian group obtained as tensor product of n copies of $\mathbb{F}_2$. In quantum information, this quantum gate is for example used to create superposition of quantum states or to perform changes of bases in order to better detect some errors. These properties make it a prime ingredient for many quantum algorithms, such as Deutsch-Josza's, Grover's, Shor's, and the Quantum Fourier transform algorithms. Our second result is Proposition 4.11: inside the groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ (again for the two aforementioned possibilities for the bilinear form B) and $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, we determine the elements whose corresponding operator under the Weil representations $\widetilde{\omega}_{\widetilde{\chi}}$, $\widetilde{\omega}_{\widetilde{\chi_k}}$ and $\widetilde{\omega}_{\chi_k}$, respectively, is exactly the Hadamard gate.

Finally, we use the explicit computations and identifications made in the third chapter when $\widetilde{W}$ is a free $\mathbb{Z}/4\mathbb{Z}$-module of rank two and W is a two-dimensional $\mathbb{F}_2$-vector space, in order to select some finite subgroups of Clifford gates on one qubit. More precisely, in section 4.2.3 we write in terms of quantum gates the generators of all the finite subgroups of Clifford gates that we obtain by means of our construction of the Weil representations. This may be useful for quantum computations, since finite groups allows us to do computations, relying on algebraic structures and properties.

To close this introduction, we mention four open questions which naturally arise from the results obtained in this thesis:

- The construction of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ in the first chapter is valid for every finite field $\mathbb{F}$ of characteristic two. In contrast, the Weil representations $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, which allow us to recover the link with the entire symplectic group $\mathrm{Sp}(W)$, are only presented for $\mathbb{F} = \mathbb{F}_2$. We may wonder how to extend the two last constructions to an arbitrary finite field of characteristic two. A promising direction mentioned by Heinrich in [Hei21] could be to use, instead of the ring $\mathbb{Z}/4\mathbb{Z}$ for $\mathbb{F}_2$, the Galois extension $\mathrm{G}(4, m)$ of degree m of the ring $\mathbb{Z}/4\mathbb{Z}$ for a general field $\mathbb{F} = \mathbb{F}_{2^m}$.

- The construction of the Weil representation is only the first step needed for further developments in different areas of mathematics. We already mentioned the Howe (or theta) correspondence, which uses the Weil representation to provide a bijection between some irreducible admissible representations of the preimages in the metaplectic group $\mathrm{Mp}(W)$ of two groups which form a reductive dual pair in the symplectic group $\mathrm{Sp}(W)$. For finite fields, this correspondence has various reincarnations. One of them is called eta correspondence, but the theory is far from complete. In odd characteristics, the Weil representation is a linear representation of $\mathrm{Sp}(W)$ itself, so the reductive dual pairs are to be searched in the symplectic group. To develop this theory in characteristic two, we first have to look for the dual pairs. Inside the pseudosymplectic group $\mathrm{Ps}(B)$ dual pairs were classified by Blasco in [Bla93]. On the other hand, to our knowledge, the problem of determining such pairs in the affine symplectic group $\mathrm{ASp}(B_W)$ and the symplectic group $\mathrm{Sp}(\widetilde{W})$ has not been solved yet. Moreover, we proved that commuting elements in these groups have preimages that do not necessarily commute in the corresponding coverings. Therefore we have to clarify how the preimage of such a dual pair behaves in the coverings $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, respectively, and this for all possible such coverings, since at present we do not know how many non-isomorphic coverings there are in general for the first two.

- At present, we only translated some of our results in the language of Quantum Information Theory in order to provide some relevant quantum gates with a finite group structure. It would be interesting to investigate more in this direction in the future, and see if these explicit constructions of the Weil representations could lead to some new applications, for example in the setting of quantum error correction, in the scope of works like [MG21] for the odd characteristic case.

- The recent preprint of Fintzen and Schwein [FS25] underlines the relevance of the Weil representation for constructing supercuspidal representations of p-adic groups. For $p \neq 2$, general constructions are known. Among them, Yu's construction [Yu01] relied on Heisenberg-Weil representations via compact induction. This construction did not apply to the situation where $p = 2$ because of the division by two in Yu's definition of the Heisenberg group and because the projective representation does not linearize on the pseudosymplectic group. In [FS25], Fintzen and Schwein generalize Yu's construction including the case $\mathbb{F} = \mathbb{F}_2$ by restricting themselves to specific subgroups where

this linearization is possible. Finding the link of their work and our explicit linearization of the projective Weil representation for the pseudosymplectic group and possibly extending their constructions to the symplectic case we study in chapters 2 and 3 seems to us quite natural and interesting future lines of investigation to work on.

Chapter 1

The case of a finite field $\mathbb{F}$ of characteristic two

1.1 The Heisenberg group $\mathrm{H}(B)$ and the pseudosymplectic group $\mathrm{Ps}(B)$

Let $\mathbb{F}$ be a finite field of characteristic two and let W be a finite dimensional $\mathbb{F}$-vector space of dimension $2n$, where $n \geqslant 1$. Since the Weil representation arises from the Heisenberg group, we first have to properly define the Heisenberg group in this setting. In fact, this group is usually defined from a nondegenerate symplectic form over W, which requires a division by two (see (7)). Since dividing by two is not possible in characteristic two, instead we use a bilinear form B (related to the nondegenerate symplectic form by (16)) to define the Heisenberg group. In particular, there are several different bilinear forms B that can be used to define this group. If $\mathbb{F} = \mathbb{F}_2$, there are only two different Heisenberg groups to consider up to isomorphism (cf. Proposition 1.6).

Definition 1.1. Let B be a nonzero bilinear form defined on W with values in $\mathbb{F}$. We define the Heisenberg group associated to B, denoted by $\mathrm{H}(B)$, as the set $W \times \mathbb{F}$ endowed with the following product:

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + B(w_1, w_2)) \tag{11}$$

for all $w_1, w_2 \in W$ and $t_1, t_2 \in \mathbb{F}$. In particular, the identity element of $\mathrm{H}(B)$ is $(0, 0)$ and the inverse of $(w, t) \in \mathrm{H}(B)$ is $(-w, -t + B(w, w))$.

Example 1.2. Let X be a finite dimensional vector space over $\mathbb{F}$ with dual space X^*. Set $W = X \times X^*$ and define a nondegenerate symplectic form on W by $\langle w_1, w_2 \rangle = x_2^*(x_1) - x_1^*(x_2)$ for all $w_1 = (x_1, x_1^*),\ w_2 = (x_2, x_2^*) \in X \times X^*$. We denote by $\mathrm{H}(B_{\mathrm{Weil}})$ the Heisenberg group associated to the bilinear form B_{Weil} defined by

$$B_{\mathrm{Weil}}(w_1, w_2) = x_2^*(x_1).$$

(We call it B_{Weil} since it is the bilinear form used by Weil in [Wei64].)

Remark 1.3. Definition 1.1 with $B = B_{\text{Weil}}$ appears with slightly different notation in the original work of Weil [Wei64], and in the article [Ger77] of Gérardin for finite fields of odd characteristic. In odd characteristic, this definition is in fact equivalent to the one given by (7) which requires a division by two, so Definition 1.1 really extends the original definition of the Heisenberg group. In characteristic two, the definition with a general bilinear form B appears for example in the works of Blasco [Bla93] and Glasby [Gla95].

We denote by $\mathscr{Q}(W)$ the set of quadratic forms defined on W with values in $\mathbb{F}$, i.e. the set of maps $f : W \to \mathbb{F}$ such that

$$(w_1, w_2) \mapsto f(w_1 + w_2) - f(w_1) - f(w_2) \text{ is a bilinear form, and} \tag{12}$$
$$f(tw) = t^2 f(w) \text{ for all } t \in \mathbb{F} \text{ and } w \in W. \tag{13}$$

We also denote by $\mathscr{Q}_a(W)$ the subset of $\mathscr{Q}(W)$ consisting of the quadratic forms which are additive.

Remark 1.4. If $\mathbb{F} = \mathbb{F}_2$ then the condition (13) is automatically satisfied by any map from W to $\mathbb{F}_2$ which satisfies (12).

The quadratic form associated to the bilinear form B is defined by $Q(w) = B(w, w)$ for all $w \in W$. Two quadratic forms Q and Q' are said to be equivalent if there exists an isomorphism σ of W such that $Q' = Q \circ \sigma$. Let $Q \in \mathscr{Q}(W)$ and let V be a vector subspace of W. Then V is said to be singular if $Q_{|V} = 0$. Following [Die63, §16], we denote by ν the Witt index of the quadratic form Q, i.e. the maximal dimension of a singular subspace V of W. We also recall the definition of the Arf invariant of a quadratic form on W (we refer to [Arf41], [LR10], [Die63, p. 34], [Dic01, p.197–199] and [Gro01, Theorem 12.9] for the original definitions and more general results). If the bilinear form $(w_1, w_2) \mapsto Q(w_1 + w_2) - Q(w_1) - Q(w_2)$ is nondegenerate, then there exists a basis $\{v_i\}_{1 \leqslant i \leqslant 2n}$ of W such that for all $w = \sum_{i=1}^{2n} \lambda_i v_i$,

$$Q(w) = \lambda_1\lambda_{n+1} + \lambda_2\lambda_{n+2} + \cdots + \lambda_{n-1}\lambda_{2n-1} + (\xi\lambda_n^2 + \lambda_n\lambda_{2n} + \xi\lambda_{2n}^2),$$

where $\xi = 0$ or ξ is such that the polynomial $\xi X^2 + X + \xi$ is irreducible over $\mathbb{F}$. Then the Arf invariant of Q is defined by $\mathrm{Arf}(Q) = \xi^2$. In particular, if $\mathbb{F} = \mathbb{F}_2$, we obtain $\mathrm{Arf}(Q) = \xi \in \{0, 1\}$. Two quadratic forms Q and Q' are equivalent if and only if $\mathrm{Arf}(Q) - \mathrm{Arf}(Q') = u^2 + u$ for some $u \in \mathbb{F}$, or equivalently, if and only if $\nu(Q) = \nu(Q')$.

Proposition 1.5. *Let B and B' be two nonzero bilinear forms from W to $\mathbb{F}$. Let Q and Q' be the quadratic forms associated to B and B', respectively.*

i. *The product* (11) *defines a group* $\mathrm{H}(B)$ *and a central extension*

$$0 \to \mathbb{F} \xrightarrow{\Phi} \mathrm{H}(B) \xrightarrow{\Psi} W \to 0, \tag{14}$$

where $\Phi(t) = (0, t)$ *and* $\Psi(w, t) = w$. *Moreover, if the alternating form*

$$(w_1, w_2) \mapsto B(w_1, w_2) - B(w_2, w_1)$$

is nondegenerate then the center of $\mathrm{H}(B)$ *is isomorphic to* $\mathbb{F}$.

ii. *B is a cocycle and the cohomology class of B in $\mathrm{H}^2(W,\mathbb{F})$ is the one associated to the isomorphism class of the central extension defined by $\mathrm{H}(B)$.*

iii. *The central extensions*

$$0 \to \mathbb{F} \xrightarrow{\Phi} \mathrm{H}(B) \xrightarrow{\Psi} W \to 0 \qquad \textit{and} \qquad 0 \to \mathbb{F} \xrightarrow{\Phi'} \mathrm{H}(B') \xrightarrow{\Psi'} W \to 0$$

are isomorphic if and only if B and B' are cohomologous. In particular, if these extensions are isomorphic then $B - B'$ is alternating, i.e. $Q = Q'$.

iv. *Suppose that the alternating forms*

$$(w_1, w_2) \mapsto B(w_1, w_2) - B(w_2, w_1)$$

and

$$(w_1, w_2) \mapsto B'(w_1, w_2) - B'(w_2, w_1)$$

are nondegenerate. Then the groups $\mathrm{H}(B)$ and $\mathrm{H}(B')$ are isomorphic if and only if there exists a map $f : W \to \mathbb{F}$, an automorphism of abelian groups $\sigma : W \to W$ and an automorphism of additive abelian groups $g : \mathbb{F} \mapsto \mathbb{F}$ such that for all $(w_1, w_2) \in W$:

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B'(\sigma(w_1), \sigma(w_2)) - g(B(w_1, w_2)). \tag{15}$$

In this case, the isomorphism is given by

$$\begin{aligned} \Phi_{(\sigma,f,g)} : \mathrm{H}(B) &\to \mathrm{H}(B') \\ (w,t) &\mapsto (\sigma(w), f(w) + g(t)). \end{aligned}$$

Any isomorphism between $\mathrm{H}(B)$ and $\mathrm{H}(B')$ is of this form.

Proof. **i.** The two group homomorphisms $\Phi : \mathbb{F} \to \mathrm{H}(B)$ and $\Psi : \mathrm{H}(B) \to W$ are such that Φ is injective, Ψ is surjective and $\mathrm{Ker}(\Psi) = \mathrm{Im}(\Phi)$. Moreover, for every $t \in \mathbb{F}$, the element $(0,t)$ is in the center of $\mathrm{H}(B)$ by (11). Let (w',t') be in the center of $\mathrm{H}(B)$. Then, for every $(w,t) \in \mathrm{H}(B)$,

$$\begin{aligned} (w,t) = (w',t')(w,t)(w',t')^{-1} &= (w'+w, t'+t+B(w',w))(-w', -t' + B(w',w')) \\ &= (w, t + B(w',w) + B(w',w') - B(w'+w,w')) \\ &= (w, t + B(w',w) - B(w,w')). \end{aligned}$$

This implies that $B(w',w) - B(w,w') = 0$ for all $w \in W$. Hence $w' = 0$ if $(w_1,w_2) \mapsto B(w_1,w_2) - B(w_2,w_1)$ is nondegenerate.

ii. B is a cocycle because B is a bilinear form: for all $w_1, w_2, w_3 \in W$,

$$B(w_1,w_2) + B(w_1+w_2,w_3) = B(w_2,w_3) + B(w_1,w_2+w_3).$$

Now, define the section $\tau : w \mapsto (w, 0)$ of the group homomorphism $\Psi : (w, t) \mapsto w$. Then for all $w_1, w_2 \in W$,

$$\begin{aligned}
&\tau(w_1)\tau(w_2)\tau(w_1 + w_2)^{-1} \\
&= (w_1, 0)(w_2, 0)(-w_1 - w_2, B(w_1 + w_2, w_1 + w_2)) \\
&= (0, B(w_1, w_2) + B(w_1 + w_2, w_1 + w_2) - B(w_1 + w_2, w_1 + w_2)) \\
&= \Phi(B(w_1, w_2)).
\end{aligned}$$

This shows that B is a cocycle that is associated to the central extension defined by $\mathrm{H}(B)$. Thus the cohomology class of B in $\mathrm{H}^2(W, \mathbb{F})$ is the one associated to the isomorphism class of the central extension defined by $\mathrm{H}(B)$.

iii. The fact that the extensions defined by $\mathrm{H}(B)$ and $\mathrm{H}(B')$ are isomorphic if and only if B and B' are cohomologous comes directly from the preceding point. Indeed, if these two extensions are isomorphic then B and B' are cohomologous. Hence, there exists a map $\delta : W \to \mathbb{F}$ such that for all $w_1, w_2 \in \mathbb{F}$,

$$B(w_1, w_2) - B'(w_1, w_2) = \delta(w_1) + \delta(w_2) - \delta(w_1 + w_2).$$

In particular, $B(w_1, w_1) - B'(w_1, w_1) = 2\delta(w_1) - \delta(2w_1) = 0$, i.e. $B - B'$ is alternating. This implies that $Q(w_1) = B(w_1, w_1) = B'(w_1, w_1) = Q'(w_1)$, hence $Q = Q'$.

iv. Suppose that there exists a map $f : W \to \mathbb{F}$, an automorphism of abelian groups $\sigma : W \to W$ and an automorphism of additive abelian groups $g : \mathbb{F} \mapsto \mathbb{F}$ such that (15) is satisfied. Then, for all $(w_1, t_1), (w_2, t_2) \in \mathrm{H}(B)$,

$$\begin{aligned}
&\Phi_{(\sigma,f,g)}(w_1, t_1)\Phi_{(\sigma,f,g)}(w_2, t_2) \\
&= (\sigma(w_1), f(w_1) + g(t_1))(\sigma(w_2), f(w_2) + g(t_2)) \\
&= (\sigma(w_1) + \sigma(w_2), f(w_1) + f(w_2) + g(t_1) + g(t_2) + B'(\sigma(w_1), \sigma(w_2))) \\
&\Phi_{(\sigma,f,g)}((w_1, t_1)(w_2, t_2)) \\
&= \Phi_{(\sigma,f,g)}(w_1 + w_2, t_1 + t_2 + B(w_1, w_2)) \\
&= (\sigma(w_1 + w_2), f(w_1 + w_2) + g(t_1 + t_2 + B(w_1, w_2))) \\
&= (\sigma(w_1) + \sigma(w_2), f(w_1) + f(w_2) + g(t_1) + g(t_2) + B'(\sigma(w_1), \sigma(w_2))) \quad \text{by (15).}
\end{aligned}$$

Hence $\Phi_{(\sigma,f,g)}((w_1, t_1)(w_2, t_2)) = \Phi_{(\sigma,f,g)}(w_1, t_1)\Phi_{(\sigma,f,g)}(w_2, t_2)$, which proves that $\Phi_{(\sigma,f,g)}$ is a group homomorphism. Moreover, for every $(w', t') \in \mathrm{H}(B')$, since σ is bijective, there exists a unique $w \in W$ such that $\sigma(w) = w'$. Since g is bijective, there exists a unique $t \in \mathbb{F}$ such that $g(t) = f(w) + t'$. Thus, we obtain a unique element $(w, t) \in \mathrm{H}(B)$ such that $\Phi_{(\sigma,f,g)}(w, t) = (\sigma(w), f(w) + g(t)) = (w', t')$. Therefore $\Phi_{(\sigma,f,g)}$ is an isomorphism.

Conversely, suppose that $s : \mathrm{H}(B) \mapsto \mathrm{H}(B')$ is a group isomorphism. Then we can decompose s as follows: $s : (w, t) \mapsto (s_1(w, t), s_2(w, t))$, where $s_1 : \mathrm{H}(B) \mapsto W$ and $s_2 : \mathrm{H}(B) \mapsto \mathbb{F}$. Hence, for every $(w, t) \in \mathrm{H}(B)$,

$$\begin{aligned}
s(w, t) = s((w, 0)(0, t)) = s(w, 0)s(0, t) &= (s_1(w, 0), s_2(w, 0))(0, s_2(0, t)) \\
&= (s_1(w, 0), s_2(w, 0) + s_2(0, t)),
\end{aligned}$$

where we used the fact that $s_1(0,t) = 0$ because s maps the center of $\mathrm{H}(B)$ to the center of $\mathrm{H}(B')$. Thus $s_1(w,t) = s_1(w,0)$, i.e. s_1 depends only on W, and $s_2(w,t) = s_2(w,0) + s_2(0,t)$. We can set $s_1(w,t) = \sigma(w)$, $s_2(w,0) = f(w)$ and $s_2(0,t) = g(t)$ with $\sigma : W \to W$, $f : W \to \mathbb{F}$ and $g : \mathbb{F} \to \mathbb{F}$. Then s acts on $\mathrm{H}(B)$ by $s(w,t) = (\sigma(w), f(w) + g(t))$. Moreover,

- $f(0) = s_2(0,0) = 0$ and $g(0) = s_2(0,0) = 0$ because $s(0,0) = 0$.
- σ is an abelian group homomorphism and (15) is satisfied. In fact, for all $w_1, w_2 \in W$, the fact that s is a group homomorphism implies the equality of the following two elements of $\mathrm{H}(B')$:

$$\begin{aligned} s((w_1,0)(w_2,0)) &= s(w_1+w_2, B(w_1,w_2)) \\ &= (\sigma(w_1+w_2), f(w_1+w_2) + g(B(w_1,w_2))) \\ s(w_1,0)s(w_2,0) &= (\sigma(w_1), f(w_1))(\sigma(w_2), f(w_2)) \\ &= (\sigma(w_1)+\sigma(w_2), f(w_1)+f(w_2)+B'(\sigma(w_1),\sigma(w_2))). \end{aligned}$$

- g is an abelian group homomorphism. In fact, for all $t_1, t_2 \in \mathbb{F}$, the equality $s((0,t_1)(0,t_2)) = s(0,t_1)s(0,t_2)$ implies that

$$(\sigma(0), f(0) + g(t_1+t_2)) = (\sigma(0), f(0)+g(t_1))(\sigma(0), f(0)+g(t_2))$$

  i.e. $(0, g(t_1+t_2)) = (0, g(t_1) + g(t_2))$.
- σ and g are bijective. In fact, let us consider s^{-1} and write it under the form $s^{-1} : (w',t') \mapsto (\sigma'(w'), f'(w') + g'(t'))$, where $\sigma' : W \to W$ is an abelian group homomorphism and $f' : W \to \mathbb{F}$ and $g' : \mathbb{F} \to \mathbb{F}$ satisfy $f'(0) = g'(0) = 0$. Then for every $(w,t) \in \mathrm{H}(B)$,

$$\begin{aligned} (w,t) = (ss^{-1})(w,t) &= s(\sigma'(w), f'(w)+g'(t)) \\ &= (\sigma(\sigma'(w)), f(\sigma'(w)) + g(f'(w)+g'(t))). \end{aligned}$$

  Hence $w = \sigma(\sigma'(w))$, i.e. σ is bijective and $\sigma' = \sigma^{-1}$. Moreover, choosing $w = 0$, we obtain $t = f(0) + g(f'(0) + g'(t)) = g(g'(t))$, i.e. g is bijective and $g^{-1} = g'$.

Therefore, every isomorphism $s : \mathrm{H}(B) \to \mathrm{H}(B')$ is of the form $s = \Phi_{(\sigma,f,g)}$ given in the statement.

□

From now on, we suppose that the bilinear form

$$(w_1, w_2) \mapsto \langle w_1, w_2 \rangle = B(w_1,w_2) - B(w_2,w_1) \tag{16}$$

is nondegenerate. This endows W with a symplectic vector space structure. We denote this space by $(W, \langle \cdot, \cdot \rangle)$.

Let B and B' be two nonzero bilinear forms on W. By Proposition 1.5.iv, if $\mathrm{H}(B)$ and $\mathrm{H}(B')$ are isomorphic, then there exists f, σ and g such that (15) is satisfied. Let Q and Q' denote the

quadratic forms associated to B and B', respectively. Then $Q'(\sigma(w)) = B'(\sigma(w), \sigma(w)) = g(B(w,w)) = g(Q(w))$ for all $w \in W$, i.e. $Q' \circ \sigma = g \circ Q$. In particular, if $\mathbb{F} = \mathbb{F}_2$, then the only possibility for g is the identity. Therefore $Q' \circ \sigma = Q$, which implies that Q and Q' are equivalent. In fact, for $\mathbb{F} = \mathbb{F}_2$, we can state a more precise result in Proposition 1.6. In this case, the Heisenberg group appears to be what is called an extraspecial 2-group. This implies that $\mathrm{H}(B)$ and $\mathrm{H}(B')$ are isomorphic groups if and only if Q and Q' are equivalent, and thus there are only two different Heisenberg groups. We refer to appendix A for the definition and structure of extraspecial 2-groups, the link with the Heisenberg group $\mathrm{H}(B)$ and a proof of Proposition 1.6.

Proposition 1.6. *Suppose that $\mathbb{F} = \mathbb{F}_2$. Let B be a nonzero bilinear form on W and Q be its associated quadratic form. Then $\mathrm{H}(B)$ is an extraspecial 2-group. In particular, there are two Heisenberg groups up to isomorphism, depending on whether $\nu(Q) = n$ (or, equivalently, $\mathrm{Arf}(Q) = 0$) or $\nu(Q) = n - 1$ (or, equivalently, $\mathrm{Arf}(Q) = 1$). This means that for two bilinear forms B and B', the groups $\mathrm{H}(B)$ and $\mathrm{H}(B')$ are isomorphic if and only if Q and Q' are equivalent.*

Proof. Cf. Proposition A.16. □

Now that the Heisenberg group $\mathrm{H}(B)$ has been defined, we are able to study its automorphisms. We write $\mathrm{Aut}(\mathrm{H}(B))$ for the group of automorphisms of $\mathrm{H}(B)$ and $\mathrm{Aut}^0(\mathrm{H}(B))$ for the subgroup of $\mathrm{Aut}(\mathrm{H}(B))$ consisting of the automorphisms of $\mathrm{H}(B)$ acting trivially on the center of $\mathrm{H}(B)$.

Definition 1.7. A vector space automorphism $\sigma : W \to W$ is said to be symplectic if it preserves the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$, i.e. for all $w_1, w_2 \in W$,

$$B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2) = B(\sigma(w_2), \sigma(w_1)) - B(w_2, w_1).$$

The set of symplectic automorphisms is a group, which we denote by $\mathrm{Sp}(W)$.
A vector space automorphism $\sigma : W \to W$ is said to be orthogonal if it preserves the quadratic form $Q : w \mapsto Q(w) = B(w,w)$, i.e. for all $w \in W$,

$$Q(\sigma(w)) = B(\sigma(w), \sigma(w)) = B(w,w) = Q(w).$$

The set of orthogonal automorphisms is a group, which we denote by $\mathrm{O}(Q)$.

Proposition 1.8. *$\mathrm{Aut}^0(\mathrm{H}(B))$ is the set of elements $s = (\sigma, f)$, where $\sigma \in \mathrm{Sp}(W)$ and $f : W \to \mathbb{F}$ are such that for all $w_1, w_2 \in W$,*

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2). \tag{17}$$

The element $s = (\sigma, f)$ acts on $\mathrm{H}(B)$ by $(\sigma, f)(w, t) = (\sigma(w), f(w) + t)$. The product of two elements of $\mathrm{Aut}^0(\mathrm{H}(B))$ is given by

$$(\sigma, f)(\sigma', f') = (\sigma\sigma', f' + f \circ \sigma'), \quad \text{where } (f' + f \circ \sigma')(w) = f'(w) + f(\sigma'(w)).$$

In particular, the inverse of $s = (\sigma, f)$ is $s^{-1} = (\sigma^{-1}, -f \circ \sigma^{-1})$.

Proof. Let $s \in \mathrm{Aut}^0(\mathrm{H}(B))$. Since s is an automorphism of $\mathrm{H}(B)$, by Proposition 1.5.iv, there exists a map $f : W \to \mathbb{F}$, an automorphism of abelian groups $\sigma : W \to W$ and an automorphism of additive abelian groups $g : \mathbb{F} \mapsto \mathbb{F}$ such that $s = \Phi_{(\sigma,f,g)}$. In particular, $s : (w,t) \mapsto (\sigma(w), f(w) + g(t))$, and for all $(w_1, w_2) \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - g(B(w_1, w_2)).$$

Since s fixes the center of $\mathrm{H}(B)$, $(0,t) = s(0,t) = (0, g(t))$ for all $t \in \mathbb{F}$, i.e. g is the identity and (17) is satisfied. Since the left-hand side of (17) is symmetric, σ is symplectic provided we prove that σ is $\mathbb{F}$-linear.

Let $\mathscr{B} = \{e_1, \dots, e_n, e_{-1}, \dots, e_{-n}\}$ be a symplectic basis of W, i.e. a basis such that $\langle e_i, e_j \rangle = \delta_{i,-j}$ for all $1 \leqslant i, j \leqslant n$. Then $\mathbb{F}e_1 = \{e_1, \dots, e_n, e_{-2}, \dots, e_{-n}\}^\perp$, where we take the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$. Set $w_1 = \sigma^{-1}(e_1)$. For every $j \neq -1$, we have that $\langle w_1, \sigma^{-1}(e_j) \rangle = \langle \sigma^{-1}(e_1), \sigma^{-1}(e_j) \rangle = \langle e_1, e_j \rangle = 0$. Hence we deduce $\mathbb{F}w_1 \subseteq \{\sigma^{-1}(e_1), \dots, \sigma^{-1}(e_n), \sigma^{-1}(e_{-2}), \dots, \sigma^{-1}(e_{-n})\}^\perp$. This implies that $\sigma(\mathbb{F}w_1) \subseteq \mathbb{F}e_1$. Then, by definition of w_1, we obtain $\sigma(\mathbb{F}w_1) \subseteq \mathbb{F}\sigma(w_1)$. In a similar way, if we set $w_{-1} = \sigma^{-1}(e_{-1})$, then $\sigma(\mathbb{F}w_{-1}) \subseteq \mathbb{F}\sigma(w_{-1})$. Thus for every $a \in \mathbb{F}$, there exist $\lambda_1(a) \in \mathbb{F}$ and $\lambda_{-1}(a) \in \mathbb{F}$ such that $\sigma(aw_1) = \lambda_1(a)\sigma(w_1)$ and $\sigma(aw_{-1}) = \lambda_{-1}(a)\sigma(w_{-1})$. Now, let $a \in \mathbb{F}$. Then

$$\langle aw_1, w_{-1} \rangle = \langle w_1, aw_{-1} \rangle = a\langle w_1, w_{-1} \rangle.$$

Hence

$$\langle \sigma(aw_1), \sigma(w_{-1}) \rangle = \langle \sigma(w_1), \sigma(aw_{-1}) \rangle = a\langle \sigma(w_1), \sigma(w_{-1}) \rangle,$$

i.e.

$$\langle \lambda_1(a)\sigma(w_1), \sigma(w_{-1}) \rangle = \langle \sigma(w_1), \lambda_{-1}(a)\sigma(w_{-1}) \rangle = a\langle \sigma(w_1), \sigma(w_{-1}) \rangle.$$

Since $\langle \sigma(w_1), \sigma(w_{-1}) \rangle = \langle e_1, e_{-1} \rangle = 1$, we conclude that

$$\lambda_1(a) = \lambda_{-1}(a) = a.$$

This proves that, for every $a \in \mathbb{F}$, $\sigma(aw_1) = a\sigma(w_1)$ and $\sigma(aw_{-1}) = a\sigma(w_{-1})$. This reasoning applies with $w_j = \sigma^{-1}(e_j)$ and $w_{-j} = \sigma^{-1}(e_{-j})$ for all $j \in \{1, \dots, n\}$. Hence $\sigma(aw_{\pm j}) = a\sigma(w_{\pm j})$. Since $\langle w_i, w_j \rangle = \delta_{i,-j}$ for all $1 \leqslant i, j \leqslant n$, the set $\{w_1, \dots, w_n, w_{-1}, \dots, w_{-n}\}$ is a basis of W. Thus $\sigma(tw) = t\sigma(w)$ for all $t \in \mathbb{F}$ and $w \in W$. □

Following Weil [Wei64, III. 31.], we define the pseudosymplectic group in our setting.

Definition-Proposition 1.9. *Let $\mathrm{Ps}(B)$ be the set of elements $(\sigma, f) \in \mathrm{Sp}(W) \times \mathscr{Q}(W)$ such that for all $w_1, w_2 \in W$,*

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2), \quad \textit{i.e. (17) is satisfied.}$$

Then $\mathrm{Ps}(B)$ is a subgroup of $\mathrm{Aut}^0(\mathrm{H}(B))$, which we call the pseudosymplectic group associated to B.

Remark 1.10. The only difference between $\mathrm{Aut}^0(\mathrm{H}(B))$ and $\mathrm{Ps}(B)$ is that an element $s = (\sigma, f)$ in the pseudosymplectic group is such that f satisfies the supplementary condition (13). In particular, Remark 1.4 implies that $\mathrm{Ps}(B) = \mathrm{Aut}^0(\mathrm{H}(B))$ if $\mathbb{F} = \mathbb{F}_2$.

Proposition 1.11. [Bla93, 1.3, Proposition b] *The group $\mathrm{Ps}(B)$ defines a short exact sequence:*

$$0 \to \mathscr{Q}_a(W) \to \mathrm{Ps}(B) \to \mathrm{O}(Q) \to 1.$$

It splits if and only if either $\dim_{\mathbb{F}}(W) = 2$ or $\mathbb{F} = \mathbb{F}_2$ and $\dim_{\mathbb{F}_2}(W) = 4$.

In the following proposition, we describe the structure of $\mathrm{Ps}(B)$. Since we could not find the complete proof in the literature, we include it here.

Proposition 1.12 (The structure of $\mathrm{Ps}(B)$).

i. *There are $|W| = |\mathbb{F}|^{2n}$ additive quadratic forms on W. More precisely, if we choose a basis $\mathscr{B} = \{v_1, \ldots, v_{2n}\}$ of W, then*

$$\mathscr{Q}_a(W) = \{f_z : \sum_{1 \leqslant j \leqslant 2n} \lambda_j v_j \mapsto \sum_{1 \leqslant j \leqslant 2n} \lambda_j^2 \langle z, v_j \rangle,\ z \in W\}.$$

If $\mathbb{F} = \mathbb{F}_2$, then $f_z = \langle z, \cdot \rangle$ for all $z \in W$. Hence $\mathscr{Q}_a(W)$ is the set of linear forms $\langle z, \cdot \rangle$, $z \in W$.

ii. *Every element of $\mathbb{F}$ is a square of an element of $\mathbb{F}$.*

iii. *Let χ be a nontrivial character of $(\mathbb{F}, +)$. Then the maps $\chi \circ f_z$, where $z \in W$, are precisely all the characters of W.*

iv. *Let $s = (\sigma, f)$ be in $\mathrm{Ps}(B)$. Then:*

- ***a.*** *σ belongs to the orthogonal group $\mathrm{O}(Q)$.*
- ***b.*** *For any choice of basis $\mathscr{B} = \{v_1, \ldots, v_{2n}\}$ of W, we can define a quadratic form on W associated to σ by*

$$f_\sigma(\sum_{1 \leqslant i \leqslant 2n} \lambda_i v_i) = \sum_{1 \leqslant i < j \leqslant 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j), \tag{18}$$

  where β_σ is the bilinear form on W defined by

$$\beta_\sigma(w_1, w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2). \tag{19}$$

  Then (σ, f_σ) is an element of $\mathrm{Ps}(B)$.

- ***c.*** *s can be uniquely written in the form $(\sigma, f_\sigma + f_z)$, where $f_z : W \to \mathbb{F}$ is an additive quadratic form.*

v. *Every element σ of the orthogonal group $\mathrm{O}(Q)$ leads to exactly $|W| = |\mathbb{F}|^{2n}$ elements of $\mathrm{Ps}(B)$. These elements can be written explicitly as $\{(\sigma, f_\sigma + f_z),\ z \in W\}$.*

vi. *Let $\mathrm{Ps}_a(B)$ be the set of elements of $\mathrm{Ps}(B)$ of the form $(1, f)$. Then $\mathrm{Ps}_a(B)$ is an abelian subgroup of $\mathrm{Ps}(B)$ and*

$$\mathrm{Ps}_a(B) = \{(1, f_z),\ z \in W\}. \tag{20}$$

Proof. **i.** If $\mathbb{F} = \mathbb{F}_2$, we already mentioned in Remark 1.4 that the only condition for a function $f : W \to \mathbb{F}_2$ to be quadratic is that $(w_1, w_2) \mapsto f(w_1 + w_2) - f(w_1) - f(w_2)$ is bilinear. Hence, if f is supposed to be quadratic and additive, it is automatically linear on $\mathbb{F}_2$. Conversely, every linear form on W is also additive and thus quadratic. Moreover, the map which sends $z \in W$ to the linear form $\langle z, \cdot \rangle$ on W is an isomorphism because the symplectic form is nondegenerate. Thus $\mathscr{Q}_a(W)$ is exactly the set of linear forms $\langle z, \cdot \rangle$, where $z \in W$.

Suppose now that $\mathbb{F}$ is an arbitrary finite field of characteristic two. Let $\mathscr{B} = \{v_1, \dots, v_{2n}\}$ be a basis of W. For every $z \in W$, define $f_z : W \to \mathbb{F}$ by

$$f_z(w) = \sum_{1 \leqslant j \leqslant 2n} \lambda_j^2 \langle z, v_j \rangle \qquad \text{for all } w = \sum_{1 \leqslant j \leqslant 2n} \lambda_j v_j.$$

Then f_z is an additive quadratic form. Moreover, $f_z \neq f_{z'}$ for all $z \neq z' \in W$. In fact, if $f_z = f_{z'}$, then in particular $\langle z, v_k \rangle = \langle z', v_k \rangle$ for all $1 \leqslant k \leqslant 2n$. This implies that the linear forms $\langle z, \cdot \rangle$ and $\langle z', \cdot \rangle$ coincide on the basis $\mathscr{B}$ of W and thus are equal. Since $\langle \cdot, \cdot \rangle$ is nondegenerate, this implies the equality $z = z'$. In this way, we defined $|W|$ additive quadratic forms.

Conversely, every additive quadratic form f is entirely determined by its images on the basis elements: for all $w = \sum_{i=1}^{2n} \lambda_i v_i$,

$$f(w) = \sum_{i=1}^{2n} \lambda_i^2 f(v_i).$$

For all $i \in \{1, \dots, 2n\}$ there are $|\mathbb{F}|$ possible values for $f(v_i)$. This means that there are exactly $|\mathbb{F}|^{2n} = |W|$ additive quadratic forms on W. This is enough to conclude that $\mathscr{Q}_a(W) = \{f_z, z \in W\}$.

ii. Let $a, b \in \mathbb{F}$ be such that $a^2 = b^2$. Then $(a + b)^2 = a^2 + b^2 = 0$. Hence $a = b$, which implies that the map $a \mapsto a^2$ is injective. Since $\mathbb{F}$ is finite, this map is bijective.

iii. Let $\mathscr{B} = \{v_1, \dots, v_{2n}\}$ be a basis of W. Recall from i that $\mathscr{Q}_a(W) = \{f_z, z \in W\}$. For every $z \in W$, $\chi \circ f_z$ is a character of W because f_z is additive and χ is a character of $\mathbb{F}$. Moreover, $\chi \circ f_z = \chi \circ f_{z'}$ implies that the image of the additive quadratic form $f_z - f_{z'}$ is contained in the kernel of χ. If we set $u = z - z'$, then $f_z - f_{z'} = f_u$. If $f_u \neq 0$, there exists $w_0 \in W$ such that $f_u(w_0) = \alpha \in \mathbb{F}^*$. Fix any $\lambda \in \mathbb{F}$. Since $\mathbb{F} = \mathbb{F}^2$ by the preceding point, we can set $\alpha = \beta^2$ and $\lambda = \mu^2$. Hence $f_u(\mu\beta^{-1}w_0) = \mu^2\beta^{-2}f_u(w_0) = \lambda$. This shows that the image of f_u is equal to $\mathbb{F}$. Since χ is nontrivial, the only possibility is $f_u = f_z - f_{z'} = 0$, i.e. $z = z'$ by i. Thus the maps $\chi \circ f_z$ define $|W|$ distinct characters of W, i.e. they describe exactly the set of characters of W.

iv. Let $s = (\sigma, f)$ be in $\mathrm{Ps}(B)$.

a. Recall that, by definition of $\mathrm{Ps}(B)$, for all $w_1, w_2 \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2).$$

If we take $w_1 = w_2$, this gives $0 = B(\sigma(w), \sigma(w)) - B(w, w)$, i.e. σ preserves the quadratic form Q.

b. Let $\mathscr{B} = \{v_1, \dots, v_{2n}\}$ be a basis of W over $\mathbb{F}$. For all $w_1 = \sum_{i=1}^{2n} \lambda_i v_i$ and $w_2 = \sum_{i=1}^{2n} \mu_i v_i$,

$$\beta_\sigma(w_1, w_2) = \sum_{1 \leqslant i,j \leqslant 2n} \lambda_i \mu_j \beta_\sigma(v_i, v_j) = \sum_{1 \leqslant i < j \leqslant 2n} (\lambda_i \mu_j + \lambda_j \mu_i) \beta_\sigma(v_i, v_j),$$

where the last equality is due to the fact that β_σ is alternating (because $\sigma \in \mathrm{O}(Q)$), and thus symmetric in characteristic two. Now, for all $w = \sum_{1 \leqslant i \leqslant 2n} \lambda_i v_i$, and for all $t \in \mathbb{F}$,

$$f_\sigma(tw) = \sum_{1 \leqslant i < j \leqslant 2n} t\lambda_i t\lambda_j \beta_\sigma(v_i, v_j) = t^2 \sum_{1 \leqslant i < j \leqslant 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j) = t^2 f_\sigma(w).$$

Moreover

$$\begin{aligned} &f_\sigma(w_1 + w_2) - f_\sigma(w_1) - f_\sigma(w_2) \\ &= \sum_{1 \leqslant i < j \leqslant 2n} (\lambda_i + \mu_i)(\lambda_j + \mu_j) \beta_\sigma(v_i, v_j) - \sum_{1 \leqslant i < j \leqslant 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j) - \sum_{1 \leqslant i < j \leqslant 2n} \mu_i \mu_j \beta_\sigma(v_i, v_j) \\ &= \sum_{1 \leqslant i < j \leqslant 2n} (\lambda_i \mu_j + \lambda_j \mu_i) \beta_\sigma(v_i, v_j) \\ &= \beta_\sigma(w_1, w_2). \end{aligned}$$

Thus f_σ is a quadratic form on W such that (σ, f_σ) belongs to $\mathrm{Ps}(B)$.

c. Since $s = (\sigma, f)$ and (σ, f_σ) both belong to $\mathrm{Ps}(B)$, then for all $w_1, w_2 \in W$,

$$f(w_1 + w_2) - f(w_1) - f(w_2) = \beta_\sigma(w_1, w_2) = f_\sigma(w_1 + w_2) - f_\sigma(w_1) - f_\sigma(w_2),$$

i.e. $(f - f_\sigma)(w_1 + w_2) - (f - f_\sigma)(w_1) - (f - f_\sigma)(w_2) = 0$. This means that $f - f_\sigma$ is an additive quadratic form on W, which we can denote by $f_z = f - f_\sigma$. Hence the choice of a basis $\mathscr{B}$ leads to a unique expression of any element of $\mathrm{Ps}(B)$ in the form $(\sigma, f) = (\sigma, f_\sigma + f_z)$ where f_σ is defined in (18) and $f_z = f - f_\sigma \in \mathscr{Q}_a(W)$.

v. If we choose a basis $\mathscr{B} = \{v_1, \dots, v_{2n}\}$ of W then $\mathscr{Q}_a(W) = \{f_z, z \in W\}$. We can thus conclude thanks to the preceding points that every element σ in $\mathrm{O}(Q)$ leads to exactly $|W| = |\mathbb{F}|^{2n}$ elements of $\mathrm{Ps}(B)$, namely the elements $(\sigma, f_\sigma + f_z)$, $z \in W$.

vi. Let $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ be in $\mathrm{Ps}_a(B)$. Observe first that $f_1, f_2 \in \mathscr{Q}_a(W)$ by (17). Moreover,

$$s_1 s_2^{-1} = (1, f_1)(1, -f_2) = (1, f_1 - f_2) = (1, -f_2)(1, f_1) = s_2^{-1} s_1.$$

Since $f_1 - f_2 \in \mathscr{Q}_a(W)$, this shows that $\mathrm{Ps}_a(B)$ is an abelian subgroup of $\mathrm{Ps}(B)$. Then i yields (20).

□

The next proposition describes the relation between the symplectic and the orthogonal groups in characteristic two.

Proposition 1.13 (Relation between $\mathrm{Sp}(W)$ and $\mathrm{O}(Q)$). *We denote by q the order of the field $\mathbb{F}$ (q is a power of the prime number 2).*

i. *The orders of the symplectic and orthogonal groups are given by:*

$$|\operatorname{Sp}(W)| = q^{n^2}\prod_{k=1}^{n}(q^{2k}-1)$$

$$|\operatorname{O}(Q)| = \begin{cases} 2q^{n(n-1)}(q^n-1)\prod_{k=1}^{n-1}(q^{2k}-1) & \text{if } \nu = n \\ 2q^{n(n-1)}(q^n+1)\prod_{k=1}^{n-1}(q^{2k}-1) & \text{if } \nu = n-1. \end{cases}$$

ii. *The orthogonal group is contained in the symplectic group, i.e. $\mathrm{O}(Q) \subseteq \mathrm{Sp}(W)$. This inclusion is strict unless $q = 2$ (i.e. $\mathbb{F} = \mathbb{F}_2$), $n = 1$ (i.e. $\dim_{\mathbb{F}_2}(W) = 2$) and $\nu = n-1 = 0$.*

Proof. **i.** Cf. [Dic01, p. 94 and p. 206].

ii. Let $\sigma \in \mathrm{O}(Q)$. Then for all $w_1, w_2 \in \mathrm{Sp}(W)$,

$$\begin{aligned} 0 &= Q(\sigma(w_1+w_2)) - Q(w_1+w_2) \\ &= Q(\sigma(w_1)) + Q(\sigma(w_2)) + \langle \sigma(w_1), \sigma(w_2)\rangle - Q(w_1) - Q(w_2) - \langle w_1, w_2\rangle \\ &= \langle \sigma(w_1), \sigma(w_2)\rangle - \langle w_1, w_2\rangle. \end{aligned}$$

This proves that σ belongs to $\mathrm{Sp}(W)$. Suppose now that $n = 1$. By i,

$$|\operatorname{Sp}(W)| = q(q^2-1) \qquad \text{and} \qquad |\operatorname{O}(Q)| = \begin{cases} 2(q-1) & \text{if } \nu = 1 \\ 2(q+1) & \text{if } \nu = 0. \end{cases}$$

Hence if $q = 2$ then

$$|\operatorname{Sp}(W)| = 6 \qquad \text{and} \qquad |\operatorname{O}(Q)| = \begin{cases} 2 & \text{if } \nu = 1 \\ 6 & \text{if } \nu = 0, \end{cases}$$

i.e. $\mathrm{O}(Q) = \mathrm{Sp}(W)$ if $\nu = 0$. If $q > 2$ then

$$\frac{|\operatorname{Sp}(W)|}{|\operatorname{O}(Q)|} = \begin{cases} \frac{q(q+1)}{2} \geqslant 10 & \text{if } \nu = 1 \\ \frac{q(q-1)}{2} \geqslant 6 & \text{if } \nu = 0, \end{cases}$$

which shows in particular that $\mathrm{O}(Q)$ is strictly contained in $\mathrm{Sp}(W)$. In the same way, if $n \geqslant 2$:

$$\frac{|\operatorname{Sp}(W)|}{|\operatorname{O}(Q)|} = \begin{cases} \frac{q^{2n}-1}{2q^{-n}(q^n-1)} = \frac{(q^n+1)q^n}{2} \geqslant 10 & \text{if } \nu = n \\ \frac{q^{2n}-1}{2q^{-n}(q^n+1)} = \frac{(q^n-1)q^n}{2} \geqslant 6 & \text{if } \nu = n-1, \end{cases} \tag{21}$$

i.e. $\mathrm{O}(Q)$ is again strictly contained in $\mathrm{Sp}(W)$.

□

Remark 1.14. Comparing the orders of the groups $\mathrm{O}(Q)$ and $\mathrm{Sp}(W)$ in (21), we realize that, in general, the pseudosymplectic group is really far from being sufficient to define a Weil representation related to the symplectic group, since $\mathrm{Ps}(B)$ only surjects to the orthogonal group $\mathrm{O}(Q)$. The construction of a metaplectic group related to the entire symplectic group over $\mathbb{F} = \mathbb{F}_2$ is provided in chapter 2. The only case for which the Weil representation constructed in chapter 1 can be related to the entire symplectic group is the one for which W is a two-dimensional $\mathbb{F}_2$-vector space and the Witt index of Q is zero (or equivalently, Q has Arf invariant one). This case is treated in detail in section 3.2.

1.2 The Heisenberg representation, the Stone-von Neumann theorem, and the projective Weil representation of $\mathrm{Ps}(B)$

Recall that $\mathbb{F}$ is a finite field of characteristic two. We denote the Pontryagin dual of $(\mathbb{F}, +)$ by $\widehat{\mathbb{F}}$. We fix a nontrivial character χ of $\mathbb{F}$. Since $\mathbb{F}$ is finite, χ takes its values in the set of complex numbers of modulus one, which we denote by $\mathrm{U}(1)$. As before, we consider a nontrivial finite dimensional $\mathbb{F}$-vector space W of dimension $2n$ equipped with a nondegenerate symplectic form $\langle \cdot, \cdot \rangle$. We also fix a complete polarisation of W, i.e. two isotropic subspaces X and Y of W (which means that $X \subseteq X^\perp$ and $Y \subseteq Y^\perp$, where the orthogonal complement is relative to the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$) of maximal dimension such that $W = X \oplus Y$. In this setup, the nondegenerate symplectic form on W is given by

$$\langle w_1, w_2 \rangle = \langle x_1 + y_1, x_2 + y_2 \rangle = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle,$$

for all $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$ in W, with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Moreover, the map which sends $y \in Y$ to the linear form $\langle y, \cdot \rangle$ on X, defines an isomorphism from Y to X^*. This enables us to identify W with $X \times X^*$, fix a bilinear form B such that

$$\langle w_1, w_2 \rangle = B(w_1, w_2) - B(w_2, w_1)$$

for all $w_1, w_2 \in W$, and define the Heisenberg group $\mathrm{H}(B)$ from section 1.1. For example, the expression of the bilinear form used by Weil in this setting is:

$$B_{\text{Weil}}(w_1, w_2) = \langle x_1, y_2 \rangle.$$

An important feature of the Heisenberg group is the fact that it has, up to isomorphism, a unique irreducible unitary representation acting as χ on its center. In our setting (finite dimensional vector spaces over a finite field of characteristic two), we can give an easy proof of this result, which is known as the Stone-von Neumann theorem. This proof is based on counting the irreducible representations of $\mathrm{H}(B)$. We begin by associating to each nontrivial character of $\mathbb{F}$ a unitary irreducible representation of $\mathrm{H}(B)$.

Lemma 1.15. ***i.*** *Every character of $\mathbb{F}$ takes its values in $\{\pm 1\}$ and can uniquely be written in the form $\chi_a : t \mapsto \chi(at)$, where $a \in \mathbb{F}$. In particular there are $|\mathbb{F}| - 1$ nontrivial characters of $\mathbb{F}$.*

ii. *χ can be extended to the commutative subgroup $X \times \mathbb{F}$ of $\mathrm{H}(B)$ in $|\mathbb{F}|^n$ different ways. Let $\widetilde{\chi}$ be such an extension. Then $\widetilde{\chi}$ takes its values in $\{\pm 1, \pm i\}$. More precisely, $\widetilde{\chi}$ takes the values $\pm i$ if and only if X is not a singular space, i.e. $Q_{|X} \neq 0$.*

Proof. Suppose that the order of $\mathbb{F}$ is equal to $q = 2^m$ with $m \geqslant 1$, i.e. $\mathbb{F} = \mathbb{F}_{2^m}$.

i. Let ψ be a character of $\mathbb{F}$. Then ψ takes its values in $\{\pm 1\}$ because for all $t \in \mathbb{F}$,

$$1 = \psi(0) = \psi(2t) = \psi(t)^2.$$

The fact that $\mathbb{F} \to \widehat{\mathbb{F}} \quad (\text{where } \chi_a : t \mapsto \chi(at))$ is an isomorphism follows from the
$a \mapsto \chi_a$
fundamental theorem for finitely generated abelian groups: $(\mathbb{F}_{2^m}, +)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ and $\widehat{\mathbb{Z}/2\mathbb{Z}} \simeq \mathbb{Z}/2\mathbb{Z}$. Hence $\widehat{F} \simeq (\widehat{\mathbb{Z}/2\mathbb{Z}})^m \simeq (\widehat{\mathbb{Z}/2\mathbb{Z}})^m \simeq (\mathbb{Z}/2\mathbb{Z})^m \simeq \mathbb{F}$.

ii. Since $\mathbb{F} = \mathbb{F}_{2^m}$, $\mathbb{F}$ is a $\mathbb{F}_2$-vector space of dimension m. Let $\{t_1, \ldots, t_m\}$ be a basis of $\mathbb{F}$ over $\mathbb{F}_2$ and let $\{x_1, \ldots, x_n\}$ be a basis of X over $\mathbb{F}$. Suppose that $\widetilde{\chi}$ is an extension of χ to $X \times \mathbb{F}$. Observe that for every integer $N \geqslant 2$ and $w_1, \ldots, w_N \in W$, we have

$$\left(\sum_{k=1}^{N} w_k, 0\right) = \left(0, -\sum_{j=1}^{N-1} \sum_{l=j+1}^{N} B(w_j, w_l)\right) \prod_{k=1}^{N} (w_k, 0) \tag{22}$$

in $\mathrm{H}(B)$. In fact, $(w_1 + w_2, 0) = (0, -B(w_1, w_2))(w_1, 0)(w_2, 0)$, so (22) is true for $N = 2$. Moreover, if we suppose it to be true for some $N \geqslant 3$, then

$$\begin{aligned}
\left(\sum_{k=1}^{N+1} w_k, 0\right) &= \left(0, -B\Big(w_1, \sum_{k=2}^{N+1} w_k\Big)\right) (w_1, 0) \left(\sum_{k=2}^{N+1} w_k, 0\right) \\
&= \left(0, -B\Big(w_1, \sum_{k=2}^{N+1} w_k\Big)\right) (w_1, 0) \left(0, -\sum_{j=2}^{N} \sum_{l=j+1}^{N+1} B(w_j, w_l)\right) \prod_{k=2}^{N+1} (w_k, 0) \\
&= \left(0, -\sum_{k=2}^{N+1} B(w_1, w_k) - \sum_{j=2}^{N} \sum_{l=j+1}^{N+1} B(w_j, w_l)\right) \prod_{k=1}^{N+1} (w_k, 0) \\
&= \left(0, -\sum_{j=1}^{N} \sum_{l=j+1}^{N+1} B(w_j, w_l)\right) \prod_{k=1}^{N+1} (w_k, 0).
\end{aligned}$$

This proves (22) for $N + 1$, and we obtain (22) for every $N \geqslant 2$ by induction. Now let (x, t) be an element of the commutative subgroup $X \times \mathbb{F}$ of $\mathrm{H}(B)$. Then there exists $\mu_{k,l} \in \mathbb{F}_2$ and $\lambda_k = \sum_{l=1}^{m} \mu_{k,l} t_l \in \mathbb{F}$ such that $(x, t) = (\sum_{k=1}^{n} \lambda_k x_k, t)$. Applying (22) to $w_k = \sum_{1 \leqslant l \leqslant m} \mu_{k,l} t_l x_k$, where $1 \leqslant k \leqslant n$, we obtain:

$$\left(\sum_{k=1}^{n} \sum_{l=1}^{m} \mu_{k,l} t_l x_k, 0\right) = (0, T) \prod_{k=1}^{n} \left(\sum_{l=1}^{m} \mu_{k,l} t_l x_k, 0\right) \tag{23}$$

where

$$T = -\sum_{j=1}^{n-1}\sum_{l=j+1}^{n} B\Big(\sum_{r=1}^{m}\mu_{j,r}t_r x_j, \sum_{s=1}^{m}\mu_{l,s}t_s x_l\Big) = -\sum_{j=1}^{n-1}\sum_{l=j+1}^{n}\Big(\sum_{r=1}^{m}\mu_{j,r}t_r\Big)\Big(\sum_{s=1}^{m}\mu_{l,s}t_s\Big)\beta_{j,l}$$

and

$$\beta_{j,l} = B(x_j, x_l).$$

Applying (22) to $v_l = \mu_{k,l}t_l x_k$, where $1 \leqslant l \leqslant m$ and k is fixed, we obtain

$$\left(\sum_{l=1}^{m}\mu_{k,l}t_l x_k, 0\right) = (0, T_k)\prod_{l=1}^{m}(\mu_{k,l}t_l x_k, 0), \tag{24}$$

where

$$T_k = -\sum_{r=1}^{m-1}\sum_{s=r+1}^{m} B(\mu_{k,r}t_r x_k, \mu_{k,s}t_s x_k) = -\Big(\sum_{r=1}^{m-1}\mu_{k,r}t_r\Big)\Big(\sum_{s=r+1}^{m}\mu_{k,s}t_s\Big)\beta_{k,k}.$$

Thus:

$$\begin{aligned}
\widetilde{\chi}(x,t) &= \widetilde{\chi}((0,t)(x,0)) \\
&= \chi(t)\widetilde{\chi}\left(\sum_{1\leqslant k\leqslant n,\ 1\leqslant l\leqslant m}\mu_{k,l}t_l x_k, 0\right) \\
&\overset{(23)}{=} \chi(t)\chi(T)\widetilde{\chi}\left(\prod_{k=1}^{n}\Big(\sum_{l=1}^{m}\mu_{k,l}t_l x_k, 0\Big)\right) \\
&= \chi(t)\chi(T)\prod_{k=1}^{n}\widetilde{\chi}\Big(\sum_{l=1}^{m}\mu_{k,l}t_l x_k, 0\Big) \\
&\overset{(24)}{=} \chi(t)\chi(T)\prod_{k=1}^{n}\chi(T_k)\prod_{l=1}^{m}\widetilde{\chi}(\mu_{k,l}t_l x_k, 0).
\end{aligned}$$

Since $\mu_{k,l} \in \mathbb{F}_2$, the character $\widetilde{\chi}$ is entirely determined by the images $\widetilde{\chi}(t_l x_k)$ for all $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$, via the formula

$$\widetilde{\chi}(x,t) = \chi(t)\chi(T)\prod_{k=1}^{n}\chi(T_k)\prod_{l=1}^{m}\widetilde{\chi}(t_l x_k, 0)^{\mu_{k,l}}. \tag{25}$$

Now let $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$. Since $(t_l x_k, 0)(t_l x_k, 0) = (0, t_l^2\beta_{k,k})$, the value of $\widetilde{\chi}(t_l x_k)$ depends entirely on $\beta_{k,k} = Q(x_k)$. In fact, if $\beta_{k,k} = 0$ then the order of $(t_l x_k, 0)$ in $X \times \mathbb{F}$ is 2, and thus $\widetilde{\chi}(t_l x_k) \in \{\pm 1\}$, whereas if $\beta_{k,k} \neq 0$ then the order of $(t_l x_k, 0)$ in $X \times \mathbb{F}$ is 4, and thus $\widetilde{\chi}(t_l x_k) \in \{\pm i\}$. In any case, there are two possibilities for $\widetilde{\chi}(t_l x_k)$. This leads to at most 2^{nm} possibilities for the extension $\widetilde{\chi}$. Conversely, any choice of $\widetilde{\chi}(t_l x_k)$ in $\{\pm 1\}$ if $\beta_{k,k} = 0$, or in $\{\pm i\}$ if $\beta_{k,k} \neq 0$, for all $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$, defines by (25) a character of $X \times \mathbb{F}$ which extends χ. Therefore, there are exactly $2^{nm} = q^n = |\mathbb{F}|^n$ extensions of χ to $X \times \mathbb{F}$, which can be explicitly computed thanks to the formula (25) above. We have also shown that every such extension $\widetilde{\chi}$ takes its values in $\{\pm 1, \pm i\}$, and more precisely the values $\pm i$ are taken if and only if there exists $x \in X$ such that $Q(x) = B(x,x) \neq 0$, i.e. if and only if X is not Q-singular. □

Now we fix the counting measures of Y and the group of characters $\widehat{Y}$, respectively, by

$$\mu_Y = \frac{1}{\sqrt{|Y|}} \sum_{y \in Y} \delta_y \quad \text{and} \quad \mu_{\widehat{Y}} = \frac{1}{\sqrt{|Y|}} \sum_{\xi \in \widehat{Y}} \delta_\xi. \tag{26}$$

Since Y is finite, the set $\mathrm{L}^2(Y)$ of square-integrable functions $\phi : Y \to \mathbb{C}$ coincides with the set of all functions $\phi : Y \to \mathbb{C}$. We denote by $\mathscr{B}_Y = \{\delta_y,\ y \in Y\}$ the natural basis of $\mathrm{L}^2(Y)$. Hence $\dim_{\mathbb{C}}(\mathrm{L}^2(Y)) = |Y| = |\mathbb{F}|^n$. Moreover, we can define a scalar product $\langle \cdot, \cdot \rangle_2$ on $\mathrm{L}^2(Y)$ by:

$$\langle \phi, \psi \rangle_2 = \int_Y \phi(y) \overline{\psi(y)}\, d\mu_Y(y), \tag{27}$$

for all $\phi, \psi \in \mathrm{L}^2(Y)$. We also define the Fourier transform $\mathscr{F}$ over $\mathrm{L}^2(Y)$ and its inverse:

$$\mathscr{F}(\phi)(\xi) = \int_Y \phi(y) \xi(y)\, d\mu_Y(y)$$

$$\phi(y) = \int_{\widehat{Y}} \mathscr{F}(\phi)(\xi) \xi(-y)\, d\mu_{\widehat{Y}}(\xi),$$

for all $\phi \in \mathrm{L}^2(Y)$, $y \in Y$ and $\xi \in \widehat{Y}$. In this way, the Plancherel theorem states that $\|\mathscr{F}(\phi)\|_2 = \|\phi\|_2$ for all $\phi \in \mathrm{L}^2(Y)$. Finally we define the adjoint of an operator $\mathrm{U} \in \mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ as the operator U^* which satisfies for all $\phi, \psi \in \mathrm{L}^2(Y)$:

$$\langle U^*(\phi), \psi \rangle_2 = \langle \phi, U(\psi) \rangle_2.$$

We denote by $\mathrm{U}(\mathrm{L}^2(Y))$ the set of unitary operators of $\mathrm{L}^2(Y)$.

Now we define the Schrödinger model of the Heisenberg representation of the group $\mathrm{H}(B)$. Then, in Lemma 1.17, we prove that it is a unitary irreducible representation of $\mathrm{H}(B)$ and present its realization on $\mathrm{L}^2(Y)$.

Definition 1.16 (Schrödinger model of the Heisenberg representation). Let $\widetilde{\chi}$ be an extension of χ to the abelian subgroup $X \times \mathbb{F}$ of $\mathrm{H}(B)$. We denote by $\mathscr{H}(X, \widetilde{\chi})$ the $\mathbb{C}$-vector space of functions $\phi : \mathrm{H}(B) \to \mathbb{C}$ such that for all $(x, t) \in X \times \mathbb{F}$ and for all $h \in \mathrm{H}(B)$,

$$\phi(h(x, t)) = \widetilde{\chi}(x, t)^{-1} \phi(h).$$

Let $\rho_{\widetilde{\chi}} : \mathrm{H}(B) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}(X, \widetilde{\chi}))$ be the map defined by $\rho_{\widetilde{\chi}}(g)\phi(h) = \phi(g^{-1}h)$. In other words, $\rho_{\widetilde{\chi}}$ is the representation $\mathrm{Ind}_{X \times \mathbb{F}}^{\mathrm{H}(B)} \widetilde{\chi}$ of $\mathrm{H}(B)$ induced from the character $\widetilde{\chi}$ of the abelian subgroup $X \times \mathbb{F}$ of $\mathrm{H}(B)$.

Lemma 1.17. *The space $\mathscr{H}(X, \widetilde{\chi})$ is isomorphic to $\mathrm{L}^2(Y)$ by restriction to Y. As a representation of $\mathrm{H}(B)$ on the Hilbert space $\mathrm{L}^2(Y)$, $\rho_{\widetilde{\chi}}$ is unitary, irreducible and acts as χ on the center. It is given by the following formula: for all $g = (x + y, t) \in \mathrm{H}(B)$, for all $\varphi \in \mathrm{L}^2(Y)$ and for all $y' \in Y$,*

$$\rho_{\widetilde{\chi}}(x + y, t)\varphi(y') = \widetilde{\chi}(x, t - Q(y) + \langle x, y' \rangle - B(x + y', y))\varphi(y' - y). \tag{28}$$

The character of $\rho_{\widetilde{\chi}}$ is given by

$$\mathrm{Trace}(\rho_{\widetilde{\chi}}(x+y,t)) = \begin{cases} \chi(t)|Y| & \text{if } x=y=0 \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

and does not depend on the choice of extension $\widetilde{\chi}$. In particular, another choice of extension $\chi^\sharp$ gives a representation $\rho_{\chi^\sharp}$ which is equivalent to $\rho_{\widetilde{\chi}}$.

Remark 1.18. The fact that $\rho_{\widetilde{\chi}}$ is a representation of $\mathrm{H}(B)$ implies in particular that for all $w, w' \in W$,

$$\rho_{\widetilde{\chi}}(w)\rho_{\widetilde{\chi}}(w') = \rho_{\widetilde{\chi}}(w+w', B(w,w')) = \chi(B(w,w'))\rho_{\widetilde{\chi}}(w+w'), \tag{30}$$

where $\rho_{\widetilde{\chi}}(w) = \rho_{\widetilde{\chi}}(w,0)$.

Remark 1.19. We may wonder why we need to extend the character χ of $\mathbb{F}$ to the abelian subgroup $X \times \mathbb{F}$ of $\mathrm{H}(B)$. In fact, we could simply try to extend trivially χ to $X \times \mathbb{F}$, and define the space $\mathscr{H}(X,\chi)$ as the set of functions $\phi : \mathrm{H}(B) \to \mathbb{C}$ such that for all $(x,t) \in X \times \mathbb{F}$ and for all $h \in \mathrm{H}(B)$,

$$\phi(h(x,t)) = \chi(t)^{-1}\phi(h).$$

However, this is only possible if $B_{|X\times X} = 0$. To see this, suppose that $\widetilde{\chi}$ extends χ trivially. Then for all $(x_1,t_1),\ (x_2,t_2) \in X \times \mathbb{F}$,

$$\begin{cases} \widetilde{\chi}(x_1,t_1)\widetilde{\chi}(x_2,t_2) = \chi(t_1)\chi(t_2) = \chi(t_1+t_2) \\ \widetilde{\chi}((x_1,t_1)(x_2,t_2)) = \widetilde{\chi}(x_1+x_2, t_1+t_2+B(x_1,x_2)) = \chi(t_1+t_2+B(x_1,x_2)). \end{cases} \tag{31}$$

Since $\widetilde{\chi}$ is a character, (31) implies that $\chi(B(x_1,x_2)) = 1$ for all $x_1,\ x_2 \in X$. If $B_{|X\times X}$ is nonzero then its image is $\mathbb{F}$. This means that $\mathbb{F} \subseteq \mathrm{Ker}(\chi)$, which is impossible since χ is nontrivial by hypothesis. Hence $B_{|X\times X} = 0$. This is true, for example, if B is B_{Weil}, but it does not hold in general. Consequently, if $B_{|X\times X}$ is nonzero then the trivial extension of χ is not a character of $X \times \mathbb{F}$. This is why, following [Bla93], in Definition 1.16, we consider arbitrary characters $\widetilde{\chi}$ of $X \times \mathbb{F}$ that restrict to χ on $\mathbb{F}$ in order to get a realization of the Heisenberg representation which is valid for every bilinear form B.

Proof of Lemma 1.17. First we show that $\rho_{\widetilde{\chi}}$ is a representation of $\mathrm{H}(B)$ on $\mathscr{H}(X,\widetilde{\chi})$ acting as χ on the center. For all $g, g', h \in \mathrm{H}(B)$, for all $(x,t) \in X \times \mathbb{F}$ and for all $\phi \in \mathscr{H}(X,\widetilde{\chi})$,

- $\rho_{\widetilde{\chi}}(g)\phi(h(x,t)) = \phi(g^{-1}h(x,t)) = \widetilde{\chi}(x,t)^{-1}\phi(g^{-1}h) = \widetilde{\chi}(x,t)^{-1}\rho_{\widetilde{\chi}}(g)\phi(h)$. Hence $\rho_{\widetilde{\chi}}(g)\phi$ belongs to $\mathscr{H}(X,\widetilde{\chi})$.
- $\rho_{\widetilde{\chi}}(0,0)\phi(h) = \phi((0,0)^{-1}h) = \phi(h)$ and

$$\rho_{\widetilde{\chi}}(gg')\phi(h) = \phi((gg')^{-1}h) = \phi(g'^{-1}g^{-1}h) = \rho_{\widetilde{\chi}}(g')\phi(g^{-1}h) = (\rho_{\widetilde{\chi}}(g)\rho_{\widetilde{\chi}}(g'))\phi(h).$$

- $\rho_{\widetilde{\chi}}(0,t)\phi(h) = \phi((0,t)^{-1}h) = \phi((0,-t)h) = \phi(h(0,-t)) = \chi(-t)^{-1}\phi(h) = \chi(t)\phi(h)$, so $\rho_{\widetilde{\chi}}(0,t) = \chi(t)\,\mathrm{Id}_{\mathscr{H}(X,\widetilde{\chi})}$.

To show that $\rho_{\widetilde{\chi}}$ is unitary and irreducible, we first prove the vector space isomorphism between $\mathscr{H}(X,\widetilde{\chi})$ and $\mathrm{L}^2(Y)$, and then we use the realization of $\rho_{\widetilde{\chi}}$ on $\mathrm{L}^2(Y)$.

Consider the linear map $\mathscr{H}(X,\widetilde{\chi}) \to \mathrm{L}^2(Y)$, where $\phi_{|Y}(y) = \phi(y,0)$ for all $y \in Y$.

$$\phi \mapsto \phi_{|Y}$$

- If $\phi \in \mathscr{H}(X,\widetilde{\chi})$ satisfies $\phi_{|Y} = 0$, then for all $h = (x+y,t) \in \mathrm{H}(B)$,

  $$\phi(h) = \phi((y,0)(x,t-B(y,x))) = \widetilde{\chi}(x,t-B(y,x))^{-1}\phi(y,0) = 0.$$

  Hence the map $\phi \mapsto \phi_{|Y}$ is injective.

- Let $\varphi \in \mathrm{L}^2(Y)$. For all $(x+y,t) \in \mathrm{H}(B)$, set $\phi_\varphi(x+y,t) = \widetilde{\chi}(x,t-B(y,x))^{-1}\varphi(y)$. Then

  $$\phi_\varphi(y,0) = \chi(B(y,0))\varphi(y) = \varphi(y),$$

  i.e. $\phi_{\varphi|Y} = \varphi$, and for all $h = (x'+y',t') \in \mathrm{H}(B)$,

  $$\begin{aligned}
  \phi_\varphi(h(x,t)) &= \phi_\varphi(x'+y'+x, t'+t+B(x'+y',x)) \\
  &= \widetilde{\chi}(x+x', t+t'+B(x'+y',x)-B(y',x+x'))^{-1}\varphi(y') \\
  &= \widetilde{\chi}((x',t'-B(y',x'))(x,t))^{-1}\varphi(y') \\
  &= \widetilde{\chi}(x,t)^{-1}\widetilde{\chi}(x',t'-B(y',x'))^{-1}\varphi(y') \\
  &= \widetilde{\chi}(x,t)^{-1}\phi_\varphi(h).
  \end{aligned}$$

  Hence $\phi_\varphi \in \mathscr{H}(X,\widetilde{\chi})$ and the map $\phi \mapsto \phi_{|Y}$ is surjective.

Thus the map $\phi \mapsto \phi_{|Y}$ defines an isomorphism between $\mathscr{H}(X,\widetilde{\chi})$ and $\mathrm{L}^2(Y)$, allowing us to realize $\rho_{\widetilde{\chi}}$ on $\mathrm{L}^2(Y)$. For every $\varphi \in \mathrm{L}^2(Y)$, for every $(x+y,t) \in \mathrm{H}(B)$ and for every $y' \in Y$,

$$\begin{aligned}
\rho_{\widetilde{\chi}}(x+y,t)\varphi(y') &= \rho_{\widetilde{\chi}}(x+y,t)\phi_\varphi(y',0) \\
&= \phi_\varphi((x+y,t)^{-1}(y',0)) \\
&= \phi_\varphi((-x-y,-t+B(x+y,x+y))(y',0)) \\
&= \phi_\varphi(-x-y+y', -t+B(x+y,x+y)-B(x+y,y')) \\
&= \widetilde{\chi}(-x,-t+B(x+y,x+y-y')-B(y-y',x))^{-1}\varphi(y'-y) \\
&= \widetilde{\chi}(x,t-Q(x)-\langle x,y-y'\rangle - B(y,x-y')-Q(y)+Q(x))\varphi(y'-y) \\
&= \widetilde{\chi}(x,t-Q(y)+\langle x,y'\rangle - B(x,y)-B(y,y'))\varphi(y'-y) \\
&= \widetilde{\chi}(x,t-Q(y)+\langle x,y'\rangle - B(x+y',y))\varphi(y'-y).
\end{aligned}$$

Suppose now that $\mathscr{V} \subseteq \mathrm{L}^2(Y)$ is a $\rho_{\widetilde{\chi}}$-invariant nonzero vector subspace, i.e. $\mathscr{V} \neq 0$ and $\rho_{\widetilde{\chi}}(h)\mathscr{V} \subseteq \mathscr{V}$ for all $h \in \mathrm{H}(B)$. If $\mathscr{V} \neq \mathrm{L}^2(Y)$ then there exists a nonzero $\varphi_0 \in \mathscr{V}$ such that $\rho_{\widetilde{\chi}}(\mathrm{H}(B))\varphi_0 \neq \mathrm{L}^2(Y)$. Thus the subspace orthogonal to $\rho_{\widetilde{\chi}}(\mathrm{H}(B))\varphi_0$ is different from $\{0\}$, i.e. there exists a nonzero $\psi_0 \in \mathrm{L}^2(Y)$ such that

$$\langle \rho_{\widetilde{\chi}}(h)\varphi_0, \psi_0\rangle_2 = 0 \quad \text{for all } h \in \mathrm{H}(B). \tag{32}$$

For all functions $\varphi, \psi \in \mathrm{L}^2(Y)$ and for all $w \in W$ we define:

$$\Lambda_{\varphi,\psi}(w) = \langle \rho_{\widetilde{\chi}}(w,0)\varphi, \psi\rangle_2,$$

i.e. for $w = x + y$, where $x \in X$ and $y \in Y$:

$$\begin{aligned}
\Lambda_{\varphi,\psi}(x+y) &= \int_Y \rho_{\widetilde{\chi}}(x+y)\varphi(y')\overline{\psi(y')}\,dy' \\
&\overset{(28)}{=} \int_Y \widetilde{\chi}(x, -Q(y) + \langle x, y'\rangle - B(x+y', y))\varphi(y'-y)\overline{\psi(y')}\,dy' \\
&= \widetilde{\chi}(x, -Q(y) - B(x,y)) \int_Y \chi(\langle x, y'\rangle - B(y', y))\varphi(y'-y)\overline{\psi(y')}\,dy' \\
&= \widetilde{\chi}(x, -Q(y) - B(x,y)) \int_Y \xi_x(y') f_{\varphi,\psi,y}(y')\,dy',
\end{aligned}$$

where $f_{\varphi,\psi,y} : y' \mapsto \chi(-B(y', y))\varphi(y'-y)\overline{\psi(y')}$ is an element of $\mathrm{L}^2(Y)$ and $\xi_x : y' \mapsto \chi(\langle x, y'\rangle)$ is a character of Y. Thus:

$$\Lambda_{\varphi,\psi}(x+y) = \widetilde{\chi}(x, -Q(y) - B(x,y))\ \mathscr{F}(f_{\varphi,\psi,y})(\xi_x).$$

Hence

$$\begin{aligned}
\|\Lambda_{\varphi,\psi}\|_2^2 &= \int_X \int_Y |\Lambda_{\varphi,\psi}(x+y)|^2\,dy\,dx \\
&= \int_X \int_Y |\widetilde{\chi}(x, -Q(y) - B(x,y))|^2\ |\mathscr{F}(f_{\varphi,\psi,y})(\xi_x)|^2\,dy\,dx \\
&= \int_X \int_Y |\mathscr{F}(f_{\varphi,\psi,y})(\xi_x)|^2\,dy\,dx \\
&= \int_{\widehat{Y}} \int_Y |\mathscr{F}(f_{\varphi,\psi,y})(\xi)|^2\,dy\,d\xi \quad \text{because } x \mapsto \xi_x \text{ is a group isomorphism between } X \text{ and } \widehat{Y} \\
&= \int_Y \|\mathscr{F}(f_{\varphi,\psi,y})\|_2^2\,dy
\end{aligned}$$

Applying Plancherel's formula, we obtain:

$$\begin{aligned}
\|\Lambda_{\varphi,\psi}\|_2^2 &= \int_Y \|f_{\varphi,\psi,y}\|_2^2\,dy \\
&= \int_{Y\times Y} |\chi(-B(y', y))\varphi(y+y')\overline{\psi(y')}|^2\,dy'\,dy \\
&= \int_{Y\times Y} |\varphi(y+y')|^2 |\overline{\psi(y')}|^2\,dy\,dy' \\
&= \int_Y |\varphi(y'')|dy'' \int_Y |\overline{\psi(y')}|^2\,dy' \\
&= \|\varphi\|_2^2\ \|\psi\|_2^2.
\end{aligned}$$

Thus

$$\|\varphi_0\|_2^2\,\|\psi_0\|_2^2 = \|\Lambda_{\varphi_0,\psi_0}\|_2^2 = \int_X \int_Y |\Lambda_{\varphi_0,\psi_0}(x+y)|^2\,dy\,dx = \int_X \int_Y |\langle \rho_{\widetilde{\chi}}(x+y)\varphi_0, \psi_0\rangle_2|^2\,dy\,dx \overset{(32)}{=} 0.$$

This contradicts the fact that φ_0 and ψ_0 are nonzero. Hence $\mathscr{V} = \mathrm{L}^2(Y)$, and thus $\rho_{\widetilde{\chi}}$ is irreducible.

Now, we prove that $\rho_{\widetilde{\chi}}$ is unitary. Let $h = (x+y,t) \in \mathrm{H}(B)$ and $\varphi \in \mathrm{L}^2(Y)$, then

$$\begin{aligned}
\|\rho_{\widetilde{\chi}}(h)\varphi\|_2^2 &= \int_Y |\rho_{\widetilde{\chi}}(x+y,t)\varphi(y')|^2\, dy' \\
&\overset{(28)}{=} \int_Y |\widetilde{\chi}(x, t-Q(y)+\langle x,y'\rangle - B(x+y',y))\varphi(y'-y)|^2\, dy' \\
&= \int_Y |\varphi(y'-y)|^2\, dy' \\
&= \int_Y |\varphi(y'')|^2\, dy'' \\
&= \|\varphi\|_2^2.
\end{aligned}$$

This shows that the operator $\rho_{\widetilde{\chi}}(h)$ preserves the norm, and thus is unitary.

The last point to prove is that if $\widetilde{\chi}$ and $\chi^\sharp$ are two different extensions of χ to $X \times \mathbb{F}$ then the representations $\rho_{\widetilde{\chi}}$ and $\rho_{\chi^\sharp}$ are equivalent. To this extent, we first determine the character of the representation $\rho_{\widetilde{\chi}}$ in the basis $\mathscr{B}_Y = \{\delta_z,\ z \in Y\}$ of $\mathrm{L}^2(Y)$. By (28), for all $x \in X$, $y, y', z \in Y$ and $t \in \mathbb{F}$,

$$\begin{aligned}
\rho_{\widetilde{\chi}}(x+y,t)\delta_z(y') &= \widetilde{\chi}(x, t-Q(y)+\langle x,y'\rangle - B(x+y',y))\delta_z(y'-y) \\
&= \widetilde{\chi}(x, t-Q(y)+\langle x,z+y\rangle - B(x+z+y,y))\delta_{z+y}(y') \\
&= \widetilde{\chi}(x, t-Q(y)+\langle x,z\rangle - B(z+y,y) - B(y,x))\delta_{z+y}(y') \\
&= \widetilde{\chi}(x, t-2Q(y)+\langle x,z\rangle - B(y,x+z))\delta_{z+y}(y')
\end{aligned}$$

i.e.

$$\rho_{\widetilde{\chi}}(x+y,t)\delta_z = \widetilde{\chi}(x, t+\langle x,z\rangle - B(y,x+z))\delta_{z+y}. \tag{33}$$

Hence $\rho_{\widetilde{\chi}}(x+y,t)\delta_z$ contributes to the trace of $\rho_{\widetilde{\chi}}(x+y,t)$ with value $\widetilde{\chi}(x,t+\langle x,z\rangle)$ if $y=0$, and 0 otherwise. Thus:

$$\mathrm{Trace}(\rho_{\widetilde{\chi}}(x+y,t)) = \begin{cases} \sum\limits_{z\in Y} \widetilde{\chi}(x,t+\langle x,z\rangle) = \widetilde{\chi}(x,t) \sum\limits_{z\in Y} \chi(\langle x,z\rangle) & \text{if } y=0 \\ 0 & \text{otherwise.} \end{cases}$$

The character $\chi(\langle x,\cdot\rangle)$ is trivial on Y if and only if $x=0$. Hence (29) follows. □

Remark 1.20. In the proof of Lemma 1.17, the irreducibility of $\rho_{\widetilde{\chi}}$ can also be proven using Lemma 1.22 below. In fact, if $\mathscr{V} \subseteq \mathrm{L}^2(Y)$ is a $\rho_{\widetilde{\chi}}$-invariant vector subspace, then $\mathscr{V}$ is invariant under the operator $\rho_{\widetilde{\chi}}(w)$ for every $w \in W$, hence $\mathscr{V}$ is invariant under all the operators of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$. Then $\mathscr{V} = \mathrm{L}^2(Y)$. A third proof can be given using Schur's orthogonality

relations. We compute (29) first and then write:

$$\begin{aligned}(\rho_{\widetilde{\chi}}, \rho_{\widetilde{\chi}}) = \frac{1}{|\,\mathrm{H}(B)|}\sum_{h\in \mathrm{H}(B)} |\,\mathrm{Trace}(\rho_{\widetilde{\chi}}(h))|^2 &= \frac{1}{|\,\mathrm{H}(B)|}\sum_{(x+y,t)\in \mathrm{H}(B)} |\chi(t)|^2|Y|^2\delta_{(0,0)}(x,y)\\ &= \frac{1}{|\,\mathrm{H}(B)|}\sum_{(0,t)\in \mathrm{H}(B)} |Y|^2\\ &= \frac{|\mathbb{F}||Y|^2}{|\,\mathrm{H}(B)|}\\ &= 1.\end{aligned}$$

Corollary 1.21. *The character of $\rho_{\widetilde{\chi}}$ is a function supported on the center of $\mathrm{H}(B)$. For $h = (w,t) \in \mathrm{H}(B)$, it is given by*

$$\mathrm{Trace}(\rho_{\widetilde{\chi}}(w,t)) = \chi(t)|Y|\delta_0(w).$$

Proof. This is exactly formula (29) in Lemma 1.17 above. □

Lemma 1.22. *The family of operators $\{\rho_{\widetilde{\chi}}(w),\ w \in W\}$ is a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$.*

Proof. Suppose that for every $w = x + y \in W$, there exists $\lambda(w) = \lambda(x,y) \in \mathbb{C}$ such that

$$\sum_{w\in W} \lambda(w)\rho_{\widetilde{\chi}}(w) = 0.$$

Then for every $y' \in Y$, by (33):

$$\begin{aligned}0 = \sum_{w\in W} \lambda(w)\rho_{\widetilde{\chi}}(w)\delta_{y'} &= \sum_{w\in W} \lambda(w)\widetilde{\chi}(x, \langle x, y'\rangle - B(y, x+y'))\delta_{y'+y}\\ &= \sum_{y\in Y} \chi(-B(y,y')) \sum_{x\in X} \lambda(x,y)\widetilde{\chi}(x, \langle x, y'\rangle - B(y,x))\delta_{y'+y}.\end{aligned}$$

Since $\{\delta_{y'+y},\ y \in Y\}$ is a basis of $\mathrm{L}^2(Y)$,

$$\sum_{x\in X} \lambda(x,y)\widetilde{\chi}(x, \langle x, y'\rangle - B(y,x)) = 0 \tag{34}$$

for all $y, y' \in Y$. Since X and Y are finite and have the same cardinality $N = |Y| = |\mathbb{F}|^n$, we can define the following matrices:

$$\begin{cases} A = (a_{y',x})_{y'\in Y, x\in X} \in \mathrm{M}_N(\mathbb{C}) & \text{by } a_{y',x} = \chi(\langle x, y'\rangle)\\ B = (b_{x,y})_{x\in X, y\in Y} \in \mathrm{M}_N(\mathbb{C}) & \text{by } b_{x,y} = \lambda(x,y)\widetilde{\chi}(x, -B(y,x)).\end{cases}$$

Then, writing (34) for all $y, y' \in Y$ is equivalent to writing $AB = 0$. Now, for all $(y, y') \in Y^2$,

$$(AA^T)_{y',y} = \sum_{x\in X} \chi(\langle x, y'\rangle)\chi(\langle y, x\rangle) = \sum_{x\in X} \chi(\langle x, y' - y\rangle) = |X|\delta_y(y'),$$

where the last equality is justified by Schur's orthogonality relations. This proves that A is invertible, its inverse being $|X|^{-1}A^T$. In particular, $AB = 0$ implies that $B = 0$. Hence, for all $x \in X$ and $y \in Y$, $\lambda(x, y)\widetilde{\chi}(x, -B(y, x)) = 0$. Since $\widetilde{\chi}$ is a character, it takes its values in the unit circle $\mathrm{U}(1)$. Thus $\lambda(w) = 0$ for all $w \in W$. This proves that $\{\rho_{\widetilde{\chi}}(w), w \in W\}$ is a family of $|W| = |Y|^2 = N^2$ linearly independent operators in $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$, which is of dimension $\dim_{\mathbb{C}}(\mathrm{L}^2(\mathrm{Y}))^2 = |Y|^2 = N^2$. This is enough to conclude that $\{\rho_{\widetilde{\chi}}(w), w \in W\}$ is a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$. □

The usual approach for constructing the metaplectic representation of a covering of the (pseudo)symplectic group is based on the Stone-von Neumann Theorem. As we shall see below, we shall not use it in our construction. However, for the sake of completeness, we now state and give a simple proof of this theorem.

Theorem 1.23 (Stone-von Neumann). *The Heisenberg group* $\mathrm{H}(B)$ *has, up to isomorphism, a unique unitary irreducible representation acting as* χ *on its center, namely the Heisenberg representation, see Definition 1.16. For every nontrivial character* χ *of* $\mathbb{F}$ *and extension* $\widetilde{\chi}$ *of* χ *to* $X \times \mathbb{F}$*, the map* $\rho_{\widetilde{\chi}}$ *defined by* (28) *is a realization of the Heisenberg representation on* $\mathrm{L}^2(Y)$*, which is therefore of dimension* $|Y| = |\mathbb{F}|^n$*.*

Proof. We begin by listing the irreducible representations of $\mathrm{H}(B)$. Notice first that for all $w, w' \in W$ and $t, t' \in \mathbb{F}$,

$$\begin{aligned}
&(w, t)(w', t')(w, t)^{-1}(w', t')^{-1} \\
&= (w + w', t + t' + B(w, w'))(-w - w', -t - t' + Q(w) + Q(w') + B(w, w')) \\
&= (0, Q(w) + Q(w') + 2B(w, w') - B(w + w', w + w')) \\
&= (0, B(w, w') - B(w', w)) \\
&= (0, \langle w, w' \rangle).
\end{aligned}$$

Hence the commutator subgroup $[\mathrm{H}(B), \mathrm{H}(B)]$ of $\mathrm{H}(B)$ is exactly the center of $\mathrm{H}(B)$, and is isomorphic to $\mathbb{F}$. The quotient $\mathrm{H}(B)/[\mathrm{H}(B), \mathrm{H}(B)]$ is thus isomorphic to W, which is an abelian group and has $|W| = |Y|^2$ one-dimensional representations. This leads to $|Y|^2$ one-dimensional representations of $\mathrm{H}(B)$. Moreover, for every nontrivial character χ, extending χ to $X \times \mathbb{F}$ and using Lemma 1.17, we provide a unitary irreducible representation of $\mathrm{H}(B)$ acting as χ on its center. This implies in particular that if χ and ψ are two distinct characters of $\mathrm{H}(B)$, then $\rho_{\widetilde{\chi}}$ and $\rho_{\widetilde{\psi}}$ are not equivalent. This leads to $|\mathbb{F}| - 1$ additional inequivalent irreducible representations of $\mathrm{H}(B)$ of dimension $|Y|$ (cf. Lemma 1.15). If we sum their dimensions squared, we obtain

$$|Y|^2 + (|\mathbb{F}| - 1)|Y|^2 = |Y|^2|\mathbb{F}| = |\,\mathrm{H}(B)|,$$

which shows (as a consequence of Schur's orthogonality relations) that we found all the pairwise inequivalent irreducible representations of $\mathrm{H}(B)$. In particular, for every nontrivial character χ of $\mathbb{F}$, $\rho_{\widetilde{\chi}}$ is up to isomorphism the unique unitary irreducible representation of $\mathrm{H}(B)$ acting as χ on its center. □

The Stone-von Neumann theorem enables us to define a projective representation of $\mathrm{Ps}(B)$. In fact, for every $s \in \mathrm{Ps}(B)$, we define the map $\rho^s_{\widetilde{\chi}} : \mathrm{H}(B) \to \mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by $\rho^s_{\widetilde{\chi}}(g) =$

$\rho_{\widetilde{\chi}}(s(g))$. Then $\rho^s_{\widetilde{\chi}}$ is a unitary irreducible representation of $\mathrm{H}(B)$ which acts as χ on the center. Therefore, the Stone-von Neumann theorem implies that $\rho_{\widetilde{\chi}}$ and $\rho^s_{\widetilde{\chi}}$ are equivalent, i.e. there exists $\omega_{\widetilde{\chi}}(s) \in \mathrm{U}(\mathrm{L}^2(Y))$ such that

$$\rho^s_{\widetilde{\chi}}(h) = \omega_{\widetilde{\chi}}(s)\rho_{\widetilde{\chi}}(h)\omega_{\widetilde{\chi}}(s)^{-1} \quad \text{for all } h \in \mathrm{H}(B). \tag{35}$$

Schur's lemma also implies that $\omega_{\widetilde{\chi}}(s)$ is unique up to multiplication by an element of the unit circle $\mathrm{U}(1)$.

Remark 1.24. Notice that proving the existence of the intertwining operators in (35) does not require the Stone-von Neumann theorem. Indeed, since the representations we consider are all finite-dimensional, it is enough to use Corollary 1.21: for every $s \in \mathrm{Ps}(B)$, $\rho_{\widetilde{\chi}}$ and $\rho^s_{\widetilde{\chi}}$ have the same character. Since these representations are irreducible, they are equivalent. Thus the existence of the intertwining operator $\omega_{\widetilde{\chi}}(s) \in \mathrm{U}(\mathrm{L}^2(Y))$ satisfying (35) follows.

Lemma 1.25. *The map $s \mapsto \omega_{\widetilde{\chi}}(s)$ defines a projective representation of $\mathrm{Ps}(B)$ on $\mathrm{L}^2(Y)$.*

Proof. By definition of $\omega_{\widetilde{\chi}}(s)$ in (35), if $s = (1, 0)$ then for every $h \in \mathrm{H}(B)$,

$$\rho_{\widetilde{\chi}}(h) = \omega_{\widetilde{\chi}}(1,0)\rho_{\widetilde{\chi}}(h)\omega_{\widetilde{\chi}}(1,0)^{-1}.$$

Hence $\omega_{\widetilde{\chi}}(1,0)$ intertwines $\rho_{\widetilde{\chi}}$ with itself. By Schur's lemma, there exists $z_0 \in \mathrm{U}(1)$ such that $\omega_{\widetilde{\chi}}(1,0) = z_0 \,\mathrm{Id}_{\mathrm{L}^2(Y)}$. In particular, $\omega_{\widetilde{\chi}}(1,0)$ belongs to the center of $\mathrm{GL}(\mathrm{L}^2(Y))$ and we can choose $z_0 = 1$.
Now let $s_1, s_2 \in \mathrm{Ps}(B)$. Then for every $h \in \mathrm{H}(B)$, by (35),

$$\begin{aligned}
\rho^{s_1 s_2}_{\widetilde{\chi}}(h)\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2) &= \rho^{s_1}_{\widetilde{\chi}}(s_2(h))\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2) \\
&= \omega_{\widetilde{\chi}}(s_1)\rho_{\widetilde{\chi}}(s_2(h))\omega_{\widetilde{\chi}}(s_2) \\
&= \omega_{\widetilde{\chi}}(s_1)\rho^{s_2}_{\widetilde{\chi}}(h)\omega_{\widetilde{\chi}}(s_2) \\
&= \omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2)\rho_{\widetilde{\chi}}(h).
\end{aligned}$$

Hence both $\omega_{\widetilde{\chi}}(s_1 s_2)$ and $\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2)$ intertwine $\rho_{\widetilde{\chi}}$ and $\rho^{s_1 s_2}_{\widetilde{\chi}}$. Thus there exists $c_{\widetilde{\chi}}(s_1, s_2) \in \mathrm{U}(1)$ such that

$$\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2) = c_{\widetilde{\chi}}(s_1, s_2)\omega_{\widetilde{\chi}}(s_1 s_2). \tag{36}$$

This proves that $s \mapsto \omega_{\widetilde{\chi}}(s)$ is a projective representation of $\mathrm{Ps}(B)$. □

Remark 1.26. The fact that $s \mapsto \omega_{\widetilde{\chi}}(s)$ is a projective representation implies that the map $c_{\widetilde{\chi}} : \mathrm{Ps}(B) \times \mathrm{Ps}(B) \to \mathrm{U}(1)$ defined in (36) is a cocycle.

1.3 Lifting the projective Weil representation to a two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$

The next step is to see if the projective representation $\omega_{\widetilde{\chi}}$ defined in Lemma 1.25 can be lifted to a linear representation, either of $\mathrm{Ps}(B)$ or of a finite nontrivial central extension of this group. To answer this question, we first construct an explicit section of $\omega_{\widetilde{\chi}}$. This allows us to

give an explicit formula for the cocycle $c_{\tilde{\chi}}$. Then we prove that the best result we can obtain is to lift $\omega_{\tilde{\chi}}$ to a non-split extension $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ of $\mathrm{Ps}(B)$ by $\mu_2(\mathbb{C}) = \{\pm 1\}$, i.e. a two-fold covering of $\mathrm{Ps}(B)$.

Lemma 1.27. *Let* $s = (\sigma, f)$ *be in* $\mathrm{Ps}(B)$. *Set* $\mathrm{K}(\sigma) = \mathrm{Ker}(1-\sigma)$ *and* $\mathrm{I}(\sigma) = \mathrm{Im}(1-\sigma)$.

i. $\mathrm{K}(\sigma)$ *and* $\mathrm{I}(\sigma)$ *are orthogonal subspaces of* W *with respect to the nondegenerate symplectic form* $\langle \cdot, \cdot \rangle$.

ii. *The restriction* $f_{|\mathrm{K}(\sigma)}$ *of* f *to* $\mathrm{K}(\sigma)$ *is additive and there exists* $w_s \in W$ *such that*

$$\chi \circ f_{|\mathrm{K}(\sigma)} = \chi \circ \langle w_s, \cdot \rangle_{|\mathrm{K}(\sigma)}. \tag{37}$$

In particular, if $\mathbb{F} = \mathbb{F}_2$ *then* $f_{|\mathrm{K}(\sigma)} = \langle w_s, \cdot \rangle_{|\mathrm{K}(\sigma)}$.

iii. $w'_s \in W$ *satisfies* (37) *if and only if* $w'_s - w_s$ *belongs to* $\mathrm{I}(\sigma)$.

iv. *If* $s \in \mathrm{Ps}_a(B)$ *then the element* w_s *satisfying* (37) *is unique. More precisely,*

- $w_{(1,0)} = 0$.
- *If* $\mathbb{F} = \mathbb{F}_2$ *then* w_s *is the unique element* v *such that* $s = (1, \langle v, \cdot \rangle)$.
- *If* $\mathbb{F}$ *is arbitrary and* $s = (1, f_z)$, *then* w_s *is the unique element* v *such that* $\chi \circ f_z = \chi \circ \langle v, \cdot \rangle$.

v. *For all* $s_1 = (\sigma_1, f_1)$ *and* $s_2 = (\sigma_2, f_2)$ *in* $\mathrm{Ps}(B)$, *and for all choices of* w_{s_1},w_{s_2} *and* $w_{s_1 s_2}$ *satisfying* (37) *for* s_1, s_2 *and* $s_1 s_2$, *respectively:*

- $\mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2) \subseteq \mathrm{K}(\sigma_1\sigma_2)$ *and* $\mathrm{I}(\sigma_1\sigma_2) \subseteq \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$.
- *There exists* $w_{s_1,s_2} \in \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$ *such that* $w_{s_1 s_2} - w_{s_1} - w_{s_2} = w_{s_1,s_2}$.

Proof. **i.** For all $u = (1-\sigma)(w) \in \mathrm{I}(\sigma)$ and $w' \in \mathrm{K}(\sigma)$,

$$\begin{aligned} \langle u, w' \rangle = \langle (1-\sigma)(w), w' \rangle &= \langle w, w' \rangle - \langle \sigma(w), w' \rangle \\ &= \langle w, w' \rangle - \langle w, \sigma^{-1}(w') \rangle \\ &= \langle w, w' \rangle - \langle w, w' \rangle \\ &= 0. \end{aligned}$$

This shows that $\mathrm{I}(\sigma) \subseteq \mathrm{K}(\sigma)^\perp$. Since $\langle \cdot, \cdot \rangle$ is nondegenerate,

$$\dim(\mathrm{K}(\sigma)^\perp) = \dim W - \dim \mathrm{K}(\sigma) = \dim \mathrm{I}(\sigma).$$

Thus $\mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$.

ii. Recall from (19) that we defined $\beta_\sigma : (w_1, w_2) \mapsto B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2)$. If w_1 and w_2 belong to $\mathrm{K}(\sigma)$, then $f(w_1 + w_2) - f(w_1) - f(w_2) = \beta_\sigma(w_1, w_2) = 0$. Hence f is additive on $\mathrm{K}(\sigma)$. We can extend $f_{|\mathrm{K}(\sigma)}$ to W by taking any vector subspace V_σ such that $W = \mathrm{K}(\sigma) \oplus V_\sigma$ and setting $F(w) = F(z+v) = f(z)$ for all $w = z + v \in \mathrm{K}(\sigma) \oplus V_\sigma$. We obtain in this way an additive quadratic form on W. If $\mathbb{F} = \mathbb{F}_2$, we can apply Proposition

1.12.i and deduce that there exists $w_s \in W$ such that $F = \langle w_s, \cdot \rangle$, which implies that $f_{|\mathrm{K}(\sigma)} = \langle w_s, \cdot \rangle_{|\mathrm{K}(\sigma)}$. Now if $\mathbb{F}$ is any finite field of characteristic two, then the fact that $f_{|\mathrm{K}(\sigma)}$ is additive implies that $\chi \circ F$ is a character of W. By Proposition 1.12.iii, there exists $w_s \in W$ such that $\chi \circ F = \chi \circ \langle w_s, \cdot \rangle$. In particular, $\chi(f(w)) = \chi(\langle w_s, w \rangle)$ for all $w \in \mathrm{K}(\sigma)$.

iii. Suppose that $w'_s \in W$ satisfies the same property (37) as w_s. Then for all $w \in \mathrm{K}(\sigma)$,

$$\chi(\langle w'_s - w_s, w \rangle) = \chi(f(w) - f(w)) = 1.$$

Hence the image of the linear form $\langle w'_s - w_s, \cdot \rangle$ defined on $\mathrm{K}(\sigma)$ is contained in the kernel of χ. If this linear form were not identically zero then we would have $\mathbb{F} \subseteq \mathrm{Ker}(\chi)$, which is impossible since χ is nontrivial by hypothesis. Hence $\langle w'_s - w_s, w \rangle = 0$ for all $w \in \mathrm{K}(\sigma)$, which is equivalent to the fact that $w'_s - w_s \in \mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$.

Conversely, if $w'_s - w_s \in \mathrm{I}(\sigma) = \mathrm{K}(\sigma)^\perp$, then $\langle w'_s - w_s, w \rangle = 0$ for all $w \in \mathrm{K}(\sigma)$. Hence $\chi(\langle w'_s, w \rangle) = \chi(\langle w_s, w \rangle) = \chi(f(w))$ for all $w \in \mathrm{K}(\sigma)$, i.e. w'_s satisfies (37).

iv. Suppose that $s = (1, f) \in \mathrm{Ps}_a(B)$. Then $\mathrm{K}(\sigma) = W$, $\mathrm{I}(\sigma) = \{0\}$ and $f \in \mathscr{Q}_a(W)$ by (17). By iii, w'_s satisfies (37) if and only if $w'_s - w_s \in \mathrm{I}(\sigma)$, i.e. $w'_s = w_s$. This shows that the element w_s satisfying (37) is unique. If $s = (1, 0)$ then (37) is written

$$1 = \chi(\langle w_s, w \rangle) \quad \text{for all } w \in W.$$

Since $\langle w_s, \cdot \rangle$ is a linear form, its image is either $\{0\}$ or $\mathbb{F}$. Since χ is nontrivial, this implies that $\langle w_s, \cdot \rangle = 0$, i.e. $w_s = 0$ because $\langle \cdot, \cdot \rangle$ is nondegenerate. Suppose now that $\mathbb{F} = \mathbb{F}_2$. By Propositions 1.12.i and 1.12.vi, there exists a unique element $z \in W$ such that $f = \langle z, \cdot \rangle$. Hence (37) is written

$$\chi(\langle z, w \rangle) = \chi(\langle w_s, w \rangle) \quad \text{for all } w \in W,$$

and we can conclude in the same way that $w_s - z = 0$, i.e. $w_s = z$. Finally, suppose that $\mathbb{F}$ is arbitrary. Then there exists a unique $z \in W$ such that $f = f_z$ by Propositions 1.12.i and 1.12.vi. Since the maps $\chi \circ f_w$, $w \in W$, describe completely the set of characters of W by Proposition 1.12.iii, there exists a unique $v \in W$ such that $\chi \circ f_z = \chi \circ \langle v, \cdot \rangle$. Then (37) is written:

$$\chi(\langle w_s, w \rangle) = \chi(f_z(w)) = \chi(\langle v, w \rangle) \quad \text{for all } w \in W.$$

Hence $w_s = v$, which concludes the proof of iv.

v. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\mathrm{Ps}(B)$.

- Let $w \in W$ such that $\sigma_1(w) = \sigma_2(w) = w$, then $(\sigma_1\sigma_2)(w) = \sigma_1(w) = w$. Hence $\mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2) \subseteq \mathrm{K}(\sigma_1\sigma_2)$. Taking the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$, we obtain $\mathrm{K}(\sigma_1\sigma_2)^\perp \subseteq (\mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2))^\perp$, i.e.

$$\mathrm{I}(\sigma_1\sigma_2) \subseteq \mathrm{K}(\sigma_1)^\perp + \mathrm{K}(\sigma_2)^\perp = \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2).$$

- By definition of w_{s_1}, w_{s_2} and $w_{s_1 s_2}$, for every $w \in \mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2) \subseteq \mathrm{K}(\sigma_1\sigma_2)$,

$$\begin{aligned}\chi((f_2 + f_1 \circ \sigma_2)(w)) &= \chi(\langle w_{s_1 s_2}, w\rangle)\\ \chi(f_2(w))\chi(f_1(w)) &= \chi(\langle w_{s_1 s_2}, w\rangle)\\ \chi(\langle w_{s_1}, w\rangle)\chi(\langle w_{s_2}, w\rangle) &= \chi(\langle w_{s_1 s_2}, w\rangle)\\ \chi(\langle w_{s_1 s_2} - w_{s_1} - w_{s_2}, w\rangle) &= 1.\end{aligned}$$

  This implies that $\langle w_{s_1 s_2} - w_{s_1} - w_{s_2}, w\rangle = 0$ for all $w \in \mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2)$, since otherwise χ would be trivial. Thus $w_{s_1 s_2} - w_{s_1} - w_{s_2} \in (\mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2))^\perp = \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$, and we can set $w_{s_1,s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2} \in \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$.

□

Definition-Proposition 1.28 (Projective Weil representation of $\mathrm{Ps}(B)$). *For every $s = (\sigma, f) \in \mathrm{Ps}(B)$ and for every $w_s \in W$ satisfying* (37)*, we define:*

$$\omega_{\tilde{\chi}}(s, w_s) = |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\tilde{\chi}}(s(w))^{-1}\rho_{\tilde{\chi}}(w_s)\rho_{\tilde{\chi}}(w) \tag{38}$$

where $\gamma(\sigma)$ is any complex number such that $|\gamma(\sigma)|^2 = |\mathbb{F}|^{-\dim \mathrm{I}(\sigma)}$. Then $\omega_{\tilde{\chi}}(s, w_s)$ satisfies the following properties:

i. *For every $h \in \mathrm{H}(B)$,*

$$\rho_{\tilde{\chi}}(s(h)) = \omega_{\tilde{\chi}}(s, w_s)\rho_{\tilde{\chi}}(h)\omega_{\tilde{\chi}}(s, w_s)^{-1}. \tag{39}$$

ii. *For every $u \in \mathrm{I}(\sigma)$, choose any $w \in W$ such that $u = (1 - \sigma)(w)$. Then*

$$\omega_{\tilde{\chi}}(s, w_s) = \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w\rangle + B(u, w + w_s))\rho_{\tilde{\chi}}(u + w_s). \tag{40}$$

iii. $\omega_{\tilde{\chi}}(s, w_s) \in \mathrm{U}(\mathrm{L}^2(Y))$.

iv. *If we make another choice of element w_s' satisfying* (37) *then $\omega_{\tilde{\chi}}(s, w_s)$ and $\omega_{\tilde{\chi}}(s, w_s')$ can only differ by a sign. More precisely:*

$$\omega_{\tilde{\chi}}(s, w_s') = \chi(-f(v) + \langle v, w_s\rangle + B(w_s' - w_s, v - w_s))\omega_{\tilde{\chi}}(s, w_s) \tag{41}$$

where v is any element such that $w_s' - w_s = (1 - \sigma)(v)$.

Thus for each choice of the set $\{w_s$ satisfying (37)*, $s \in \mathrm{Ps}(B)\}$, the map $s \mapsto \omega_{\tilde{\chi}}(s, w_s)$ is an explicit section of the projective Weil representation on $\mathrm{L}^2(Y)$.*

Remark 1.29. To be completely rigorous in the definition (38) of the operator $\omega_{\tilde{\chi}}(s, w_s)$, we should consider the parameter $\gamma(\sigma)$ as depending not only on σ but also on $s = (\sigma, f)$ and the choice of w_s. In other words, we should write $\gamma(s, w_s)$ instead of $\gamma(\sigma)$. However, our goal is to lift this projective representation to a linear representation of the smallest possible extension of $\mathrm{Ps}(B)$. By replacing $\gamma(\sigma)$ with $\gamma(s, w_s)$ in Corollary 1.33, we can show in the same way

that c_χ is not a coboundary. Therefore the best result we could obtain with $\gamma(s, w_s)$ is to lift $s \mapsto \omega_{\tilde{\chi}}(s, w_s)$ to a two-fold covering of $\mathrm{Ps}(B)$. Since by Proposition 1.36, we are already able to construct such two-fold covering with $\gamma(\sigma)$, we have chosen to restrict ourselves to the normalizing factors $\gamma(\sigma)$ rather than $\gamma(s, w_s)$ in order to simplify the computations and the notation. (The only effect that the parameter $\gamma(s, w_s)$ could provide is some more non-isomorphic two-fold coverings of $\mathrm{Ps}(B)$ in Proposition 1.36.)

Proof of Proposition 1.28. **i.** Let $s = (\sigma, f) \in \mathrm{Ps}(B)$, $w_s \in W$ satisfying (37) and $h = (v, t) \in \mathrm{H}(B)$. Then:

$$\begin{aligned}\rho_{\tilde{\chi}}(s(h))\omega_{\tilde{\chi}}(s, w_s) &= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w\in W}\rho_{\tilde{\chi}}(s(h))\rho_{\tilde{\chi}}(s(w))^{-1}\rho_{\tilde{\chi}}(w_s)\rho_{\tilde{\chi}}(w)\\ &= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w\in W}\rho_{\tilde{\chi}}(s(h)s(w)^{-1}w_s w).\end{aligned}$$

Setting $w' = w - v$, we see that

$$\begin{aligned}&s(h)s(w)^{-1}w_s w\\ &= (\sigma(v), f(v) + t)(-\sigma(w), -f(w) + Q(\sigma(w)))w_s w\\ &= (\sigma(v - w), f(v) - f(w) + t + Q(\sigma(w)) - B(\sigma(v), \sigma(w)))w_s w\\ &= (\sigma(v - w), -f(w - v) - B(\sigma(w - v), \sigma(v)) + B(w - v, v) + t + Q(\sigma(w)) - B(\sigma(v), \sigma(w)))w_s w\\ &= (-\sigma(w'), -f(w') - B(\sigma(w'), \sigma(v)) + B(w', v) + t + Q(\sigma(w')) + Q(\sigma(v)) + \langle\sigma(w'), \sigma(v)\rangle\\ &\quad - B(\sigma(v), \sigma(w' + v)))w_s w\\ &= (-\sigma(w'), -f(w') + B(w', v) + t + Q(\sigma(w')) + \langle\sigma(w'), \sigma(v)\rangle + \langle\sigma(v), \sigma(w')\rangle - 2B(\sigma(v), \sigma(w')))\\ &\quad \cdot w_s w\\ &= (-\sigma(w'), -f(w') + B(w', v) + t + Q(\sigma(w')))w_s(w' + v)\\ &= (-\sigma(w'), -f(w') + Q(\sigma(w')))(0, B(w', v) + t)w_s w'(v, -B(w', v))\\ &= s(w')^{-1}w_s w'(v, t)\\ &= s(w')^{-1}w_s w'h.\end{aligned}$$

Hence:

$$\begin{aligned}\rho_{\tilde{\chi}}(s(h))\omega_{\tilde{\chi}}(s, w_s) &= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w\in W}\rho_{\tilde{\chi}}(s(h)s(w)^{-1}w_s w)\\ &= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w'\in W}\rho_{\tilde{\chi}}(s(w')^{-1}w_s w'h)\\ &= \omega_{\tilde{\chi}}(s, w_s)\rho_{\tilde{\chi}}(h).\end{aligned}$$

ii. Let $s = (\sigma, f) \in \mathrm{Ps}(B)$ and $w_s \in W$ satisfying (37). Then:

$$\omega_{\tilde{\chi}}(s, w_s) = |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w\in W}\rho_{\tilde{\chi}}(s(w)^{-1}w_s w),$$

where

$$\begin{aligned}
&s(w)^{-1}w_s w\\
&= (-\sigma(w), -f(w)+Q(\sigma(w)))(w_s+w, B(w_s,w))\\
&= ((1-\sigma)(w)+w_s, -f(w)+Q(\sigma(w))+B(w_s,w)-B(\sigma(w),w_s+w))\\
&= ((1-\sigma)(w)+w_s, -f(w)+Q(\sigma(w))-Q(w)+\langle w_s,w\rangle+B((1-\sigma)(w),w_s+w))\\
&= ((1-\sigma)(w)+w_s, f(w)+\langle w_s,w\rangle+B((1-\sigma)(w),w_s+w))
\end{aligned} \tag{42}$$

since $0 = f(2w) = 2f(w)+Q(\sigma(w))-Q(w) = Q(\sigma(w))-Q(w)$ by (17). Hence

$$\rho_{\tilde{\chi}}(s(w)^{-1}w_s w) = \chi(f(w)+\langle w_s,w\rangle+B((1-\sigma)(w),w_s+w))\rho_{\tilde{\chi}}((1-\sigma)(w)+w_s). \tag{43}$$

Now, for every $v \in \mathrm{K}(\sigma)$, by (43):

$$\begin{aligned}
&\rho_{\tilde{\chi}}(s(w+v)^{-1}w_s(w+v))\\
&= \chi(f(w+v)+\langle w_s,w+v\rangle+B((1-\sigma)(w+v),w_s+w+v))\\
&\quad \rho_{\tilde{\chi}}((1-\sigma)(w+v)+w_s)\\
&= \chi(f(w)+f(v)+B(\sigma(w),\sigma(v))-B(w,v)+\langle w_s,w\rangle+\langle w_s,v\rangle\\
&\quad +B((1-\sigma)(w),w_s+w)+B((1-\sigma)(w),v))\rho_{\tilde{\chi}}((1-\sigma)(w)+w_s)\\
&= \chi(f(w)+f(v)+\langle w_s,w\rangle+\langle w_s,v\rangle+B((1-\sigma)(w),w_s+w))\rho_{\tilde{\chi}}((1-\sigma)(w)+w_s)\\
&= \chi(f(v)+\langle w_s,v\rangle)\chi(f(w)+\langle w_s,w\rangle+B((1-\sigma)(w),w_s+w)))\rho_{\tilde{\chi}}((1-\sigma)(w)+w_s)\\
&= \chi(f(v)+\langle w_s,v\rangle)\rho_{\tilde{\chi}}(s(w)^{-1}w_s w),
\end{aligned}$$

i.e. by (37):

$$\rho_{\tilde{\chi}}(s(w+v)^{-1}w_s(w+v)) = \rho_{\tilde{\chi}}(s(w)^{-1}w_s w).$$

This shows that $\rho_{\tilde{\chi}}(s(w)^{-1}w_s w)$ does not depend on the left coset of w modulo $\mathrm{K}(\sigma)$. This together with (43) enables us to write:

$$\begin{aligned}
&\omega_{\tilde{\chi}}(s,w_s)\\
&= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w\in W}\rho_{\tilde{\chi}}(s(w)^{-1}w_s w)\\
&= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma)\sum_{w+\mathrm{K}(\sigma)\in W/\,\mathrm{K}(\sigma)}|\,\mathrm{K}(\sigma)|\rho_{\tilde{\chi}}(s(w)^{-1}w_s w)\\
&= \gamma(\sigma)\sum_{w+\mathrm{K}(\sigma)\in W/\,\mathrm{K}(\sigma)}\chi(f(w)+\langle w_s,w\rangle+B((1-\sigma)(w),w_s+w))\rho_{\tilde{\chi}}((1-\sigma)(w)+w_s)\\
&= \gamma(\sigma)\sum_{u\in \mathrm{I}(\sigma)}\chi(f(w)+\langle w_s,w\rangle+B(u,w_s+w))\rho_{\tilde{\chi}}(u+w_s),
\end{aligned}$$

where $w \in W$ is any element such that $u = (1-\sigma)(w)$, and we use the isomorphism between $W/\,\mathrm{K}(\sigma)$ and $\mathrm{I}(\sigma)$.

iii. Because of the intertwining property (39) and Schur's lemma, in order to verify that $\omega_{\tilde{\chi}}(s,w_s)$ is invertible, it is enough to show that it is not the zero operator. The expression (40) of $\omega_{\tilde{\chi}}(s,w_s)$ implies that $\omega_{\tilde{\chi}}(s,w_s)$ has coefficients equal to ± 1 in the basis

$\{\rho_{\tilde{\chi}}(w), w \in W\}$ of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ (cf. Lemmas 1.15.i and 1.22). Thus $\omega_{\tilde{\chi}}(s, w_s)$ cannot be the zero operator. Furthermore, using the unitarity of $\rho_{\tilde{\chi}}$, we can write

$$\begin{aligned}\omega_{\tilde{\chi}}(s, w_s)\omega_{\tilde{\chi}}(s, w_s)^* &= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w\in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w) \sum_{w'\in W} \rho_{\tilde{\chi}}(s(w')^{-1}w_s w')^* \\ &= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w,w'\in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w)\rho_{\tilde{\chi}}(w'^{-1}w_s^{-1}s(w')).\end{aligned}$$

To compute the expression on the right-hand side, observe first that by (42):

$$\begin{aligned}&s(w)^{-1}w_s w w'^{-1}w_s^{-1}s(w') \\ &= ((1-\sigma)(w) + w_s, f(w) + \langle w_s, w\rangle + B((1-\sigma)(w), w_s + w)) \\ &\quad \cdot (-(1-\sigma)(w') - w_s, -f(w') - \langle w_s, w'\rangle - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s)) \\ &= ((1-\sigma)(w-w'), f(w) - f(w') + \langle w_s, w - w'\rangle + B((1-\sigma)(w), w_s + w) \\ &\quad - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s) - B((1-\sigma)(w) + w_s, (1-\sigma)(w') + w_s)) \\ &= ((1-\sigma)(w-w'), T(w, w', w_s)),\end{aligned}$$

where, if we set $z = w - w'$,

$$\begin{aligned}&T(w, w', w_s) \\ &= f(w-w') + B(\sigma(w-w'), \sigma(w')) - B(w-w', w') + \langle w_s, w-w'\rangle + B((1-\sigma)(w), w_s + w) \\ &\quad - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s) - B((1-\sigma)(w) + w_s, (1-\sigma)(w') + w_s) \\ &= f(z) + B(\sigma(z), \sigma(w')) - B(z, w') + \langle w_s, z\rangle + B((1-\sigma)(z+w'), w_s + z + w') \\ &\quad - B((1-\sigma)(w'), w_s + w') + Q((1-\sigma)(w') + w_s) - B((1-\sigma)(z+w') + w_s, (1-\sigma)(w') + w_s) \\ &= f(z) + Q(z) + \langle w_s, z\rangle - B(\sigma(z), z) + \langle z, (1-\sigma)(w')\rangle.\end{aligned}$$

Thus, by (43),

$$\begin{aligned}\omega_{\tilde{\chi}}(s, w_s)\omega_{\tilde{\chi}}(s, w_s)^* &= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w,w'\in W} \rho_{\tilde{\chi}}(s(w)^{-1}w_s w)\rho_{\tilde{\chi}}(w'^{-1}w_s^{-1}s(w')) \\ &= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z\in W} \chi(f(z) + Q(z) + \langle w_s, z\rangle - B(\sigma(z), z)) \\ &\qquad \sum_{w'\in W} \chi(\langle z, (1-\sigma)(w')\rangle)\rho_{\tilde{\chi}}((1-\sigma)(z)) \\ &= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z\in W} \chi(f(z) + Q(z) + \langle w_s, z\rangle - B(\sigma(z), z)) \\ &\qquad \sum_{w'\in W} \chi(\langle (1-\sigma^{-1})(z), w'\rangle)\rho_{\tilde{\chi}}((1-\sigma)(z)) \\ &= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z\in \mathrm{K}(\sigma)} |W|\rho_{\tilde{\chi}}(0) \\ &= |\,\mathrm{K}(\sigma)|^{-1}|\gamma(\sigma)|^2|W|\,\mathrm{Id}_{\mathrm{L}^2(Y)} \\ &= |\mathbb{F}|^{-\dim \mathrm{K}(\sigma)}|\mathbb{F}|^{-\dim \mathrm{I}(\sigma)}|\mathbb{F}|^{\dim \mathrm{K}(\sigma)+\dim \mathrm{I}(\sigma)}\,\mathrm{Id}_{\mathrm{L}^2(Y)} \\ &= \mathrm{Id}_{\mathrm{L}^2(Y)},\end{aligned}$$

where the three last equalities are justified by the fact that $w' \mapsto \chi(\langle (1-\sigma^{-1})(z), w'\rangle)$ is a character of W which is trivial if and only if $z \in \mathrm{K}(\sigma)$, Schur's orthogonality relations, the definition of $\gamma(\sigma)$ and the isomorphism between $W/\mathrm{K}(\sigma)$ and $\mathrm{I}(\sigma)$. Therefore $\omega_{\widetilde{\chi}}(s, w_s)$ is a unitary operator of $\mathrm{L}^2(Y)$.

iv. Let $w_s' \in W$ be such that (37) is satisfied, i.e. $\chi \circ f_{|\mathrm{K}(\sigma)} = \chi \circ \langle w_s', \cdot\rangle_{|\mathrm{K}(\sigma)}$. Then Lemma 1.27.iii implies that $w_s' - w_s \in \mathrm{I}(\sigma)$. Hence we can set $w_s' - w_s = (1-\sigma)(v)$ for some $v \in W$. Then

$$\omega_{\widetilde{\chi}}(s, w_s') = |\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w\in W} \rho_{\widetilde{\chi}}(s(w)^{-1}w_s'w),$$

where, setting $z = w + v$ and applying (42), we find that

$$\begin{aligned}
s(w)^{-1}w_s'w &= ((1-\sigma)(w) + w_s', f(w) + \langle w_s', w\rangle + B((1-\sigma)(w), w_s' + w)) \\
&= ((1-\sigma)(w) + w_s + (1-\sigma)(v), f(w) + \langle w_s + (1-\sigma)(v), w\rangle \\
&\quad + B((1-\sigma)(w), w_s + (1-\sigma)(v) + w)) \\
&= ((1-\sigma)(z) + w_s, f(z-v) + \langle w_s + (1-\sigma)(v), z - v\rangle \\
&\quad + B((1-\sigma)(z-v), w_s + (1-\sigma)(v) + z - v)) \\
&= ((1-\sigma)(z) + w_s, f(z) + \langle w_s, z\rangle + B((1-\sigma)(z), w_s + z) - f(v) \\
&\quad - \langle w_s, v\rangle - B((1-\sigma)(v), w_s - v)) \\
&= ((1-\sigma)(z) + w_s, f(z) + \langle w_s, z\rangle + B((1-\sigma)(z), w_s + z)) \\
&\quad \cdot (0, -f(v) - \langle w_s, v\rangle + B(w_s' - w_s, v - w_s)) \\
&\overset{(42)}{=} s(z)^{-1}w_s z(0, -f(v) - \langle w_s, v\rangle + B(w_s' - w_s, v - w_s))
\end{aligned}$$

Thus the change of variables $z = w + v$ in the sum over $w \in W$ leads to

$$\begin{aligned}
\omega_{\widetilde{\chi}}(s, w_s') &= |\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{z\in W} \chi(-f(v) + \langle v, w_s\rangle + B(w_s' - w_s, v - w_s))\rho_{\widetilde{\chi}}(s(z)^{-1}w_s z) \\
&= \chi(-f(v) + \langle v, w_s\rangle + B(w_s' - w_s, v - w_s))\omega_{\widetilde{\chi}}(s, w_s).
\end{aligned}$$

□

Notation 1.30. In Proposition 1.28, we defined an operator $\omega_{\widetilde{\chi}}(s, w_s)$ for all $s = (\sigma, f) \in \mathrm{Ps}(B)$ and for all w_s satisfying (37). From now on, to simplify the notation, if it is clear that a choice of w_s has been made, then we will write indifferently $\omega_{\widetilde{\chi}}(s, w_s)$ or $\omega_{\widetilde{\chi}}(s)$.

Corollary 1.31 (Computation of the cocycle $c_{\widetilde{\chi}}$). *We keep the same notations as in Lemma 1.27 and Proposition 1.28, and we assume that a choice of* $w_s \in W$ *satisfying* (37) *has been made for all* $s \in \mathrm{Ps}(B)$*. The cocycle* $c_{\widetilde{\chi}}$ *associated to the projective Weil representation* $\omega_{\widetilde{\chi}}$ *is given by the following formula. For all* $s_1 = (\sigma_1, f_1)$ *and* $s_2 = (\sigma_2, f_2)$ *in* $\mathrm{Ps}(B)$*,*

$$c_{\widetilde{\chi}}(s_1, s_2) = C_{1,2} \sum_{\substack{u_1\in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1+u_2 = w_{s_1,s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1\rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2\rangle) \\ \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2})) \tag{44}$$

where $C_{1,2} = \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1}$*,* $w_{s_1,s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2}$ *and* w_i *is any element such that* $(1-\sigma_i)(w_i) = u_i$ *for* $i \in \{1, 2\}$*. In particular, the cocycle* $c_{\widetilde{\chi}}$ *does not depend on the choice of the extension* $\widetilde{\chi}$ *of* χ*: we can denote it by* c_χ *instead of* $c_{\widetilde{\chi}}$*.*

Proof. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\mathrm{Ps}(B)$. The fact that $\omega_{\widetilde{\chi}}$ is a projective representation implies that $c_{\widetilde{\chi}}$ satisfies

$$\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2) = c_{\widetilde{\chi}}(s_1, s_2)\omega_{\widetilde{\chi}}(s_1 s_2), \tag{45}$$

where we recall from Remark 1.30 that $\omega_{\widetilde{\chi}}(s_1) = \omega_{\widetilde{\chi}}(s_1, w_{s_1})$ and $\omega_{\widetilde{\chi}}(s_2) = \omega_{\widetilde{\chi}}(s_2, w_{s_2})$. Now we compute separately $\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2)$ and $\omega_{\widetilde{\chi}}(s_1 s_2)$, using the expression (40) found in Proposition 1.28 and (30):

$$\begin{aligned}
&\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2) \\
&= \gamma(\sigma_1)\gamma(\sigma_2) \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1) \\ u_2 \in \mathrm{I}(\sigma_2)}} \chi(f_1(w_1) + \langle w_{s_1}, w_1\rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2\rangle) \\
&\qquad\qquad \chi(B(u_2, w_2 + w_{s_2}))\rho_{\widetilde{\chi}}(u_1 + w_{s_1})\rho_{\widetilde{\chi}}(u_2 + w_{s_2}) \\
&= \gamma(\sigma_1)\gamma(\sigma_2) \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1) \\ u_2 \in \mathrm{I}(\sigma_2)}} \chi(f_1(w_1) + \langle w_{s_1}, w_1\rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2\rangle) \\
&\qquad\qquad \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2}))\rho_{\widetilde{\chi}}(u_1 + w_{s_1} + u_2 + w_{s_2})
\end{aligned}$$

and

$$\omega_{\widetilde{\chi}}(s_1 s_2) = \gamma(\sigma_1\sigma_2) \sum_{u \in \mathrm{I}(\sigma_1\sigma_2)} \chi((f_2 + f_1 \circ \sigma_2)(w) + \langle w_{s_1 s_2}, w\rangle + B(u, w + w_{s_1 s_2}))\rho_{\widetilde{\chi}}(u + w_{s_1 s_2}).$$

Now, using the fact that the operators $\rho_{\widetilde{\chi}}(w)$, $w \in W$, form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ (cf. Lemma 1.22) and identifying the coefficients of $\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2)$ and $\omega_{\widetilde{\chi}}(s_1 s_2)$ in this basis, we obtain for all $u \in \mathrm{I}(\sigma_1\sigma_2)$:

$$\begin{aligned}
&c_{\widetilde{\chi}}(s_1, s_2)\gamma(\sigma_1\sigma_2)\chi((f_2 + f_1 \circ \sigma_2)(w) + \langle w_{s_1 s_2}, w\rangle + B(u, w + w_{s_1 s_2})) \\
&= \gamma(\sigma_1)\gamma(\sigma_2) \\
&\quad \cdot \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1 + u_2 + w_{s_1} + w_{s_2} = u + w_{s_1 s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1\rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2\rangle) \\
&\qquad\qquad \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2})).
\end{aligned}$$

Taking $u = 0$ and replacing $w_{s_1 s_2} - w_{s_1} - w_{s_2}$ with w_{s_1, s_2} (defined in Lemma 1.27.v) in the previous equation, we finally obtain:

$$\begin{aligned}
c_{\widetilde{\chi}}(s_1, s_2) &= \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1} \\
&\quad \cdot \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1, s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1\rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2\rangle) \\
&\qquad\qquad \chi(B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2})).
\end{aligned}$$

In particular, this expression does not depend on the choice of the extension $\widetilde{\chi}$ of χ, so we can write c_χ instead of $c_{\widetilde{\chi}}$. □

Now that we have determined an explicit formula for the cocycle c_χ, the next goal is to find its range in order to construct the smallest possible extension of $\mathrm{Ps}(B)$. In fact, the range of c_χ will depend on the choice made for $\gamma(\sigma)$ in the definition (38) of the operator $\omega_{\widetilde{\chi}}(s, w_s)$, for every $s = (\sigma, f) \in \mathrm{Ps}(B)$. Recall from Proposition 1.28 that the only condition on $\gamma(\sigma)$ is that $|\gamma(\sigma)|^2 = |\mathbb{F}|^{-\dim \mathrm{I}(\sigma)}$ to ensure that $\omega_{\widetilde{\chi}}(s, w_s)$ is a unitary operator. This means that

$$\gamma(\sigma) = |\mathbb{F}|^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)}, \quad \text{where} \quad \theta(\sigma) \in \mathbb{R}. \tag{46}$$

Remark 1.32. Recall that we want to lift the projective Weil representation of $\mathrm{Ps}(B)$ defined in Proposition 1.28 to a linear representation of either $\mathrm{Ps}(B)$ or a smallest possible extension of $\mathrm{Ps}(B)$. The range of c_χ will therefore play a crucial role, since it determines the size of the extension we construct. Before examining the range of this cocycle, we can first highlight in Corollary 1.33 the fact that this cocycle is not a coboundary. This is of critical importance for the construction of the Weil representation since it shows that we need to define a nontrivial central extension of $\mathrm{Ps}(B)$ in order to lift the projective Weil representation to a linear one. In fact, we will see in Corollary 1.35 that the best result we can obtain is to lift the projective representation $s \mapsto \omega_{\tilde{\chi}}(s)$ to an extension of $\mathrm{Ps}(B)$ by two elements. This should be put into perspective with the other classical cases: for a finite field of odd characteristic, we do not need any nontrivial covering to linearize the projective Weil representation of $\mathrm{Sp}(W)$, whereas in the real case we also have to define a two-fold metaplectic covering of $\mathrm{Sp}(W)$ (cf. for example [AP14]).

Corollary 1.33. *The cocycle c_χ restricted to the abelian subgroup $\mathrm{Ps}_a(B)$ of $\mathrm{Ps}(B)$ is not symmetric. In particular, c_χ is not a coboundary.*

Proof. Recall from Proposition 1.12.vi that $\mathrm{Ps}_a(B)$ denotes the abelian subgroup of $\mathrm{Ps}(B)$ consisting of the elements of the form $(1, f)$. Let w_1 and w_2 in W be such that $\chi(\langle w_1, w_2\rangle) \neq 1$. Such a choice is possible if $W \neq \{0\}$. For example, if we pick an arbitrary w_1 in $W\backslash\{0\}$, then $\langle w_1, \cdot\rangle$ is a linear form on W with image $\mathbb{F}$. Hence there exists w_2 such that $\chi(\langle w_1, w_2\rangle) \neq 1$ because χ is nontrivial. In turn, $\chi(\langle w_1, w_2\rangle) \neq 1$ is equivalent to have $\chi(B(w_1, w_2)) \neq \chi(B(w_2, w_1))$. Moreover, $\chi(\langle w_1, \cdot\rangle)$ and $\chi(\langle w_2, \cdot\rangle)$ are two distinct characters of W. By Proposition 1.12.iii, there exists f_{z_1} and f_{z_2} in $\mathscr{Q}_a(W)$ such that $\chi(\langle w_1, \cdot\rangle) = \chi \circ f_{z_1}$, $\chi(\langle w_2, \cdot\rangle) = \chi \circ f_{z_2}$ and $z_1 \neq z_2$. Define now $s_1 = (1, f_{z_1})$ and $s_2 = (1, f_{z_2})$, which are in $\mathrm{Ps}_a(B)$ by definition. Then by Lemma 1.27.iv, the two elements w_{s_1} and w_{s_2} satisfying (37) for s_1 and s_2, respectively, are such that $w_{s_1} = w_1$ and $w_{s_2} = w_2$. Thus if we compute the cocycle with the formula (44) found in Corollary 1.31, we obtain

$$c_\chi(s_1, s_2) = \gamma(1)\chi(B(w_{s_1}, w_{s_2})) = \gamma(1)\chi(B(w_1, w_2))$$

and

$$c_\chi(s_2, s_1) = \gamma(1)\chi(B(w_{s_2}, w_{s_1})) = \gamma(1)\chi(B(w_2, w_1)).$$

In particular, $c_\chi(s_1, s_2) \neq c_\chi(s_2, s_1)$ because we supposed that $\chi(\langle w_1, w_2\rangle) \neq 1$. Therefore, c_χ is not symmetric on $\mathrm{Ps}_a(B)$.
Now if we suppose that c_χ is a coboundary, then there exists $\nu : \mathrm{Ps}(B) \to \mathrm{U}(1)$ such that for all $s_1, s_2 \in \mathrm{Ps}(B)$,

$$c_\chi(s_1, s_2) = \nu(s_1)\nu(s_2)\nu(s_1 s_2)^{-1}. \tag{47}$$

In particular, (47) has to be satisfied for all elements of $\mathrm{Ps}_a(B)$. We therefore have for all $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ in $\mathrm{Ps}_a(B)$:

$$c_\chi(s_2, s_1) = \nu(s_2)\nu(s_1)\nu(s_2 s_1)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1 s_2)^{-1} = c_\chi(s_1, s_2).$$

This means that the cocycle c_χ would be symmetric on $\mathrm{Ps}_a(B)$, which we have just proven to be false. □

Remark 1.34. We proved in Corollary 1.33 that the cocycle c_χ is not symmetric on the abelian subgroup $\mathrm{Ps}_a(B)$ of $\mathrm{Ps}(B)$. Since the product law in the covering of $\mathrm{Ps}(B)$ that we will define in Proposition 1.36 is entirely determined by this cocycle, this implies that there are commuting elements in $\mathrm{Ps}(B)$ whose preimages in the covering do not commute.

Corollary 1.35 (Range of the cocycle c_χ). *The cocycle c_χ takes its values in $\mathrm{U}(1)$. Furthermore, c_χ takes its values in $\mu_2(\mathbb{C}) = \{\pm 1\}$ if and only if*

$$\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \pi \qquad \text{for all } \sigma_1, \sigma_2 \in \mathrm{O}(Q), \tag{48}$$

where the map $\sigma \mapsto \theta(\sigma)$ is defined by (46)*. In particular, this is the case if $\gamma(\sigma) \in \mathbb{R}$ for all $\sigma \in \mathrm{O}(Q)$, for example if $\gamma(\sigma) = |\mathbb{F}|^{-\frac{\dim \mathrm{I}(\sigma)}{2}}$.*

Proof. Equation (45), together with the unitarity of the operators $\omega_{\tilde\chi}(s)$ for all $s \in \mathrm{Ps}(B)$, implies that c_χ takes its values in $\mathrm{U}(1)$. Since χ takes its values in $\mu_2(\mathbb{C})$, formula (44) implies that c_χ takes its values in $\mathbb{R}$ if and only if $C_{1,2} = \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1}$ is real for all $\sigma_1, \sigma_2 \in \mathrm{O}(Q)$. By (46),

$$\begin{aligned} C_{1,2} \in \mathbb{R} \qquad &\Longleftrightarrow \qquad e^{i(\theta(\sigma_1)+\theta(\sigma_2)-\theta(\sigma_1\sigma_2))} \in \mathbb{R} \\ &\Longleftrightarrow \qquad \theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \pi. \end{aligned}$$

□

Now that we have established that lifting the projective Weil representation to a two-fold covering of $\mathrm{Ps}(B)$ is the best we can do, we define such a covering in the following proposition.

Definition-Proposition 1.36 (Two-fold covering of $\mathrm{Ps}(B)$). *We keep the notations and assumptions from Lemma 1.27 and Proposition 1.28. In particular, we suppose that a choice of w_s satisfying* (37) *has been made for all $s \in \mathrm{Ps}(B)$. We fix also $\gamma(\sigma)$, defined in Proposition 1.28, such that* (46) *and* (48) *are satisfied. Set*

$$\widetilde{\mathrm{Ps}(B)}_{\tilde\chi} = \{(s, (-1)^l \omega_{\tilde\chi}(s)),\ s \in \mathrm{Ps}(B),\ l \in \mathbb{Z}\}. \tag{49}$$

Then we have the following results.

i. *$\widetilde{\mathrm{Ps}(B)}_{\tilde\chi}$ is a subgroup of $\mathrm{Ps}(B) \times \mathrm{U}(\mathrm{L}^2(Y))$.*

ii. *$\widetilde{\mathrm{Ps}(B)}_{\tilde\chi}$ is a two-fold covering of $\mathrm{Ps}(B)$, i.e. it defines a non-split central exact sequence*

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\tilde\chi}} \widetilde{\mathrm{Ps}(B)}_{\tilde\chi} \xrightarrow{\Psi_{\tilde\chi}} \mathrm{Ps}(B) \longrightarrow 1,$$

and c_χ is a cocycle associated to this extension. Moreover, $\widetilde{\mathrm{Ps}(B)}_{\tilde\chi}$ is a stem extension of $\mathrm{Ps}(B)$ by $\mu_2(\mathbb{C})$, i.e. $\Phi_{\tilde\chi}(\mu_2(\mathbb{C}))$ is contained in the derived subgroup $[\widetilde{\mathrm{Ps}(B)}_{\tilde\chi}, \widetilde{\mathrm{Ps}(B)}_{\tilde\chi}]$ of $\widetilde{\mathrm{Ps}(B)}_{\tilde\chi}$.

iii. *Different choices of $\{w_s, s \in \mathrm{Ps}(B)\}$ lead to isomorphic extensions of $\mathrm{Ps}(B)$.*

iv. *Different choices of extensions* $\widetilde{\chi}$ *of* χ *lead to isomorphic extensions of* $\mathrm{Ps}(B)$.

v. *Let* $\{\gamma'(\sigma), \sigma \in \mathrm{O}(Q)\}$ *be another set of elements satisfying* (46) *and* (48). *Then the extensions* $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ *and* $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ *of* $\mathrm{Ps}(B)$ *(associated to the choices* $\gamma(\sigma)$ *and* $\gamma'(\sigma)$*, respectively) are isomorphic if and only if the map*

$$(s_1, s_2) \mapsto \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1} \tag{50}$$

defines a coboundary of $\mathrm{Ps}(B)$. *In particular, this is the case if*

$$\theta'(\sigma) \equiv \theta(\sigma) \mod \pi \qquad \text{for all } \sigma \in \mathrm{O}(Q). \tag{51}$$

For example, (51) *is satisfied if* $\gamma(\sigma)$ *and* $\gamma'(\sigma)$ *are chosen to be real for all* $\sigma \in \mathrm{O}(Q)$.

vi. *Let* $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ *be the two-fold covering of* $\mathrm{Ps}(B)$ *defined by the extension* $\widetilde{\chi}$ *of* χ *and the choices* $\{w_s,\ s \in \mathrm{Ps}(B)\}$ *and* $\{\gamma(\sigma), \sigma \in \mathrm{O}(Q)\}$ *such that* (37), (46) *and* (48) *are satisfied. Define similarly* $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$ *by the extension* $\chi^\sharp$ *of* χ *and the choices* $\{w'_s,\ s \in \mathrm{Ps}(B)\}$ *and* $\{\gamma'(\sigma), \sigma \in \mathrm{O}(Q)\}$ *such that* (37), (46) *and* (48) *are satisfied. If* (51) *is satisfied then*

$$\begin{aligned} \mathscr{I} : \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} &\to \widetilde{\mathrm{Ps}(B)}_{\chi^\sharp} \\ (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s, \gamma(\sigma))) &\mapsto (s, (-1)^l \nu(s)^{-1}\gamma(\sigma)\gamma'(\sigma)^{-1}\omega_{\chi^\sharp}(s, w'_s, \gamma'(\sigma))) \end{aligned} \tag{52}$$

is an isomorphism of central extensions, where ν *is defined by* (53). *This means that the following diagram is commutative:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} & \xrightarrow{\Psi_{\widetilde{\chi}}} & \mathrm{Ps}(B) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathscr{I}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\chi^\sharp}} & \widetilde{\mathrm{Ps}(B)}_{\chi^\sharp} & \xrightarrow{\Psi_{\chi^\sharp}} & \mathrm{Ps}(B) & \longrightarrow & 1 \end{array} \tag{D_{14}}$$

Proof. **i.** Let $\widetilde{s_1} = (s_1, (-1)^{l_1}\omega_{\widetilde{\chi}}(s_1))$ and $\widetilde{s_2} = (s_2, (-1)^{l_2}\omega_{\widetilde{\chi}}(s_2))$ be in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Then, by definition, $\widetilde{s_1}$ and $\widetilde{s_2}$ belong to $\mathrm{Ps}(B) \times \mathrm{U}(\mathrm{L}^2(Y))$. Moreover,

$$\begin{aligned} \widetilde{s_1}\widetilde{s_2}^{-1} &= (s_1, (-1)^{l_1}\omega_{\widetilde{\chi}}(s_1))(s_2^{-1}, (-1)^{-l_2}\omega_{\widetilde{\chi}}(s_2)^{-1}) \\ &= (s_1 s_2^{-1}, (-1)^{l_1-l_2}\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2)^{-1}) \\ &= (s_1 s_2^{-1}, (-1)^{l_1-l_2}c_\chi(s_2, s_2^{-1})^{-1}\omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2^{-1})) \qquad \text{by (45)} \\ &= (s_1 s_2^{-1}, (-1)^{l_1-l_2}c_\chi(s_2, s_2^{-1})^{-1}c_\chi(s_1, s_2^{-1})\omega_{\widetilde{\chi}}(s_1 s_2^{-1})) \quad \text{by (45).} \end{aligned}$$

Since (48) is satisfied, Corollary 1.35 implies that $(-1)^{l_1-l_2}c_\chi(s_2, s_2^{-1})^{-1}c_\chi(s_1, s_2^{-1})$ belongs to $\mu_2(\mathbb{C}) = \{\pm 1\}$. Hence $\widetilde{s_1}\widetilde{s_2}^{-1} \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, and thus $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is a subgroup of $\mathrm{Ps}(B) \times \mathrm{U}(\mathrm{L}^2(Y))$.

ii. Define $\Phi_{\widetilde{\chi}} : \mu_2(\mathbb{C}) \to \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\Psi_{\widetilde{\chi}} : \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \to \mathrm{Ps}(B)$. Then $\Phi_{\widetilde{\chi}}$ $z \mapsto ((1,0), z\,\mathrm{Id}_{\mathrm{L}^2(Y)})$ $(s,(-1)^l\omega_{\widetilde{\chi}}(s)) \mapsto s$ and $\Psi_{\widetilde{\chi}}$ are group homomorphisms such that:

- if $\Phi_{\widetilde{\chi}}(z) = ((1,0), \mathrm{Id}_{\mathrm{L}^2(Y)})$ then $z = 1$ by definition of $\Phi_{\widetilde{\chi}}$, and thus $\Phi_{\widetilde{\chi}}$ is injective.
- For every $s \in \mathrm{Ps}(B)$, we proved in Proposition 1.28 that $\omega_{\widetilde{\chi}}(s)$ is a unitary operator on $\mathrm{L}^2(Y)$. Hence $(s, \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Moreover, $\Psi_{\widetilde{\chi}}(s, \omega_{\widetilde{\chi}}(s)) = s$, which implies that $\Psi_{\widetilde{\chi}}$ is surjective.
- If $\Psi_{\widetilde{\chi}}(s,(-1)^l\omega_{\widetilde{\chi}}(s)) = (1,0)$ then $s = (1,0)$. Thus $(s,(-1)^l\omega_{\widetilde{\chi}}(s)) \in \mathrm{Im}(\Phi_{\widetilde{\chi}})$ by definition of $\Phi_{\widetilde{\chi}}$, $\Psi_{\widetilde{\chi}}$ and $\omega_{\widetilde{\chi}}$. Conversely, it is clear that $\mathrm{Im}(\Phi_{\widetilde{\chi}}) \subseteq \mathrm{Ker}(\Psi_{\widetilde{\chi}})$. Therefore $\mathrm{Im}(\Phi_{\widetilde{\chi}}) = \mathrm{Ker}(\Psi_{\widetilde{\chi}})$.

This shows that $1 \to \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi}}} \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \xrightarrow{\Psi_{\widetilde{\chi}}} \mathrm{Ps}(B) \to 1$ is a central extension. Now we want to show that this extension does not split. First we remark that c_χ is a cocycle associated to this extension. In fact, define $\tau(s) = (s, \omega_{\widetilde{\chi}}(s))$ for all $s \in \mathrm{Ps}(B)$. Then τ is a section of $\Psi_{\widetilde{\chi}}$, which satisfies for all $s_1, s_2 \in \mathrm{Ps}(B)$:

$$\begin{aligned}\tau(s_1)\tau(s_2) = (s_1, \omega_{\widetilde{\chi}}(s_1))(s_2, \omega_{\widetilde{\chi}}(s_2)) = (s_1s_2, \omega_{\widetilde{\chi}}(s_1)\omega_{\widetilde{\chi}}(s_2)) &= (s_1s_2, c_\chi(s_1,s_2)\omega_{\widetilde{\chi}}(s_1s_2)) \\ &= \Phi_{\widetilde{\chi}}(c_\chi(s_1,s_2))\tau(s_1s_2).\end{aligned}$$

As a consequence, the fact that this extension does not split follows directly from Corollary 1.33.

Finally, it remains to prove that $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is a stem extension, i.e. that $\Phi_{\widetilde{\chi}}(-1)$ is in the commutator subgroup $[\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}, \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}]$. As in Corollary 1.33, let $w_1, w_2 \in W$ be such that $\chi(\langle w_1, w_2\rangle) \neq 1$. Consider again $s_1 = (1, f_{z_1})$ and $s_2 = (1, f_{z_2})$ in $\mathrm{Ps}_a(B)$. Then $w_{s_1} = w_1$, $w_{s_2} = w_2$, and we can set $\widetilde{s_1} = (s_1, \omega_{\widetilde{\chi}}(s_1, w_1))$ and $\widetilde{s_2} = (s_2, \omega_{\widetilde{\chi}}(s_2, w_2))$. Then by (40), $\widetilde{s_1}$ and $\widetilde{s_2}$ are two elements of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ such that

$$\begin{aligned}\widetilde{s_1}\widetilde{s_2}\widetilde{s_1}^{-1}\widetilde{s_2}^{-1} &= (s_1s_2s_1^{-1}s_2^{-1}, \omega_{\widetilde{\chi}}(s_1,w_1)\omega_{\widetilde{\chi}}(s_2,w_2)\omega_{\widetilde{\chi}}(s_1,w_1)^{-1}\omega_{\widetilde{\chi}}(s_2,w_2)^{-1}) \\ &= ((1,0), \rho_{\widetilde{\chi}}(w_1)\rho_{\widetilde{\chi}}(w_2)\rho_{\widetilde{\chi}}(w_1)^{-1}\rho_{\widetilde{\chi}}(w_2)^{-1}) \quad \text{since } \mathrm{Ps}_a(B) \text{ is commutative} \\ &= ((1,0), \rho_{\widetilde{\chi}}(w_1)\rho_{\widetilde{\chi}}(w_2)\rho_{\widetilde{\chi}}(-w_1, Q(w_1))\rho_{\widetilde{\chi}}(-w_2, Q(w_2))) \\ &= ((1,0), \rho_{\widetilde{\chi}}(0, B(w_1,w_2) + Q(w_1) + Q(w_2) + B(w_1,w_2) - Q(w_1+w_2))) \\ &= ((1,0), \rho_{\widetilde{\chi}}(0, -\langle w_1, w_2\rangle)) \\ &= ((1,0), \chi(-\langle w_1, w_2\rangle)\,\mathrm{Id}_{\mathrm{L}^2(Y)})\end{aligned}$$

Since $\chi(\langle w_1, w_2\rangle) \neq 1$ and χ takes its values in $\{\pm 1\}$ (see Lemma 1.15.i), we obtain that $\widetilde{s_1}\widetilde{s_2}\widetilde{s_1}^{-1}\widetilde{s_2}^{-1} = ((1,0), -\mathrm{Id}_{\mathrm{L}^2(Y)}) = \Phi_{\widetilde{\chi}}(-1)$.

iii. Suppose that for every $s \in \mathrm{Ps}(B)$, w'_s also satisfies (37). Denote by $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ the set of elements $\{(s,(-1)^l\omega_{\widetilde{\chi}}(s,w'_s), s \in \mathrm{Ps}(B), l \in \mathbb{Z}\}$. Then by (41) in Proposition 1.28, for every $s \in \mathrm{Ps}(B)$,

$$\omega_{\widetilde{\chi}}(s,w'_s) = \chi(-f(v) + \langle v, w_s\rangle + B(w'_s - w_s, v - w_s))\omega_{\widetilde{\chi}}(s,w_s),$$

where v is any element such that $w'_s - w_s = (1-\sigma)(v)$. Since χ takes its values in $\mu_2(\mathbb{C})$, the set of operators $(-1)^l\omega_{\widetilde{\chi}}(s, w'_s)$ is exactly the same as the set of operators $(-1)^l\omega_{\widetilde{\chi}}(s, w_s)$, where $l \in \mathbb{Z}$. Hence $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ are equal as sets.

Let c'_χ denote the cocycle associated to the projective representation $s \mapsto \omega_{\widetilde{\chi}}(s, w'_s)$ (and then also associated to the extension $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$). Then we can prove that c_χ and c'_χ are cohomologous. In fact, $\omega_{\widetilde{\chi}}(s, w_s)$ and $\omega_{\widetilde{\chi}}(s, w'_s)$ satisfy the same intertwining property: for all $s \in \mathrm{Ps}(B)$ and $h \in \mathrm{H}(B)$,

$$\rho_{\widetilde{\chi}}(s(h))\omega_{\widetilde{\chi}}(s) = \omega_{\widetilde{\chi}}(s)\rho_{\widetilde{\chi}}(h).$$

Schur's lemma then implies the existence of $\nu : \mathrm{Ps}(B) \to \mathrm{U}(1)$ such that $\omega_{\widetilde{\chi}}(s, w'_s) = \nu(s)\omega_{\widetilde{\chi}}(s, w_s)$. We computed such a map ν in Proposition 1.28 (41), since we proved that

$$\omega_{\widetilde{\chi}}(s, w'_s) = \chi(-f(v) + \langle v, w_s\rangle + B(w'_s - w_s, v - w_s))\omega_{\widetilde{\chi}}(s, w_s),$$

where v is any element such that $w'_s - w_s = (1-\sigma)(v)$. Hence we obtained

$$\nu(s) = \chi(-f(v) + \langle v, w_s\rangle + B(w'_s - w_s, v - w_s)) \in \{\pm 1\}. \tag{53}$$

Thus for all $s_1, s_2 \in \mathrm{Ps}(B)$,

$$\begin{aligned}
\omega_{\widetilde{\chi}}(s_1, w'_{s_1})\omega_{\widetilde{\chi}}(s_2, w'_{s_2}) &= \nu(s_1)\nu(s_2)\omega_{\widetilde{\chi}}(s_1, w_{s_1})\omega_{\widetilde{\chi}}(s_2, w_{s_2}) \\
c'_\chi(s_1, s_2)\omega_{\widetilde{\chi}}(s_1s_2, w'_{s_1s_2}) &= \nu(s_1)\nu(s_2)c_\chi(s_1, s_2)\omega_{\widetilde{\chi}}(s_1s_2, w_{s_1s_2}) \\
c'_\chi(s_1, s_2)\nu(s_1s_2)\omega_{\widetilde{\chi}}(s_1s_2, w_{s_1s_2}) &= \nu(s_1)\nu(s_2)c_\chi(s_1, s_2)\omega_{\widetilde{\chi}}(s_1s_2, w_{s_1s_2}).
\end{aligned}$$

Hence

$$c'_\chi(s_1, s_2)c_\chi(s_1, s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}. \tag{54}$$

Since ν takes its values in $\{\pm 1\}$, this shows that c_χ and c'_χ are cohomologous. Therefore, $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ are extensions of $\mathrm{Ps}(B)$ whose associated cocycles are cohomologous: they are isomorphic extensions. In particular, the groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ are isomorphic.

iv. Formula (44) found for the cocycle c_χ in Corollary 1.31 does not depend on the choice of extension $\widetilde{\chi}$ of χ. In particular, two different extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$ corresponding to the extensions $\widetilde{\chi}$ and $\chi^\sharp$ of χ, respectively, are associated to the same cocycle c_χ. This implies that the extensions (hence also the groups) $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$ are isomorphic.

v. Let $\{\gamma'(\sigma), \sigma \in \mathrm{O}(Q)\}$ be another set of elements satisfying (46) and (48). We denote by $\omega'_{\widetilde{\chi}}(s)$ and c'_χ the operators and the cocycle associated to this choice. Then formula (44) implies that for all $s_1, s_2 \in \mathrm{Ps}(B)$,

$$c'_\chi(s_1, s_2) = \gamma'(\sigma_1)\gamma'(\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\gamma(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)c_\chi(s_1, s_2),$$

i.e.

$$c'_\chi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \tag{55}$$

Hence the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ are isomorphic if and only if the map $(s_1,s_2)\mapsto \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}$ is a coboundary of $\mathrm{Ps}(B)$. In particular, if $\theta'(\sigma)\equiv\theta(\sigma) \mod \pi$ for all $\sigma\in \mathrm{O}(Q)$, then

$$\gamma'(\sigma)\gamma(\sigma)^{-1} = e^{i(\theta'(\sigma)-\theta(\sigma))}\in\{\pm 1\},$$

which shows that the map $\sigma\mapsto\gamma'(\sigma)\gamma(\sigma)^{-1}$ takes its values in $\{\pm1\}$. Thus $c'_\chi c_\chi^{-1}$ is a coboundary by (55). This implies that the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ are isomorphic.

vi. Denote by c_χ and c'_χ the cocycles associated to the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$ of $\mathrm{Ps}(B)$ defined in the statement, respectively. (Recall that $\chi^\sharp$ does not appear in the notation of these cocycles since the cocycle c_χ does not depend on the choice of extension of χ by Corollary 1.31.) Let $\widetilde{s_1}=(s_1,(-1)^{l_1}\omega_{\widetilde{\chi}}(s_1,w_{s_1},\gamma(\sigma_1)))$ and $\widetilde{s_2}=(s_2,(-1)^{l_2}\omega_{\widetilde{\chi}}(s_2,w_{s_2},\gamma(\sigma_2)))$ be in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Then

$$\begin{aligned}
\mathscr{I}(\widetilde{s_1})\mathscr{I}(\widetilde{s_2}) &= (s_1,(-1)^{l_1}\nu(s_1)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\omega_{\chi^\sharp}(s_1,w'_{s_1},\gamma'(\sigma_1)))\\
&\quad\cdot(s_2,(-1)^{l_2}\nu(s_2)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\omega_{\chi^\sharp}(s_2,w'_{s_2},\gamma'(\sigma_2)))\\
&= (s_1s_2,(-1)^{l_1+l_2}\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\omega_{\chi^\sharp}(s_1,w'_{s_1},\gamma'(\sigma_1))\\
&\quad\omega_{\chi^\sharp}(s_2,w'_{s_2},\gamma'(\sigma_2)))\\
&= (s_1s_2,(-1)^{l_1+l_2}\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}c'_\chi(s_1,s_2)\\
&\quad\omega_{\chi^\sharp}(s_1s_2,w'_{s_1s_2},\gamma'(\sigma_1\sigma_2)))
\end{aligned} \tag{56}$$

$$\begin{aligned}
\mathscr{I}(\widetilde{s_1}\widetilde{s_2}) &= \mathscr{I}(s_1s_2,(-1)^{l_1+l_2}\omega_{\widetilde{\chi}}(s_1,w_{s_1},\gamma(\sigma_1))\omega_{\widetilde{\chi}}(s_2,w_{s_2},\gamma(\sigma_2))\\
&= \mathscr{I}(s_1s_2,(-1)^{l_1+l_2}c_\chi(s_1,s_2)\omega_{\widetilde{\chi}}(s_1s_2,w_{s_1s_2},\gamma(\sigma_1\sigma_2)))\\
&= (s_1s_2,(-1)^{l_1+l_2}c_\chi(s_1,s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\omega_{\chi^\sharp}(s_1s_2,w'_{s_1s_2},\gamma'(\sigma_1\sigma_2))).
\end{aligned} \tag{57}$$

Hence $\mathscr{I}$ is a group homomorphism if and only if

$$\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}c'_\chi(s_1,s_2) = c_\chi(s_1,s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1},$$

i.e.

$$c'_\chi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\gamma'(\sigma_1)\gamma(\sigma_2)^{-1}\gamma'(\sigma_2). \tag{58}$$

Denote by c''_χ the cocycle associated to the two-fold covering of $\mathrm{Ps}(B)$ which is defined with the extension $\widetilde{\chi}$ of χ and the choices $\{w'_s,\ s\in\mathrm{Ps}(B)\}$ and $\{\gamma(\sigma),\sigma\in\mathrm{O}(Q)\}$ such that (37), (46) and (48) are satisfied. By (54),

$$c''_\chi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}. \tag{59}$$

By (55), since (51) is satisfied,

$$c'_\chi(s_1,s_2)c''_\chi(s_1,s_2)^{-1} = \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \quad (60)$$

Since $c'_\chi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = c'_\chi(s_1,s_2)c''_\chi(s_1,s_2)^{-1}c''_\chi(s_1,s_2)c_\chi(s_1,s_2)^{-1}$, (59) and (60) imply (58). Let $\widetilde{s} = (s,(-1)^l\omega_{\widetilde{\chi}}(s,w_s,\gamma(\sigma))) \in \mathrm{Ker}(\mathscr{I})$. Then $s = (1,0)$ and

$$\begin{aligned}\mathrm{Id}_{\mathrm{L}^2(Y)} &= (-1)^l\nu(s)^{-1}\gamma(\sigma)\gamma'(\sigma)^{-1}\omega_{\chi^\sharp}(s,w'_s,\gamma'(\sigma))\\ &= (-1)^l\nu(1,0)^{-1}\gamma(1)\gamma'(1)^{-1}\omega_{\chi^\sharp}((1,0),0,\gamma'(1))\\ &= (-1)^l\gamma(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)}\,.\end{aligned}$$

Hence $(-1)^l\gamma(1) = 1$. This enables us to conclude that

$$\begin{aligned}(s,(-1)^l\omega_{\widetilde{\chi}}(s,w_s,\gamma(\sigma))) &= ((1,0),(-1)^l\omega_{\widetilde{\chi}}((1,0),0,\gamma(1)))\\ &= ((1,0),(-1)^l\gamma(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)})\\ &= ((1,0),\mathrm{Id}_{\mathrm{L}^2(Y)}),\end{aligned}$$

i.e. $\mathscr{I}$ is injective. Since $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$ have the same order, $\mathscr{I}$ is an isomorphism. Now, it only remains to prove that the diagram (D_{14}) commutes. First we have that

$$\begin{aligned}(\mathscr{I}\circ\Phi_{\widetilde{\chi}})(-1) = \mathscr{I}((1,0),-\mathrm{Id}_{\mathrm{L}^2(Y)}) &= \mathscr{I}((1,0),-\gamma(1)^{-1}\omega_{\widetilde{\chi}}((1,0),0,\gamma(1)))\\ &= ((1,0),-\gamma(1)^{-1}\gamma(1)\gamma'(1)^{-1}\omega_{\chi^\sharp}((1,0),0,\gamma'(1)))\\ &= ((1,0),-\gamma'(1)^{-1}\gamma'(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)})\\ &= \Phi_{\chi^\sharp}(-1).\end{aligned}$$

In addition, let $\widetilde{s} = (s,(-1)^l\omega_{\widetilde{\chi}}(s,w_s,\gamma(\sigma))) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Then

$$(\Psi_{\chi^\sharp}\circ\mathscr{I})(\widetilde{s}) = \Psi_{\chi^\sharp}(s,(-1)^l\nu(s)^{-1}\gamma(\sigma)\gamma'(\sigma)^{-1}\omega_{\chi^\sharp}(s,w'_s,\gamma'(\sigma))) = s = \Psi_{\widetilde{\chi}}(s).$$

This proves that the diagram (D_{14}) is commutative.

□

Remark 1.37.

- Proposition 1.36.v provides a necessary and sufficient condition for the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$, associated to the choices $\gamma(\sigma)$ and $\gamma'(\sigma)$, respectively, to be isomorphic. In particular, if (51) is satisfied then the map defined by (50) defines a coboundary. For example, (51) is satisfied if we choose $\gamma(\sigma)$ and $\gamma'(\sigma)$ to be real, which is consistent with the fact that χ is real. At present, we do not know if the converse holds in general, i.e if the fact that the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ are isomorphic implies (51). Nevertheless, we prove that this equivalence holds in the two-dimensional case for $\mathbb{F} = \mathbb{F}_2$, see Remarks 3.3 and 3.29.
- We may wonder whether we can obtain all the possible non-split stem extensions of $\mathrm{Ps}(B)$ by two elements with all possible choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{Ps}(B)\}$ satisfying (48). This is false in general, since as we will show in Corollary 1.38, if $\mathbb{F} = \mathbb{F}_2$ then $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ contains a copy of the Heisenberg group $\mathrm{H}(B)$. In particular, in the two-dimensional case for $\mathbb{F} = \mathbb{F}_2$, we obtain all the possible groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ which correspond to non-split stem extensions of $\mathrm{Ps}(B)$ by two elements which contain a copy of the Heisenberg group $\mathrm{H}(B)$ (cf. Remarks 3.3 and 3.29).

Corollary 1.38. ***i.*** *The map*

$$\begin{aligned}\Phi_{\mathrm{H}(B)} : \mathrm{H}(B) &\to \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \\ (w,t) &\mapsto ((1,f_z), \chi(t)\gamma(1)\omega_{\widetilde{\chi}}((1,f_z),w))\end{aligned} \tag{61}$$

is a group homomorphism, where $(1,f_z) \in \mathrm{Ps}_a(B)$ is such that $f_z \in \mathscr{Q}_a(W)$ and $\chi \circ f_z = \chi \circ \langle w, \cdot \rangle$ for all $w \in W$ (see Proposition 1.12). Its kernel is the set $\{(0,t),\ t \in \mathrm{Ker}(\chi)\}$ and its image is a normal subgroup of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$.

ii. *If $\mathbb{F} = \mathbb{F}_2$, then $\Phi_{\mathrm{H}(B)}$ is injective. Furthermore, the two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ contains a copy of the Heisenberg group if and only if $\mathbb{F} = \mathbb{F}_2$.*

iii. *If $\mathbb{F} = \mathbb{F}_2$, then the group $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ lies in the following short exact sequence:*

$$1 \longrightarrow \mathrm{H}(B) \xrightarrow{\Phi_{\mathrm{H}(B)}} \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \xrightarrow{\pi_{\widetilde{\mathrm{Ps}}}} \mathrm{O}(Q) \longrightarrow 1, \tag{62}$$

where $\pi_{\widetilde{\mathrm{Ps}}}$ is the projection onto $\mathrm{O}(Q)$.

Proof. **i.** For every $w \in W$, we proved in Corollary 1.33 that there exists $z \in W$ and $s = (1,f_z) \in \mathrm{Ps}_a(B)$ such that $w_s = w$ is the unique possibility for w_s. Then, by (40) in Proposition 1.28,

$$\omega_{\widetilde{\chi}}(s,w) = \gamma(1) \sum_{u \in I(1)} \chi(f_z(v) + \langle w, v \rangle + B(u, v+w))\rho_{\widetilde{\chi}}(u+w) = \gamma(1)\rho_{\widetilde{\chi}}(w), \tag{63}$$

where v is any element such that $u = (1-1)(v) = 0$. Since χ takes its values in $\mu_2(\mathbb{C})$ and $\gamma(1)^2 = 1$ by (46) and (48), the map $\Phi_{\mathrm{H}(B)}$ is well-defined by the description of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ in Proposition 1.36. Now, let (w_1,t_1) and (w_2,t_2) be in $\mathrm{H}(B)$. Then there exist f_{z_1} and f_{z_2} in $\mathscr{Q}_a(W)$ such that $\chi \circ f_{z_i} = \chi \circ \langle w_i, \cdot \rangle$ for $i \in \{1,2\}$. Hence

$$\begin{aligned}&\Phi_{\mathrm{H}(B)}(w_1,t_1)\Phi_{\mathrm{H}(B)}(w_2,t_2) \\ &= ((1,f_{z_1}), \chi(t_1)\gamma(1)\omega_{\widetilde{\chi}}((1,f_{z_1}),w_1))((1,f_{z_2}), \chi(t_2)\gamma(1)\omega_{\widetilde{\chi}}((1,f_{z_2}),w_2)) \\ &= ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2)\gamma(1)^2\omega_{\widetilde{\chi}}((1,f_{z_1}),w_1)\omega_{\widetilde{\chi}}((1,f_{z_2}),w_2)) \\ &\overset{(63)}{=} ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2)\gamma(1)^4\rho_{\widetilde{\chi}}(w_1)\rho_{\widetilde{\chi}}(w_2)) \\ &= ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2)\rho_{\widetilde{\chi}}(w_1+w_2, B(w_1,w_2))) \\ &= ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2+B(w_1,w_2))\rho_{\widetilde{\chi}}(w_1+w_2)).\end{aligned}$$

Observe that $\chi \circ (f_{z_1}+f_{z_2}) = \chi \circ (\langle w_1, \cdot \rangle + \langle w_2, \cdot \rangle) = \chi \circ \langle w_1+w_2, \cdot \rangle$, which means that the additive quadratic form f_z corresponding to the element w_1+w_2 is exactly $f_{z_1}+f_{z_2}$. Then

$$\begin{aligned}&\Phi_{\mathrm{H}(B)}((w_1,t_1)(w_2,t_2)) \\ &= \Phi_{\mathrm{H}(B)}(w_1+w_2, t_1+t_2+B(w_1,w_2)) \\ &= ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2+B(w_1,w_2))\gamma(1)\omega_{\widetilde{\chi}}((1,f_{z_1}+f_{z_2}),w_1+w_2)) \\ &\overset{(63)}{=} ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2+B(w_1,w_2))\gamma(1)^2\rho_{\widetilde{\chi}}(w_1+w_2)) \\ &= ((1,f_{z_1}+f_{z_2}), \chi(t_1+t_2+B(w_1,w_2))\rho_{\widetilde{\chi}}(w_1+w_2)).\end{aligned}$$

This shows that $\Phi_{\mathrm{H}(B)}((w_1,t_1)(w_2,t_2)) = \Phi_{\mathrm{H}(B)}(w_1,t_1)\Phi_{\mathrm{H}(B)}(w_2,t_2)$, which implies that $\Phi_{\mathrm{H}(B)}$ is a group homomorphism.

Moreover, the image of $\mathrm{H}(B)$ by $\Phi_{\mathrm{H}(B)}$ is a normal subgroup of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. In fact, let $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s))$ be in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, where $s = (\sigma, f)$, and let (w,t) be in $\mathrm{H}(B)$. Then

$$\begin{aligned}
&\widetilde{s}\Phi_{\mathrm{H}(B)}(w,t)\widetilde{s}^{-1} \\
&= (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s))((1, f_z), \chi(t)\gamma(1)\omega_{\widetilde{\chi}}((1, f_z), w))(s^{-1}, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)^{-1}) \\
&= (s(1, f_z)s^{-1}, \chi(t)\omega_{\widetilde{\chi}}(s, w_s)\omega_{\widetilde{\chi}}((1, f_z), w)\omega_{\widetilde{\chi}}(s, w_s)^{-1}).
\end{aligned}$$

By definition of the product law in $\mathrm{Ps}(B)$,

$$s(1, f_z)s^{-1} = (\sigma, f)(1, f_z)(\sigma^{-1}, -f \circ \sigma^{-1}) = (\sigma, f_z + f)(\sigma^{-1}, -f \circ \sigma^{-1}) = (1, f_z \circ \sigma^{-1}),$$

where for all $v \in W$, $\chi((f_z \circ \sigma^{-1})(v)) = \chi(f_z(\sigma^{-1}(v))) = \chi(\langle w, \sigma^{-1}(v)\rangle) = \chi(\langle \sigma(w), v\rangle)$, i.e. $f_z \circ \sigma^{-1}$ is the additive quadratic form which corresponds to $\sigma(w)$. This implies that $w_{s(1,f_z)s^{-1}} = \sigma(w)$. By definition of the projective Weil representation,

$$\omega_{\widetilde{\chi}}(s, w_s)\omega_{\widetilde{\chi}}(s^{-1}, w_{s^{-1}}) = c_\chi(s, s^{-1})\omega_{\widetilde{\chi}}(1, 0) = c_\chi(s, s^{-1})\gamma(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Thus

$$\begin{aligned}
&\widetilde{s}\Phi_{\mathrm{H}(B)}(w,t)\widetilde{s}^{-1} \\
&= (s(1, f_z)s^{-1}, \chi(t)\omega_{\widetilde{\chi}}(s, w_s)\omega_{\widetilde{\chi}}((1, f_z), w)\omega_{\widetilde{\chi}}(s, w_s)^{-1}) \\
&= ((1, f_z \circ \sigma^{-1}), \chi(t)c_\chi(s, (1, f_z))c_\chi(s, s^{-1})^{-1}\gamma(1)^{-1}\omega_{\widetilde{\chi}}(s(1, f_z), w_{s(1,f_z)})\omega_{\widetilde{\chi}}(s^{-1}, w_{s^{-1}})) \\
&= ((1, f_z \circ \sigma^{-1}), \\
&\qquad \chi(t)c_\chi(s, (1, f_z))c_\chi(s, s^{-1})^{-1}\gamma(1)^{-1}c_\chi(s(1, f_z), s^{-1})\omega_{\widetilde{\chi}}(s(1, f_z)s^{-1}, w_{s(1,f_z)s^{-1}})) \\
&= ((1, f_z \circ \sigma^{-1}), \\
&\qquad \chi(t)c_\chi(s, (1, f_z))c_\chi(s, s^{-1})^{-1}\gamma(1)c_\chi(s(1, f_z), s^{-1})\omega_{\widetilde{\chi}}((1, f_z \circ \sigma^{-1}), \sigma(w))).
\end{aligned}$$

Since $\chi(t)c_\chi(s, (1, f_z))c_\chi(s, s^{-1})^{-1}c_\chi(s(1, f_z), s^{-1}) \in \mu_2(\mathbb{C}) = \{\pm 1\}$, there exists $t' \in \mathbb{F}$ such that $\chi(t)c_\chi(s, (1, f_z))c_\chi(s, s^{-1})^{-1}c_\chi(s(1, f_z), s^{-1}) = \chi(t')$. Therefore, we obtain $\widetilde{s}\Phi_{\mathrm{H}(B)}(w,t)\widetilde{s}^{-1} = \Phi_{\mathrm{H}(B)}(\sigma(w), t') \in \Phi_{\mathrm{H}(B)}(\mathrm{H}(B))$.

Now, suppose that $(w,t) \in \mathrm{H}(B)$ belongs to the kernel of $\Phi_{\mathrm{H}(B)}$. Then $(1, f_z) = (1, 0)$ and $\chi(t)\gamma(1)\omega_{\widetilde{\chi}}((1, f_z), w) = \mathrm{Id}_{\mathrm{L}^2(Y)}$. This implies that $\chi \circ \langle w, \cdot\rangle = \chi \circ f_z = 1$, i.e. $w = 0$. Hence

$$\begin{aligned}
\mathrm{Id}_{\mathrm{L}^2(Y)} = \chi(t)\gamma(1)\omega_{\widetilde{\chi}}((1, f_z), w) = \chi(t)\gamma(1)\omega_{\widetilde{\chi}}((1, 0), 0) &= \chi(t)\gamma(1)^2\,\mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= \chi(t)\,\mathrm{Id}_{\mathrm{L}^2(Y)},
\end{aligned}$$

which implies that $t \in \mathrm{Ker}(\chi)$. Conversely, $\Phi_{\mathrm{H}(B)}(0, t) = ((1, 0), \mathrm{Id}_{\mathrm{L}^2(Y)})$ for all $t \in \mathrm{Ker}(\chi)$. Thus the kernel of $\Phi_{\mathrm{H}(B)}$ is exactly the set $\{(0, t),\ t \in \mathrm{Ker}(\chi)\}$.

ii. If $\mathbb{F} = \mathbb{F}_2$, since χ is faithful, then the group homomorphism $\Phi_{\mathrm{H}(B)}$ is injective by the preceding point. This shows that the image of $\Phi_{\mathrm{H}(B)}$ is a subgroup of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ which is isomorphic to $\mathrm{H}(B)$. In contrast, if $\mathbb{F} \neq \mathbb{F}_2$, the pseudosymplectic covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ does not contain $\mathrm{H}(B)$ as a subgroup. In fact, the center of $\mathrm{H}(B)$ is isomorphic to $\mathbb{F}$, whereas the center of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ only contains two elements.

iii. Suppose that $\mathbb{F} = \mathbb{F}_2$. Then by the preceding point, $\Phi_{\mathrm{H}(B)}$ is a group monomorphism. Moreover, if $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s))$ is an element of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ such that $s = (\sigma, f)$ and $\pi_{\widetilde{\mathrm{Ps}}}(\widetilde{s}) = 1$, then $\sigma = 1$. By Proposition 1.12, there exist $w \in W$ and $z \in W$ such that $f = f_z$ and $\chi \circ f_z = \chi \circ \langle w, \cdot \rangle$. Hence $s = (1, f_z)$ and $w_s = w$. Since there exists $t \in \mathbb{F}_2$ such that $\chi(t)\gamma(1) = (-1)^l$, this implies that $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)) = ((1, f_z), (-1)^l \omega_{\widetilde{\chi}}((1, f_z), w)) = ((1, f_z), \chi(t)\gamma(1)\omega_{\widetilde{\chi}}((1, f_z), w)) = \Phi_{\mathrm{H}(B)}(w, t)$. Conversely, the image of $\Phi_{\mathrm{H}(B)}$ is included in the kernel of $\pi_{\widetilde{\mathrm{Ps}}}$ by definition. This proves that $1 \longrightarrow \mathrm{H}(B) \xrightarrow{\Phi_{\mathrm{H}(B)}} \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \xrightarrow{\pi_{\widetilde{\mathrm{Ps}}}} \mathrm{O}(Q) \longrightarrow 1$ is a short exact sequence.

□

Now we have everything we need to lift the projective Weil representation of $\mathrm{Ps}(B)$ to a linear representation of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$.

Proposition 1.39 (The Weil representation of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$). ***i.*** *The projective Weil representation $\omega_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ can be lifted to a linear representation $\widetilde{\omega}_{\widetilde{\chi}}$ of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, which will be called the Weil representation. The representation $\widetilde{\omega}_{\widetilde{\chi}}$ is defined by*

$$\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}) = (-1)^l \omega_{\widetilde{\chi}}(s) \quad \textit{for all } \widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}. \tag{64}$$

ii. *$\widetilde{\omega}_{\widetilde{\chi}}$ is unitary, faithful and irreducible.*

iii. *The image of $\widetilde{\omega}_{\widetilde{\chi}}$ contains all the operators $\rho_{\widetilde{\chi}}(w)$, $w \in W$ (which form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$). More precisely, for every $w \in W$,*

$$\rho_{\widetilde{\chi}}(w) = \widetilde{\omega}_{\widetilde{\chi}}((1, f_z), \gamma(1)\omega_{\widetilde{\chi}}((1, f_z), w)), \tag{65}$$

where $f_z \in \mathscr{Q}_a(W)$ is such that $\chi \circ f_z = \chi(\langle w, \cdot \rangle)$, and $\gamma(1)^2 = 1$.

iv. *If $\mathbb{F} = \mathbb{F}_2$, then $(\widetilde{\omega}_{\widetilde{\chi}} \circ \Phi_{\mathrm{H}(B)})(\mathrm{H}(B))$ is a subgroup of the image of $\widetilde{\omega}_{\widetilde{\chi}}$ which is isomorphic to $\mathrm{H}(B)$, where $\Phi_{\mathrm{H}(B)}$ is defined in Corollary 1.38. If $\mathbb{F} \neq \mathbb{F}_2$, then the image of $\widetilde{\omega}_{\widetilde{\chi}}$ does not contain a copy of the Heisenberg group $\mathrm{H}(B)$.*

Remark 1.40. Observe that the fact that $\widetilde{\omega}_{\widetilde{\chi}}$ is irreducible and contains all the operators $\rho_{\widetilde{\chi}}(w)$, $w \in W$, as well as the fact that $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ contains a copy of the Heisenberg group $\mathrm{H}(B)$ for $\mathbb{F} = \mathbb{F}_2$, are properties that do not arise for finite fields of odd characteristic or p-adic fields or for the reals.

Proof. Define the map $\widetilde{\omega}_{\widetilde{\chi}}$ by (64).

i. Observe first that for $s = (1, 0)$, since $w_s = 0$ by Lemma 1.27.iv, we obtain by (38) that

$$\omega_{\widetilde{\chi}}((1, 0), 0) = |\,\mathrm{K}(1)|^{-1}\gamma(1) \sum_{w \in W} \rho_{\widetilde{\chi}}(w)^{-1}\rho_{\widetilde{\chi}}(0)\rho_{\widetilde{\chi}}(w) = \gamma(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Then, since $\gamma(1)^2 = 1$ by (46) and (48), we have that

$$\widetilde{\omega}_{\widetilde{\chi}}((1, 0), \mathrm{Id}_{\mathrm{L}^2(Y)}) = \widetilde{\omega}_{\widetilde{\chi}}((1, 0), \gamma(1)\omega_{\widetilde{\chi}}((1, 0), 0)) = \mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Moreover, for all $\widetilde{s_1} = (s_1, (-1)^{l_1}\omega_{\widetilde{\chi}}(s_1))$ and $\widetilde{s_2} = (s_2, (-1)^{l_2}\omega_{\widetilde{\chi}}(s_2))$ in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$,

$$\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s_1})\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s_2}) = (-1)^{l_1}\omega_{\widetilde{\chi}}(s_1)(-1)^{l_2}\omega_{\widetilde{\chi}}(s_2) = c_\chi(s_1, s_2)(-1)^{l_1+l_2}\omega_{\widetilde{\chi}}(s_1 s_2)$$

and

$$\begin{aligned}\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s_1}\widetilde{s_2}) &= \widetilde{\omega}_{\widetilde{\chi}}(s_1 s_2, (-1)^{l_1}\omega_{\widetilde{\chi}}(s_1)(-1)^{l_2}\omega_{\widetilde{\chi}}(s_2)) \\ &= \widetilde{\omega}_{\widetilde{\chi}}(s_1 s_2, (-1)^{l_1+l_2}c_\chi(s_1, s_2)\omega_{\widetilde{\chi}}(s_1 s_2)) \\ &= c_\chi(s_1, s_2)(-1)^{l_1+l_2}\omega_{\widetilde{\chi}}(s_1 s_2).\end{aligned}$$

Thus $\widetilde{\omega}_{\widetilde{\chi}}$ is a linear representation of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, which lifts the projective Weil representation of $\mathrm{Ps}(B)$.

ii. The unitarity of $\widetilde{\omega}_{\widetilde{\chi}}$ comes directly from Proposition 1.28. Let $s = (\sigma, f) \in \mathrm{Ps}(B)$ and $\widetilde{s} = (s, (-1)^l\omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Suppose that $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}) = \mathrm{Id}_{\mathrm{L}^2(Y)}$, then $(-1)^l\omega_{\widetilde{\chi}}(s) = \mathrm{Id}_{\mathrm{L}^2(Y)}$. Then by (40) in Proposition 1.28,

$$(-1)^l\gamma(\sigma)\sum_{u\in\mathrm{I}(\sigma)}\chi(f(w) + \langle w_s, w\rangle + B(u, w + w_s))\rho_{\widetilde{\chi}}(u + w_s) = \mathrm{Id}_{\mathrm{L}^2(Y)} = \rho_{\widetilde{\chi}}(0),$$

where $w \in W$ is any element such that $u = (1-\sigma)(w)$. The operators $\rho_{\widetilde{\chi}}(w)$ form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ according to Lemma 1.22, and χ takes its values in $\{\pm 1\}$. Hence only one term $\rho_{\widetilde{\chi}}(u + w_s)$ can appear in the sum, the one for which $u + w_s = 0$. Moreover, since $u \in \mathrm{I}(\sigma)$, this means that $\mathrm{I}(\sigma)$ is a vector space which contains only one element in this case. Hence $\mathrm{I}(\sigma) = \{0\}$ and $w_s = u = 0$. This implies that $\mathrm{K}(\sigma) = W$, i.e. $\sigma = 1$. In addition, $w_s = 0$ implies by (37) that $\chi \circ f_{|\mathrm{K}(\sigma)} = \chi \circ f = \chi \circ \langle w_s, \cdot\rangle = 1$, i.e. $\chi \circ f = 1$. If $f \neq 0$ then $f(W) = \mathbb{F}$, which implies that $\mathbb{F} \subseteq \mathrm{Ker}(\chi)$. This is impossible because χ is nontrivial by hypothesis. Hence $f = 0$ and $s = (1, 0)$. In particular, $\omega_{\widetilde{\chi}}(s) = \gamma(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)}$. Since $(-1)^l\omega_{\widetilde{\chi}}(s) = \mathrm{Id}_{\mathrm{L}^2(Y)}$ by hypothesis, this implies that $(-1)^l\gamma(1) = 1$, i.e. $\widetilde{s} = ((1, 0), \mathrm{Id}_{\mathrm{L}^2(Y)})$. This finally proves that $\widetilde{\omega}_{\widetilde{\chi}}$ is faithful. We prove the irreducibility of $\widetilde{\omega}_{\widetilde{\chi}}$ in the next point.

iii. For every $w \in W$, by (63), $\omega_{\widetilde{\chi}}((1, f_z), w) = \gamma(1)\rho_{\widetilde{\chi}}(w)$, where $\chi \circ f_z = \chi \circ \langle w, \cdot\rangle$. Since $\gamma(1)^2 = 1$ by (46) and (48), this shows that $\widetilde{\omega}_{\widetilde{\chi}}((1, f_z), \gamma(1)\omega_{\widetilde{\chi}}((1, f_z), w)) = \rho_{\widetilde{\chi}}(w)$. Thus $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ contains the operators $\rho_{\widetilde{\chi}}(w)$ for all $w \in W$. Now, if $\mathscr{V} \subseteq \mathrm{L}^2(Y)$ is a vector subspace invariant under $\widetilde{\omega}_{\widetilde{\chi}}$ then, in particular, $\mathscr{V}$ is invariant under all the operators $\rho_{\widetilde{\chi}}(w)$, $w \in W$. Since these operators form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by Lemma 1.22, then either $\mathscr{V} = \{0\}$ or $\mathscr{V} = \mathrm{L}^2(Y)$, i.e. $\widetilde{\omega}_{\widetilde{\chi}}$ is irreducible.

iv. This follows directly from Corollary 1.38 and the fact that $\widetilde{\omega}_{\widetilde{\chi}}$ is faithful.

□

Corollary 1.41 (Character of the Weil representation of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$). *Let $\Theta_{\widetilde{\chi}}$ be the character of $\widetilde{\omega}_{\widetilde{\chi}}$. We suppose as before that a choice of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{w_s,\ s \in \mathrm{Ps}(B)\}$ has been made. Then, for every $\widetilde{s} = (s, (-1)^l\omega_{\widetilde{\chi}}(s, w_s))$ in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$,*

$$\Theta_{\widetilde{\chi}}(\widetilde{s}) = (-1)^l\gamma(\sigma)|Y|\chi(f(w) - B(w_s + w, w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s), \tag{66}$$

where $\mathbb{1}_{\mathrm{I}(\sigma)}$ is the indicator function of $\mathrm{I}(\sigma)$, and $w \in W$ is any element such that $(\sigma-1)(w) = w_s$. In particular, $\Theta_{\widetilde{\chi}}$ does not depend on the choice of extension $\widetilde{\chi}$ of χ and we can denote it by Θ_χ. Moreover, Θ_χ satisfies the following properties:

i. *The support of Θ_χ is*

$$\mathrm{supp}(\Theta_\chi) = \{\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}},\ s = (\sigma, f) \text{ with } \chi \circ f_{|\,\mathrm{K}(\sigma)} = 1\}.$$

Hence it does not depend on the choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{w_s,\ s \in \mathrm{Ps}(B)\}$.

ii. *For every $\sigma \in \mathrm{O}(Q)$, there exists $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ such that the projection of s on $\mathrm{O}(Q)$ is σ and $\Theta_\chi(\widetilde{s}) \neq 0$.*

iii. *Let $\sigma \in \mathrm{O}(Q)$. Then the following assertions are equivalent:*

- *$\Theta_\chi(\widetilde{s}) \neq 0$ for all $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ such that the projection of s on $\mathrm{O}(Q)$ is σ.*
- *$1 - \sigma$ is invertible.*

Proof. **i.** Let $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, where $s = (\sigma, f) \in \mathrm{Ps}(B)$. By definition of $\widetilde{\omega}_{\widetilde{\chi}}$, we have that $\Theta_{\widetilde{\chi}}(\widetilde{s}) = \mathrm{Trace}(\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s})) = (-1)^l \mathrm{Trace}(\omega_{\widetilde{\chi}}(s))$. Recall the expression (40) proved in Proposition 1.28:

$$\omega_{\widetilde{\chi}}(s) = \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w\rangle + B(u, w + w_s)) \rho_{\widetilde{\chi}}(u + w_s),$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$. We can first compute the trace of every operator of the form $\rho_{\widetilde{\chi}}(w)$, $w \in W$. In fact, this computation has already been done in Lemma 1.17 in the basis $\mathscr{B}_Y = \{\delta_y, y \in Y\}$ of $\mathrm{L}^2(Y)$, and the result is given by (29): for all $w_0 \in W$,

$$\mathrm{Trace}(\rho_{\widetilde{\chi}}(w_0)) = |Y| \delta_0(w_0).$$

Then we obtain

$$\begin{aligned}
\mathrm{Trace}(\omega_{\widetilde{\chi}}(s)) &= \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w\rangle + B(u, w + w_s)) \,\mathrm{Trace}(\rho_{\widetilde{\chi}}(u + w_s)) \\
&= \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w\rangle + B(u, w + w_s)) |Y| \delta_0(u + w_s) \\
&= \gamma(\sigma) |Y| \chi(f(w) + \langle w_s, w\rangle - B(w_s, w + w_s)) \mathbb{1}_{\mathrm{I}(\sigma)}(w_s) \\
&= \gamma(\sigma) |Y| \chi(f(w) - B(w_s + w, w_s)) \mathbb{1}_{\mathrm{I}(\sigma)}(w_s),
\end{aligned}$$

where $w \in W$ is any element such that $w_s = (\sigma - 1)(w)$. This proves (66). Now, $\Theta_\chi(\widetilde{s})$ is nonzero if and only if $w_s \in \mathrm{I}(\sigma)$, which does not depend on the choice made for $\gamma(\sigma)$. By Lemma 1.27.iii, the fact that w_s belongs to $\mathrm{I}(\sigma)$ is independent of the choice of w_s. Moreover, since $\mathrm{K}(\sigma)$ and $\mathrm{I}(\sigma)$ are orthogonal with respect to the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$ (Lemma 1.27), the fact that w_s belongs to $\mathrm{I}(\sigma)$ is equivalent to have $\chi \circ f_{|\,\mathrm{K}(\sigma)} = 1$ by (37).

ii. Let σ be an element of $\mathrm{O}(Q)$. Let $\{v_1, \dots, v_l\}$ be a basis of $\mathrm{K}(\sigma)$ and complete it into a basis $\mathscr{B} = \{v_1, \dots, v_{2n}\}$ of W. Then we can associate to σ an element $s_\sigma = (\sigma, f_\sigma) \in \mathrm{Ps}(B)$ by Proposition 1.12, where f_σ and β_σ are given by (18) and (19):

$$f_\sigma(w = \sum_{1 \leqslant i \leqslant 2n} \lambda_i v_i) = \sum_{1 \leqslant i < j \leqslant 2n} \lambda_i \lambda_j \beta_\sigma(v_i, v_j),$$

and for all $w_1, w_2 \in W$,

$$\beta_\sigma(w_1, w_2) = B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2).$$

In particular, if $w = \sum_{1 \leqslant i \leqslant l} \lambda_i v_i$ belongs to $\mathrm{K}(\sigma)$, then $\beta_\sigma(v_i, v_j) = 0$ for all $1 \leqslant i < j \leqslant l$, and thus $f_\sigma(w) = 0$. Then (37) is written

$$1 = \chi(f_\sigma(w)) = \chi(\langle w_{s_\sigma}, w \rangle)$$

for all $w \in \mathrm{K}(\sigma)$. This implies that w_{s_σ} belongs to $\mathrm{I}(\sigma)$. In fact, this is automatically true if $\mathrm{I}(\sigma) = W$, and otherwise $\mathrm{K}(\sigma) \neq \{0\}$ and the image $\mathbb{F}$ of the linear map $\langle w_{s_\sigma}, \cdot \rangle_{|\mathrm{K}(\sigma)}$ would be contained in the kernel of χ. If we set $\widetilde{s}_\sigma = (s_\sigma, \omega_{\widetilde{\chi}}(s_\sigma, w_{s_\sigma})) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, then we obtain that $\Theta_\chi(\widetilde{s}) \neq 0$ by (66).

Remark 1.42. This proof shows that, for every $\sigma \in \mathrm{O}(Q)$, there always exists a basis of W such that $f_{\sigma|\mathrm{K}(\sigma)} = 0$. However, it requires to fix a basis of $\mathrm{K}(\sigma)$ and then complete it into a basis of W. There is in fact an alternative solution if we do not want to fix these bases. We can define f_σ from any basis of W with (18). Then, since $(\sigma, f_\sigma) \in \mathrm{Ps}(B)$, we know that f_σ is additive on $\mathrm{K}(\sigma)$ by (17). Hence there exists $v \in W$ such that $\chi \circ f_\sigma = \chi \circ \langle v, \cdot \rangle$ on $\mathrm{K}(\sigma)$ by Proposition 1.12.i. Set $s = (\sigma, f = f_\sigma - \langle v, \cdot \rangle)$. Then $s \in \mathrm{Ps}(B)$ and $\chi \circ f = 1$ on $\mathrm{K}(\sigma)$. This enables to conclude by (37) that $w_s \in \mathrm{I}(\sigma)$, and we also obtain ii in this way.

iii. Let $\sigma \in \mathrm{O}(Q)$. If $1 - \sigma$ is invertible, then $\mathrm{I}(\sigma) = W$. Then every w_s satisfying (37) belongs to $\mathrm{I}(\sigma)$. By (66), this implies that $\Theta_\chi(\widetilde{s}) \neq 0$ for all $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ such that the projection of s on $\mathrm{O}(Q)$ is σ. Conversely, suppose that $1 - \sigma$ has a nontrivial kernel $\mathrm{K}(\sigma)$. Define first f_σ as in the proof of ii above, i.e. with a basis $\mathscr{B} = \{v_1, \dots, v_{2n}\}$ of W such that $\{v_1, \dots, v_l\}$ is a basis of $\mathrm{K}(\sigma)$, with $l \geqslant 1$. In particular, f_σ satisfies $f_{\sigma|\mathrm{K}(\sigma)} = 0$. Since $\mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$, $1 - \sigma$ is not invertible and $\langle \cdot, \cdot \rangle$ is nondegenerate, then $\mathrm{K}(\sigma)^\perp \neq W$. Hence there exists an element $w_0 \notin \mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$. In particular, the map $\langle w_0, \cdot \rangle_{|\mathrm{K}(\sigma)}$ is not identically zero. Then the image of the linear form $\langle w_0, \cdot \rangle_{|\mathrm{K}(\sigma)}$ is equal to $\mathbb{F}$. It follows that $\chi \circ \langle w_0, \cdot \rangle_{|\mathrm{K}(\sigma)} \neq 1$ since χ is nontrivial by hypothesis. Now, by Proposition 1.12.iii, there exists $z \in W$ such that $\chi \circ f_z = \chi \circ \langle w_0, \cdot \rangle$, where $f_z \in \mathscr{Q}_a(W)$. Set $s = (\sigma, f_\sigma + f_z) \in \mathrm{Ps}(B)$. Then for all $w \in \mathrm{K}(\sigma)$:

$$\chi(\langle w_0, w \rangle) = \chi(f_z(w)) = \chi((f_\sigma + f_z)(w)) = \chi(f(w)),$$

i.e. w_0 satisfies (37) for $s = (\sigma, f)$ with $f = f_\sigma + f_z$. Thus, by Lemma 1.27.iii, $w_s - w_0 \in \mathrm{I}(\sigma)$, which implies that w_s does not belong to $\mathrm{I}(\sigma)$. This shows that $\Theta_\chi(\widetilde{s}) = 0$ for $\widetilde{s} = (s, \omega_{\widetilde{\chi}}(s, w_s))$. Since the projection of s on $\mathrm{O}(Q)$ is σ, this completes the proof by contraposition.

□

Corollary 1.43. *The dual representation $\widetilde{\omega}_{\widetilde{\chi}}^*$ of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ determined by the family $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ is the Weil representation determined by the family $\{\overline{\gamma(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$. In particular, if $\gamma(\sigma)$ is real for all $\sigma \in \mathrm{O}(Q)$, then $\widetilde{\omega}_{\widetilde{\chi}}$ is self-dual.*

Proof. Let $\widetilde{\omega}_{\widetilde{\chi}}^*$ be the dual representation of $\widetilde{\omega}_{\widetilde{\chi}}$ and denote by Θ_{χ}^* its character. Since $\widetilde{\omega}_{\widetilde{\chi}}$ is unitary, for every $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$,

$$\Theta_{\chi}^*(\widetilde{s}) = \mathrm{Trace}(\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}^{-1})^T) = \mathrm{Trace}((\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s})^*)^T) = \overline{\mathrm{Trace}(\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}))} = \overline{\Theta_{\chi}(\widetilde{s})}.$$

If $\gamma(\sigma)$ is real for all $\sigma \in \mathrm{O}(Q)$ then the character Θ_{χ} of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ is real by (66). In particular, the character Θ_{χ}^* of the dual representation $\widetilde{\omega}_{\widetilde{\chi}}^*$ is exactly equal to the character Θ_{χ} of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$. □

Remark 1.44. In the real case, p-adic case and for finite fields of odd characteristic, up to isomorphism there are precisely two Weil representations, one dual to the other. As Corollary 1.43 shows, the case of finite fields of characteristic two is completely different.

Corollary 1.45. *Let $\widetilde{\chi}$ and $\chi^\sharp$ be two extensions of χ to $X \times \mathbb{F}$. For every $\sigma \in \mathrm{O}(Q)$, choose $\gamma(\sigma)$ and $\gamma'(\sigma)$ satisfying* (46)*. For every $s \in \mathrm{Ps}(B)$, choose w_s and w_s' satisfying* (37)*. Suppose that* (48) *and* (51) *are satisfied. Then the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$, obtained with the choices $\widetilde{\chi}$, $\{w_s$ satisfying* (37)*, $s \in \mathrm{Ps}(B)\}$ and $\{\gamma(\sigma)$ satisfying* (46)*, $\sigma \in \mathrm{O}(Q)\}$, is isomorphic to the Weil representation $\widetilde{\omega}_{\chi^\sharp} \circ \mathscr{I}$, obtained with the choices $\chi^\sharp$, $\{w_s'$ satisfying* (37)*, $s \in \mathrm{Ps}(B)\}$ and $\{\gamma'(\sigma)$ satisfying* (46)*, $\sigma \in \mathrm{O}(Q)\}$. In other terms, there exists an isomorphism $T \in \mathrm{U}(\mathrm{L}^2(Y))$ such that the following diagram commutes:*

$$\widetilde{\omega}_{\chi^\sharp}(\mathscr{I}(\widetilde{s})) = T \circ \widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}) \circ T^{-1} \quad \textit{for all } \widetilde{s} \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}, \quad \textit{i.e.} \quad \begin{array}{ccc} \mathrm{L}^2(Y) & \xrightarrow{\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s})} & \mathrm{L}^2(Y) \\ \downarrow T & & \downarrow T \\ \mathrm{L}^2(Y) & \xrightarrow{\widetilde{\omega}_{\chi^\sharp}(\mathscr{I}(\widetilde{s}))} & \mathrm{L}^2(Y) \end{array} \qquad (D_{15})$$

where $\mathscr{I}$ is defined by (52)*.*

Proof. Since (51) is satisfied, by Proposition 1.36.vi, the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$ of $\mathrm{Ps}(B)$ associated to the choices $(\widetilde{\chi}, \gamma(\sigma), w_s)$ and $(\chi^\sharp, \gamma'(\sigma), w_s')$, respectively, are isomorphic. An explicit isomorphism $\mathscr{I}$ is given by (52). Hence, $\mathscr{I}$ is also a group isomorphism between $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\chi^\sharp}$. Now, we denote by Θ_{χ} and Θ_{χ}' the characters of $\widetilde{\omega}_{\widetilde{\chi}}$ and $\widetilde{\omega}_{\chi^\sharp}$, respectively. Then by definition of $\mathscr{I}$, Θ_{χ} and Θ_{χ}', for every $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s, \gamma(\sigma))) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$,

$$\begin{aligned}
&\mathrm{Trace}(\widetilde{\omega}_{\chi^\sharp}(\mathscr{I}(\widetilde{s}))) \\
&= \mathrm{Trace}((-1)^l \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \omega_{\chi^\sharp}(s, w_s', \gamma'(\sigma))) \\
&= (-1)^l \nu(s)^{-1} \gamma(\sigma) \gamma'(\sigma)^{-1} \Theta_{\chi}'(s, \omega_{\chi^\sharp}(s, w_s', \gamma'(\sigma))) \\
&\overset{(53),(66)}{=} (-1)^l \gamma(\sigma) \gamma'(\sigma)^{-1} \chi(f(v) - \langle v, w_s \rangle - B(w_s' - w_s, v - w_s)) \gamma'(\sigma) |Y| \\
&\qquad \chi(f(w') - B(w_s' + w', w_s')) \mathbb{1}_{\mathrm{I}(\sigma)}(w_s') \\
&= (-1)^l \gamma(\sigma) |Y| \chi(f(v) - \langle v, w_s \rangle - B(w_s' - w_s, v - w_s) + f(w') - B(w_s' + w', w_s')) \\
&\qquad \mathbb{1}_{\mathrm{I}(\sigma)}(w_s'),
\end{aligned}$$

where v is any element such that $w'_s - w_s = (1-\sigma)(v)$ and $w' \in W$ is any element such that $(\sigma - 1)(w') = w'_s$. By Lemma 1.27.iii, $\mathbb{1}_{\mathrm{I}(\sigma)}(w'_s) = \mathbb{1}_{\mathrm{I}(\sigma)}(w_s)$. Moreover,

$$\begin{aligned}
&f(v) - \langle v, w_s\rangle - B(w'_s - w_s, v - w_s) + f(w') - B(w'_s + w', w'_s)\\
&= f(v+w') - B(\sigma(v), \sigma(w')) + B(v, w') - B(v, w_s) + B(w_s, v) - B(w'_s, v - w_s)\\
&\quad - B(w_s, w_s) + B(w_s, v) - B(w'_s, w'_s) - B(w', (1-\sigma)(v) + w_s)\\
&= f(v+w') - B(w_s + v + w', w_s) - B(\sigma(v), \sigma(w')) + B(v, w') - B(w'_s, v - w_s + w'_s)\\
&\quad - B(w', (1-\sigma)(v))\\
&= f(v+w') - B(w_s + v + w', w_s) - B(\sigma(v), \sigma(w')) + B(v, w') + B((\sigma-1)(w'), \sigma(v))\\
&\quad - B(w', (1-\sigma)(v))\\
&= f(v+w') - B(w_s + v + w', w_s) - \langle \sigma(v), \sigma(w')\rangle + \langle v, w'\rangle\\
&= f(v+w') - B(w_s + v + w', w_s) \quad \text{since } \sigma \in \mathrm{Sp}(W).
\end{aligned}$$

Hence

$$\mathrm{Trace}(\widetilde{\omega}_{\chi^\sharp}(\mathscr{I}(\widetilde{s}))) = (-1)^l \gamma(\sigma)|Y|\chi(f(v+w') - B(w_s + v + w', w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s),$$

where $(\sigma - 1)(v + w') = w_s$. Therefore, by Corollary 1.41, we obtain $\mathrm{Trace}(\widetilde{\omega}_{\chi^\sharp}(\mathscr{I}(\widetilde{s}))) = \Theta_\chi(\widetilde{s})$. This proves that $\widetilde{\omega}_{\widetilde{\chi}}$ and $\widetilde{\omega}_{\chi^\sharp} \circ \mathscr{I}$ are two irreducible representations with the same character, hence they are isomorphic. □

We proved in Proposition 1.36.vi that the two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ we construct does not depend on the choices made for the extension $\widetilde{\chi}$ of χ, $\{w_s$ satisfying (37), $s \in \mathrm{Ps}(B)\}$ and $\{\gamma(\sigma)$ satisfying (46) and (48), $\sigma \in \mathrm{O}(Q)\}$, provided (51) is satisfied. However, this two-fold covering of $\mathrm{Ps}(B)$ depends a priori on the choice of nontrivial character χ of $\mathbb{F}$ that we fixed in the beginning of section 1.2. In fact, we can prove that different choices of characters lead to isomorphic extensions of $\mathrm{Ps}(B)$, provided (51) is satisfied.

Proposition 1.46. *Let ψ be a nontrivial character of $\mathbb{F}$. We suppose that a choice of w_s satisfying* (37) *for χ has been made for all $s \in \mathrm{Ps}(B)$. We fix also $\gamma(\sigma)$ satisfying* (46) *for all $\sigma \in \mathrm{O}(Q)$ and* (48)*. Then we have the following results.*

i. *There exists $a = b^2 \in \mathbb{F}\backslash\{0\}$ such that $\psi(t) = \chi(at)$ for all $t \in \mathbb{F}$.*

ii. *For all $s = (\sigma, f) \in \mathrm{Ps}(B)$, the element $b^{-1}w_s$ satisfies* (37) *for ψ, i.e.*

$$\psi \circ f_{|\mathrm{K}(\sigma)} = \psi \circ \langle b^{-1}w_s, \cdot\rangle_{|\mathrm{K}(\sigma)}.$$

iii. *Let $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ be the two-fold covering of $\mathrm{Ps}(B)$ obtained from the extension $\widetilde{\chi}$ of χ to $X \times \mathbb{F}$ and the choices $\{w_s,\ s \in \mathrm{Ps}(B)\}$ and $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$. Let $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\psi}}$ be the two-fold covering of $\mathrm{Ps}(B)$ obtained from the extension $\widetilde{\psi}$ of ψ to $X \times \mathbb{F}$ and the choices $\{b^{-1}w_s,\ s \in \mathrm{Ps}(B)\}$ mentioned in ii, and $\{\gamma'(\sigma),\ \sigma \in \mathrm{O}(Q)\}$. If* (51) *is satisfied then the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}'_{\widetilde{\psi}}$ of $\mathrm{Ps}(B)$ are isomorphic.*

iv. *Let $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ be the extension defined in iii. Let $\widetilde{\mathrm{Ps}(B)}_{\tilde{\psi}}$ be the two-fold covering of $\mathrm{Ps}(B)$ obtained from the extension $\tilde{\psi}$ of ψ to $X \times \mathbb{F}$ and the choices $\{v_s,\ s \in \mathrm{Ps}(B)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ such that* (37)*,* (46) *and* (48) *are satisfied. If* (51) *is satisfied then the extensions $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\tilde{\psi}}$ of $\mathrm{Ps}(B)$ are isomorphic. An explicit isomorphism of central extensions is*

$$\begin{aligned} \mathscr{J} : \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} &\to \widetilde{\mathrm{Ps}(B)}_{\tilde{\psi}} \\ (s, (-1)^l \omega_{\tilde{\chi}}(s, w_s, \gamma(\sigma))) &\mapsto (s, (-1)^l \nu(s)^{-1} \gamma(\sigma) \delta(\sigma)^{-1} \omega_{\tilde{\psi}}(s, v_s, \delta(\sigma))) \end{aligned} \tag{67}$$

where ν relates the choices $\{b^{-1}w_s, s \in \mathrm{Ps}(B)\}$ and $\{v_s,\ s \in \mathrm{Ps}(B)\}$ by (53)*, i.e.*

$$\nu(s) = \psi(-f(v) + \langle v, b^{-1}w_s\rangle + B(v_s - b^{-1}w_s, v - b^{-1}w_s)), \tag{68}$$

where v is any element such that $v_s - b^{-1}w_s = (1-\sigma)(v)$. This means that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathscr{J}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\psi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\psi}} & \xrightarrow{\Psi_{\tilde{\psi}}} & \mathrm{Ps}(B) & \longrightarrow & 1 \end{array} \tag{D_{16}}$$

Proof. **i.** This follows from Proposition 1.12.ii and Lemma 1.15.

ii. For every $w \in \mathrm{K}(\sigma)$, by i and the fact that f is quadratic,

$$\psi(f(w)) = \chi(af(w)) = \chi(f(bw)) = \chi(\langle w_s, bw\rangle) = \chi(a\langle b^{-1}w_s, w\rangle) = \psi(\langle b^{-1}w_s, w\rangle),$$

where the third equality comes from the fact that w_s satisfies (37) for χ. This proves that $b^{-1}w_s$ satisfies (37) for ψ.

iii. To show that the extensions $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ (associated to $\{w_s, s \in \mathrm{Ps}(B)\}$ and $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$) and $\widetilde{\mathrm{Ps}(B)}'_{\tilde{\psi}}$ (associated to $\{b^{-1}w_s, s \in \mathrm{Ps}(B)\}$ and $\{\gamma'(\sigma),\ \sigma \in \mathrm{O}(Q)\}$) are isomorphic, it is enough to show that the cocycles c_χ and c'_ψ associated to these extensions are cohomologous. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\mathrm{Ps}(B)$. Then, by (44) in Corollary 1.31,

$$c_\chi(s_1, s_2) = C_{1,2} \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1,s_2}}} \chi(f_1(w_1) + \langle w_{s_1}, w_1\rangle + B(u_1, w_1 + w_{s_1}) + f_2(w_2)) \\ \chi(\langle w_{s_2}, w_2\rangle + B(u_2, w_2 + w_{s_2}) + B(u_1 + w_{s_1}, u_2 + w_{s_2}))$$

where $C_{1,2} = \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1}$, $(1-\sigma)(w_i) = u_i$ for $i \in \{1,2\}$, and $w_{s_1,s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2}$. This implies that $b^{-1}w_{s_1,s_2} = b^{-1}w_{s_1 s_2} - b^{-1}w_{s_1} - b^{-1}w_{s_2}$. If we set

$u_i' = b^{-1}u_i$ and $w_i' = b^{-1}w_i$, then $(1-\sigma)(w_i') = u_i'$ and

$$
\begin{aligned}
&c_\chi(s_1,s_2)\\
&= C_{1,2} \sum_{\substack{u_1'\in \mathrm{I}(\sigma_1), u_2'\in \mathrm{I}(\sigma_2)\\ u_1'+u_2' = b^{-1}w_{s_1,s_2}}} \chi(f_1(bw_1') + \langle w_{s_1}, bw_1'\rangle + B(bu_1', bw_1' + w_{s_1}) + f_2(bw_2'))\\
&\qquad \chi(\langle w_{s_2}, bw_2'\rangle + B(bu_2', bw_2' + w_{s_2}) + B(bu_1' + w_{s_1}, bu_2' + w_{s_2}))\\
&= C_{1,2} \sum_{\substack{u_1'\in \mathrm{I}(\sigma_1), u_2'\in \mathrm{I}(\sigma_2)\\ u_1'+u_2' = b^{-1}w_{s_1,s_2}}} \chi(af_1(w_1') + a\langle b^{-1}w_{s_1}, w_1'\rangle + aB(u_1', w_1' + b^{-1}w_{s_1}) + af_2(w_2'))\\
&\qquad \chi(a\langle b^{-1}w_{s_2}, w_2'\rangle + aB(u_2', w_2' + b^{-1}w_{s_2}) + aB(u_1' + b^{-1}w_{s_1}, u_2' + b^{-1}w_{s_2}))\\
&= C_{1,2} \sum_{\substack{u_1'\in \mathrm{I}(\sigma_1), u_2'\in \mathrm{I}(\sigma_2)\\ u_1'+u_2' = b^{-1}w_{s_1,s_2}}} \psi(f_1(w_1') + \langle b^{-1}w_{s_1}, w_1'\rangle + B(u_1', w_1' + b^{-1}w_{s_1}) + f_2(w_2'))\\
&\qquad \psi(\langle b^{-1}w_{s_2}, w_2'\rangle + B(u_2', w_2' + b^{-1}w_{s_2}) + B(u_1' + b^{-1}w_{s_1}, u_2' + b^{-1}w_{s_2}))\\
&= \gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)^{-1}\gamma'(\sigma_1\sigma_2)c_\psi'(s_1,s_2),
\end{aligned}
$$

i.e.

$$c_\chi(s_1,s_2)c_\psi'(s_1,s_2)^{-1} = \gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)^{-1}\gamma'(\sigma_1\sigma_2). \tag{69}$$

Since $\sigma \mapsto \gamma(\sigma)\gamma'(\sigma)^{-1}$ takes its values in $\{\pm 1\}$ by (51), $c_\chi c_\psi'^{-1}$ is a coboundary.

iv. Consider again the extension $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ defined in iii. Let $\{\delta(\sigma),\ \sigma\in \mathrm{O}(Q)\}$ be such that (46) and (48) are satisfied. Suppose also that (51) is satisfied. Let $\widetilde{\mathrm{Ps}(B)}''_{\widetilde{\psi}}$ be the extension defined from $\widetilde{\psi}$, $\{b^{-1}w_s,\ s\in \mathrm{Ps}(B)\}$ and $\{\delta(\sigma),\ \sigma\in \mathrm{O}(Q)\}$. By iii, the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}''_{\widetilde{\psi}}$ of $\mathrm{Ps}(B)$ are isomorphic. Define now the extension $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\psi}}$ from the extension $\widetilde{\psi}$ of ψ and the choices $\{v_s,\ s\in \mathrm{Ps}(B)\}$ and $\{\delta(\sigma),\ \sigma\in \mathrm{O}(Q)\}$. Then the extensions $\widetilde{\mathrm{Ps}(B)}''_{\widetilde{\psi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\psi}}$ of $\mathrm{Ps}(B)$ are isomorphic by Proposition 1.36.iii. This is enough to conclude that the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\psi}}$ of $\mathrm{Ps}(B)$ are isomorphic. More precisely, define $\mathscr{J}$ as in (67). We denote by c_ψ'' and c_ψ the cocycles associated to $\widetilde{\mathrm{Ps}(B)}''_{\widetilde{\psi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\psi}}$, respectively. Let $\widetilde{s_1} = (s_1, (-1)^{l_1}\omega_{\widetilde{\chi}}(s_1, w_{s_1}, \gamma(\sigma_1)))$, $\widetilde{s_2} = (s_2, (-1)^{l_2}\omega_{\widetilde{\chi}}(s_2, w_{s_2}, \gamma(\sigma_2))) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. As in Proposition 1.36.vi, we prove that

$$
\begin{aligned}
\mathscr{J}(\widetilde{s_1})\mathscr{J}(\widetilde{s_2}) = (s_1s_2, (-1)^{l_1+l_2}&\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\delta(\sigma_1)^{-1}\gamma(\sigma_2)\delta(\sigma_2)^{-1}c_\psi(s_1,s_2)\\
&\omega_{\widetilde{\psi}}(s_1s_2, v_{s_1s_2}, \delta(\sigma_1\sigma_2)))
\end{aligned}
\tag{70}
$$

$$\mathscr{J}(\widetilde{s_1}\widetilde{s_2}) = (s_1s_2, (-1)^{l_1+l_2}c_\chi(s_1,s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\delta(\sigma_1\sigma_2)^{-1}\omega_{\widetilde{\psi}}(s_1s_2, v_{s_1s_2}, \delta(\sigma_1\sigma_2))). \tag{71}$$

Hence $\mathscr{J}$ is a group homomorphism if and only if

$$c_\psi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\delta(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\delta(\sigma_1)\gamma(\sigma_2)^{-1}\delta(\sigma_2). \tag{72}$$

By (54),

$$c_\psi(s_1,s_2)c_\psi''(s_1,s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}. \tag{73}$$

By (69),

$$c''_\psi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = \delta(\sigma_1)\gamma(\sigma_1)^{-1}\delta(\sigma_2)\gamma(\sigma_2)^{-1}(\delta(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \tag{74}$$

Since $c_\psi(s_1,s_2)c_\chi(s_1,s_2)^{-1} = c_\psi(s_1,s_2)c''_\psi(s_1,s_2)^{-1}c''_\psi(s_1,s_2)c_\chi(s_1,s_2)^{-1}$, (73) and (74) imply (72). The fact that $\mathscr{J}$ is an isomorphism such that the diagram (D_{16}) commutes can be proven exactly in the same way as for $\mathscr{I}$ in Proposition 1.36.vi.

□

Corollary 1.47. *Let χ and ψ be two nontrivial characters of $\mathbb{F}$ (such that $\psi(t) = \chi(at)$ for all $t \in \mathbb{F}$, where $a = b^2 \in \mathbb{F}\backslash\{0\}$, cf. Proposition 1.46.i). Let $\widetilde{\chi}$ and $\widetilde{\psi}$ be two extensions of χ and ψ to $X \times \mathbb{F}$, respectively. For every $\sigma \in \mathrm{O}(Q)$, choose $\gamma(\sigma)$ and $\delta(\sigma)$ satisfying* (46) *with χ and ψ, respectively. For every $s \in \mathrm{Ps}(B)$, choose w_s and v_s satisfying* (37) *with χ and ψ, respectively. Suppose that* (48) *and* (51) *are satisfied. Then the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$, obtained with $\widetilde{\chi}$, $\{w_s$ satisfying* (37)*, $s \in \mathrm{Ps}(B)\}$ *and $\{\gamma(\sigma)$ satisfying* (46)*, $\sigma \in \mathrm{O}(Q)\}$, is isomorphic to the Weil representation $\widetilde{\omega}_{\widetilde{\psi}} \circ \mathscr{J}$, obtained with $\widetilde{\psi}$, $\{v_s$ satisfying* (37)*, $s \in \mathrm{Ps}(B)\}$ *and $\{\delta(\sigma)$ satisfying* (46)*, $\sigma \in \mathrm{O}(Q)\}$. In other terms, there exists an isomorphism $U \in \mathrm{U}(\mathrm{L}^2(Y))$ such that the following diagram commutes:*

$$\widetilde{\omega}_{\widetilde{\psi}}(\mathscr{J}(\widetilde{s})) = U \circ \widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}) \circ U^{-1} \quad \textit{for all } \widetilde{s} \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}, \quad \textit{i.e.} \quad \begin{array}{ccc} \mathrm{L}^2(Y) & \xrightarrow{\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s})} & \mathrm{L}^2(Y) \\ \downarrow U & & \downarrow U \\ \mathrm{L}^2(Y) & \xrightarrow{\widetilde{\omega}_{\widetilde{\psi}}(\mathscr{J}(\widetilde{s}))} & \mathrm{L}^2(Y) \end{array} \tag{D_{17}}$$

where $\mathscr{J}$ is defined by (67).

Proof. Since (51) is satisfied, by Proposition 1.46.iv, the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\psi}}$ of $\mathrm{Ps}(B)$ associated to the choices $(\widetilde{\chi}, \gamma(\sigma), w_s)$ and $(\widetilde{\psi}, \delta(\sigma), v_s)$, respectively, are isomorphic. An explicit isomorphism $\mathscr{J}$ is given by (67). Hence, $\mathscr{J}$ is also a group isomorphism between $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\psi}}$. Now, we denote by Θ_χ and Θ_ψ the characters of $\widetilde{\omega}_{\widetilde{\chi}}$ and $\widetilde{\omega}_{\widetilde{\psi}}$, respectively. Then by definition of $\mathscr{J}$, Θ_χ and Θ_ψ, for every $\widetilde{s} = (s, (-1)^l\omega_{\widetilde{\chi}}(s, w_s, \gamma(\sigma))) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, we have that

$$\begin{aligned}
&\mathrm{Trace}(\widetilde{\omega}_{\widetilde{\psi}}(\mathscr{J}(\widetilde{s}))) \\
&= \mathrm{Trace}((-1)^l\nu(s)^{-1}\gamma(\sigma)\delta(\sigma)^{-1}\omega_{\widetilde{\psi}}(s, v_s, \delta(\sigma))) \\
&= (-1)^l\nu(s)^{-1}\gamma(\sigma)\delta(\sigma)^{-1}\Theta_\psi(s, \omega_{\widetilde{\psi}}(s, v_s, \delta(\sigma))) \\
&\overset{(66),(68)}{=} (-1)^l\gamma(\sigma)\delta(\sigma)^{-1}\psi(f(v) - \langle v, b^{-1}w_s\rangle - B(v_s - b^{-1}w_s, v - b^{-1}w_s))\delta(\sigma)|Y| \\
&\qquad \psi(f(w) - B(v_s + w, v_s))\mathbb{1}_{\mathrm{I}(\sigma)}(v_s) \\
&= (-1)^l\gamma(\sigma)|Y|\psi(f(v) - \langle v, b^{-1}w_s\rangle - B(v_s - b^{-1}w_s, v - b^{-1}w_s) + f(w) - B(v_s + w, v_s)) \\
&\qquad \mathbb{1}_{\mathrm{I}(\sigma)}(v_s),
\end{aligned}$$

where v is any element such that $v_s - b^{-1}w_s = (1 - \sigma)(v)$ and $w \in W$ is any element such that $(\sigma - 1)(w) = v_s$. As in Corollary 1.45, we can prove that

$$\begin{aligned}
&f(v) - \langle v, b^{-1}w_s\rangle - B(v_s - b^{-1}w_s, v - b^{-1}w_s) + f(w) - B(v_s + w, v_s) \\
&= f(v + w) - B(b^{-1}w_s + v + w, b^{-1}w_s).
\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Trace}(\widetilde{\omega}_{\tilde{\psi}}(\mathscr{J}(\widetilde{s}))) &= (-1)^l\gamma(\sigma)|Y|\psi(f(v+w)-B(b^{-1}w_s+v+w,b^{-1}w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s)\\ &= (-1)^l\gamma(\sigma)|Y|\chi(af(v+w)-aB(b^{-1}w_s+v+w,b^{-1}w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s)\\ &= (-1)^l\gamma(\sigma)|Y|\chi(f(bv+bw)-B(w_s+bv+bw,w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s),\end{aligned}$$

where $(\sigma-1)(bv+bw) = w_s - bv_s + bv_s = w_s$. Therefore, by Corollary 1.41, we obtain $\mathrm{Trace}(\widetilde{\omega}_{\tilde{\psi}}(\mathscr{J}(\widetilde{s}))) = \Theta_\chi(\widetilde{s})$. This proves that $\widetilde{\omega}_{\tilde{\chi}}$ and $\widetilde{\omega}_{\tilde{\psi}}\circ\mathscr{J}$ are two irreducible representations with the same character, hence they are isomorphic. □

Chapter 2

Extension from the field $\mathbb{F} = \mathbb{F}_2$ to the ring $\mathbb{Z}/4\mathbb{Z}$

In chapter 1, we considered a finite-dimensional symplectic vector space $(W, \langle \cdot, \cdot \rangle)$ over a finite field $\mathbb{F}$ of characteristic two and a bilinear form B associated to the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$ by (16). We defined the Weil representation as a linear representation of a two-fold covering of the pseudosymplectic group $\mathrm{Ps}(B)$. The main drawback of the pseudosymplectic group is that it projects onto the orthogonal group $\mathrm{O}(Q)$, which is almost always a proper subgroup of the symplectic group $\mathrm{Sp}(W)$, the only exception being $W \simeq (\mathbb{F}_2)^2$ and Q of Witt index 0, according to Proposition 1.13. In other settings, for example in the real case, if $\mathbb{F}$ is a p-adic field with p odd or a finite field of odd characteristic, we can always come back to the symplectic group. In fact, we can define the (linear) Weil representation of either $\mathrm{Sp}(W)$ for a finite field of odd characteristic or a two-fold covering of $\mathrm{Sp}(W)$ in the real or p-adic case (see e.g. [AP14]). It is therefore a natural question to know what kind of result related to $\mathrm{Sp}(W)$ we can obtain for a finite field of characteristic two. We examine in this section the case $\mathbb{F} = \mathbb{F}_2$.

We start by considering a free $\mathbb{Z}/4\mathbb{Z}$-module $(\widetilde{W}, \langle \cdot, \cdot \rangle_{\widetilde{W}})$ of finite rank, endowed with a nondegenerate and alternating bilinear form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$. We also say that $(\widetilde{W}, \langle \cdot, \cdot \rangle_{\widetilde{W}})$ is a free symplectic module. Its rank is even and we denote it by $2n$, where $n \geqslant 1$. We choose a bilinear form $B_{\widetilde{W}}$ associated to $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ by

$$\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} = B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) - B_{\widetilde{W}}(\widetilde{w}_2, \widetilde{w}_1). \tag{75}$$

Since there exists a symplectic basis for $(\widetilde{W}, \langle \cdot, \cdot \rangle_{\widetilde{W}})$, we can fix a decomposition $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$, where $\widetilde{X}$ and $\widetilde{Y}$ are two submodules of $\widetilde{W}$ of rank n such that $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ vanishes on $\widetilde{X} \times \widetilde{X}$ and on $\widetilde{Y} \times \widetilde{Y}$. Then for all $\widetilde{w}_1 = \widetilde{x}_1 + \widetilde{y}_1$ and $\widetilde{w}_2 = \widetilde{x}_2 + \widetilde{y}_2$ in $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$,

$$\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} = \langle \widetilde{x}_1 + \widetilde{y}_1, \widetilde{x}_2 + \widetilde{y}_2 \rangle_{\widetilde{W}} = \langle \widetilde{x}_1, \widetilde{y}_2 \rangle_{\widetilde{W}} + \langle \widetilde{y}_1, \widetilde{x}_2 \rangle_{\widetilde{W}}.$$

Thanks to the following lemma, we recover the settings of chapter 1 by constructing a symplectic $\mathbb{F}_2$-vector space W of dimension $2n$ from the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$.

Lemma 2.1. *Denote by R the ring $\mathbb{Z}/4\mathbb{Z}$.*

i. *There is an abelian group isomorphism $\kappa : 2R \to \mathbb{F}_2$. Explicitly, $\kappa(0) = 0$ and $\kappa(2) = 1$.*

ii. *Let $W = \widetilde{W}/2\widetilde{W}$. For all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$, we define*

$$\langle w_1, w_2\rangle_W = \langle \widetilde{w}_1 + 2\widetilde{W}, \widetilde{w}_2 + 2\widetilde{W}\rangle_W = 2\langle \widetilde{w}_1, \widetilde{w}_2\rangle_{\widetilde{W}}. \tag{76}$$

Then W is a free $R/2R$-module of rank $2n$ and $\langle \cdot, \cdot \rangle_W$ is a well-defined, nondegenerate and alternating bilinear form on W with values in $2R$.

iii. *We can identify W with an $\mathbb{F}_2$-vector space. If we set $\langle \cdot, \cdot \rangle = \kappa(\langle \cdot, \cdot \rangle_W)$ then $\langle \cdot, \cdot \rangle_W$ can be identified with the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$ defined on the vector space W with values in $\mathbb{F}_2$. In this way, $(W, \langle \cdot, \cdot \rangle)$ becomes a symplectic $\mathbb{F}_2$-vector space of dimension $2n$.*

Proof. **i.** $2R$ is an ideal of R, and $R/2R$ is a ring which has only two elements. Hence the map $\phi : R/2R \to \mathbb{F}_2$, defined by $\phi(2R) = 0$ and $\phi(1 + 2R) = 1$, is a field isomorphism. Consider now the projection map $\psi : R \to R/2R$ and the map $\gamma : R/2R \to 2R$ which is defined by $\gamma(\psi(r)) = 2r$. Since $\gamma(2R) = 0$, $\gamma(1 + 2R) = 2$ and $\gamma(\psi(r) + \psi(r')) = \gamma(\psi(r + r')) = 2(r + r') = \gamma(\psi(r)) + \gamma(\psi(r'))$, then γ is an abelian group isomorphism. Then $\kappa = \phi \circ \gamma^{-1} : 2R \to \mathbb{F}_2$ is an abelian group isomorphism such that $\kappa(0) = \phi(2R) = 0$ and $\kappa(2) = \phi(1 + 2R) = 1$. (Observe that $2R$ is not a ring so we cannot hope for κ to be a ring isomorphism).

ii. Let $\{\widetilde{u}_1, \dots, \widetilde{u}_{2n}\}$ be a basis of the free R-module $\widetilde{W}$. Then $\{\widetilde{u}_1 + 2\widetilde{W}, \dots, \widetilde{u}_{2n} + 2\widetilde{W}\}$ is clearly a generating set of the $R/2R$-module $W = \widetilde{W}/2\widetilde{W}$. Moreover, if $\lambda_1, \dots, \lambda_{2n}$ are elements of R such that $\sum_{1 \leqslant i \leqslant 2n} (\lambda_i + 2R)(\widetilde{u}_i + 2\widetilde{W}) = 0$, then $\sum_{1 \leqslant i \leqslant 2n} \lambda_i \widetilde{u}_i$ belongs to $2\widetilde{W}$. Hence $2 \sum_{1 \leqslant i \leqslant 2n} \lambda_i \widetilde{u}_i = 0$, which implies that $2\lambda_i = 0$ for every index i, since $\{\widetilde{u}_i,\ 1 \leqslant i \leqslant 2n\}$ is a basis of $\widetilde{W}$. This proves that $\lambda_i \in 2R$ for every i, and this is enough to conclude that $\{\widetilde{u}_1 + 2\widetilde{W}, \dots, \widetilde{u}_{2n} + 2\widetilde{W}\}$ is a basis of W. Now, let $\widetilde{w}_1$, $\widetilde{w}_2$ and $\widetilde{v}_1$, $\widetilde{v}_2$ be in $\widetilde{W}$ such that $\widetilde{w}_i - \widetilde{v}_i \in 2\widetilde{W}$ for $i = 1, 2$. Then $2(\widetilde{w}_i - \widetilde{v}_i) = 0$, which implies that $2\langle \widetilde{w}_1, \widetilde{w}_2\rangle_{\widetilde{W}} = 2\langle \widetilde{v}_1, \widetilde{v}_2\rangle_{\widetilde{W}}$. Hence $\langle \cdot, \cdot \rangle_W$ is well-defined. This form is also bilinear and alternating since $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ satisfies these properties. Now, let $\widetilde{w}_1 \in \widetilde{W}$ and suppose that for every $\widetilde{w}_2 \in \widetilde{W}$, $\langle \widetilde{w}_1 + 2\widetilde{W}, \widetilde{w}_2 + 2\widetilde{W}\rangle_W = 2\langle \widetilde{w}_1, \widetilde{w}_2\rangle_{\widetilde{W}} = 0$. Since $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ is nondegenerate, this implies that $2\widetilde{w}_1 = 0$, i.e. $\widetilde{w}_1 \in 2\widetilde{W}$. This proves that $\langle \cdot, \cdot \rangle_W$ is nondegenerate.

iii. Since $R/2R$ is a field isomorphic to $\mathbb{F}_2$, the $R/2R$-module $W = \widetilde{W}/2\widetilde{W}$ of rank $2n$ can be identified with a $\mathbb{F}_2$-vector space of dimension $2n$. The result follows from the two preceding points.

□

Define the bilinear form B_W on W by $B_W(w_1, w_2) = 2B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)$ for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$. Then

$$\langle w_1, w_2\rangle_W = 2\langle \widetilde{w}_1, \widetilde{w}_2\rangle_{\widetilde{W}} = 2(B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) - B_{\widetilde{W}}(\widetilde{w}_2, \widetilde{w}_1)) = B_W(w_1, w_2) - B_W(w_2, w_1). \tag{77}$$

If we set $B = \kappa \circ B_W$, where κ is defined in Lemma 2.1, then B is a bilinear form on W with values in $\mathbb{F}_2$ and

$$\langle w_1, w_2 \rangle = \kappa(\langle w_1, w_2 \rangle_W) = \kappa(B_W(w_1, w_2)) - \kappa(B_W(w_2, w_1)) = B(w_1, w_2) - B(w_2, w_1). \tag{78}$$

Set $X = \widetilde{X}/2\widetilde{X}$ and $Y = \widetilde{Y}/2\widetilde{Y}$. Then X and Y can be identified with two $\mathbb{F}_2$-vector spaces of dimension n such that $W = X \oplus Y$ and $\langle \cdot, \cdot \rangle$ vanishes on $X \times X$ and $Y \times Y$. Finally, we define $Q_W(w) = B_W(w, w)$ and $Q(w) = B(w, w)$ for all $w \in W$.

In this way, we recover the setup of chapter 1. In particular, we can consider $\mathrm{O}(Q)$ and $\mathrm{Sp}(W)$, the groups of $\mathbb{F}_2$-vector space automorphisms preserving the quadratic form Q and the nondegenerate symplectic form $\langle \cdot, \cdot \rangle$, respectively. In the same way, we define $\mathrm{Sp}(\widetilde{W})$ as the group of $\mathbb{Z}/4\mathbb{Z}$-module automorphisms preserving the nondegenerate symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$. Using the complete polarizations $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$ and $W = X \oplus Y$ given above, we denote the elements of $\mathrm{Sp}(W)$ and $\mathrm{Sp}(\widetilde{W})$, respectively, by $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where α, β, γ, $\delta \in \mathrm{M}_n(\mathbb{F}_2)$, and $\widetilde{\sigma} = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix}$ where $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{\delta} \in \mathrm{M}_n(\mathbb{Z}/4\mathbb{Z})$. Similarly, the matrices of the nondegenerate symplectic forms of W and $\widetilde{W}$ with respect to symplectic bases compatible with the decompositions $X \oplus Y$ and $\widetilde{X} \oplus \widetilde{Y}$, are given by $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{F}_2)$ and $\widetilde{\Omega} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{Z}/4\mathbb{Z})$, respectively.

Now we follow the same path as in chapter 1. We start by defining a Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ associated to the bilinear form B_W. We study the automorphism group of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, acting trivially on the center, i.e. the affine symplectic group $\mathrm{ASp}(B_W)$, and construct a natural projective representation of $\mathrm{ASp}(B_W)$. Then we study the coverings of $\mathrm{ASp}(B_W)$ which will enable us to construct the corresponding Weil representation as a linearization of the previously mentioned projective representation. Here we are using the term "coverings" in the plural, as we will be considering two different approaches. Our initial focus, as in chapter 1, will be on finding the smallest possible covering. Except for the case in which Q has Witt index zero, our general construction shows that the smallest possible extension is a four-fold covering of $\mathrm{ASp}(B_W)$. This result is consistent with the construction of a four-fold covering studied by Gurevich-Hadani in [GH12] and Genestier-Lysenko in [GL12]. A variation of our construction leads to a two-fold covering when $\dim_{\mathbb{F}_2}(W) = 2$ and the Witt index of Q is one.

A second approach consists in studying the link between the covering and Weil representation defined in chapter 1 and the one presented below. In other words, we will examine how the following construction connects to Weil's original work considered in chapter 1. In this context, we will have to consider an eight-fold covering of $\mathrm{ASp}(B_W)$. This extension, even if is not minimal, turns out to be structurally consistent with the construction we did for $\mathrm{Ps}(B)$.

Finally, we will present a third construction of the Weil representation, which is directly related to the symplectic group $\mathrm{Sp}(\widetilde{W})$, bypassing an extension of the symplectic group (such

as $\mathrm{ASp}(B_W)$). To this extent, we will define a Heisenberg group $\mathrm{H}(\widetilde{W})$, associated to the symplectic form $\langle\cdot,\cdot\rangle_{\widetilde{W}}$, on which the symplectic group $\mathrm{Sp}(\widetilde{W})$ acts by automorphisms. Then we will define a natural projective representation of $\mathrm{Sp}(\widetilde{W})$, and we will lift this projective representation to a linear representation of a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. This provides the construction of a representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ that is independent of the affine metaplectic covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Although this representation is not faithful in all but one case (and hence it behaves differently from all classical incarnations of the Weil representation), we will continue to refer to it as "the Weil representation" of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. To complete our study, we will investigate the links between $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ for $B = B_{\mathrm{Weil}}$, by examining the commutativity of the diagram (D_{18}). This will also shed light on the results of Gurevich and Hadani in [GH12].

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{Hom}_{\mathbb{Z}}(W,\mathbb{Z}/4\mathbb{Z}) & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & \overset{\mathscr{S}}{\curvearrowleft} & \mathrm{U}(\mathrm{L}^2(Y)) & \overset{\eta}{\curvearrowleft} & \mathrm{Sp}(W) & \longrightarrow & 1 \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\chi_k}} & & \uparrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W})_{\chi_k} & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) & \longrightarrow & 1
\end{array}
\qquad (D_{18})
$$

2.1 The Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and the affine symplectic group $\mathrm{ASp}(B_W)$

Let $B_{\widetilde{W}}$ be a bilinear form satisfying (75). Then the bilinear form $B = \kappa \circ B_W$, where κ is defined in Lemma 2.1, satisfies (78). Using B_W, we first define a Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ that is analog to the group $\mathrm{H}(B)$ defined in 1.1. Notice that if, for example, we choose $B_{\widetilde{W}}(\widetilde{w}_1,\widetilde{w}_2) = \langle\widetilde{x}_1,\widetilde{y}_2\rangle_{\widetilde{W}}$, then $B = \kappa \circ B_W$ corresponds to the bilinear form used by Weil in [Wei64]. The definition of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is used by Glasby [Gla95] and Genestier-Lysenko [GL12] for a general bilinear form B_W. Gurevich and Hadani [GH12] mention the general definition, but they fix B_W so that $B = \kappa \circ B_W$ corresponds to B_{Weil}.

Definition-Proposition 2.2. *Define $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ to be the set $W \times \mathbb{Z}/4\mathbb{Z}$ endowed with the following product:*

$$(w_1,t_1)(w_2,t_2) = (w_1+w_2, t_1+t_2+B_W(w_1,w_2)). \tag{79}$$

Then:

i. *$\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is a group whose center is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Its identity element is $(0,0)$ and the inverse of (w,t) in $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is $(-w, -t + B_W(w,w))$.*

ii. *The product* (79) *defines a central extension*

$$0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\Phi} \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \xrightarrow{\Psi} W \to 0,$$

where $\Phi(t) = (0,t)$ and $\Psi(w,t) = w$. Moreover, B_W is a cocycle and its cohomology class in $\mathrm{H}^2(W, \mathbb{Z}/4\mathbb{Z})$ is the one associated to the isomorphism class of the central extension defined by $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$.

iii. *Let $B_{\widetilde{W}}$ and $B'_{\widetilde{W}}$ be two bilinear forms satisfying* (75).

- *The central extensions defined by $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ are isomorphic if and only if B_W and B'_W are cohomologous.*
- *The groups $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ are isomorphic if and only if there exists a map $f : W \to \mathbb{Z}/4\mathbb{Z}$, an automorphism of abelian groups $\sigma : W \to W$ and an automorphism of additive abelian groups $g : \mathbb{Z}/4\mathbb{Z} \mapsto \mathbb{Z}/4\mathbb{Z}$ such that for all $(w_1, w_2) \in W$:*

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B'_W(\sigma(w_1), \sigma(w_2)) - g(B_W(w_1, w_2)). \tag{80}$$

In this case, the isomorphism is given by

$$\begin{aligned} \Phi_{(\sigma,f,g)} : \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) &\to \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W) \\ (w,t) &\mapsto (\sigma(w), f(w) + g(t)). \end{aligned} \tag{81}$$

Every isomorphism between $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ is of this form.

Proof. This proof is very similar to the one of Proposition 1.5. In particular, we use the fact that $(w_1, w_2) \mapsto \langle w_1, w_2 \rangle_W = B_W(w_1, w_2) - B_W(w_2, w_1)$ is nondegenerate. □

Let $B_{\widetilde{W}}$ and $B'_{\widetilde{W}}$ be two bilinear forms satisfying (75). Proposition 2.2.iii provides a necessary and sufficient condition for $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ to be isomorphic. In fact, as we are going to prove, these two groups are always isomorphic. One fact to notice (and to compare with the similar result that $\mathrm{H}(B)$ is extraspecial, see Proposition 1.6) is that the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is an almost extraspecial 2-group (see appendix A for the definition and structure of almost extraspecial groups). The following proposition shows that this fact implies that, up to isomorphism, there is only one Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ to consider. This differs from the situation encountered in the previous section where, in the case $\mathbb{F} = \mathbb{F}_2$, there were two non-isomorphic Heisenberg groups $\mathrm{H}(B)$, depending on the Witt index (or equivalently, Arf invariant) of B.

Proposition 2.3. *Let $B_{\widetilde{W}}$ be a bilinear form satisfying* (75). *Then $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is an almost extraspecial 2-group. In particular, there is only one Heisenberg group of order 2^{2n+2} up to isomorphism.*

Proof. Cf. Proposition A.17. □

Remark 2.4. Suppose that $\mathbb{F} = \mathbb{F}_2$, and let B and B' be two bilinear forms which satisfy (16), and such that their Witt indices are different. By Proposition 2.3, the groups $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ are isomorphic. By Proposition 2.2, we can fix an isomorphism $\Phi_{(\sigma,f,g)} : \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \to \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ as in (81). Observe that $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \simeq \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ contains both $\mathrm{H}(B)$ and $\mathrm{H}(B')$ as subgroups. (In the terms used in appendix A, the almost extraspecial 2-group of order 2^{2n+2} contains a copy of each of the two extraspecial 2-groups of order 2^{2n+1}.) In fact, by definition of κ in Lemma 2.1, and the product laws (11) and (79) of the two Heisenberg groups, the map $\iota_{\mathrm{H}} : \mathrm{H}(B) \to \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ defined by $\iota_{\mathrm{H}}(w,t) = (w, \kappa^{-1}(t))$ for all $w \in W$ and $t \in \mathbb{F}_2$, defines a group monomorphism. This shows that $\mathrm{H}(B)$ is contained in $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \simeq \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$. The fact that $\mathrm{H}(B')$ is contained in $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \simeq \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B'_W)$ then follows from the fact that $\Phi_{(\sigma,f,g)} \circ \iota_{\mathrm{H}}$ is a group monomorphism.

We now proceed with the study of the automorphisms of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ that fix its center.

Definition-Proposition 2.5. *The group of automorphisms of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ acting trivially on its center is the set of elements $s = (\sigma, f)$, where $\sigma \in \mathrm{Sp}(W)$ and $f : W \to \mathbb{Z}/4\mathbb{Z}$ are such that for all $w_1, w_2 \in W$,*

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B_W(\sigma(w_1), \sigma(w_2)) - B_W(w_1, w_2). \tag{82}$$

We denote this group by $\mathrm{ASp}(B_W)$ and call it the affine symplectic group. The element $s = (\sigma, f)$ acts on $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ by $(\sigma, f)(w,t) = (\sigma(w), f(w)+t)$. The product of two elements in $\mathrm{ASp}(B_W)$ is given by

$$(\sigma_1, f_1)(\sigma_2, f_2) = (\sigma_1\sigma_2, f_2 + f_1 \circ \sigma_2), \quad \text{where } (f_2 + f_1 \circ \sigma_2)(w) = f_2(w) + f_1(\sigma_2(w)).$$

In particular, the inverse of $s = (\sigma, f)$ is $s^{-1} = (\sigma^{-1}, -f \circ \sigma^{-1})$.

Proof. This proof is the same as the one of Proposition 1.8, except that the $\mathbb{F}_2$-linearity of σ is automatically satisfied for every $s = (\sigma, f) \in \mathrm{ASp}(B_W)$ (since $\sigma : W \to W$ is an automorphism of abelian groups by Proposition 2.2.iii). □

Observe that the name "affine symplectic group" comes from [GH12], where this group is considered only for the particular bilinear form B_W which corresponds to B_{Weil}. In fact, $\mathrm{ASp}(B_W)$ is the same for every bilinear form B_W satisfying (77), since there is only one Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ by Proposition 2.3. This group was already studied by Glasby [Gla95] in the framework of (almost) extraspecial 2-groups.

Now, as in the first section, we construct a projective representation of the affine symplectic group $\mathrm{ASp}(B_W)$, show that it is not linear on $\mathrm{ASp}(B_W)$, and we then linearize it to the smallest possible central extension of $\mathrm{ASp}(B_W)$ in order to obtain a linear Weil representation of the corresponding covering. Before doing this, we specify the structure of $\mathrm{ASp}(B_W)$ and its relation with the pseudosymplectic group $\mathrm{Ps}(B)$ defined in Proposition 1.9.

Proposition 2.6. *Let $B = \kappa \circ B_W$, where κ is defined in Lemma 2.1. Then $\mathrm{ASp}(B_W)$ contains the pseudosymplectic group $\mathrm{Ps}(B)$. More precisely,*

$$\begin{aligned} \Gamma : \mathrm{Ps}(B) &\to \mathrm{ASp}(B_W) \\ (\sigma, f) &\mapsto (\sigma, \kappa^{-1} \circ f) \end{aligned}$$

is a group monomorphism.

Proof. Let $(\sigma, f) \in \mathrm{Ps}(B)$. Then $\sigma \in \mathrm{O}(Q) \subseteq \mathrm{Sp}(W)$. By (17) and since κ is an abelian group homomorphism, for all $w_1, w_2 \in W$,

$$\begin{aligned}(\kappa^{-1} \circ f)(w_1 + w_2) - (\kappa^{-1} \circ f)(w_1) - (\kappa^{-1} \circ f)(w_2) &= \kappa^{-1}(f(w_1 + w_2) - f(w_1) - f(w_2)) \\ &= \kappa^{-1}(B(\sigma(w_1), \sigma(w_2)) - B(w_1, w_2)) \\ &= B_W(\sigma(w_1), \sigma(w_2)) - B_W(w_1, w_2).\end{aligned}$$

Hence $\Gamma(\sigma, f) = (\sigma, \kappa^{-1} \circ f)$ belongs to $\mathrm{ASp}(B_W)$. Now let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\mathrm{Ps}(B)$. Then

$$\begin{aligned}\Gamma(s_1)\Gamma(s_2) = (\sigma_1, \kappa^{-1} \circ f_1)(\sigma_2, \kappa^{-1} \circ f_2) &= (\sigma_1\sigma_2, \kappa^{-1} \circ f_2 + (\kappa^{-1} \circ f_1) \circ \sigma_2) \\ &= (\sigma_1\sigma_2, \kappa^{-1}(f_2 + f_1 \circ \sigma_2)) \\ &= \Gamma(s_1 s_2),\end{aligned}$$

which shows that Γ is a group homomorphism. Finally, let $s = (\sigma, f) \in \mathrm{Ps}(B)$ be such that $\Gamma(s) = (1, 0)$. Then $\sigma = 1$ and $\kappa^{-1} \circ f = 0$, i.e. $f = 0$ since κ is an isomorphism. Thus $(\sigma, f) = (1, 0)$ and Γ is injective. □

Proposition 2.7 (The structure of $\mathrm{Sp}(\widetilde{W})$ and of $\mathrm{ASp}(B_W)$). ***i.*** *As an abelian group,* $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$ *is isomorphic to* W*. More precisely,* $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) = \{\langle v, \cdot\rangle_W, v \in W\}$.

ii. $\mathrm{Sp}(W)$ *is generated by the following sets:*

- *translations* $T = \left\{ t_\beta = \begin{pmatrix} I_n & \beta \\ 0 & I_n \end{pmatrix},\ \beta = \beta^T \right\}$
- *rotations* $R = \left\{ r_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^T)^{-1} \end{pmatrix},\ \det\alpha \neq 0 \right\}$
- *semi-involutions* $S = \left\{ s_Q = \begin{pmatrix} Q & I_n - Q \\ Q - I_n & Q \end{pmatrix},\ Q \text{ diagonal matrix} \right\}$.

iii. *Define* $\pi : \mathrm{Sp}(\widetilde{W}) \to \mathrm{Sp}(W)$*, where for every* $w = \widetilde{w} + 2\widetilde{W}$*, the map* σ *is given by* $\widetilde{\sigma} \mapsto \sigma$ $\sigma(\widetilde{w} + 2\widetilde{W}) = \widetilde{\sigma}(\widetilde{w}) + 2\widetilde{W}$*. Then* π *is a group epimorphism. If the rank of* $\widetilde{W}$ *is* $2n$ *then*

$$|\mathrm{Sp}(\widetilde{W})| = 2^{2n^2+n}|\mathrm{Sp}(W)| = 2^{3n^2+n}\prod_{k=1}^{n}(2^{2k} - 1). \tag{83}$$

iv. *Let* $\widetilde{v} \in \widetilde{W}$ *and* $v \in W$*. We define the transvections*

$$\begin{aligned} \sigma_v : W &\to W \\ w &\mapsto w + \langle w, v\rangle v \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{\sigma}_{\widetilde{v}} : \widetilde{W} &\to \widetilde{W} \\ \widetilde{w} &\mapsto \widetilde{w} + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} \widetilde{v} \end{aligned} . \tag{84}$$

Every such transvection belongs to $\mathrm{Sp}(W)$ *and* $\mathrm{Sp}(\widetilde{W})$*, respectively. If* $v = \widetilde{v} + 2\widetilde{W}$*, then* $\pi(\widetilde{\sigma}_{\widetilde{v}}) = \sigma_v$*. In addition,* $\mathrm{Sp}(W)$ *and* $\mathrm{Sp}(\widetilde{W})$ *are generated by the set of transvections* $\{\sigma_v,\ v \in W\}$ *and* $\{\widetilde{\sigma}_{\widetilde{v}},\ \widetilde{v} \in \widetilde{W}\}$*, respectively.*

v. *Recall that a group G is perfect if it coincides with its derived subgroup $[G, G]$. The symplectic groups $\mathrm{Sp}(W)$ and $\mathrm{Sp}(\widetilde{W})$ are perfect for $n \geqslant 3$. If $n = 2$ then*

$$\mathrm{Sp}(W)/[\mathrm{Sp}(W), \mathrm{Sp}(W)] \simeq \mathbb{Z}/2\mathbb{Z} \quad \textit{and} \quad \mathrm{Sp}(\widetilde{W})/[\mathrm{Sp}(\widetilde{W}), \mathrm{Sp}(\widetilde{W})] \simeq \mathbb{Z}/2\mathbb{Z}. \tag{85}$$

If $n = 1$ then

$$\mathrm{Sp}(W)/[\mathrm{Sp}(W), \mathrm{Sp}(W)] \simeq \mathbb{Z}/2\mathbb{Z} \quad \textit{and} \quad \mathrm{Sp}(\widetilde{W})/[\mathrm{Sp}(\widetilde{W}), \mathrm{Sp}(\widetilde{W})] \simeq \mathbb{Z}/4\mathbb{Z}. \tag{86}$$

vi. *For every $\sigma \in \mathrm{Sp}(W)$, there exists $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and $f_{\widetilde{\sigma}} : W \to \mathbb{Z}/4\mathbb{Z}$ such that $\pi(\widetilde{\sigma}) = \sigma$ and $(\sigma, f_{\widetilde{\sigma}})$ belongs to $\mathrm{ASp}(B_W)$. Moreover, the map $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ is a group homomorphism.* $\widetilde{\sigma} \mapsto (\sigma, f_{\widetilde{\sigma}})$

vii. *Suppose that for every $\sigma \in \mathrm{Sp}(W)$, we fix $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ such that $\pi(\widetilde{\sigma}) = \sigma$. Then $\mathrm{ASp}(B_W)$ is the set of elements of the form $(\sigma, f_{\widetilde{\sigma}} + \langle v, \cdot\rangle_W)$, $v \in W$. Hence the order of $\mathrm{ASp}(B_W)$ is*

$$|\,\mathrm{ASp}(B_W)| = |\,\mathrm{Sp}(W)||W| = 2^{n^2+2n}\prod_{k=1}^{n}(2^{2k} - 1). \tag{87}$$

viii. *Let $\mathrm{ASp}_a(B_W)$ be the set of elements of $\mathrm{ASp}(B_W)$ of the form $(1, f)$. Then $\mathrm{ASp}_a(B_W)$ is an abelian subgroup of $\mathrm{ASp}(B_W)$ and*

$$\mathrm{ASp}_a(B_W) = \{(1, \langle v, \cdot\rangle_W), v \in W\}. \tag{88}$$

ix. *$0 \to \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \xrightarrow{\Phi} \mathrm{ASp}(B_W) \xrightarrow{\Psi} \mathrm{Sp}(W) \to 1$ is a short exact sequence, where $\Phi(\langle v, \cdot\rangle_W) = (1, \langle v, \cdot\rangle_W)$ and $\Psi(\sigma, f) = \sigma$. This exact sequence does not split, unless $n = 1$.*

x. *The group homomorphism $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ is surjective. In particular, we have that $|\,\mathrm{Ker}(\eta)| = 2^{2n^2-n}$.* $\widetilde{\sigma} \mapsto (\sigma, f_{\widetilde{\sigma}})$

Proof. **i.** Suppose that $f : W \to \mathbb{Z}/4\mathbb{Z}$ is an additive map. Then, in particular, for every $w \in W$, we have $0 = f(w+w) = 2f(w)$, which implies that f takes its values in $2\mathbb{Z}/4\mathbb{Z}$. If we choose a basis $\mathscr{B} = \{v_1, \ldots, v_{2n}\}$ of W, then f is completely determined by the values of $f(v_i)$ for all $1 \leqslant i \leqslant 2n$. There are two possibilities for each value of $f(v_i)$, namely 0 or 2. Thus there are at most 2^{2n} elements in $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Moreover, every map $\langle v, \cdot\rangle_W$ is additive and takes its values in $2\mathbb{Z}/4\mathbb{Z}$. Since $\langle \cdot, \cdot\rangle_W$ is nondegenerate, this leads to $|W| = 2^{2n}$ distinct elements of $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Thus we conclude that $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) = \{\langle v, \cdot\rangle_W, v \in W\}$. In particular, $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq W$ and $|\,\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})| = 2^{2n} = |W|$.

ii. Essentially, we follow the proof given in [HR49] by adapting it to the case of $\mathrm{Sp}_{2n}(\mathbb{F}_2)$. It is enough to show that taking any element $\sigma \in \mathrm{Sp}_{2n}(\mathbb{F}_2)$ and multiplying it by elements

of R, S, T, we obtain an element of R, S, T. Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{F}_2)$, i.e such that $\sigma^T \Omega \sigma = \Omega$, or equivalently

$$\begin{cases} \alpha^T\gamma - \gamma^T\alpha &= 0 \\ \alpha^T\delta - \gamma^T\beta &= I_n \\ \beta^T\delta - \delta^T\beta &= 0. \end{cases} \tag{89}$$

(Observe that this directly verifies that R, S and T are subsets of $\mathrm{Sp}(W)$). Since (89) implies that $(\alpha, \beta) \neq (0,0)$, and $\sigma s_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} -\beta & \alpha \\ -\delta & \gamma \end{pmatrix}$, we can suppose that $\alpha \neq 0$. In particular α is of rank $m > 0$. Hence there exists $P_1, P_2 \in \mathrm{GL}_m(\mathbb{F}_2)$ such that $P_1\alpha P_2 = J_m$, where $J_m = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$r_{P_1}\sigma r_{P_2} = \begin{pmatrix} P_1 & 0 \\ 0 & (P_1^T)^{-1} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} P_2 & 0 \\ 0 & (P_2^T)^{-1} \end{pmatrix} = \begin{pmatrix} P_1\alpha P_2 & * \\ * & * \end{pmatrix} = \begin{pmatrix} J_m & * \\ * & * \end{pmatrix}.$$

Hence we can suppose that $\alpha = J_m$. Now we decompose $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$ such that $\beta_1 \in \mathrm{M}_m(\mathbb{F}_2)$. The fact that σ is in $\mathrm{Sp}_{2n}(\mathbb{F}_2)$ implies that $\sigma^T \in \mathrm{Sp}_{2n}(\mathbb{F}_2)$, so $\alpha\beta^T = \beta\alpha^T$ by (89) applied to σ^T. Hence $\beta_1^T = \beta_1$ and $\beta_3 = 0$. If we set $B = \begin{pmatrix} -\beta_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_n(\mathbb{F}_2)$ then $B^T = B$ and

$$\sigma t_B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \alpha & \alpha B + \beta \\ \gamma & \gamma B + \delta \end{pmatrix},$$

where $\alpha B + \beta = J_m B + \beta = \begin{pmatrix} -\beta_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & \beta_2 \\ 0 & \beta_4 \end{pmatrix}$. This proves that we can suppose that $\beta_1 = 0$, and now we have $\sigma = \begin{pmatrix} J_m & \beta \\ \gamma & \delta \end{pmatrix}$ with $\beta = \begin{pmatrix} 0 & \beta_2 \\ 0 & \beta_4 \end{pmatrix}$. Hence

$$\begin{aligned} s_0\sigma s_{I_n - J_m} &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} J_m & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} I_n - J_m & J_m \\ -J_m & I_n - J_m \end{pmatrix}. \\ &= \begin{pmatrix} * & * \\ -J_m(I_n - J_m) + \beta J_m & * \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{aligned}$$

We can therefore suppose that $\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$. Using (89) again, we obtain $\alpha^T\delta = I_n$ and $\beta^T\delta = \delta^T\beta$, i.e. $\alpha \in \mathrm{GL}_n(\mathbb{F}_2)$, $\delta = (\alpha^T)^{-1}$ and $\beta = \nu(\alpha^T)^{-1}$ with $\nu = \alpha\beta^T$ symmetric. Thus

$$t_\nu r_\alpha = \begin{pmatrix} I_n & \nu \\ 0 & I_n \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^T)^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \sigma,$$

which completes the proof.

iii. We reason in terms of matrices: each coefficient of $\pi(\widetilde{\sigma})$ is equal to the corresponding coefficient of $\widetilde{\sigma}$ taken modulo 2. Then π is a group homomorphism because the modulo 2 map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2$ is a homomorphism of rings.

Recall that $\widetilde{\sigma} = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix}$ belongs to $\mathrm{Sp}(\widetilde{W})$ if and only if $\widetilde{\sigma}^T \widetilde{\Omega} \widetilde{\sigma} = \widetilde{\Omega}$, i.e.

$$\begin{cases} \widetilde{\alpha}^T \widetilde{\gamma} - \widetilde{\gamma}^T \widetilde{\alpha} &= 0 \\ \widetilde{\alpha}^T \widetilde{\delta} - \widetilde{\gamma}^T \widetilde{\beta} &= I_n \\ \widetilde{\beta}^T \widetilde{\delta} - \widetilde{\delta}^T \widetilde{\beta} &= 0 \end{cases}. \tag{90}$$

Let $r_\alpha \in R$, $t_\beta \in T$ and $s_Q \in S$. Let $\widetilde{r_\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^T)^{-1} \end{pmatrix}$, $\widetilde{t_\beta} = \begin{pmatrix} I_n & \beta \\ 0 & I_n \end{pmatrix}$ and $\widetilde{s_Q} = \begin{pmatrix} Q & I_n - Q \\ Q - I_n & Q \end{pmatrix}$, where the coefficients of the matrices α, β and Q are now considered in $\mathbb{Z}/4\mathbb{Z}$. Then

- $\widetilde{r_\alpha} \in \mathrm{Sp}(\widetilde{W})$ because $\alpha^T (\alpha^T)^{-1} = I_n$. Moreover $\pi(\widetilde{r_\alpha}) = r_\alpha$.
- $\widetilde{t_\beta} \in \mathrm{Sp}(\widetilde{W})$ because $\beta^T - \beta = 0$. Moreover $\pi(\widetilde{t_\beta}) = t_\beta$.
- $\widetilde{s_Q} \in \mathrm{Sp}(\widetilde{W})$ because

$$\begin{cases} Q^T(Q - I_n) - (Q - I_n)^T Q = Q(Q - I_n) - (Q - I_n)Q = 0 \\ Q^T Q - (Q - I_n)^T (I_n - Q) = Q^2 + I_n - 2Q + Q^2 = I_n - 2Q + 2Q = I_n \\ (I_n - Q)^T Q - Q^T (I_n - Q) = (I_n - Q)Q - Q(I_n - Q) = 0 \end{cases},$$

  where we use that diagonal matrices are invariant under transposition and commute, and $Q^2 = Q$ since the coefficients of Q are valued in $\{0, 1\}$. Moreover $\pi(\widetilde{s_Q}) = s_Q$.

By iii, this is enough to show the surjectivity of π because every generator of $\mathrm{Sp}(W)$ has an antecedent by the map π.

Now we use this result to compute the order of the group $\mathrm{Sp}(\widetilde{W})$. Let $\widetilde{\sigma} = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix}$ be in $\mathrm{M}_{2n}(\mathbb{Z}/4\mathbb{Z})$. Then $\widetilde{\sigma} \in \mathrm{Ker}(\pi)$ if and only if $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and there exists $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in M_n(\mathbb{Z}/4\mathbb{Z})$ such that $\widetilde{\sigma} = \begin{pmatrix} I_n + 2\widetilde{A} & 2\widetilde{B} \\ 2\widetilde{C} & I_n + 2\widetilde{D} \end{pmatrix}$. Equivalently, by (90), $\widetilde{\sigma}$ is in $\mathrm{Ker}(\pi)$ if and only if $\widetilde{\sigma} = \begin{pmatrix} I_n + 2\widetilde{A} & 2\widetilde{B} \\ 2\widetilde{C} & I_n + 2\widetilde{D} \end{pmatrix}$, where $\begin{cases} (I_n + 2\widetilde{A})^T 2\widetilde{C} - 2\widetilde{C}^T (I_n + 2\widetilde{A}) &= 0 \\ (I_n + 2\widetilde{A})^T (I_n + 2\widetilde{D}) - 2\widetilde{C}^T 2\widetilde{B} &= I_n \\ 2\widetilde{B}^T (I_n + 2\widetilde{D})^T - (I_n + 2\widetilde{D}) 2\widetilde{B} &= 0 \end{cases}$,

i.e. $\begin{cases} 2\widetilde{C} - 2\widetilde{C}^T &= 0 \\ 2\widetilde{A}^T + 2\widetilde{D} &= 0 \\ 2\widetilde{B}^T - 2\widetilde{B} &= 0 \end{cases}$. This gives $2^{\frac{n(n+1)}{2}}$ possibilities for $2\widetilde{B}$, $2^{\frac{n(n+1)}{2}}$ possibilities for

$2\widetilde{C}$ and 2^{n^2} possibilities for $2\widetilde{A}$. Moreover, there is one possibility for $2\widetilde{D}$ once $2\widetilde{A}$ is fixed. Thus there are $2^{\frac{n(n+1)}{2}}2^{\frac{n(n+1)}{2}}2^{n^2} = 2^{2n^2+n}$ possibilities for $\widetilde{\sigma}$. It follows that $|\operatorname{Ker}(\pi)| = 2^{2n^2+n}$. Using the cardinality of $\mathrm{Sp}(W)$ given in Proposition 1.13.i, the first isomorphism theorem and the surjectivity of π, we get:

$$|\operatorname{Sp}(\widetilde{W})| = |\operatorname{Ker}(\pi)||\operatorname{Sp}(W)| = 2^{3n^2+n}\prod_{k=1}^{n}(2^{2k}-1).$$

iv. Let v, w_1 and w_2 be in W. Then

$$\begin{aligned}\langle \sigma_v(w_1), \sigma_v(w_2)\rangle &= \langle w_1 + \langle w_1, v\rangle v, w_2 + \langle w_2, v\rangle v\rangle \\ &= \langle w_1, w_2\rangle + \langle w_2, v\rangle\langle w_1, v\rangle + \langle w_1, v\rangle\langle v, w_2\rangle \\ &= \langle w_1, w_2\rangle.\end{aligned}$$

This shows that σ_v belongs to $\mathrm{Sp}(W)$. Similarly, we can prove that $\widetilde{\sigma}_{\widetilde{v}}$ belongs to $\mathrm{Sp}(\widetilde{W})$ for every $\widetilde{v} \in \widetilde{W}$. The groups $\mathrm{Sp}(W)$ and $\mathrm{Sp}(\widetilde{W})$ are generated by the sets of transvections $\{\sigma_v,\ v \in W\}$ and $\{\widetilde{\sigma}_{\widetilde{v}},\ \widetilde{v} \in \widetilde{W}\}$ by [OMe78, Theorem 2.1.9] and [Ish89], respectively. Now, let $v \in W$ and $\widetilde{v} \in \widetilde{W}$ be such that $v = \widetilde{v} + 2\widetilde{W}$. For all $\widetilde{w} \in \widetilde{W}$, recall that

$$\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}) = \widetilde{w} + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}}\widetilde{v}.$$

The element $\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}) + 2\widetilde{W} = \widetilde{w} + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}}\widetilde{v} + 2\widetilde{W}$ is identified with $\sigma_v(w)$ in the $\mathbb{F}_2$-vector space W because the reduction modulo 2 of $\langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}}$ is exactly $\langle w, v\rangle$. Thus, by definition of π, this implies that $\pi(\widetilde{\sigma}_{\widetilde{v}}) = \sigma_v$.

v. For $n \geqslant 2$, this follows from [LSTX17, Proposition 1.b.]. The case $n = 1$ is studied in detail in chapter 3; $\mathrm{Sp}(W) = \mathrm{SL}_2(\mathbb{F}_2)$ is isomorphic to the symmetric group S_3 and $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is isomorphic to the group $\mathrm{A}_4 \rtimes \mathrm{C}_4$.

vi. Let σ be in $\mathrm{Sp}(W)$. Choose any element $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ such that $\pi(\widetilde{\sigma}) = \sigma$ and define

$$\beta_{\widetilde{\sigma}}(\widetilde{w}_1, \widetilde{w}_2) = B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) - B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) \tag{91}$$

for all $\widetilde{w}_1$ and $\widetilde{w}_2$ in $\widetilde{W}$. Define also $F_{\widetilde{\sigma}} : \widetilde{W} \to \mathbb{Z}/4\mathbb{Z}$ by

$$F_{\widetilde{\sigma}}(\widetilde{w}) = \beta_{\widetilde{\sigma}}(\widetilde{w}, \widetilde{w}). \tag{92}$$

Then for all $\widetilde{w}_1, \widetilde{w}_2$ such that $\widetilde{w}_1 - \widetilde{w}_2 = 2\widetilde{v} \in 2\widetilde{W}$:

$$\begin{aligned}F_{\widetilde{\sigma}}(\widetilde{w}_1) &= F_{\widetilde{\sigma}}(\widetilde{w}_2 + 2\widetilde{v}) \\ &= \beta_{\widetilde{\sigma}}(\widetilde{w}_2 + 2\widetilde{v}, \widetilde{w}_2 + 2\widetilde{v}) \\ &= \beta_{\widetilde{\sigma}}(\widetilde{w}_2, \widetilde{w}_2) + 2\beta_{\widetilde{\sigma}}(\widetilde{w}_2, \widetilde{v}) + 2\beta_{\widetilde{\sigma}}(\widetilde{v}, \widetilde{w}_2) + 4\beta_{\widetilde{\sigma}}(\widetilde{v}, \widetilde{v}) \\ &= \beta_{\widetilde{\sigma}}(\widetilde{w}_2, \widetilde{w}_2) + 2\langle \widetilde{\sigma}(\widetilde{w}_2), \widetilde{\sigma}(\widetilde{v})\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_2, \widetilde{v}\rangle_{\widetilde{W}} \\ &= \beta_{\widetilde{\sigma}}(\widetilde{w}_2, \widetilde{w}_2) \quad \text{because } \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W}) \\ &= F_{\widetilde{\sigma}}(\widetilde{w}_2).\end{aligned}$$

This shows that the map

$$\begin{aligned} f_{\widetilde{\sigma}} : W &\to \mathbb{Z}/4\mathbb{Z} \ , \\ w &\mapsto F_{\widetilde{\sigma}}(\widetilde{w}) \end{aligned} \tag{93}$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $w = \widetilde{w} + 2\widetilde{W}$, is well-defined. Now for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$:

$$\begin{aligned} &f_{\widetilde{\sigma}}(w_1 + w_2) - f_{\widetilde{\sigma}}(w_1) - f_{\widetilde{\sigma}}(w_2) \\ &= F_{\widetilde{\sigma}}(\widetilde{w}_1 + \widetilde{w}_2) - F_{\widetilde{\sigma}}(\widetilde{w}_1) - F_{\widetilde{\sigma}}(\widetilde{w}_2) \\ &= \beta_{\widetilde{\sigma}}(\widetilde{w}_1 + \widetilde{w}_2, \widetilde{w}_1 + \widetilde{w}_2) - \beta_{\widetilde{\sigma}}(\widetilde{w}_1, \widetilde{w}_1) - \beta_{\widetilde{\sigma}}(\widetilde{w}_2, \widetilde{w}_2) \\ &= \beta_{\widetilde{\sigma}}(\widetilde{w}_1, \widetilde{w}_2) + \beta_{\widetilde{\sigma}}(\widetilde{w}_2, \widetilde{w}_1) \\ &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) - B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) + B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_2), \widetilde{\sigma}(\widetilde{w}_1)) - B_{\widetilde{W}}(\widetilde{w}_2, \widetilde{w}_1) \\ &= 2B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) - 2B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) - B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) + B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) \\ &\quad + B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_2), \widetilde{\sigma}(\widetilde{w}_1)) - B_{\widetilde{W}}(\widetilde{w}_2, \widetilde{w}_1) \\ &= B_W(\sigma(w_1), \sigma(w_2)) - B_W(w_1, w_2) - \langle \widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2) \rangle_{\widetilde{W}} + \langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} \\ &= B_W(\sigma(w_1), \sigma(w_2)) - B_W(w_1, w_2). \end{aligned}$$

Hence (82) is satisfied, i.e. $(\sigma, f_{\widetilde{\sigma}})$ is in $\mathrm{ASp}(B_W)$ and η is well-defined.

Now we check that η is a group homomorphism. Let $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ be in $\mathrm{Sp}(\widetilde{W})$. Then for every $w = \widetilde{w} + 2\widetilde{W}$,

$$\begin{aligned} &(f_{\widetilde{\sigma}_2} + f_{\widetilde{\sigma}_1} \circ \sigma_2)(w) \\ &= f_{\widetilde{\sigma}_2}(w) + f_{\widetilde{\sigma}_1}(\sigma_2(w)) \\ &= F_{\widetilde{\sigma}_2}(\widetilde{w}) + F_{\widetilde{\sigma}_1}(\widetilde{\sigma}_2(\widetilde{w})) \\ &= \beta_{\widetilde{\sigma}_2}(\widetilde{w}, \widetilde{w}) + \beta_{\widetilde{\sigma}_1}(\widetilde{\sigma}_2(\widetilde{w}), \widetilde{\sigma}_2(\widetilde{w})) \\ &= B_{\widetilde{W}}(\widetilde{\sigma}_2(\widetilde{w}), \widetilde{\sigma}_2(\widetilde{w})) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) + B_{\widetilde{W}}(\widetilde{\sigma}_1(\widetilde{\sigma}_2(\widetilde{w})), \widetilde{\sigma}_1(\widetilde{\sigma}_2(\widetilde{w}))) - B_{\widetilde{W}}(\widetilde{\sigma}_2(\widetilde{w}), \widetilde{\sigma}_2(\widetilde{w})) \\ &= f_{\widetilde{\sigma}_1\widetilde{\sigma}_2}(w). \end{aligned}$$

Thus $\eta(\widetilde{\sigma}_1)\eta(\widetilde{\sigma}_2) = (\widetilde{\sigma}_1, f_{\widetilde{\sigma}_1})(\widetilde{\sigma}_2, f_{\widetilde{\sigma}_2}) = (\widetilde{\sigma}_1\widetilde{\sigma}_2, f_{\widetilde{\sigma}_2} + f_{\widetilde{\sigma}_1} \circ \sigma_2) = (\widetilde{\sigma}_1\widetilde{\sigma}_2, f_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) = \eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)$.

vii. For every $\sigma \in \mathrm{Sp}(W)$, we choose $\widetilde{\sigma}$ in $\mathrm{Sp}(\widetilde{W})$ in the preimage of σ by π. Then by vi, we obtain an element $(\sigma, f_{\widetilde{\sigma}})$ in $\mathrm{ASp}(B_W)$. Now by (82), for every map $f : W \to \mathbb{Z}/4\mathbb{Z}$, the element (σ, f) is in $\mathrm{ASp}(B_W)$ if and only if $f - f_{\widetilde{\sigma}}$ is additive, i.e. by i, there exists $v \in W$ such that $f - f_{\widetilde{\sigma}} = \langle v, \cdot \rangle_W$. This is enough to conclude that $\mathrm{ASp}(B_W)$ is the set of elements of the form $(\sigma, f_{\widetilde{\sigma}} + \langle v, \cdot \rangle_W)$, $v \in W$. In particular, the order of $\mathrm{ASp}(B_W)$ is exactly the product of the orders of $\mathrm{Sp}(W)$ and $|W|$.

viii. Let $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ be in $\mathrm{ASp}_a(B_W)$. Observe first by (82) that $f_1, f_2 \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Moreover,

$$s_1 s_2^{-1} = (1, f_1)(1, -f_2) = (1, f_1 - f_2) = (1, -f_2)(1, f_1) = s_2^{-1} s_1.$$

Since $f_1 - f_2 \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$, this shows that $\mathrm{ASp}_a(B_W)$ is an abelian subgroup of $\mathrm{ASp}(B_W)$. Using i, we obtain the explicit expression of the elements of $\mathrm{ASp}_a(B_W)$ given by (88).

ix. The fact that Φ and Ψ are group homomorphisms comes from the product law of $\mathrm{ASp}(B_W)$ defined in Definition-Proposition 2.5. The nondegeneracy of the symplectic form $\langle\cdot,\cdot\rangle_W$ implies that Φ is injective, and vi implies that Ψ is surjective. Finally, the image of Φ and the kernel of Ψ are both equal to the subgroup $\mathrm{ASp}_a(B_W)$ by (88). This shows that

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \xrightarrow{\Phi} \mathrm{ASp}(B_W) \xrightarrow{\Psi} \mathrm{Sp}(W) \to 1$$

is a short exact sequence.

In chapter 3, we prove that if $n = 1$, then $\mathrm{ASp}(B_W)$ is isomorphic to the symmetric group S_4, which implies that this short exact sequence splits (see Remark 3.6).

Suppose now that $n \geqslant 2$. If we prove that $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$ is isomorphic to the set of inner automorphisms of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, then the fact that this short exact sequence does not split follows directly from [Gri73, Corollaries 2 and 3] and the fact that $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is isomorphic to $\mathrm{H}(B) \circ \mathbb{Z}/4\mathbb{Z}$; see Proposition A.17. Let $(w,t) \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Then for all $(w',t') \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$,

$$\begin{aligned}(w,t)(w',t')(w,t)^{-1} &= (w+w', t+t'+B_W(w,w'))(-w, -t+B_W(w,w)) \\ &= (w', t'+B_W(w,w')+B_W(w,w)-B_W(w+w',w)) \\ &= (w', t'+\langle w, w'\rangle_W) \\ &= s_w(w',t'),\end{aligned} \tag{94}$$

where $\langle w,\cdot\rangle_W$ belongs to $\mathrm{Hom}_{\mathbb{Z}}(W,\mathbb{Z}/4\mathbb{Z})$ by i, and $s_w = (1, \langle w,\cdot\rangle_W)$ belongs to $\mathrm{ASp}(B_W)$. Thus (94) shows that $\mathrm{Hom}_{\mathbb{Z}}(W,\mathbb{Z}/4\mathbb{Z})$ is isomorphic to the group of inner automorphisms of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, which concludes the proof.

x. We already proved that $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ is a group homomorphism. Let (σ, f)

$$\widetilde{\sigma} \mapsto (\sigma, f_{\widetilde{\sigma}})$$

be in $\mathrm{ASp}(B_W)$ and let $\widetilde{\sigma}$ be in $\mathrm{Sp}(\widetilde{W})$ such that $\pi(\widetilde{\sigma}) = \sigma$. Then we proved that $(\sigma, f_{\widetilde{\sigma}})$ belongs to $\mathrm{ASp}(B_W)$ and there exists $v \in W$ such that $f = f_{\widetilde{\sigma}} + \langle v,\cdot\rangle_W$. Now, set $\widetilde{\tau} = \widetilde{\sigma}\widetilde{\sigma}^2_{\widetilde{v}}$. Then for every $w = \widetilde{w} + 2\widetilde{W}$,

$$\begin{aligned}&f_{\widetilde{\tau}}(w) \\ &= B_{\widetilde{W}}((\widetilde{\sigma}\widetilde{\sigma}^2_{\widetilde{v}})(\widetilde{w}), (\widetilde{\sigma}\widetilde{\sigma}^2_{\widetilde{v}})(\widetilde{w})) - B_{\widetilde{W}}(\widetilde{w},\widetilde{w}) \\ &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}+2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}}\widetilde{v}), \widetilde{\sigma}(\widetilde{w}+2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}}\widetilde{v})) - B_{\widetilde{W}}(\widetilde{w},\widetilde{w}) \\ &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}),\widetilde{\sigma}(\widetilde{w})) + 2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}),\widetilde{\sigma}(\widetilde{v})) + 2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{v}),\widetilde{\sigma}(\widetilde{w})) \\ &\quad - B_{\widetilde{W}}(\widetilde{w},\widetilde{w}) \\ &= f_{\widetilde{\sigma}}(w) + 2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}),\widetilde{\sigma}(\widetilde{v})) - 2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{v}),\widetilde{\sigma}(\widetilde{w})) \\ &= f_{\widetilde{\sigma}}(w) + 2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}}\langle\widetilde{\sigma}(\widetilde{w}),\widetilde{\sigma}(\widetilde{v})\rangle_{\widetilde{W}} \\ &= f_{\widetilde{\sigma}}(w) + 2\langle\widetilde{w},\widetilde{v}\rangle^2_{\widetilde{W}} \\ &= f_{\widetilde{\sigma}}(w) + 2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}} \\ &= f_{\widetilde{\sigma}}(w) + \langle v, w\rangle_W \\ &= f(w).\end{aligned}$$

Moreover, since $\widetilde{\sigma}_{\widetilde{v}}^2(\widetilde{w})+2\widetilde{W} = \widetilde{w}+2\langle\widetilde{w},\widetilde{v}\rangle_{\widetilde{W}}\widetilde{v}+2\widetilde{W} = \widetilde{w}+2\widetilde{W}$, we have that $\pi(\widetilde{\sigma}_{\widetilde{v}}^2) = 1$. Therefore, $\eta(\widetilde{\tau}) = \eta(\widetilde{\sigma}\widetilde{\sigma}_{\widetilde{v}}^2) = (\pi(\widetilde{\sigma}\widetilde{\sigma}_{\widetilde{v}}^2), f_{\widetilde{\tau}}) = (\sigma, f_{\widetilde{\tau}}) = (\sigma, f)$, which shows that η is surjective. Hence, using the first isomorphism theorem and the preceding points, we obtain

$$|\operatorname{Ker}(\eta)| = |\operatorname{Sp}(\widetilde{W})||\operatorname{ASp}(B_W)|^{-1} = 2^{3n^2+n-(n^2+2n)} = 2^{2n^2-n}.$$

□

2.2 The Heisenberg representation, the Stone-von Neumann theorem, and the projective Weil representation of $\operatorname{ASp}(B_W)$

In section 1.2 we proved that, given any nontrivial character χ of a finite field of characteristic two, the Heisenberg group has up to isomorphism a unique irreducible unitary representation acting as χ on its center. We can wonder if this property remains true for the new Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. If χ is a faithful character of $\mathbb{Z}/4\mathbb{Z}$, the answer is positive. In fact, we can provide a proof of the Stone-von Neumann theorem which is similar to the one of the first section: to this extent, we begin by associating to each faithful character χ of $\mathbb{Z}/4\mathbb{Z}$ (there are two) an extension of χ to the abelian subgroup $X \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, and then we construct a unitary irreducible representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$.

Lemma 2.8. ***i.*** *The additive group $\mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\}$ has four characters, namely $\chi_k : t \mapsto \exp(\frac{2i\pi kt}{4})$, $0 \leqslant k \leqslant 3$. In particular, every character of $\mathbb{Z}/4\mathbb{Z}$ takes its values in $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$. The only faithful characters of $\mathbb{Z}/4\mathbb{Z}$ are χ_1 and χ_3.*

ii. *Every character of $\mathbb{Z}/4\mathbb{Z}$ can be extended to $X \times \mathbb{Z}/4\mathbb{Z}$ in 2^n different ways.*

Proof. **i.** Let χ be a character of $\mathbb{Z}/4\mathbb{Z}$. Then χ takes its values in $\mu_4(\mathbb{C})$ because for every $t \in \mathbb{Z}/4\mathbb{Z}$,

$$1 = \psi(0) = \psi(4t) = \psi(t)^4.$$

Moreover, χ is entirely determined by $\chi(1)$ because $\chi(t) = \chi(1)^t$ for every $t \in \mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\}$. This gives four possibilities for $\chi(1)$: $1, i, -1$ and $-i$, which lead to the characters χ_0, χ_1, χ_2 and χ_3, respectively.

ii. Let χ be a character of $\mathbb{Z}/4\mathbb{Z}$ and let $\{x_1, \ldots, x_n\}$ be a basis of X over $\mathbb{F}_2$. Suppose that $\widetilde{\chi}$ is an extension of χ to $X \times \mathbb{Z}/4\mathbb{Z}$. Formula (22) is only a consequence of the definition of the product in the Heisenberg group, and therefore it holds for B_W instead of B. Let (x,t) be in the subgroup $X \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ be such that $x = \sum_{i=1}^{n} \lambda_i x_i$, where

$\lambda_i \in \mathbb{F}_2$. Taking $w_i = \lambda_i x_i$ in (22), we obtain

$$\begin{aligned}\widetilde{\chi}(x,t) &= \widetilde{\chi}((0,t)(x,0)) \\ &= \chi(t)\widetilde{\chi}\left(\sum_{1\leqslant i\leqslant n}\lambda_i x_i, 0\right) \\ &= \chi(t)\widetilde{\chi}\left(0, -\sum_{j=1}^{n-1}\sum_{l=j+1}^{n} B_W(\lambda_j x_j, \lambda_l x_l)\right)\prod_{1\leqslant i\leqslant n}\widetilde{\chi}(\lambda_i x_i, 0).\end{aligned}$$

Then, if we set $\beta_{j,l}^W = B_W(x_j, x_l)$ and use the fact that $\lambda_i \in \mathbb{F}_2$,

$$\widetilde{\chi}(x,t) = \chi\left(t - \sum_{j=1}^{n-1}\sum_{l=j+1}^{n}\lambda_j\lambda_l\beta_{j,l}^W\right)\prod_{1\leqslant i\leqslant n}\widetilde{\chi}(x_i,0)^{\lambda_i}. \tag{95}$$

This shows that $\widetilde{\chi}$ is entirely determined by the images $\widetilde{\chi}(x_i, 0)$, which can take the values ± 1 if $\beta_{i,i}^W = B_W(x_i, x_i) = 0$ and $\pm i$ otherwise, i.e. if $\beta_{i,i}^W = B_W(x_i, x_i) = 2$ (the proof is the same as in Lemma 1.15.ii). Hence there are two possibilities for every $\widetilde{\chi}(x_i, 0)$. Conversely, any choice between these two possibilities for every $1 \leqslant i \leqslant n$ defines an extension $\widetilde{\chi}$ by the formula (95) above. We deduce that there are exactly 2^n extensions of χ to $X \times \mathbb{Z}/4\mathbb{Z}$.

□

Now we fix a faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ (i.e. $\chi_k = \chi_1$ or $\chi_k = \chi_3 = \overline{\chi_1}$).

Lemma 2.9. *The map* $W \to \widehat{W}$ *is a group isomorphism.*
$w \mapsto \chi_k(\langle w, \cdot\rangle_W)$

Proof. For every $w \in W$, define the map $\xi_w : v \mapsto \chi_k(\langle w, v\rangle_W)$. Then ξ_w is a character of W, i.e. the map in the statement is well-defined. Let w_1 and w_2 be in W. Then for all $v \in W$,

$$\xi_{w_1+w_2}(v) = \chi_k(\langle w_1 + w_2, v\rangle_W) = \chi_k(\langle w_1, v\rangle_W)\chi_k(\langle w_2, v\rangle_W),$$

which implies that $w \mapsto \xi_w$ is a group homomorphism.

Now, suppose that $\chi_k(\langle w, v\rangle_W) = 1$ for every $v \in W$. Since χ_k is supposed to be faithful, then $\langle w, v\rangle_W = 0$ for every $v \in W$. Thus $w = 0$ and $w \mapsto \xi_w$ is injective. Since W is a finite abelian group, $|W| = |\widehat{W}|$ and this is enough to show that $w \mapsto \xi_w$ is a group isomorphism. □

Now (similarly to Definition 1.16 and Lemma 1.17) we define the Schrödinger model of the Heisenberg representation of the group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Then, in Lemma 2.11, we prove that it is a unitary irreducible representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and present its realization on $\mathrm{L}^2(Y)$. We keep the notations from section 1.2.

Definition 2.10 (Schrödinger model of the Heisenberg representation). Let $\widetilde{\chi_k}$ be an extension of χ_k to the abelian subgroup $X \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. We denote by $\mathscr{H}(X, \widetilde{\chi_k})$ the $\mathbb{C}$-vector space of functions $\phi : \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \to \mathbb{C}$ such that for all $(x,t) \in X \times \mathbb{Z}/4\mathbb{Z}$ and for all $h \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$,

$$\phi(h(x,t)) = \widetilde{\chi_k}(x,t)^{-1}\phi(h).$$

Let $\rho_{\widetilde{\chi_k}} : \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}(X, \widetilde{\chi_k}))$ be the map defined by $\rho_{\widetilde{\chi_k}}(g)\phi(h) = \phi(g^{-1}h)$.

Lemma 2.11. *The space $\mathscr{H}(X, \widetilde{\chi_k})$ is isomorphic to $\mathrm{L}^2(Y)$ by restriction to Y. As a representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ on the Hilbert space $\mathrm{L}^2(Y)$, $\rho_{\widetilde{\chi_k}}$ is unitary, irreducible and acts as χ_k on the center. It is given by the following formula: for all $g = (x+y, t) \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, for all $\varphi \in \mathrm{L}^2(Y)$ and for all $y' \in Y$,*

$$\rho_{\widetilde{\chi_k}}(x+y,t)\varphi(y') = \widetilde{\chi_k}(x, t - Q_W(y) + \langle x, y'\rangle_W - B_W(x+y', y))\varphi(y'-y). \tag{96}$$

Another choice of extension $\chi_k^\sharp$ gives a representation $\rho_{\chi_k^\sharp}$ which is equivalent to $\rho_{\widetilde{\chi_k}}$.

Proof. The proof is essentially the same as the proof of Lemma 1.17, where we replace χ, $\widetilde{\chi}$, $\langle\cdot,\cdot\rangle$, B and Q by χ_k, $\widetilde{\chi_k}$, $\langle\cdot,\cdot\rangle_W$, B_W and Q_W, respectively. In particular, we use the fact that $2B_W$ and $2Q_W$ vanish on W since B_W takes its values in $2\mathbb{Z}/4\mathbb{Z}$.

□

Remark 2.12.

- As in Remark 1.18, the fact that $\rho_{\widetilde{\chi_k}}$ is a representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ implies in particular that for all $w, w' \in W$,

$$\rho_{\widetilde{\chi_k}}(w)\rho_{\widetilde{\chi_k}}(w') = \rho_{\widetilde{\chi_k}}(w+w', B_W(w,w')) = \chi_k(B_W(w,w'))\rho_{\widetilde{\chi_k}}(w+w'), \tag{97}$$

  where $\rho_{\widetilde{\chi_k}}(w) = \rho_{\widetilde{\chi_k}}(w, 0)$.

- Remark 1.19 can be reproduced here to explain why we need to define an extension $\widetilde{\chi_k}$ of χ_k which may be nontrivial in general.

Corollary 2.13. *The character of $\rho_{\widetilde{\chi_k}}$ is a function supported on the center of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. For $h = (w,t) \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, it is given by*

$$\mathrm{Trace}(\rho_{\widetilde{\chi_k}}(w,t)) = \chi_k(t)|Y|\delta_0(w).$$

Proof. The proof is similar to the one of (29). □

Remark 2.14. By Corollaries 1.21 and 2.13, the characters of the representations $\rho_{\widetilde{\chi}}$ and $\rho_{\widetilde{\chi_k}}$ coincide on W.

Lemma 2.15. *The family of operators $\{\rho_{\widetilde{\chi_k}}(w),\ w \in W\}$ is a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$.*

Proof. This is exactly the same proof as the one of Lemma 1.22. □

In the same way as in the first section, even if our construction of the Weil representation will not require it (as we shall see in Remark 2.17), we now give a proof of the Stone-von Neumann theorem for the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$.

Theorem 2.16 (Stone-von Neumann). *The Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ has, up to isomorphism, a unique unitary irreducible representation acting as χ_k on its center, which will be called the Heisenberg representation. For every faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ and extension $\widetilde{\chi_k}$ of χ_k to $X \times \mathbb{Z}/4\mathbb{Z}$, the map $\rho_{\widetilde{\chi_k}}$ defined by* (96) *is a realization of the Heisenberg representation on $\mathrm{L}^2(Y)$, which is therefore of dimension $|Y| = 2^n$.*

Proof. This is also a slightly modified proof of the Stone-von Neumann theorem for $\mathrm{H}(B)$. We begin by listing the irreducible representations of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. For all $w, w' \in W$,

$$\begin{aligned}
&(w,t)(w',t')(w,t)^{-1}(w',t')^{-1}\\
&= (w+w', t+t'+B_W(w,w'))(-w-w', -t-t'+Q_W(w)+Q_W(w')+B_W(w,w'))\\
&= (0, Q_W(w)+Q_W(w')+2B_W(w,w')-B_W(w+w', w+w'))\\
&= (0, \langle w, w'\rangle_W).
\end{aligned}$$

Hence the commutator subgroup $[\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W), \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)]$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is isomorphic to the image of the nondegenerate symplectic form $\langle\cdot,\cdot\rangle_W$, i.e $2\mathbb{Z}/4\mathbb{Z}$. As a consequence, the quotient $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)/[\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W), \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)]$ is isomorphic to the direct product $W \times \mathbb{Z}/2\mathbb{Z}$, which is an abelian group and has $2|W| = 2|Y|^2$ one-dimensional representations. This leads to $2|Y|^2$ one-dimensional representations of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Moreover, for every faithful character χ_k and extension $\widetilde{\chi_k}$ of χ_k to $X \times \mathbb{Z}/4\mathbb{Z}$, Lemma 2.11 provides a unitary irreducible representation $\rho_{\widetilde{\chi_k}}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ acting as χ_k on its center. This action on the center implies in particular that $\rho_{\widetilde{\chi_1}}$ and $\rho_{\widetilde{\chi_3}}$ are not equivalent. This leads to two additional inequivalent irreducible representations of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ of dimension $|Y|$ (cf. Lemma 2.8). If we sum their dimensions squared, we obtain

$$2|Y|^2 + 2|Y|^2 = 4|Y|^2 = |\,\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)|.$$

This shows (as a consequence of Schur's orthogonality relations) that we found all the pairwise inequivalent irreducible representations of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. In particular, for every faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ and extension $\widetilde{\chi_k}$, $\rho_{\widetilde{\chi_k}}$ is up to isomorphism the unique unitary irreducible representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ acting as χ_k on its center. □

The Stone-von Neumann theorem enables us, as in section 1.2, to define a projective representation of $\mathrm{ASp}(B_W)$. For every $s \in \mathrm{ASp}(B_W)$, we define the map $\rho^s_{\widetilde{\chi_k}} : \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \to \mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by $\rho^s_{\widetilde{\chi_k}}(g) = \rho_{\widetilde{\chi_k}}(s(g))$. We obtain in this way a unitary irreducible representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ which acts as χ_k on the center. The Stone-von Neumann theorem implies therefore that $\rho_{\widetilde{\chi_k}}$ and $\rho^s_{\widetilde{\chi_k}}$ are equivalent, i.e. there exists $\omega_{\widetilde{\chi_k}}(s) \in \mathrm{U}(\mathrm{L}^2(Y))$ such that

$$\rho^s_{\widetilde{\chi_k}}(h) = \omega_{\widetilde{\chi_k}}(s)\rho_{\widetilde{\chi_k}}(h)\omega_{\widetilde{\chi_k}}(s)^{-1} \quad \text{for all } h \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W). \tag{98}$$

Schur's lemma also implies that $\omega_{\widetilde{\chi_k}}(s)$ is unique up to multiplication by an element of $\mathrm{U}(1)$.

Remark 2.17. Remark 1.24 can be reproduced here: we do not need the Stone-von Neumann theorem in our setting to assert the existence of the intertwining operators $\omega_{\widetilde{\chi_k}}(s)$ for every $s \in \mathrm{ASp}(B_W)$. In fact, we prove that $\rho_{\widetilde{\chi_k}}$ and $\rho^s_{\widetilde{\chi_k}}$ share the same character (cf. Corollary 2.13).

Lemma 2.18. *The map* $s \mapsto \omega_{\widetilde{\chi_k}}(s)$ *defines a projective representation of* $\mathrm{ASp}(B_W)$ *on* $\mathrm{L}^2(Y)$.

Proof. Exactly the same proof as the one of Lemma 1.25. □

Remark 2.19. Since $s \mapsto \omega_{\widetilde{\chi_k}}(s)$ is a projective representation, we can fix an operator $\omega_{\widetilde{\chi_k}}(s)$ for every $s \in \mathrm{ASp}(B_W)$. Then there is a cocycle $c_{\widetilde{\chi_k}} : \mathrm{ASp}(B_W) \times \mathrm{ASp}(B_W) \to \mathrm{U}(1)$ defined by

$$\omega_{\widetilde{\chi_k}}(s_1)\omega_{\widetilde{\chi_k}}(s_2) = c_{\widetilde{\chi_k}}(s_1, s_2)\omega_{\widetilde{\chi_k}}(s_1 s_2). \tag{99}$$

2.3 Lifting the projective Weil representation to a covering $\mathrm{G}_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$

Now that we have defined the projective representation $\omega_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ in Lemma 2.18, our next goal is to see if $\omega_{\widetilde{\chi_k}}$ can be lifted to a linear representation either of $\mathrm{ASp}(B_W)$ or of a finite nontrivial central extension of this group. To answer this question, we first construct an explicit section of $\omega_{\widetilde{\chi_k}}$. Then we prove that we need to construct a nontrivial central extension $\mathrm{G}_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ to lift $\omega_{\widetilde{\chi_k}}$ to a linear representation. We give two possibilities for $\mathrm{G}_{\widetilde{\chi_k}}$: we extend $\mathrm{ASp}(B_W)$ either by eight or four elements, obtaining in this way the groups $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, respectively. Finally, we show that $\omega_{\widetilde{\chi_k}}$ can be lifted to the linear Weil representation of either $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ or $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, depending on whether we extend by $\widetilde{\mu_8(\mathbb{C})} = \{e^{\frac{il\pi}{4}}, 0 \leqslant l \leqslant 7\}$ or $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$. Both constructions have their own interest: $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ can be related (in the sense of the commutativity of the diagram (D_{22}), see Proposition 2.39) to the pseudosymplectic group and its two-fold covering, hence it directly extends the original work of Weil; $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is the smallest possible covering but it has the drawback of losing in general the link with the pseudosymplectic covering. To give a precise and complete description of our construction, it remains to mention the case in which the Witt index of Q is equal to zero. In this case (and only this one), we are able to provide a third possibility for the covering $\mathrm{G}_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$, in fact a two-fold covering. However, this additional covering reduces to the pseudosymplectic covering considered in chapter 1 in that very exceptional case, see Propositions 3.28 and 3.30 for a detailed analysis.

Lemma 2.20. *Let $s = (\sigma, f)$ be in $\mathrm{ASp}(B_W)$. Recall the notations $\mathrm{K}(\sigma) = \mathrm{Ker}(1-\sigma)$ and $\mathrm{I}(\sigma) = \mathrm{Im}(1-\sigma)$. Recall also that $\kappa \circ \langle \cdot, \cdot \rangle_W = \langle \cdot, \cdot \rangle$, where κ is defined in Lemma 2.1.*

i. *$f_{|\mathrm{K}(\sigma)}$ extends to an element of $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$ and there exists $w_s \in W$ such that*

$$f_{|\mathrm{K}(\sigma)} = \langle w_s, \cdot \rangle_{W|\mathrm{K}(\sigma)}. \tag{100}$$

ii. *$w'_s \in W$ satisfies (100) if and only if $w'_s - w_s$ belongs to $\mathrm{I}(\sigma)$.*

iii. *If $s \in \mathrm{ASp}_a(B_W)$ then the element w_s satisfying (100) is unique. More precisely, w_s is the unique element v such that $s = (1, \langle v, \cdot \rangle_W)$. In particular, $w_{(1,0)} = 0$.*

iv. *Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\mathrm{ASp}(B_W)$. For all choices of w_{s_1},w_{s_2} and $w_{s_1 s_2}$ satisfying (100) for s_1, s_2 and $s_1 s_2$, respectively, there exists $w_{s_1,s_2} \in \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$ such that $w_{s_1 s_2} - w_{s_1} - w_{s_2} = w_{s_1,s_2}$.*

v. *Let $v \in W$ and let σ_v be the transvection in $\mathrm{Sp}(W)$ defined by $\sigma_v(w) = w + \langle w, v \rangle v$, for every $w \in W$.*

- *If $v \notin \mathrm{I}(\sigma)$ then $\mathrm{I}(\sigma_v \sigma) = \mathrm{I}(\sigma_v) \oplus \mathrm{I}(\sigma)$.*
- *Suppose that $v \in \mathrm{I}(\sigma)$, i.e. there exists $z \in W$ such that $(1-\sigma)(z) = v$. If $\langle z, v \rangle \neq 1$ then $\mathrm{I}(\sigma_v \sigma) = \mathrm{I}(\sigma)$, and if $\langle z, v \rangle = 1$ then $\mathrm{I}(\sigma_v \sigma) = \mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^{\perp}$.*

Proof. Let $s = (\sigma, f)$ be in $\mathrm{ASp}(B_W)$.

i. Let w_1 and w_2 be in $\mathrm{K}(\sigma)$. Then

$$f(w_1 + w_2) - f(w_1) - f(w_2) = B_W(\sigma(w_1), \sigma(w_2)) - B_W(w_1, w_2) = 0,$$

which implies that f is additive on $\mathrm{K}(\sigma)$. We extend $f_{|\mathrm{K}(\sigma)}$ to W by taking any vector subspace V_σ such that $W = \mathrm{K}(\sigma) \oplus V_\sigma$ and setting $F(z + v) = f(z)$. We obtain in this way an additive map on W, i.e. $F \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Proposition 2.7.i then implies the existence of $w_s \in W$ such that $F = \langle w_s, \cdot \rangle_W$. Hence $f_{|\mathrm{K}(\sigma)} = \langle w_s, \cdot \rangle_{W|\mathrm{K}(\sigma)}$.

ii. Suppose that $w'_s \in W$ satisfies the same condition (100) as w_s. Then for every $w \in \mathrm{K}(\sigma)$,

$$\langle w'_s - w_s, w \rangle_W = f(w) - f(w) = 0.$$

Hence $0 = \kappa(\langle w'_s - w_s, w \rangle_W) = \langle w'_s - w_s, w \rangle$ for every $w \in \mathrm{K}(\sigma)$, which is equivalent to $w'_s - w_s \in \mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$ (cf. Lemma 1.27).

Conversely, if $w'_s - w_s \in \mathrm{I}(\sigma) = \mathrm{K}(\sigma)^\perp$ then $\langle w'_s - w_s, w \rangle = 0$ for every $w \in \mathrm{K}(\sigma)$. This implies that $\langle w'_s - w_s, w \rangle_W = \kappa^{-1}(\langle w'_s - w_s, w \rangle) = \kappa^{-1}(0) = 0$ for every $w \in \mathrm{K}(\sigma)$, i.e. w'_s satisfies (100).

iii. Suppose that $s = (1, f) \in \mathrm{ASp}_a(B_W)$. Then $\mathrm{K}(\sigma) = W$, $\mathrm{I}(\sigma) = \{0\}$ and f belongs to $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$ by (82). Hence, by ii, w'_s satisfies (100) if and only if $w'_s - w_s \in \mathrm{I}(\sigma)$, i.e. $w'_s = w_s$. This shows that the element w_s satisfying (100) is unique. Moreover, by Proposition 2.7.i, there exists a unique element $v \in W$ such that $f = \langle v, \cdot \rangle_W$. Hence (100) becomes

$$\langle v, w \rangle_W = \langle w_s, w \rangle_W$$

for every $w \in W$. Since $\langle \cdot, \cdot \rangle_W$ is nondegenerate, this implies that $w_s = v$. In particular, if $s = (1, 0)$ then $w_s = v = 0$.

iv. By definition of w_{s_1}, w_{s_2} and $w_{s_1 s_2}$, for every $w \in \mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2) \subseteq \mathrm{K}(\sigma_1\sigma_2)$,

$$\langle w_{s_1 s_2}, w \rangle_W = (f_2 + f_1 \circ \sigma_2)(w) = f_2(w) + f_1(w) = \langle w_{s_2}, w \rangle_W + \langle w_{s_1}, w \rangle_W,$$

that is

$$\langle w_{s_1 s_2} - w_{s_1} - w_{s_2}, w \rangle_W = 0.$$

This implies that $\langle w_{s_1 s_2} - w_{s_1} - w_{s_2}, w \rangle = 0$ for every $w \in \mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2)$. Thus $w_{s_1 s_2} - w_{s_1} - w_{s_2} \in (\mathrm{K}(\sigma_1) \cap \mathrm{K}(\sigma_2))^\perp = \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$ (cf. Lemma 1.27) and we can set $w_{s_1, s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2} \in \mathrm{I}(\sigma_1) + \mathrm{I}(\sigma_2)$.

v. Let $v \in W$. The transvection σ_v is defined by $\sigma_v(w) = w + \langle w, v \rangle v$ for every $w \in W$. If $v = 0$ then σ_v is the identity and the result follows directly. Hence we can suppose that $v \neq 0$.

- Suppose that $v \notin \mathrm{I}(\sigma)$. Then $\mathrm{I}(\sigma_v) \cap \mathrm{I}(\sigma) = \{0\}$ because $\mathrm{I}(\sigma_v) = \mathrm{Span}\{v\}$. Taking the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$, we know that showing that

$\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma_v) + \mathrm{I}(\sigma)$ is equivalent to prove that $\mathrm{K}(\sigma_v\sigma) = \mathrm{K}(\sigma_v) \cap \mathrm{K}(\sigma)$. The inclusion $\mathrm{K}(\sigma_v) \cap \mathrm{K}(\sigma) \subseteq \mathrm{K}(\sigma_v\sigma)$ is obvious. Now let w be in $\mathrm{K}(\sigma_v\sigma)$. Then

$$\sigma(w) = \sigma_v^{-1}(w) = w - \langle w, v\rangle v. \tag{101}$$

Hence $(1-\sigma)(w) = \langle w, v\rangle v = 0$, since by hypothesis $v \notin \mathrm{I}(\sigma)$. This implies that $w \in \mathrm{K}(\sigma)$ and $\langle w, v\rangle = 0$, which in turn implies that $w \in \mathrm{K}(\sigma_v)$.

- Suppose that $v \in \mathrm{I}(\sigma)$ and let $z \in W$ be such that $(1-\sigma)(z) = v$. Let $w \in \mathrm{K}(\sigma_v\sigma)$. Then replacing v by $(1-\sigma)(z)$ in (101), we obtain that

$$(1-\sigma)(w) = \langle w, v\rangle v = \langle w, v\rangle (1-\sigma)(z).$$

Hence $w - \langle w, v\rangle z$ belongs to $\mathrm{K}(\sigma)$, i.e. $w \in \mathrm{K}(\sigma) + \mathrm{Span}\{z\}$. Taking orthogonal complements, this shows that $\mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp \subseteq \mathrm{I}(\sigma_v\sigma)$. On the other hand, if $u \in \mathrm{I}(\sigma_v\sigma)$ then there exists $w \in W$ such that

$$u = (1-\sigma_v\sigma)(w) = w - \sigma(w) - \langle \sigma(w), v\rangle v = (1-\sigma)(w - \langle \sigma(w), v\rangle z),$$

i.e. $u \in \mathrm{I}(\sigma)$. Hence $\mathrm{I}(\sigma_v\sigma) \subseteq \mathrm{I}(\sigma)$. Therefore we obtain

$$\mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp \subseteq \mathrm{I}(\sigma_v\sigma) \subseteq \mathrm{I}(\sigma). \tag{102}$$

Moreover, $\mathrm{I}(\sigma_v\sigma) \subseteq \mathrm{Span}(z)^\perp$ is equivalent to $z \in \mathrm{K}(\sigma_v\sigma)$. Since

$$(1-\sigma_v\sigma)(z) = z - \sigma_v(z-v) = v - \langle z-v, v\rangle v = v - \langle z, v\rangle v = (1 - \langle z, v\rangle)v,$$

we have that $\mathrm{I}(\sigma_v\sigma) \subseteq \mathrm{Span}(z)^\perp$ if and only if $\langle z, v\rangle = 1$. Thus by (102), if $\langle z, v\rangle = 1$ then $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp$. Suppose now that $\langle z, v\rangle \neq 1$. Then by (102), $\mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp \subset \mathrm{I}(\sigma_v\sigma) \subseteq \mathrm{I}(\sigma)$. Moreover, since $v \neq 0$ then $z \notin \mathrm{K}(\sigma)$. Hence

$$\begin{aligned}\dim(\mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp) &= \dim W - \dim(\mathrm{K}(\sigma) + \mathrm{Span}\{z\}) \\ &= \dim W - \dim \mathrm{K}(\sigma) - 1 + \dim(\mathrm{K}(\sigma) \cap \mathrm{Span}\{z\}) \\ &= \dim \mathrm{I}(\sigma) - 1.\end{aligned}$$

This is enough to conclude that $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma)$.

□

Now we are ready to define an explicit section of the projective representation $s \mapsto \omega_{\widetilde{\chi_k}}(s)$ and compute the associated cocycle $c_{\widetilde{\chi_k}}$.

Definition-Proposition 2.21 (Projective Weil representation). *For every $s = (\sigma, f) \in \mathrm{ASp}(B_W)$ and for every w_s in W satisfying* (100)*, we define*

$$\omega_{\widetilde{\chi_k}}(s, w_s) = |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\widetilde{\chi_k}}(s(w))^{-1}\rho_{\widetilde{\chi_k}}(w_s)\rho_{\widetilde{\chi_k}}(w), \tag{103}$$

where $|\gamma(\sigma)|^2 = 2^{-\dim \mathrm{I}(\sigma)}$. Then $\omega_{\widetilde{\chi_k}}(s, w_s)$ has the following properties:

i. *For every* $h \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$,

$$\rho_{\widetilde{\chi_k}}(s(h)) = \omega_{\widetilde{\chi_k}}(s, w_s)\rho_{\widetilde{\chi_k}}(h)\omega_{\widetilde{\chi_k}}(s, w_s)^{-1}. \tag{104}$$

ii. *For every* $u \in \mathrm{I}(\sigma)$, *choose any* $w \in W$ *such that* $u = (1-\sigma)(w)$. *Then*

$$\omega_{\widetilde{\chi_k}}(s, w_s) = \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi_k(f(w) + \langle w_s, w\rangle_W + B_W(u, w+w_s))\rho_{\widetilde{\chi_k}}(u+w_s). \tag{105}$$

iii. $\omega_{\widetilde{\chi_k}}(s, w_s) \in \mathrm{U}(\mathrm{L}^2(Y))$.

iv. *If we make another choice of element* w_s' *satisfying* (100) *then* $\omega_{\widetilde{\chi_k}}(s, w_s)$ *and* $\omega_{\widetilde{\chi_k}}(s, w_s')$ *can differ only by a fourth root of unity. More precisely:*

$$\omega_{\widetilde{\chi_k}}(s, w_s') = \chi_k(-f(v) + \langle v, w_s\rangle_W + B_W(w_s' - w_s, v - w_s))\omega_{\widetilde{\chi_k}}(s, w_s) \tag{106}$$

where v *is any element such that* $w_s' - w_s = (1-\sigma)(v)$.

Thus for each choice of the set $\{w_s$ *satisfying* (100)*,* $s \in \mathrm{ASp}(B_W)\}$, *the map* $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ *is an explicit section of the projective Weil representation of* $\mathrm{ASp}(B_W)$ *on* $\mathrm{L}^2(Y)$.

Remark 2.22 (Related to Remark 2.36)**.** As noted in Remark 1.29, the parameter $\gamma(\sigma)$ in (103) should in fact depend on $s = (\sigma, f)$ and the specific choice of w_s. For this reason, we should consider $\gamma(s, w_s)$ instead of $\gamma(\sigma)$. If we replace $\gamma(\sigma)$ by $\gamma(s, w_s)$ in Corollary 2.26, we can apply the same proof to show that c_{χ_k} is not a coboundary. So the smallest central extension of $\mathrm{ASp}(B_W)$ we could construct using the cocycle c_{χ_k} defined by the parameters $\gamma(s, w_s)$, in order to lift the projective representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ to a linear representation, is a two-fold covering of $\mathrm{ASp}(B_W)$. In particular, as we shall see in Corollary 2.28.iii, we do not need to consider the more general parameter $\gamma(s, w_s)$ if $\dim_{\mathbb{F}_2}(W) = 2$ and the Witt index of Q is zero. In fact, in this case, we prove a necessary and sufficient condition on the parameter $\gamma(\sigma)$ so that the cocycle c_{χ_k} takes its values in $\mu_2(\mathbb{C})$, and then construct two non-isomorphic two-fold coverings of $\mathrm{ASp}(B_W)$ (see Propositions 2.29 and 3.30). The fact that χ_k takes its values in $\mu_4(\mathbb{C})$ makes it generally difficult to determine the conditions under which the cocycle c_{χ_k}, corresponding to the projective Weil representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$, takes its values in $\mu_2(\mathbb{C})$ rather than in $\mu_4(\mathbb{C})$. To simplify our notation and computations, we have chosen to keep the parameters $\gamma(\sigma)$ as a function of $\sigma \in \mathrm{Sp}(W)$ only. With this choice of parameters, we can determine the smallest possible range of the cocycle c_{χ_k}, see Corollary 2.28. Indeed, using the parameter $\gamma(\sigma)$, we prove that the smallest covering of $\mathrm{ASp}(B_W)$ to linearize $\omega_{\widetilde{\chi_k}}$ is a two-fold covering if the Witt index of Q is zero, and a four-fold covering otherwise. At present, the possibility of lowering the degree of the covering of $\mathrm{ASp}(B_W)$ from four to two by a suitable choice of more general parameters $\gamma(s, w_s)$ (necessarily depending on $s = (\sigma, f)$ and w_s and not only on σ) remains an open problem. Nevertheless, we can provide a complete solution when $\dim_{\mathbb{F}_2}(W) = 2$ and the Witt index of Q is one. In this case, using cohomological arguments independent of our construction, we show that $\omega_{\widetilde{\chi_k}}$ linearizes to a two-fold covering. Then we exhibit parameters $\gamma(s, w_s)$ leading to two non-isomorphic two-fold coverings of $\mathrm{ASp}(B_W)$ that linearize $\omega_{\widetilde{\chi_k}}$. This is discussed in full in Remark 3.8.

Gurevich-Hadani [GH12] and Genestier-Lysenko [GL12] do not address neither the question of the minimal possible covering of $\mathrm{ASp}(B_W)$ nor the question of existence of non-isomorphic coverings of $\mathrm{ASp}(B_W)$ of fixed degree. They work with one realization of a covering that linearizes ω. On the other hand, people working on group cohomology try to find out how many non-isomorphic coverings exist for a given degree. Hence the question above is relevant and we answer it in some cases.

Proof. **i.** Let $s = (\sigma, f) \in \mathrm{ASp}(B_W)$, $w_s \in W$ satisfying (100) and $h = (v, t) \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Then, exactly as in the proof of Proposition 1.28.i, only replacing $\langle \cdot, \cdot \rangle$, B and Q by $\langle \cdot, \cdot \rangle_W$, B_W and Q_W, respectively, we show that

$$\rho_{\widetilde{\chi_k}}(s(h))\omega_{\widetilde{\chi_k}}(s, w_s) = \omega_{\widetilde{\chi_k}}(s, w_s)\rho_{\widetilde{\chi_k}}(h).$$

ii. Let $s = (\sigma, f) \in \mathrm{ASp}(B_W)$ and let $w_s \in W$ satisfying (100). Then

$$\omega_{\widetilde{\chi_k}}(s, w_s) = |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\widetilde{\chi_k}}(s(w)^{-1} w_s w),$$

where

$$\begin{aligned} &s(w)^{-1} w_s w \\ &= ((1-\sigma)(w) + w_s, -f(w) + \langle w_s, w \rangle_W + B_W((1-\sigma)(w), w + w_s) + Q_W(\sigma(w)) - Q_W(w)) \end{aligned} \tag{107}$$

(this is again the same computation as in Proposition 1.28.ii). Moreover by (82),

$$0 = f(2w) = 2f(w) + Q_W(\sigma(w)) - Q_W(w), \quad \text{i.e. } -f(w) + Q_W(\sigma(w)) - Q_W(w) = f(w)$$

since $-Q_W = Q_W$ (recall that Q_W takes its values in $2\mathbb{Z}/4\mathbb{Z}$). Hence

$$s(w)^{-1} w_s w = ((1-\sigma)(w) + w_s, f(w) + \langle w_s, w \rangle_W + B_W((1-\sigma)(w), w + w_s). \tag{108}$$

Now the proof can be completed as the one of Proposition 1.28.ii. Using that $\sigma \in \mathrm{Sp}(W)$ and $2B_W = 0$, we show that the expression $\rho_{\widetilde{\chi_k}}(s(w)^{-1} w_s w)$ does not depend on the left coset of w modulo $\mathrm{K}(\sigma)$. Then we obtain

$$\begin{aligned} \omega_{\widetilde{\chi_k}}(s, w_s) &= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\widetilde{\chi_k}}(s(w)^{-1} w_s w) \\ &= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w + \mathrm{K}(\sigma) \in W/\,\mathrm{K}(\sigma)} |\,\mathrm{K}(\sigma)| \rho_{\widetilde{\chi_k}}(s(w)^{-1} w_s w) \\ &= \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi_k(f(w) + \langle w_s, w \rangle_W + B_W(u, w + w_s)) \rho_{\widetilde{\chi_k}}(u + w_s), \end{aligned}$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$, and we use the isomorphism between $W/\,\mathrm{K}(\sigma)$ and $\mathrm{I}(\sigma)$.

iii. To show that $\omega_{\widetilde{\chi_k}}(s, w_s)$ is invertible, by the intertwining property (104) and Schur's lemma, it is enough to show that this operator is not identically zero. The expression (105) of $\omega_{\widetilde{\chi_k}}(s, w_s)$ implies that $\omega_{\widetilde{\chi_k}}(s, w_s)$ has coefficients in $\mu_4(\mathbb{C})$ in the basis

$\{\rho_{\widetilde{\chi_k}}(w), w \in W\}$ of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ (cf. Lemma 2.15). Thus $\omega_{\widetilde{\chi_k}}(s, w_s)$ cannot be identically zero, hence is invertible. Furthermore we can write

$$\omega_{\widetilde{\chi_k}}(s, w_s)\omega_{\widetilde{\chi_k}}(s, w_s)^* = |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w,w' \in W} \rho_{\widetilde{\chi_k}}(s(w)^{-1}w_s w)\rho_{\widetilde{\chi_k}}(s(w')^{-1}w_s w')^*.$$

Using the unitarity of $\rho_{\widetilde{\chi_k}}$ and setting $z = w - w'$, we show as in Proposition 1.28.iii that

$$\begin{aligned}
&\omega_{\widetilde{\chi_k}}(s, w_s)\omega_{\widetilde{\chi_k}}(s, w_s)^* \\
&= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w,w' \in W} \rho_{\widetilde{\chi_k}}(s(w)^{-1}w_s w)\rho_{\widetilde{\chi_k}}(w'^{-1}w_s^{-1}s(w')) \\
&= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{w',z \in W} \rho_{\widetilde{\chi_k}}((1-\sigma)(z), f(z) + Q_W(\sigma(z)) + \langle w_s, z\rangle_W - B_W(\sigma(z), z) \\
&\qquad\qquad + \langle z, (1-\sigma)(w')\rangle_W) \\
&= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z \in W} \rho_{\widetilde{\chi_k}}((1-\sigma)(z), f(z) + Q_W(\sigma(z)) + \langle w_s, z\rangle_W - B_W(\sigma(z), z)) \\
&\qquad\qquad \sum_{w' \in W} \chi_k(\langle z, (1-\sigma)(w')\rangle_W) \\
&= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z \in W} \rho_{\widetilde{\chi_k}}((1-\sigma)(z), f(z) + Q_W(\sigma(z)) + \langle w_s, z\rangle_W - B_W(\sigma(z), z)) \\
&\qquad\qquad \sum_{w' \in W} \chi_k(\langle (1-\sigma^{-1})(z), w'\rangle_W) \\
&= |\,\mathrm{K}(\sigma)|^{-2}|\gamma(\sigma)|^2 \sum_{z \in \mathrm{K}(\sigma)} |W|\rho_{\widetilde{\chi_k}}(0) \\
&= |\,\mathrm{K}(\sigma)|^{-1}|\gamma(\sigma)|^2 |W|\,\mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= 2^{-\dim \mathrm{K}(\sigma)} 2^{-\dim \mathrm{I}(\sigma)} 2^{\dim \mathrm{K}(\sigma) + \dim \mathrm{I}(\sigma)}\,\mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= \mathrm{Id}_{\mathrm{L}^2(Y)},
\end{aligned}$$

where the three last equalities are justified by the fact that $w \mapsto \chi_k(\langle (1-\sigma^{-1})(z), w\rangle_W)$ is a character of W which is trivial if and only if $z \in \mathrm{K}(\sigma)$, Schur's orthogonality relations, the definition of $\gamma(\sigma)$ and the isomorphism between $W/\,\mathrm{K}(\sigma)$ and $\mathrm{I}(\sigma)$. Therefore $\omega_{\widetilde{\chi_k}}(s, w_s)$ is a unitary operator of $\mathrm{L}^2(Y)$.

iv. Let $w_s' \in W$ be such that (100) is satisfied, i.e. $f_{|\,\mathrm{K}(\sigma)} = \langle w_s, \cdot\rangle_{W|\,\mathrm{K}(\sigma)}$. Then Lemma 2.20 implies that $w_s' - w_s \in \mathrm{I}(\sigma)$ and we can set $w_s' - w_s = (1-\sigma)(v)$ for some $v \in W$. Then

$$\omega_{\widetilde{\chi_k}}(s, w_s') = |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{w \in W} \rho_{\widetilde{\chi_k}}(s(w)^{-1}w_s' w)$$

where, using again (108) and setting $z = w + v$, we obtain as in Proposition 1.28.iv that

$$\begin{aligned}
&\omega_{\widetilde{\chi_k}}(s, w_s') \\
&= |\,\mathrm{K}(\sigma)|^{-1}\gamma(\sigma) \sum_{z \in W} \chi_k(-f(v) + \langle v, w_s\rangle_W + B_W(w_s' - w_s, v - w_s))\rho_{\widetilde{\chi_k}}(s(z)^{-1}w_s z) \\
&= \chi_k(-f(v) + \langle v, w_s\rangle_W + B_W(w_s' - w_s, v - w_s))\omega_{\widetilde{\chi_k}}(s, w_s)
\end{aligned}$$

□

Remark 2.23. In Proposition 2.21, we defined an operator $\omega_{\widetilde{\chi_k}}(s, w_s)$ for all $s = (\sigma, f) \in \mathrm{ASp}(B_W)$ and for all w_s satisfying (100). From now on, to simplify the notation, if it is clear that a choice of w_s has been made, then we will write indifferently $\omega_{\widetilde{\chi_k}}(s, w_s)$ or $\omega_{\widetilde{\chi_k}}(s)$.

Corollary 2.24 (Computation of the cocycle $c_{\widetilde{\chi_k}}$). *We use the same notations as in Lemma 2.20 and Proposition 2.21, and we assume that a choice of $w_s \in W$ satisfying* (100) *has been made for all $s \in \mathrm{ASp}(B_W)$. The cocycle $c_{\widetilde{\chi_k}}$ associated to the projective Weil representation $\omega_{\widetilde{\chi_k}}$ is given by the following formula. For all $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ in $\mathrm{ASp}(B_W)$,*

$$c_{\widetilde{\chi_k}}(s_1, s_2) = C_{1,2} \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1, s_2}}} \chi_k(f_1(w_1) + \langle w_{s_1}, w_1 \rangle_W + B_W(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle_W + B_W(u_2, w_2 + w_{s_2}) + B_W(u_1 + w_{s_1}, u_2 + w_{s_2})) \tag{109}$$

where $C_{1,2} = \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1}$, $w_{s_1,s_2} = w_{s_1 s_2} - w_{s_1} - w_{s_2}$ and w_i is any element such that $(1 - \sigma_i)(w_i) = u_i$ for $i \in \{1, 2\}$. In particular, the cocycle $c_{\widetilde{\chi_k}}$ does not depend on the choice of the extension $\widetilde{\chi_k}$ of χ_k: we can denote it by c_{χ_k} instead of $c_{\widetilde{\chi_k}}$.

Proof. Let $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ be in $\mathrm{ASp}(B_W)$. The fact that $\omega_{\widetilde{\chi_k}}$ is a projective representation implies that $c_{\widetilde{\chi_k}}$ satisfies

$$\omega_{\widetilde{\chi_k}}(s_1)\omega_{\widetilde{\chi_k}}(s_2) = c_{\widetilde{\chi_k}}(s_1, s_2)\omega_{\widetilde{\chi_k}}(s_1 s_2), \tag{110}$$

where we recall from Remark 2.23 that $\omega_{\widetilde{\chi_k}}(s_1) = \omega_{\widetilde{\chi_k}}(s_1, w_{s_1})$ and $\omega_{\widetilde{\chi_k}}(s_2) = \omega_{\widetilde{\chi_k}}(s_2, w_{s_2})$. Exactly as in the proof of Corollary 1.31 (only replacing $\langle \cdot, \cdot \rangle$ and B by $\langle \cdot, \cdot \rangle_W$ and B_W, respectively), we compute separately $\omega_{\widetilde{\chi_k}}(s_1)\omega_{\widetilde{\chi_k}}(s_2)$ and $\omega_{\widetilde{\chi_k}}(s_1 s_2)$, using the expression (105) found in Proposition 2.21 and (97). Then, using the fact that the operators $\rho_{\widetilde{\chi_k}}(w)$, $w \in W$, form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ (cf. Lemma 2.15), we identify the coefficients of $\omega_{\widetilde{\chi_k}}(s_1)\omega_{\widetilde{\chi_k}}(s_2)$ and $\omega_{\widetilde{\chi_k}}(s_1 s_2)$ in this basis. We obtain for every $u \in \mathrm{I}(\sigma_1\sigma_2)$,

$$\begin{aligned} &c_{\widetilde{\chi_k}}(s_1, s_2)\gamma(\sigma_1\sigma_2)\chi_k((f_2 + f_1 \circ \sigma_2)(w) + \langle w_{s_1 s_2}, w \rangle_W + B_W(u, w + w_{s_1 s_2})) \\ &= \gamma(\sigma_1)\gamma(\sigma_2) \\ &\quad \cdot \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1 + u_2 + w_{s_1} + w_{s_2} = u + w_{s_1 s_2}}} \chi_k(f_1(w_1) + \langle w_{s_1}, w_1 \rangle_W + B_W(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle_W \\ &\qquad + B_W(u_2, w_2 + w_{s_2}) + B_W(u_1 + w_{s_1}, u_2 + w_{s_2})). \end{aligned} \tag{111}$$

Taking $u = 0$ and replacing $w_{s_1 s_2} - w_{s_1} - w_{s_2}$ by w_{s_1, s_2} (defined in Lemma 2.20) in the previous equation, we finally obtain:

$$\begin{aligned} c_{\widetilde{\chi_k}}(s_1, s_2) &= \gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1} \\ &\quad \cdot \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2) \\ u_1 + u_2 = w_{s_1, s_2}}} \chi_k(f_1(w_1) + \langle w_{s_1}, w_1 \rangle_W + B_W(u_1, w_1 + w_{s_1}) + f_2(w_2) + \langle w_{s_2}, w_2 \rangle_W \\ &\qquad + B_W(u_2, w_2 + w_{s_2}) + B_W(u_1 + w_{s_1}, u_2 + w_{s_2})). \end{aligned}$$

In particular this expression does not depend on the choice of the extension $\widetilde{\chi_k}$ of χ_k, so we can write c_{χ_k} instead of $c_{\widetilde{\chi_k}}$. □

Now that we have determined the formula for the cocycle c_{χ_k}, as in section 1.3 the next goal is to know its range in order to construct the smallest possible extension of $\mathrm{ASp}(B_W)$.

In fact, the range of c_{χ_k} will depend on the choice made for the parameters $\gamma(\sigma)$ appearing in the definition (103) of the operator $\omega_{\widetilde{\chi_k}}(s, w_s)$ for every $s = (\sigma, f) \in \mathrm{ASp}(B_W)$. Recall from Proposition 2.21 that the only condition on $\gamma(\sigma)$ is that $|\gamma(\sigma)|^2 = 2^{-\dim \mathrm{I}(\sigma)}$, which ensures that $\omega_{\widetilde{\chi_k}}(s, w_s)$ is a unitary operator. This means that we can write

$$\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)}, \quad \text{where } \theta(\sigma) \in \mathbb{R}. \tag{112}$$

Remark 2.25. Just before investigating the range of the cocycle c_{χ_k}, we recall that our goal is to lift the projective Weil representation of $\mathrm{ASp}(B_W)$ defined in Proposition 2.18 to a smallest possible covering of $\mathrm{ASp}(B_W)$. To this extent, the range of c_{χ_k} plays a crucial role. Corollary 2.26 states a particularly important result: by showing that the cocycle is not a coboundary for every choice of the parameter set $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$, we demonstrate that the projective Weil representation cannot be linearized on $\mathrm{ASp}(B_W)$, i.e. we need a nontrivial central extension of $\mathrm{ASp}(B_W)$ in order to lift $s \mapsto \omega_{\widetilde{\chi_k}}(s)$ to a linear representation. The question of finding the optimal size of this extension will be addressed in Corollary 2.28.

Corollary 2.26. *The cocycle* c_{χ_k} *restricted to the abelian subgroup* $\mathrm{ASp}_a(B_W)$ *of* $\mathrm{ASp}(B_W)$ *is not symmetric. In particular,* c_{χ_k} *is not a coboundary.*

Proof. Recall from Proposition 2.7.viii that $\mathrm{ASp}_a(B_W)$ denotes the abelian subgroup of $\mathrm{ASp}(B_W)$ consisting of the elements of the form $(1, f)$. Let w_1 and w_2 in W be such that $\chi_k(\langle w_1, w_2\rangle_W) \neq 1$ (such a choice is always possible if $W \neq \{0\}$ since $\langle \cdot, \cdot \rangle_W$ is nondegenerate and χ_k is faithful). This is equivalent to have $\chi_k(B_W(w_1, w_2)) \neq \chi_k(B_W(w_2, w_1))$. Recall that we proved in Proposition 2.7 that $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$ is the set of maps $\langle w, \cdot \rangle_W$, where $w \in W$. Let $f_1 = \langle w_1, \cdot \rangle_W$ and $f_2 = \langle w_2, \cdot \rangle_W$. Then by definition of w_{s_1} and w_{s_2} satisfying (100), $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ are two elements of $\mathrm{ASp}_a(B_W)$ such that for every $w \in W$,

$$\langle w_{s_i}, w \rangle_W = f_i(w) = \langle w_i, w \rangle_W.$$

Hence $w_{s_i} = w_i$ for $i \in \{1, 2\}$. Now, if we compute the cocycle with the formula (109) found in Corollary 2.24, since $\mathrm{I}(1) = \{0\}$, then we obtain:

$$c_{\chi_k}(s_1, s_2) = \gamma(1)\chi_k(B_W(w_{s_1}, w_{s_2})) = \gamma(1)\chi_k(B_W(w_1, w_2)) \tag{113}$$

and

$$c_{\chi_k}(s_2, s_1) = \gamma(1)\chi_k(B_W(w_{s_2}, w_{s_1})) = \gamma(1)\chi_k(B_W(w_2, w_1)).$$

In particular $c_{\chi_k}(s_1, s_2) \neq c_{\chi_k}(s_2, s_1)$ due to our hypothesis on $\chi_k(\langle w_1, w_2 \rangle_W)$, which implies that the cocycle c_{χ_k} is not symmetric on $\mathrm{ASp}_a(B_W)$.

Now, if we suppose that c_{χ_k} is a coboundary, then there exists $\nu : \mathrm{ASp}(B_W) \to \mathrm{U}(1)$ such that for all $s_1, s_2 \in \mathrm{ASp}(B_W)$,

$$c_{\chi_k}(s_1, s_2) = \nu(s_1)\nu(s_2)\nu(s_1 s_2)^{-1}. \tag{114}$$

In particular, (114) has to be satisfied for all elements of $\mathrm{ASp}_a(B_W)$. We therefore have for all $s_1 = (1, f_1)$ and $s_2 = (1, f_2)$ in $\mathrm{ASp}_a(B_W)$:

$$c_{\chi_k}(s_2, s_1) = \nu(s_2)\nu(s_1)\nu(s_2 s_1)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1 s_2)^{-1} = c_{\chi_k}(s_1, s_2),$$

i.e. the cocycle c_{χ_k} would be symmetric on $\mathrm{ASp}_a(B_W)$. □

Remark 2.27 (Related to Remark 1.34)**.** We proved in Corollary 2.26 that the cocycle c_{χ_k} is not symmetric on the abelian subgroup $\mathrm{ASp}_a(B_W)$ of $\mathrm{ASp}(B_W)$. Since the product law in the coverings of $\mathrm{ASp}(B_W)$ that we will define in Proposition 2.29 is entirely determined by this cocycle, this implies that there are commuting elements in $\mathrm{ASp}(B_W)$ whose preimages in the coverings do not commute.

Corollary 2.28 (Range of the cocycle c_{χ_k}). *The cocycle c_{χ_k} takes its values in $\mathrm{U}(1)$. Moreover, depending on the choice made for $\gamma(\sigma)$ for every $\sigma \in \mathrm{Sp}(W)$, we have the following properties concerning the range of c_{χ_k}.*

i. *The cocycle c_{χ_k} takes its values in $\mu_8(\mathbb{C}) = \{e^{\frac{il\pi}{4}}, 0 \leqslant l \leqslant 7\}$ if and only if*

$$\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \frac{\pi}{4} \qquad \text{for all } \sigma_1, \sigma_2 \in \mathrm{Sp}(W). \tag{115}$$

In particular, this is the case if $\gamma(\sigma) \in \mathbb{R}$ for every $\sigma \in \mathrm{Sp}(W)$, for example if $\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}}$.

ii. *The cocycle c_{χ_k} takes its values in $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$ if and only if*

$$\begin{cases} \theta(1) \equiv 0 \mod \frac{\pi}{2} \\ \theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv \begin{cases} 0 \mod \frac{\pi}{2} & \text{if } v \notin \mathrm{I}(\sigma) \text{ or } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 1 \\ \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 0 \end{cases} \end{cases} \tag{116}$$

for every $\sigma \in \mathrm{Sp}(W)$ and every transvection $\sigma_v \in \mathrm{Sp}(W)$ such that $v \neq 0$. In particular, this is the case if $\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{-\frac{i\pi \dim \mathrm{I}(\sigma)}{4}} = (1+i)^{-\dim \mathrm{I}(\sigma)}$ for every $\sigma \in \mathrm{Sp}(W)$.

If W is a two-dimensional space, then (116) *can be rewritten as follows: c_{χ_k} takes its values in $\mu_4(\mathbb{C}) = \{\pm 1, \pm i\}$ if and only if*

$$\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \frac{\pi}{2} \qquad \text{for all } \sigma_1, \sigma_2 \in \mathrm{Sp}(W). \tag{117}$$

In particular, this is the case if $\gamma(\sigma) \in \mathbb{R}$ for every $\sigma \in \mathrm{Sp}(W)$, for example if $\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}}$.

iii. *Suppose that the Witt index of Q is zero, in which case W is a two-dimensional space. Then c_{χ_k} takes its values in $\mu_2(\mathbb{C}) = \{\pm 1\}$ if and only if*

$$\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \pi \qquad \text{for all } \sigma_1, \sigma_2 \in \mathrm{Sp}(W). \tag{118}$$

In particular, this is the case if $\gamma(\sigma) \in \mathbb{R}$ for every $\sigma \in \mathrm{Sp}(W)$, for example if $\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}}$.

iv. *Regardless of the choice of $\{\gamma(\sigma), \sigma \in \mathrm{Sp}(W)\}$, the smallest possible range of c_{χ_k} is:*

$$\begin{cases} \mu_2(\mathbb{C}) = \{\pm 1\} & \text{if the Witt index of } Q \text{ is zero (in which case } \dim_{\mathbb{F}_2}(W) = 2) \\ \mu_4(\mathbb{C}) = \{\pm 1, \pm i\} & \text{otherwise, i.e. if the Witt index of } Q \text{ is nonzero.} \end{cases}$$

Proof. Equation (110), together with the unitarity of the operators $\omega_{\widetilde{\chi_k}}(s)$ for all $s \in \mathrm{ASp}(B_W)$, implies that c_{χ_k} takes its values in $\mathrm{U}(1)$. The group $\mathrm{Sp}(W)$ is generated by the transvections $\sigma_v : w \mapsto w + \langle w, v\rangle v$, $v \in W$ (cf. [OMe78, Th. 2.1.9]). Let $v \in W$ and $\widetilde{v} \in \widetilde{W}$ be such that $v = \widetilde{v} + 2\widetilde{W}$. Recall that, for every $\widetilde{w} \in \widetilde{W}$,

$$\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}) = \widetilde{w} + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} \widetilde{v}.$$

Then, by Proposition 2.7.iii, $\pi(\widetilde{\sigma}_{\widetilde{v}}) = \sigma_v$. Let $f_{\widetilde{\sigma}_{\widetilde{v}}} : W \to \mathbb{Z}/4\mathbb{Z}$ be defined according to (93) with $\widetilde{\sigma_v} = \widetilde{\sigma}_{\widetilde{v}}$, i.e.

$$f_{\widetilde{\sigma}_{\widetilde{v}}}(w) = B_{\widetilde{W}}(\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}), \widetilde{\sigma}_{\widetilde{v}}(\widetilde{w})) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w})$$

for every $w \in W$. Then $(\sigma_v, f_{\widetilde{\sigma}_{\widetilde{v}}})$ belongs to $\mathrm{ASp}(B_W)$. Hence $\mathrm{ASp}(B_W)$ is generated by the set of elements $(\sigma_v, f_{\widetilde{\sigma}_{\widetilde{v}}})$, $v \in W$, together with the elements of $\mathrm{ASp}_a(B_W) = \{(1, \langle v, \cdot\rangle_W),\, v \in W\}$. We denote this set of generators of $\mathrm{ASp}(B_W)$ by $\mathscr{G}$. Now let r and s be in $\mathrm{ASp}(B_W)$. Then there exist $r_1, \ldots, r_p \in \mathscr{G}$ such that $r = r_1 \cdots r_p$. Since c_{χ_k} is a cocycle, we can write

$$c_{\chi_k}(r_1, r_2 \cdots r_p) c_{\chi_k}(r, s) = c_{\chi_k}(r_1, r_2 \cdots r_p s) c_{\chi_k}(r_2 \cdots r_p, s). \tag{119}$$

So, using (119) by induction, we see that showing that c_{χ_k} takes its values in $\mu_8(\mathbb{C})$ (or $\mu_4(\mathbb{C})$) is equivalent to showing that $c_{\chi_k}(r, s)$ belongs to $\mu_8(\mathbb{C})$ (or $\mu_4(\mathbb{C})$) for all $s \in \mathrm{ASp}(B_W)$ and $r \in \mathscr{G}$.

Let $s = (\sigma, f) \in \mathrm{ASp}(B_W)$, $r_a = (1, \langle a, \cdot\rangle_W) \in \mathscr{G}$ and $r = (\sigma_v, f_{\widetilde{\sigma}_{\widetilde{v}}}) \in \mathscr{G}$, where $v \neq 0$. Then $\mathrm{I}(\sigma_v) = \mathrm{Span}\{v\}$ and $\mathrm{K}(\sigma_v) = \mathrm{Span}\{v\}^\perp$. We suppose that a choice for $w_{s'}$ for every $s' \in \mathrm{ASp}(B_W)$ has been made, and we recall from Lemma 2.20.iii that $w_{r_a} = a$. By (112), $\gamma(\sigma_v) = 2^{\frac{-\dim \mathrm{I}(\sigma_v)}{2}} e^{i\theta(\sigma_v)} = 2^{-\frac{1}{2}} e^{i\theta(\sigma_v)}$ and $\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)}$.

- First we study $c_{\chi_k}(r_a, s)$. By (109) and since $\mathrm{I}(1) = \{0\}$,

$$\begin{aligned} c_{\chi_k}(r_a, s) &= C \sum_{\substack{u_1 \in \mathrm{I}(1), u_2 \in \mathrm{I}(\sigma) \\ u_1 + u_2 = w_{r_a, s}}} \chi_k(\langle a, w_1\rangle_W + \langle a, w_1\rangle_W + B_W(u_1, w_1 + a) + f(w_2) \\ &\qquad + \langle w_s, w_2\rangle_W + B_W(u_2, w_2 + w_s) + B_W(u_1 + a, u_2 + w_s)) \\ &= C \sum_{\substack{u_2 \in \mathrm{I}(\sigma) \\ u_2 = w_{r_a, s}}} \chi_k(f(w_2) + \langle w_s, w_2\rangle_W + B_W(u_2, w_2 + w_s) + B_W(a, u_2 + w_s)) \\ &= C\, \chi_k(f(w_2) + \langle w_s, w_2\rangle_W + B_W(w_{r_a,s}, w_2 + w_s) + B_W(a, w_{r_a,s} + w_s)) \end{aligned}$$

  where $(1 - \sigma)(w_2) = w_{r_a,s} = w_{r_a s} - a - w_s$ and $C = \gamma(1)\gamma(\sigma)\gamma(\sigma)^{-1} = \gamma(1)$. Hence

$$c_{\chi_k}(r_a, s) = \gamma(1)\chi_k(f(w_2) + \langle w_s, w_2\rangle_W + B_W(w_{r_a,s}, w_2 + w_s) + B_W(a, w_{r_a,s} + w_s)), \tag{120}$$

  i.e. $c_{\chi_k}(r_a, s) \in \mu_8(\mathbb{C})$ (or $\mu_4(\mathbb{C})$) if and only if $\gamma(1) \in \mu_8(\mathbb{C})$ (or $\mu_4(\mathbb{C})$), since χ_k takes its values in $\mu_4(\mathbb{C})$.

- We now study $c_{\chi_k}(r, s)$ under the hypothesis that v does not belong to $\mathrm{I}(\sigma)$. Using again (109), we obtain

$$\begin{aligned} c_{\chi_k}(r, s) = C_{r,s} \sum_{\substack{u_1 \in \mathrm{I}(\sigma_v), u_2 \in \mathrm{I}(\sigma) \\ u_1 + u_2 = w_{r,s}}} &\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + \langle w_r, w_1\rangle_W + B_W(u_1, w_1 + w_r) + f(w_2) \\ &+ \langle w_s, w_2\rangle_W + B_W(u_2, w_2 + w_s) + B_W(u_1 + w_r, u_2 + w_s)) \end{aligned} \tag{121}$$

where $C_{r,s} = \gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1}$ and $w_{r,s} = w_{rs} - w_r - w_s$. Since $w_{r,s} \in \mathrm{I}(\sigma_v) + \mathrm{I}(\sigma)$ and this sum is direct (cf. Lemma 2.20.v), there is only one way to decompose $w_{r,s}$ in $\mathrm{I}(\sigma_v) + \mathrm{I}(\sigma)$. Hence the sum in the expression (121) of the cocycle contains only one term, which is an element of $\mu_4(\mathbb{C})$. Moreover, since $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma_v) \oplus \mathrm{I}(\sigma)$ by Lemma 2.20.v,

$$\begin{aligned} C_{r,s} = \gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1} &= 2^{-\frac{1}{2}} e^{i\theta(\sigma_v)} 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)} 2^{\frac{\dim \mathrm{I}(\sigma_v\sigma)}{2}} e^{-i\theta(\sigma_v\sigma)} \\ &= e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}. \end{aligned}$$

Hence, since B_W takes its values in $2\mathbb{Z}/4\mathbb{Z}$ and $\chi_k(2\mathbb{Z}/4\mathbb{Z}) \subseteq \{\pm 1\}$,

$$c_{\chi_k}(r,s) = \pm e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))} \chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + f(w_2)), \tag{122}$$

where $u_1 \in \mathrm{I}(\sigma_v)$, $u_2 \in \mathrm{I}(\sigma)$, $u_1 + u_2 = w_{r,s}$, $(1-\sigma_v)(w_1) = u_1$ and $(1-\sigma)(w_2) = u_2$. This implies the two following conditions:

$$\begin{cases} c_{\chi_k}(r,s) \in \mu_8(\mathbb{C}) & \iff \quad \theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{4} \\ c_{\chi_k}(r,s) \in \mu_4(\mathbb{C}) & \iff \quad \theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{2}. \end{cases} \tag{123}$$

- Now we study $c_{\chi_k}(r,s)$ under the hypothesis that $v \neq 0$ (if $v = 0$ then $\sigma_v = 1$ and we have already treated this case above) belongs to $\mathrm{I}(\sigma)$ and more precisely that $v = (1-\sigma)(z)$ with $\langle z, v\rangle \neq 1$. Hence $\langle z, v\rangle = 0$ since $\langle \cdot, \cdot \rangle$ takes its values in $\mathbb{F}_2$. Then $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma)$ (cf. Lemma 2.20.v). Moreover, by (93), for every $w = \widetilde{w} + 2\widetilde{W}$,

$$\begin{aligned} f_{\widetilde{\sigma}_{\widetilde{v}}}(w) &= B_{\widetilde{W}}(\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}), \widetilde{\sigma}_{\widetilde{v}}(\widetilde{w})) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) \\ &= B_{\widetilde{W}}(\widetilde{w} + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} \widetilde{v}, \widetilde{w} + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} \widetilde{v}) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) \\ &= \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} (B_{\widetilde{W}}(\widetilde{w}, \widetilde{v}) + B_{\widetilde{W}}(\widetilde{v}, \widetilde{w})) + \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}}^2 B_{\widetilde{W}}(\widetilde{v}, \widetilde{v}) \\ &= \langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}}^2 (1 + B_{\widetilde{W}}(\widetilde{v}, \widetilde{v})) + 2\langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{v}, \widetilde{w}). \end{aligned} \tag{124}$$

If $w \in \mathrm{K}(\sigma_v)$ then $\langle w, v\rangle = 0$, which implies that $2\langle \widetilde{w}, \widetilde{v}\rangle_{\widetilde{W}} = 0$ and thus $f_{\widetilde{\sigma}_{\widetilde{v}}}(w) = 0$. This shows that

$$f_{\widetilde{\sigma}_{\widetilde{v}} | \mathrm{K}(\sigma_v)} = 0. \tag{125}$$

So we can suppose that $w_r = 0$ because for any other choice of $w_r \neq 0$, $\omega_{\widetilde{\chi_k}}(r, w_r)$ and $\omega_{\widetilde{\chi_k}}(r,0)$ differ only by a fourth root of unity by Proposition 2.21.iv. This simplification is possible because now we are only concerned in finding a necessary and sufficient condition for c_{χ_k} to take its values in a set which is not smaller than $\mu_4(\mathbb{C})$ (which in fact will turn out to be the optimal range of this cocycle).

As we are going to see, we can also suppose that $f_{|\mathrm{K}(\sigma)} = 0$ and thus choose $w_s = 0$. In fact, $f_{|\mathrm{K}(\sigma)} = \langle w_s, \cdot \rangle_{W|\mathrm{K}(\sigma)}$ by (100). Set $\eta = (1, -\langle w_s, \cdot\rangle_W) \in \mathrm{ASp}_a(W)$. Now

$$c_{\chi_k}(r,s) c_{\chi_k}(rs, \eta) = c_{\chi_k}(r, s\eta) c_{\chi_k}(s, \eta),$$

where $c_{\chi_k}(rs, \eta) \in \mu_4(\mathbb{C})$ and $c_{\chi_k}(s, \eta) \in \mu_4(\mathbb{C})$ by a similar argument to the one proving (120). Hence, to show that $c_{\chi_k}(r,s)$ belongs to $\mu_8(\mathbb{C})$ and $\mu_4(\mathbb{C})$, respectively, it is enough

to show that $c_{\chi_k}(r,s\eta)$ belongs to the same set. Since $s\eta = (\sigma, -\langle w_s,\cdot\rangle_W + f)$ and $-\langle w_s,\cdot\rangle_W + f$ vanishes over $\mathrm{K}(\sigma)$, we can then suppose that $f_{|\,\mathrm{K}(\sigma)} = 0$ and thus $w_s = 0$. So, from now on, we assume that $w_r = w_s = 0$. Observe that $w_{r,s} = w_{rs} - w_r - w_s = w_{rs}$ belongs to $\mathrm{I}(\sigma_v) + \mathrm{I}(\sigma) = \mathrm{I}(\sigma)$ (since $v \in \mathrm{I}(\sigma)$, and $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma)$ by Lemma 2.20.v. Hence, using the computations made in the proof of Corollary 2.24, we can set $u = w_{rs}$ in (111). Since $\mathrm{I}(\sigma_v) = \{0, v\}$, this gives

$$\begin{aligned}
&c_{\chi_k}(r,s)\chi_k((f + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma)(w) + \langle w_{rs}, w\rangle_W + B_W(w_{rs}, w + w_{rs}))\\
&= C_{r,s} \sum_{\substack{u_1\in \mathrm{I}(\sigma_v), u_2 \in \mathrm{I}(\sigma)\\ u_1+u_2=0}} \chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + B_W(u_1,w_1) + f(w_2) + B_W(u_2,w_2) + B_W(u_1,u_2))\\
&= C_{r,s}(1 + \chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + B_W(v,w_1) + f(z) + B_W(v,z) + B_W(v,v))),
\end{aligned}$$

where $C_{r,s} = \gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1}$, $(1-\sigma_v\sigma)(w) = w_{rs}$ and $(1-\sigma_v)(w_1) = v$. (Notice that we can choose $w_2 = z$ since by definition $(1-\sigma)(z) = v$.) Since $\chi_k(2\mathbb{Z}/4\mathbb{Z}) \subseteq \{\pm 1\}$,

$$c_{\chi_k}(r,s)\chi_k((f+f_{\widetilde{\sigma}_{\widetilde{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(w_{rs},w+w_{rs})) = C_{r,s}(1\pm\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1)+f(z))).$$

Now we examine $f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + f(z)$.

- Let $\widetilde{w}_1 \in \widetilde{W}$ be such that $w_1 = \widetilde{w}_1 + 2\widetilde{W}$. Then by (124),

  $$f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) = \langle \widetilde{w}_1, \widetilde{v}\rangle^2_{\widetilde{W}}(1 + B_{\widetilde{W}}(\widetilde{v},\widetilde{v})) + 2\langle \widetilde{w}_1, \widetilde{v}\rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{v},\widetilde{w}_1).$$

  Since $(1-\sigma_v)(w_1) = v$, we have that $\langle w_1, v\rangle = -1$, i.e. $\langle w_1, v\rangle_W = 2$, which implies that $\langle \widetilde{w}_1, \widetilde{v}\rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}$ and in particular $\langle \widetilde{w}_1, \widetilde{v}\rangle^2_{\widetilde{W}} = 1$. This shows that

  $$f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) = 1 + B_{\widetilde{W}}(\widetilde{v},\widetilde{v}) + B_W(v, w_1). \tag{126}$$

  Thus $f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) \in 2\mathbb{Z}/4\mathbb{Z}$ if and only if $B_{\widetilde{W}}(\widetilde{v},\widetilde{v}) \notin 2\mathbb{Z}/4\mathbb{Z}$, i.e. $Q(v) = B(v,v) = 1$.
- By (82),

  $$0 = f(2z) = 2f(z) + Q_W(\sigma(z)) - Q_W(z).$$

  Since $(1-\sigma)(z) = v$ and $\langle z, v\rangle = 0$ by hypothesis, this yields

  $$2f(z) = Q_W(z) - Q_W(z-v) = -Q_W(v) + \langle z, v\rangle_W = -Q_W(v).$$

  Since $f(z) \in 2\mathbb{Z}/4\mathbb{Z}$ is equivalent to $2f(z) = 0$, this shows that $f(z) \in 2\mathbb{Z}/4\mathbb{Z}$ if and only if $Q_W(v) = 0$, i.e. $Q(v) = B(v,v) = 0$.

Then in any case ($Q(v) = 0$ or $Q(v) = 1$), the sum $f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + f(z)$ does not belong to $2\mathbb{Z}/4\mathbb{Z}$, which implies that $\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + f(z)) \in \{\pm i\}$. Hence

$$c_{\chi_k}(r,s)\chi_k((f + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma)(w) + \langle w_{rs}, w\rangle_W + B_W(w_{rs}, w + w_{rs})) = C_{r,s}(1 \pm i). \tag{127}$$

Since $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma)$ by Lemma 2.20.v),

$$\begin{aligned}
C_{r,s} = \gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1} &= 2^{-\frac{1}{2}} e^{i\theta(\sigma_v)} 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)} 2^{\frac{\dim \mathrm{I}(\sigma_v\sigma)}{2}} e^{-i\theta(\sigma_v\sigma)}\\
&= 2^{-\frac{1}{2}} e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}.
\end{aligned}$$

Hence

$$c_{\chi_k}(r,s)\chi_k((f+f_{\widetilde{\sigma}_{\widetilde{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(w_{rs},w+w_{rs}))=\frac{(1\pm i)}{\sqrt{2}}e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}.$$

Thus we obtain the following two conditions:

$$\begin{cases} c_{\chi_k}(r,s)\in\mu_8(\mathbb{C}) & \iff \quad \theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma)\equiv 0 \mod \frac{\pi}{4} \\ c_{\chi_k}(r,s)\in\mu_4(\mathbb{C}) & \iff \quad \theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma)\equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \end{cases} \tag{128}$$

- Finally we study $c_{\chi_k}(r,s)$ under the hypothesis that v belongs to $\mathrm{I}(\sigma)$, and more precisely that $v=(1-\sigma)(z)$ with $\langle z,v\rangle=1$. In particular, since $(1-\sigma_v)(z)=-\langle z,v\rangle v$, we also have that $v=(1-\sigma_v)(z)$. As in the previous case for which $\langle z,v\rangle=0$, we can suppose that $w_r=w_s=0$. Moreover, $\mathrm{I}(\sigma_v\sigma)=\mathrm{I}(\sigma)\cap\mathrm{Span}\{z\}^\perp$ (cf. Lemma 2.20.v) and $w_{r,s}=w_{rs}-w_r-w_s=w_{rs}$ belongs to $\mathrm{I}(\sigma_v)+\mathrm{I}(\sigma)=\mathrm{I}(\sigma)$. Now we distinguish between two cases: $w_{rs}\in\mathrm{I}(\sigma_v\sigma)$ and $w_{rs}\notin\mathrm{I}(\sigma_v\sigma)$.

  - Suppose first that $w_{rs}\in\mathrm{I}(\sigma_v\sigma)$. Then we can set $u=w_{rs}$ in (111) and obtain

$$\begin{aligned} &c_{\chi_k}(r,s)\chi_k((f+f_{\widetilde{\sigma}_{\widetilde{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(w_{rs},w+w_{rs})) \\ &=C_{r,s}\sum_{\substack{u_1\in\mathrm{I}(\sigma_v),u_2\in\mathrm{I}(\sigma)\\ u_1+u_2=0}}\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1)+B_W(u_1,w_1)+f(w_2)+B_W(u_2,w_2)+B_W(u_1,u_2)) \\ &=C_{r,s}(1+\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(z)+B_W(v,z)+f(z)+B_W(v,z)+B_W(v,v))) \\ &=C_{r,s}(1+\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(z)+f(z)+B_W(v,v))), \end{aligned}$$

    where $C_{r,s}=\gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1}$, $(1-\sigma_v\sigma)(w)=w_{rs}$ and we already proved that $w_1=w_2=z$ is a valid choice to have $(1-\sigma_v)(w_1)=v$ and $(1-\sigma)(w_2)=v$. Since $\chi_k(2\mathbb{Z}/4\mathbb{Z})\subseteq\{\pm1\}$, we can write

$$c_{\chi_k}(r,s)\chi_k((f+f_{\widetilde{\sigma}_{\widetilde{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(w_{rs},w+w_{rs}))=C_{r,s}(1\pm\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(z)+f(z))).$$

    Now we examine in detail the sum $f_{\widetilde{\sigma}_{\widetilde{v}}}(z)+f(z)$.

    - The same reasoning done to obtain (126) can be repeated here by replacing w_1 by z. This leads to

$$f_{\widetilde{\sigma}_{\widetilde{v}}}(z)=1+B_{\widetilde{W}}(\widetilde{v},\widetilde{v})+B_W(v,z).$$

      Hence $f_{\widetilde{\sigma}_{\widetilde{v}}}(z)\in2\mathbb{Z}/4\mathbb{Z}$ if and only if $B_{\widetilde{W}}(\widetilde{v},\widetilde{v})\notin2\mathbb{Z}/4\mathbb{Z}$, i.e. $Q(v)=B(v,v)=1$.

    - By (82),

$$0=f(2z)=2f(z)+Q_W(\sigma(z))-Q_W(z).$$

      Since $(1-\sigma)(z)=v$ and $\langle z,v\rangle=1$ by hypothesis, this gives

$$2f(z)=Q_W(z)-Q_W(z-v)=-Q_W(v)+\langle z,v\rangle_W=2-Q_W(v).$$

      Since $f(z)\in2\mathbb{Z}/4\mathbb{Z}$ is equivalent to $2f(z)=0$, this shows that $f(z)\in2\mathbb{Z}/4\mathbb{Z}$ if and only if $Q_W(v)=2$, i.e. $Q(v)=B(v,v)=1$.

Thus in any case ($Q(v) = 0$ or $Q(v) = 1$), the sum $f_{\tilde{\sigma}_{\hat{v}}}(z) + f(z)$ belongs to $2\mathbb{Z}/4\mathbb{Z}$, which implies that $\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z) + f(z)) \in \{\pm 1\}$. Hence

$$c_{\chi_k}(r,s)\chi_k((f+f_{\tilde{\sigma}_{\hat{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(w_{rs},w+w_{rs})) = C_{r,s}(1\pm 1) = 2C_{r,s}, \tag{129}$$

where the last equality comes from the fact that c_{χ_k} does not vanish. Moreover, since $\mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp$ by Lemma 2.20.v and $z \notin \mathrm{K}(\sigma)$,

$$\begin{aligned} C_{r,s} = \gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1} &= 2^{-\frac{1}{2}}e^{i\theta(\sigma_v)}2^{-\frac{\dim \mathrm{I}(\sigma)}{2}}e^{i\theta(\sigma)}2^{\frac{\dim \mathrm{I}(\sigma_v\sigma)}{2}}e^{-i\theta(\sigma_v\sigma)} \\ &= 2^{-\frac{1}{2}}e^{i\theta(\sigma_v)}2^{-\frac{\dim \mathrm{I}(\sigma)}{2}}e^{i\theta(\sigma)}2^{\frac{\dim \mathrm{I}(\sigma)-1}{2}}e^{-i\theta(\sigma_v\sigma)} \\ &= 2^{-1}e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}. \end{aligned} \tag{130}$$

This implies that

$$c_{\chi_k}(r,s)\chi_k((f+f_{\tilde{\sigma}_{\hat{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(w_{rs},w+w_{rs})) = e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}. \tag{131}$$

Thus we have the following two conditions:

$$\begin{cases} c_{\chi_k}(r,s) \in \mu_8(\mathbb{C}) & \iff \quad \theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{4} \\ c_{\chi_k}(r,s) \in \mu_4(\mathbb{C}) & \iff \quad \theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{2}. \end{cases} \tag{132}$$

- Suppose now that $w_{rs} \notin \mathrm{I}(\sigma_v\sigma)$. We can decompose W as the direct sum $W = \mathrm{Span}\{v\} \oplus \mathrm{Span}\{z\}^\perp$. In fact, $\langle z, v\rangle = 1$ implies that the intersection of these two subspaces is trivial and a dimension argument is enough to conclude. Hence there exists $\lambda \in \mathbb{F}_2$ and $u_{rs} \in \mathrm{Span}\{z\}^\perp$ such that $w_{rs} = \lambda v + u_{rs}$. Since w_{rs} belongs to $\mathrm{I}(\sigma)$ and $w_{rs} \notin \mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^\perp$ by hypothesis, we know that $\lambda \neq 0$. Therefore $\lambda = 1$ since λ is in $\mathbb{F}_2$. Hence $w_{rs} = v + u_{rs}$. Now we can set $u = -u_{rs}$ in (111) in order to obtain

$$\begin{aligned} &c_{\chi_k}(r,s)\chi_k((f+f_{\tilde{\sigma}_{\hat{v}}}\circ\sigma)(w)+\langle w_{rs},w\rangle_W+B_W(-u_{rs},w+w_{rs})) \\ &= C_{r,s}\sum_{\substack{u_1\in\mathrm{I}(\sigma_v),u_2\in\mathrm{I}(\sigma)\\ u_1+u_2=-u_{rs}+w_{rs}=v}} \chi_k(f_{\tilde{\sigma}_{\hat{v}}}(w_1)+B_W(u_1,w_1)+f(w_2)+B_W(u_2,w_2)+B_W(u_1,u_2)) \\ &= C_{r,s}(\chi_k(f(z)+B_W(v,z))+\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z)+B_W(v,z))) \end{aligned}$$

where $C_{r,s} = \gamma(\sigma_v)\gamma(\sigma)\gamma(\sigma_v\sigma)^{-1}$, $(1-\sigma_v\sigma)(w) = -u_{rs}$, and we already proved that z can be chosen as w_1 such that $u_1 = v = (1-\sigma_v)(w_1)$ and as w_2 such that $u_2 = v = (1-\sigma)(w_2)$. Since $\chi_k(2\mathbb{Z}/4\mathbb{Z}) \subseteq \{\pm 1\}$,

$$c_{\chi_k}(r,s)\chi_k((f+f_{\tilde{\sigma}_{\hat{v}}}\circ\sigma)(w)) = \pm C_{r,s}(\chi_k(f(z))+\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z))).$$

In the previous case, where $w_{rs} \in \mathrm{I}(\sigma_v\sigma)$, we proved that $f(z) \in 2\mathbb{Z}/4\mathbb{Z}$ if and only if $f_{\tilde{\sigma}_{\hat{v}}}(z) \in 2\mathbb{Z}/4\mathbb{Z}$ without using the hypothesis that $w_{rs} \in \mathrm{I}(\sigma_v\sigma)$. So we can use this result here and conclude that $\chi_k(f(z)) = \pm\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z))$. This implies that

$$\begin{aligned} c_{\chi_k}(r,s)\chi_k((f+f_{\tilde{\sigma}_{\hat{v}}}\circ\sigma)(w)) &= \pm C_{r,s}(\pm\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z))+\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z))) \\ &= \pm 2C_{r,s}\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z)) \\ &= \pm e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}\chi_k(f_{\tilde{\sigma}_{\hat{v}}}(z)), \end{aligned} \tag{133}$$

where the last two equalities come from the fact that c_{χ_k} does not vanish and from (130), which is also valid here. Hence

$$c_{\chi_k}(r,s) = \pm e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}\chi_k(f_{\tilde{\sigma}_{\tilde{v}}}(z) - (f + f_{\tilde{\sigma}_{\tilde{v}}} \circ \sigma)(w)). \tag{134}$$

This gives the following two conditions:

$$\begin{cases} c_{\chi_k}(r,s) \in \mu_8(\mathbb{C}) & \iff \quad \theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{4} \\ c_{\chi_k}(r,s) \in \mu_4(\mathbb{C}) & \iff \quad \theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{2}. \end{cases} \tag{135}$$

Combining equations (123), (128), (132) and (135), we obtain the following two results:

- c_{χ_k} takes its values in $\mu_8(\mathbb{C})$ if and only if $\theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{4}$ for every $\sigma \in \mathrm{Sp}(W)$ and every transvection $\sigma_v \in \mathrm{Sp}(W)$. This is equivalent to $\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \frac{\pi}{4}$ for all $\sigma_1, \sigma_2 \in \mathrm{Sp}(W)$. In fact, since the transvections generate $\mathrm{Sp}(W)$ (cf. [OMe78, Th. 2.1.9]), we can decompose $\sigma_1 = \sigma_{v_1} \cdots \sigma_{v_q}$ with $q \geqslant 1$. Supposing that $\theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \frac{\pi}{4}$ for every $\sigma \in \mathrm{Sp}(W)$ and every transvection $\sigma_v \in \mathrm{Sp}(W)$, we can write

$$\begin{aligned} \theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) &\equiv \theta(\sigma_{v_1} \cdots \sigma_{v_q}) + \theta(\sigma_2) - \theta(\sigma_{v_1} \cdots \sigma_{v_q}\sigma_2) \mod \frac{\pi}{4} \\ &\equiv \sum_{j=1}^{q} \theta(\sigma_{v_j}) + \theta(\sigma_2) - \left(\sum_{j=1}^{q} \theta(\sigma_{v_j}) + \theta(\sigma_2) \right) \mod \frac{\pi}{4} \\ &\equiv 0 \mod \frac{\pi}{4}. \end{aligned} \tag{136}$$

This proves the equivalence (115) announced in i:

c_{χ_k} takes its values in $\mu_8(\mathbb{C}) \iff \theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \frac{\pi}{4}$ for all $\sigma_1, \sigma_2 \in \mathrm{Sp}(W)$.

This condition holds for example if $\gamma(\sigma)$ is chosen to be real. (This choice may seem more natural at first glance since it is related to the choice of $\gamma(\sigma)$ made in Corollary 1.35; more details about this are given in Proposition 2.39 below.)

- c_{χ_k} takes its values in $\mu_4(\mathbb{C})$ if and only if $\theta(1) \equiv 0 \mod \frac{\pi}{2}$ and

$$\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma) \equiv \begin{cases} 0 \mod \frac{\pi}{2} & \text{if } v \notin \mathrm{I}(\sigma) \text{ or } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 1 \\ \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 0 \end{cases} \tag{137}$$

for every $\sigma \in \mathrm{Sp}(W)$ and every transvection $\sigma_v \in \mathrm{Sp}(W)$ such that $v \neq 0$. This proves the equivalence (116) announced in ii. In particular, the condition (137) is satisfied if we set $\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{-\frac{i\pi \dim \mathrm{I}(\sigma)}{4}} = (1+i)^{-\dim \mathrm{I}(\sigma)}$ for every $\sigma \in \mathrm{Sp}(W)$. In fact, we have in this case $\theta(1) = 0$ and by Lemma 2.20.v,

$$\begin{aligned} &\theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \\ &= -\frac{\pi}{4}(\dim \mathrm{I}(\sigma_v) + \dim \mathrm{I}(\sigma) - \dim \mathrm{I}(\sigma_v\sigma)) \\ &= \begin{cases} 0 & \text{if } v \notin \mathrm{I}(\sigma), \text{ since } \mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma_v) \oplus \mathrm{I}(\sigma) \\ -\frac{\pi}{4} \dim \mathrm{I}(\sigma_v) = -\frac{\pi}{4} & \text{if } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 0, \text{ since } \mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma) \\ -\frac{\pi}{4}(\dim \mathrm{I}(\sigma_v) + 1) = -\frac{\pi}{2} & \text{if } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 1, \text{ since } \mathrm{I}(\sigma_v\sigma) = \mathrm{I}(\sigma) \cap \mathrm{Span}\{z\}^{\perp}. \end{cases} \end{aligned}$$

We now consider the case $\dim_{\mathbb{F}_2}(W) = 2$. Let $\sigma \in \mathrm{Sp}(W)$ and $v \in W\backslash\{0\}$ be such that $v \in \mathrm{I}(\sigma)$, i.e. there exists $z \in W$ such that $v = (1-\sigma)(z)$. Since $\dim_{\mathbb{F}_2}(W) = 2$, the only nonzero element of W which is orthogonal to v with respect to the symplectic form $\langle \cdot, \cdot \rangle$ is v itself. This means that if $\langle z, v \rangle = 0$ then $z = v$. Hence $v = (1-\sigma)(z) = (1-\sigma)(v) = 0$, which is impossible by hypothesis. Thus, if $v \in \mathrm{I}(\sigma)$ then $\langle z, v \rangle = 1$, i.e. the second case in (137) does not occur. The same reasoning as to prove (136) enables us to rewrite (137) for a two-dimensional space W as follows: c_{χ_k} takes its values in $\mu_4(\mathbb{C})$ if and only if

$$\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \frac{\pi}{2} \qquad \text{for all } \sigma_1, \sigma_2 \in \mathrm{Sp}(W).$$

Finally, it remains to prove iii and iv. Suppose first that the Witt index of Q is zero, in which case $\dim_{\mathbb{F}_2}(W) = 2$. According to Proposition 1.13, this is the only case in which $\mathrm{Sp}(W) = \mathrm{O}(Q)$. In particular, the right member of (82) vanishes for $w_1 = w_2$: for every $w \in W$,

$$f(2w) - 2f(w) = Q_W(\sigma(w)) - Q_W(w) = 0. \tag{138}$$

Hence every element $(\sigma, f) \in \mathrm{ASp}(B_W)$ is such that f takes its values in $2\mathbb{Z}/4\mathbb{Z}$. By (120), this implies that $c_{\chi_k}(r_a, s) = \pm 1$ for all $a \in W$ and $s \in \mathrm{ASp}(B_W)$. Now let $v \in W\backslash\{0\}$, $r = (\sigma_v, f_{\widetilde{\sigma_{\widetilde{v}}}})$ and $s \in \mathrm{ASp}(B_W)$. If $v \notin \mathrm{I}(\sigma)$ then by (122),

$$c_{\chi_k}(r, s) = \pm e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}\chi_k(f_{\widetilde{\sigma_{\widetilde{v}}}}(w_1) + f(w_2)) = \pm e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))},$$

where $u_1 \in \mathrm{I}(\sigma_v)$, $u_2 \in \mathrm{I}(\sigma)$, $u_1 + u_2 = w_{r,s}$, $(1-\sigma_v)(w_1) = u_1$ and $(1-\sigma)(w_2) = u_2$. Therefore $c_{\chi_k}(r, s) \in \mu_2(\mathbb{C})$ if and only if $\theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \pi$. Now if $v \in \mathrm{I}(\sigma)$ then there exists $z \in W$ such that $v = (1-\sigma)(z)$, and this implies that $\langle z, v \rangle = 1$. In fact, there are only four elements in W and $\mathrm{Span}\{v\}^\perp = \{0, v\}$. We already proved that $\langle z, v \rangle = 0$ would imply $v = 0$, which is impossible by assumption. Then by (131) and (134),

$$c_{\chi_k}(r, s) = \pm e^{i(\theta(\sigma_v)+\theta(\sigma)-\theta(\sigma_v\sigma))}.$$

Thus $c_{\chi_k}(r, s) \in \mu_2(\mathbb{C})$ if and only if $\theta(\sigma_v) + \theta(\sigma) - \theta(\sigma_v\sigma) \equiv 0 \mod \pi$. Using the same reasoning as to prove (136), we obtain iii:

$$c_{\chi_k} \text{ takes its values in } \mu_2(\mathbb{C}) \iff \theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) \equiv 0 \mod \pi \text{ for all } \sigma_1, \sigma_2 \in \mathrm{Sp}(W).$$

In order to prove iv, we are left to find the smallest possible range of the cocycle c_{χ_k} . Recall that by Corollary 2.26, for every choice of $\{w_s$ satisfying (100), $s \in \mathrm{ASp}(B_W)\}$ and $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$, the cocycle c_{χ_k} is not a coboundary. This implies that the projective representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ is not a linear representation, hence the range of the cocycle is at least $\mu_2(\mathbb{C})$. By iii, we know that $\mu_2(\mathbb{C})$ can actually be obtained if the Witt index of Q is zero (in which case W is two-dimensional). So $\mu_2(\mathbb{C})$ is the smallest possible range of c_{χ_k} in this case, and we can suppose from now on that the Witt index of Q is nonzero. Furthermore, we know by ii that the smallest possible range of c_{χ_k} is smaller than $\mu_4(\mathbb{C})$. So we can suppose that c_{χ_k} takes its values in $\mu_4(\mathbb{C})$, in which case (116) is satisfied. Since the Witt index of Q is nonzero by hypothesis, there exists $v \in W\backslash\{0\}$ such that $Q(v) = 0$. Let $\widetilde{v} \in \widetilde{W}$ be such that $v = \widetilde{v} + 2\widetilde{W}$.

Since $\langle \cdot, \cdot \rangle$ is nondegenerate, there exists $a \in W$ such that $\langle a, v \rangle = 1$. Set $s_1 = (\sigma_v, f_{\widetilde{\sigma}_{\widetilde{v}}})$ and $s_2 = (\sigma_v, f_{\widetilde{\sigma}_{\widetilde{v}}} + \langle a, \cdot \rangle_W)$. Recall from (124) that for every $w \in W$,

$$f_{\widetilde{\sigma}_{\widetilde{v}}}(w) = \langle \widetilde{w}, \widetilde{v} \rangle^2_{\widetilde{W}}(1 + B_{\widetilde{W}}(\widetilde{v}, \widetilde{v})) + 2\langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} B_{\widetilde{W}}(\widetilde{v}, \widetilde{w}). \tag{139}$$

Since $\sigma_v^2 = 1$, we have that $s_1^2 = (1, f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v)$, where $f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Hence, by Proposition 2.7.i, there exists $a_{\widetilde{v}} \in W$ such that $f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v = \langle a_{\widetilde{v}}, \cdot \rangle_W$. Since $f_{\widetilde{\sigma}_{\widetilde{v}}|\,\mathrm{K}(\sigma_v)} = 0$ by (125), we have that $(f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v)_{|\,\mathrm{K}(\sigma_v)} = 0$, which implies that $a_{\widetilde{v}} \in \mathrm{K}(\sigma_v)^\perp = \mathrm{I}(\sigma_v) = \{0, v\}$. Hence either $f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v = 0$ or $f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v = \langle v, \cdot \rangle_W$. Since $\langle a, v \rangle = 1$ by hypothesis, then $\sigma_v(a) = a + v$. Hence, since $f_{\widetilde{\sigma}_{\widetilde{v}}}(v) = 0$ by (139) and since $Q(v) = 0$,

$$\begin{aligned}
(f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v)(a) = f_{\widetilde{\sigma}_{\widetilde{v}}}(a) + f_{\widetilde{\sigma}_{\widetilde{v}}}(a + v) &\overset{(82)}{=} 2f_{\widetilde{\sigma}_{\widetilde{v}}}(a) + f_{\widetilde{\sigma}_{\widetilde{v}}}(v) + B_W(\sigma_v(a), \sigma_v(v)) - B_W(a, v) \\
&= 2f_{\widetilde{\sigma}_{\widetilde{v}}}(a) + Q_W(v) \\
&= 2f_{\widetilde{\sigma}_{\widetilde{v}}}(a) \\
&\overset{(139)}{=} 2(1 + B_{\widetilde{W}}(\widetilde{v}, \widetilde{v})) \\
&= 2 + Q_W(v) \\
&= 2.
\end{aligned} \tag{140}$$

This shows that $(f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v)(a) = 2$ is nonzero, and therefore $f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v = \langle v, \cdot \rangle_W$. Hence

$$\begin{cases}
s_1^2 = (1, f_{\widetilde{\sigma}_{\widetilde{v}}} + f_{\widetilde{\sigma}_{\widetilde{v}}} \circ \sigma_v) = (1, \langle v, \cdot \rangle_W) & \\
s_2^2 = (1, f_{\widetilde{\sigma}_{\widetilde{v}}} + \langle a, \cdot \rangle_W + (f_{\widetilde{\sigma}_{\widetilde{v}}} + \langle a, \cdot \rangle_W) \circ \sigma_v) & = (1, \langle v, \cdot \rangle_W + \langle a, \cdot \rangle_W + \langle a, \cdot \rangle_W \circ \sigma_v) \\
& = (1, \langle v + a + \sigma_v^{-1}(a), \cdot \rangle_W) = (1, 0).
\end{cases}$$

Thus, Corollary 2.26 implies that $w_{s_1^2} = v$ and $w_{s_2^2} = 0$. Moreover, by Lemma 2.20.iv, we have that $w_{s_1, s_1} = w_{s_1^2} - w_{s_1} - w_{s_1} = w_{s_1^2} = v$, and similarly $w_{s_2, s_2} = w_{s_2^2} - w_{s_2} - w_{s_2} = w_{s_2^2} = 0$. Now using the formula (109) that gives the expression of the cocycle, and the fact that $\chi_k(2\mathbb{Z}/4\mathbb{Z}) = \{\pm 1\}$, we have for $j \in \{1, 2\}$:

$$c_{\chi_k}(s_j, s_j) = \gamma(\sigma_v)^2 \gamma(1)^{-1} \sum_{\substack{u_1, u_2 \in \mathrm{I}(\sigma_v) \\ u_1 + u_2 = w_{s_j, s_j}}} \pm \chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(w_1) + f_{\widetilde{\sigma}_{\widetilde{v}}}(w_2))$$

where $w_1, w_2 \in W$ are such that $(1 - \sigma_v)(w_1) = u_1$ and $(1 - \sigma_v)(w_2) = u_2$. Since $\mathrm{I}(\sigma_v) = \{0, v\}$, $w_{s_1, s_1} = v$ and $w_{s_2, s_2} = 0$, then there are two terms in the sum: if $j = 1$ then $(u_1, u_2) = (0, v)$ or $(u_1, u_2) = (v, 0)$, and if $j = 2$ then $(u_1, u_2) = (0, 0)$ or $(u_1, u_2) = (v, v)$. For $l \in \{1, 2\}$, we take $w_l = 0$ if $u_l = 0$ and $w_l = a$ if $u_l = v$, which is a valid choice since $\langle a, v \rangle = 1$. Then

$$\begin{cases}
c_{\chi_k}(s_1, s_1) = \gamma(\sigma_v)^2 \gamma(1)^{-1} (\pm \chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(0) + f_{\widetilde{\sigma}_{\widetilde{v}}}(a)) \pm \chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(a) + f_{\widetilde{\sigma}_{\widetilde{v}}}(0))) \\
c_{\chi_k}(s_2, s_2) = \gamma(\sigma_v)^2 \gamma(1)^{-1} (\pm \chi_k(2f_{\widetilde{\sigma}_{\widetilde{v}}}(0)) \pm \chi_k(2f_{\widetilde{\sigma}_{\widetilde{v}}}(a))).
\end{cases}$$

We proved in (140) that $2f_{\widetilde{\sigma}_{\widetilde{v}}}(a) = 2$. This implies that $f_{\widetilde{\sigma}_{\widetilde{v}}}(a) \notin 2\mathbb{Z}/4\mathbb{Z}$, and thus $\chi_k(f_{\widetilde{\sigma}_{\widetilde{v}}}(a)) = \pm i$. Using also the fact that the cocycle does not vanish, we obtain

$$\begin{cases}
c_{\chi_k}(s_1, s_1) = \pm 2i \gamma(\sigma_v)^2 \gamma(1)^{-1} \\
c_{\chi_k}(s_2, s_2) = \pm 2 \gamma(\sigma_v)^2 \gamma(1)^{-1}.
\end{cases} \tag{141}$$

Now the condition (116) implies that $2\theta(\sigma_v) - \theta(1) \equiv 0 \mod \frac{\pi}{2}$. Then

$$\gamma(\sigma_v)^2\gamma(1)^{-1} = 2^{-\dim \mathrm{I}(\sigma_v)} 2^{\frac{\dim(\mathrm{I}(1))}{2}} e^{i(2\theta(\sigma_v)-\theta(1))} = 2^{-1} e^{i(2\theta(\sigma_v)-\theta(1))},$$

where $2\theta(\sigma_v) - \theta(1) \equiv 0 \mod \frac{\pi}{2}$. By (141),

- if $2\theta(\sigma_v) - \theta(1) \equiv 0 \mod \pi$ then $c_{\chi_k}(s_1, s_1) = \pm i$ and $c_{\chi_k}(s_2, s_2) = \pm 1$,
- if $2\theta(\sigma_v) - \theta(1) \equiv \frac{\pi}{2} \mod \pi$ then $c_{\chi_k}(s_1, s_1) = \pm 1$ and $c_{\chi_k}(s_2, s_2) = \pm i$.

This concludes the proof by showing that, independently of the choice of $\gamma(\sigma_v)$, the smallest possible range of c_{χ_k} is $\mu_4(\mathbb{C})$ if the Witt index of Q is nonzero. □

Now we can proceed to the construction of coverings of $\mathrm{ASp}(B_W)$ that correspond to the possible ranges of c_{χ_k} stated in Corollary 2.28. Regardless of the choice of $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$, we proved that the smallest possible central extension of $\mathrm{ASp}(B_W)$ on which $s \mapsto \omega_{\widetilde{\chi_k}}(s)$ lifts to a linear representation is an extension by two elements if the Witt index of Q is zero and by four elements if the Witt index of Q is nonzero. We construct these extensions in Proposition 2.29. Unfortunately, if the Witt index of Q is greater than one, in spite of its optimal size, the extension by four elements is too small to be related (in a sense that we will make precise in Proposition 2.39) to the pseudosymplectic extension $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ defined in Proposition 1.36, hence to Weil's work. This is why we also construct an additional extension of $\mathrm{ASp}(B_W)$ by eight elements.

Definition-Proposition 2.29 (Coverings of $\mathrm{ASp}(B_W)$). *We keep the notations and assumptions from Lemma 2.20 and Proposition 2.21. In particular, we suppose that a choice of* w_s *satisfying* (100) *has been made for every* $s \in \mathrm{ASp}(B_W)$*. We fix also for every* $\sigma \in \mathrm{Sp}(W)$ *the constant* $\gamma(\sigma)$*, defined in Proposition 2.21, such that* (112) *is satisfied. We define the following sets* $\mathrm{G}^0_{\widetilde{\chi_k}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *and* $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$*:*

$$\begin{cases} \mathrm{G}^0_{\widetilde{\chi_k}} & = \{(s, (-1)^l \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\} \\ \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} & = \{(s, i^l \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\} \\ \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}} & = \{(s, e^{\frac{il\pi}{4}} \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}. \end{cases}$$

We use the notation $\mathrm{G}_{\widetilde{\chi_k}}$ *to refer to each of these sets, provided additional conditions:*

$$\mathrm{G}_{\widetilde{\chi_k}} = \begin{cases} \mathrm{G}^0_{\widetilde{\chi_k}} & \text{if the Witt index of } Q \text{ is zero and (118) is satisfied} \\ \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} & \text{if (116) is satisfied} \\ \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}} & \text{if (115) is satisfied.} \end{cases}$$

We will also use the following notation:

$$(z, N) = \begin{cases} (-1, 2) & \text{if } \mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}} \\ (i, 4) & \text{if } \mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \\ (e^{\frac{i\pi}{4}}, 8) & \text{if } \mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}} \end{cases}. \tag{142}$$

Then we have the following results.

i. $\mathrm{G}_{\widetilde{\chi_k}}$ *is a subgroup of* $\mathrm{ASp}(B_W) \times \mathrm{U}(\mathrm{L}^2(Y))$.

ii. $\mathrm{G}_{\widetilde{\chi_k}}$ *is a two-fold, four-fold, and eight-fold covering of* $\mathrm{ASp}(B_W)$*, respectively. This means that there is a non-split central exact sequence:*

$$1 \longrightarrow \mu_N(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{G}_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(B_W) \longrightarrow 1,$$

where $N = 2, 4$ *and* 8*, respectively. Moreover,* c_{χ_k} *is a cocycle associated to this extension.*

iii. *Different choices of* $\{w_s, s \in \mathrm{ASp}(B_W)\}$ *lead to isomorphic two-fold, four-fold and eight-fold coverings of* $\mathrm{ASp}(B_W)$*, respectively.*

iv. *Different choices of extensions* $\widetilde{\chi_k}$ *of* χ_k *lead to isomorphic two-fold, four-fold and eight-fold coverings of* $\mathrm{ASp}(B_W)$*, respectively.*

v. *Let* $\{\gamma'(\sigma), \sigma \in \mathrm{Sp}(W)\}$ *be another set of elements satisfying* (118)*,* (116) *and* (115)*, respectively, and also* (112)*. Then the extensions* $\mathrm{G}_{\widetilde{\chi_k}}$ *and* $\mathrm{G}'_{\widetilde{\chi_k}}$ *of* $\mathrm{ASp}(B_W)$ *(associated to the choices* $\gamma(\sigma)$ *and* $\gamma'(\sigma)$*, respectively) are isomorphic if and only if the map*

$$(s_1, s_2) \mapsto \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1} \tag{143}$$

defines a coboundary of $\mathrm{ASp}(B_W)$*. In particular, this is the case if*

$$\theta'(\sigma) - \theta(\sigma) \equiv 0 \mod \frac{2\pi}{N} \qquad \text{for every } \sigma \in \mathrm{Sp}(W). \tag{144}$$

If $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ *or* $\mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$*, then* (144) *is for example satisfied if* $\gamma(\sigma)$ *and* $\gamma'(\sigma)$ *are chosen to be real for every* $\sigma \in \mathrm{Sp}(W)$*.*

vi. *Let* $\mathrm{G}_{\widetilde{\chi_k}}$ *be the two-fold, four-fold and eight-fold covering of* $\mathrm{ASp}(B_W)$*, respectively, defined with the extension* $\widetilde{\chi_k}$ *of* χ_k *and the choices* $\{w_s,\ s \in \mathrm{ASp}(B_W)\}$ *and* $\{\gamma(\sigma), \sigma \in \mathrm{Sp}(W)\}$ *such that* (100) *and* (112) *are satisfied, and also* (118)*,* (116) *and* (115)*, respectively. Define similarly* $\mathrm{G}_{\chi_k^\sharp}$ *with the extension* $\chi_k^\sharp$ *of* χ_k *and the choices* $\{w'_s,\ s \in \mathrm{ASp}(B_W)\}$ *and* $\{\gamma'(\sigma), \sigma \in \mathrm{Sp}(W)\}$ *such that* (100)*,* (112)*, and* (118)*,* (116) *and* (115)*, respectively, are satisfied. If* (144) *is satisfied then*

$$\begin{aligned} \mathscr{I} : \mathrm{G}_{\widetilde{\chi_k}} &\to \mathrm{G}_{\chi_k^\sharp} \\ (s, z^l\omega_{\widetilde{\chi_k}}(s, w_s, \gamma(\sigma))) &\mapsto (s, z^l\nu(s)^{-1}\gamma(\sigma)\gamma'(\sigma)^{-1}\omega_{\chi_k^\sharp}(s, w'_s, \gamma'(\sigma))) \end{aligned} \tag{145}$$

is an isomorphism of central extensions, where ν *is defined by* (146) *and* $z = -1, i$ *and* $e^{\frac{i\pi}{4}}$ *if* $N = 2, 4$ *and* 8*, respectively. This means that the following diagram is commutative:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_N(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{G}_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\ & & \downarrow \mathrm{id} & & \downarrow \mathscr{I} & & \downarrow \mathrm{id} & & \\ 1 & \longrightarrow & \mu_N(\mathbb{C}) & \xrightarrow{\Phi_{\chi_k^\sharp}} & \mathrm{G}_{\chi_k^\sharp} & \xrightarrow{\Psi_{\chi_k^\sharp}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \end{array} \tag{D_{19}}$$

Proof. **i.** Let $\widetilde{s_1} = (s_1, z^{l_1}\omega_{\widetilde{\chi_k}}(s_1))$ and $\widetilde{s_2} = (s_2, z^{l_2}\omega_{\widetilde{\chi_k}}(s_2))$ be in $\mathrm{G}_{\widetilde{\chi_k}}$. Then by definition $\widetilde{s_1}$ and $\widetilde{s_2}$ belong to $\mathrm{ASp}(B_W) \times \mathrm{U}(\mathrm{L}^2(Y))$. Moreover, by (110), we compute exactly as in the proof of Proposition 1.36 that

$$\widetilde{s_1}\widetilde{s_2}^{-1} = (s_1 s_2^{-1}, z^{l_1-l_2} c_{\chi_k}(s_2, s_2^{-1})^{-1} c_{\chi_k}(s_1, s_2^{-1})\omega_{\widetilde{\chi_k}}(s_1 s_2^{-1})).$$

Corollary 2.28 implies that $z^{l_1-l_2} c_{\chi_k}(s_2, s_2^{-1})^{-1} c_{\chi_k}(s_1, s_2^{-1}) \in \mu_N(\mathbb{C})$. Hence $\widetilde{s_1}\widetilde{s_2}^{-1} \in \mathrm{G}_{\widetilde{\chi_k}}$ and thus $\mathrm{G}_{\widetilde{\chi_k}}$ is a subgroup of $\mathrm{ASp}(B_W) \times \mathrm{U}(\mathrm{L}^2(Y))$.

ii. Define $\Phi_{\widetilde{\chi_k}} : \mu_N(\mathbb{C}) \to \mathrm{G}_{\widetilde{\chi_k}}$ and $\Psi_{\widetilde{\chi_k}} : \mathrm{G}_{\widetilde{\chi_k}} \to \mathrm{ASp}(B_W)$. Then,

$$t \mapsto ((1,0), t\,\mathrm{Id}_{\mathrm{L}^2(Y)}) \qquad (s, z^l\omega_{\widetilde{\chi_k}}(s)) \mapsto s$$

exactly as in Proposition 1.36, we prove that $\Phi_{\widetilde{\chi_k}}$ and $\Psi_{\widetilde{\chi_k}}$ are group homomorphisms such that $\Phi_{\widetilde{\chi_k}}$ is injective, $\Psi_{\widetilde{\chi_k}}$ is surjective and $\mathrm{Im}(\Phi_{\widetilde{\chi_k}}) = \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})$. This shows that $\Phi_{\widetilde{\chi_k}}$ and $\Psi_{\widetilde{\chi_k}}$ define a central extension $1 \to \mu_N(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{G}_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(B_W) \to 1$. Now we want to show that this extension does not split. First we observe that c_{χ_k} is a cocycle associated to this extension. Define $\tau(s) = (s, \omega_{\widetilde{\chi_k}}(s))$ for every $s \in \mathrm{ASp}(B_W)$. Then τ is a section of $\Psi_{\widetilde{\chi_k}}$ which satisfies for all $s_1, s_2 \in \mathrm{ASp}(B_W)$:

$$\begin{aligned}\tau(s_1)\tau(s_2) = (s_1, \omega_{\widetilde{\chi_k}}(s_1))(s_2, \omega_{\widetilde{\chi_k}}(s_2)) &= (s_1 s_2, \omega_{\widetilde{\chi_k}}(s_1)\omega_{\widetilde{\chi_k}}(s_2)) \\ &= (s_1 s_2, c_{\chi_k}(s_1, s_2)\omega_{\widetilde{\chi_k}}(s_1 s_2)) \\ &= \Phi_{\widetilde{\chi_k}}(c_{\chi_k}(s_1, s_2))\tau(s_1 s_2).\end{aligned}$$

The fact that this extension splits is then equivalent to the fact that c_{χ_k} is a coboundary, which is false by Corollary 2.26.

iii. Suppose that for all $s \in \mathrm{ASp}(B_W)$, w_s' also satisfies (100), and denote by $\mathrm{G}'_{\widetilde{\chi_k}}$ the set of elements $\{(s, z^l\omega_{\widetilde{\chi_k}}(s, w_s'), s \in \mathrm{ASp}(B_W), l \in \mathbb{Z}\}$. Then by (106) in Proposition 2.21, for every $s = (\sigma, f) \in \mathrm{ASp}(B_W)$,

$$\omega_{\widetilde{\chi_k}}(s, w_s') = \chi_k(-f(v) + \langle v, w_s\rangle_W + B_W(w_s' - w_s, v - w_s))\omega_{\widetilde{\chi_k}}(s, w_s)$$

where v is any element such that $w_s' - w_s = (1-\sigma)(v)$. Suppose first that $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ or $\mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$. Since χ_k takes its values in $\mu_4(\mathbb{C})$, the set of operators $z^l\omega_{\widetilde{\chi_k}}(s, w_s')$ is exactly the same as the set of operators $z^l\omega_{\widetilde{\chi_k}}(s, w_s)$, where $l \in \mathbb{Z}$. Now if $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$, since $\mathrm{Sp}(W) = \mathrm{O}(Q)$, we proved in (138) that f takes its values in $2\mathbb{Z}/4\mathbb{Z}$. Hence

$$\omega_{\widetilde{\chi_k}}(s, w_s') = \chi_k(-f(v) + \langle v, w_s\rangle_W + B_W(w_s' - w_s, v - w_s))\omega_{\widetilde{\chi_k}}(s, w_s) = \pm\omega_{\widetilde{\chi_k}}(s, w_s).$$

Thus we also have that the sets of operators $z^l\omega_{\widetilde{\chi_k}}(s, w_s')$ and $z^l\omega_{\widetilde{\chi_k}}(s, w_s)$, where $l \in \mathbb{Z}$, are the same if $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$. Hence $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ are equal as sets.

Let c'_{χ_k} denote the cocycle associated to the projective representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s')$ (and then also associated to the extension $\mathrm{G}'_{\widetilde{\chi_k}}$). Then we can show that c_{χ_k} and c'_{χ_k} are cohomologous. In fact $\omega_{\widetilde{\chi_k}}(s, w_s)$ and $\omega_{\widetilde{\chi_k}}(s, w_s')$ satisfy the same intertwining property: for all $s \in \mathrm{ASp}(B_W)$ and $h \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$,

$$\rho_{\widetilde{\chi_k}}(s(h))\omega_{\widetilde{\chi_k}}(s) = \omega_{\widetilde{\chi_k}}(s)\rho_{\widetilde{\chi_k}}(h).$$

Schur's lemma then implies the existence of $\nu : \mathrm{ASp}(B_W) \to \mathrm{U}(1)$ such that $\omega_{\widetilde{\chi_k}}(s, w'_s) = \nu(s)\omega_{\widetilde{\chi_k}}(s, w_s)$. We computed such a map ν in Proposition 2.21 (106), since we proved that

$$\omega_{\widetilde{\chi_k}}(s, w'_s) = \chi_k(-f(v) + \langle v, w_s\rangle_W + B_W(w'_s - w_s, v - w_s))\omega_{\widetilde{\chi_k}}(s, w_s),$$

where v is any element such that $w'_s - w_s = (1 - \sigma)(v)$. Hence we obtained

$$\nu(s) = \chi_k(-f(v)+\langle v, w_s\rangle_W + B_W(w'_s - w_s, v - w_s)) \in \begin{cases} \{\pm 1, \pm i\} & \text{if } N = 4 \text{ or } N = 8 \\ \{\pm 1\} & \text{if } N = 2. \end{cases} \tag{146}$$

Thus we prove as in Proposition 1.36 that for all $s_1, s_2 \in \mathrm{ASp}(B_W)$,

$$c'_{\chi_k}(s_1, s_2)c_{\chi_k}(s_1, s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1 s_2)^{-1}. \tag{147}$$

Since ν takes its values in $\mu_N(\mathbb{C})$, this shows that c_{χ_k} and c'_{χ_k} are cohomologous. Therefore, $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ are extensions of $\mathrm{ASp}(B_W)$ whose associated cocycles are cohomologous: they are isomorphic extensions. In particular, the groups $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ are isomorphic.

iv. Formula (109) found for the cocycle c_{χ_k} in Corollary 2.24 does not depend on the choice of extension $\widetilde{\chi_k}$ of χ_k. In particular, two different coverings $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}_{\chi_k^\sharp}$ corresponding to the extensions $\widetilde{\chi_k}$ and $\chi_k^\sharp$ of χ_k, respectively, are associated to the same cocycle c_{χ_k}. Thus the extensions (hence also the groups) $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}_{\chi_k^\sharp}$ are isomorphic.

v. Let $\{\gamma'(\sigma), \sigma \in \mathrm{Sp}(W)\}$ be another set of elements satisfying (118), (116) and (115), respectively, and also (112). We denote by $\omega'_{\widetilde{\chi_k}}(s)$ and c'_{χ_k} the operators and the cocycle associated to this choice. Then formula (109) implies that for all $s_1, s_2 \in \mathrm{ASp}(B_W)$,

$$c'_{\chi_k}(s_1, s_2) = \gamma'(\sigma_1)\gamma'(\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\gamma(\sigma_2)^{-1}\gamma(\sigma_1\sigma_2)c_{\chi_k}(s_1, s_2),$$

i.e.

$$c'_{\chi_k}(s_1, s_2)c_{\chi_k}(s_1, s_2)^{-1} = \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \tag{148}$$

Hence the extensions $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ are isomorphic if and only if the map $(s_1, s_2) \mapsto \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}$ defines a coboundary of $\mathrm{ASp}(B_W)$. In particular, if (144) is satisfied for every $\sigma \in \mathrm{Sp}(W)$, then

$$\gamma'(\sigma)\gamma(\sigma)^{-1} = e^{i(\theta'(\sigma)-\theta(\sigma))} \in \mu_N(\mathbb{C}),$$

which shows that the map $\sigma \mapsto \gamma'(\sigma)\gamma(\sigma)^{-1}$ takes its values in $\mu_N(\mathbb{C})$. Thus $c'_{\chi_k}c_{\chi_k}^{-1}$ is a coboundary by (148). This implies that the extensions $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ are isomorphic.

vi. Denote by c_{χ_k} and c'_{χ_k} the cocycles associated to the extensions $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}_{\chi_k^\sharp}$ of $\mathrm{ASp}(B_W)$ defined in the statement, respectively. (Recall that $\chi_k^\sharp$ does not appear in the notation of

these cocycles since they do not depend on the choice of extension of χ_k by Corollary 2.24.) Let $\widetilde{s_1} = (s_1, z^{l_1}\omega_{\widetilde{\chi_k}}(s_1, w_{s_1}, \gamma(\sigma_1)))$ and $\widetilde{s_2} = (s_2, z^{l_2}\omega_{\widetilde{\chi_k}}(s_2, w_{s_2}, \gamma(\sigma_2)))$ in $\mathrm{G}_{\widetilde{\chi_k}}$. Then, as in Proposition 1.36, we can write

$$\begin{aligned}\mathscr{I}(\widetilde{s_1})\mathscr{I}(\widetilde{s_2}) = (s_1s_2, z^{l_1+l_2}\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}c'_{\chi_k}(s_1,s_2)\\ \omega_{\chi_k^\sharp}(s_1s_2, w'_{s_1s_2}, \gamma'(\sigma_1\sigma_2)))\end{aligned} \tag{149}$$

$$\mathscr{I}(\widetilde{s_1}\widetilde{s_2}) = (s_1s_2, z^{l_1+l_2}c_{\chi_k}(s_1,s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\omega_{\chi_k^\sharp}(s_1s_2, w'_{s_1s_2}, \gamma'(\sigma_1\sigma_2))). \tag{150}$$

Hence $\mathscr{I}$ is a group homomorphism if and only if

$$\nu(s_1)^{-1}\nu(s_2)^{-1}\gamma(\sigma_1)\gamma'(\sigma_1)^{-1}\gamma(\sigma_2)\gamma'(\sigma_2)^{-1}c'_{\chi_k}(s_1,s_2) = c_{\chi_k}(s_1,s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1},$$

i.e.

$$c'_{\chi_k}(s_1,s_2)c_{\chi_k}(s_1,s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}\gamma(\sigma_1\sigma_2)\gamma'(\sigma_1\sigma_2)^{-1}\gamma(\sigma_1)^{-1}\gamma'(\sigma_1)\gamma(\sigma_2)^{-1}\gamma'(\sigma_2). \tag{151}$$

Denote by c''_{χ_k} the cocycle associated to the covering of $\mathrm{ASp}(B_W)$ which is defined with the extension $\widetilde{\chi_k}$ of χ_k and the choices $\{w'_s,\ s \in \mathrm{ASp}(B_W)\}$ and $\{\gamma(\sigma), \sigma \in \mathrm{Sp}(W)\}$ such that (100), (112), and (118), (116) and (115) are satisfied, respectively. By (147),

$$c''_{\chi_k}(s_1,s_2)c_{\chi_k}(s_1,s_2)^{-1} = \nu(s_1)\nu(s_2)\nu(s_1s_2)^{-1}. \tag{152}$$

By (148), since (144) is satisfied,

$$c'_{\chi_k}(s_1,s_2)c''_{\chi_k}(s_1,s_2)^{-1} = \gamma'(\sigma_1)\gamma(\sigma_1)^{-1}\gamma'(\sigma_2)\gamma(\sigma_2)^{-1}(\gamma'(\sigma_1\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}. \tag{153}$$

Since $c'_{\chi_k}(s_1,s_2)c_{\chi_k}(s_1,s_2)^{-1} = c'_{\chi_k}(s_1,s_2)c''_{\chi_k}(s_1,s_2)^{-1}c''_{\chi_k}(s_1,s_2)c_{\chi_k}(s_1,s_2)^{-1}$, (152) and (153) imply (151). Finally, we prove exactly as in Proposition 1.36 that $\mathscr{I}$ is an isomorphism and the diagram (D_{19}) commutes.

□

Remark 2.30 (Related to Remark 1.37).

- Proposition 2.29.v provides a necessary and sufficient condition for the extensions $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$, associated to the choices $\gamma(\sigma)$ and $\gamma'(\sigma)$, respectively, to be isomorphic. In particular, if (144) is satisfied then the map defined by (143) defines a coboundary. At present, we do not know if the converse holds in general, i.e if the fact that the extensions $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}'_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ are isomorphic implies (144). Nevertheless, we prove that this equivalence holds in the two-dimensional case, see Remarks 3.7 and 3.33.

- We may wonder whether we can obtain all the possible non-split central extensions of $\mathrm{ASp}(B_W)$ by $\mu_N(\mathbb{C})$ with all possible choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (118), (116) and (115), respectively. This is false in general. Indeed, as we will show in Corollary 2.31, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ contain a copy of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. In particular, if W is a two-dimensional space, we obtain in Propositions 3.5 and 3.30 exactly two non-isomorphic coverings $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, and we prove in Remarks 3.7 and 3.33 that we obtain this way all the possible non-split central extensions of $\mathrm{ASp}(B_W)$

by four elements which contain the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. For $N = 2$, the possibilities for $\mathrm{G}^0_{\widetilde{\chi_k}}$ describe all the non-split stem central extensions of $\mathrm{ASp}(B_W)$ by two elements (see Remark 3.33).

Corollary 2.31. ***i.*** *Suppose that* $\mathrm{G}_{\widetilde{\chi_k}} \in \{\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}, \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}\}$. *Then the map*

$$\begin{aligned} \Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)} : \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) &\to \mathrm{G}_{\widetilde{\chi_k}} \\ (w,t) &\mapsto (s_w, \chi_k(t)\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w,w)) \end{aligned} \tag{154}$$

is a group monomorphism, where $s_w = (1, \langle w, \cdot\rangle_W) \in \mathrm{ASp}_a(B_W)$ *for all* $w \in W$ *(see Proposition 2.7). Its image is a normal subgroup of* $\mathrm{G}_{\widetilde{\chi_k}}$. *In particular,* $\mathrm{G}_{\widetilde{\chi_k}}$ *contains the Heisenberg group* $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ *as a normal subgroup.*

ii. *The group* $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *lies in the following short exact sequence:*

$$1 \longrightarrow \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \overset{\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}}{\longrightarrow} \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \overset{\pi_{\mathrm{AMp}}}{\longrightarrow} \mathrm{Sp}(W) \longrightarrow 1, \tag{155}$$

where π_{AMp} *is the projection onto* $\mathrm{Sp}(W)$.

Proof. **i.** For every $w \in W$, we proved in Corollary 2.26 that $s = (1, \langle w, \cdot\rangle_W)$ is an element of $\mathrm{ASp}_a(B_W)$ such that $w_s = w$ is the unique possibility for w_s. Then, by (105) in Proposition 2.18,

$$\omega_{\widetilde{\chi_k}}(s,w) = \gamma(1) \sum_{u \in \mathrm{I}(1)} \chi_k(\langle w, v\rangle_W + \langle w, v\rangle_W + B_W(u, v+w))\rho_{\widetilde{\chi_k}}(u+w) = \gamma(1)\rho_{\widetilde{\chi_k}}(w), \tag{156}$$

where v is any element such that $u = (1-1)(v) = 0$. Since χ_k takes its values in $\mu_4(\mathbb{C})$ and $\gamma(1)^4 = 1$ by (112) and (116), the map $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ is well-defined by the description of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ in Proposition 2.29. The proof of the fact that $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ is a group homomorphism whose image is a normal subgroup of $\mathrm{G}_{\widetilde{\chi_k}}$ is the same as in Corollary 1.38, only replacing f_z by $\langle w, \cdot\rangle_W$ for all $w, z \in W$, and χ, c_χ and $\omega_{\widetilde{\chi}}$ by χ_k, c_{χ_k} and $\omega_{\widetilde{\chi_k}}$, respectively.

Now, suppose that $(w,t) \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ belongs to the kernel of $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$. Then $s_w = (1, \langle w, \cdot\rangle_W) = (1,0)$ and $\chi_k(t)\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}((1, \langle w, \cdot\rangle_W), w) = \mathrm{Id}_{\mathrm{L}^2(Y)}$. Thus $w = 0$, which implies that

$$\mathrm{Id}_{\mathrm{L}^2(Y)} = \chi_k(t)\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}((1, \langle w, \cdot\rangle_W), w) = \chi_k(t)\gamma(1)^{-1}\omega_{\widetilde{\chi}}((1,0),0) = \chi_k(t)\,\mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Thus $t = 0$ since χ_k is faithful. This shows that the group homomorphism $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ is injective. Therefore, the image of $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ is a normal subgroup of $\mathrm{G}_{\widetilde{\chi_k}}$ which is isomorphic to $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$.

ii. By the preceding point, $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ is a group monomorphism. Moreover, if the element $\widetilde{s} = (s, i^l\omega_{\widetilde{\chi_k}}(s, w_s))$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is such that $s = (\sigma, f)$ and $\pi_{\mathrm{AMp}}(\widetilde{s}) = 1$, then $\sigma = 1$. By Proposition 2.7, there exists $w \in W$ such that $s = s_w = (1, \langle w, \cdot\rangle_W)$, and $w_s = w$. Since there exists $t \in \mathbb{Z}/4\mathbb{Z}$ such that $\chi_k(t)\gamma(1)^{-1} = i^l$, this implies that $\widetilde{s} =$

$(s, i^l\omega_{\widetilde{\chi_k}}(s, w_s)) = (s_w, i^l\omega_{\widetilde{\chi_k}}(s_w, w)) = (s_w, \chi_k(t)\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w, w)) = \Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}(w, t)$. Conversely, the image of $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ is included in the kernel of π_{AMp} by definition. This proves that $1 \longrightarrow \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \overset{\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}}{\longrightarrow} \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \overset{\pi_{\mathrm{AMp}}}{\longrightarrow} \mathrm{Sp}(W) \longrightarrow 1$ is a short exact sequence.

□

We now have everything we need to lift the projective Weil representation of $\mathrm{ASp}(B_W)$ to a linear representation of $\mathrm{G}_{\widetilde{\chi_k}}$.

Proposition 2.32 (The Weil representation of $\mathbf{G}_{\widetilde{\chi_k}}$). *Recall that* $\mathrm{G}_{\widetilde{\chi_k}}$ *denotes one of the groups* $\mathrm{G}^0_{\widetilde{\chi_k}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *and* $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$, *and* $z = -1$, i *and* $e^{\frac{i\pi}{4}}$, *provided* (118), (116) *and* (115) *are satisfied, respectively.*

- ***i.*** *The projective Weil representation* $\omega_{\widetilde{\chi_k}}$ *of* $\mathrm{ASp}(B_W)$ *can be lifted to a linear representation* $\widetilde{\omega}_{\widetilde{\chi_k}}$ *of* $\mathrm{G}_{\widetilde{\chi_k}}$, *which will be called the Weil representation of* $\mathrm{G}_{\widetilde{\chi_k}}$. *The representation* $\widetilde{\omega}_{\widetilde{\chi_k}}$ *is defined by*

$$\widetilde{\omega}_{\widetilde{\chi_k}}(\widetilde{s}) = z^l\omega_{\widetilde{\chi_k}}(s) \quad \textit{for every } \widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s)) \in \mathrm{G}_{\widetilde{\chi_k}}\,. \tag{157}$$

- ***ii.*** $\widetilde{\omega}_{\widetilde{\chi_k}}$ *is unitary, faithful and irreducible.*

- ***iii.*** *The image of* $\widetilde{\omega}_{\widetilde{\chi_k}}$ *contains all the operators* $\rho_{\widetilde{\chi_k}}(w)$, $w \in W$ *(which form a basis of the vector space* $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$*). More precisely, for every* $w \in W$,

$$\rho_{\widetilde{\chi_k}}(w) = \widetilde{\omega}_{\widetilde{\chi_k}}((1, \langle w, \cdot\rangle_W), \gamma(1)^{-1}\omega_{\widetilde{\chi_k}}((1, \langle w, \cdot\rangle_W), w)), \tag{158}$$

  where $\gamma(1) \in \mu_N(\mathbb{C})$.

- ***iv.*** *If* $\mathrm{G}_{\widetilde{\chi_k}} \in \{\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}, \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}\}$, *then* $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)})(\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W))$ *is a subgroup of the image of* $\widetilde{\omega}_{\widetilde{\chi_k}}$ *which is isomorphic to* $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, *where* $\Phi_{\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)}$ *is defined in Corollary 2.31.*

Proof. **i.** Define the map $\widetilde{\omega}_{\widetilde{\chi_k}}$ by (157). Observe first that for $s = (1, 0)$, since $w_s = 0$ by Lemma 2.20.iii, we obtain by (103) that $\omega_{\widetilde{\chi_k}}(1, 0) = \gamma(1)\,\mathrm{Id}_{\mathrm{L}^2(Y)}$. By (118), (116) and (115), $\gamma(1) \in \mu_N(\mathbb{C})$. Hence we have that

$$\widetilde{\omega}_{\widetilde{\chi_k}}((1, 0), \mathrm{Id}_{\mathrm{L}^2(Y)}) = \widetilde{\omega}_{\widetilde{\chi_k}}((1, 0), \gamma(1)^{-1}\omega_{\widetilde{\chi_k}}((1, 0), 0)) = \mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Then we prove exactly as in Proposition 1.39 that $\widetilde{\omega}_{\widetilde{\chi_k}}$ is a linear representation of $\mathrm{G}_{\widetilde{\chi_k}}$ which lifts the projective Weil representation of $\mathrm{ASp}(B_W)$.

ii. Let $s = (\sigma, f) \in \mathrm{ASp}(B_W)$ and $\widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s)) \in \mathrm{G}_{\widetilde{\chi_k}}$. We already proved in Proposition 2.21 that $\omega_{\widetilde{\chi_k}}(s)$ is a unitary operator. Suppose that $\widetilde{\omega}_{\widetilde{\chi_k}}(\widetilde{s}) = \mathrm{Id}_{\mathrm{L}^2(Y)}$, then $z^l\omega_{\widetilde{\chi_k}}(s) = \mathrm{Id}_{\mathrm{L}^2(Y)}$. We prove that $\widetilde{s} = 1$ in a similar way than Proposition 1.39, using (100) in Proposition 2.21 and Lemma 2.15. Thus $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful. We prove the irreducibility of $\omega_{\widetilde{\chi_k}}(s)$ in the next point.

iii. For every $w \in W$, we proved in Corollary 2.26 that $s = (1, \langle w, \cdot \rangle_W)$ is an element of $\mathrm{ASp}_a(B_W)$ such that $w_s = w$ is the unique possibility for w_s. Then, by (156),

$$\omega_{\widetilde{\chi_k}}(s, w) = \gamma(1)\rho_{\widetilde{\chi_k}}(w).$$

Hence we have that

$$\widetilde{\omega}_{\widetilde{\chi_k}}((1, \langle w, \cdot \rangle_W), \gamma(1)^{-1}\omega_{\widetilde{\chi_k}}((1, \langle w, \cdot \rangle_W), w)) = \rho_{\widetilde{\chi_k}}(w).$$

This shows that $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{G}_{\widetilde{\chi_k}})$ contains the operator $\rho_{\widetilde{\chi_k}}(w)$ for every $w \in W$. Now if $V \subseteq \mathrm{L}^2(Y)$ is a vector subspace invariant under $\widetilde{\omega}_{\widetilde{\chi_k}}$ then, in particular, V is invariant under all the operators $\rho_{\widetilde{\chi_k}}(w)$, $w \in W$. Since these operators form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by Lemma 2.15, then either $V = \{0\}$ or $V = \mathrm{L}^2(Y)$, i.e. $\widetilde{\omega}_{\widetilde{\chi_k}}$ is irreducible.

iv. This follows directly from Corollary 2.31 and the fact that $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful.

□

Corollary 2.33 (Character of the Weil representation of $\mathrm{G}_{\widetilde{\chi_k}}$). *Let $\Theta_{\widetilde{\chi_k}}$ be the character of $\widetilde{\omega}_{\widetilde{\chi_k}}$. We suppose as before that a choice of $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ and $\{w_s$ satisfying (100), $s \in \mathrm{ASp}(B_W)\}$ has been made. Then, for every $\widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s))$ in $\mathrm{G}_{\widetilde{\chi_k}}$,*

$$\Theta_{\widetilde{\chi_k}}(\widetilde{s}) = z^l\gamma(\sigma)|Y|\chi_k(f(w) - B_W(w_s + w, w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s), \tag{159}$$

where $\mathbb{1}_{\mathrm{I}(\sigma)}$ is the indicator function of $\mathrm{I}(\sigma)$, and $w \in W$ is any element such that $(\sigma - 1)(w) = w_s$. In particular, $\Theta_{\widetilde{\chi_k}}$ does not depend on the choice of extension $\widetilde{\chi_k}$ of χ_k and we can denote it by Θ_{χ_k}. Moreover, Θ_{χ_k} satisfies the following properties:

i. *The support of Θ_{χ_k} is*

$$\mathrm{supp}(\Theta_{\chi_k}) = \{\widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s, w_s)) \in \mathrm{G}_{\widetilde{\chi_k}},\ s = (\sigma, f) \text{ with } f_{|\mathrm{K}(\sigma)} = 0\}.$$

Hence it does not depend on the choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ and $\{w_s,\ s \in \mathrm{ASp}(B_W)\}$.

ii. *For every $\sigma \in \mathrm{Sp}(W)$, there exists $\widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s, w_s)) \in \mathrm{G}_{\widetilde{\chi_k}}$ such that the projection of s on $\mathrm{Sp}(W)$ is σ and $\Theta_{\chi_k}(\widetilde{s}) \neq 0$.*

iii. *Let $\sigma \in \mathrm{Sp}(W)$. Then the following assertions are equivalent:*

- *$\Theta_{\chi_k}(\widetilde{s}) \neq 0$ for every $\widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s, w_s)) \in \mathrm{G}_{\widetilde{\chi_k}}$ such that $s = (\sigma, f)$.*
- *$1 - \sigma$ is invertible.*

Proof. **i.** Let $\widetilde{s} = (s, z^l\omega_{\widetilde{\chi_k}}(s)) \in \mathrm{G}_{\widetilde{\chi_k}}$, where $s = (\sigma, f) \in \mathrm{ASp}(B_W)$. By definition of $\widetilde{\omega}_{\widetilde{\chi_k}}$, we have that $\Theta_{\widetilde{\chi_k}}(\widetilde{s}) = \mathrm{Trace}(\widetilde{\omega}_{\widetilde{\chi_k}}(\widetilde{s})) = z^l\,\mathrm{Trace}(\omega_{\widetilde{\chi_k}}(s))$. Recall the expression (105) proved in Proposition 2.21:

$$\omega_{\widetilde{\chi_k}}(s) = \gamma(\sigma)\sum_{u \in \mathrm{I}(\sigma)} \chi_k(f(w) + \langle w_s, w \rangle_W + B_W(u, w_s + w))\rho_{\widetilde{\chi_k}}(u + w_s),$$

where $w \in W$ is any element such that $u = (1-\sigma)(w)$. Hence we can use the formula for the trace of every operator of the form $\rho_{\widetilde{\chi_k}}(w)$, $w \in W$, given in Corollary 2.13 in the basis $\mathscr{B}_Y = \{\delta_y, y \in Y\}$ of $\mathrm{L}^2(Y)$. For every $w_0 = x_0 + y_0 \in W = X \oplus Y$,

$$\mathrm{Trace}(\rho_{\widetilde{\chi_k}}(w_0)) = |Y|\delta_0(w_0).$$

Then we obtain (159) with a computation which is similar to the one of the proof of Corollary 1.41:

$$\mathrm{Trace}(\omega_{\widetilde{\chi_k}}(s)) = \gamma(\sigma)|Y|\chi_k(f(w) - B_W(w_s + w, w_s))\mathbb{1}_{\mathrm{I}(\sigma)}(w_s),$$

where $w \in W$ is any element such that $w_s = (\sigma - 1)(w)$. Then $\Theta_{\chi_k}(\widetilde{s})$ is nonzero if and only if w_s belongs to $\mathrm{I}(\sigma)$, and this condition is independent of the choices of $\gamma(\sigma)$ and w_s by Lemma 2.20.ii. Since $\mathrm{K}(\sigma)$ and $\mathrm{I}(\sigma)$ are orthogonal with respect to the nondegenerate symplectic form $\langle \cdot, \cdot \rangle_W$, the fact that w_s is in $\mathrm{I}(\sigma)$ is equivalent to have $f_{|\,\mathrm{K}(\sigma)} = 0$ by (100).

ii. Let σ be an element of $\mathrm{Sp}(W)$. Choose any element $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ such that $\pi(\widetilde{\sigma}) = \sigma$. Recall the definition (93) of $f_{\widetilde{\sigma}}$:

$$\begin{aligned} f_{\widetilde{\sigma}} : W &\to \mathbb{Z}/4\mathbb{Z} \\ w &\mapsto F_{\widetilde{\sigma}}(\widetilde{w}) = \beta_{\widetilde{\sigma}}(\widetilde{w}, \widetilde{w}), \end{aligned}$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $w = \widetilde{w} + 2\widetilde{W}$, and for all $\widetilde{w}_1$ and $\widetilde{w}_2$ in $\widetilde{W}$,

$$\beta_{\widetilde{\sigma}}(\widetilde{w}_1, \widetilde{w}_2) = B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) - B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2).$$

Let $s_{\widetilde{\sigma}} = (\sigma, f_{\widetilde{\sigma}})$. Then $s_{\widetilde{\sigma}} \in \mathrm{ASp}(B_W)$ by Proposition 2.7. By (100),

$$f_{\widetilde{\sigma}\,|\,\mathrm{K}(\sigma)} = \langle w_{s_{\widetilde{\sigma}}}, \cdot \rangle_{W\,|\,\mathrm{K}(\sigma)}.$$

Now, set $s = (\sigma, f_{\widetilde{\sigma}} - \langle w_{s_{\widetilde{\sigma}}}, \cdot \rangle_W)$. Then by (100), for every $w \in \mathrm{K}(\sigma)$,

$$\langle w_s, w \rangle_W = f_{\widetilde{\sigma}}(w) - \langle w_{s_{\widetilde{\sigma}}}, w \rangle_W = \langle w_{s_{\widetilde{\sigma}}}, w \rangle_W - \langle w_{s_{\widetilde{\sigma}}}, w \rangle_W = 0.$$

Thus $w_s \in \mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$. If we set $\widetilde{s} = (s, \omega_{\widetilde{\chi_k}}(s)) \in \mathrm{G}_{\widetilde{\chi_k}}$, then $\Theta_{\chi_k}(\widetilde{s}) \neq 0$ by (159).

iii. Let $\sigma \in \mathrm{Sp}(W)$. If $1 - \sigma$ is invertible then $\mathrm{I}(\sigma) = W$, and thus by (159), $\Theta_{\chi_k}(\widetilde{s}) \neq 0$ for every $\widetilde{s} = (s, z^l \omega_{\widetilde{\chi_k}}(s)) \in \mathrm{G}_{\widetilde{\chi_k}}$ such that the projection of s on $\mathrm{Sp}(W)$ is σ. Conversely, suppose that $1 - \sigma$ has a nontrivial kernel $\mathrm{K}(\sigma)$. First we choose $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ such that $\pi(\widetilde{\sigma}) = \sigma$. Then, by the computations done in the proof of ii above, we know that $f = f_{\widetilde{\sigma}} - \langle w_{s_{\widetilde{\sigma}}}, \cdot \rangle_W$ is such that $f_{|\,\mathrm{K}(\sigma)} = 0$. Now, since $\langle \cdot, \cdot \rangle_W$ is nondegenerate, $\mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$ and $1 - \sigma$ is not invertible, we know that $\mathrm{K}(\sigma)^\perp \neq W$. Hence there exists an element $w_0 \notin \mathrm{K}(\sigma)^\perp = \mathrm{I}(\sigma)$, which means that $\langle w_0, \cdot \rangle_{W\,|\,\mathrm{K}(\sigma)}$ is not identically zero. Set $s_0 = (\sigma, f + \langle w_0, \cdot \rangle_W) = (\sigma, f_{\widetilde{\sigma}} - \langle w_{s_{\widetilde{\sigma}}} - w_0, \cdot \rangle_W) \in \mathrm{ASp}(B_W)$. Then for every $w \in \mathrm{K}(\sigma)$, since $f_{|\,\mathrm{K}(\sigma)} = 0$,

$$\langle w_0, w \rangle_W = \langle w_0, w \rangle_W + f_{\widetilde{\sigma}}(w) - \langle w_{s_{\widetilde{\sigma}}}, w \rangle_W,$$

i.e. w_0 satisfies (100) for s_0. Thus, by Lemma 2.20.ii, any other element w_{s_0} satisfying (100) for s_0 does not belong to $\mathrm{I}(\sigma)$. This shows that $\Theta_{\chi_k}(\widetilde{s_0}) = 0$ for every $\widetilde{s_0} = (s_0, z^l\omega_{\widetilde{\chi_k}}(s_0))$. Since the projection of s_0 on $\mathrm{Sp}(W)$ is σ, this completes the proof by contraposition.

□

Corollary 2.34. *Let $\widetilde{\chi_k}$ and $\chi_k^\sharp$ be two extensions of χ_k to $X \times \mathbb{Z}/4\mathbb{Z}$. For every $\sigma \in \mathrm{Sp}(W)$, choose $\gamma(\sigma)$ and $\gamma'(\sigma)$ satisfying* (112)*. For every $s \in \mathrm{ASp}(B_W)$, choose w_s and w'_s satisfying* (100)*. Suppose that* (118)*,* (116) *and* (115)*, respectively, are satisfied, and* (144) *too. Then the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$, obtained with the choices $\widetilde{\chi_k}$, $\{w_s$ satisfying* (100)*, $s \in \mathrm{ASp}(B_W)\}$ and $\{\gamma(\sigma)$ satisfying* (112)*, $\sigma \in \mathrm{Sp}(W)\}$, is isomorphic to the Weil representation $\widetilde{\omega}_{\chi_k^\sharp} \circ \mathscr{I}$, obtained with the choices $\chi_k^\sharp$, $\{w'_s$ satisfying* (100)*, $s \in \mathrm{ASp}(B_W)\}$ and $\{\gamma'(\sigma)$ satisfying* (46)*, $\sigma \in \mathrm{Sp}(W)\}$. In other terms, there exists an isomorphism $T \in \mathrm{U}(\mathrm{L}^2(Y))$ such that the following diagram commutes:*

$$\widetilde{\omega}_{\chi_k^\sharp}(\mathscr{I}(\widetilde{s})) = T \circ \widetilde{\omega}_{\widetilde{\chi_k}}(\widetilde{s}) \circ T^{-1} \quad \textit{for every } \widetilde{s} \in \mathrm{G}_{\widetilde{\chi_k}}, \quad \textit{i.e.} \quad \begin{array}{ccc} \mathrm{L}^2(Y) & \xrightarrow{\widetilde{\omega}_{\widetilde{\chi_k}}(\widetilde{s})} & \mathrm{L}^2(Y) \\ \downarrow T & & \downarrow T \\ \mathrm{L}^2(Y) & \xrightarrow{\widetilde{\omega}_{\chi_k^\sharp}(\mathscr{I}(\widetilde{s}))} & \mathrm{L}^2(Y) \end{array} \qquad (D_{20})$$

where $\mathscr{I}$ is defined by (145)*.*

Proof. Since (144) is satisfied, by Proposition 2.29.vi, the extensions $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}_{\chi_k^\sharp}$ of $\mathrm{ASp}(B_W)$ associated to the choices $(\widetilde{\chi_k}, \gamma(\sigma), w_s)$ and $(\chi_k^\sharp, \gamma'(\sigma), w'_s)$, respectively, are isomorphic. An explicit isomorphism $\mathscr{I}$ is given by (145). Hence, $\mathscr{I}$ is also a group isomorphism between $\mathrm{G}_{\widetilde{\chi_k}}$ and $\mathrm{G}_{\chi_k^\sharp}$. Now, we denote by Θ_{χ_k} and Θ'_{χ_k} the characters of $\widetilde{\omega}_{\widetilde{\chi_k}}$ and $\widetilde{\omega}_{\chi_k^\sharp}$, respectively. Then by definition of $\mathscr{I}$, Θ_{χ_k} and Θ'_{χ_k}, for every $\widetilde{s} = (s, z^l\omega_{\widetilde{\chi}}(s, w_s, \gamma(\sigma))) \in \mathrm{G}_{\widetilde{\chi_k}}$, we prove exactly as in Corollary 1.45 that $\mathrm{Trace}(\widetilde{\omega}_{\chi_k^\sharp}(\mathscr{I}(\widetilde{s}))) = \Theta_{\chi_k}(\widetilde{s})$. This proves that $\widetilde{\omega}_{\widetilde{\chi_k}}$ and $\widetilde{\omega}_{\chi_k^\sharp} \circ \mathscr{I}$ are two irreducible representations with the same character, hence they are isomorphic. □

To construct the Weil representation of $\mathrm{G}_{\widetilde{\chi_k}}$, we made a choice between the two characters $\chi_1 : t \mapsto i^t$ and $\chi_3 : t \mapsto i^{3t} = i^{-t} = (-i)^t$. Notice that $\chi_3 = \overline{\chi_1}$. We may wonder whether these choices give rise to isomorphic coverings of $\mathrm{ASp}(B_W)$. The next result gives a necessary and sufficient condition for $\mathrm{G}_{\widetilde{\chi_1}}$ and $\mathrm{G}_{\widetilde{\chi_3}}$ to be isomorphic.

Proposition 2.35. *Let $\widetilde{\chi_1}$ and $\widetilde{\chi_3}$ be two extensions of χ_1 and χ_3, respectively, to the abelian subgroup $X \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Let $\{w_s,\ s \in \mathrm{ASp}(B_W)\}$ be a choice of elements satisfying* (100)*. Let $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ and $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ be two sets of elements satisfying* (112)*. Suppose also that* (118)*,* (116) *and* (115) *are satisfied, respectively. Let $\mathrm{G}_{\widetilde{\chi_1}}$ and $\mathrm{G}_{\widetilde{\chi_3}}$ be the coverings of $\mathrm{ASp}(B_W)$ defined in Proposition 2.29 with the choices $(\widetilde{\chi_1}, w_s, \gamma(\sigma))$ and $(\widetilde{\chi_3}, w_s, \delta(\sigma))$, respectively. Then the extensions $\mathrm{G}_{\widetilde{\chi_1}}$ and $\mathrm{G}_{\widetilde{\chi_3}}$ of $\mathrm{ASp}(B_W)$ are isomorphic if and only if the map*

$$(s_1, s_2) \mapsto \delta(\sigma_1)^{-1}\overline{\gamma(\sigma_1)}\delta(\sigma_2)^{-1}\overline{\gamma(\sigma_2)}\delta(\sigma_1\sigma_2)\overline{\gamma(\sigma_1\sigma_2)^{-1}}c_{\chi_1}(s_1, s_2)^2 \qquad (160)$$

is a coboundary. More precisely, we have the following results.

i. *If* $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ *for* $k \in \{1,3\}$*, then* $\mathrm{G}_{\widetilde{\chi_1}}$ *and* $\mathrm{G}_{\widetilde{\chi_3}}$ *are isomorphic if and only if the map*

$$(s_1, s_2) \mapsto \delta(\sigma_1)^{-1}\overline{\gamma(\sigma_1)}\delta(\sigma_2)^{-1}\overline{\gamma(\sigma_2)}\delta(\sigma_1\sigma_2)\overline{\gamma(\sigma_1\sigma_2)^{-1}} \tag{161}$$

is a coboundary. In particular, this is the case if

$$\nu(\sigma) + \theta(\sigma) \equiv 0 \mod \pi \qquad \text{for all } \sigma \in \mathrm{Sp}(W), \tag{162}$$

For example, (162) *is satisfied if we set* $\delta(\sigma) = \overline{\gamma(\sigma)}$ *for every* $\sigma \in \mathrm{Sp}(W)$*, and in this case* $c_{\chi_3} = c_{\chi_1}$*.*

ii. *If* $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *for* $k \in \{1,3\}$ *or* $\mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ *for* $k \in \{1,3\}$*, and we set* $\delta(\sigma) = \overline{\gamma(\sigma)}$*, then* $c_{\chi_3} = \overline{c_{\chi_1}}$ *and the extensions* $\mathrm{G}_{\widetilde{\chi_1}}$ *and* $\mathrm{G}_{\widetilde{\chi_3}}$ *of* $\mathrm{ASp}(B_W)$ *are isomorphic if and only if* $c^2_{\chi_1}$ *is a coboundary.*

Proof. The fixed choice $\{w_s,\ s \in \mathrm{ASp}(B_W)\}$ is suitable for both constructions of $\mathrm{G}_{\widetilde{\chi_1}}$ and $\mathrm{G}_{\widetilde{\chi_3}}$ because (100) does not depend on the chosen character. Moreover, for all s_1 and s_2 in $\mathrm{ASp}(B_W)$, using (109) in Corollary 2.24 and the fact that $\chi_3 = \overline{\chi_1}$, we obtain:

$$\begin{aligned}
&c_{\chi_3}(s_1, s_2)\\
&= \delta(\sigma_1)\delta(\sigma_2)\delta(\sigma_1\sigma_2)^{-1} \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2)\\ u_1+u_2 = w_{s_1}, s_2}} \chi_3(f_1(w_1) + \langle w_{s_1}, w_1\rangle_W + B_W(u_1, w_1 + w_{s_1}) + f_2(w_2)\\
&\qquad + \langle w_{s_2}, w_2\rangle_W + B_W(u_2, w_2 + w_{s_2})\\
&\qquad + B_W(u_1 + w_{s_1}, u_2 + w_{s_2}))\\
&= \delta(\sigma_1)\delta(\sigma_2)\delta(\sigma_1\sigma_2)^{-1} \sum_{\substack{u_1 \in \mathrm{I}(\sigma_1), u_2 \in \mathrm{I}(\sigma_2)\\ u_1+u_2 = w_{s_1}, s_2}} \overline{\chi_1(f_1(w_1) + \langle w_{s_1}, w_1\rangle_W + B_W(u_1, w_1 + w_{s_1}) + f_2(w_2)}\\
&\qquad \overline{+\langle w_{s_2}, w_2\rangle_W + B_W(u_2, w_2 + w_{s_2})}\\
&\qquad \overline{+B_W(u_1 + w_{s_1}, u_2 + w_{s_2}))}\\
&= \delta(\sigma_1)\delta(\sigma_2)\delta(\sigma_1\sigma_2)^{-1}\overline{(\gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}c_{\chi_1}(s_1, s_2)}\\
&= \delta(\sigma_1)\overline{\gamma(\sigma_1)^{-1}}\delta(\sigma_2)\overline{\gamma(\sigma_2)^{-1}}\delta(\sigma_1\sigma_2)^{-1}\overline{\gamma(\sigma_1\sigma_2)c_{\chi_1}(s_1, s_2)}.
\end{aligned}$$

Then, since c_{χ_1} takes its values in $\mathrm{U}(1)$,

$$\begin{aligned}
c_{\chi_1}(s_1, s_2)c_{\chi_3}(s_1, s_2)^{-1} &= c_{\chi_1}(s_1, s_2)\delta(\sigma_1)^{-1}\overline{\gamma(\sigma_1)}\delta(\sigma_2)^{-1}\overline{\gamma(\sigma_2)}\delta(\sigma_1\sigma_2)\overline{\gamma(\sigma_1\sigma_2)^{-1}c_{\chi_1}(s_1, s_2)}^{-1}\\
&= \delta(\sigma_1)^{-1}\overline{\gamma(\sigma_1)}\delta(\sigma_2)^{-1}\overline{\gamma(\sigma_2)}\delta(\sigma_1\sigma_2)\overline{\gamma(\sigma_1\sigma_2)^{-1}}c_{\chi_1}(s_1, s_2)^2.
\end{aligned}$$

Since by Proposition 2.29.ii, c_{χ_1} and c_{χ_3} are associated to the extensions $\mathrm{G}_{\widetilde{\chi_1}}$ and $\mathrm{G}_{\widetilde{\chi_3}}$ of $\mathrm{ASp}(B_W)$, respectively, then these extensions are isomorphic if and only if the map

$$(s_1, s_2) \mapsto \delta(\sigma_1)^{-1}\overline{\gamma(\sigma_1)}\delta(\sigma_2)^{-1}\overline{\gamma(\sigma_2)}\delta(\sigma_1\sigma_2)\overline{\gamma(\sigma_1\sigma_2)^{-1}}c_{\chi_1}(s_1, s_2)^2 \tag{163}$$

is a coboundary. Moreover, we can set $\delta(\sigma) = \overline{\gamma(\sigma)}$ for every $\sigma \in \mathrm{Sp}(W)$ since this choice also satisfies (118), (116) and (115). If we choose to set $\delta(\sigma) = \gamma(\sigma)$ then $c_{\chi_3} = \overline{c_{\chi_1}}$ by the first computation above, and the map (163) becomes the map $(s_1, s_2) \mapsto c_{\chi_1}(s_1, s_2)^2$. This proves ii. Finally, suppose that $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ for $k \in \{1,3\}$. Then c_{χ_1} is real and takes its values in $\mu_2(\mathbb{C})$ by (118), so $c^2_{\chi_1} = 1$. Hence $c_{\chi_3} = c_{\chi_1}$, the condition (160) is equivalent to (161), and (161) is satisfied if $\nu(\sigma) + \theta(\sigma) \equiv 0 \mod \pi$, since then $\delta(\sigma)^{-1}\overline{\gamma(\sigma)} \in \mu_2(\mathbb{C})$. This proves i. □

Remark 2.36 (Related to Remark 2.22).

- By Proposition 2.35, if (162) is satisfied then the map defined by (161) is a coboundary, hence the extensions $\mathrm{G}^0_{\widetilde{\chi_1}}$ and $\mathrm{G}^0_{\widetilde{\chi_3}}$ of $\mathrm{ASp}(B_W)$ are isomorphic. We mention in Remark 3.34 that this is in fact an equivalence, i.e. $\mathrm{G}^0_{\widetilde{\chi_1}}$ and $\mathrm{G}^0_{\widetilde{\chi_3}}$ are isomorphic if and only if (162) is satisfied.
- Remarks 3.8 and 3.34 show that the extensions $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$ of $\mathrm{ASp}(B_W)$, defined with the choices $\gamma(\sigma)$ and $\delta(\sigma)$, respectively, are isomorphic if and only if $\theta(\sigma) + \nu(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for all $\sigma \in \mathrm{Sp}(W)$. It is an open problem to know if this equivalence is true in general.
- The two preceding points raise a question that we already mentioned in Remark 2.22. In fact, in the two-dimensional case, we shall prove that the cocycle $c^2_{\chi_k}$, defined with a choice of $\{\gamma(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (116), is always a coboundary. As explained precisely in Remark 3.8, this enables us to "twist" the projective Weil representation with a parameter that depends on s and w_s, in order to obtain a new associated cocycle c'_{χ_k} which leads to a two-fold covering of $\mathrm{ASp}(B_W)$, instead of a four-fold covering. So, as in Remark 2.22, the question is whether considering the more general parameters $\gamma(s, w_s)$ could provide a two-fold covering of $\mathrm{ASp}(B_W)$ associated to c'_{χ_k}. Answering this question requires some additional tools because it is difficult to determine if the cocycle $c^2_{\chi_k}$ is a coboundary out of the explicit formula for c_{χ_k}. The argument used in the two-dimensional example is that every element of the second cohomology group $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z})$ is of order at most two (see Remark 3.8). So a possible approach could be to determine $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z})$ for W of arbitrary dimension.

2.4 Relation between the constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}_{\widetilde{\chi_k}}$

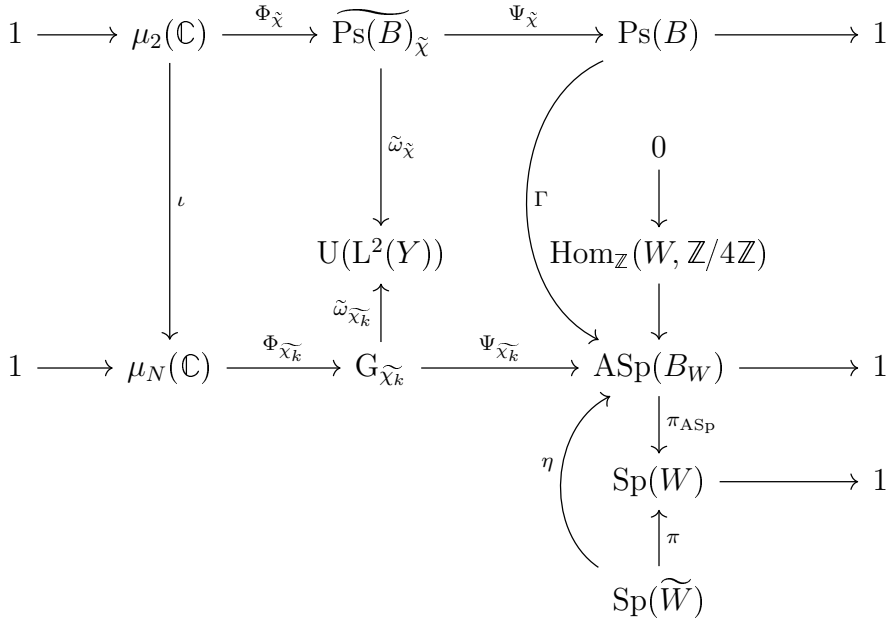
We keep the same notations as in chapter 1 and the beginning of chapter 2. Since we work on $\mathbb{F} = \mathbb{F}_2$, there is only one nontrivial character χ of $\mathbb{F}_2$. Recall that we consider a free symplectic module $(\widetilde{W}, \langle\cdot,\cdot\rangle_{\widetilde{W}})$ of rank $2n$, and we identify $(\widetilde{W}/2\widetilde{W}, \langle\cdot,\cdot\rangle_W)$ with a $2n$-dimensional symplectic vector space $(W, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle = \kappa(\langle\cdot,\cdot\rangle_W)$ (cf. Lemma 2.1) and $n \geqslant 1$. Recall also that we fixed a complete polarization $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$, where $\widetilde{X}$ and $\widetilde{Y}$ are two submodules of $\widetilde{W}$ of rank n such that $\langle\cdot,\cdot\rangle_{\widetilde{W}|\widetilde{X}\times\widetilde{X}} = \langle\cdot,\cdot\rangle_{\widetilde{W}|\widetilde{Y}\times\widetilde{Y}} = 0$. We also fixed a complete polarization $W = X \oplus Y$, where $X = \widetilde{X}/2\widetilde{X}$ and $Y = \widetilde{Y}/2\widetilde{Y}$ are identified with two $\mathbb{F}_2$-vector spaces of dimension n such that $\langle\cdot,\cdot\rangle_{|X\times X} = \langle\cdot,\cdot\rangle_{|Y\times Y} = 0$. We chose a bilinear form $B_{\widetilde{W}}$ such that (75) is satisfied and we defined the bilinear form $B = \kappa \circ B_W$ on W, where $B_W(w_1, w_2) = 2B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)$ for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$. Then B satisfies (78). Finally we recall the quadratic forms defined by $Q_W(w) = B_W(w, w)$ and $Q(w) = B(w, w)$ for every $w \in W$.

Now we fix an extension $\widetilde{\chi}$ of χ to the abelian subgroup $X \times \mathbb{F}$ of $\mathrm{H}(B)$. We fix also a faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ and we choose an extension $\widetilde{\chi_k}$ of χ_k to the abelian subgroup $X \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ such that

$$\widetilde{\chi} \text{ and } \widetilde{\chi_k} \text{ coincide on } X \tag{164}$$

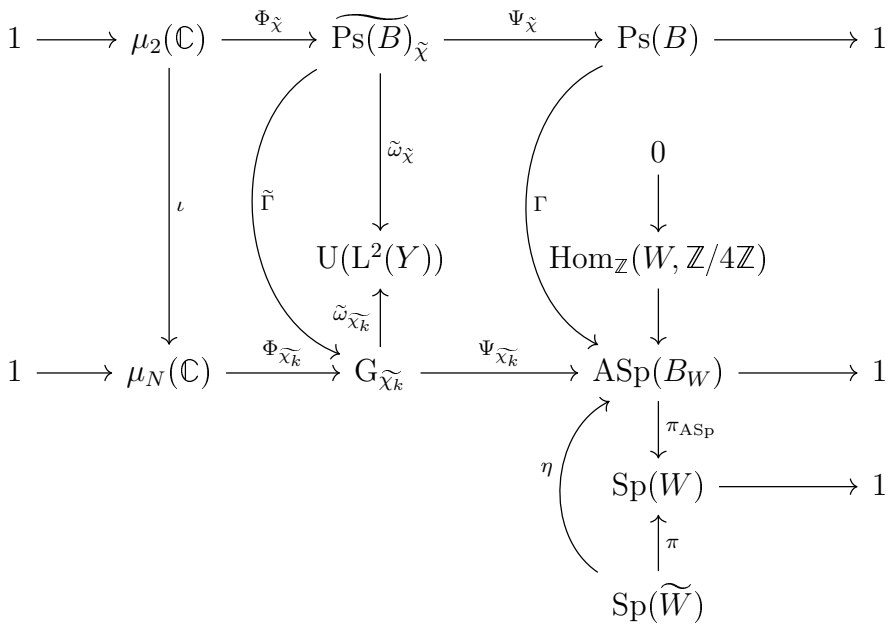
(recall from Propositions 1.36 and 2.29 that the isomorphism classes of the coverings $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ and $\mathrm{G}_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ do not depend on the choices of extensions $\widetilde{\chi}$ and $\widetilde{\chi_k}$, respectively). This choice is always possible. In fact, according to (25) and (95), if we choose a basis $\{x_1, \ldots, x_n\}$ of X over $\mathbb{F}_2$, then $\widetilde{\chi}$ and $\widetilde{\chi_k}$ are entirely determined on X by the values of $\widetilde{\chi}(x_j, 0)$ and $\widetilde{\chi_k}(x_j, 0)$, respectively, for every $1 \leqslant j \leqslant n$. For every such j, the same two possible values occur for $\widetilde{\chi}(x_j, 0)$ and $\widetilde{\chi_k}(x_j, 0)$, depending on whether $B(x_j, x_j)$ and $B_W(x_j, x_j)$, respectively, vanish or not. So we can choose $\widetilde{\chi_k}(x_j, 0) = \widetilde{\chi}(x_j, 0)$ for every $1 \leqslant j \leqslant n$. For example if X is Q-singular (i.e. $Q_{|X} = 0$) we can choose $\widetilde{\chi}$ and $\widetilde{\chi_k}$ to be the trivial extensions, i.e. $\widetilde{\chi}(x, t) = \chi(t)$ and $\widetilde{\chi_k}(x, t') = \chi_k(t')$ for all $x \in X$, $t \in \mathbb{F}_2$ and $t' \in \mathbb{Z}/4\mathbb{Z}$. In particular, this choice is possible if $B = B_{\mathrm{Weil}}$.

Recall from Proposition 2.29 that $\mathrm{G}_{\widetilde{\chi_k}}$ denotes one of the groups $\mathrm{G}^0_{\widetilde{\chi_k}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ depending on whether (118), (116) or (115) is satisfied. So far, we have two different constructions. The first one is motivated by the works of Weil [Wei64], Blasco [Bla93] and Glasby [Gla95]. It provides a Weil representation (unitary, faithful and irreducible) of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, which is a two-fold covering of $\mathrm{Ps}(B)$ (related to the orthogonal group $\mathrm{O}(Q)$). The second one is motivated by the articles of Glasby [Gla95], Gurevich-Hadani [GH12] and Genestier-Lysenko [GL12], and is a way to recover a link with the entire symplectic group $\mathrm{Sp}(W)$. Here we define a Weil representation (unitary, faithful and irreducible) of $\mathrm{G}_{\widetilde{\chi_k}}$, which is a covering of $\mathrm{ASp}(B_W)$ (related to the symplectic group $\mathrm{Sp}(W)$) by two, four and eight elements if (118), (116) and (115) are satisfied, respectively. We already know by Proposition 2.6 that there is a group monomorphism Γ from $\mathrm{Ps}(B)$ to $\mathrm{ASp}(B_W)$. As a result, we have the following diagram:



(D_{21})

Now we can ask whether these two constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}_{\widetilde{\chi_k}}$ are compatible, i.e. if there exists a group homomorphism $\widetilde{\Gamma} : \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \to \mathrm{G}_{\widetilde{\chi_k}}$ such that the following diagram (D_{22}) commutes:



(D_{22})

This section is dedicated to find out the conditions under which such group homomorphism $\widetilde{\Gamma}$ exists. First we prove in Lemma 2.37 that the commutativity of (D_{22}) entirely determines the expression of $\widetilde{\Gamma}$, and also implies its injectivity. This lemma also shows that the commutativity of (D_{22}) can be expressed as a relation between the choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ used to define the operators $\omega_{\widetilde{\chi}}(s)$ and $\omega_{\widetilde{\chi_k}}(s)$ in Propositions 1.28 and 2.21, respectively. Then, except for the case in which Q has Witt index one and $\dim_{\mathbb{F}_2}(W) = 4$, Proposition 2.39 elucidates the question. Provided suitable choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$, the group monomorphism $\widetilde{\Gamma}$ making the diagram (D_{22}) commute exists only if $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ (extension of $\mathrm{ASp}(B_W)$ by two elements, where Q has Witt index zero), $\mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$ (extension of $\mathrm{ASp}(B_W)$ by eight elements), or $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ with W a two-dimensional vector space (in which case Q has Witt index zero or one). This means in particular that if the Witt index of Q is greater than one, in spite of its minimal size, the affine metaplectic covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ gives rise to a Weil representation which is not compatible (in the sense we just specified) with the one of the pseudosymplectic covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. The case of a four-dimensional vector space W with a quadratic form Q of Witt index one remains open for now.

By Propositions 1.36 and 2.29, the extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ and $\mathrm{G}_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ do not depend on the choice of w_s satisfying (37) and (100) for every s in $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$, respectively. Hence we can first fix a choice $\{w_s \in W$ satisfying (37), $s \in \mathrm{Ps}(B)\}$. Then, since for every $s = (\sigma, f) \in \mathrm{Ps}(B)$, and for every $w \in \mathrm{K}(\sigma)$,

$$\langle w_s, w\rangle = f(w) \quad \Longleftrightarrow \quad \kappa(\langle w_s, w\rangle_W) = f(w) \quad \Longleftrightarrow \quad \langle w_s, w\rangle_W = \kappa^{-1}(f(w)),$$

we have that w_s satisfies (100) for $\Gamma(s) = (\sigma, \kappa^{-1} \circ f)$. To construct $\mathrm{G}_{\widetilde{\chi_k}}$, we can thus fix a choice $\{v_s \in W$ satisfying (100), $s \in \mathrm{ASp}(B_W)\}$ such that

$$v_{\Gamma(s)} = w_s \text{ for every } s \in \mathrm{Ps}(B). \tag{165}$$

Moreover, in order to simplify the notation, we may write $\omega_{\widetilde{\chi}}(s)$ and $\omega_{\widetilde{\chi_k}}(s)$ instead of $\omega_{\widetilde{\chi}}(s, w_s)$ and $\omega_{\widetilde{\chi_k}}(s, v_s)$, respectively. In the following lemma, we begin by proving a necessary and sufficient condition for the diagram (D_{22}) to commute.

Lemma 2.37. *We choose* $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ *satisfying* (46) *and* (48)*, and we choose* $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ *satisfying* (112)*. Suppose also that* $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ *satisfies* (118)*,* (116) *and* (115)*, respectively. If* $\widetilde{\Gamma} : \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \to \mathrm{G}_{\widetilde{\chi_k}}$ *is a group homomorphism such that the diagram* (D_{22}) *commutes, then* $\widetilde{\Gamma}$ *is injective and is defined by:*

$$\begin{aligned} \widetilde{\Gamma} : \qquad & \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \to \mathrm{G}_{\widetilde{\chi_k}} \\ & (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \mapsto (\Gamma(s), (-1)^l \omega_{\widetilde{\chi}}(s)). \end{aligned} \tag{166}$$

Moreover, the commutativity of (D_{22}) *is equivalent to the following condition:*

$$\nu(\sigma) \equiv \theta(\sigma) \mod \frac{2\pi}{N} \qquad \textit{for every } \sigma \in \mathrm{O}(Q). \tag{167}$$

Proof. Let

$$\begin{aligned} \widetilde{\Gamma} : \qquad & \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \to \mathrm{G}_{\widetilde{\chi_k}} \subseteq \mathrm{ASp}(B_W) \times \mathrm{U}(\mathrm{L}^2(Y)) \\ & \widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \mapsto (\widetilde{\Gamma}_1(\widetilde{s}), \widetilde{\Gamma}_2(\widetilde{s})) \end{aligned}$$

be a group homomorphism such that (D_{22}) commutes. Let $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Then the commutativity of (D_{22}) implies:

- $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{s}) = (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\Gamma})(\widetilde{s}) = \widetilde{\omega}_{\widetilde{\chi_k}}(\widetilde{\Gamma}_1(\widetilde{s}), \widetilde{\Gamma}_2(\widetilde{s})) = \widetilde{\Gamma}_2(\widetilde{s})$, so $\widetilde{\Gamma}_2(\widetilde{s}) = (-1)^l \omega_{\widetilde{\chi}}(s)$.
- $(\Gamma \circ \Psi_{\widetilde{\chi}})(\widetilde{s}) = (\Psi_{\widetilde{\chi_k}} \circ \widetilde{\Gamma})(\widetilde{s}) = \widetilde{\Gamma}_1(\widetilde{s})$, so $\widetilde{\Gamma}_1(\widetilde{s}) = \Gamma(s)$.

This entirely determines the expression of $\widetilde{\Gamma}$ and proves that this map makes the diagram commute, but it remains to prove that $\widetilde{\Gamma}$ is a well-defined and injective group homomorphism. The fact that $\widetilde{\Gamma}$ is a group monomorphism comes from the definition of $\widetilde{\Gamma}$, the expression of the product in $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}_{\widetilde{\chi_k}}$, and the fact that Γ is a group monomorphism. Therefore, the commutativity of (D_{22}) is equivalent to the fact that $\widetilde{\Gamma}$ is well-defined. Now, by definition of $\mathrm{G}_{\widetilde{\chi_k}}$, the map $\widetilde{\Gamma}$ is well-defined if and only if for every $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, the operator $\omega_{\widetilde{\chi}}(s)$ belongs to the set $\mu_N(\mathbb{C})\widetilde{\omega}_{\widetilde{\chi_k}}(\Gamma(s))$. Let $\widetilde{s} = (s, (-1)^l \omega_{\widetilde{\chi}}(s)) \in \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. By Propositions 1.28 and 2.21, and by (165),

$$\begin{aligned} \omega_{\widetilde{\chi_k}}(\Gamma(s)) &= \delta(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi_k((\kappa^{-1} \circ f)(w) + \langle v_{\Gamma(s)}, w \rangle_W + B_W(u, w + v_{\Gamma(s)})) \rho_{\widetilde{\chi_k}}(u + v_{\Gamma(s)}) \\ &= \delta(\sigma) \sum_{u \in \mathrm{I}(\sigma)} (\chi_k \circ \kappa^{-1})(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\widetilde{\chi_k}}(u + w_s) \\ &= \delta(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\widetilde{\chi_k}}(u + w_s), \end{aligned}$$

where $w \in W$ is any element such that $u = (1-\sigma)(w)$ and $\chi_k \circ \kappa^{-1}$ is the only nontrivial character χ of $\mathbb{F}_2$. For all $\varphi \in \mathrm{L}^2(Y)$ and $y' \in Y$, if we write $u + w_s = x + y$ and use Lemmas 1.17 and 2.11, then we obtain

$$\begin{aligned}
&\rho_{\widetilde{\chi_k}}(u+w_s)\varphi(y') \\
&= \widetilde{\chi_k}(x, -Q_W(y) + \langle x, y'\rangle_W - B_W(x+y', y))\varphi(y'-y) \\
&= \widetilde{\chi_k}(x,0)\chi_k(-Q_W(y) + \langle x, y'\rangle_W - B_W(x+y', y))\varphi(y'-y) \\
&= \widetilde{\chi_k}(x,0)\widetilde{\chi}(x,0)^{-1}\widetilde{\chi}(x,0)(\chi_k \circ \kappa^{-1})(-Q(y) + \langle x, y'\rangle - B(x+y', y))\varphi(y'-y) \\
&= \widetilde{\chi_k}(x,0)\widetilde{\chi}(x,0)^{-1}\widetilde{\chi}(x, -Q(y) + \langle x, y'\rangle - B(x+y', y))\varphi(y'-y) \\
&= \widetilde{\chi_k}(x,0)\widetilde{\chi}(x,0)^{-1}\rho_{\widetilde{\chi}}(u+w_s)\varphi(y').
\end{aligned}$$

This shows that $\rho_{\widetilde{\chi_k}}(u+w_s) = \widetilde{\chi_k}(x,0)\widetilde{\chi}(x,0)^{-1}\rho_{\widetilde{\chi}}(u+w_s)$, where $u + w_s = x + y$. This is the point where we need $\widetilde{\chi}$ and $\widetilde{\chi_k}$ to be related. Since by (164), we chose them such that they coincide on X, we have precisely that $\rho_{\widetilde{\chi_k}}(u+w_s) = \rho_{\widetilde{\chi}}(u+w_s)$, and thus $\omega_{\widetilde{\chi_k}}(\Gamma(s)) = \delta(\sigma)\gamma(\sigma)^{-1}\omega_{\widetilde{\chi}}(s)$. Hence $\omega_{\widetilde{\chi}}(s) \in \mu_N(\mathbb{C})\widetilde{\omega}_{\widetilde{\chi_k}}(\Gamma(s))$ if and only if $\gamma(\sigma)\delta(\sigma)^{-1} \in \mu_N(\mathbb{C})$, which is equivalent to the condition (167). □

Now we investigate under which conditions (167) is satisfied. In particular, we are going to prove that if $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ (i.e. if (116) is satisfied) and the Witt index of Q is greater than one, then, regardless of the choices made for $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (46) and (112), respectively, it is impossible that the diagram (D_{22}) commutes. First we put together all the conditions required to have a commutative diagram. If we use Lemma 2.37 in the case in which $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, we obtain the following condition relating the commutativity of (D_{22}) and the choices $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$: the diagram (D_{22}) commutes if and only if

$$\begin{cases}
\theta(\sigma_1) + \theta(\sigma_2) - \theta(\sigma_1\sigma_2) & \equiv 0 \mod \pi \\
\nu(1) \equiv 0 \mod \frac{\pi}{2} & \\
\nu(\sigma_v) + \nu(\sigma) - \nu(\sigma_v\sigma) & \equiv \begin{cases} 0 \mod \frac{\pi}{2} & \text{if } v \notin \mathrm{I}(\sigma) \text{ or } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 1 \\ \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } v = (1-\sigma)(z) \text{ with } \langle z, v\rangle = 0 \end{cases} \\
\nu(\sigma_1) - \theta(\sigma_1) & \equiv 0 \mod \frac{\pi}{2}
\end{cases} \tag{168}$$

for all $\sigma_1, \sigma_2 \in \mathrm{O}(Q)$, $\sigma \in \mathrm{Sp}(W)$ and every transvection $\sigma_v :\ w \mapsto w + \langle w, v\rangle v$ in $\mathrm{Sp}(W)$ such that $v \neq 0$ (i.e. (48), (116) and (167) are satisfied).

Lemma 2.38. *We choose $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying* (46) *and* (48)*, and we choose $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying* (112)*. Suppose also that* (118)*,* (116) *and* (115) *are satisfied, respectively.*

- ***i.*** *Let $\sigma \in \mathrm{O}(Q)$ such that $\sigma^2 = 1$. Then $\theta(\sigma) \equiv 0 \mod \frac{\pi}{2}$.*
- ***ii.*** *Let $v \in W$. Then $\sigma_v \in \mathrm{O}(Q)$ if and only if $v = 0$ or $Q(v) = 1$.*
- ***iii.*** *Let $v_1, v_2 \in W \backslash \{0\}$ such that $v_1 \neq v_2$. Then the transvections σ_{v_1} and σ_{v_2} commute if and only if $\langle v_1, v_2\rangle = 0$. Moreover, $\mathrm{I}(\sigma_{v_1}\sigma_{v_2}) = \mathrm{Span}\{v_1, v_2\}$ and any element $z_j \in W$ such*

that $v_j = (1-\sigma_{v_1}\sigma_{v_2})(z_j)$ *satisfies* $\langle z_j, v_j\rangle = 1$ *for* $j \in \{1,2\}$. *Furthermore,* $\sigma_{v_1}\sigma_{v_2}$ *belongs to* $\mathrm{O}(Q)$ *if and only if* $Q(v_1) = Q(v_2) = 1$.

iv. *Let* $v_1, v_2 \in W\backslash\{0\}$ *such that* $v_1 \neq v_2$. *Then:*

- $\mathrm{I}(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1}) = \mathrm{Span}\{\langle v_1, v_2\rangle v_1 + v_2\}$. *Moreover,* $\sigma_{v_1}\sigma_{v_2}\sigma_{v_1}$ *belongs to* $\mathrm{O}(Q)$ *if and only if*

$$Q(v_1) = Q(v_2) = 1 \quad \textit{or} \quad \begin{cases} (Q(v_1), Q(v_2)) & = (0,1) \\ \langle v_1, v_2\rangle & = 0 \end{cases} \quad \textit{or} \quad \begin{cases} Q(v_1) = Q(v_2) & = 0 \\ \langle v_1, v_2\rangle & = 1. \end{cases} \tag{169}$$

- $\mathrm{I}(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) = \mathrm{Span}\{(1 + \langle v_1, v_2\rangle)v_1, v_2\}$. *Moreover,* $\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}$ *belongs to* $\mathrm{O}(Q)$ *if and only if*

$$\begin{cases} Q(v_1) = Q(v_2) & = 1 \\ \langle v_1, v_2\rangle & = 1 \end{cases} \textit{or} \begin{cases} (Q(v_1), Q(v_2)) & = (0,1) \\ \langle v_1, v_2\rangle & = 1 \end{cases} \textit{or} \begin{cases} Q(v_1) = Q(v_2) & = 0 \\ \langle v_1, v_2\rangle & = 0. \end{cases} \tag{170}$$

v. (168) *implies that for every transvection* σ_v *in* $\mathrm{O}(Q)$ *and* $\mathrm{Sp}(W)$, *respectively,*

$$\begin{cases} \theta(\sigma_v) & \equiv 0 \mod \frac{\pi}{2} \\ \nu(\sigma_v) & \equiv 0 \mod \frac{\pi}{2} \end{cases}. \tag{171}$$

Proof. **i.** Let $\sigma \in \mathrm{O}(Q)$. If $\sigma^2 = 1$ then by (48),

$$\theta(\sigma) + \theta(\sigma) - \theta(\sigma^2) \equiv 0 \mod \pi,$$

i.e. $\theta(\sigma) \equiv 0 \mod \frac{\pi}{2}$ since $\theta(1) \equiv 0 \mod \pi$ by (38).

ii. Let $v \in W$. If $v = 0$ then $\sigma_v = 1$ so σ_v belongs to $\mathrm{O}(Q)$. Suppose now that $v \neq 0$. Then for every $z \in W$,

$$Q(\sigma_v(z)) = Q(z + \langle z, v\rangle v) = Q(z) + \langle z, v\rangle^2 Q(v) + \langle z, v\rangle^2 = Q(z) + \langle z, v\rangle(1 + Q(v)).$$

This implies that $\sigma \in \mathrm{O}(Q)$ if and only if $Q(v) = 1$.

iii. Let $v_1, v_2 \in W\backslash\{0\}$ such that $v_1 \neq v_2$. Then for every $z \in W$,

$$(\sigma_{v_1}\sigma_{v_2})(z) = z + \langle z, v_2\rangle v_2 + \langle z + \langle z, v_2\rangle v_2, v_1\rangle v_1 = z + \langle z, v_2\rangle v_2 + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)v_1. \tag{172}$$

Hence

$$(\sigma_{v_1}\sigma_{v_2})(z) - (\sigma_{v_2}\sigma_{v_1})(z) = \langle z, v_2\rangle\langle v_2, v_1\rangle v_1 - \langle z, v_1\rangle\langle v_1, v_2\rangle v_2 = \langle v_2, v_1\rangle(\langle z, v_2\rangle v_1 + \langle z, v_1\rangle v_2). \tag{173}$$

Since $v_1, v_2 \neq 0$ and $v_1 \neq v_2$, we obtain by (173) that σ_{v_1} and σ_{v_2} commute if and only if $\langle v_1, v_2\rangle = 0$. Moreover, (172) implies that for every $z \in W$,

$$(1-\sigma_{v_1}\sigma_{v_2})(z) = 0 \iff \begin{cases} \langle z, v_2\rangle & = 0 \\ \langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle & = 0 \end{cases} \iff \langle z, v_1\rangle = \langle z, v_2\rangle = 0.$$

This implies that $\mathrm{K}(\sigma_{v_1}\sigma_{v_2}) = \mathrm{Span}\{v_1\}^\perp \cap \mathrm{Span}\{v_2\}^\perp$, i.e. $\mathrm{I}(\sigma_{v_1}\sigma_{v_2}) = \mathrm{Span}\{v_1, v_2\}$. Furthermore, by (172),

$$\begin{cases} (1-\sigma_{v_1}\sigma_{v_2})(z) = v_1 & \iff \begin{cases} \langle z, v_2\rangle & = 0 \\ \langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle & = 1 \end{cases} \iff \begin{cases} \langle z, v_2\rangle & = 0 \\ \langle z, v_1\rangle & = 1 \end{cases} \\ (1-\sigma_{v_1}\sigma_{v_2})(z) = v_2 & \iff \begin{cases} \langle z, v_2\rangle & = 1 \\ \langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle & = 0 \end{cases} \iff \begin{cases} \langle z, v_2\rangle & = 1 \\ \langle z, v_1\rangle & = \langle v_1, v_2\rangle. \end{cases} \end{cases}$$

Now by (172), for every $z \in W$,

$$\begin{aligned} &Q((\sigma_{v_1}\sigma_{v_2})(z)) \\ &= Q(z) + Q(\langle z, v_2\rangle v_2 + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)v_1) + \langle z, \langle z, v_2\rangle v_2 + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)v_1\rangle \\ &= Q(z) + \langle z, v_2\rangle Q(v_2) + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)Q(v_1) + \langle z, v_2\rangle(\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)\langle v_2, v_1\rangle \\ &\quad + \langle z, \langle z, v_2\rangle v_2 + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)v_1\rangle \\ &= Q(z) + \langle z, v_2\rangle Q(v_2) + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)Q(v_1) + \langle z, v_2\rangle(\langle z, v_1\rangle + 1)\langle v_2, v_1\rangle \\ &\quad + \langle z, v_2\rangle + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)\langle z, v_1\rangle \\ &= Q(z) + \langle z, v_2\rangle Q(v_2) + (\langle z, v_1\rangle + \langle z, v_2\rangle\langle v_2, v_1\rangle)Q(v_1) + \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_1\rangle. \end{aligned} \tag{174}$$

If $(Q(v_1), Q(v_2)) = (1, 1)$ then by ii, $\sigma_{v_1}\sigma_{v_2} \in \mathrm{O}(Q)$. Similarly if $(Q(v_1), Q(v_2)) \in \{(0, 1), (1, 0)\}$, then $\sigma_{v_1}\sigma_{v_2} \notin \mathrm{O}(Q)$. Suppose now that $(Q(v_1), Q(v_2)) = (0, 0)$. Then (174) becomes

$$Q((\sigma_{v_1}\sigma_{v_2})(z)) = Q(z) + \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_1\rangle. \tag{175}$$

If $\sigma_{v_1}\sigma_{v_2}$ were in $\mathrm{O}(Q)$ then by (175), $\mathrm{Span}\{(1 + \langle v_1, v_2\rangle)v_2 + v_1\}^\perp$ would be the whole space W, which is impossible since $v_1 \neq 0$ and $v_1 \neq v_2$ by assumption. Thus $\sigma_{v_1}\sigma_{v_2}$ does not belong to $\mathrm{O}(Q)$. Finally we conclude that $\sigma_{v_1}\sigma_{v_2}$ belongs to $\mathrm{O}(Q)$ if and only if $Q(v_1) = Q(v_2) = 1$.

iv. Let $v_1, v_2 \in W\backslash\{0\}$ such that $v_1 \neq v_2$. Let $v_3 \in W$. Then for every $z \in W$, by (172),

$$\begin{aligned} &(\sigma_{v_1}\sigma_{v_2}\sigma_{v_3})(z) \\ &= \sigma_{v_1}(z + \langle z, v_3\rangle v_3 + (\langle z, v_2\rangle + \langle z, v_3\rangle\langle v_3, v_2\rangle)v_2) \\ &= z + \langle z, v_3\rangle v_3 + (\langle z, v_2\rangle + \langle z, v_3\rangle\langle v_3, v_2\rangle)v_2 + \langle z + \langle z, v_3\rangle v_3 + (\langle z, v_2\rangle \\ &\quad + \langle z, v_3\rangle\langle v_3, v_2\rangle)v_2, v_1\rangle v_1 \\ &= z + \langle z, v_3\rangle v_3 + (\langle z, v_2\rangle + \langle z, v_3\rangle\langle v_3, v_2\rangle)v_2 \\ &\quad + (\langle z, v_1\rangle + \langle z, v_3\rangle\langle v_3, v_1\rangle + (\langle z, v_2\rangle + \langle z, v_3\rangle\langle v_3, v_2\rangle)\langle v_2, v_1\rangle)v_1 \\ &= z + \lambda_3 v_3 + \lambda_2 v_2 + \lambda_1 v_1, \end{aligned} \tag{176}$$

where

$$\begin{cases} \lambda_1 & = \langle z, v_1\rangle + \langle z, v_3\rangle\langle v_3, v_1\rangle + (\langle z, v_2\rangle + \langle z, v_3\rangle\langle v_3, v_2\rangle)\langle v_2, v_1\rangle \\ & = \langle z, v_1\rangle + \lambda_3\langle v_3, v_1\rangle + \lambda_2\langle v_2, v_1\rangle \\ \lambda_2 & = \langle z, v_2\rangle + \langle z, v_3\rangle\langle v_3, v_2\rangle = \langle z, v_2\rangle + \lambda_3\langle v_3, v_2\rangle \\ \lambda_3 & = \langle z, v_3\rangle. \end{cases} \tag{177}$$

Hence

$$\begin{aligned} &Q((\sigma_{v_1}\sigma_{v_2}\sigma_{v_3})(z)) \\ &= Q(z) + \lambda_3 Q(v_3) + \lambda_2 Q(v_2) + \lambda_1 Q(v_1) + \langle \lambda_2 v_2, \lambda_1 v_1\rangle + \langle \lambda_3 v_3, \lambda_2 v_2 + \lambda_1 v_1\rangle \\ &\quad + \langle z, \lambda_3 v_3 + \lambda_2 v_2 + \lambda_1 v_1\rangle \\ &= Q(z) + \lambda_3 Q(v_3) + \lambda_2 Q(v_2) + \lambda_1 Q(v_1) + \lambda_1\lambda_2\langle v_2, v_1\rangle + \lambda_3\langle v_3, \lambda_2 v_2 + \lambda_1 v_1\rangle \\ &\quad + \langle z, \lambda_3 v_3 + \lambda_2 v_2 + \lambda_1 v_1\rangle. \end{aligned} \tag{178}$$

- Suppose that $v_3 = v_1$, then by (177),

$$\begin{cases} \lambda_1 &= \langle z, v_1\rangle + (\langle z, v_2\rangle + \langle z, v_1\rangle\langle v_1, v_2\rangle)\langle v_2, v_1\rangle = \langle z, v_1\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_2\rangle\langle v_1, v_2\rangle \\ \lambda_2 &= \langle z, v_2\rangle + \langle z, v_1\rangle\langle v_1, v_2\rangle \\ \lambda_3 &= \langle z, v_1\rangle. \end{cases} \tag{179}$$

This implies by (176) that

$$(1 - \sigma_{v_1}\sigma_{v_2}\sigma_{v_1})(z) = (\lambda_1 + \lambda_3)v_1 + \lambda_2 v_2 = \langle z, v_1 + v_2\rangle\langle v_1, v_2\rangle v_1 + (\langle z, v_2\rangle + \langle z, v_1\rangle\langle v_1, v_2\rangle)v_2. \tag{180}$$

Since $v_1, v_2 \neq 0$ and $v_1 \neq v_2$,

$$z \in \mathrm{K}(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1}) \iff \begin{cases} \langle z, v_1 + v_2\rangle\langle v_1, v_2\rangle &= 0 \\ \langle z, v_2\rangle + \langle z, v_1\rangle\langle v_1, v_2\rangle &= 0 \end{cases} \iff \begin{cases} \langle z, v_2\rangle = 0 & \text{if } \langle v_1, v_2\rangle = 0 \\ \langle z, v_1 + v_2\rangle = 0 & \text{if } \langle v_1, v_2\rangle = 1. \end{cases}$$

This shows that $\mathrm{I}(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1}) = \mathrm{Span}\{\langle v_1, v_2\rangle v_1 + v_2\}$. Now by (177), (178) and (179),

$$\begin{aligned} &Q((\sigma_{v_1}\sigma_{v_2}\sigma_{v_1})(z)) \\ &= Q(z) + (\lambda_1 + \lambda_3)Q(v_1) + \lambda_2 Q(v_2) + \lambda_2(\lambda_1 + \lambda_3)\langle v_1, v_2\rangle + \langle z, (\lambda_1 + \lambda_3)v_1 + \lambda_2 v_2\rangle \\ &= Q(z) + \lambda_2\langle v_1, v_2\rangle Q(v_1) + \lambda_2 Q(v_2) + \lambda_2\lambda_2\langle v_1, v_2\rangle\langle v_1, v_2\rangle + \langle z, \lambda_2\langle v_1, v_2\rangle v_1 + \lambda_2 v_2\rangle \\ &= Q(z) + \lambda_2(\langle v_1, v_2\rangle Q(v_1) + Q(v_2) + \langle v_1, v_2\rangle + \langle v_1, v_2\rangle\langle z, v_1\rangle + \langle z, v_2\rangle) \\ &= Q(z) + \lambda_2(\langle v_1, v_2\rangle Q(v_1) + Q(v_2) + \langle v_1, v_2\rangle + \lambda_2) \\ &= Q(z) + \lambda_2(\langle v_1, v_2\rangle(1 + Q(v_1)) + Q(v_2) + 1). \end{aligned} \tag{181}$$

Hence $\sigma_{v_1}\sigma_{v_2}\sigma_{v_1}$ belongs to $\mathrm{O}(Q)$ if and only if for every $z \in W$,

$$\lambda_2 = 0 \qquad \text{or} \qquad \langle v_1, v_2\rangle(1 + Q(v_1)) + Q(v_2) + 1 = 0. \tag{182}$$

Since $\lambda_2 = \langle z, v_2\rangle + \langle z, v_1\rangle\langle v_1, v_2\rangle = \langle z, v_2 + \langle v_1, v_2\rangle v_1\rangle$ and $v_2 + \langle v_1, v_2\rangle v_1$ is nonzero, there exists $z \in W$ such that λ_2 is nonzero. Thus (182) is satisfied if and only if $\langle v_1, v_2\rangle(1 + Q(v_1)) + Q(v_2) = 1$, i.e. $Q(v_1) = Q(v_2) = 1$, or $(Q(v_1), Q(v_2)) = (0, 1)$ and $\langle v_1, v_2\rangle = 0$, or $(Q(v_1), Q(v_2)) = (0, 0)$ and $\langle v_1, v_2\rangle = 1$.

- Suppose now that $v_3 = v_1 + v_2$. Then by (177),

$$\begin{cases} \lambda_1 &= \langle z, v_1\rangle + \langle z, v_2\rangle\langle v_1, v_2\rangle \\ \lambda_2 &= \langle z, v_1\rangle\langle v_1, v_2\rangle + \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle) \\ \lambda_3 &= \langle z, v_1 + v_2\rangle. \end{cases} \tag{183}$$

Hence by (176),

$$\begin{aligned}(1 - \sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2})(z) &= (\lambda_1 + \lambda_3)v_1 + (\lambda_2 + \lambda_3)v_2 \\ &= \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle)v_1 + (\langle z, v_1\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_2\rangle\langle v_1, v_2\rangle)v_2.\end{aligned} \tag{184}$$

Since $v_1, v_2 \neq 0$ and $v_1 \neq v_2$,

$$\begin{aligned} z \in \mathrm{K}(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) &\iff \begin{cases} \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle) & = 0 \\ \langle z, v_1\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_2\rangle\langle v_1, v_2\rangle & = 0 \end{cases} \\ &\iff \begin{cases} \langle z, v_1\rangle = \langle z, v_2\rangle = 0 & \text{if } \langle v_1, v_2\rangle = 0 \\ \langle z, v_2\rangle = 0 & \text{if } \langle v_1, v_2\rangle = 1. \end{cases} \end{aligned}$$

This shows that $\mathrm{I}(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) = \mathrm{Span}\{(1 + \langle v_1, v_2\rangle)v_1, v_2\}$. Now by (178) and (183),

$$\begin{aligned} &Q((\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2})(z)) \\ &= Q(z) + (\lambda_1 + \lambda_3)Q(v_1) + (\lambda_2 + \lambda_3)Q(v_2) + (\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)\langle v_1, v_2\rangle \\ &\quad + \langle z, (\lambda_1 + \lambda_3)v_1 + (\lambda_2 + \lambda_3)v_2\rangle \\ &= Q(z) + \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle)Q(v_1) + (\langle z, v_1\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_2\rangle\langle v_1, v_2\rangle)Q(v_2) \\ &\quad + (1 + \langle v_1, v_2\rangle)\langle z, v_1\rangle\langle z, v_2\rangle\langle v_1, v_2\rangle + \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle)\langle z, v_1\rangle \\ &\quad + (\langle z, v_1\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_2\rangle\langle v_1, v_2\rangle)\langle z, v_2\rangle \\ &= Q(z) + \langle z, v_2\rangle(1 + \langle v_1, v_2\rangle)Q(v_1) + (\langle z, v_1\rangle(1 + \langle v_1, v_2\rangle) + \langle z, v_2\rangle\langle v_1, v_2\rangle)Q(v_2) \\ &\quad + \langle z, v_2\rangle\langle v_1, v_2\rangle. \end{aligned} \tag{185}$$

Hence $\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}$ belongs to $\mathrm{O}(Q)$ if and only if for every $z \in W$:

$$\langle z, v_2\rangle(1+\langle v_1, v_2\rangle)Q(v_1)+(\langle z, v_1\rangle(1+\langle v_1, v_2\rangle)+\langle z, v_2\rangle\langle v_1, v_2\rangle)Q(v_2)+\langle z, v_2\rangle\langle v_1, v_2\rangle = 0. \tag{186}$$

If $(Q(v_1), Q(v_2)) = (0, 0)$ then (186) is satisfied if and only if $\langle v_1, v_2\rangle = 0$, since v_2 is nonzero. If $(Q(v_1), Q(v_2)) = (0, 1)$ then (186) is satisfied if and only if $\langle v_1, v_2\rangle = 1$, since v_1 is nonzero. If $(Q(v_1), Q(v_2)) = (1, 0)$ then (186) is not satisfied since v_2 is nonzero. Finally, if $(Q(v_1), Q(v_2)) = (1, 1)$, then (186) is satisfied if and only if $\langle v_1, v_2\rangle = 1$, since $v_1 + v_2$ is nonzero.

v. Let $v \in W$ such that the transvection σ_v belongs to $\mathrm{O}(Q)$, which is equivalent to $v = 0$ or $Q(v) = 1$ by ii. Since $\sigma_v^2 = 1$, by i, $\theta(\sigma_v) \equiv 0 \mod \frac{\pi}{2}$. Then by (168), $\nu(\sigma_v) \equiv 0 \mod \frac{\pi}{2}$. So it remains to prove (171) if v is nonzero and $Q(v) = 0$. Since v is nonzero,

there exists $w \in W$ such that $\langle w, v\rangle = 1$. If $Q(w) = 0$ then we set $v_1 = w$, and if $Q(w) = 1$ then we set $v_1 = w + v$. In this way, $Q(v_1) = Q(v) = 0$ and $\langle v_1, v\rangle = 1$. According to iv, $v_1 \notin \mathrm{I}(\sigma_{v_1}\sigma_v\sigma_{v_1}) = \mathrm{Span}\{v_1 + v\}$. Then by (168),

$$\nu(\sigma_{v_1}) + \nu(\sigma_{v_1}\sigma_v\sigma_{v_1}) - \nu(\sigma_{v_1}\sigma_{v_1}\sigma_v\sigma_{v_1}) \equiv 0 \mod \frac{\pi}{2},$$

i.e.

$$\nu(\sigma_{v_1}\sigma_v\sigma_{v_1}) \equiv \nu(\sigma_v\sigma_{v_1}) - \nu(\sigma_{v_1}) \mod \frac{\pi}{2}. \tag{187}$$

Since $v \notin \mathrm{I}(\sigma_{v_1}) = \mathrm{Span}\{v_1\}$, again by (168):

$$\nu(\sigma_v) + \nu(\sigma_{v_1}) - \nu(\sigma_v\sigma_{v_1}) \equiv 0 \mod \frac{\pi}{2}.$$

Then (187) becomes

$$\nu(\sigma_{v_1}\sigma_v\sigma_{v_1}) \equiv \nu(\sigma_v) \mod \frac{\pi}{2}. \tag{188}$$

Since by iv, the map $\sigma_{v_1}\sigma_v\sigma_{v_1}$ belongs to $\mathrm{O}(Q)$, then by (168):

$$\nu(\sigma_{v_1}\sigma_v\sigma_{v_1}) \equiv \theta(\sigma_{v_1}\sigma_v\sigma_{v_1}) \mod \frac{\pi}{2}. \tag{189}$$

Since $(\sigma_{v_1}\sigma_v\sigma_{v_1})^2 = 1$, $\theta(\sigma_{v_1}\sigma_v\sigma_{v_1}) \equiv 0 \mod \frac{\pi}{2}$ by i. Therefore, (188) and (189) finally imply that $\nu(\sigma_v) \equiv 0 \mod \frac{\pi}{2}$, which concludes the proof.

□

Now we can finally state the following results concerning the compatibility of the constructions of $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ and $\mathrm{G}_{\widetilde{\chi_k}}$, in terms of the existence of the commutative diagram (D_{22}).

Proposition 2.39 (Link between $\widetilde{\mathbf{Ps}(B)}_{\tilde{\chi}}$ and $\mathbf{G}_{\widetilde{\chi_k}}$). *We choose* $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ *satisfying* (46) *and* (48)*, and we choose* $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ *satisfying* (112)*. Consider* $\widetilde{\Gamma}$ *as in* (166)*.*

i. *Suppose that* $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$*, i.e. that* Q *has Witt index zero,* $\dim_{\mathbb{F}_2}(W) = 2$ *and* (118) *is satisfied. Then the diagram* (D_{22}) *commutes if and only if*

$$\nu(\sigma) \equiv \theta(\sigma) \mod \pi \qquad \textit{for every } \sigma \in \mathrm{O}(Q). \tag{190}$$

In particular, this is the case if $\gamma(\sigma)$ *and* $\delta(\sigma)$ *are chosen to be real for every* σ *in* $\mathrm{O}(Q) = \mathrm{Sp}(W)$*.*

ii. *Suppose that* $\mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$*, i.e. that* (115) *is satisfied. Then the diagram* (D_{22}) *commutes if and only if*

$$\nu(\sigma) \equiv \theta(\sigma) \mod \frac{\pi}{4} \qquad \textit{for every } \sigma \in \mathrm{O}(Q). \tag{191}$$

In particular, this is the case if $\gamma(\sigma)$ *and* $\delta(\sigma)$ *are chosen to be real for every* σ *in* $\mathrm{O}(Q)$ *and* $\mathrm{Sp}(W)$*, respectively.*

iii. *Suppose that $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, i.e. that (116) is satisfied.*

- *If $\dim_{\mathbb{F}_2}(W) = 2$ (in which case Q has Witt index zero or one) then the diagram (D_{22}) commutes if and only if*

$$\nu(\sigma) \equiv \theta(\sigma) \mod \frac{\pi}{2} \qquad \text{for every } \sigma \in \mathrm{O}(Q). \tag{192}$$

  In particular, this is the case if $\gamma(\sigma)$ and $\delta(\sigma)$ are chosen to be real for every σ in $\mathrm{O}(Q)$ and $\mathrm{Sp}(W)$, respectively.
- *If the Witt index of Q is greater than one then it is impossible to choose $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ such that the diagram (D_{22}) commutes.*

Remark 2.40. The only case which is not treated in Proposition 2.39 is the one in which Q has Witt index one and $\dim_{\mathbb{F}_2}(W) = 4$. At present, this case remains unsettled.

Proof. **i.** This result comes directly from (167) in Lemma 2.37 since $N = 2$ in this case. This example is studied more deeply in section 3.2.

ii. This result is also due to (167) in Lemma 2.37, with $N = 8$ in this case.

iii. Suppose that $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, i.e. that (116) is satisfied. If $\dim_{\mathbb{F}_2}(W) = 2$ (in which case Q has Witt index zero or one) then the result is directly due to (167) in Lemma 2.37 (with $N = 4$ in this case), and the fact that (116) is equivalent to (117), which implies that $\delta(\sigma)$ can be chosen to be real for every σ in $\mathrm{Sp}(W)$. This is also more detailed in sections 3.1 and 3.2.
Suppose now that the Witt index of Q is greater than one and the diagram (D_{22}) commutes, i.e. (168) is satisfied. Then there exists $V = \mathrm{Span}\{v_1, v_2\}$ a Q-singular subspace of dimension two. Hence v_1 and v_2 are nonzero and satisfy $v_1 \neq v_2$, $Q(v_1) = Q(v_2) = 0$ and $\langle v_1, v_2\rangle = Q(v_1 + v_2) - Q(v_1) - Q(v_2) = 0$. By Lemma 2.38.iii, $\mathrm{I}(\sigma_{v_2}\sigma_{v_1+v_2}) = \mathrm{Span}\{v_2, v_1 + v_2\} = \mathrm{Span}\{v_1, v_2\}$ so $v_1 \in \mathrm{I}(\sigma_{v_2}\sigma_{v_1+v_2})$. Let $z \in W$ such that $(1 - \sigma_{v_2}\sigma_{v_1+v_2})(z) = v_1$. By (172) and since $\langle v_1, v_2\rangle = 0$,

$$\begin{aligned}(1 - \sigma_{v_2}\sigma_{v_1+v_2})(z) &= \langle z, v_1 + v_2\rangle(v_1 + v_2) + (\langle z, v_2\rangle + \langle z, v_1 + v_2\rangle\langle v_1 + v_2, v_2\rangle)v_2 \\ &= \langle z, v_1 + v_2\rangle v_1 + (\langle z, v_2\rangle + \langle z, v_1 + v_2\rangle(1 + \langle v_1, v_2\rangle))v_2 \\ &= \langle z, v_1 + v_2\rangle v_1 + \langle z, v_1\rangle v_2.\end{aligned}$$

Since $v_1 \neq v_2$, this implies that $\langle z, v_1\rangle = 0$ and $\langle z, v_1 + v_2\rangle = 1$. Then by (168), since $(1 - \sigma_{v_2}\sigma_{v_1+v_2})(z) = v_1$ with $\langle z, v_1\rangle = 0$:

$$\nu(\sigma_{v_1}) + \nu(\sigma_{v_2}\sigma_{v_1+v_2}) - \nu(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \tag{193}$$

Moreover, since $v_2 \notin \mathrm{I}(\sigma_{v_1+v_2}) = \mathrm{Span}\{v_1 + v_2\}$, by (168),

$$\nu(\sigma_{v_2}) + \nu(\sigma_{v_1+v_2}) - \nu(\sigma_{v_2}\sigma_{v_1+v_2}) \equiv 0 \mod \frac{\pi}{2}. \tag{194}$$

Combining equations (193) and (194) together, we obtain

$$\nu(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) \equiv \nu(\sigma_{v_1}) + \nu(\sigma_{v_2}) + \nu(\sigma_{v_1+v_2}) - \frac{\pi}{4} \mod \frac{\pi}{2}.$$

By Lemma 2.38.v, this provides

$$\nu(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) \equiv -\frac{\pi}{4} \mod \frac{\pi}{2}. \tag{195}$$

Now since $\langle v_1, v_2\rangle = 0$, the transvections σ_{v_1}, σ_{v_2} and $\sigma_{v_1+v_2}$ commute by Lemma 2.38.iii. This implies that $(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2})^2 = 1$. Moreover, by Lemma 2.38.iv, $\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}$ belongs to $\mathrm{O}(Q)$. Thus by Lemma 2.38.i,

$$\theta(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) \equiv 0 \mod \frac{\pi}{2}.$$

This contradicts (195) since by (168), we should have $\nu(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) \equiv \theta(\sigma_{v_1}\sigma_{v_2}\sigma_{v_1+v_2}) \mod \frac{\pi}{2}$. Therefore it is impossible to choose $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ and $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ such that the diagram (D_{22}) commutes.

□

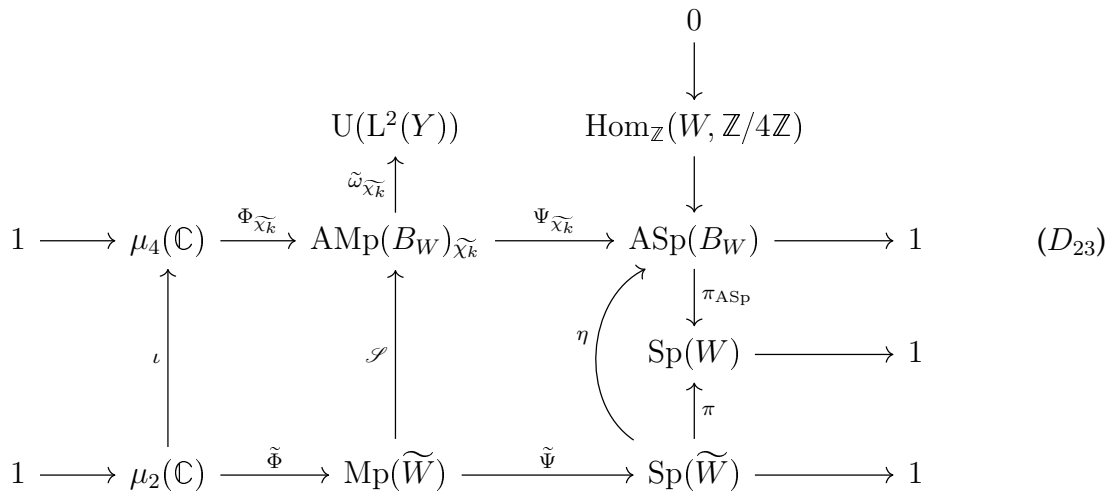
Remark 2.41. In chapter 1, the linear Weil representation arising from the study of the orthogonal group $\mathrm{O}(Q)$ has been settled as a linear representation of a two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$. Now if we study the symplectic group $\mathrm{Sp}(W)$, then Proposition 2.39 provides two different paths to follow, depending on the properties we want the linear Weil representation of $\mathrm{G}_{\widetilde{\chi_k}}$ to satisfy:

- If our goal is to define a linear Weil representation of the smallest possible extension of $\mathrm{ASp}(B_W)$, then we consider the Weil representation of $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ if Q has Witt index zero, and $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ otherwise. Nevertheless, if the Witt index of Q is greater than one, we lose the compatibility of this construction with the one of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$. Here compatibility means the commutativity of the diagram (D_{22}).

- If our goal is to define a linear Weil representation for which the diagram (D_{22}) commutes, then we should consider $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ if Q has Witt index zero, and $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ if Q has Witt index one and $\dim_{\mathbb{F}_2}(W) = 2$ (consequently, the size of the extension is also minimal in these cases). However, if the Witt index of Q is greater than one, then we need to consider the linear Weil representation of $\mathrm{G}_{\widetilde{\chi_k}} = \widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$, and unfortunately we lose the minimality of the extension's size.

2.5 The two-fold metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$

In [GH12], Gurevich and Hadani use the bilinear forms $B_{\widetilde{W}}$ and B_W corresponding to B_{Weil}. They construct a central extension $1 \to \mu_2(\mathbb{C}) \to \mathrm{Mp}(\widetilde{W}) \to \mathrm{Sp}(\widetilde{W}) \to 1$, a homomorphism of central extensions $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and then a "splitting" of the Weil representation, i.e. a Weil representation of the metaplectic group $\mathrm{Mp}(\widetilde{W})$. This result appears as Theorem 1.4 in [GH12] and suggests another commutative diagram (D_{23}):



(D_{23})

Since $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ should be a lift to the metaplectic group $\mathrm{Mp}(\widetilde{W})$ of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, we would expect that $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ stays irreducible and faithful. Unfortunately, as shows the following result, this is in general not possible.

Lemma 2.42 (Non-injectivity of $\mathscr{S}$). *Recall that $\dim_{\mathbb{F}_2}(W) = 2n$, where $n \geqslant 1$.*

- ***i.*** *Suppose that $n = 1$. Then the only way to obtain a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and an injective group homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is to construct $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3)$ and $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{U}_2(\mathbb{F}_3)$. In this case, $\mathscr{S}$ is an isomorphism. Moreover, we define in this way a Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of the metaplectic group $\mathrm{Mp}(\widetilde{W})$ which stays unitary, irreducible and faithful. (We can even set $\mathscr{S} = \mathrm{id}$, and in this case the lift $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is the original representation $\widetilde{\omega}_{\widetilde{\chi_k}}$.)*
- ***ii.*** *If $n \geqslant 2$, then it is not possible to find a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ such that there exists an injective group homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. More generally, there is no injective group homomorphism from a two-fold covering of $\mathrm{Sp}(\widetilde{W})$ to a four-fold covering of $\mathrm{ASp}(B_W)$. Hence it is not possible to lift the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ to the metaplectic group $\mathrm{Mp}(\widetilde{W})$ by means of $\mathscr{S}$, if we want to keep the injectivity of the representation.*

Proof. The case $n = 1$ is treated in detail in chapter 3. More precisely, a first approach, which only uses the construction of the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, is done in sections 3.1.4 and 3.2.4, see in particular Remarks 3.18 and 3.39 for a summary of the situation. A second approach, which uses only the construction of the Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ defined in Proposition 2.65, is presented in section 3.3. So we can suppose in the rest of the proof that $n \geqslant 2$. We use an argument of cardinality by showing that $|\mathrm{Mp}(\widetilde{W})| > |\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}|$. In fact, using Proposition 2.7.vii, we have on one hand

$$|\mathrm{Mp}(\widetilde{W})| = 2|\mathrm{Sp}(\widetilde{W})| = 2^{2n^2+n+1}|\mathrm{Sp}(W)|,$$

and on the other hand

$$|\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}| = 4|\mathrm{ASp}(B_W)| = 4|W||\mathrm{Sp}(W)| = 2^{2n+2}|\mathrm{Sp}(W)|.$$

This implies that

$$|\mathrm{Mp}(\widetilde{W})| = 2^{2n^2+n+1-(2n+2)}|\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}| = 2^{2n^2-n-1}|\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}|,$$

where $2n^2 - n - 1 > 0$ for all $n \geqslant 2$. Since $|\mathrm{Mp}(\widetilde{W})| > |\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}|$, there is no way to construct a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and a group homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that $\mathscr{S}$ is injective. Since the entire argument depends only on the cardinality of the group $\mathrm{Sp}(W)$ and the required degrees of the coverings, a construction of a faithful representation of a two-fold covering of $\mathrm{Sp}(\widetilde{W})$ as an injective splitting of a faithful representation of a four-fold covering of $\mathrm{ASp}(B_W)$ is impossible. □

Remark 2.43. Let $(W, \langle \cdot, \cdot \rangle)$ be a symplectic $\mathbb{F}_2$-vector space, and B be a bilinear form such that $\langle w_1, w_2 \rangle = B(w_1, w_2) - B(w_2, w_1)$ for all $w_1, w_2 \in W$. Then Lemma 2.42 shows the following fact. If we want to construct a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ as Gurevich and Hadani did, and a group homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that $\widetilde{\omega}_{\widetilde{\chi_k}}$ lifts to a (linear) Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$, then, unless $\dim_{\mathbb{F}_2}(W) = 2$, the lifted representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ will never be faithful, unlike $\widetilde{\omega}_{\widetilde{\chi_k}}$. Therefore, we can only hope to construct a group homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that the diagram (D_{23}) is commutative. More generally, a construction of a faithful representation of a two-fold covering of $\mathrm{Sp}(\widetilde{W})$ as an injective splitting of a faithful representation of a four-fold covering of $\mathrm{ASp}(B_W)$ is impossible.

Before investigating further the existence of $\mathrm{Mp}(\widetilde{W})$ and the homomorphism of central extensions $\mathscr{S}$, we make some comments about [GH12]. In this article, the existence of $\mathscr{S}$ and the Weil representation of $\mathrm{Mp}(\widetilde{W})$ rely on an argument which appears in [GH12, A.6] (proof of Theorem 3.3), and which we reproduce in the next lemma. Lysenko suggested to us that this argument might not hold in general for finite fields of characteristic two, and this issue was later confirmed by Gurevich in a conversation with Przebinda. In fact, we now give a counterexample to this argument in small dimension. This adds further motivation to find an explicit construction of a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$, in order to confirm or refute the existence of a group homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ making the diagram (D_{23}) commute.

Lemma 2.44. [GH12, A.6, Proof of Theorem 3.3.] *Let W be a symplectic vector space defined over a finite field $\mathbb{F}$ of characteristic two. Let N and L be two lagrangians of W. Let M_1 and M_2 be two lagrangians of W such that $W = M_1 \oplus N = L \oplus M_1 = M_2 \oplus N = L \oplus M_2$. Then there exists a lagrangian M_3 of W such that $M_1 \oplus M_3 = M_2 \oplus M_3 = M_3 \oplus N = L \oplus M_3$.*

Remark 2.45. Gurevich and Hadani use the statement of Lemma 2.44 for enhanced Lagrangians instead of Lagrangians. However, since every Lagrangian is the domain of an enhanced Lagrangian (see [GH12, Lemma 2.2]), the non-existence of a Lagrangian implies the non-existence of an enhanced Lagrangian.

Counterexample 2.46. Suppose that $\mathbb{F} = \mathbb{F}_2$. If the dimension of W over $\mathbb{F}_2$ is two, then we can consider the ordered basis $\{(1,0),(0,1)\}$ of W, and the lagrangians of W are the subspaces

$\mathbb{F}_2(1,0)$, $\mathbb{F}_2(0,1)$ and $\mathbb{F}_2(1,1)$. Hence the dimension of W should at least be equal to four in Lemma 2.44 to ensure the existence of enough lagrangians. Suppose now that $W \simeq \mathbb{F}_2^4$. Then W has 16 elements and a symplectic basis of W is given by

$$\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\},$$

which means that a complete polarization of W is for example $W = X \oplus Y$, where $X = \mathrm{Span}_{\mathbb{F}_2}\{(1,0,0,0),(0,1,0,0)\}$ and $Y = \mathrm{Span}_{\mathbb{F}_2}\{(0,0,1,0),(0,0,0,1)\}$. The nondegenerate symplectic form is then given by

$$\langle (x_1,x_2,y_1,y_2),(x_1',x_2',y_1',y_2')\rangle = x_1y_1' + x_2y_2' - y_1x_1' - y_2x_2'.$$

To simplify the notation, the element (x,y,z,t) of W will be denoted by w_k, where $k = 2^3x + 2^2y + 2z + t \in \{0,\dots,15\}$, i.e. (x,y,z,t) corresponds to the binary expression of the number k. The two-dimensional subspaces of dimension two of W are the spaces $W_{\{k,l\}} = \mathrm{Span}_{\mathbb{F}_2}\{w_k,w_l\}$, where $k \neq l$ and $k,l \neq 0$. The lagrangians of W are the spaces $W_{\{k,l\}}$ ($k \neq l$ and $k,l \neq 0$) for which $\langle w_k, w_l\rangle = 0$. These spaces are the following ones:

$W_{\{1,2\}}, W_{\{1,3\}}, W_{\{1,8\}}, W_{\{1,9\}}, W_{\{1,10\}}, W_{\{1,11\}}, W_{\{2,3\}}, W_{\{2,4\}}, W_{\{2,5\}}, W_{\{2,6\}}, W_{\{2,7\}}, W_{\{3,12\}}, W_{\{3,13\}},$
$W_{\{3,14\}}, W_{\{3,15\}}, W_{\{4,6\}}, W_{\{4,8\}}, W_{\{4,10\}}, W_{\{4,12\}}, W_{\{4,14\}}, W_{\{5,7\}}, W_{\{5,8\}}, W_{\{5,10\}}, W_{\{5,13\}}, W_{\{5,15\}},$
$W_{\{6,9\}}, W_{\{6,11\}}, W_{\{6,13\}}, W_{\{6,15\}}, W_{\{7,9\}}, W_{\{7,11\}}, W_{\{7,12\}}, W_{\{7,14\}}, W_{\{8,9\}}, W_{\{8,12\}}, W_{\{8,13\}}, W_{\{9,14\}},$
$W_{\{9,15\}}, W_{\{10,11\}}, W_{\{10,14\}}, W_{\{10,15\}}, W_{\{11,12\}}, W_{\{11,13\}}, W_{\{12,15\}}, W_{\{13,14\}}.$

These lagrangian subspaces are not all distinct. In fact, suppose that $i,j \in \{1,\dots,15\}$ are such that $i \neq j$ and consider $w_k = w_i + w_j$. Then $W_{\{i,j\}} = \{0,w_i,w_j,w_k\}$ and thus $W_{\{i,j\}} = W_{\{i,k\}} = W_{\{j,k\}}$. This reduces the previous list of lagrangian subspaces of W to the following list:

$W_{\{1,2\}}, W_{\{1,8\}}, W_{\{1,10\}}, W_{\{2,4\}}, W_{\{2,5\}}, W_{\{3,12\}}, W_{\{3,13\}}, W_{\{4,8\}}, W_{\{4,10\}}, W_{\{5,8\}}, W_{\{5,10\}}, W_{\{6,9\}},$
$W_{\{6,11\}}, W_{\{7,9\}}, W_{\{7,11\}}.$

Choose for example $L = X = W_{\{4,8\}}$ and $N = Y = W_{\{1,2\}}$. The list $\mathscr{U}(L)$ of lagrangians M such that $W = L \oplus M$ is the following:

$$\mathscr{U}(L) = \{W_{\{1,2\}}, W_{\{1,10\}}, W_{\{2,5\}}, W_{\{3,13\}}, W_{\{5,10\}}, W_{\{6,9\}}, W_{\{6,11\}}, W_{\{7,9\}}\},$$

and similarly

$$\mathscr{U}(N) = \{W_{\{4,8\}}, W_{\{4,10\}}, W_{\{5,8\}}, W_{\{5,10\}}, W_{\{6,9\}}, W_{\{6,11\}}, W_{\{7,9\}}, W_{\{7,11\}}\}.$$

In the statement of Lemma 2.44, M_1 and M_2 belong by hypothesis to $\mathscr{U}(L) \cap \mathscr{U}(N) = \{W_{\{5,10\}}, W_{\{6,9\}}, W_{\{6,11\}}, W_{\{7,9\}}\}$. For example, we can choose $M_1 = W_{\{5,10\}}$ and $M_2 = W_{\{6,9\}}$. Then by Lemma 2.44, there should exist a lagrangian M_3 in $\mathscr{U}(L) \cap \mathscr{U}(N) \cap \mathscr{U}(M_1) \cap \mathscr{U}(M_2)$. However, the fact that $M_3 \in \mathscr{U}(M_1) \cap \mathscr{U}(M_2)$ implies that w_5, w_6, w_9, w_{10} and w_{15} do not belong to M_3. This is incompatible with the condition $M_3 \in \mathscr{U}(L) \cap \mathscr{U}(N)$. Thus $\mathscr{U}(L) \cap \mathscr{U}(N) \cap \mathscr{U}(M_1) \cap \mathscr{U}(M_2) = \varnothing$, which means that M_3 does not exist.

Now, following the idea of [GH12], we would like to construct explicitly a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and a homomorphism of central extensions $\mathscr{S}$ such that (D_{23}) commutes. This would allow us to extend the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to the group $\mathrm{Mp}(\widetilde{W})$. Recall that the constructions of the Weil representations $\widetilde{\omega}_{\widetilde{\chi}}$ in section 1.3 and $\widetilde{\omega}_{\widetilde{\chi_k}}$ in section 2.3 are the lifts to central non-split extensions of the projective Weil representations of $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$, respectively, which are (sub)groups of automorphisms of the Heisenberg groups $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, respectively. Thus to hope for the construction of $\mathrm{Mp}(\widetilde{W})$, we first look for a Heisenberg group $\mathrm{H}(\widetilde{W})$ on which $\mathrm{Sp}(\widetilde{W})$ acts by automorphisms. Then we define a projective representation ω_{χ_k} of $\mathrm{Sp}(\widetilde{W})$ in a similar way to the previous sections, and we compute the associated cocycle $\widetilde{c}_{\chi_k}$. Finally, in order to linearize this projective representation to the smallest possible central extension of $\mathrm{Sp}(\widetilde{W})$, we look for the smallest possible range of this cocycle. This way, we prove that $\widetilde{c}_{\chi_k}$ is not a coboundary, and we define a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. More precisely, a fact which is important to highlight here (and to compare with the constructions of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ in chapter 1 and $\mathrm{G}_{\widetilde{\chi_k}}$ in chapter 2) is that there are exactly two possibilities for $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ if the rank of $\widetilde{W}$ is two, and only one if the rank of $\widetilde{W}$ is greater or equal to six (see Proposition 2.63). This allows us to construct a Weil representation $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, which is completely independent of the construction of the affine metaplectic group $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. However, we still investigate the homomorphism of central extensions $\mathscr{S}$. To study the existence of $\mathscr{S}$, we consider the following bilinear forms

$$\begin{aligned} B_{\widetilde{W}} : \widetilde{W} &\to \mathbb{Z}/4\mathbb{Z} \\ (\widetilde{x}_1+\widetilde{x}_2, \widetilde{y}_1+\widetilde{y}_2) &\mapsto \langle \widetilde{x}_1, \widetilde{y}_2\rangle_{\widetilde{W}} \end{aligned} \quad \text{and} \quad \begin{aligned} B = B_{\mathrm{Weil}} : W &\to \mathbb{F}_2 \\ (x_1+x_2, y_1+y_2) &\mapsto \langle x_1, y_2\rangle. \end{aligned}$$

These are the forms studied by Gurevich and Hadani. Then we find a necessary and sufficient condition so that the entire diagram (D_{25}) commutes. This condition implies the existence and the uniqueness of $\mathscr{S}$.

Definition 2.47. The Heisenberg group $\mathrm{H}(\widetilde{W})$ is defined by the set $\widetilde{W}\times\mathbb{Z}/4\mathbb{Z}$, endowed with the following product: for all $\widetilde{w}_1,\ \widetilde{w}_2 \in \widetilde{W}$ and for all $t_1,\ t_2 \in \mathbb{Z}/4\mathbb{Z}$,

$$(\widetilde{w}_1, t_1)(\widetilde{w}_2, t_2) = (\widetilde{w}_1+\widetilde{w}_2, t_1+t_2+\langle \widetilde{w}_1, \widetilde{w}_2\rangle_{\widetilde{W}}). \tag{196}$$

The identity is $(0,0)$ and the inverse is given by $(\widetilde{w}, t)^{-1} = (-\widetilde{w}, -t)$.

Proposition 2.48.

- *The center of* $\mathrm{H}(\widetilde{W})$ *is* $2\widetilde{W}\times\mathbb{Z}/4\mathbb{Z}$ *and the commutator subgroup of* $\mathrm{H}(\widetilde{W})$ *is isomorphic to* $\mathbb{F}_2$.
- *The product* (196) *defines a central non-split extension* $0\to\mathbb{Z}/4\mathbb{Z}\to\mathrm{H}(\widetilde{W})\to\widetilde{W}\to 0$. *The nondegenerate symplectic form* $\langle\cdot,\cdot\rangle_{\widetilde{W}}$ *is a cocycle of* $\widetilde{W}$ *and its cohomology class in* $\mathrm{H}^2(\widetilde{W}, \mathbb{Z}/4\mathbb{Z})$ *is the one associated to the isomorphism class of the central extension defined by* $\mathrm{H}(\widetilde{W})$.
- $\mathrm{Sp}(\widetilde{W})$ *acts by automorphisms on* $\mathrm{H}(\widetilde{W})$, *leaving* $\{0\}\times\mathbb{Z}/4\mathbb{Z}$ *invariant. This action is given by*

$$\widetilde{\sigma}\cdot(\widetilde{w}, t) = (\widetilde{\sigma}(\widetilde{w}), t), \tag{197}$$

for every $\widetilde{\sigma}\in\mathrm{Sp}(\widetilde{W})$ *and* $(\widetilde{w}, t)\in\mathrm{H}(\widetilde{W})$.

Proof. The proof of the first two points is similar to the one of Proposition 1.5. For the last point, for all $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \mathrm{Sp}(\widetilde{W})$ and for every $(\widetilde{w}, t) \in \mathrm{H}(\widetilde{W})$, we have that $\mathrm{id} \cdot (\widetilde{w}, t) = (\widetilde{w}, t)$ and $\widetilde{\sigma}_1 \cdot (\widetilde{\sigma}_2 \cdot (\widetilde{w}, t)) = \widetilde{\sigma}_1 \cdot (\widetilde{\sigma}_2(\widetilde{w}), t) = ((\widetilde{\sigma}_1\widetilde{\sigma}_2)(\widetilde{w}), t) = (\widetilde{\sigma}_1\widetilde{\sigma}_2)(\widetilde{w}, t)$. Finally, since $\widetilde{\sigma}_1$ is invertible, there exists a unique $\widetilde{w}' \in \widetilde{W}$ such that $\widetilde{w} = \widetilde{\sigma}_1(\widetilde{w}')$, hence $(\widetilde{w}, t) = \widetilde{\sigma}_1 \cdot (\widetilde{w}', t)$. This shows that $\mathrm{Sp}(\widetilde{W})$ acts by automorphisms on $\mathrm{H}(\widetilde{W})$ by (197). □

Remark 2.49. The way of defining a Heisenberg group with the nondegenerate symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ by (196), instead of B or B_W as for $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, respectively, is similar to the "classical" definition of the Heisenberg group for a symplectic real vector space $(V, \langle \cdot, \cdot \rangle)$, where the product law uses $\frac{1}{2}\langle \cdot, \cdot \rangle$. In our setting, we cannot divide by 2, hence we only use the nondegenerate symplectic form in (196). Observe that, since $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ is nonsymmetric, this defines a noncommutative group. This could not be done in the previous sections because the nondegenerate symplectic forms $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_W$ are symmetric, hence (196) would have defined an abelian group, which is not of interest. Moreover, following the previous sections, we could also replace $\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}}$ by $B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)$. However, in this case, the group $\mathrm{Sp}(\widetilde{W})$ would not act on this Heisenberg group as in (197), whereas we want an action of $\mathrm{Sp}(\widetilde{W})$ on the Heisenberg group we construct, so that $\mathrm{Sp}(\widetilde{W})$ plays the role of $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$ in the present context.

Now, the classical path would be to look for a version of the Stone-von Neumann theorem which is available for the Heisenberg group $\mathrm{H}(\widetilde{W})$, then to fix a model for the Heisenberg representation, and this should lead to a projective representation of $\mathrm{Sp}(\widetilde{W})$. This is not exactly what we do. Recall that we did not need Stone von Neumann theorem in the previous sections to construct a projective representation of $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$, respectively. Moreover, we cannot use the Schrödinger model of the Heisenberg representation given in Lemma 2.11 without some adjustments, which will appear in the following statements.

Definition 2.50. Let χ_k be a faithful character of $\mathbb{Z}/4\mathbb{Z}$. Since $\widetilde{X}$ is isotropic, χ_k can be trivially extended to the abelian subgroup $\widetilde{X} \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}(\widetilde{W})$. We denote by $\mathscr{H}(\widetilde{X}, \chi_k)$ the $\mathbb{C}$-vector space of functions $\phi : \mathrm{H}(\widetilde{W}) \to \mathbb{C}$ such that for all $(\widetilde{x}, t) \in \widetilde{X} \times \mathbb{Z}/4\mathbb{Z}$ and for all $h \in \mathrm{H}(\widetilde{W})$,

$$\phi(h(\widetilde{x}, t)) = \chi_k(\widetilde{x}, t)^{-1}\phi(h) = \chi_k(-t)\phi(h).$$

Let $\widetilde{\rho}_{\chi_k} : \mathrm{H}(\widetilde{W}) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}(\widetilde{X}, \chi_k))$ be the map defined by $\widetilde{\rho}_{\chi_k}(g)\phi(h) = \phi(g^{-1}h)$. In other words, $\widetilde{\rho}_{\chi_k}$ is the representation $\mathrm{Ind}_{\widetilde{X} \times \mathbb{Z}/4\mathbb{Z}}^{\mathrm{H}(\widetilde{W})} \chi_k$ of $\mathrm{H}(\widetilde{W})$ induced from the character χ_k of the abelian subgroup $\widetilde{X} \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}(\widetilde{W})$.

Lemma 2.51. *The space $\mathscr{H}(\widetilde{X}, \chi_k)$ is isomorphic to $\mathrm{L}^2(\widetilde{Y})$ by restriction to $\widetilde{Y}$. As a representation of $\mathrm{H}(\widetilde{W})$ on the space $\mathrm{L}^2(\widetilde{Y})$, $\widetilde{\rho}_{\chi_k}$ is unitary and acts as χ_k on $\{0\} \times \mathbb{Z}/4\mathbb{Z}$. It is given by the following formula: for all $g = (\widetilde{x} + \widetilde{y}, t) \in \mathrm{H}(\widetilde{W})$, for all $\varphi \in \mathrm{L}^2(\widetilde{Y})$ and for all $\widetilde{z} \in \widetilde{Y}$,*

$$\widetilde{\rho}_{\chi_k}(\widetilde{x} + \widetilde{y}, t)\varphi(\widetilde{z}) = \chi_k(t + 2\langle \widetilde{x}, \widetilde{z} \rangle_{\widetilde{W}} - \langle \widetilde{x}, \widetilde{y} \rangle_{\widetilde{W}})\varphi(\widetilde{z} - \widetilde{y}). \tag{198}$$

Hence, the character of $\widetilde{\rho}_{\chi_k}$ is given by

$$\mathrm{Trace}(\widetilde{\rho}_{\chi_k}(\widetilde{x} + \widetilde{y}, t)) = \chi_k(t)|\widetilde{Y}|\delta_0(\widetilde{y})\mathbb{1}_{2\widetilde{X}}(\widetilde{x}), \tag{199}$$

for all $(\widetilde{w} = \widetilde{x} + \widetilde{y}, t) \in \mathrm{H}(\widetilde{W})$. Moreover, $\widetilde{\rho}_{\chi_k}$ is reducible.

Proof. As in Lemma 2.11, the map $\mathscr{H}(\widetilde{X},\chi_k)\to \mathrm{L}^2(\widetilde{Y})$ is an isomorphism. Moreover, its

$$\phi\mapsto \phi_{|\widetilde{Y}}$$

inverse is given by $\varphi\mapsto\phi_\varphi$, where $\phi_\varphi(\widetilde{x}+\widetilde{y},t)=\chi_k(t-\langle\widetilde{y},\widetilde{x}\rangle_{\widetilde{W}})^{-1}\varphi(\widetilde{y})$. Then, for all $g=(\widetilde{x}+\widetilde{y},t)\in \mathrm{H}(\widetilde{W})$, for all $\varphi\in \mathrm{L}^2(\widetilde{Y})$ and for all $\widetilde{z}\in\widetilde{Y}$,

$$\begin{aligned}\widetilde{\rho}_{\chi_k}(\widetilde{x}+\widetilde{y},t)\varphi(\widetilde{z})=\widetilde{\rho}_{\chi_k}(\widetilde{x}+\widetilde{y},t)\phi_\varphi(\widetilde{z})&=\phi_\varphi((\widetilde{x}+\widetilde{y},t)^{-1}\widetilde{z})\\&=\phi_\varphi((-\widetilde{x}-\widetilde{y}+\widetilde{z},-t-\langle\widetilde{x}+\widetilde{y},\widetilde{z}\rangle_{\widetilde{W}})\\&=\chi_k(-t-\langle\widetilde{x}+\widetilde{y},\widetilde{z}\rangle_{\widetilde{W}}+\langle\widetilde{z}-\widetilde{y},\widetilde{x}\rangle_{\widetilde{W}})^{-1}\varphi(\widetilde{z}-\widetilde{y})\\&=\chi_k(t-2\langle\widetilde{z},\widetilde{x}\rangle_{\widetilde{W}}+\langle\widetilde{y},\widetilde{x}\rangle_{\widetilde{W}})\varphi(\widetilde{z}-\widetilde{y})\\&=\chi_k(t+2\langle\widetilde{x},\widetilde{z}\rangle_{\widetilde{W}}-\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\varphi(\widetilde{z}-\widetilde{y}).\end{aligned}$$

In particular, if $\varphi=\delta_{\widetilde{z}}$, then

$$\widetilde{\rho}_{\chi_k}(\widetilde{x}+\widetilde{y},t)\delta_{\widetilde{z}}=\chi_k(t+2\langle\widetilde{x},\widetilde{z}\rangle_{\widetilde{W}}+\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\delta_{\widetilde{z}+\widetilde{y}}. \tag{200}$$

Hence, we can compute the trace of the operator $\widetilde{\rho}_{\chi_k}(\widetilde{x}+\widetilde{y},t)$ in the basis $\{\delta_{\widetilde{z}},\ \widetilde{z}\in\widetilde{Y}\}$ of $\mathrm{L}^2(\widetilde{Y})$:

$$\mathrm{Trace}(\widetilde{\rho}_{\chi_k}(\widetilde{x}+\widetilde{y},t))=\sum_{\widetilde{z}\in\widetilde{Y}}\chi_k(t+2\langle\widetilde{x},\widetilde{z}\rangle_{\widetilde{W}}+\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\delta_0(\widetilde{y})=\chi_k(t)|\widetilde{Y}|\delta_0(\widetilde{y})\mathbb{1}_{2\widetilde{X}}(\widetilde{x}).$$

Then,

$$\begin{aligned}(\widetilde{\rho}_{\chi_k},\widetilde{\rho}_{\chi_k})=\frac{1}{|\,\mathrm{H}(\widetilde{W})|}\sum_{h\in\mathrm{H}(\widetilde{W})}|\,\mathrm{Trace}(\widetilde{\rho}_{\chi_k}(h))|^2&=\frac{1}{|\,\mathrm{H}(\widetilde{W})|}\sum_{(\widetilde{x}+\widetilde{y},t)\in\mathrm{H}(\widetilde{W})}|\chi_k(t)|^2|\widetilde{Y}|^2\delta_0(\widetilde{y})\mathbb{1}_{2\widetilde{X}}(\widetilde{x})\\&=\frac{4|\widetilde{Y}|^2|2\widetilde{X}|}{4|\widetilde{Y}|^2}=|2\widetilde{X}|=2^n.\end{aligned}$$

Since $(\widetilde{\rho}_{\chi_k},\widetilde{\rho}_{\chi_k})\neq 1$, the representation $\widetilde{\rho}_{\chi_k}$ is reducible. □

We now use (198) in order to obtain an irreducible representation of $\mathrm{H}(\widetilde{W})$.

Definition-Proposition 2.52. *For all* $g=(\widetilde{x}+\widetilde{y},t)\in\mathrm{H}(\widetilde{W})$*, for all* $\varphi\in\mathrm{L}^2(Y)$ *and for all* $z\in Y$*, we define*

$$\rho_{\chi_k}(\widetilde{x}+\widetilde{y},t)\varphi(z)=\chi_k(t+\langle x,z\rangle_W-\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\varphi(z-y), \tag{201}$$

where $x=\widetilde{x}+2\widetilde{X}$*. This formula defines a unitary and irreducible representation of* $\mathrm{H}(\widetilde{W})$*, which acts as* χ_k *on the center* $2\widetilde{W}\times\mathbb{Z}/4\mathbb{Z}$ *of* $\mathrm{H}(\widetilde{W})$*. Moreover, the character of* ρ_{χ_k} *is given by*

$$\mathrm{Trace}(\rho_{\chi_k}(\widetilde{w},t))=\chi_k(t)|Y|\mathbb{1}_{2\widetilde{W}}(\widetilde{w}), \tag{202}$$

for all $(\widetilde{w},t)\in\mathrm{H}(\widetilde{W})$.

Proof. The fact that ρ_{χ_k} is unitary can be proven exactly as in Lemma 1.17. Moreover, for all $g = (\widetilde{x} + \widetilde{y}, t) \in \mathrm{H}(\widetilde{W})$ and $z \in Y$, by (201),

$$\rho_{\chi_k}(\widetilde{x} + \widetilde{y}, t)\delta_z = \chi_k(t + \langle x, z\rangle_W + \langle \widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})\delta_{z+y}. \tag{203}$$

Hence the trace of the operator $\rho_{\chi_k}(\widetilde{x} + \widetilde{y}, t)$ in the basis $\{\delta_z,\ z \in Y\}$ of $\mathrm{L}^2(Y)$ is given by

$$\begin{aligned}\mathrm{Trace}(\rho_{\chi_k}(\widetilde{x} + \widetilde{y}, t)) = \sum_{z\in Y} \chi_k(t + \langle x, z\rangle_W + \langle \widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})\mathbb{1}_{2\widetilde{Y}}(\widetilde{y}) &= \chi_k(t + \langle \widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})|Y|\mathbb{1}_{2\widetilde{Y}}(\widetilde{y})\mathbb{1}_{2\widetilde{X}}(\widetilde{x}) \\ &= \chi_k(t)|Y|\mathbb{1}_{2\widetilde{W}}(\widetilde{w}).\end{aligned}$$

Then,

$$\begin{aligned}(\rho_{\chi_k}, \rho_{\chi_k}) = \frac{1}{|\,\mathrm{H}(\widetilde{W})|}\sum_{h\in\mathrm{H}(\widetilde{W})} |\,\mathrm{Trace}(\rho_{\chi_k}(h))|^2 &= \frac{1}{|\,\mathrm{H}(\widetilde{W})|}\sum_{(\widetilde{w},t)\in\mathrm{H}(\widetilde{W})} |\chi_k(t)|^2|Y|^2\mathbb{1}_{2\widetilde{W}}(\widetilde{w}) \\ &= \frac{4|Y|^2|2\widetilde{W}|}{4|\widetilde{Y}|^2} = \frac{|Y|^4}{|\widetilde{Y}|^2} = 1.\end{aligned}$$

Since $(\rho_{\chi_k}, \rho_{\chi_k}) = 1$, the representation ρ_{χ_k} is irreducible. □

Now, recall that $\mathrm{Sp}(\widetilde{W})$ acts on $\mathrm{H}(\widetilde{W})$ by (197). Then, for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, the map $\rho^{\widetilde{\sigma}}_{\chi_k} :\ h \mapsto \rho_{\chi_k}(\widetilde{\sigma}(h))$ defines a unitary irreducible representation of $\mathrm{H}(\widetilde{W})$, which acts as χ_k on $2\widetilde{W} \times \mathbb{Z}/4\mathbb{Z}$. Moreover, by (202), the representations ρ_{χ_k} and $\rho^{\widetilde{\sigma}}_{\chi_k}$ have the same character. Hence, they are equivalent. Thus, by Schur's lemma, for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, there exists a unique (up to multiplication by an element of $\mathrm{U}(1)$) unitary operator $\omega_{\chi_k}(\widetilde{\sigma})$ such that for all $h \in \mathrm{H}(\widetilde{W})$,

$$\rho^{\widetilde{\sigma}}_{\chi_k}(h) = \omega_{\chi_k}(\widetilde{\sigma}) \circ \rho_{\chi_k}(h) \circ \omega_{\chi_k}(\widetilde{\sigma})^{-1}. \tag{204}$$

Therefore, (204) provides a projective representation of $\mathrm{Sp}(\widetilde{W})$. Suppose that we have fixed a section of the projective representation ω_{χ_k}. Then for all $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \mathrm{Sp}(\widetilde{W})$, there exists $\widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2) \in \mathrm{U}(1)$ such that

$$\omega_{\chi_k}(\widetilde{\sigma}_1)\omega_{\chi_k}(\widetilde{\sigma}_2) = \widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2)\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2). \tag{205}$$

Our next goal will be to derive explicit formulas for the operators $\omega_{\chi_k}(\widetilde{\sigma})$, for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, and for the cocycle $\widetilde{c}_{\chi_k}$. Before doing all this, we need several preliminary results.

Lemma 2.53. *Let $\{\widetilde{w}_j,\ 1 \leqslant j \leqslant |W|\}$ be a system of representatives of $\widetilde{W}/2\widetilde{W}$. Then the family of operators $\{\rho_{\chi_k}(\widetilde{w}_j),\ 1 \leqslant j \leqslant |W|\}$ is a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$.*

Proof. For every $1 \leqslant j \leqslant |W|$, we write $\widetilde{w}_j = \widetilde{x}_j + \widetilde{y}_j$, where $\widetilde{x}_j \in \widetilde{X}$ and $\widetilde{y}_j \in \widetilde{Y}$. Let $\lambda(\widetilde{w}_j) \in \mathbb{C}$ for every $1 \leqslant j \leqslant |W|$. Suppose that

$$\sum_{1\leqslant j\leqslant |W|} \lambda(\widetilde{w}_j)\rho_{\chi_k}(\widetilde{w}_j) = 0.$$

Then, for every $z \in Y$,

$$\begin{aligned} 0 &= \left(\sum_{1 \leqslant j \leqslant |W|} \lambda(\widetilde{w}_j) \rho_{\chi_k}(\widetilde{w}_j) \right) \delta_z \\ &= \sum_{1 \leqslant j \leqslant |W|} \lambda(\widetilde{w}_j) \chi_k(\langle x_j, z \rangle_W + \langle \widetilde{x}_j, \widetilde{y}_j \rangle_{\widetilde{W}}) \delta_{z+y_j} \qquad \text{by (203), where } \begin{cases} x_j = \widetilde{x}_j + 2\widetilde{X} \\ y_j = \widetilde{y}_j + 2\widetilde{Y} \end{cases} \\ &= \sum_{y \in Y} \sum_{\substack{1 \leqslant j \leqslant |W| \\ \widetilde{y}_j + 2\widetilde{Y} = y}} \lambda(\widetilde{w}_j) \chi_k(\langle x_j, z \rangle_W + \langle \widetilde{x}_j, \widetilde{y}_j \rangle_{\widetilde{W}}) \delta_{z+y}. \end{aligned}$$

Since $\{\delta_{z+y},\ y \in Y\}$ is a basis of $\mathrm{L}^2(Y)$, for every $y \in Y$,

$$\sum_{\substack{1 \leqslant j \leqslant |W| \\ \widetilde{y}_j + 2\widetilde{Y} = y}} \lambda(\widetilde{w}_j) \chi_k(\langle x_j, z \rangle_W + \langle \widetilde{x}_j, \widetilde{y}_j \rangle_{\widetilde{W}}) = 0 \qquad \text{for every } z \in Y. \tag{206}$$

Now, for every $y \in Y$, the set $S_y = \{1 \leqslant j \leqslant |W|,\ \widetilde{y}_j + 2\widetilde{Y} = y\}$ is finite, and contains exactly $N = |X| = |Y|$ elements. In fact, consider the map

$$\begin{aligned} \phi_y :\ S_y &\to X \\ j &\mapsto x_j = \widetilde{x}_j + 2\widetilde{X} \end{aligned} .$$

If $j, l \in S_y$ are such that $\phi_y(j) = \phi_y(l)$, then $\widetilde{x}_j - \widetilde{x}_l \in 2\widetilde{X}$. Since $\widetilde{y}_j - \widetilde{y}_l \in 2\widetilde{Y}$ by definition of S_y, this implies that $\widetilde{w}_j - \widetilde{w}_l \in 2\widetilde{W}$, i.e. $j = l$ because $\{\widetilde{w}_j, 1 \leqslant j \leqslant |W|\}$ is a system of representatives of $\widetilde{W}/2\widetilde{W}$. Hence, ϕ_y is injective. Now, for every $x \in X$, there exists $j_0 \in \{1, \dots, |W|\}$ such that $x + y = \widetilde{w}_{j_0} + 2\widetilde{W} = \widetilde{x}_{j_0} + 2\widetilde{X} + \widetilde{y}_{j_0} + 2\widetilde{Y}$. This shows that $j_0 \in S_y$ and $\phi_y(j_0) = x$. Therefore ϕ_y is surjective. The fact that ϕ_y is a bijection implies that S_y has $N = |X| = |Y|$ elements, hence we can write it as $S_y = \{j_l,\ 1 \leqslant l \leqslant N\}$. Moreover, since ϕ_y is bijective, the set $\{x_{j_l},\ 1 \leqslant l \leqslant N\}$ is exactly the set X. Now, if we set

$$\begin{cases} A_y = (a_{z,l})_{z \in Y,\ 1 \leqslant l \leqslant N} \in \mathrm{M}_N(\mathbb{C}) & \text{by } a_{z,l} = \chi_k(\langle x_{j_l}, z \rangle_W) \\ V_y = (v_l)_{1 \leqslant l \leqslant N} \in \mathrm{M}_{N,1}(\mathbb{C}) & \text{by } v_l = \lambda(\widetilde{w}_{j_l}) \chi_k(\langle \widetilde{x}_{j_l}, \widetilde{y}_{j_l} \rangle_{\widetilde{W}}) \end{cases},$$

then (206) is equivalent to write $A_y V_y = 0$. Besides, for every $(z, z') \in Y^2$,

$$(A_y A_y^T)_{z,z'} = \sum_{1 \leqslant l \leqslant N} \chi_k(\langle x_{j_l}, z \rangle_W) \chi_k(\langle x_{j_l}, z' \rangle_W) = \sum_{1 \leqslant l \leqslant N} \chi_k(\langle x_{j_l}, z + z' \rangle_W).$$

Moreover, we already proved that the set $\{x_{j_l},\ 1 \leqslant l \leqslant N\}$ is exactly equal to X. By Schur's orthogonality relations, we can then deduce that $(A_y A_y^T)_{z,z'} = |Y| \delta_z(z')$. This implies that A_y is invertible, its inverse being $\dfrac{1}{|Y|} A^T$. In particular, the only possibility for V_y such that $A_y V_y = 0$ is $V_y = 0$. Hence, $\lambda(\widetilde{w}_{j_l}) = 0$ for all $1 \leqslant l \leqslant N$. Since this is true for every $y \in Y$, this shows that $\lambda(\widetilde{w}_j) = 0$ for all $1 \leqslant j \leqslant |W|$. Therefore, $\{\rho_{\chi_k}(\widetilde{w}_j),\ 1 \leqslant j \leqslant |W|\}$ is a family of $|W| = \dim(\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y)))$ linearly independent operators, hence a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$. □

Similarly to Lemma 1.27, for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, we define $\mathrm{K}(\widetilde{\sigma}) = \mathrm{Ker}(1-\widetilde{\sigma})$ and $\mathrm{I}(\widetilde{\sigma}) = \mathrm{Im}(1-\widetilde{\sigma})$. Then $\mathrm{K}(\widetilde{\sigma})$ and $\mathrm{I}(\widetilde{\sigma})$ are submodules of $\widetilde{W}$ which are orthogonal with respect to the nondegenerate symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$.

Lemma 2.54. *Let $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$.*

i. *Let $\widetilde{V}$ be a submodule of $\widetilde{W}$. Then the map $\widetilde{V}/\widetilde{V} \cap 2\widetilde{W} \to 2\widetilde{V}$ is bijective. In particular,*

$$\widetilde{v} + \widetilde{V} \cap 2\widetilde{W} \mapsto 2\widetilde{v}$$

we have that $|\widetilde{V}||\widetilde{V} \cap 2\widetilde{W}|^{-1} = |2\widetilde{V}|$.

ii. *Let $\mathrm{K}'(\widetilde{\sigma}) = \{\widetilde{w} \in \widetilde{W},\ 2\widetilde{w} \in \mathrm{K}(\widetilde{\sigma})\}$. Then $\mathrm{K}'(\widetilde{\sigma})$ is a $\mathbb{Z}/4\mathbb{Z}$-module and its orthogonal complement with respect to the nondegenerate symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ is $2\,\mathrm{I}(\widetilde{\sigma})$.*

iii. *There exists $\widetilde{w}_{\widetilde{\sigma}} \in \widetilde{W}$ such that $2\widetilde{w}_{\widetilde{\sigma}} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}$ and*

$$\langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w} \rangle_{\widetilde{W}} = 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}} \quad \text{for every } \widetilde{w} \in \mathrm{K}'(\widetilde{\sigma}). \tag{207}$$

iv. *$\widetilde{w}'_{\widetilde{\sigma}} \in \widetilde{W}$ satisfies (207) if and only if $2\widetilde{w}'_{\widetilde{\sigma}} - 2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})$.*

v. *Let $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ be in $\mathrm{Sp}(\widetilde{W})$. We set $\widetilde{w}_{\widetilde{\sigma}_1,\widetilde{\sigma}_2} = \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - \widetilde{w}_{\widetilde{\sigma}_1} - \widetilde{w}_{\widetilde{\sigma}_2}$. Then $2\widetilde{w}_{\widetilde{\sigma}_1,\widetilde{\sigma}_2}$ belongs to $2\,\mathrm{I}(\widetilde{\sigma}_1) + 2\,\mathrm{I}(\widetilde{\sigma}_2)$.*

vi. *Let $\widetilde{v} \in \widetilde{W}$ and let $\widetilde{\sigma}_{\widetilde{v}}$ be the transvection in $\mathrm{Sp}(\widetilde{W})$ defined by $\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}) = \widetilde{w} + \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v}$, for every $\widetilde{w} \in W$.*

- *If $2\widetilde{v} \notin 2\,\mathrm{I}(\widetilde{\sigma})$ then $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \oplus 2\,\mathrm{I}(\widetilde{\sigma})$.*
- *Suppose that $2\widetilde{v} \in 2\,\mathrm{I}(\widetilde{\sigma})$, i.e. there exists $\widetilde{z} \in \widetilde{W}$ such that $(1-\widetilde{\sigma})(2\widetilde{z}) = 2\widetilde{v}$. If $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}$ then $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma})$ and if $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}$ then $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}) \cap (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp}$.*

Proof. Let $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$.

i. Define the map $\widetilde{V} \to 2\widetilde{V}$. Let $\widetilde{v}_1$ and $\widetilde{v}_2$ be in $\widetilde{V}$. Then $\widetilde{v}_1 - \widetilde{v}_2 \in 2\widetilde{W}$ if and only if

$$\widetilde{v} \mapsto 2\widetilde{v}$$

$2(\widetilde{v}_1 - \widetilde{v}_2) = 0$, i.e. $2\widetilde{v}_1 = 2\widetilde{v}_2$. This shows that the map $\widetilde{V}/\widetilde{V} \cap 2\widetilde{W} \to 2\widetilde{V}$ is well-

$$\widetilde{v} + \widetilde{V} \cap 2\widetilde{W} \mapsto 2\widetilde{v}$$

defined and injective. Moreover, since by definition every element in $2\widetilde{V}$ can be written in the form $2\widetilde{v}$ for a certain $\widetilde{v} \in \widetilde{V}$, this map is also clearly surjective. Thus, it is bijective and we obtain $|\widetilde{V}||\widetilde{V} \cap 2\widetilde{W}|^{-1} = |2\widetilde{V}|$.

ii. Let $2\widetilde{u} \in 2\,\mathrm{I}(\widetilde{\sigma})$. For all $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma})$,

$$\langle \widetilde{w}, 2\widetilde{u} \rangle_{\widetilde{W}} = \langle \underbrace{2\widetilde{w}}_{\in \mathrm{K}(\widetilde{\sigma})}, \underbrace{\widetilde{u}}_{\in \mathrm{I}(\widetilde{\sigma})} \rangle_{\widetilde{W}} = 0$$

since $\mathrm{K}(\tilde{\sigma})$ and $\mathrm{I}(\tilde{\sigma})$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widetilde{W}}$. Hence, $2\,\mathrm{I}(\tilde{\sigma}) \subseteq \mathrm{K}'(\tilde{\sigma})^{\perp}$. Now, let $\tilde{w}$ be in $(2\,\mathrm{I}(\tilde{\sigma}))^{\perp}$, i.e. such that for every $2\tilde{u} \in 2\,\mathrm{I}(\tilde{\sigma})$,

$$0 = \langle \tilde{w}, 2\tilde{u}\rangle_{\widetilde{W}} = \langle 2\tilde{w}, \tilde{u}\rangle_{\widetilde{W}}.$$

Then $\langle 2\tilde{w}, \tilde{u}\rangle_{\widetilde{W}} = 0$ for all $\tilde{u} \in \mathrm{I}(\tilde{\sigma})$, which means exactly that $2\tilde{w} \in \mathrm{I}(\tilde{\sigma})^{\perp} = \mathrm{K}(\tilde{\sigma})$, and thus $\tilde{w} \in \mathrm{K}'(\tilde{\sigma})$. Taking the orthogonal spaces, we obtain that $\mathrm{K}'(\tilde{\sigma})^{\perp} \subseteq 2\,\mathrm{I}(\tilde{\sigma})$ and this concludes the proof.

iii. Let $\tilde{w}_1, \tilde{w}_2 \in \mathrm{K}'(\tilde{\sigma})$. Then, for every $\lambda \in \mathbb{Z}/4\mathbb{Z}$, $\langle \tilde{\sigma}(\lambda\tilde{w}_1), \lambda\tilde{w}_1\rangle_{\widetilde{W}} = \lambda^2\langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_1\rangle_{\widetilde{W}}$. Since

$$\tilde{w}_1 \in \mathrm{K}'(\tilde{\sigma}) \iff 2\tilde{w}_1 \in \mathrm{K}(\tilde{\sigma}) \iff (1-\tilde{\sigma})(\tilde{w}_1) \in \mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}, \tag{208}$$

we see that

$$\langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_1\rangle_{\widetilde{W}} = \langle \tilde{\sigma}(\tilde{w}_1) - \tilde{w}_1, \tilde{w}_1\rangle_{\widetilde{W}} = -\langle \underbrace{(1-\tilde{\sigma})(\tilde{w}_1)}_{\in 2\widetilde{W}}, \tilde{w}_1\rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}. \tag{209}$$

Hence, $\lambda^2\langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_1\rangle_{\widetilde{W}} = \lambda\langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_1\rangle_{\widetilde{W}}$ for every $\lambda \in \mathbb{Z}/4\mathbb{Z}$. Moreover,

$$\begin{aligned}\langle \tilde{\sigma}(\tilde{w}_1+\tilde{w}_2), \tilde{w}_1+\tilde{w}_2\rangle_{\widetilde{W}} &= \langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_1\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_2), \tilde{w}_2\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_2\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_2), \tilde{w}_1\rangle_{\widetilde{W}} \\ &= \langle \tilde{\sigma}(\tilde{w}_1), \tilde{w}_1\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_2), \tilde{w}_2\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_1) - \tilde{\sigma}^{-1}(\tilde{w}_1), \tilde{w}_2\rangle_{\widetilde{W}}.\end{aligned}$$

By (208), there exists $2\tilde{z}_1 \in \widetilde{W}$ such that $(1-\tilde{\sigma})(\tilde{w}_1) = 2\tilde{z}_1$. Hence $\tilde{\sigma}(\tilde{w}_1) = \tilde{w}_1 - 2\tilde{z}_1$ and $\tilde{\sigma}^{-1}(\tilde{w}_1) = \tilde{w}_1 + 2\tilde{\sigma}^{-1}(\tilde{z}_1)$. This shows that

$$\langle \tilde{\sigma}(\tilde{w}_1) - \tilde{\sigma}^{-1}(\tilde{w}_1), \tilde{w}_2\rangle_{\widetilde{W}} = \langle -2\tilde{z}_1 - 2\tilde{\sigma}^{-1}(\tilde{z}_1), \tilde{w}_2\rangle_{\widetilde{W}} = \langle \underbrace{(1-\tilde{\sigma}^{-1})(2\tilde{z}_1)}_{\in 2\,\mathrm{I}(\tilde{\sigma}^{-1})}, \underbrace{\tilde{w}_2}_{\in \mathrm{K}'(\tilde{\sigma})}\rangle_{\widetilde{W}} = 0,$$

where the last equality comes from the fact that $\mathrm{K}'(\tilde{\sigma}) = \mathrm{K}'(\tilde{\sigma}^{-1})$, and as proved in the first point above, $\mathrm{K}'(\tilde{\sigma}^{-1})$ and $2\,\mathrm{I}(\tilde{\sigma}^{-1})$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widetilde{W}}$. Thus we have proved that $\mathrm{K}'(\tilde{\sigma}) \to 2\mathbb{Z}/4\mathbb{Z}$ is a $\mathbb{Z}/4\mathbb{Z}$-module homomorphism. In particular,

$$\tilde{w} \mapsto \langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}}$$

since $\mathrm{K}'(\tilde{\sigma})^{\perp} = 2\,\mathrm{I}(\tilde{\sigma})$, then for all $\tilde{w} \in \mathrm{K}'(\tilde{\sigma})$ and $\tilde{u} \in \widetilde{W}$,

$$\begin{aligned}\langle \tilde{\sigma}(\tilde{w}+2\tilde{u}), \tilde{w}+2\tilde{u}\rangle_{\widetilde{W}} &= \langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}), 2\tilde{u}\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(2\tilde{u}), \tilde{w}\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(2\tilde{u}), 2\tilde{u}\rangle_{\widetilde{W}} \\ &= \langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(2\tilde{w}), \tilde{u}\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(2\tilde{u}), \tilde{w}\rangle_{\widetilde{W}} \\ &= \langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}} + \langle 2\tilde{w}, \tilde{u}\rangle_{\widetilde{W}} + \langle \tilde{\sigma}(2\tilde{u}), \tilde{w}\rangle_{\widetilde{W}} \\ &= \langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}} + \langle \tilde{w}, 2(1-\tilde{\sigma})(\tilde{u})\rangle_{\widetilde{W}} \\ &= \langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}}.\end{aligned}$$

This shows that the term $\langle \tilde{\sigma}(\tilde{w}), \tilde{w}\rangle_{\widetilde{W}}$ does not depend on the left coset of $\tilde{w}$ modulo $2\widetilde{W}$. Furthermore, let $\tilde{w} \in \widetilde{W}$ and set $w = \tilde{w} + 2\widetilde{W}$, where $w \in W$. Observe that $\tilde{w} \in \mathrm{K}'(\tilde{\sigma})$

if and only if $w \in \mathrm{K}(\sigma)$. In fact, by definition of π in Proposition 2.7, $(1-\sigma)(w) = 0$ is equivalent to $(1-\tilde{\sigma})(\tilde{w}) \in 2\widetilde{W}$. Therefore, the following map is well-defined:

$$\begin{aligned} \mathrm{K}(\sigma) &\to \mathbb{Z}/4\mathbb{Z} \qquad \text{, where } \tilde{w} \in \widetilde{W} \text{ is any element such that } w = \tilde{w} + 2\widetilde{W}. \\ w &\mapsto \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} \end{aligned} \tag{210}$$

Hence the map (210) is additive and it can be extended to an element of $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. By Proposition 2.7, there exists $w_{\tilde{\sigma}} \in W$ such that

$$\langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} = \langle w_{\tilde{\sigma}}, w \rangle_W \qquad \text{for every } w \in \mathrm{K}(\sigma).$$

Finally, by definition of $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ and $\langle \cdot, \cdot \rangle_W$, if we choose any $\tilde{w}_{\tilde{\sigma}}$ such that $w_{\tilde{\sigma}} = \tilde{w}_{\tilde{\sigma}} + 2\widetilde{W}$, then $\langle w_{\tilde{\sigma}}, w \rangle_W = 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}$. This proves the existence of $\tilde{w}_{\tilde{\sigma}} \in \widetilde{W}$ such that for all $\tilde{w} \in \mathrm{K}'(\tilde{\sigma})$,

$$\langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} = \langle w_{\tilde{\sigma}}, w \rangle_W = \langle 2\tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}.$$

Furthermore, for every $\tilde{w} \in \mathrm{K}(\tilde{\sigma}) \subseteq \mathrm{K}'(\tilde{\sigma})$, $\langle 2\tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} = \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} = \langle \tilde{w}, \tilde{w} \rangle_{\widetilde{W}} = 0$. This shows that $2\tilde{w}_{\tilde{\sigma}}$ belongs to $\mathrm{K}(\tilde{\sigma})^{\perp} = \mathrm{I}(\tilde{\sigma})$ and concludes the proof.

Remark 2.55. Another proof of the existence of $\tilde{w}_{\tilde{\sigma}}$ can be made using (280) and (281), but it requires to prove (100) first.

iv. Let $\tilde{w}'_{\tilde{\sigma}} \in \widetilde{W}$ be such that (207) is satisfied. Then, for all $\tilde{w} \in \mathrm{K}'(\tilde{\sigma})$,

$$\langle 2\tilde{w}'_{\tilde{\sigma}} - 2\tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} = \langle 2\tilde{w}'_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} - \langle 2\tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} = \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} = 0. \tag{211}$$

Hence, $2\tilde{w}'_{\tilde{\sigma}} - 2\tilde{w}_{\tilde{\sigma}} \in \mathrm{K}'(\tilde{\sigma})^{\perp} = 2\,\mathrm{I}(\tilde{\sigma})$. The converse follows from a similar argument, using (211).

v. Let $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ be in $\mathrm{Sp}(\widetilde{W})$. Let $\tilde{w} \in \mathrm{K}'(\tilde{\sigma}_1) \cap \mathrm{K}'(\tilde{\sigma}_2) \subseteq \mathrm{K}'(\tilde{\sigma}_1\tilde{\sigma}_2)$. Since $(1-\tilde{\sigma}_1)(2\tilde{w}) = 0$ and $(1-\tilde{\sigma}_2)(2\tilde{w}) = 0$, there exist $\tilde{w}_1, \tilde{w}_2 \in \widetilde{W}$ such that $\tilde{\sigma}_1(\tilde{w}) = \tilde{w} + 2\tilde{w}_1$ and $\tilde{\sigma}_2(\tilde{w}) = \tilde{w} + 2\tilde{w}_2$. Hence

$$(\tilde{\sigma}_1\tilde{\sigma}_2)(\tilde{w}) = \tilde{\sigma}_1(\tilde{w} + 2\tilde{w}_2) = \tilde{w} + 2\tilde{w}_1 + \tilde{\sigma}_1(2\tilde{w}_2).$$

By (207),

$$\begin{aligned} 2\langle \tilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2} - \tilde{w}_{\tilde{\sigma}_1} - \tilde{w}_{\tilde{\sigma}_2}, \tilde{w} \rangle_{\widetilde{W}} &= \langle (\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_1 - \tilde{\sigma}_2)(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} \\ &= \langle \tilde{w} + 2\tilde{w}_1 + \tilde{\sigma}_1(2\tilde{w}_2) - \tilde{w} - 2\tilde{w}_1 - \tilde{w} - 2\tilde{w}_2, \tilde{w} \rangle_{\widetilde{W}} \\ &= \langle \tilde{\sigma}_1(2\tilde{w}_2), \tilde{w} \rangle_{\widetilde{W}} - \langle 2\tilde{w}_2, \tilde{w} \rangle_{\widetilde{W}} \\ &= \langle \tilde{w}_2, \tilde{\sigma}_1^{-1}(2\tilde{w}) \rangle_{\widetilde{W}} - \langle 2\tilde{w}_2, \tilde{w} \rangle_{\widetilde{W}} \\ &= \langle \tilde{w}_2, 2\tilde{w} \rangle_{\widetilde{W}} - \langle 2\tilde{w}_2, \tilde{w} \rangle_{\widetilde{W}} \\ &= 0. \end{aligned}$$

This implies that $2\tilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2} = 2\tilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2} - 2\tilde{w}_{\tilde{\sigma}_1} - 2\tilde{w}_{\tilde{\sigma}_2}$ belongs to $(\mathrm{K}'(\tilde{\sigma}_1) \cap \mathrm{K}'(\tilde{\sigma}_2))^{\perp} = 2\,\mathrm{I}(\tilde{\sigma}_1) + 2\,\mathrm{I}(\tilde{\sigma}_2)$.

vi. Let $\widetilde{v} \in \widetilde{W}$ and $\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}) = \widetilde{w} + \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v}$ for all $\widetilde{w} \in \widetilde{W}$. If $\widetilde{v} \in 2\widetilde{W}$ then $\widetilde{\sigma}_{\widetilde{v}} = 1$ and the result is obvious, so we can suppose that $\widetilde{v} \notin 2\widetilde{W}$.

- Suppose that $2\widetilde{v} \notin 2\,\mathrm{I}(\widetilde{\sigma})$. Then $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \cap 2\,\mathrm{I}(\widetilde{\sigma}) = \{0\}$ because $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) = 2\mathbb{Z}/4\mathbb{Z}\widetilde{v}$. By taking the orthogonal complement with respect to the symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$, we see that showing that $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) + 2\,\mathrm{I}(\widetilde{\sigma})$ is equivalent to prove that $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}) \cap \mathrm{K}'(\widetilde{\sigma})$. The inclusion $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}) \cap \mathrm{K}'(\widetilde{\sigma}) \subseteq \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$ is obvious. Now let $\widetilde{w}$ be in $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$. Then $(1 - \widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})(2\widetilde{w}) = 0$, i.e.

$$\widetilde{\sigma}(2\widetilde{w}) = \widetilde{\sigma}_{\widetilde{v}}^{-1}(2\widetilde{w}) = 2\widetilde{w} - \langle 2\widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v}. \tag{212}$$

  Hence $(1 - \widetilde{\sigma})(2\widetilde{w}) = \langle 2\widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v} = 0$ since by hypothesis $2\widetilde{v} \notin 2\,\mathrm{I}(\widetilde{\sigma})$. This implies that $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma})$ and $2\langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} = 0$, which in turn implies that $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}})$.

- Suppose that $2\widetilde{v} \in 2\,\mathrm{I}(\widetilde{\sigma})$ and let $\widetilde{z} \in \widetilde{W}$ be such that $(1 - \widetilde{\sigma})(2\widetilde{z}) = 2\widetilde{v}$. Let $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$. Then, replacing $2\widetilde{v}$ by $(1 - \widetilde{\sigma})(2\widetilde{z})$ in (212), we obtain that

$$(1 - \widetilde{\sigma})(2\widetilde{w}) = \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} 2\widetilde{v} = \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} (1 - \widetilde{\sigma})(2\widetilde{z}),$$

  i.e. $\widetilde{w} - \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{z}$ belongs to $\mathrm{K}'(\widetilde{\sigma})$, i.e. $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma}) + \mathbb{Z}/4\mathbb{Z}\widetilde{z}$. Taking orthogonal complements, it follows that $2\,\mathrm{I}(\widetilde{\sigma}) \cap (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp} \subseteq 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$. On the other hand, if $2\widetilde{u} \in 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$ then there exists $\widetilde{w} \in \widetilde{W}$ such that

$$2\widetilde{u} = (1 - \widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})(2\widetilde{w}) = 2\widetilde{w} - \widetilde{\sigma}(2\widetilde{w}) - \langle \widetilde{\sigma}(2\widetilde{w}), \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v} = (1 - \widetilde{\sigma})(2\widetilde{w} - \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{v} \rangle_{\widetilde{W}} 2\widetilde{z}),$$

  i.e. $2\widetilde{u} \in 2\,\mathrm{I}(\widetilde{\sigma})$. Hence $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq 2\,\mathrm{I}(\widetilde{\sigma})$. So, we obtain

$$2\,\mathrm{I}(\widetilde{\sigma}) \cap (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp} \subseteq 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq 2\,\mathrm{I}(\widetilde{\sigma}). \tag{213}$$

  Moreover, $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp}$ is equivalent to $\widetilde{z} \in \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$. Also

$$\begin{aligned}(1 - \widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})(2\widetilde{z}) = 2\widetilde{z} - \widetilde{\sigma}_{\widetilde{v}}(2\widetilde{z} - 2\widetilde{v}) = 2\widetilde{v} - \langle 2\widetilde{z} - 2\widetilde{v}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v} &= 2\widetilde{v} - 2\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v} \\ &= (1 - \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}}) 2\widetilde{v}.\end{aligned} \tag{214}$$

  Since $\widetilde{v} \notin 2\widetilde{W}$, this shows that $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp}$ if and only if $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}$. Thus, if $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}$, then by (213), $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}) \cap (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp}$. Suppose now that $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}$. Then by (213), $2\,\mathrm{I}(\widetilde{\sigma}) \cap (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp} \subset 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq 2\,\mathrm{I}(\widetilde{\sigma})$. Taking orthogonal complements, it follows that $\mathrm{K}'(\widetilde{\sigma}) \subseteq \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subset \mathrm{K}'(\widetilde{\sigma}) + \mathbb{Z}/4\mathbb{Z}\widetilde{z}$. Now, let $\widetilde{w}'$ be in $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$. Since $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subset \mathrm{K}'(\widetilde{\sigma}) + \mathbb{Z}/4\mathbb{Z}\widetilde{z}$, there exist $\widetilde{a} \in \mathrm{K}'(\widetilde{\sigma})$ and $\lambda \in \mathbb{Z}/4\mathbb{Z}$ such that $\widetilde{w}' = \widetilde{a} + \lambda\widetilde{z}$. Hence, from $\mathrm{K}'(\widetilde{\sigma}) \subseteq \mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})$ and (214), we see that

$$0 = (1 - \widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})(2\widetilde{w}') = (1 - \widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})(2\widetilde{a} + 2\lambda\widetilde{z}) = \lambda(1 - \widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma})(2\widetilde{z}) = \lambda(1 - \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}}) 2\widetilde{v}.$$

  Since $2\widetilde{v} \neq 0$, this implies that $\lambda(1 - \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}}) \in 2\mathbb{Z}/4\mathbb{Z}$, i.e. $\lambda \in 2\mathbb{Z}/4\mathbb{Z}$ because $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}$ by hypothesis. This shows that $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq \mathrm{K}'(\widetilde{\sigma}) + 2\mathbb{Z}/4\mathbb{Z}\widetilde{z}$. More precisely, since $2\widetilde{W} \subseteq \mathrm{K}'(\widetilde{\tau})$ for every $\widetilde{\tau}$ in $\mathrm{Sp}(\widetilde{W})$, we have that $2\mathbb{Z}/4\mathbb{Z}\widetilde{z} \subseteq \mathrm{K}'(\widetilde{\sigma})$, and thus $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) \subseteq \mathrm{K}'(\widetilde{\sigma})$. Therefore, we conclude that $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = \mathrm{K}'(\widetilde{\sigma})$, i.e. $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma})$. □

We are now ready to define an explicit section of the projective representation $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma})$ of $\mathrm{Sp}(\widetilde{W})$ and then compute the associated cocycle $\widetilde{c}_{\chi_k}$.

Definition-Proposition 2.56 (Projective Weil representation of $\mathrm{Sp}(\widetilde{W})$). *For every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, and for every $\widetilde{w}_{\widetilde{\sigma}} \in \widetilde{W}$ such that* (207) *is satisfied, we define*

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\,\mathrm{K}(\widetilde{\sigma})|^{-1} |\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w} \in \widetilde{W}} \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1} \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}), \tag{215}$$

where $|\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 = |2\,\mathrm{I}(\widetilde{\sigma})|^{-1}$. Then $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ has the following properties:

i. *For every $h \in \mathrm{H}(\widetilde{W})$,*

$$\rho_{\chi_k}(\widetilde{\sigma}(h)) = \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(h) \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}. \tag{216}$$

ii. *For every $\widetilde{u} \in \mathrm{I}(\widetilde{\sigma})$, choose any $\widetilde{w} \in W$ such that $\widetilde{u} = (1 - \widetilde{\sigma})(\widetilde{w})$. Then*

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma})} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}). \tag{217}$$

iii. *Let $S_{\widetilde{\sigma}}$ be a system of representatives of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$. Then*

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}), \tag{218}$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$.

iv. *$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mathrm{U}(\mathrm{L}^2(Y))$.*

v. *If $\widetilde{w}'_{\widetilde{\sigma}}$ is another element satisfying* (207) *then*

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}}) \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1} \chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a} - 2\widetilde{v} \rangle_{\widetilde{W}} - \langle (1 - \widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a} \rangle_{\widetilde{W}}) \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}), \tag{219}$$

where $\widetilde{a}$ and $\widetilde{v}$ are such that $\widetilde{w}'_{\widetilde{\sigma}} - \widetilde{w}_{\widetilde{\sigma}} = (1 - \widetilde{\sigma})(\widetilde{v}) + 2\widetilde{a}$.

Therefore, for each choice $\{\widetilde{w}_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$, the map $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ is an explicit section of the projective Weil representation of $\mathrm{Sp}(\widetilde{W})$ on $\mathrm{L}^2(Y)$.

Proof. **i.** Let $h = (\widetilde{v}, t) \in \mathrm{H}(\widetilde{W})$ and $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$. Then

$$\begin{aligned} &\rho_{\chi_k}(\widetilde{\sigma}(h)) \circ \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\ &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\,\mathrm{K}(\widetilde{\sigma})|^{-1} |\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w} \in \widetilde{W}} \rho_{\chi_k}(\widetilde{\sigma}(h)) \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1} \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}), \end{aligned} \tag{220}$$

where

$$\begin{aligned}
&\rho_{\chi_k}(\tilde{\sigma}(h))\rho_{\chi_k}(\tilde{\sigma}(\tilde{w}))^{-1}\rho_{\chi_k}(\tilde{w}_{\tilde{\sigma}})\rho_{\chi_k}(\tilde{w})\\
&= \rho_{\chi_k}((\tilde{\sigma}(\tilde{v}),t)(-\tilde{\sigma}(\tilde{w}),0)(\tilde{w}_{\tilde{\sigma}},0)(\tilde{w},0))\\
&= \rho_{\chi_k}((\tilde{\sigma}(\tilde{v}-\tilde{w}),t-\langle\tilde{\sigma}(\tilde{v}),\tilde{\sigma}(\tilde{w})\rangle_{\widetilde{W}})(\tilde{w}_{\tilde{\sigma}}+\tilde{w},\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}}))\\
&= \rho_{\chi_k}(\tilde{\sigma}(\tilde{v}-\tilde{w})+\tilde{w}_{\tilde{\sigma}}+\tilde{w},t-\langle\tilde{\sigma}(\tilde{v}),\tilde{\sigma}(\tilde{w})\rangle_{\widetilde{W}}+\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}}+\langle\tilde{\sigma}(\tilde{v}-\tilde{w}),\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}(\tilde{\sigma}(\tilde{v}-\tilde{w})+\tilde{w}_{\tilde{\sigma}}+\tilde{w},t-\langle\tilde{v},\tilde{w}\rangle_{\widetilde{W}}+\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}}+\langle\tilde{\sigma}(\tilde{v}-\tilde{w}),\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}(\tilde{\sigma}(\tilde{v}-\tilde{w})+\tilde{w}_{\tilde{\sigma}}+\tilde{w},t+\langle\tilde{w}_{\tilde{\sigma}}-\tilde{v},\tilde{w}\rangle_{\widetilde{W}}+\langle\tilde{\sigma}(\tilde{v}-\tilde{w}),\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}}).
\end{aligned}$$

If we set $\tilde{w}' = \tilde{w} - \tilde{v}$, then

$$\begin{aligned}
&(\tilde{\sigma}(\tilde{v}-\tilde{w})+\tilde{w}_{\tilde{\sigma}}+\tilde{w},t+\langle\tilde{w}_{\tilde{\sigma}}-\tilde{v},\tilde{w}\rangle_{\widetilde{W}}+\langle\tilde{\sigma}(\tilde{v}-\tilde{w}),\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}})\\
&= (-\tilde{\sigma}(\tilde{w}')+\tilde{w}_{\tilde{\sigma}}+\tilde{w}'+\tilde{v},t+\langle\tilde{w}_{\tilde{\sigma}}-\tilde{v},\tilde{w}'+\tilde{v}\rangle_{\widetilde{W}}-\langle\tilde{\sigma}(\tilde{w}'),\tilde{w}_{\tilde{\sigma}}+\tilde{w}'+\tilde{v}\rangle_{\widetilde{W}})\\
&= (-\tilde{\sigma}(\tilde{w}'),0)(\tilde{w}_{\tilde{\sigma}}+\tilde{w}'+\tilde{v},t+\langle\tilde{w}_{\tilde{\sigma}}-\tilde{v},\tilde{w}'+\tilde{v}\rangle_{\widetilde{W}})\\
&= (-\tilde{\sigma}(\tilde{w}'),0)(\tilde{w}_{\tilde{\sigma}},0)(\tilde{w}'+\tilde{v},t-\langle\tilde{v},\tilde{w}'+\tilde{v}\rangle_{\widetilde{W}})\\
&= (-\tilde{\sigma}(\tilde{w}'),0)(\tilde{w}_{\tilde{\sigma}},0)(\tilde{w}',0)(\tilde{v},t).
\end{aligned}$$

Hence, (220) becomes

$$\begin{aligned}
&\rho_{\chi_k}(\tilde{\sigma}(h))\circ\omega_{\chi_k}(\tilde{\sigma},\tilde{w}_{\tilde{\sigma}})\\
&= \gamma(\tilde{\sigma},\tilde{w}_{\tilde{\sigma}})|\,\mathrm{K}(\tilde{\sigma})|^{-1}|\,\mathrm{I}(\tilde{\sigma})\cap 2\widetilde{W}|^{-1}\sum_{\tilde{w}'\in\widetilde{W}}\rho_{\chi_k}((-\tilde{\sigma}(\tilde{w}'),0)(\tilde{w}_{\tilde{\sigma}},0)(\tilde{w}',0)(\tilde{v},t))\\
&= \gamma(\tilde{\sigma},\tilde{w}_{\tilde{\sigma}})|\,\mathrm{K}(\tilde{\sigma})|^{-1}|\,\mathrm{I}(\tilde{\sigma})\cap 2\widetilde{W}|^{-1}\sum_{\tilde{w}'\in\widetilde{W}}\rho_{\chi_k}(\tilde{\sigma}(\tilde{w}'))^{-1}\rho_{\chi_k}(\tilde{w}_{\tilde{\sigma}})\rho_{\chi_k}(\tilde{w}')\rho_{\chi_k}(h)\\
&= \omega_{\chi_k}(\tilde{\sigma},\tilde{w}_{\tilde{\sigma}})\circ\rho_{\chi_k}(h).
\end{aligned}$$

ii. Let $\tilde{\sigma}$ be an element of $\mathrm{Sp}(\widetilde{W})$. Then

$$\begin{aligned}
&\rho_{\chi_k}(\tilde{\sigma}(\tilde{w}))^{-1}\rho_{\chi_k}(\tilde{w}_{\tilde{\sigma}})\rho_{\chi_k}(\tilde{w})\\
&= \rho_{\chi_k}(-\tilde{\sigma}(\tilde{w})+\tilde{w}_{\tilde{\sigma}}+\tilde{w},-\langle\tilde{\sigma}(\tilde{w}),\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}}+\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}((1-\tilde{\sigma})(\tilde{w})+\tilde{w}_{\tilde{\sigma}},\langle(1-\tilde{\sigma})(\tilde{w}),\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}}+2\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}(\tilde{u}+\tilde{w}_{\tilde{\sigma}},\langle\tilde{u},\tilde{w}_{\tilde{\sigma}}+\tilde{w}\rangle_{\widetilde{W}}+2\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}})\\
&= \chi_k(\langle\tilde{u},\tilde{w}+\tilde{w}_{\tilde{\sigma}}\rangle_{\widetilde{W}}+2\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}(\tilde{u}+\tilde{w}_{\tilde{\sigma}}),
\end{aligned}\tag{221}$$

where $\tilde{u} = (1-\tilde{\sigma})(\tilde{w}) \in \mathrm{I}(\tilde{\sigma})$. Now, let $\tilde{v} \in \mathrm{K}(\tilde{\sigma})$. Then, since $2\tilde{w}_{\tilde{\sigma}} \in \mathrm{I}(\tilde{\sigma})$ by (207), and $\mathrm{I}(\tilde{\sigma})$ and $\mathrm{K}(\tilde{\sigma})$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widetilde{W}}$,

$$\begin{aligned}
&\chi_k(\langle(1-\tilde{\sigma})(\tilde{w}+\tilde{v}),\tilde{w}+\tilde{v}+\tilde{w}_{\tilde{\sigma}}\rangle_{\widetilde{W}}+2\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}+\tilde{v}\rangle_{\widetilde{W}})\rho_{\chi_k}((1-\tilde{\sigma})(\tilde{w}+\tilde{v})+\tilde{w}_{\tilde{\sigma}})\\
&= \chi_k(\langle(1-\tilde{\sigma})(\tilde{w}),\tilde{w}+\tilde{v}+\tilde{w}_{\tilde{\sigma}}\rangle_{\widetilde{W}}+2\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}((1-\tilde{\sigma})(\tilde{w})+\tilde{w}_{\tilde{\sigma}})\\
&= \chi_k(\langle(1-\tilde{\sigma})(\tilde{w}),\tilde{w}+\tilde{w}_{\tilde{\sigma}}\rangle_{\widetilde{W}}+2\langle\tilde{w}_{\tilde{\sigma}},\tilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}((1-\tilde{\sigma})(\tilde{w})+\tilde{w}_{\tilde{\sigma}}).
\end{aligned}$$

This proves that the term $\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w})$ does not depend on the left coset of $\widetilde{w}$ modulo $\mathrm{K}(\widetilde{\sigma})$. Thus, we obtain

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{K}(\widetilde{\sigma})|^{-1}|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w}\in\widetilde{W}} \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{K}(\widetilde{\sigma})|^{-1}|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w}+\mathrm{K}(\widetilde{\sigma})\in\widetilde{W}/\,\mathrm{K}(\widetilde{\sigma})} |\,\mathrm{K}(\widetilde{\sigma})|\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w}) \\
&\overset{(221)}{=} \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w}+\mathrm{K}(\widetilde{\sigma})\in\widetilde{W}/\,\mathrm{K}(\widetilde{\sigma})} \chi_k(\langle\widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} + 2\langle\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{u}\in\mathrm{I}(\widetilde{\sigma})} \chi_k(\langle\widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} + 2\langle\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}),
\end{aligned}$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$, and we use the isomorphism between $\widetilde{W}/\,\mathrm{K}(\widetilde{\sigma})$ and $\mathrm{I}(\widetilde{\sigma})$.

iii. We proved in the preceding point that for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$,

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{u}\in\mathrm{I}(\widetilde{\sigma})} \chi_k(\langle\widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} + 2\langle\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}), \tag{222}$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$. However, the formula (222) is not completely satisfactory for what we will need in later computations, since the operator $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ is not decomposed in a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$. In fact, the operators $\rho_{\chi_k}(\widetilde{u})$, $\widetilde{u} \in \mathrm{I}(\widetilde{\sigma})$, may be linearly dependent. More precisely, let $\widetilde{u}_1 = \widetilde{x}_1 + \widetilde{y}_1 \in \widetilde{W}$ and $\widetilde{u}_2 = \widetilde{x}_2 + \widetilde{y}_2 \in \widetilde{W}$, where we recall that $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$. By Lemma 2.53, $\rho_{\chi_k}(\widetilde{u}_1)$ and $\rho_{\chi_k}(\widetilde{u}_2)$ are linearly dependent if and only if $\widetilde{u}_1 - \widetilde{u}_2 \in 2\widetilde{W}$. Suppose that $\widetilde{u}_1 - \widetilde{u}_2 = 2\widetilde{u} = 2\widetilde{x} + 2\widetilde{y}$. Then by (201), for all $\varphi \in \mathrm{L}^2(Y)$ and for all $z \in Y$,

$$\begin{aligned}
\rho_{\chi_k}(\widetilde{u}_1)\varphi(z) &= \rho_{\chi_k}(\widetilde{u}_2 + 2\widetilde{u})\varphi(z) \\
&= \chi_k(\langle x_2, z\rangle_W - \langle\widetilde{x}_2 + 2\widetilde{x}, \widetilde{y}_2 + 2\widetilde{y}\rangle_{\widetilde{W}})\varphi(z - y_2) \\
&= \chi_k(\langle x_2, z\rangle_W - \langle\widetilde{x}_2, \widetilde{y}_2\rangle_{\widetilde{W}} - \langle\widetilde{x}_2, 2\widetilde{y}\rangle_{\widetilde{W}} - \langle 2\widetilde{x}, \widetilde{y}_2\rangle_{\widetilde{W}})\varphi(z - y_2) \\
&= \chi_k(\langle x_2, z\rangle_W - \langle\widetilde{x}_2, \widetilde{y}_2\rangle_{\widetilde{W}} - \langle x_2, y\rangle_W - \langle x, y_2\rangle_W)\varphi(z - y_2) \\
&= \chi_k(\langle x_2, y\rangle_W + \langle y_2, x\rangle_W)\chi_k(\langle x_2, z\rangle_W - \langle\widetilde{x}_2, \widetilde{y}_2\rangle_{\widetilde{W}})\varphi(z - y_2) \\
&= \chi_k(\langle u_2, u\rangle_W)\rho_{\chi_k}(\widetilde{u}_2)\varphi(z) \\
&= \chi_k(\langle\widetilde{u}_2, 2\widetilde{u}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u}_2)\varphi(z) \\
&= \chi_k(\langle\widetilde{u}_2, \widetilde{u}_1\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u}_2)\varphi(z),
\end{aligned} \tag{223}$$

where $x_2 = \widetilde{x}_2 + 2\widetilde{X} = \widetilde{x}_2 + 2\widetilde{x} + 2\widetilde{X}$, $y_2 = \widetilde{y}_2 + 2\widetilde{Y} = \widetilde{y}_2 + 2\widetilde{y} + 2\widetilde{Y}$, $x = \widetilde{x} + 2\widetilde{X}$ and $y = \widetilde{y} + 2\widetilde{Y}$. Now, for every $\widetilde{u}_2$ in $\mathrm{I}(\widetilde{\sigma})$, set $S_{\widetilde{u}_2} = \{\widetilde{u}_1 \in \mathrm{I}(\widetilde{\sigma}),\ \widetilde{u}_1 - \widetilde{u}_2 \in 2\widetilde{W}\}$. By definition of $S_{\widetilde{u}_2}$, the map

$$\begin{aligned}
\phi_{\widetilde{u}_2} : S_{\widetilde{u}_2} &\to \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W} \\
\widetilde{u}_1 &\mapsto \widetilde{u}_1 - \widetilde{u}_2
\end{aligned}$$

is well-defined and bijective. Hence, if we set $N_{\tilde{\sigma}} = |\mathrm{I}(\tilde{\sigma})/(\mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W})|$ and choose $\{\tilde{u}_j,\ 1 \leqslant j \leqslant N_{\tilde{\sigma}}\}$ as a system of representatives of $\mathrm{I}(\tilde{\sigma})/(\mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W})$, then (222) becomes

$$
\begin{aligned}
&\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})^{-1} |\mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}| \omega_{\chi_k}(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}}) \\
&= \sum_{\tilde{u} \in \mathrm{I}(\tilde{\sigma})} \chi_k(\langle \tilde{u}, \tilde{w} + \tilde{w}_{\tilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\tilde{u} + \tilde{w}_{\tilde{\sigma}}) \\
&= \sum_{j=1}^{N_{\tilde{\sigma}}} \sum_{\tilde{u} \in S_{\tilde{u}_j}} \chi_k(\langle \tilde{u}, \tilde{w} + \tilde{w}_{\tilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\tilde{u} + \tilde{w}_{\tilde{\sigma}}) \\
&= \sum_{j=1}^{N_{\tilde{\sigma}}} \sum_{2\tilde{v} \in \mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \tilde{u}_j + 2\tilde{v}, \tilde{w} + \tilde{w}_{\tilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\tilde{u}_j + 2\tilde{v} + \tilde{w}_{\tilde{\sigma}}) \\
&\overset{(223)}{=} \sum_{j=1}^{N_{\tilde{\sigma}}} \sum_{2\tilde{v} \in \mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \tilde{u}_j + 2\tilde{v}, \tilde{w} + \tilde{w}_{\tilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}) \chi_k(\langle \tilde{u}_j + \tilde{w}_{\tilde{\sigma}}, 2\tilde{v} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\tilde{u}_j + \tilde{w}_{\tilde{\sigma}}) \\
&= \sum_{j=1}^{N_{\tilde{\sigma}}} \chi_k(\langle \tilde{u}_j, \tilde{w}_{\tilde{\sigma}} \rangle_{\widetilde{W}}) \sum_{2\tilde{v} \in \mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \tilde{u}_j, \tilde{w} + 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{v} + \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\tilde{u}_j + \tilde{w}_{\tilde{\sigma}}),
\end{aligned}
$$

where $\tilde{w} \in \widetilde{W}$ is any element such that $(1 - \tilde{\sigma})(\tilde{w}) = \tilde{u} = \tilde{u}_j + 2\tilde{v}$ and $w = \tilde{w} + 2\widetilde{W}$. Now, we fix $j \in \{1, \ldots, N_{\tilde{\sigma}}\}$ and we examine the sum

$$
\sum_{2\tilde{v} \in \mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \tilde{u}_j, \tilde{w} + 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{v} + \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}}).
$$

Let $\tilde{w}_j \in \widetilde{W}$ be any element such that $(1 - \tilde{\sigma})(\tilde{w}_j) = \tilde{u}_j$, let $\tilde{w} \in \widetilde{W}$ be any element such that $(1 - \tilde{\sigma})(\tilde{w}) = \tilde{u}_j + 2\tilde{v}$, and let $2\tilde{v} \in \mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W}$. Then, since $\tilde{w} - \tilde{w}_j \in \mathrm{K}'(\tilde{\sigma})$, by (207),

$$
\begin{aligned}
&\langle \tilde{\sigma}(\tilde{w} - \tilde{w}_j), \tilde{w} - \tilde{w}_j \rangle_{\widetilde{W}} = 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} - \tilde{w}_j \rangle_{\widetilde{W}} \\
&\implies \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}), \tilde{w}_j \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}_j), \tilde{w} \rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_j), \tilde{w}_j \rangle_{\widetilde{W}} = 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} - \tilde{w}_j \rangle_{\widetilde{W}} \\
&\implies 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} = 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w}_j \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}), \tilde{w}_j \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}_j), \tilde{w} \rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_j), \tilde{w}_j \rangle_{\widetilde{W}} \\
&\implies 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} - \langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} = 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w}_j \rangle_{\widetilde{W}} + \langle \tilde{w}, (\tilde{\sigma} - \tilde{\sigma}^{-1})(\tilde{w}_j) \rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_j), \tilde{w}_j \rangle_{\widetilde{W}}.
\end{aligned} \tag{224}
$$

Hence

$$
\begin{aligned}
&\langle \tilde{u}_j, \tilde{w} + 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{v} + \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} \\
&= \langle \tilde{u}_j + 2\tilde{v}, \tilde{w} \rangle_{\widetilde{W}} + \langle \tilde{u}_j, 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} \\
&= \langle (1 - \tilde{\sigma})(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} + \langle \tilde{u}_j, 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} \\
&= -\langle \tilde{\sigma}(\tilde{w}), \tilde{w} \rangle_{\widetilde{W}} + \langle \tilde{u}_j, 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w} \rangle_{\widetilde{W}} \\
&\overset{(224)}{=} \langle \tilde{u}_j, 2\tilde{v} \rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}}, \tilde{w}_j \rangle_{\widetilde{W}} + \langle \tilde{w}, (\tilde{\sigma} - \tilde{\sigma}^{-1})(\tilde{w}_j) \rangle_{\widetilde{W}} + \langle \tilde{\sigma}(\tilde{w}_j), \tilde{w}_j \rangle_{\widetilde{W}}.
\end{aligned} \tag{225}
$$

Moreover,

$$\begin{aligned}
&\langle \widetilde{u}_j, 2\widetilde{v}\rangle_{\widetilde{W}} + \langle \widetilde{w}, (\widetilde{\sigma} - \widetilde{\sigma}^{-1})(\widetilde{w}_j)\rangle_{\widetilde{W}} \\
&= \langle (1-\widetilde{\sigma})(\widetilde{w}_j), (1-\widetilde{\sigma})(\widetilde{w} - \widetilde{w}_j)\rangle_{\widetilde{W}} + \langle \widetilde{w}, \widetilde{\sigma}(\widetilde{w}_j)\rangle_{\widetilde{W}} - \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w}_j\rangle_{\widetilde{W}} \\
&= \langle (1-\widetilde{\sigma})(\widetilde{w}_j), (1-\widetilde{\sigma})(\widetilde{w})\rangle_{\widetilde{W}} + \langle \widetilde{w}, \widetilde{\sigma}(\widetilde{w}_j)\rangle_{\widetilde{W}} - \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w}_j\rangle_{\widetilde{W}} \\
&= 2\langle \widetilde{w}_j, \widetilde{w}\rangle_{\widetilde{W}} - 2\langle \widetilde{\sigma}(\widetilde{w}_j), \widetilde{w}\rangle_{\widetilde{W}} \\
&= \langle 2(1-\widetilde{\sigma})(\widetilde{w}_j), \widetilde{w}\rangle_{\widetilde{W}} \\
&= \langle 2\widetilde{u}_j, \widetilde{w}\rangle_{\widetilde{W}}.
\end{aligned}$$

Furthermore, since $2\widetilde{u}_j \in 2\,\mathrm{I}(\widetilde{\sigma})$, $\widetilde{w} - \widetilde{w}_j \in \mathrm{K}'(\widetilde{\sigma})$ and $\mathrm{K}'(\widetilde{\sigma})^\perp = 2\,\mathrm{I}(\widetilde{\sigma})$ by Lemma 2.54, we have that $\langle 2\widetilde{u}_j, \widetilde{w} - \widetilde{w}_j\rangle_{\widetilde{W}} = 0$. Hence

$$\langle \widetilde{u}_j, 2\widetilde{v}\rangle_{\widetilde{W}} + \langle \widetilde{w}, (\widetilde{\sigma} - \widetilde{\sigma}^{-1})(\widetilde{w}_j)\rangle_{\widetilde{W}} = \langle 2\widetilde{u}_j, \widetilde{w}\rangle_{\widetilde{W}} = \langle 2\widetilde{u}_j, \widetilde{w}_j\rangle_{\widetilde{W}}. \tag{226}$$

Now, (225) and (226) imply that

$$\sum_{2\widetilde{v}\in \mathrm{I}(\widetilde{\sigma})\cap 2\widetilde{W}} \chi_k(\langle \widetilde{u}_j, \widetilde{w}+2\widetilde{v}\rangle_{\widetilde{W}} + 2\langle \widetilde{v}+\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}}) = |\,\mathrm{I}(\widetilde{\sigma})\cap 2\widetilde{W}|\chi_k(2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_j\rangle_{\widetilde{W}} + \langle \widetilde{\sigma}(\widetilde{w}_j), \widetilde{w}_j\rangle_{\widetilde{W}} + \langle 2\widetilde{u}_j, \widetilde{w}_j\rangle_{\widetilde{W}}). \tag{227}$$

This proves that

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{j=1}^{N_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}_j, \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}}) \sum_{\substack{2\widetilde{v}\in \mathrm{I}(\widetilde{\sigma})\cap 2\widetilde{W} \\ \rho_{\chi_k}(\widetilde{u}_j + \widetilde{w}_{\widetilde{\sigma}})}} \chi_k(\langle \widetilde{u}_j, \widetilde{w} + 2\widetilde{v}\rangle_{\widetilde{W}} + 2\langle \widetilde{v} + \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{j=1}^{N_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}_j, \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|\chi_k(2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_j\rangle_{\widetilde{W}} + \langle \widetilde{\sigma}(\widetilde{w}_j), \widetilde{w}_j\rangle_{\widetilde{W}}) \\
&\qquad \chi_k(\langle 2\widetilde{u}_j, \widetilde{w}_j\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u}_j + \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{j=1}^{N_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}_j, \widetilde{w}_j + \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_j\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u}_j + \widetilde{w}_{\widetilde{\sigma}}).
\end{aligned} \tag{228}$$

The last thing we can check is that this expression does not depend on the choice of $\widetilde{w}_j$ which satisfies $(1-\widetilde{\sigma})(\widetilde{w}_j) = \widetilde{u}_j$. In fact, if $\widetilde{w}'_j$ is such that $(1-\widetilde{\sigma})(\widetilde{w}'_j) = \widetilde{u}_j$, then $\widetilde{w}_j - \widetilde{w}'_j \in \mathrm{K}(\widetilde{\sigma})$. Since $\widetilde{u}_j \in \mathrm{I}(\widetilde{\sigma})$ by definition, $2\widetilde{w}_{\widetilde{\sigma}} \in \mathrm{I}(\widetilde{\sigma})$ by (207) and $\mathrm{K}(\widetilde{\sigma})^\perp = \mathrm{I}(\widetilde{\sigma})$, we have that $\langle \widetilde{u}_j, \widetilde{w}_j - \widetilde{w}'_j\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_j - \widetilde{w}'_j\rangle_{\widetilde{W}} = 0$. Finally, rewriting (228), we obtain that for every system of representatives $S_{\widetilde{\sigma}}$ of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$,

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u}\in S_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}),$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1-\widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$.

iv. Since ρ_{χ_k} is a unitary representation,

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})^*\\
&= |\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})|^2|\,\mathrm{K}(\widetilde{\sigma})|^{-2}|\,\mathrm{I}(\widetilde{\sigma})\cap 2\widetilde{W}|^{-2}\\
&\quad\sum_{\widetilde{w}\in\widetilde{W}}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w})\sum_{\widetilde{w}'\in\widetilde{W}}\rho_{\chi_k}(\widetilde{w}')^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})^{-1}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}'))\\
&= |\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})|^2|\,\mathrm{K}(\widetilde{\sigma})|^{-2}|\,\mathrm{I}(\widetilde{\sigma})\cap 2\widetilde{W}|^{-2}\\
&\quad\sum_{\widetilde{w},\widetilde{w}'\in\widetilde{W}}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w})\rho_{\chi_k}(\widetilde{w}')^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})^{-1}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}')),
\end{aligned}$$

where, if we set $\widetilde{z}=\widetilde{w}-\widetilde{w}'$,

$$\begin{aligned}
&\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w})\rho_{\chi_k}(\widetilde{w}')^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})^{-1}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}'))\\
&= \rho_{\chi_k}(-\widetilde{\sigma}(\widetilde{w})+\widetilde{w}_{\widetilde{\sigma}}+\widetilde{w},\langle\widetilde{w}+\widetilde{w}_{\widetilde{\sigma}},\widetilde{\sigma}(\widetilde{w})\rangle_{\widetilde{W}}+\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{w}\rangle_{\widetilde{W}})\\
&\quad\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}')-\widetilde{w}'-\widetilde{w}_{\widetilde{\sigma}},\langle\widetilde{\sigma}(\widetilde{w}'),\widetilde{w}'+\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}}+\langle\widetilde{w}',\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}'-\widetilde{w})+\widetilde{w}-\widetilde{w}',\langle\widetilde{w}+\widetilde{w}_{\widetilde{\sigma}},\widetilde{\sigma}(\widetilde{w})\rangle_{\widetilde{W}}+\langle\widetilde{\sigma}(\widetilde{w}'),\widetilde{w}'+\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}}+\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{w}-\widetilde{w}'\rangle_{\widetilde{W}}\\
&\quad+\langle\widetilde{w}+\widetilde{w}_{\widetilde{\sigma}}-\widetilde{\sigma}(\widetilde{w}),\widetilde{\sigma}(\widetilde{w}')-\widetilde{w}'-\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{w}'+\widetilde{z}+\widetilde{w}_{\widetilde{\sigma}},\widetilde{\sigma}(\widetilde{w}'+\widetilde{z})\rangle_{\widetilde{W}}+\langle\widetilde{\sigma}(\widetilde{w}'),\widetilde{w}'+\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}}+\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}}\\
&\quad-\langle(1-\widetilde{\sigma})(\widetilde{w}'+\widetilde{z})+\widetilde{w}_{\widetilde{\sigma}},(1-\widetilde{\sigma})(\widetilde{w}')+\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{w}',\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+\langle\widetilde{z},\widetilde{\sigma}(\widetilde{w}')\rangle_{\widetilde{W}}+\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{\sigma}(\widetilde{z})+\widetilde{z}\rangle_{\widetilde{W}}\\
&\quad-\langle(1-\widetilde{\sigma})(\widetilde{z}),(1-\widetilde{\sigma})(\widetilde{w}')+\widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+2\langle\widetilde{z},\widetilde{\sigma}(\widetilde{w}')\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}}-2\langle\widetilde{z},\widetilde{w}'\rangle_{\widetilde{W}})\\
&= \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}}+2\langle\widetilde{w}',(1-\widetilde{\sigma}^{-1})(\widetilde{z})\rangle_{\widetilde{W}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})|^{-2}|\,\mathrm{K}(\widetilde{\sigma})|^2|\,\mathrm{I}(\widetilde{\sigma})\cap 2\widetilde{W}|^2\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})^*\\
&= \sum_{\widetilde{w},\widetilde{w}'\in\widetilde{W}}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w})\rho_{\chi_k}(\widetilde{w}')^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})^{-1}\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}'))\\
&= \sum_{\widetilde{z},\widetilde{w}'\in\widetilde{W}}\rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}}+2\langle\widetilde{w}',(1-\widetilde{\sigma}^{-1})(\widetilde{z})\rangle_{\widetilde{W}})\\
&= \sum_{\widetilde{z}\in\widetilde{W}}\rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}})\sum_{\widetilde{w}'\in\widetilde{W}}\chi_k(2\langle\widetilde{w}',(1-\widetilde{\sigma}^{-1})(\widetilde{z})\rangle_{\widetilde{W}})\\
&= \sum_{\widetilde{z}\in\widetilde{W}}\rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}})|\widetilde{W}|\mathbb{1}_{2\widetilde{W}}((1-\widetilde{\sigma}^{-1})(\widetilde{z}))\\
&= |\widetilde{W}|\sum_{\widetilde{z}\in\widetilde{W}}\rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}})\mathbb{1}_{2\widetilde{W}}((1-\widetilde{\sigma})(\widetilde{z})),
\end{aligned}$$

where the fourth equality is due to Schur's orthogonality relations. Since $2\widetilde{w}_{\widetilde{\sigma}}\in\mathrm{I}(\widetilde{\sigma})$ by (207), the term $\rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}),\langle\widetilde{z},\widetilde{\sigma}(\widetilde{z}))\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\widetilde{\sigma}},\widetilde{z}\rangle_{\widetilde{W}})\mathbb{1}_{2\widetilde{W}}((1-\widetilde{\sigma})(\widetilde{z}))$ does not

depend on the left coset of $\widetilde{z}$ modulo $\mathrm{K}(\widetilde{\sigma})$. Thus, it follows that

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^* \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{K}(\widetilde{\sigma})|^{-2} |\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} |\widetilde{W}| \\
&\qquad \sum_{\widetilde{z}+\mathrm{K}(\widetilde{\sigma}) \in \widetilde{W}/\mathrm{K}(\widetilde{\sigma})} |\mathrm{K}(\widetilde{\sigma})| \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}), \langle \widetilde{z}, \widetilde{\sigma}(\widetilde{z})\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \mathbb{1}_{2\widetilde{W}}((1-\widetilde{\sigma})(\widetilde{z})) \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma})} \rho_{\chi_k}(\widetilde{u}, \langle \widetilde{u}, \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \mathbb{1}_{2\widetilde{W}}(\widetilde{u}) \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}} \rho_{\chi_k}(\widetilde{u}, \langle \widetilde{u}, \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \\
&\overset{(223)}{=} |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \widetilde{u}, \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \rho_{\chi_k}(0) \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \widetilde{u}, \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \, \mathrm{Id}_{\mathrm{L}^2(Y)},
\end{aligned}$$

where $\widetilde{z}$ is any element of $\widetilde{W}$ such that $(1-\widetilde{\sigma})(\widetilde{z}) = \widetilde{u}$, and we use the isomorphism between $\widetilde{W}/\mathrm{K}(\widetilde{\sigma})$ and $\mathrm{I}(\widetilde{\sigma})$. Now, for all $\widetilde{u} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}$ and $\widetilde{z}$ such that $(1-\widetilde{\sigma})(\widetilde{z}) = \widetilde{u}$, by (208), we have that $\widetilde{z} \in \mathrm{K}'(\widetilde{\sigma})$. Then, by (207),

$$\langle \widetilde{u}, \widetilde{z}\rangle_{\widetilde{W}} = \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{z}\rangle_{\widetilde{W}} = -\langle \widetilde{\sigma}(\widetilde{z}), \widetilde{z}\rangle_{\widetilde{W}} = -2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}.$$

Thus

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^* \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}} \chi_k(\langle \widetilde{u}, \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \, \mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}} \chi_k(-2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}) \, \mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-2} |\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}| \, \mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= |\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 |2\,\mathrm{I}(\widetilde{\sigma})| \, \mathrm{Id}_{\mathrm{L}^2(Y)} \\
&= \mathrm{Id}_{\mathrm{L}^2(Y)},
\end{aligned}$$

since $|\mathrm{I}(\widetilde{\sigma})| \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} = |2\,\mathrm{I}(\widetilde{\sigma})|$ by Lemma 2.54 and $|\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 = |2\,\mathrm{I}(\widetilde{\sigma})|^{-1}$ by hypothesis.

v. Let $\widetilde{w}'_{\widetilde{\sigma}} \in \widetilde{W}$ such that (207) is satisfied. By Lemma 2.54, there exists $\widetilde{v} \in \widetilde{W}$ such that $2\widetilde{w}'_{\widetilde{\sigma}} - 2\widetilde{w}_{\widetilde{\sigma}} = 2(1-\widetilde{\sigma})(\widetilde{v})$. In particular, $2(\widetilde{w}'_{\widetilde{\sigma}} - \widetilde{w}_{\widetilde{\sigma}} - (1-\widetilde{\sigma})(\widetilde{v})) = 0$, i.e. there exists $\widetilde{a} \in \widetilde{W}$ such that $\widetilde{w}'_{\widetilde{\sigma}} - \widetilde{w}_{\widetilde{\sigma}} - (1-\widetilde{\sigma})(\widetilde{v}) = 2\widetilde{a}$. Using (221), and setting $\widetilde{z} = \widetilde{w} + \widetilde{v}$, we

write

$$
\begin{aligned}
&\widetilde{\sigma}(\widetilde{w})^{-1}\widetilde{w}'_{\widetilde{\sigma}}\widetilde{w}\\
&= ((1-\widetilde{\sigma})(\widetilde{w}) + \widetilde{w}'_{\widetilde{\sigma}}, \langle (1-\widetilde{\sigma})(\widetilde{w}), \widetilde{w}'_{\widetilde{\sigma}} + \widetilde{w}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}'_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}})\\
&= ((1-\widetilde{\sigma})(\widetilde{w}) + \widetilde{w}_{\widetilde{\sigma}} + (1-\widetilde{\sigma})(\widetilde{v}) + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{w}), \widetilde{w}_{\widetilde{\sigma}} + (1-\widetilde{\sigma})(\widetilde{v}) + 2\widetilde{a} + \widetilde{w}\rangle_{\widetilde{W}}\\
&\quad + 2\langle \widetilde{w}_{\widetilde{\sigma}} + (1-\widetilde{\sigma})(\widetilde{v}) + 2\widetilde{a}, \widetilde{w}\rangle_{\widetilde{W}})\\
&= ((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{z}-\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + (1-\widetilde{\sigma})(\widetilde{v}) + 2\widetilde{a} + \widetilde{z} - \widetilde{v}\rangle_{\widetilde{W}}\\
&\quad + 2\langle \widetilde{w}_{\widetilde{\sigma}} + (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{z}-\widetilde{v}\rangle_{\widetilde{W}})\\
&= ((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a} - \widetilde{\sigma}(\widetilde{v})\rangle_{\widetilde{W}}\\
&\quad - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} - \widetilde{\sigma}(\widetilde{v}) + 2\widetilde{a} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}} + 2\langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{z}-\widetilde{v}\rangle_{\widetilde{W}})\\
&= ((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}} + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a}\rangle_{\widetilde{W}}\\
&\quad + \langle \widetilde{z}, (1-\widetilde{\sigma})(\widetilde{v})\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{z}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} - \widetilde{\sigma}(\widetilde{v}) + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}}\\
&\quad + 2\langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{\sigma}(\widetilde{v}), \widetilde{v}\rangle_{\widetilde{W}})\\
&= ((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}} + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a}\rangle_{\widetilde{W}}\\
&\quad - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} - \widetilde{\sigma}(\widetilde{v}) + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}} + 2\langle \widetilde{\sigma}(\widetilde{v}), \widetilde{v}\rangle_{\widetilde{W}})\\
&= ((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}} + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a}\rangle_{\widetilde{W}}\\
&\quad - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}})
\end{aligned}
$$

Then

$$
\begin{aligned}
&\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w})^{-1}\widetilde{w}'_{\widetilde{\sigma}}\widetilde{w})\\
&= \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{a}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}} + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a}\rangle_{\widetilde{W}}\\
&\quad - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}})\\
&\overset{(223)}{=} \chi_k(\langle (1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a}\rangle_{\widetilde{W}})\rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}}\\
&\quad + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}})\\
&= \chi_k(\langle (1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a}\rangle_{\widetilde{W}} + \langle (1-\widetilde{\sigma})(\widetilde{z}), 2\widetilde{a}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}})\\
&\quad \rho_{\chi_k}((1-\widetilde{\sigma})(\widetilde{z}) + \widetilde{w}_{\widetilde{\sigma}}, \langle (1-\widetilde{\sigma})(\widetilde{z}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{z}\rangle_{\widetilde{W}})\\
&= \chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}} - 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{v}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{z})^{-1}\widetilde{w}_{\widetilde{\sigma}}\widetilde{z})\\
&= \chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a} - 2\widetilde{v}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{z})^{-1}\widetilde{w}_{\widetilde{\sigma}}\widetilde{z}),
\end{aligned}
$$

where we used again (221) in order to obtain the second to last equality. Now, by (215),

$$
\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}})\\
&= |\,\mathrm{K}(\widetilde{\sigma})|^{-1}\gamma(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}}) \sum_{\widetilde{w}\,\in\,\widetilde{W}} \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1}\rho_{\chi_k}(\widetilde{w}'_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w})\\
&= |\,\mathrm{K}(\widetilde{\sigma})|^{-1}\gamma(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}}) \sum_{\widetilde{z}\,\in\,\widetilde{W}} \chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a} - 2\widetilde{v}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{z})^{-1}\widetilde{w}_{\widetilde{\sigma}}\widetilde{z})\\
&= \gamma(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}})\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, 2\widetilde{a} - 2\widetilde{v}\rangle_{\widetilde{W}} - \langle (1-\widetilde{\sigma})(\widetilde{v}), \widetilde{w}_{\widetilde{\sigma}} + \widetilde{v} + 2\widetilde{a}\rangle_{\widetilde{W}})\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}).
\end{aligned}
$$

□

Remark 2.57. In Proposition 2.56, we defined an operator $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and for all $\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207). From now on, to simplify the notation, if it is clear that a choice of $\widetilde{w}_{\widetilde{\sigma}}$ has been made, then we will write indifferently $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ or $\omega_{\chi_k}(\widetilde{\sigma})$.

Corollary 2.58 (Computation of the cocycle $\widetilde{c}_{\chi_k}$). *We use the same notations as in Lemma 2.54 and Proposition 2.56, and we assume that a choice of $\widetilde{w}_{\widetilde{\sigma}} \in \widetilde{W}$ satisfying* (207) *has been made for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$. Then the cocycle $\widetilde{c}_{\chi_k}$ associated to the projective Weil representation ω_{χ_k} is given by the following formula. Let $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ be in $\mathrm{Sp}(\widetilde{W})$, and let $S_{\widetilde{\sigma}_1}$ and $S_{\widetilde{\sigma}_2}$ be two systems of representatives of $\mathrm{I}(\widetilde{\sigma}_1)/(\mathrm{I}(\widetilde{\sigma}_1) \cap 2\widetilde{W})$ and $\mathrm{I}(\widetilde{\sigma}_2)/(\mathrm{I}(\widetilde{\sigma}_2) \cap 2\widetilde{W})$, respectively. Then*

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2) = C_{1,2} \sum_{\substack{\widetilde{u}_1 \in S_{\widetilde{\sigma}_1} \\ \widetilde{u}_2 \in S_{\widetilde{\sigma}_2} \\ \widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_1, \widetilde{\sigma}_2} \in 2\widetilde{W}}} \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_1} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\sigma}_2} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_2 \rangle_{\widetilde{W}} + \langle \widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\sigma}_2} \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_1 \widetilde{\sigma}_2}, \widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_1, \widetilde{\sigma}_2} \rangle_{\widetilde{W}}). \tag{229}$$

where $C_{1,2} = \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}$, $\widetilde{w}_{\widetilde{\sigma}_1, \widetilde{\sigma}_2} = \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - \widetilde{w}_{\widetilde{\sigma}_1} - \widetilde{w}_{\widetilde{\sigma}_2}$, and for $i \in \{1, 2\}$, $\widetilde{w}_i \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma}_i)(\widetilde{w}_i) = \widetilde{u}_i$.

Proof. Let $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ be in $\mathrm{Sp}(\widetilde{W})$. Since ω_{χ_k} is a projective representation, the cocycle $\widetilde{c}_{\chi_k}$ satisfies:

$$\omega_{\chi_k}(\widetilde{\sigma}_1)\omega_{\chi_k}(\widetilde{\sigma}_2) = \widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2)\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2), \tag{230}$$

where we recall from Remark 2.57 that $\omega_{\chi_k}(\widetilde{\sigma}_1) = \omega_{\chi_k}(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})$ and $\omega_{\chi_k}(\widetilde{\sigma}_2) = \omega_{\chi_k}(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})$. Now we compute separately $\omega_{\chi_k}(\widetilde{\sigma}_1)\omega_{\chi_k}(\widetilde{\sigma}_2)$ and $\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2)$, using the expression (218) found in Proposition 2.56. Let $S_{\widetilde{\sigma}_1}$, $S_{\widetilde{\sigma}_2}$ and $S_{\widetilde{\sigma}_1\widetilde{\sigma}_2}$ be three systems of representatives of $\mathrm{I}(\widetilde{\sigma}_1)/(\mathrm{I}(\widetilde{\sigma}_1) \cap 2\widetilde{W})$, $\mathrm{I}(\widetilde{\sigma}_2)/(\mathrm{I}(\widetilde{\sigma}_2) \cap 2\widetilde{W})$ and $\mathrm{I}(\widetilde{\sigma}_1\widetilde{\sigma}_2)/(\mathrm{I}(\widetilde{\sigma}_1\widetilde{\sigma}_2) \cap 2\widetilde{W})$. Then

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}_1)\omega_{\chi_k}(\widetilde{\sigma}_2) \\
&= \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}) \\
&\quad \cdot \sum_{\substack{\widetilde{u}_1 \in S_{\widetilde{\sigma}_1} \\ \widetilde{u}_2 \in S_{\widetilde{\sigma}_2}}} \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_1} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\sigma}_2} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_2 \rangle_{\widetilde{W}}) \\
&\qquad \rho_{\chi_k}(\widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_1})\rho_{\chi_k}(\widetilde{u}_2 + \widetilde{w}_{\widetilde{\sigma}_2}) \\
&= \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}) \\
&\quad \cdot \sum_{\substack{\widetilde{u}_1 \in S_{\widetilde{\sigma}_1} \\ \widetilde{u}_2 \in S_{\widetilde{\sigma}_2}}} \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_1} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\sigma}_2} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_2 \rangle_{\widetilde{W}} \\
&\qquad + \langle \widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\sigma}_2} \rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_1} + \widetilde{u}_2 + \widetilde{w}_{\widetilde{\sigma}_2})
\end{aligned}$$

where, for $j \in \{1, 2\}$, $\widetilde{w}_j \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma}_j)(\widetilde{w}_j) = \widetilde{u}_j$. Moreover,

$$\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2) = \gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}_1\widetilde{\sigma}_2}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, \widetilde{w} \rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}),$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma}_1\widetilde{\sigma}_2)(\widetilde{w}) = \widetilde{u}$. By Lemma 2.53, we know that the family of operators $\rho_{\chi_k}(\widetilde{v})$, for all $\widetilde{v}$ in a system of representatives of $\widetilde{W}/2\widetilde{W}$, form a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$. Hence, we can identify the coefficients of $\omega_{\chi_k}(\widetilde{\sigma}_1)\omega_{\chi_k}(\widetilde{\sigma}_2)$ and $\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2)$ in this basis. In fact, for all $\widetilde{u} \in S_{\widetilde{\sigma}_1\widetilde{\sigma}_2}$, $\widetilde{u}_1 \in S_{\widetilde{\sigma}_1}$ and $\widetilde{u}_2 \in S_{\widetilde{\sigma}_2}$, the operators $\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})$ and

$\rho_{\chi_k}(\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2})$ are linearly dependent if and only if $\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}-(\widetilde{u}+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2})$ is in $2\widetilde{W}$. In this case, by (223),

$$\rho_{\chi_k}(\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2})=\chi_k(\underbrace{\langle\widetilde{u}+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2},\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}}_{\in 2\mathbb{Z}/4\mathbb{Z}})\rho_{\chi_k}(\widetilde{u}+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2}).$$

So, for every $\widetilde{u}\in S_{\tilde{\sigma}_1\tilde{\sigma}_2}$,

$$\begin{aligned}&\widetilde{c}_{\chi_k}(\tilde{\sigma}_1,\tilde{\sigma}_2)\gamma(\tilde{\sigma}_1\tilde{\sigma}_2,\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2})\chi_k(\langle\widetilde{u},\widetilde{w}+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2},\widetilde{w}\rangle_{\widetilde{W}})\\&=\gamma(\tilde{\sigma}_1,\widetilde{w}_{\tilde{\sigma}_1})\gamma(\tilde{\sigma}_2,\widetilde{w}_{\tilde{\sigma}_2})\\&\cdot\sum_{\substack{\widetilde{u}_1\in S_{\tilde{\sigma}_1}\\ \widetilde{u}_2\in S_{\tilde{\sigma}_2}\\ \widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}-(\widetilde{u}+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2})\in 2\widetilde{W}}}\chi_k(\langle\widetilde{u}_1,\widetilde{w}_1+\widetilde{w}_{\tilde{\sigma}_1}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_1},\widetilde{w}_1\rangle_{\widetilde{W}}+\langle\widetilde{u}_2,\widetilde{w}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_2},\widetilde{w}_2\rangle_{\widetilde{W}}\\&\qquad+\langle\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1},\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}+\langle\widetilde{u}+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2},\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}).\end{aligned}\tag{231}$$

Now, since this formula is valid for every system of representatives of $\mathrm{I}(\tilde{\sigma}_1\tilde{\sigma}_2)/(\mathrm{I}(\tilde{\sigma}_1\tilde{\sigma}_2)\cap 2\widetilde{W})$, and since there exists a unique element $\widetilde{u}_0$ in $S_{\tilde{\sigma}_1\tilde{\sigma}_2}$ such that $\widetilde{u}_0\in\mathrm{I}(\tilde{\sigma}_1\tilde{\sigma}_2)\cap 2\widetilde{W}$, we can suppose that $\widetilde{u}_0=0$ and write (231) for $\widetilde{u}=\widetilde{u}_0$. In particular, we can also suppose that $\widetilde{w}_0=0$. Hence $\langle\widetilde{u}_0,\widetilde{w}_0+\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2},\widetilde{w}_0\rangle_{\widetilde{W}}=0$. Thus, we obtain

$$\begin{aligned}&\widetilde{c}_{\chi_k}(\tilde{\sigma}_1,\tilde{\sigma}_2)\gamma(\tilde{\sigma}_1\tilde{\sigma}_2,\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2})\\&=\gamma(\tilde{\sigma}_1,\widetilde{w}_{\tilde{\sigma}_1})\gamma(\tilde{\sigma}_2,\widetilde{w}_{\tilde{\sigma}_2})\\&\cdot\sum_{\substack{\widetilde{u}_1\in S_{\tilde{\sigma}_1}\\ \widetilde{u}_2\in S_{\tilde{\sigma}_2}\\ \widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}-\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2}\in 2\widetilde{W}}}\chi_k(\langle\widetilde{u}_1,\widetilde{w}_1+\widetilde{w}_{\tilde{\sigma}_1}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_1},\widetilde{w}_1\rangle_{\widetilde{W}}+\langle\widetilde{u}_2,\widetilde{w}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_2},\widetilde{w}_2\rangle_{\widetilde{W}}\\&\qquad+\langle\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1},\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}+\langle\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2},\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1}+\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}).\end{aligned}$$

If we set $C_{1,2}=\gamma(\tilde{\sigma}_1,\widetilde{w}_{\tilde{\sigma}_1})\gamma(\tilde{\sigma}_2,\widetilde{w}_{\tilde{\sigma}_2})\gamma(\tilde{\sigma}_1\tilde{\sigma}_2,\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2})^{-1}$ and $\widetilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2}=\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2}-\widetilde{w}_{\tilde{\sigma}_1}-\widetilde{w}_{\tilde{\sigma}_2}$, then we finally obtain

$$\begin{aligned}\widetilde{c}_{\chi_k}(\tilde{\sigma}_1,\tilde{\sigma}_2)=C_{1,2}\sum_{\substack{\widetilde{u}_1\in S_{\tilde{\sigma}_1}\\ \widetilde{u}_2\in S_{\tilde{\sigma}_2}\\ \widetilde{u}_1+\widetilde{u}_2-\widetilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2}\in 2\widetilde{W}}}&\chi_k(\langle\widetilde{u}_1,\widetilde{w}_1+\widetilde{w}_{\tilde{\sigma}_1}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_1},\widetilde{w}_1\rangle_{\widetilde{W}}+\langle\widetilde{u}_2,\widetilde{w}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}+2\langle\widetilde{w}_{\tilde{\sigma}_2},\widetilde{w}_2\rangle_{\widetilde{W}}\\&+\langle\widetilde{u}_1+\widetilde{w}_{\tilde{\sigma}_1},\widetilde{u}_2+\widetilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}}+\langle\widetilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2},\widetilde{u}_1+\widetilde{u}_2-\widetilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2}\rangle_{\widetilde{W}}).\end{aligned}$$

□

Now, as for the constructions of the coverings of $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$ from the previous sections, the next step is to look for the smallest possible range of the cocycle $\widetilde{c}_{\chi_k}$, in order to define the smallest possible central extension of $\mathrm{Sp}(\widetilde{W})$, and then to linearize the projective representation ω_{χ_k} to this extension. The range of this cocycle depends entirely on the choice made for $\gamma(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})$ for every $\tilde{\sigma}\in\mathrm{Sp}(\widetilde{W})$. Recall that $\gamma(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})$ is defined in Proposition 2.56, and it only has to satisfy $|\gamma(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})|^2=|2\,\mathrm{I}(\tilde{\sigma})|^{-1}$ to ensure that $\omega_{\chi_k}(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})$ is a unitary operator. Hence, we can set

$$\gamma(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})=|2\,\mathrm{I}(\tilde{\sigma})|^{-\frac{1}{2}}e^{i\theta(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})},\quad\text{where }\theta(\tilde{\sigma},\widetilde{w}_{\tilde{\sigma}})\in\mathbb{R}.\tag{232}$$

The first question is if we can choose $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ so that $\widetilde{c}_{\chi_k}$ is a coboundary, which would mean that we do not need any extension of $\mathrm{Sp}(\widetilde{W})$ to lift ω_{χ_k} to a linear representation. The next result shows that there is no such choice.

Corollary 2.59. *The cocycle $\widetilde{c}_{\chi_k}$ is not a coboundary.*

Proof. Since the operator $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ is unitary for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, we know that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mathrm{U}(1)$. If $\widetilde{c}_{\chi_k}$ is a coboundary, then there exists $\nu : \mathrm{Sp}(\widetilde{W}) \to \mathrm{U}(1)$ such that for all $\widetilde{\sigma}$ and $\widetilde{\tau}$ in $\mathrm{Sp}(\widetilde{W})$,

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau}) = \nu(\widetilde{\sigma})\nu(\widetilde{\tau})\nu(\widetilde{\sigma}\widetilde{\tau})^{-1}. \tag{233}$$

Then if $\widetilde{\sigma}$ and $\widetilde{\tau}$ commute, we have that $\widetilde{c}_{\chi_k}(\widetilde{\tau}, \widetilde{\sigma}) = \widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau})$. So to prove that $\widetilde{c}_{\chi_k}$ is not a coboundary, it is enough to find two elements $\widetilde{\sigma}$ and $\widetilde{\tau}$ in $\mathrm{Sp}(\widetilde{W})$ which commute and such that the cocycle does not satisfy this equality. We use the notation of chapter 4, in particular a symplectic basis $\{\widetilde{e}_1, \dots, \widetilde{e}_n, \widetilde{f}_1, \dots, \widetilde{f}_n\}$ of $\widetilde{W}$ and the decomposition $\widetilde{W} = \widetilde{W}_1 \oplus \cdots \oplus \widetilde{W}_n$ of $\widetilde{W}$ in submodules of rank 2 such that $\widetilde{W}_i = \mathbb{Z}/4\mathbb{Z}\widetilde{e}_i \oplus \mathbb{Z}/4\mathbb{Z}\widetilde{f}_i$, and $\widetilde{W}_i$ and $\widetilde{W}_j$ are orthogonal with respect to the symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ for all $1 \leqslant i \neq j \leqslant n$. Let $\widetilde{\sigma} = (\widetilde{\sigma}_1, 1, \dots, 1) \in \mathrm{Sp}(\widetilde{W}_1) \times \cdots \times \mathrm{Sp}(\widetilde{W}_n)$, where $\widetilde{\sigma}_1 \in \mathrm{Sp}(\widetilde{W}_1)$ is defined by $\widetilde{\sigma}_1(\widetilde{x}_1\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1) = \widetilde{x}_1\widetilde{e}_1 + (2\widetilde{x}_1 + \widetilde{y}_1)\widetilde{f}_1$ for all $\widetilde{w}_1 = \widetilde{x}_1\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1 \in \widetilde{W}_1$. By (420), for all $\widetilde{w} = \widetilde{w}_1 + \cdots + \widetilde{w}_n = \sum_{1 \leqslant i \leqslant n} (\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i) \in \widetilde{W}$,

$$\widetilde{\sigma}(\widetilde{w}_1 + \cdots + \widetilde{w}_n) = \widetilde{\sigma}_1(\widetilde{w}_1) + \widetilde{w}_2 + \cdots + \widetilde{w}_n = \widetilde{x}_1\widetilde{e}_1 + (2\widetilde{x}_1 + \widetilde{y}_1)\widetilde{f}_1 + \sum_{2 \leqslant i \leqslant n} (\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i). \tag{234}$$

Then

$$\begin{cases} \mathrm{K}(\widetilde{\sigma}) & = \mathrm{K}(\widetilde{\sigma}_1) \oplus \widetilde{W}_2 \oplus \cdots \oplus \widetilde{W}_n = \{(\overbrace{(2\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)}^{\in \widetilde{X}}, \overbrace{(\widetilde{y}_1, \widetilde{y}_2, \dots, \widetilde{y}_n)}^{\in \widetilde{Y}}),\ \widetilde{x}_i, \widetilde{y}_i \in \mathbb{Z}/4\mathbb{Z}\} \\ & \qquad = 2\mathbb{Z}/4\mathbb{Z}\widetilde{e}_1 \oplus \mathbb{Z}/4\mathbb{Z}\widetilde{f}_1 \oplus \bigoplus_{2 \leqslant i \leqslant n} (\mathbb{Z}/4\mathbb{Z}\widetilde{e}_i \oplus \mathbb{Z}/4\mathbb{Z}\widetilde{f}_i) \\ \mathrm{I}(\widetilde{\sigma}) & = \mathrm{I}(\widetilde{\sigma}_1) \oplus \{0\} \oplus \cdots \oplus \{0\} = \{((0, 0, \dots, 0), (2\widetilde{y}_1, 0, \dots, 0)),\ \widetilde{y}_1 \in \mathbb{Z}/4\mathbb{Z}\} = 2\mathbb{Z}/4\mathbb{Z}\widetilde{f}_1 \\ \mathrm{K}'(\widetilde{\sigma}) & = \mathrm{K}'(\widetilde{\sigma}_1) \oplus \widetilde{W}_2 \oplus \cdots \oplus \widetilde{W}_n = \widetilde{W}. \end{cases}$$

Hence a system of representatives of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$ is for example $S_{\widetilde{\sigma}} = \{0\}$. Moreover, let $\widetilde{w}_{\widetilde{\sigma}} = \sum_{1 \leqslant i \leqslant n} (\widetilde{\alpha}_i\widetilde{e}_i + \widetilde{\beta}_i\widetilde{f}_i) \in \widetilde{W}_1 \oplus \cdots \oplus \widetilde{W}_n$, where $\widetilde{\alpha}_i,\ \widetilde{\beta}_i \in \mathbb{Z}/4\mathbb{Z}$. Since $\widetilde{W}_i$ and $\widetilde{W}_j$ are orthogonal with respect to the symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ for $i \neq j$, then $\widetilde{w}_{\widetilde{\sigma}}$ satisfies (207) if and only if for all $\widetilde{w} = \sum_{1 \leqslant i \leqslant n} (\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i)$ in $\mathrm{K}'(\widetilde{\sigma}) = \widetilde{W}$,

$$\begin{aligned} & 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}} = \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w} \rangle_{\widetilde{W}} \\ & \iff 2 \sum_{1 \leqslant i \leqslant n} \langle \widetilde{\alpha}_i\widetilde{e}_i + \widetilde{\beta}_i\widetilde{f}_i, \widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i \rangle_{\widetilde{W}_i} = \langle \underbrace{\widetilde{\sigma}_1(\widetilde{x}_1\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1)}_{\widetilde{x}_1\widetilde{e}_1 + (2\widetilde{x}_1 + \widetilde{y}_1)\widetilde{f}_1}, \widetilde{x}_1\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1 \rangle_{\widetilde{W}_1} \\ & \iff \sum_{1 \leqslant i \leqslant n} 2(\widetilde{\alpha}_i\widetilde{y}_i - \widetilde{\beta}_i\widetilde{x}_i) = 2\widetilde{x}_1^2. \end{aligned}$$

This is equivalent to have $2\widetilde{\alpha}_1 = 0$, $2\widetilde{\beta}_1 = 2$ and $2\widetilde{\alpha}_i = 2\widetilde{\beta}_i = 0$ for all $i \geqslant 2$. Hence $\widetilde{w}_{\widetilde{\sigma}}$ satisfies (207) if and only if we can write it under the following form:

$$\widetilde{w}_{\widetilde{\sigma}} = 2\widetilde{\alpha}_1\widetilde{e}_1 + \widetilde{\beta}_1\widetilde{f}_1 + \sum_{2\leqslant i\leqslant n}(2\widetilde{\alpha}_i\widetilde{e}_i + 2\widetilde{\beta}_i\widetilde{f}_i), \quad \text{where } \begin{cases} \widetilde{\beta}_1 \in (\mathbb{Z}/4\mathbb{Z})\backslash(2\mathbb{Z}/4\mathbb{Z}) \\ \widetilde{\alpha}_1, \widetilde{\alpha}_i, \widetilde{\beta}_i \in \mathbb{Z}/4\mathbb{Z} \quad \text{for } i > 1, \end{cases} \tag{235}$$

Now we define $\widetilde{\tau} = (\widetilde{\tau}_1, 1, \dots, 1) \in \mathrm{Sp}(\widetilde{W}_1) \times \cdots \times \mathrm{Sp}(\widetilde{W}_n)$, where $\widetilde{\tau}_1 \in \mathrm{Sp}(\widetilde{W}_1)$ is defined by $\widetilde{\tau}_1(\widetilde{x}_1\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1) = (\widetilde{x}_1 + 2\widetilde{y}_1)\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1$ for all $\widetilde{w}_1 = \widetilde{x}_1\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1 \in \widetilde{W}_1$. By (420), for all $\widetilde{w} = \widetilde{w}_1 + \cdots + \widetilde{w}_n = \sum_{1\leqslant i\leqslant n}(\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i) \in \widetilde{W}$,

$$\widetilde{\tau}(\widetilde{w}_1 + \cdots + \widetilde{w}_n) = \widetilde{\tau}_1(\widetilde{w}_1) + \widetilde{w}_2 + \cdots + \widetilde{w}_n = (\widetilde{x}_1 + 2\widetilde{y}_1)\widetilde{e}_1 + \widetilde{y}_1\widetilde{f}_1 + \sum_{2\leqslant i\leqslant n}(\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i). \tag{236}$$

Then $\widetilde{\sigma}\widetilde{\tau} = (\widetilde{\sigma}_1\widetilde{\tau}_1, 1, \dots, 1) = (\widetilde{\tau}_1\widetilde{\sigma}_1, 1, \dots, 1) = \widetilde{\tau}\widetilde{\sigma}$ and we prove in the same way as for $\widetilde{\sigma}$ that two systems of representatives of $\mathrm{I}(\widetilde{\tau})/(\mathrm{I}(\widetilde{\tau}) \cap 2\widetilde{W})$ and $\mathrm{I}(\widetilde{\sigma}\widetilde{\tau})/(\mathrm{I}(\widetilde{\sigma}\widetilde{\tau}) \cap 2\widetilde{W})$ are $S_{\widetilde{\tau}} = \{0\}$ and $S_{\widetilde{\sigma}\widetilde{\tau}} = \{0\}$. We prove also in a similar way that $\widetilde{w}_{\widetilde{\tau}}$ and $\widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}$ satisfy (207) for $\widetilde{\tau}$ and $\widetilde{\sigma}\widetilde{\tau}$, respectively, if and only if they can be written under the following form:

$$\begin{aligned} \widetilde{w}_{\widetilde{\tau}} &= \widetilde{\gamma}_1\widetilde{e}_1 + 2\widetilde{\delta}_1\widetilde{f}_1 + \sum_{2\leqslant i\leqslant n}(2\widetilde{\gamma}_i\widetilde{e}_i + 2\widetilde{\delta}_i\widetilde{f}_i), \quad \text{where } \begin{cases} \widetilde{\gamma}_1 \in (\mathbb{Z}/4\mathbb{Z})\backslash(2\mathbb{Z}/4\mathbb{Z}) \\ \widetilde{\delta}_1, \widetilde{\gamma}_i, \widetilde{\delta}_i \in \mathbb{Z}/4\mathbb{Z} \quad \text{for } i > 1 \end{cases} \\ \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}} &= \widetilde{\lambda}_1\widetilde{e}_1 + \widetilde{\mu}_1\widetilde{f}_1 + \sum_{2\leqslant i\leqslant n}(2\widetilde{\lambda}_i\widetilde{e}_i + 2\widetilde{\mu}_i\widetilde{f}_i), \quad \text{where } \begin{cases} \widetilde{\lambda}_1, \widetilde{\mu}_1 \in (\mathbb{Z}/4\mathbb{Z})\backslash(2\mathbb{Z}/4\mathbb{Z}) \\ \widetilde{\lambda}_i, \widetilde{\mu}_i \in \mathbb{Z}/4\mathbb{Z} \quad \text{for } i > 1. \end{cases} \end{aligned} \tag{237}$$

Then by (229),

$$\begin{aligned} \widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau}) &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} - \langle \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma},\widetilde{\tau}}\rangle_{\widetilde{W}}) \\ &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} - \langle \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}} - \widetilde{w}_{\widetilde{\sigma}} - \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}}) \\ &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma}} + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}}). \end{aligned}$$

In the same way, since $\widetilde{\sigma}$ and $\widetilde{\tau}$ commute,

$$\begin{aligned} \widetilde{c}_{\chi_k}(\widetilde{\tau}, \widetilde{\sigma}) &= \gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}\widetilde{\sigma}, \widetilde{w}_{\widetilde{\tau}\widetilde{\sigma}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} - \langle \widetilde{w}_{\widetilde{\tau}\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau},\widetilde{\sigma}}\rangle_{\widetilde{W}}) \\ &= \gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma}} + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}}). \end{aligned}$$

This implies by (235) and (237) that

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau})\widetilde{c}_{\chi_k}(\widetilde{\tau}, \widetilde{\sigma})^{-1} = \chi_k(2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}}) = \chi_k(2\widetilde{\beta}_1\widetilde{\gamma}_1) = \chi_k(2) = -1. \tag{238}$$

Therefore, $\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau}) \neq \widetilde{c}_{\chi_k}(\widetilde{\tau}, \widetilde{\sigma})$, and this allows to conclude that c_{χ_k} is not a coboundary. □

Remark 2.60. As in Remarks 1.34 and 2.27, Corollary 2.59 shows that there are commuting elements in $\mathrm{Sp}(\widetilde{W})$ whose preimages in the metaplectic covering that we will define in Proposition 2.63 do not commute.

Corollary 2.59 shows that $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ cannot be lifted to a linear representation of $\mathrm{Sp}(\widetilde{W})$ itself. However, there is an important subgroup of $\mathrm{Sp}(\widetilde{W})$ on which the cocycle is a coboundary, hence on which the projective representation $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ can be defined as a linear representation. This does not happen for the Weil representation over the reals.

Corollary 2.61. *Let $\widetilde{\mathrm{P}}$ be the subgroup of $\mathrm{Sp}(\widetilde{W})$ generated by the elements $\widetilde{\sigma}$ which preserve $\widetilde{X}$, i.e.*

$$\widetilde{\mathrm{P}} = \left\{ \widetilde{\sigma} = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ 0 & (\widetilde{\alpha}^T)^{-1} \end{pmatrix},\ \widetilde{\alpha} \in \mathrm{GL}_n(\mathbb{Z}/4\mathbb{Z}), \widetilde{\beta} \in \mathrm{M}_n(\mathbb{Z}/4\mathbb{Z}),\ \widetilde{\beta}^T = \widetilde{\alpha}^{-1}\widetilde{\beta}\widetilde{\alpha}^T \right\}. \tag{239}$$

There is a choice of $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207)*, $\widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying* (232)*, $\widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$ such that for all $\widetilde{\sigma} \in \widetilde{\mathrm{P}}$, $\varphi \in \mathrm{L}^2(Y)$ and $z \in Y$,*

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\varphi(z) = \chi_k(\langle \widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha^T z), \tag{240}$$

where $\widetilde{z} \in \widetilde{Y}$ is any element such that $z = \widetilde{z} + 2\widetilde{Y}$, and $\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & (\alpha^T)^{-1} \end{pmatrix} = \pi(\widetilde{\sigma}) \in \mathrm{Sp}(W)$ (recall that π is defined in Proposition 2.7.iii). This implies that, with these choices, $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ is a linear representation of $\widetilde{\mathrm{P}}$. It follows in particular that for every choice of $\{\widetilde{w}_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}), \widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$, the restriction of the cocycle $\widetilde{c}_{\chi_k}$ to the subgroup $\widetilde{\mathrm{P}}$ of $\mathrm{Sp}(\widetilde{W})$ is a coboundary.

Proof. Let $\widetilde{\sigma} = \begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ 0 & (\widetilde{\alpha}^T)^{-1} \end{pmatrix}$ be an element of $\widetilde{\mathrm{P}}$. The condition $\widetilde{\beta}^T = \widetilde{\alpha}^{-1}\widetilde{\beta}\widetilde{\alpha}^T$ ensures that the map $Y \to \mathbb{Z}/4\mathbb{Z}$, $z \mapsto \langle \widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}}$, where $\widetilde{z} \in \widetilde{Y}$ is any element such that $z = \widetilde{z} + 2\widetilde{Y}$, is well-defined.
Then we can define the operator $\Omega_{\chi_k}(\widetilde{\sigma})$ by the following formula: for all $\varphi \in \mathrm{L}^2(Y)$, $z \in Y$ and $\widetilde{z} \in \widetilde{Y}$ such that $z = \widetilde{z} + 2\widetilde{Y}$,

$$\Omega_{\chi_k}(\widetilde{\sigma})\varphi(z) = \chi_k(\langle \widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha^T z), \tag{241}$$

where $\sigma = \begin{pmatrix} \alpha & \beta \\ 0 & (\alpha^T)^{-1} \end{pmatrix} = \pi(\widetilde{\sigma}) \in \mathrm{Sp}(W)$. Then by (26) and (27),

$$\begin{aligned}
\|\Omega_{\chi_k}(\widetilde{\sigma})\varphi\|_2^2 = \int_Y |\chi_k(\langle \widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha^T z)|^2\, dz &= \int_Y |\varphi(\alpha^T z)|^2\, dz \\
&= \frac{1}{\sqrt{|Y|}} \sum_{z \in Y} |\varphi(\alpha^T z)|^2 \\
&= \frac{1}{\sqrt{|Y|}} \sum_{y \in Y} |\varphi(y)|^2 \qquad \text{if we set } y = \alpha^T z \\
&= \int_Y |\varphi(y)|^2\, dy \\
&= \|\varphi\|_2^2.
\end{aligned}$$

This proves that $\Omega_{\chi_k}(\widetilde{\sigma})$ is a unitary operator of $\mathrm{L}^2(Y)$. Moreover, we compute easily that

$$\Omega_{\chi_k}(\widetilde{\sigma})^{-1}\varphi(z) = \chi_k(-\langle\widetilde{\beta}\widetilde{z}, (\widetilde{\alpha}^T)^{-1}\widetilde{z}\rangle_{\widetilde{W}})\varphi((\alpha^T)^{-1}z).$$

Hence for all $(\widetilde{w}, t) = ((\widetilde{x}, \widetilde{y}), t) \in \mathrm{H}(\widetilde{W})$, by (201),

$$\begin{aligned}
&(\Omega_{\chi_k}(\widetilde{\sigma}) \circ \rho_{\chi_k}(\widetilde{w}, t) \circ \Omega_{\chi_k}(\widetilde{\sigma})^{-1})\varphi \\
&= (\Omega_{\chi_k}(\widetilde{\sigma}) \circ \rho_{\chi_k}(\widetilde{w}, t))(z \mapsto \chi_k(-\langle\widetilde{\beta}\widetilde{z}, (\widetilde{\alpha}^T)^{-1}\widetilde{z}\rangle_{\widetilde{W}})\varphi((\alpha^T)^{-1}z)) \\
&= \Omega_{\chi_k}(\widetilde{\sigma})(z \mapsto \chi_k(t + 2\langle\widetilde{x}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})\chi_k(-\langle\widetilde{\beta}(\widetilde{z} - \widetilde{y}), (\widetilde{\alpha}^T)^{-1}(\widetilde{z} - \widetilde{y})\rangle_{\widetilde{W}})\varphi((\alpha^T)^{-1}(z - y))) \\
&= (z \mapsto \chi_k(\langle\widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\chi_k(t + 2\langle\widetilde{x}, \widetilde{\alpha}^T\widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})\chi_k(-\langle\widetilde{\beta}\widetilde{\alpha}^T\widetilde{z} - \widetilde{\beta}\widetilde{y}, \widetilde{z} - (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}}) \\
&\quad \varphi(z - (\alpha^T)^{-1}y)) \\
&= (z \mapsto \chi_k(t + 2\langle\widetilde{\alpha}\widetilde{x}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}} + \underbrace{\langle\widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}, (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}}}_{=\widetilde{z}^T\widetilde{\alpha}\widetilde{\beta}^T(\widetilde{\alpha}^T)^{-1}\widetilde{y}=\widetilde{y}^T\widetilde{\alpha}^{-1}\widetilde{\beta}\widetilde{\alpha}^T\widetilde{z}=\widetilde{y}^T\widetilde{\beta}^T\widetilde{z}} + \langle\widetilde{\beta}\widetilde{y}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{\beta}\widetilde{y}, (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}}) \\
&\quad \varphi(z - (\alpha^T)^{-1}y)) \\
&= (z \mapsto \chi_k(t + 2\langle\widetilde{\alpha}\widetilde{x}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}} + 2\langle\widetilde{\beta}\widetilde{y}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{\beta}\widetilde{y}, (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}})\varphi(z - (\alpha^T)^{-1}y)) \\
&= (z \mapsto \chi_k(t + 2\langle\widetilde{\alpha}\widetilde{x} + \widetilde{\beta}\widetilde{y}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}} - \langle\widetilde{\beta}\widetilde{y}, (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}})\varphi(z - (\alpha^T)^{-1}y)).
\end{aligned}$$

On the other hand, again by (201),

$$\begin{aligned}
\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}, t))\varphi(z) &= \chi_k(t + 2\langle\widetilde{\alpha}\widetilde{x} + \widetilde{\beta}\widetilde{y}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{\alpha}\widetilde{x} + \widetilde{\beta}\widetilde{y}, (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}})\varphi(z - (\alpha^T)^{-1}y) \\
&= \chi_k(t + 2\langle\widetilde{\alpha}\widetilde{x} + \widetilde{\beta}\widetilde{y}, \widetilde{z}\rangle_{\widetilde{W}} - \langle\widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}} - \langle\widetilde{\beta}\widetilde{y}, (\widetilde{\alpha}^T)^{-1}\widetilde{y}\rangle_{\widetilde{W}})\varphi(z - (\alpha^T)^{-1}y).
\end{aligned}$$

This shows that

$$\Omega_{\chi_k}(\widetilde{\sigma}) \circ \rho_{\chi_k}(\widetilde{w}, t) \circ \Omega_{\chi_k}(\widetilde{\sigma})^{-1} = \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}, t))$$

for all $(\widetilde{w}, t) \in \mathrm{H}(\widetilde{W})$. Since by Schur's lemma, the unitary operator satisfying this intertwining property is unique up to multiplication by an element of $\mathrm{U}(1)$, if we make any choice of two sets $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207), $\widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$, then there exists $\nu(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}, \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \in \mathrm{U}(1)$ such that for all $\widetilde{\sigma} \in \widetilde{\mathrm{P}}$,

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \nu(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}, \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))\Omega_{\chi_k}(\widetilde{\sigma}). \tag{242}$$

Now, for all $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \widetilde{\mathrm{P}}$, and for all $\varphi \in \mathrm{L}^2(Y)$,

$$\begin{aligned}
(\Omega_{\chi_k}(\widetilde{\sigma}_1) \circ \Omega_{\chi_k}(\widetilde{\sigma}_2))\varphi &= \Omega_{\chi_k}(\widetilde{\sigma}_1)(z \mapsto \chi_k(\langle\widetilde{\beta}_2\widetilde{\alpha}_2^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha_2^T z)) \\
&= (z \mapsto \chi_k(\langle\widetilde{\beta}_1\widetilde{\alpha}_1^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}} + \langle\widetilde{\beta}_2\widetilde{\alpha}_2^T\widetilde{\alpha}_1^T\widetilde{z}, \widetilde{\alpha}_1^T\widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha_2^T\alpha_1^T z))
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2) &= \chi_k(\langle(\widetilde{\alpha}_1\widetilde{\beta}_2 + \widetilde{\beta}_1(\widetilde{\alpha}_2^T)^{-1})(\widetilde{\alpha}_1\widetilde{\alpha}_2)^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi((\alpha_1\alpha_2)^T z) \\
&= \chi_k(\langle\widetilde{\alpha}_1\widetilde{\beta}_2\widetilde{\alpha}_2^T\widetilde{\alpha}_1^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}} + \langle\widetilde{\beta}_1\widetilde{\alpha}_1^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha_2^T\alpha_1^T z) \\
&= \chi_k(\langle\widetilde{\beta}_2\widetilde{\alpha}_2^T\widetilde{\alpha}_1^T\widetilde{z}, \widetilde{\alpha}_1^T\widetilde{z}\rangle_{\widetilde{W}} + \langle\widetilde{\beta}_1\widetilde{\alpha}_1^T\widetilde{z}, \widetilde{z}\rangle_{\widetilde{W}})\varphi(\alpha_2^T\alpha_1^T z).
\end{aligned}$$

Thus $\Omega_{\chi_k}(\widetilde{\sigma}_1) \circ \Omega_{\chi_k}(\widetilde{\sigma}_2) = \Omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2)$ for all $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \widetilde{\mathrm{P}}$. This implies by (242) that

$$\begin{aligned}
&\widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2)\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) \\
&= \omega_{\chi_k}(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\omega_{\chi_k}(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}) \\
&= \nu(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}, \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}))\nu(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}, \gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}))\Omega_{\chi_k}(\widetilde{\sigma}_1)\Omega_{\chi_k}(\widetilde{\sigma}_2) \\
&= \nu(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}, \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}))\nu(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}, \gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}))\Omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2) \\
&= \nu(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}, \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}))\nu(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}, \gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}))\nu(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, \gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}))^{-1}\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}),
\end{aligned}$$

i.e.

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2) = \nu(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}, \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}))\nu(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}, \gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}))\nu(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, \gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}))^{-1}. \tag{243}$$

This shows that the restriction of $\widetilde{c}_{\chi_k}$ to the subgroup $\widetilde{\mathrm{P}}$ of $\mathrm{H}(\widetilde{W})$ is a coboundary. Moreover, if we redefine the operators $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ with the choices $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207), $\widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$ and $\{\nu(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}, \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))^{-1}\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}), \widetilde{\sigma} \in \widetilde{\mathrm{P}}\}$, then (242) shows that

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \Omega_{\chi_k}(\widetilde{\sigma}),$$

i.e. (240) is satisfied and the map $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ is a linear representation of $\widetilde{\mathrm{P}}$. □

Corollary 2.62 (Range of the cocycle $\widetilde{c}_{\chi_k}$). *The cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mathrm{U}(1)$. Moreover, the following properties hold.*

i. *Suppose that for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, we have made a choice of $\widetilde{w}_{\widetilde{\sigma}}$ satisfying* (207) *and of $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying* (232)*. Then $\widetilde{c}_{\chi_k}$ takes its values in $\mu_4(\mathbb{C})$ if and only if $\theta(1, \widetilde{w}_1) \equiv 0 \mod \frac{\pi}{2}$ and*

$$\begin{aligned}
&\theta(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}) \\
&\equiv \begin{cases} 0 \mod \frac{\pi}{2} & \text{if } 2\widetilde{v} \notin 2\,\mathrm{I}(\widetilde{\tau}) \text{ or } 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{z}) \text{ with } \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z} \\ \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{z}) \text{ with } \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z} \end{cases}
\end{aligned} \tag{244}$$

for every $\widetilde{\tau} \in \mathrm{Sp}(\widetilde{W})$ and every transvection $\widetilde{\sigma}_{\widetilde{v}} \in \mathrm{Sp}(\widetilde{W})$ such that $\widetilde{v} \notin 2\widetilde{W}$. For example, this is the case if $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{-\frac{i\pi \dim \mathrm{I}(\sigma)}{4}} = (1+i)^{-\dim \mathrm{I}(\sigma)}$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, where $\sigma = \pi(\widetilde{\sigma})$ (recall that π is defined in Proposition 2.7.iii).

ii. *The smallest possible range of the cocycle $\widetilde{c}_{\chi_k}$ is $\mu_2(\mathbb{C})$.*

Proof. **i.** Equation (230), together with the unitarity of the operators $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, implies that $\widetilde{c}_{\chi_k}$ takes its values in $\mathrm{U}(1)$. The group $\mathrm{Sp}(\widetilde{W})$ is generated by the transvections $\widetilde{\sigma}_{\widetilde{v}} : \widetilde{w} \mapsto \widetilde{w} + \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v}$, where $\widetilde{v} \in \widetilde{W}$ (cf. [Ish89]). Let $\widetilde{\sigma}$ and $\widetilde{\tau}$ be in $\mathrm{Sp}(\widetilde{W})$. Then there exist $\widetilde{v}_1, \ldots, \widetilde{v}_p \in \widetilde{W}$ such that $\widetilde{\sigma} = \widetilde{\sigma}_{\widetilde{v}_1} \cdots \widetilde{\sigma}_{\widetilde{v}_p}$. Since $\widetilde{c}_{\chi_k}$ is a cocycle, we can write

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}_1}, \widetilde{\sigma}_{\widetilde{v}_2} \cdots \widetilde{\sigma}_{\widetilde{v}_p})\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau}) = \widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}_1}, \widetilde{\sigma}_{\widetilde{v}_2} \cdots \widetilde{\sigma}_{\widetilde{v}_p}\widetilde{\tau})\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}_2} \cdots \widetilde{\sigma}_{\widetilde{v}_p}, \widetilde{\tau}). \tag{245}$$

So, by induction with (245), showing that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_4(\mathbb{C})$ is equivalent to show that $\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau})$ belongs to $\mu_4(\mathbb{C})$ for all $\widetilde{v} \in \widetilde{W}$ and $\widetilde{\tau} \in \mathrm{Sp}(\widetilde{W})$.

Let $\widetilde{v} \in \widetilde{W}$ and $\widetilde{\tau} \in \mathrm{Sp}(\widetilde{W})$. If $\widetilde{v} \in 2\widetilde{W}$ then $\widetilde{\sigma}_{\widetilde{v}} = 1$. Hence $\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) = \{0\}$, $\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \in 2\widetilde{W}$ and $\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} = \widetilde{w}_{\widetilde{\tau}} - \widetilde{w}_1 - \widetilde{w}_{\widetilde{\tau}} = -\widetilde{w}_1$. Then by (229), if we set $S_{\widetilde{\sigma}_{\widetilde{v}}} = \{0\}$,

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau}) = C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \sum_{\substack{\widetilde{u}_2 \in S_{\widetilde{\tau}} \\ \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\widetilde{W}}} \chi_k(\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_2 \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}, \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \rangle_{\widetilde{W}}),$$

where $C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} = \gamma(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}})^{-1} = \gamma(1, \widetilde{w}_1)$ and $(1 - \widetilde{\tau})(\widetilde{w}_2) = \widetilde{u}_2$. Hence there is only one term in this sum, which is a fourth root of unity because χ_k takes its values in $\mu_4(\mathbb{C})$. This implies that $\widetilde{c}_{\chi_k}(1, \widetilde{\tau})$ belongs to $\mu_4(\mathbb{C})$ if and only if $\gamma(1, \widetilde{w}_1) \in \mu_4(\mathbb{C})$, i.e. $\theta(1) \equiv 0 \mod \frac{\pi}{2}$.

So from now on, we can suppose that $\widetilde{v} \notin 2\widetilde{W}$. Then $\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) = \mathbb{Z}/4\mathbb{Z}\widetilde{v} = \{0, \widetilde{v}, 2\widetilde{v}, 3\widetilde{v} = -\widetilde{v}\}$ and $\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \cap 2\widetilde{W} = \{0, 2\widetilde{v}\} = 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}})$. Hence a system of representatives of $\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}})/\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \cap 2\widetilde{W}$ is for example given by $S_{\widetilde{\sigma}_{\widetilde{v}}} = \{0, \widetilde{v}\}$. Moreover, $\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}$ satisfies (207) if and only if, for every $\widetilde{w}$ such that $2\widetilde{w} \in \mathrm{K}(\widetilde{\sigma}_{\widetilde{v}})$,

$$\langle 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{w} \rangle_{\widetilde{W}} = \langle \widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}), \widetilde{w} \rangle_{\widetilde{W}} = \langle \widetilde{w} + \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v}, \widetilde{w} \rangle_{\widetilde{W}} = -\langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}}^2 = 0,$$

since $2\widetilde{w} \in \mathrm{K}(\widetilde{\sigma}_{\widetilde{v}})$ implies that $\langle 2\widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} = 0$, thus $\langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}$ and $\langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}}^2 = 0$. By Lemma 2.54, this shows that $\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}$ satisfies (207) if and only if $2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}$ is in $\mathrm{K}'(\widetilde{\sigma}_{\widetilde{v}})^{\perp} = 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) = \{0, 2\widetilde{v}\}$. Suppose that a choice has been made for $\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}$ and $\widetilde{w}_{\widetilde{\tau}}$. Then, by (229),

$$\begin{aligned}\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau}) = C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \sum_{\substack{\widetilde{u}_1 \in S_{\widetilde{\sigma}_{\widetilde{v}}} \\ \widetilde{u}_2 \in S_{\widetilde{\tau}} \\ \widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\widetilde{W}}} & \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{w}_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_2 \rangle_{\widetilde{W}} \\ & + \langle \widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}, \widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \rangle_{\widetilde{W}}),\end{aligned} \tag{246}$$

where $C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} = \gamma(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}})^{-1}$, $(1 - \widetilde{\sigma}_{\widetilde{v}})(\widetilde{w}_1) = \widetilde{u}_1$ and $(1 - \widetilde{\tau})(\widetilde{w}_2) = \widetilde{u}_2$.

- We suppose first that $2\widetilde{v} \notin 2\,\mathrm{I}(\widetilde{\tau})$. Then, by Lemma 2.54, we have that $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}) = 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \oplus 2\,\mathrm{I}(\widetilde{\tau})$. For $\widetilde{u}_1 \in S_{\widetilde{\sigma}_{\widetilde{v}}}$ and $\widetilde{u}_2 \in S_{\widetilde{\tau}}$, the condition $\widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\widetilde{W}$ is equivalent to $2\widetilde{u}_1 + 2\widetilde{u}_2 - 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} = 0$. Hence the existence of two elements $\widetilde{u}_1$ and $\widetilde{u}_2$ such that $\widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\widetilde{W}$ (which is guaranteed by the fact that the cocycle is nonzero) is equivalent to have $2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) + 2\,\mathrm{I}(\widetilde{\tau})$. Since this sum is direct, there is only one way to decompose $2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}$ in $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \oplus 2\,\mathrm{I}(\widetilde{\tau})$. Thus there exist unique elements $\widetilde{u}_1$ and $\widetilde{u}_2$ such that $\widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\widetilde{W}$. Then

$$\begin{aligned}\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau}) &= C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{w}_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_2 \rangle_{\widetilde{W}} \\ &\qquad + \langle \widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}, \widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \rangle_{\widetilde{W}}) \\ &= \pm C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + \langle \widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}}).\end{aligned} \tag{247}$$

Since $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}) = 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) \oplus 2\,\mathrm{I}(\widetilde{\tau})$, by (232),

$$\begin{aligned} C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} &= e^{i(\theta(\widetilde{\sigma}_{\widetilde{v}},\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}})+\theta(\widetilde{\tau},\widetilde{w}_{\widetilde{\tau}})-\theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau},\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}))}|2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}})|^{-\frac{1}{2}}|2\,\mathrm{I}(\widetilde{\tau})|^{-\frac{1}{2}}|2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau})|^{\frac{1}{2}} \\ &= e^{i(\theta(\widetilde{\sigma}_{\widetilde{v}},\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}})+\theta(\widetilde{\tau},\widetilde{w}_{\widetilde{\tau}})-\theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau},\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}))}. \end{aligned} \tag{248}$$

Since χ_k takes its values in $\mu_4(\mathbb{C})$, this shows that

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}) \in \mu_4(\mathbb{C}) \quad \Longleftrightarrow \quad \theta(\widetilde{\sigma}_{\widetilde{v}},\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau},\widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau},\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}) \equiv 0 \mod \frac{\pi}{2}. \tag{249}$$

- Suppose now that $2\widetilde{v} \in 2\,\mathrm{I}(\widetilde{\tau})$, i.e. that there exists $\widetilde{z} \in \widetilde{W}$ such that $(1-\widetilde{\tau})(2\widetilde{z}) = 2\widetilde{v}$. Then, by Lemma 2.54, we have that $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}) \subseteq 2\,\mathrm{I}(\widetilde{\tau})$. Since $2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \in 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) + 2\,\mathrm{I}(\widetilde{\tau}) = 2\,\mathrm{I}(\widetilde{\tau})$ and $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}) = \{0, 2\widetilde{v}\}$, there are exactly two ways to decompose $2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}$ as $2\widetilde{u}_1 + 2\widetilde{u}_2$, where $\widetilde{u}_1 \in 2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}})$ and $\widetilde{u}_2 \in 2\,\mathrm{I}(\widetilde{\tau})$:

$$\begin{cases} 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} = 2\widetilde{u}_1 + 2\widetilde{u}_2 & \text{where } 2\widetilde{u}_1 = 0 \text{ and } 2\widetilde{u}_2 = 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \\ 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} = 2\widetilde{u}_1 + 2\widetilde{u}_2 & \text{where } 2\widetilde{u}_1 = 2\widetilde{v} \text{ and } 2\widetilde{u}_2 = 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} - 2\widetilde{v}. \end{cases} \tag{250}$$

For each of these decompositions, there are unique elements $\widetilde{u}_1$ in $S_{\widetilde{\sigma}_{\widetilde{v}}}$ and $\widetilde{u}_2$ in $S_{\widetilde{\tau}}$ satisfying these equalities. Hence by (246),

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}) = C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}(\chi_k(A) + \chi_k(B)), \tag{251}$$

where

$$\begin{cases} A &= \langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{w}_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_2 \rangle_{\widetilde{W}} \\ & \qquad + \langle \widetilde{u}_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}, \widetilde{u}_1 + \widetilde{u}_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \rangle_{\widetilde{W}} \\ B &= \langle \widetilde{u}'_1, \widetilde{w}'_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{w}'_1 \rangle_{\widetilde{W}} + \langle \widetilde{u}'_2, \widetilde{w}'_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}'_2 \rangle_{\widetilde{W}} \\ & \qquad + \langle \widetilde{u}'_1 + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}'_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}, \widetilde{u}'_1 + \widetilde{u}'_2 - \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \rangle_{\widetilde{W}}, \end{cases}$$

with $2\widetilde{u}_1 = 0$, $2\widetilde{u}'_1 = 2\widetilde{v}$, $2\widetilde{u}_2 = 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}$, $2\widetilde{u}'_2 = 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} - 2\widetilde{v}$, $(1-\widetilde{\sigma}_{\widetilde{v}})(\widetilde{w}_1) = \widetilde{u}_1$, $(1-\widetilde{\sigma}_{\widetilde{v}})(\widetilde{w}'_1) = \widetilde{u}'_1$, $(1-\widetilde{\tau})(\widetilde{w}_2) = \widetilde{u}_2$ and $(1-\widetilde{\tau})(\widetilde{w}'_2) = \widetilde{u}'_2$. Then

$$\begin{cases} 2\widetilde{u}'_2 = 2\widetilde{u}_2 - 2\widetilde{v} \\ (1-\widetilde{\tau})(2\widetilde{w}'_2) = 2\widetilde{u}'_2 = 2\widetilde{u}_2 - 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{w}_2 - 2\widetilde{z}) \quad \text{i.e. } 2\widetilde{w}'_2 - 2\widetilde{w}_2 + 2\widetilde{z} \in \mathrm{K}(\widetilde{\tau}). \end{cases} \tag{252}$$

This implies that

$$2A = 2\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}},$$

and since $\mathrm{K}(\widetilde{\tau})$ and $\mathrm{I}(\widetilde{\tau})$ are orthogonal with respect to the symplectic form,

$$\begin{aligned}
2B &= 2\langle \widetilde{u}_1', \widetilde{w}_1' + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}\rangle_{\widetilde{W}} + 2\langle \widetilde{u}_2', \widetilde{w}_2' + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + 2\langle \widetilde{u}_1' + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2' + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} \\
&= 2\langle \widetilde{v}, \widetilde{w}_1' + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}\rangle_{\widetilde{W}} + 2\langle \widetilde{u}_2 - \widetilde{v}, \widetilde{w}_2' + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + 2\langle \widetilde{v} + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 - \widetilde{v} + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} \\
&= 2\langle \widetilde{v}, \widetilde{w}_1'\rangle_{\widetilde{W}} + \underbrace{2\langle \widetilde{v}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}\rangle_{\widetilde{W}}}_{=0 \text{ since } 2\widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}} \in \{0, 2\widetilde{v}\}} + 2\langle \widetilde{u}_2 - \widetilde{v}, \widetilde{w}_2 - \widetilde{z} + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} \\
&\quad + 2\langle \widetilde{v} + \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 - \widetilde{v} + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} \\
&= 2\langle \widetilde{v}, \widetilde{w}_1'\rangle_{\widetilde{W}} + 2\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} - 2\langle \widetilde{u}_2, \widetilde{z}\rangle_{\widetilde{W}} - \langle 2\widetilde{v}, \widetilde{w}_2 - \widetilde{z}\rangle_{\widetilde{W}} + 2\langle \widetilde{v}, \widetilde{u}_2\rangle_{\widetilde{W}} \\
&\quad + 2\langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} \\
&= 2A + \underbrace{2\langle \widetilde{v}, \widetilde{w}_1'\rangle_{\widetilde{W}}}_{=2 \text{ since } (1-\widetilde{\sigma}_{\widetilde{v}})(2\widetilde{w}_1') = -2\langle \widetilde{w}_1', \widetilde{v}\rangle_{\widetilde{W}}\widetilde{v} = 2\widetilde{v} \neq 0} + 2\langle \widetilde{v} + \widetilde{z}, \widetilde{u}_2\rangle_{\widetilde{W}} - \langle 2\widetilde{v}, \widetilde{w}_2 - \widetilde{z}\rangle_{\widetilde{W}} \\
&= 2A + 2 - 2\langle \widetilde{\tau}(\widetilde{z}), (1-\widetilde{\tau})(\widetilde{w}_2)\rangle_{\widetilde{W}} - 2\langle (1-\widetilde{\tau})(\widetilde{z}), \widetilde{w}_2\rangle_{\widetilde{W}} + 2\langle \widetilde{v}, \widetilde{z}\rangle_{\widetilde{W}} \\
&= 2A + 2 + 2\langle \widetilde{v}, \widetilde{z}\rangle_{\widetilde{W}}.
\end{aligned}$$

Thus by (251),

$$\begin{aligned}
\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau}) = C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} \big(&\pm\chi_k(\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}}) \\
&\pm\chi_k(\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + 1 + \langle \widetilde{v}, \widetilde{z}\rangle_{\widetilde{W}})\big) \\
= C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}\chi_k(\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}})&(\pm 1 \pm \chi_k(1 + \langle \widetilde{v}, \widetilde{z}\rangle_{\widetilde{W}}))
\end{aligned}$$

Since the cocycle does not vanish, we obtain

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau}) = \begin{cases} C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}\chi_k(\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}})(\pm 1 \pm i) & \text{if } \langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z} \\ \pm C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}}2\chi_k(\langle \widetilde{u}_2, \widetilde{w}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}, \widetilde{u}_2 + \widetilde{w}_{\widetilde{\tau}}\rangle_{\widetilde{W}}) & \text{if } \langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}. \end{cases} \tag{253}$$

Now, since $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}) = 2\,\mathrm{I}(\widetilde{\tau})$ if $\langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}$ and $2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}) = 2\,\mathrm{I}(\widetilde{\tau}) \cap (\mathbb{Z}/4\mathbb{Z}\widetilde{z})^{\perp}$ if $\langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}$ by Lemma 2.54,

$$\begin{aligned}
C_{\widetilde{\sigma}_{\widetilde{v}},\widetilde{\tau}} &= e^{i(\theta(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}))} |2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}})|^{-\frac{1}{2}} |2\,\mathrm{I}(\widetilde{\tau})|^{-\frac{1}{2}} |2\,\mathrm{I}(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau})|^{\frac{1}{2}} \\
&= \begin{cases} \frac{1}{\sqrt{2}} e^{i(\theta(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}))} & \text{if } \langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z} \\ \frac{1}{2} e^{i(\theta(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}))} & \text{if } \langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}. \end{cases}
\end{aligned} \tag{254}$$

Finally by (253) and (254), $\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{\tau})$ belongs to $\mu_4(\mathbb{C})$ if and only if

$$\theta(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}) \equiv \begin{cases} \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } \langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z} \\ 0 \mod \frac{\pi}{2} & \text{if } \langle \widetilde{z}, \widetilde{v}\rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}. \end{cases} \tag{255}$$

Combining (249) and (255), we obtain (244).

Now, suppose that $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{-\frac{i\pi \dim \mathrm{I}(\sigma)}{4}} = (1+i)^{-\dim \mathrm{I}(\sigma)}$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, where $\sigma = \pi(\widetilde{\sigma})$. First this choice is valid because $|\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 = 2^{-\dim \mathrm{I}(\sigma)} = |\,\mathrm{I}(\sigma)|^{-1}$,

$|\,\mathrm{I}(\sigma)| = |\,\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})|$ by definition of π and $|\,\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})| = |2\,\mathrm{I}(\widetilde{\sigma})|$ by Lemma 2.54. Moreover, $\gamma(1, \widetilde{w}_1) = (1+i)^{-\dim \mathrm{I}(1)} = 1$, which implies in particular that $\theta(1, \widetilde{w}_1) \equiv 0 \mod \frac{\pi}{2}$. Let $\widetilde{v} \in \widetilde{W} \backslash 2\widetilde{W}$ and $\widetilde{\tau} \in \mathrm{Sp}(\widetilde{W})$. Then we have that $\pi(\widetilde{\sigma}_{\widetilde{v}}) = \sigma_v$ by Proposition 2.7.iv, and we set $\tau = \pi(\widetilde{\tau})$. Observe that

$$\begin{aligned} 2\widetilde{v} \in 2\,\mathrm{I}(\widetilde{\tau}) &\iff \exists \widetilde{z} \in \widetilde{W},\ 2(1-\widetilde{\tau})(\widetilde{z}) = 2\widetilde{v} \\ &\iff \exists \widetilde{z} \in \widetilde{W},\ (1-\widetilde{\tau})(\widetilde{z}) - \widetilde{v} \in 2\widetilde{W} \\ &\iff \exists \widetilde{z} \in \widetilde{W},\ (1-\widetilde{\tau})(\widetilde{z}) + 2\widetilde{W} = \widetilde{v} + 2\widetilde{W} \\ &\iff \exists z = \widetilde{z} + 2\widetilde{W} \in W,\ (1-\tau)(z) = v \\ &\iff v \in \mathrm{I}(\tau), \end{aligned}$$

and in particular $\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}$ if and only if $\langle z, v \rangle = \kappa(2\langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}}) = 0$. Thus we have that

$$\begin{cases} 2\widetilde{v} \notin \mathrm{I}(\widetilde{\tau}) & \iff v \notin \mathrm{I}(\tau) \\ 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{z}) \text{ with } \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z} & \iff v = (1-\tau)(z) \text{ with } \langle z, v \rangle = 0 \\ 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{z}) \text{ with } \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z} & \iff v = (1-\tau)(z) \text{ with } \langle z, v \rangle = 1. \end{cases} \tag{256}$$

Then by Lemma 2.20.v,

$$\begin{aligned} &\theta(\widetilde{\sigma}_{\widetilde{v}}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}}) + \theta(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) - \theta(\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}_{\widetilde{v}}\widetilde{\tau}}) \\ &= -\frac{\pi}{4}(\dim \mathrm{I}(\sigma_v) + \dim \mathrm{I}(\tau) - \dim \mathrm{I}(\sigma_v\tau)) \\ &= \begin{cases} 0 & \text{if } 2\widetilde{v} \notin \mathrm{I}(\widetilde{\tau}), \text{ since } \mathrm{I}(\sigma_v\tau) = \mathrm{I}(\sigma_v) \oplus \mathrm{I}(\tau) \\ -\frac{\pi}{4}\dim \mathrm{I}(\sigma_v) = -\frac{\pi}{4} & \text{if } 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{z}) \text{ with } \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \in 2\mathbb{Z}/4\mathbb{Z}, \text{ since } \mathrm{I}(\sigma_v\tau) = \mathrm{I}(\tau) \\ -\frac{\pi}{4}(\dim \mathrm{I}(\sigma_v) + 1) = -\frac{\pi}{2} & \text{if } 2\widetilde{v} = (1-\widetilde{\tau})(2\widetilde{z}) \text{ with } \langle \widetilde{z}, \widetilde{v} \rangle_{\widetilde{W}} \notin 2\mathbb{Z}/4\mathbb{Z}, \\ & \text{since } \mathrm{I}(\sigma_v\tau) = \mathrm{I}(\tau) \cap \mathrm{Span}\{z\}^{\perp}. \end{cases} \end{aligned}$$

Therefore, this choice of $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfies (244).

ii. We suppose first that the rank $2n$ of the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ is such that $n \geqslant 3$. We start by computing $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z})$. By [FP20], the Schur multiplier of $\mathrm{Sp}(\widetilde{W})$ in this case is $\mathrm{H}_2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Then the universal coefficient theorem provides the following short split exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) \to \mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) \to 0.$$

By Proposition 2.7.v, $\mathrm{Sp}(\widetilde{W})$ is perfect. Thus $\mathrm{H}_1(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}) \simeq \mathrm{Sp}(\widetilde{W})/[\mathrm{Sp}(\widetilde{W}), \mathrm{Sp}(\widetilde{W})]$ is trivial. Moreover, $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. This shows that

$$\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}. \tag{257}$$

Now, suppose that we choose $\widetilde{w}_{\widetilde{\sigma}}$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ so that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mu_4(\mathbb{C})$, i.e. (244) is satisfied. By Corollary 2.59, $\widetilde{c}_{\chi_k}$ is not a coboundary. Then by (257), $[\widetilde{c}_{\chi_k}] \in \mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z})$ has order two. This means that $\widetilde{c}^2_{\chi_k}$ is a

coboundary, i.e. there exists $\Lambda : \mathrm{Sp}(\widetilde{W}) \to \mu_4(\mathbb{C})$ such that for all $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ in $\mathrm{Sp}(\widetilde{W})$,

$$\tilde{c}_{\chi_k}(\tilde{\sigma}_1, \tilde{\sigma}_2)^2 = \Lambda(\tilde{\sigma}_1)\Lambda(\tilde{\sigma}_2)\Lambda(\tilde{\sigma}_1\tilde{\sigma}_2)^{-1}. \tag{258}$$

For every $\tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, we fix $\lambda(\tilde{\sigma})$ to be a square root of $\Lambda(\tilde{\sigma})$ in $\mathbb{C}$. Since $\Lambda(\tilde{\sigma}) \in \mu_4(\mathbb{C})$, we have that $\lambda(\tilde{\sigma}) \in \mu_8(\mathbb{C})$, and in particular $\lambda(\tilde{\sigma}) \in \mathrm{U}(1)$. Consequently, we can consider the cocycle c'_{χ_k} associated to the projective representation $\tilde{\sigma} \mapsto \lambda(\tilde{\sigma})^{-1}\omega_{\chi_k}(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ (this corresponds to replacing $\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ by $\lambda(\tilde{\sigma})^{-1}\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ in (215), which is possible since $|\lambda(\tilde{\sigma})^{-1}\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})|^2 = |\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})|^2$). Then

$$\tilde{c}'_{\chi_k}(\tilde{\sigma}_1, \tilde{\sigma}_2) = \lambda(\tilde{\sigma}_1)^{-1}\lambda(\tilde{\sigma}_2)^{-1}\lambda(\tilde{\sigma}_1\tilde{\sigma}_2)\tilde{c}_{\chi_k}(\tilde{\sigma}_1, \tilde{\sigma}_2), \tag{259}$$

i.e. $\tilde{c}'_{\chi_k}(\tilde{\sigma}_1, \tilde{\sigma}_2)^2 = 1$ by (258). This implies that $\tilde{c}'_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$. Since $\tilde{c}'_{\chi_k}$ is not a coboundary by Corollary 2.59, we conclude that the range of $\tilde{c}'_{\chi_k}$ is $\mu_2(\mathbb{C})$.

Now, suppose that $n = 2$. Let $\widetilde{W}'$ be a free symplectic $\mathbb{Z}/4\mathbb{Z}$-module of rank $2(m-2)$ and $\tilde{V} = \widetilde{W} \oplus \widetilde{W}'$. Then $\tilde{V}$ is a free symplectic module of rank $2m$, where $m \geqslant 3$, and $\widetilde{W}$ and $\widetilde{W}'$ are two submodules of $\tilde{V}$ which are orthogonal with respect to the symplectic form $\langle \cdot, \cdot \rangle_{\tilde{V}}$. This way, every element $\tilde{\sigma}$ of $\mathrm{Sp}(\widetilde{W})$ is identified with the element $(\tilde{\sigma}, 1) \in \mathrm{Sp}(\widetilde{W}) \times \mathrm{Sp}(\widetilde{W}') \subseteq \mathrm{Sp}(\tilde{V})$, where

$$(\tilde{\sigma}, 1)(\tilde{w} + \tilde{w}') = \tilde{\sigma}(\tilde{w}) + \tilde{w}' \quad \text{for all } \tilde{w} + \tilde{w}' \in \widetilde{W} \oplus \widetilde{W}' = \tilde{V}.$$

In particular, $\mathrm{I}((\tilde{\sigma}, 1)) = \mathrm{I}(\tilde{\sigma}) \oplus \{0\}$ and $\mathrm{K}((\tilde{\sigma}, 1)) = \mathrm{K}(\tilde{\sigma}) \oplus \widetilde{W}'$. Thus if $S_{\tilde{\sigma}}$ is a system of representatives of $\mathrm{I}(\tilde{\sigma})/(\mathrm{I}(\tilde{\sigma}) \cap 2\widetilde{W})$, then $S_{\tilde{\sigma}}$ is also a system of representatives of $\mathrm{I}((\tilde{\sigma}, 1))/(\mathrm{I}((\tilde{\sigma}, 1)) \cap 2\tilde{V})$. Moreover, if we denote by $\tilde{w}_{\tilde{\sigma}} \in \widetilde{W}$ and $\tilde{w}_{(\tilde{\sigma},1)} = \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma},1)} + \tilde{w}^{\widetilde{W}'}_{(\tilde{\sigma},1)} \in \widetilde{W} \oplus \widetilde{W}' = \tilde{V}$ two elements satisfying (207) for $\tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and $(\tilde{\sigma}, 1) \in \mathrm{Sp}(\tilde{V})$, respectively, then we obtain that $2\tilde{w}^{\widetilde{W}}_{(\tilde{\sigma},1)} - 2\tilde{w}_{\tilde{\sigma}} \in 2\,\mathrm{I}(\tilde{\sigma})$ and $2\tilde{w}^{\widetilde{W}'}_{(\tilde{\sigma},1)} = 0$. Then by Lemma 2.54.iv, one can choose $\tilde{w}_{\tilde{\sigma}} = \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma},1)}$. Now, for a given choice of $\{\tilde{w}_{\tilde{\tau}}$ satisfying (207), $\tilde{\tau} \in \mathrm{Sp}(\tilde{V})\}$ and $\{\gamma(\tilde{\tau}, \tilde{w}_{\tilde{\tau}})$ satisfying (232), $\tilde{\tau} \in \mathrm{Sp}(\tilde{V})\}$, we denote by $\tilde{c}^{\tilde{V}}_{\chi_k}$ the cocycle associated to the projective representation $\tilde{\tau} \mapsto \omega_{\chi_k}(\tilde{\tau}, \tilde{w}_{\tilde{\tau}})$ of $\mathrm{Sp}(\tilde{V})$. In the same way, we denote by $\tilde{c}_{\chi_k}$ the cocyle associated to the projective representation $\tilde{\sigma} \mapsto \omega_{\chi_k}(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ of $\mathrm{Sp}(\widetilde{W})$ for a choice of $\{\tilde{w}_{\tilde{\sigma}}$ satisfying (207), $\tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and $\{\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ satisfying (232), $\tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that for every $\tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, $\tilde{w}_{\tilde{\sigma}} = \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma},1)}$ and $\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}}) = \gamma((\tilde{\sigma}, 1), \tilde{w}_{(\tilde{\sigma},1)})$. Let $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ be in $\mathrm{Sp}(\widetilde{W})$. By (229), all of this enables us to write

$$\begin{aligned}
&\tilde{c}^{\tilde{V}}_{\chi_k}((\tilde{\sigma}_1, 1), (\tilde{\sigma}_2, 1)) \\
&= C_{1,2} \sum_{\substack{\tilde{u}_1 \in S_{\tilde{\sigma}_1} \\ \tilde{u}_2 \in S_{\tilde{\sigma}_2} \\ \tilde{u}_1 + \tilde{u}_2 - \tilde{w}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)} \in 2\widetilde{W}}} \chi_k(\langle \tilde{u}_1, \tilde{w}_1 + \tilde{w}_{(\tilde{\sigma}_1,1)} \rangle_{\tilde{V}} + 2\langle \tilde{w}_{(\tilde{\sigma}_1,1)}, \tilde{w}_1 \rangle_{\tilde{V}} + \langle \tilde{u}_2, \tilde{w}_2 + \tilde{w}_{(\tilde{\sigma}_2,1)} \rangle_{\tilde{V}} \\
&\qquad + 2\langle \tilde{w}_{(\tilde{\sigma}_2,1)}, \tilde{w}_2 \rangle_{\tilde{V}} + \langle \tilde{u}_1 + \tilde{w}_{(\tilde{\sigma}_1,1)}, \tilde{u}_2 + \tilde{w}_{(\tilde{\sigma}_2,1)} \rangle_{\tilde{V}} \\
&\qquad + \langle \tilde{w}_{(\tilde{\sigma}_1\tilde{\sigma}_2,1)}, \tilde{u}_1 + \tilde{u}_2 - \tilde{w}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)} \rangle_{\tilde{V}}).
\end{aligned} \tag{260}$$

where $C_{1,2} = \gamma(\tilde{\sigma}_1, \tilde{w}_{\tilde{\sigma}_1})\gamma(\tilde{\sigma}_2, \tilde{w}_{\tilde{\sigma}_2})\gamma(\tilde{\sigma}_1\tilde{\sigma}_2, \tilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2})^{-1}$, $\tilde{w}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)} = \tilde{w}_{(\tilde{\sigma}_1\tilde{\sigma}_2,1)} - \tilde{w}_{(\tilde{\sigma}_1,1)} - \tilde{w}_{(\tilde{\sigma}_2,1)}$, and for $i \in \{1,2\}$, $\tilde{w}_i \in \widetilde{W} \subseteq \tilde{V}$ is any element such that $(1-(\tilde{\sigma}_i,1))(\tilde{w}_i) = (1-\tilde{\sigma}_i)(\tilde{w}_i) = \tilde{u}_i$.

By hypothesis, we also have that $2\tilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2} = 2\tilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2} - 2\tilde{w}_{\tilde{\sigma}_1} - 2\tilde{w}_{\tilde{\sigma}_2} = 2\tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1\tilde{\sigma}_2,1)} - 2\tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1,1)} - 2\tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_2,1)} = 2\tilde{w}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)}$. Hence for $\tilde{u}_1 \in S_{\tilde{\sigma}_1}$ and $\tilde{u}_2 \in S_{\tilde{\sigma}_2}$, the conditions $\tilde{u}_1 + \tilde{u}_2 - \tilde{w}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)} \in 2\widetilde{W}$ and $\tilde{u}_1 + \tilde{u}_2 - \tilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2} \in 2\widetilde{W}$ are equivalent. Moreover, since $\tilde{u}_1$, $\tilde{w}_1$, $\tilde{u}_2$ and $\tilde{w}_2$ belong to $\widetilde{W}$,

$$
\begin{aligned}
&\langle \tilde{u}_1, \tilde{w}_1 + \tilde{w}_{(\tilde{\sigma}_1,1)}\rangle_{\tilde{V}} + 2\langle \tilde{w}_{(\tilde{\sigma}_1,1)}, \tilde{w}_1\rangle_{\tilde{V}} + \langle \tilde{u}_2, \tilde{w}_2 + \tilde{w}_{(\tilde{\sigma}_2,1)}\rangle_{\tilde{V}} + 2\langle \tilde{w}_{(\tilde{\sigma}_2,1)}, \tilde{w}_2\rangle_{\tilde{V}} \\
&\quad + \langle \tilde{u}_1 + \tilde{w}_{(\tilde{\sigma}_1,1)}, \tilde{u}_2 + \tilde{w}_{(\tilde{\sigma}_2,1)}\rangle_{\tilde{V}} + \langle \tilde{w}_{(\tilde{\sigma}_1\tilde{\sigma}_2,1)}, \tilde{u}_1 + \tilde{u}_2 - \tilde{w}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)}\rangle_{\tilde{V}} \\
&= \langle \tilde{u}_1, \tilde{w}_1 + \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1,1)}\rangle_{\widetilde{W}} + 2\langle \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1,1)}, \tilde{w}_1\rangle_{\widetilde{W}} + \langle \tilde{u}_2, \tilde{w}_2 + \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_2,1)}\rangle_{\widetilde{W}} + 2\langle \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_2,1)}, \tilde{w}_2\rangle_{\widetilde{W}} \\
&\quad + \langle \tilde{u}_1 + \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1,1)}, \tilde{u}_2 + \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_2,1)}\rangle_{\widetilde{W}} + \langle \underbrace{\tilde{w}^{\widetilde{W}'}_{(\tilde{\sigma}_1,1)}}_{\in 2\widetilde{W}'}, \underbrace{\tilde{w}^{\widetilde{W}'}_{(\tilde{\sigma}_2,1)}}_{\in 2\widetilde{W}'}\rangle_{\widetilde{W}'} \\
&\quad + \langle \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1\tilde{\sigma}_2,1)}, \tilde{u}_1 + \tilde{u}_2 - \tilde{w}^{\widetilde{W}}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)}\rangle_{\widetilde{W}} - \langle \underbrace{\tilde{w}^{\widetilde{W}'}_{(\tilde{\sigma}_1\tilde{\sigma}_2,1)}}_{\in 2\widetilde{W}'}, \underbrace{\tilde{w}^{\widetilde{W}'}_{(\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)}}_{\in 2\widetilde{W}'}\rangle_{\widetilde{W}'} \\
&= \langle \tilde{u}_1, \tilde{w}_1 + \tilde{w}_{\tilde{\sigma}_1}\rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}_1}, \tilde{w}_1\rangle_{\widetilde{W}} + \langle \tilde{u}_2, \tilde{w}_2 + \tilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}} + 2\langle \tilde{w}_{\tilde{\sigma}_2}, \tilde{w}_2\rangle_{\widetilde{W}} \\
&\quad + \langle \tilde{u}_1 + \tilde{w}_{\tilde{\sigma}_1}, \tilde{u}_2 + \tilde{w}_{\tilde{\sigma}_2}\rangle_{\widetilde{W}} + \langle \tilde{w}_{\tilde{\sigma}_1\tilde{\sigma}_2}, \tilde{u}_1 + \tilde{u}_2 - \tilde{w}_{\tilde{\sigma}_1,\tilde{\sigma}_2}\rangle_{\widetilde{W}}.
\end{aligned}
$$

By (229) and (260), this shows that $\tilde{c}^{\tilde{V}}_{\chi_k}((\tilde{\sigma}_1,1),(\tilde{\sigma}_2,1)) = \tilde{c}_{\chi_k}(\tilde{\sigma}_1,\tilde{\sigma}_2)$ for all $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \mathrm{Sp}(\widetilde{W})$. Now, suppose that the choice $\{\gamma(\tilde{\tau}, \tilde{w}_{\tilde{\tau}})$ satisfying (232), $\tilde{\tau} \in \mathrm{Sp}(\tilde{V})\}$ guarantees that the cocycle $\tilde{c}^{\tilde{V}}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$ (such choice is possible since the rank $2m$ of $\tilde{V}$ is greater than four). The equality between the two cocycles above finally allows us to conclude that the cocycle $\tilde{c}_{\chi_k}$ of $\widetilde{W}$ takes its values in $\mu_2(\mathbb{C})$.

Finally, if $n = 1$, then the existence of a choice of parameter $\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ can be proven exactly as in the case $n = 2$ above. However, the case $n = 1$ will be treated in detail in section 3.3. More precisely, Proposition 3.49 provides two explicit choices of $\{\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}}),\ \tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the associated cocycle's range is $\mu_2(\mathbb{C})$.

□

We are now ready to construct the long-awaited object from this section, that is the two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$.

Definition-Proposition 2.63 (The two-fold metaplectic covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$). *We use the notations of Lemma 2.54 and Proposition 2.56. We suppose that a choice of $\tilde{w}_{\tilde{\sigma}}$ satisfying* (207) *has been made for every $\tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$, and we fix a choice of $\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ defined by Proposition 2.56 and* (232)*. We further suppose that this choice of $\{\gamma(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}}),\ \tilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is such that the range of the cocycle $\tilde{c}_{\chi_k}$ is $\mu_2(\mathbb{C})$ (such a choice exists by Corollary 2.62, see Remark 2.64 for more details). Set*

$$\mathrm{Mp}(\widetilde{W})_{\chi_k} = \{(\tilde{\sigma}, (-1)^l \omega_{\chi_k}(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})),\ \tilde{\sigma} \in \mathrm{Sp}(\widetilde{W}),\ l \in \mathbb{Z}\}. \tag{261}$$

This group has the following properties.

i. $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ *is a subgroup of* $\mathrm{Sp}(\widetilde{W}) \times \mathrm{U}(\mathrm{L}^2(Y))$.

ii. $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ *is a two-fold covering of* $\mathrm{Sp}(\widetilde{W})$*, i.e it defines a non-split central exact sequence*

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W})_{\chi_k} \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \longrightarrow 1.$$

Moreover, $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ *is a stem extension of* $\mathrm{Sp}(\widetilde{W})$ *and* $\widetilde{c}_{\chi_k}$ *is a cocycle associated to this extension.*

iii. *If n=1, the two possibilities for the two-fold covering* $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ *are* $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ *and* $\mathrm{U}_2(\mathbb{F}_3)$.

iv. *Suppose that* $n \geqslant 3$*. Let* $\mathrm{Mp}(\widetilde{W})'_{\chi_k}$ *be the two-fold covering of* $\mathrm{Sp}(\widetilde{W})$ *defined with the choices* $\{\widetilde{w}'_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ *and* $\{\gamma'(\widetilde{\sigma}, \widetilde{w}'_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *such that the associated cocycle* $\widetilde{c}'_{\chi_k}$ *takes its values in* $\mu_2(\mathbb{C})$*. Then the extensions of* $\mathrm{Sp}(\widetilde{W})$ *defined by* $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ *and* $\mathrm{Mp}(\widetilde{W})'_{\chi_k}$ *are isomorphic. In other words, the two-fold metaplectic covering* $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ *of* $\mathrm{Sp}(\widetilde{W})$ *is unique up to isomorphism.*

Proof. **i.** This proof is similar to the first part of Proposition 1.36.

ii. Define $\widetilde{\Phi} : \mu_2(\mathbb{C}) \to \mathrm{Mp}(\widetilde{W})_{\chi_k}$ and $\widetilde{\Psi} : \mathrm{Mp}(\widetilde{W})_{\chi_k} \to \mathrm{Sp}(\widetilde{W})$. Then we

$$z \mapsto (1, z\,\mathrm{Id}_{\mathrm{L}^2(Y)}) \qquad (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \mapsto \widetilde{\sigma}$$

prove as in Proposition 1.36 that $1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W})_{\chi_k} \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \longrightarrow 1$ is a central extension, and that $\widetilde{c}_{\chi_k}$ is an associated cocycle. In particular, this extension does not split because $\widetilde{c}_{\chi_k}$ is not a coboundary by Corollary 2.59. It only remains to prove that this extension is stem, i.e. that $\widetilde{\Phi}(-1)$ belongs to $[\mathrm{Mp}(\widetilde{W})_{\chi_k}, \mathrm{Mp}(\widetilde{W})_{\chi_k}]$. Consider again the elements $\widetilde{\sigma}$ and $\widetilde{\tau}$ defined by (234) and (236), respectively. Using the arguments appearing in the proof of Corollary 2.59, in particular the fact that we can set $S_{\widetilde{\sigma}} = \{0\}$ and $S_{\widetilde{\tau}} = \{0\}$, we compute $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ and $\omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})$ by (218):

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \qquad \text{and} \qquad \omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) = \gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) \rho_{\chi_k}(\widetilde{w}_{\widetilde{\tau}}).$$

Then, since $\widetilde{\sigma}$ and $\widetilde{\tau}$ commute,

$$\begin{aligned}
&(\widetilde{\sigma}, \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))(\widetilde{\tau}, \omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}))(\widetilde{\sigma}, \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))^{-1}(\widetilde{\tau}, \omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}))^{-1} \\
&= (\widetilde{\sigma}\widetilde{\tau}\widetilde{\sigma}^{-1}\widetilde{\tau}^{-1}, \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}\omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})^{-1}) \\
&= (1, \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w}_{\widetilde{\tau}})\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})^{-1}\rho_{\chi_k}(\widetilde{w}_{\widetilde{\tau}})^{-1}) \\
&= (1, \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(\widetilde{w}_{\widetilde{\tau}})\rho_{\chi_k}(-\widetilde{w}_{\widetilde{\sigma}})\rho_{\chi_k}(-\widetilde{w}_{\widetilde{\tau}})) \\
&= (1, \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}} + \widetilde{w}_{\widetilde{\tau}}, \langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}})\rho_{\chi_k}(-\widetilde{w}_{\widetilde{\sigma}} - \widetilde{w}_{\widetilde{\tau}}, \langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}})) \\
&= (1, \rho_{\chi_k}(0, 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} - \langle \widetilde{w}_{\widetilde{\sigma}} + \widetilde{w}_{\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma}} + \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}})) \\
&= (1, \chi_k(2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}})\,\mathrm{Id}_{\mathrm{L}^2(Y)}).
\end{aligned}$$

Thus by (238),

$$\begin{aligned}
(\widetilde{\sigma}, \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))(\widetilde{\tau}, \omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}))(\widetilde{\sigma}, \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))^{-1}(\widetilde{\tau}, \omega_{\chi_k}(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}))^{-1} &= (1, -\mathrm{Id}_{\mathrm{L}^2(Y)}) \\
&= \widetilde{\Phi}(-1).
\end{aligned}$$

This shows that $\mathrm{Im}(\widetilde{\Phi}) \subseteq [\mathrm{Mp}(\widetilde{W})_{\chi_k}, \mathrm{Mp}(\widetilde{W})_{\chi_k}]$ and concludes the proof.

iii. See Proposition 3.49 and Remark 3.50.

iv. Suppose that the rank $2n$ of the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ is such that $n \geqslant 3$. Then we can do the same reasoning as we did in Corollary 2.62.ii to compute $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z})$, but replacing $\mathbb{Z}/4\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$. In this way, we prove that

$$\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}. \tag{262}$$

Hence there is a unique non-split two-fold covering of $\mathrm{Sp}(\widetilde{W})$ up to isomorphism. Thus, every choice of the sets $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$ leads to the same extension $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ up to isomorphism. This extension is the unique non-split two-fold covering of $\mathrm{Sp}(\widetilde{W})$. In particular, all the groups $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ that we define with any of these choices are isomorphic.

□

Remark 2.64.

- Proposition 2.63 provides an explicit construction of a two-fold metaplectic covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. The only unspecified element in this construction is the way to choose a set $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$. In fact, Corollary 2.62 provides a necessary and sufficient condition on $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ so that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_4(\mathbb{C})$. We deduced this fact directly from the explicit expression of the cocycle found in Corollary 2.58. However, improving from $\mu_4(\mathbb{C})$ to $\mu_2(\mathbb{C})$ out of this explicit expression is much more difficult. Instead we used the known equality $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ to ensure the existence of a choice $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$ (recall that this equality holds if $\widetilde{W}$ is a $\mathbb{Z}/4\mathbb{Z}$-module of rank $2n$ at least equal to six, but it allowed us to deduce that the smallest posssible range of the cocycle is $\mu_2(\mathbb{C})$ in any case). The next step is to find one choice of $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$. To this purpose, we can either directly find a choice that works (we are able to provide two such choices only if $n = 1$, see Proposition 3.49) or we can follow the procedure explained in Corollary 2.62.ii: we start with any choice $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ satisfying (244) (there is an explicit example right after (244)) and we obtain the cocycle $\widetilde{c}_{\chi_k}$ with values in $\mu_4(\mathbb{C})$ (notice here that if $n \geqslant 3$ then every such choice also leads to the only non-split four-fold covering of $\mathrm{Sp}(\widetilde{W})$ since $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ by (257)), then we square it and find a coboundary associated to this cocycle by (258), and finally we renormalize the operators $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ so that the new associated cocycle satisfies (259), and thus has a range equal to $\mu_2(\mathbb{C})$.
- Proposition 2.63 proves that, up to isomorphism, there are exactly two possibilities for the two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ if $n = 1$, and only one if $n \geqslant 3$. It is an open problem to know how many such two-fold coverings there are for $n = 2$.

We are now able to lift the projective representation $\widetilde{\sigma} \mapsto \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ of $\mathrm{Sp}(\widetilde{W})$ to the metaplectic group $\mathrm{Mp}(\widetilde{W})_{\chi_k}$.

Proposition 2.65 (The Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$). *We use the notations and assumptions of Proposition 2.63.*

i. *The projective Weil representation ω_{χ_k} of $\mathrm{Sp}(\widetilde{W})$ can be lifted to a linear representation $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, which will be called the Weil representation. It is defined by*

$$\widetilde{\omega}_{\chi_k}(\widetilde{s}) = (-1)^l \omega_{\chi_k}(\widetilde{\sigma}) \quad \textit{for all } \widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \in \mathrm{Mp}(\widetilde{W})_{\chi_k}. \tag{263}$$

ii. *$\widetilde{\omega}_{\chi_k}$ is unitary and irreducible. Moreover, its kernel is*

$$\begin{aligned}
&\mathrm{Ker}(\widetilde{\omega}_{\chi_k}) \\
&= \{(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \textit{ such that } \widetilde{\sigma} \in \mathrm{Ker}(\pi),\ \widetilde{w}_{\widetilde{\sigma}} \in 2\widetilde{W},\ \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C}) \textit{ and } (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 1\} \\
&= \{(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \textit{ such that } \widetilde{\sigma} \in \mathrm{Ker}(\eta),\ \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C}) \textit{ and } (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 1\},
\end{aligned} \tag{264}$$

where π and η are defined in Proposition 2.7. In particular,

- *If $n = 1$, then the Weil representation $\widetilde{\omega}_{\chi_k}$ constructed with the choices $\{\widetilde{w}_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ given by* (372) *and* (373) *in Proposition 3.49 is faithful, whereas its kernel is $\{(1, \mathrm{Id}_{\mathrm{L}^2(Y)}), (-1, \mathrm{Id}_{\mathrm{L}^2(Y)})\}$ if we use* (374) *instead of* (373).
- *If $n \geqslant 2$, then $\widetilde{\omega}_{\chi_k}$ is not faithful.*

iii. *The image of $\widetilde{\omega}_{\chi_k}$ contains a unitary multiple of each operator belonging to the following set:*

$$\left\{\rho_{\chi_k}(\widetilde{e}_{j_1} + \cdots + \widetilde{e}_{j_s} + \widetilde{f}_{i_1} + \cdots + \widetilde{f}_{i_r}), \textit{ where } 1 \leqslant r, s \leqslant n \textit{ and } i_1, \ldots, i_r, j_1, \ldots, j_s \in \{1, \ldots, n\}\right\},$$

cf. (270) *and* (271). *(Recall from Lemma 2.53 that this family of operators forms a basis of* $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$*.)*

Remark 2.66. We call the representation $\widetilde{\omega}_{\chi_k}$ constructed in Proposition 2.65 the Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$. However, it is not faithful in general, which turns out to be a special feature of characteristic two (recall from (201) that the representation ρ_{χ_k} of $\mathrm{H}(\widetilde{W})$ is already non-faithful). The motivation to still call this representation the Weil representation is because it is a realization of the projective Weil representation, as we did for the Weil representations of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.

Proof. **i.** Define the map $\widetilde{\omega}_{\chi_k}$ by (263). Observe first that for $\widetilde{\sigma} = 1$, since $\widetilde{w}_1 \in 2\widetilde{W}$ by (207) and we can set $S_1 = \{0\}$, we obtain by (218) and (201) that

$$\omega_{\chi_k}(1, \widetilde{w}_1) = \gamma(1, \widetilde{w}_1) \rho_{\chi_k}(\widetilde{w}_1) = \gamma(1, \widetilde{w}_1)\, \mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Moreover, by (229), $\widetilde{c}_{\chi_k}(1, 1) = \gamma(1, \widetilde{w}_1)$. Since we made a choice $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}), \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ satisfying (232) and such that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$, this implies that $\gamma(1, \widetilde{w}_1) \in \mu_2(\mathbb{C})$. Hence

$$\widetilde{\omega}_{\chi_k}(1, \mathrm{Id}_{\mathrm{L}^2(Y)}) = \widetilde{\omega}_{\chi_k}(1, \gamma(1, \widetilde{w}_1)^{-1} \omega_{\chi_k}(1, \widetilde{w}_1)) = \mathrm{Id}_{\mathrm{L}^2(Y)}\,.$$

Then we prove exactly as in Proposition 1.39 that $\widetilde{\omega}_{\chi_k}$ is a linear representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ that lifts the projective Weil representation of $\mathrm{Sp}(\widetilde{W})$.

ii. The unitarity of $\widetilde{\omega}_{\chi_k}$ follows directly from Proposition 2.56. We will prove its irreducibility in the next point, so we study only its kernel here. Let $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \in \mathrm{Mp}(\widetilde{W})_{\chi_k}$. If $\widetilde{\omega}_{\chi_k}(\widetilde{s}) = \mathrm{Id}_{\mathrm{L}^2(Y)}$ then by (218), for every system of representatives $S_{\widetilde{\sigma}}$ of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$,

$$\mathrm{Id}_{\mathrm{L}^2(Y)} = (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}),$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$. This implies that $S_{\widetilde{\sigma}}$ contains only one element, i.e $\mathrm{I}(\widetilde{\sigma}) = \mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W} \subseteq 2\widetilde{W}$, and we can suppose that $S_{\widetilde{\sigma}} = \{0\}$. Then we have only one term in the sum, for which we can set $\widetilde{u} = 0$ and $\widetilde{w} = 0$. Hence

$$\mathrm{Id}_{\mathrm{L}^2(Y)} = (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}).$$

By Lemma 2.53, this implies that $\widetilde{w}_{\widetilde{\sigma}} \in 2\widetilde{W}$ and $(-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 1$. Then by (207), for all $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma}) = (2\,\mathrm{I}(\widetilde{\sigma}))^{\perp} = \widetilde{W}$,

$$0 = 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}} = \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w} \rangle_{\widetilde{W}}. \tag{265}$$

Moreover, $\mathrm{I}(\widetilde{\sigma}) \subseteq 2\widetilde{W}$ is equivalent to $\pi(\widetilde{\sigma}) = 1$. Hence by Proposition 2.7.vi, (280) and (265), $\eta(\widetilde{\sigma}) = (\pi(\widetilde{\sigma}), f_{\widetilde{\sigma}}) = (1, 0)$, i.e. $\widetilde{\sigma} \in \mathrm{Ker}(\eta)$. Thus $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}))$ with $\widetilde{\sigma} \in \mathrm{Ker}(\eta)$ and $(-1)^l = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}$, which requires that $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C})$. Conversely, suppose that $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}))$ is an element of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ such that $\widetilde{\sigma} \in \mathrm{Ker}(\eta)$, $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C})$ and $(-1)^l = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}$. Then $\pi(\widetilde{\sigma}) = 1$, which implies that $\mathrm{I}(\widetilde{\sigma}) \subseteq 2\widetilde{W}$. Hence we can set $S_{\widetilde{\sigma}} = \{0\}$, and since $\mathrm{K}'(\widetilde{\sigma}) = \widetilde{W}$ and $f_{\widetilde{\sigma}} = 0$, we have that $\widetilde{w}_{\widetilde{\sigma}} \in 2\widetilde{W}$ by (207) and (280). Thus we compute by (218) that $(-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) = \mathrm{Id}_{\mathrm{L}^2(Y)}$, i.e. $\widetilde{\omega}_{\chi_k}(\widetilde{s}) = \mathrm{Id}_{\mathrm{L}^2(Y)}$. Therefore, we conclude that

$$\begin{aligned}
&\mathrm{Ker}(\widetilde{\omega}_{\chi_k}) \\
&= \{(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \text{ such that } \widetilde{\sigma} \in \mathrm{Ker}(\pi),\ \widetilde{w}_{\widetilde{\sigma}} \in 2\widetilde{W},\ \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C}) \text{ and } (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 1\} \\
&= \{(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \text{ such that } \widetilde{\sigma} \in \mathrm{Ker}(\eta),\ \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C}) \text{ and } (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = 1\}.
\end{aligned} \tag{266}$$

Observe that the condition $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_2(\mathbb{C})$ in (266) is necessary since it may not be satisfied by every element $\widetilde{\sigma} \in \mathrm{Ker}(\eta)$. In fact, if $\widetilde{\sigma} \in \mathrm{Ker}(\eta)$ then we have that $\widetilde{\sigma}^2 = 1$, which implies by (229) that $\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\sigma}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^2 \gamma(1, \widetilde{w}_1)^{-1}$. Since $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$, we already obtained that $\gamma(1, \widetilde{w}_1) \in \mu_2(\mathbb{C})$ in the first point of the proof above. So we know that $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^2 \in \mu_2(\mathbb{C})$, which only implies that $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_4(\mathbb{C})$. For example, if the rank of $\widetilde{W}$ is 2 and we consider the choice of $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ given by (373) in Proposition 3.49, then the kernel of η has only two elements: 1 and -1. Since $\gamma(-1, \widetilde{w}_{-1})^2 = -1$, there is no element $(-1, (-1)^l \omega_{\chi_k}(-1, \widetilde{w}_{-1}))$ which belongs to the kernel of $\widetilde{\omega}_{\chi_k}$. In this example, the only element in the kernel of $\widetilde{\omega}_{\chi_k}$ is $(1, \mathrm{Id}_{\mathrm{L}^2(Y)})$, which implies that the representation $\widetilde{\omega}_{\chi_k}$ is faithful. This is the only case where this representation is faithful. In fact, by Proposition 3.51, the kernel of $\widetilde{\omega}_{\chi_k}$

is $\{(1, \mathrm{Id}_{\mathrm{L}^2(Y)}), (-1, \mathrm{Id}_{\mathrm{L}^2(Y)})\}$ if we use (374) instead of (373), so we can suppose that the rank of $\widetilde{W}$ is $2n \geqslant 4$. By Proposition 2.7.vi, the kernel of η contains $2^{2n^2-n} \geqslant 2^6$ elements. Let $\widetilde{\sigma}$ and $\widetilde{\tau}$ be in the kernel of η such that $\widetilde{\sigma}, \widetilde{\tau} \neq 1$ and $\widetilde{\sigma} \neq \widetilde{\tau}$. Suppose that $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \notin \mu_2(\mathbb{C})$ and $\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) \notin \mu_2(\mathbb{C})$, i.e. they both belong to $\{\pm i\}$. Then by (229),

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})^{-1}\chi_k(\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\tau}} \rangle_{\widetilde{W}} - \langle \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}, \widetilde{w}_{\widetilde{\sigma},\widetilde{\tau}} \rangle_{\widetilde{W}}).$$

Since $\widetilde{w}_{\widetilde{\sigma}}$, $\widetilde{w}_{\widetilde{\tau}}$ and $\widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}$ are in $2\widetilde{W}$, we obtain

$$\widetilde{c}_{\chi_k}(\widetilde{\sigma}, \widetilde{\tau}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}})\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})^{-1}.$$

The fact that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\widetilde{\tau}, \widetilde{w}_{\widetilde{\tau}}) \in \mu_2(\mathbb{C})$ implies that $\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}) \in \mu_2(\mathbb{C})$. Now set $l \in \mathbb{Z}$ such that $(-1)^l\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}) = 1$. Then

$$\widetilde{\omega}_{\chi_k}(\widetilde{\sigma}\widetilde{\tau}, (-1)^l\omega_{\chi_k}(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})) = (-1)^l\gamma(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}}) = \mathrm{Id}_{\mathrm{L}^2(Y)},$$

i.e. $(\widetilde{\sigma}\widetilde{\tau}, (-1)^l\omega_{\chi_k}(\widetilde{\sigma}\widetilde{\tau}, \widetilde{w}_{\widetilde{\sigma}\widetilde{\tau}})) \in \mathrm{Ker}(\widetilde{\omega}_{\chi_k})$ with $\widetilde{\sigma}\widetilde{\tau} \neq 1$.

iii. We use the notation of chapter 4, in particular a symplectic basis $\{\widetilde{e}_1, \ldots, \widetilde{e}_n, \widetilde{f}_1, \ldots, \widetilde{f}_n\}$ of $\widetilde{W}$ and the decomposition $\widetilde{W} = \widetilde{W}_1 \oplus \cdots \oplus \widetilde{W}_n$ of $\widetilde{W}$ in submodules of rank 2 such that $\widetilde{W}_i = \mathbb{Z}/4\mathbb{Z}\widetilde{e}_i \oplus \mathbb{Z}/4\mathbb{Z}\widetilde{f}_i$, and $\widetilde{W}_i$ and $\widetilde{W}_j$ are orthogonal with respect to the symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ for all $1 \leqslant i \neq j \leqslant n$. Exactly as in Corollary 2.59, we define for every $1 \leqslant j \leqslant n$ the elements $\widetilde{\sigma}_j$ and $\widetilde{\tau}_j$ of $\mathrm{Sp}(\widetilde{W})$ by $\widetilde{\sigma}_j = \left(1, \ldots, 1, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, 1, \ldots, 1\right) \in \mathrm{Sp}(\widetilde{W}_1) \times \cdots \times \mathrm{Sp}(\widetilde{W}_n)$ and $\widetilde{\tau}_j = \left(1, \ldots, 1, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, 1, \ldots, 1\right) \in \mathrm{Sp}(\widetilde{W}_1) \times \cdots \times \mathrm{Sp}(\widetilde{W}_n)$, i.e. by (420), for all $\widetilde{w} = \widetilde{w}_1 + \cdots + \widetilde{w}_n = \sum_{1 \leqslant i \leqslant n} (\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i) \in \widetilde{W}$,

$$\widetilde{\sigma}_j(\widetilde{w}_1 + \cdots + \widetilde{w}_n) = \widetilde{x}_j\widetilde{e}_j + (2\widetilde{x}_j + \widetilde{y}_j)\widetilde{f}_j + \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq j}} (\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i) \tag{267}$$

and

$$\widetilde{\tau}_j(\widetilde{w}_1 + \cdots + \widetilde{w}_n) = (\widetilde{x}_j + 2\widetilde{y}_j)\widetilde{e}_j + \widetilde{y}_j\widetilde{f}_j + \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq j}} (\widetilde{x}_i\widetilde{e}_i + \widetilde{y}_i\widetilde{f}_i). \tag{268}$$

Then we prove as in Corollary 2.59 that

$$\begin{cases} \widetilde{w}_{\widetilde{\sigma}_j} = 2\widetilde{\alpha}_j\widetilde{e}_j + \widetilde{\beta}_j\widetilde{f}_j + \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq j}} (2\widetilde{\alpha}_i\widetilde{e}_i + 2\widetilde{\beta}_i\widetilde{f}_i), \quad \text{where } \begin{cases} \widetilde{\beta}_j \in (\mathbb{Z}/4\mathbb{Z})\backslash(2\mathbb{Z}/4\mathbb{Z}) \\ \widetilde{\alpha}_j, \widetilde{\alpha}_i, \widetilde{\beta}_i \in \mathbb{Z}/4\mathbb{Z} \quad \text{for } i \neq j, \end{cases} \\ \widetilde{w}_{\widetilde{\tau}_j} = \widetilde{\gamma}_j\widetilde{e}_j + 2\widetilde{\delta}_j\widetilde{f}_j + \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq j}} (2\widetilde{\gamma}_i\widetilde{e}_i + 2\widetilde{\delta}_i\widetilde{f}_i), \quad \text{where } \begin{cases} \widetilde{\gamma}_j \in (\mathbb{Z}/4\mathbb{Z})\backslash(2\mathbb{Z}/4\mathbb{Z}) \\ \widetilde{\delta}_j, \widetilde{\gamma}_i, \widetilde{\delta}_i \in \mathbb{Z}/4\mathbb{Z} \quad \text{for } i \neq j. \end{cases} \end{cases} \tag{269}$$

Since two systems of representatives of $\mathrm{I}(\widetilde{\sigma}_j)/(\mathrm{I}(\widetilde{\sigma}_j) \cap 2\widetilde{W})$ and $\mathrm{I}(\widetilde{\tau}_j)/(\mathrm{I}(\widetilde{\tau}_j) \cap 2\widetilde{W})$ are $S_{\widetilde{\sigma}_j} = \{0\}$ and $S_{\widetilde{\tau}_j} = \{0\}$, by (218) and (223),

$$\begin{cases} \omega_{\chi_k}(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j}) = \gamma(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j})\rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}_j}) &= \gamma(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j})\chi_k(\langle \widetilde{\beta}_j \widetilde{f}_j, \widetilde{w}_{\widetilde{\sigma}_j} \rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{\beta}_j \widetilde{f}_j) \\ &= \gamma(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j})\chi_k(2\widetilde{\alpha}_j \widetilde{\beta}_j)\rho_{\chi_k}(\widetilde{\beta}_j \widetilde{f}_j) \\ &= \gamma(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j})\chi_k(2\widetilde{\alpha}_j)\rho_{\chi_k}(\widetilde{f}_j) \\ \omega_{\chi_k}(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j}) = \gamma(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j})\rho_{\chi_k}(\widetilde{w}_{\widetilde{\tau}_j}) &= \gamma(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j})\chi_k(\langle \widetilde{\gamma}_j \widetilde{e}_j, \widetilde{w}_{\widetilde{\tau}_j} \rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{\gamma}_j \widetilde{e}_j) \\ &= \gamma(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j})\chi_k(2\widetilde{\gamma}_j \widetilde{\delta}_j)\rho_{\chi_k}(\widetilde{\gamma}_j \widetilde{e}_j) \\ &= \gamma(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j})\chi_k(2\widetilde{\delta}_j)\rho_{\chi_k}(\widetilde{e}_j). \end{cases}$$

This implies that

$$\begin{cases} \widetilde{\omega}_{\chi_k}(\widetilde{\sigma}_j, \chi_k(2\widetilde{\alpha}_j)\omega_{\chi_k}(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j})) &= \gamma(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j})\rho_{\chi_k}(\widetilde{f}_j) \\ \widetilde{\omega}_{\chi_k}(\widetilde{\tau}_j, \chi_k(2\widetilde{\delta}_j)\omega_{\chi_k}(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j})) &= \gamma(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j})\rho_{\chi_k}(\widetilde{e}_j), \end{cases} \tag{270}$$

where $\gamma(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j}),\ \gamma(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j}) \in \mu_4(\mathbb{C})$ since $\widetilde{\sigma}_j^2 = 1$, $\widetilde{\tau}_j^2 = 1$ and $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$ (we already proved this result in the second point of the proof above). Now we set $\widetilde{s}_j = (\widetilde{\sigma}_j, \chi_k(2\widetilde{\alpha}_j)\omega_{\chi_k}(\widetilde{\sigma}_j, \widetilde{w}_{\widetilde{\sigma}_j}))$ and $\widetilde{r}_j = (\widetilde{\tau}_j, \chi_k(2\widetilde{\delta}_j)\omega_{\chi_k}(\widetilde{\tau}_j, \widetilde{w}_{\widetilde{\tau}_j}))$ for all $1 \leqslant j \leqslant n$. Let r and s be two integers such that $1 \leqslant r, s \leqslant n$. Let $i_1, \ldots, i_r \in \{1, \ldots, n\}$ and $j_1, \ldots, j_s \in \{1, \ldots, n\}$. Using (270) and the expression of the product law in $\mathrm{H}(\widetilde{W})$, we obtain that

$$\begin{aligned} &\widetilde{\omega}_{\chi_k}(\widetilde{r}_{j_1} \cdots \widetilde{r}_{j_s} \widetilde{s}_{i_1} \cdots \widetilde{s}_{i_r}) \\ &= \widetilde{\omega}_{\chi_k}(\widetilde{r}_{j_1}) \cdots \widetilde{\omega}_{\chi_k}(\widetilde{r}_{j_s}) \widetilde{\omega}_{\chi_k}(\widetilde{s}_{i_1}) \cdots \widetilde{\omega}_{\chi_k}(\widetilde{s}_{i_r}) \\ &= \gamma(\widetilde{\tau}_{j_1}, \widetilde{w}_{\widetilde{\tau}_{j_1}}) \cdots \gamma(\widetilde{\tau}_{j_s}, \widetilde{w}_{\widetilde{\tau}_{j_s}}) \gamma(\widetilde{\sigma}_{i_1}, \widetilde{w}_{\widetilde{\sigma}_{i_1}}) \cdots \gamma(\widetilde{\sigma}_{i_r}, \widetilde{w}_{\widetilde{\sigma}_{i_r}}) \\ &\qquad \chi_k\left(\sum_{1 \leqslant k < l \leqslant s} \langle \widetilde{e}_{j_k}, \widetilde{e}_{j_l} \rangle_{\widetilde{W}} + \sum_{1 \leqslant k' < l' \leqslant r} \langle \widetilde{f}_{i_{k'}}, \widetilde{f}_{i_{l'}} \rangle_{\widetilde{W}} \right) \rho_{\chi_k}(\widetilde{e}_{j_1} + \cdots + \widetilde{e}_{j_s}) \rho_{\chi_k}(\widetilde{f}_{i_1} + \cdots + \widetilde{f}_{i_r}) \\ &= \gamma(\widetilde{\tau}_{j_1}, \widetilde{w}_{\widetilde{\tau}_{j_1}}) \cdots \gamma(\widetilde{\tau}_{j_s}, \widetilde{w}_{\widetilde{\tau}_{j_s}}) \gamma(\widetilde{\sigma}_{i_1}, \widetilde{w}_{\widetilde{\sigma}_{i_1}}) \cdots \gamma(\widetilde{\sigma}_{i_r}, \widetilde{w}_{\widetilde{\sigma}_{i_r}}) \\ &\qquad \chi_k\left(\sum_{1 \leqslant k < l \leqslant s} \langle \widetilde{e}_{j_k}, \widetilde{e}_{j_l} \rangle_{\widetilde{W}} + \sum_{1 \leqslant k' < l' \leqslant r} \langle \widetilde{f}_{i_{k'}}, \widetilde{f}_{i_{l'}} \rangle_{\widetilde{W}} + \langle \widetilde{e}_{j_1} + \cdots + \widetilde{e}_{j_s}, \widetilde{f}_{i_1} + \cdots + \widetilde{f}_{i_r} \rangle_{\widetilde{W}} \right) \\ &\qquad \rho_{\chi_k}(\widetilde{e}_{j_1} + \cdots + \widetilde{e}_{j_s} + \widetilde{f}_{i_1} + \cdots + \widetilde{f}_{i_r}) \\ &\in \mu_4(\mathbb{C}) \rho_{\chi_k}(\widetilde{e}_{j_1} + \cdots + \widetilde{e}_{j_s} + \widetilde{f}_{i_1} + \cdots + \widetilde{f}_{i_r}). \end{aligned} \tag{271}$$

Thus the image of $\widetilde{\omega}_{\chi_k}$ contains a unitary multiple of the operator $\rho_{\chi_k}(\widetilde{e}_{j_1} + \cdots + \widetilde{e}_{j_s} + \widetilde{f}_{i_1} + \cdots + \widetilde{f}_{i_r})$, for all possible choices of indices $i_1, \ldots, i_r \in \{1, \ldots, n\}$ and $j_1, \ldots, j_s \in \{1, \ldots, n\}$ such that $1 \leqslant r, s \leqslant n$. Since this family of operators forms a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by Lemma 2.53, the representation $\widetilde{\omega}_{\chi_k}$ is irreducible. In fact, if $\mathscr{V} \subseteq \mathrm{L}^2(Y)$ is a vector subspace invariant under $\widetilde{\omega}_{\chi_k}$, then $\mathscr{V}$ is in particular invariant under all the operators of this list. Thus either $\mathscr{V} = \{0\}$ or $\mathscr{V} = \mathrm{L}^2(Y)$.

□

Corollary 2.67 (Character of the Weil representation of $\mathbf{Mp}(\widetilde{W})_{\chi_k}$). *Let* $\widetilde{\Theta}_{\chi_k}$ *be the character of* $\widetilde{\omega}_{\chi_k}$*. Then, for every* $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))$ *in* $\mathrm{Mp}(\widetilde{W})_{\chi_k}$*,*

$$\widetilde{\Theta}_{\chi_k}(\widetilde{s}) = (-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |Y| \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \mathbb{1}_{2\,\mathrm{I}(\widetilde{\sigma})}(2\widetilde{w}_{\widetilde{\sigma}}), \tag{272}$$

where $\widetilde{u}$ *is any element of* $\mathrm{I}(\widetilde{\sigma})$ *such that* $2\widetilde{u} = 2\widetilde{w}_{\widetilde{\sigma}}$*, and* $\widetilde{w} \in \widetilde{W}$ *is any element such that* $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$*. Moreover, the support of* $\widetilde{\Theta}_{\chi_k}$ *is*

$$\begin{aligned} \mathrm{supp}(\widetilde{\Theta}_{\chi_k}) &= \{\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \in \mathrm{Mp}(\widetilde{W})_{\chi_k}, 2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})\} \\ &= \{\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \in \mathrm{Mp}(\widetilde{W})_{\chi_k}, f_{\widetilde{\sigma}|\,\mathrm{K}(\pi(\widetilde{\sigma}))} = 0\}, \end{aligned}$$

where π *is defined in Proposition 2.7.iii. It does not depend on the choices of* $\{\widetilde{w}_{\widetilde{\sigma}}$ *satisfying* (207)*,* $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *and* $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ *satisfying* (232)*,* $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *such that* $\widetilde{c}_{\chi_k}$ *takes its values in* $\mu_2(\mathbb{C})$*. In particular, if* $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ *is such that* $1 - \widetilde{\sigma}$ *is invertible, then* $\widetilde{\Theta}_{\chi_k}(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \neq 0$ *for all* $l \in \mathbb{Z}$*.*

Proof. Let $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma})) \in \mathrm{Mp}(\widetilde{W})_{\chi_k}$, where $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$. By definition of $\widetilde{\omega}_{\chi_k}$, we have that $\widetilde{\Theta}_{\chi_k}(\widetilde{s}) = \mathrm{Trace}(\widetilde{\omega}_{\chi_k}(\widetilde{s})) = (-1)^l \,\mathrm{Trace}(\omega_{\chi_k}(\widetilde{\sigma}))$. Recall the expression (218) proved in Proposition 2.56:

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}),$$

where $S_{\widetilde{\sigma}}$ is any system of representatives of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$ and $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$. Then by (202),

$$\begin{aligned} \mathrm{Trace}(\omega_{\chi_k}(\widetilde{\sigma})) &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \,\mathrm{Trace}(\rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}})) \\ &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}}} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) |Y| \mathbb{1}_{2\widetilde{W}}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}) \end{aligned}$$

The condition $\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}} \in 2\widetilde{W}$ is equivalent to $2\widetilde{u} + 2\widetilde{w}_{\widetilde{\sigma}} = 0$. If there is an element $\widetilde{u} \in S_{\widetilde{\sigma}}$ such that this condition is satisfied, then $2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})$ since $\widetilde{u} \in \mathrm{I}(\widetilde{\sigma})$ by definition of $S_{\widetilde{\sigma}}$. Conversely, if $2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})$ then there exists $\widetilde{v} \in \mathrm{I}(\widetilde{\sigma})$ such that $2\widetilde{w}_{\widetilde{\sigma}} = 2\widetilde{v}$. Since $S_{\widetilde{\sigma}}$ is a system of representatives of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$, the element $\widetilde{u}$ in $S_{\widetilde{\sigma}}$ such that $2\widetilde{u} = 2\widetilde{w}_{\widetilde{\sigma}}$ is unique. So, the existence (and unicity) of $\widetilde{u} \in S_{\widetilde{\sigma}}$ such that $\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}} \in 2\widetilde{W}$ is equivalent to the condition $2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})$. Hence we obtain

$$\mathrm{Trace}(\omega_{\chi_k}(\widetilde{\sigma})) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |Y| \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \mathbb{1}_{2\,\mathrm{I}(\widetilde{\sigma})}(2\widetilde{w}_{\widetilde{\sigma}}),$$

where $\widetilde{u}$ is the unique element of $S_{\widetilde{\sigma}}$ such that $2\widetilde{u} = 2\widetilde{w}_{\widetilde{\sigma}}$, and $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$. Since this is true for every system of representatives of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$, we can take for $\widetilde{u}$ in the last formula any element of $\mathrm{I}(\widetilde{\sigma})$ such that $2\widetilde{u} = 2\widetilde{w}_{\widetilde{\sigma}}$. Thus we obtain (272). Now, $\widetilde{\Theta}_{\chi_k}(\widetilde{s})$ is nonzero if and only if $2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})$. This condition does not depend

on the choice of $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$, and also not on the choice of $\widetilde{w}_{\widetilde{\sigma}}$ by Lemma 2.54. Moreover, by definition of $\widetilde{w}_{\widetilde{\sigma}}$ and by (280), for all $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma})$,

$$2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}} = \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w} \rangle_{\widetilde{W}} = f_{\widetilde{\sigma}}(w), \tag{273}$$

where $w = \widetilde{w} + 2\widetilde{W}$ and $f_{\widetilde{\sigma}}$ is defined by (93). Since $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma})$ if and only if $w \in \mathrm{K}(\pi(\widetilde{\sigma}))$, and $\mathrm{K}'(\widetilde{\sigma})$ and $2\,\mathrm{I}(\widetilde{\sigma})$ are orthogonal with respect to the symplectic form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$, we obtain that $2\widetilde{w}_{\widetilde{\sigma}} \in 2\,\mathrm{I}(\widetilde{\sigma})$ if and only if the restriction of $f_{\widetilde{\sigma}}$ to $\mathrm{K}(\pi(\widetilde{\sigma}))$ vanishes.
If $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ is such that $1 - \widetilde{\sigma}$ is invertible, then $2\,\mathrm{I}(\widetilde{\sigma}) = 2\widetilde{W}$. This implies that every element $\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207) is such that $2\widetilde{w}_{\widetilde{\sigma}}$ belongs to $2\,\mathrm{I}(\widetilde{\sigma})$. In particular, this proves that $\widetilde{\Theta}_{\chi_k}(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \neq 0$ for every $l \in \mathbb{Z}$ and every choice of $\widetilde{w}_{\widetilde{\sigma}}$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ such that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$. □

We proved in Proposition 2.63 that if the rank $2n$ of $\widetilde{W}$ is such that $n \geqslant 3$, then the isomorphism class of the two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ we construct does not depend on the choices made for $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the cocycle $\widetilde{c}_{\chi_k}$ takes its values in $\mu_2(\mathbb{C})$, and by (262) it does not depend on the chosen faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ either. The next result specifies the link between $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$, where we recall that $\chi_1(1) = i$ and $\chi_3 = \overline{\chi_1}$.

Proposition 2.68. *We suppose that we made a choice of the set $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying* (207)*, $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$. Let $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and $\{\delta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ be two choices satisfying* (232) *and such that the cocycles $\widetilde{c}_{\chi_1}$ obtained with the choices $(\widetilde{w}_{\widetilde{\sigma}}, \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))$, and $\widetilde{c}_{\chi_3}$ obtained with the choices $(\widetilde{w}_{\widetilde{\sigma}}, \delta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))$, take their values in $\mu_2(\mathbb{C})$. Then we have the following results.*

- ***i.*** *The extensions $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ of $\mathrm{Sp}(\widetilde{W})$ are isomorphic if and only if the map*

$$(\widetilde{\sigma}_1, \widetilde{\sigma}_2) \mapsto \delta(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})^{-1}\overline{\gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})}\delta(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})^{-1}\overline{\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})}\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})\overline{\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})}^{-1} \tag{274}$$

  defines a coboundary. In particular, this is the case if the rank $2n$ of the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ is such that $n \geqslant 3$: the extensions $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ of $\mathrm{Sp}(\widetilde{W})$ are isomorphic to the only two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$.

- ***ii.*** *If $\{\overline{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is a choice which leads to a cocycle with values in $\mu_2(\mathbb{C})$ and we set $\delta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \overline{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})}$, then $\widetilde{c}_{\chi_3} = \widetilde{c}_{\chi_1}$, i.e. $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ are isomorphic extensions. More precisely, the map*

$$\begin{aligned} \mathscr{J} : \mathrm{Mp}(\widetilde{W})_{\chi_1} &\to \mathrm{Mp}(\widetilde{W})_{\chi_3} \\ (\widetilde{\sigma}, (-1)^l \omega_{\chi_1}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) &\mapsto (\widetilde{\sigma}, (-1)^l \omega_{\chi_3}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \end{aligned} \tag{275}$$

  is an isomorphism of central extensions. This means that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\widetilde{\Phi}_1} & \mathrm{Mp}(\widetilde{W})_{\chi_1} & \xrightarrow{\widetilde{\Psi}_1} & \mathrm{Sp}(\widetilde{W}) & \longrightarrow & 1 \\ & & \downarrow \mathrm{id} & & \downarrow \mathscr{J} & & \downarrow \mathrm{id} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\widetilde{\Phi}_3} & \mathrm{Mp}(\widetilde{W})_{\chi_3} & \xrightarrow{\widetilde{\Psi}_3} & \mathrm{Sp}(\widetilde{W}) & \longrightarrow & 1 \end{array} \tag{D_{24}}$$

Moreover, the Weil representation $\widetilde{\omega}_{\chi_3} \circ \mathscr{J}$ *is isomorphic to the dual representation of* $\widetilde{\omega}_{\chi_1}$.

Proof. **i.** First observe that (207) does not depend on the choice of character χ_k, $k \in \{1,3\}$, so we can use use the same choice for both constructions of $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$. By (229) and since $\chi_3 = \overline{\chi_1}$ and $\widetilde{c}_{\chi_1}$ takes its values in $\mu_2(\mathbb{C})$, for all $\widetilde{\sigma}_1,\ \widetilde{\sigma}_2 \in \mathrm{Sp}(\widetilde{W})$,

$$
\begin{aligned}
&\widetilde{c}_{\chi_3}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)\\
&= \delta(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})\delta(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}\\
&\qquad \sum_{\substack{\widetilde{u}_1\in S_{\widetilde{\sigma}_1}\\ \widetilde{u}_2\in S_{\widetilde{\sigma}_2}\\ \widetilde{u}_1+\widetilde{u}_2-\widetilde{w}_{\widetilde{\sigma}_1,\widetilde{\sigma}_2}\in 2\widetilde{W}}} \chi_3(\langle \widetilde{u}_1,\widetilde{w}_1+\widetilde{w}_{\widetilde{\sigma}_1}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_1},\widetilde{w}_1\rangle_{\widetilde{W}} + \langle \widetilde{u}_2,\widetilde{w}_2+\widetilde{w}_{\widetilde{\sigma}_2}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_2},\widetilde{w}_2\rangle_{\widetilde{W}}\\
&\qquad\qquad + \langle \widetilde{u}_1+\widetilde{w}_{\widetilde{\sigma}_1},\widetilde{u}_2+\widetilde{w}_{\widetilde{\sigma}_2}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2},\widetilde{u}_1+\widetilde{u}_2-\widetilde{w}_{\widetilde{\sigma}_1,\widetilde{\sigma}_2}\rangle_{\widetilde{W}})\\
&= \delta(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})\delta(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}\\
&\qquad \sum_{\substack{\widetilde{u}_1\in S_{\widetilde{\sigma}_1}\\ \widetilde{u}_2\in S_{\widetilde{\sigma}_2}\\ \widetilde{u}_1+\widetilde{u}_2-\widetilde{w}_{\widetilde{\sigma}_1,\widetilde{\sigma}_2}\in 2\widetilde{W}}} \overline{\chi_1(\langle \widetilde{u}_1,\widetilde{w}_1+\widetilde{w}_{\widetilde{\sigma}_1}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_1},\widetilde{w}_1\rangle_{\widetilde{W}} + \langle \widetilde{u}_2,\widetilde{w}_2+\widetilde{w}_{\widetilde{\sigma}_2}\rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}_2},\widetilde{w}_2\rangle_{\widetilde{W}}}\\
&\qquad\qquad \overline{+\langle \widetilde{u}_1+\widetilde{w}_{\widetilde{\sigma}_1},\widetilde{u}_2+\widetilde{w}_{\widetilde{\sigma}_2}\rangle_{\widetilde{W}} + \langle \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2},\widetilde{u}_1+\widetilde{u}_2-\widetilde{w}_{\widetilde{\sigma}_1,\widetilde{\sigma}_2}\rangle_{\widetilde{W}})}\\
&= \delta(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})\delta(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}\overline{(\gamma(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})\gamma(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1})^{-1}\widetilde{c}_{\chi_1}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)}\\
&= \delta(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})\overline{\gamma(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})^{-1}}\delta(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})\overline{\gamma(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})^{-1}}\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}\overline{\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})}\widetilde{c}_{\chi_1}(\widetilde{\sigma}_1,\widetilde{\sigma}_2).
\end{aligned}
$$

Then

$$
\begin{aligned}
&c_{\chi_1}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)c_{\chi_3}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)^{-1}\\
&= \delta(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})^{-1}\overline{\gamma(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})}\delta(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})^{-1}\overline{\gamma(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})}\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})\overline{\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})}^{-1}.
\end{aligned}
\tag{276}
$$

Then we obtain that $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ are isomorphic extensions if and only if the map

$$
(\widetilde{\sigma}_1,\widetilde{\sigma}_2) \mapsto \delta(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})^{-1}\overline{\gamma(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})}\delta(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})^{-1}\overline{\gamma(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})}\delta(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})\overline{\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})}^{-1}
$$

defines a coboundary. Recall that we have denoted by $2n$ the rank of $\widetilde{W}$ as a $\mathbb{Z}/4\mathbb{Z}$-module. If $n \geqslant 3$, we know that the extensions $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ of $\mathrm{Sp}(\widetilde{W})$ are isomorphic to the only non-split two-fold covering of $\mathrm{Sp}(\widetilde{W})$ since $\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}),\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ by (262), so the last map is in fact a coboundary.

ii. If we set $\delta(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}}) = \overline{\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})}$ for all $\widetilde{\sigma}$ in $\mathrm{Sp}(\widetilde{W})$, then $\widetilde{c}_{\chi_1} = \widetilde{c}_{\chi_3}$ by (276). If $\mathscr{J}$ is defined by (275), then for all $\widetilde{s}_1 = (\widetilde{\sigma}_1,(-1)^{l_1}\omega_{\chi_1}(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1}))$ and $\widetilde{s}_2 = (\widetilde{\sigma}_2,(-1)^{l_2}\omega_{\chi_1}(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2}))$ in $\mathrm{Mp}(\widetilde{W})_{\chi_1}$,

$$
\begin{aligned}
\mathscr{J}(\widetilde{s}_1)\mathscr{J}(\widetilde{s}_2) &= (\widetilde{\sigma}_1,(-1)^{l_1}\omega_{\chi_3}(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1}))(\widetilde{\sigma}_2,(-1)^{l_2}\omega_{\chi_3}(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2}))\\
&= (\widetilde{\sigma}_1\widetilde{\sigma}_2,(-1)^{l_1+l_2}\widetilde{c}_{\chi_3}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)\omega_{\chi_3}(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}))
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{J}(\widetilde{s}_1\widetilde{s}_2) &= \mathscr{J}(\widetilde{\sigma}_1\widetilde{\sigma}_2,(-1)^{l_1+l_2}\widetilde{c}_{\chi_1}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)\omega_{\chi_1}(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}))\\
&= (\widetilde{\sigma}_1\widetilde{\sigma}_2,(-1)^{l_1+l_2}\widetilde{c}_{\chi_1}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)\omega_{\chi_3}(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})).
\end{aligned}
$$

Since $\widetilde{c}_{\chi_3} = \widetilde{c}_{\chi_1}$, the map $\mathscr{J}$ is a group homomorphism. It is clearly surjective by definition, so it is an isomorphism since $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ have the same order. Moreover, if $\delta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \overline{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})}$ for all $\widetilde{\sigma}$ in $\mathrm{Sp}(\widetilde{W})$, then by (272), we obtain for all $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_1}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))$,

$$\begin{aligned}\widetilde{\Theta}_{\chi_3}(\mathscr{J}(\widetilde{s})) &= (-1)^l \overline{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})} |Y| \chi_3(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \mathbb{1}_{2\,\mathrm{I}(\widetilde{\sigma})}(2\widetilde{w}_{\widetilde{\sigma}}) \\ &= \overline{(-1)^l \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |Y| \chi_1(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \mathbb{1}_{2\,\mathrm{I}(\widetilde{\sigma})}(2\widetilde{w}_{\widetilde{\sigma}})} \\ &= \overline{\widetilde{\Theta}_{\chi_1}(\widetilde{s})},\end{aligned} \tag{277}$$

where $\widetilde{u}$ is any element of $\mathrm{I}(\widetilde{\sigma})$ such that $2\widetilde{u} = 2\widetilde{w}_{\widetilde{\sigma}}$, and $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$. Denote by $\widetilde{\Theta}^*_{\chi_1}$ the character of the representation which is dual to $\widetilde{\omega}_{\chi_1}$. Then this shows that

$$\begin{aligned}\widetilde{\Theta}^*_{\chi_1}(\widetilde{s}) = \mathrm{Trace}(\widetilde{\omega}_{\chi_1}(\widetilde{s}^{-1})^T) = \mathrm{Trace}((\widetilde{\omega}_{\chi_1}(\widetilde{s})^*)^T) = \overline{\mathrm{Trace}(\widetilde{\omega}_{\chi_1}(\widetilde{s}))} &= \overline{\widetilde{\Theta}_{\chi_1}(\widetilde{s})} \\ &= \widetilde{\Theta}_{\chi_3}(\mathscr{J}(\widetilde{s})),\end{aligned}$$

i.e. the representation $\widetilde{\omega}_{\chi_3} \circ \mathscr{J}$ is isomorphic to the dual representation of $\widetilde{\omega}_{\chi_1}$.

□

Remark 2.69. If the rank $2n$ of the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ is two, then we only define two choices (373) and (374) which lead to two non-isomorphic two-fold coverings $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ in Proposition 3.49. By Remark 3.50, these are in fact the only two possibilities, and the extensions $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ of $\mathrm{Sp}(\widetilde{W})$ are isomorphic if and only if they are defined with the same choice between (373) and (374).

Now that we have constructed the two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$, we can look for a group homomorphism $\mathscr{S}$ which makes the diagram (D_{23}) commute. Since $\mathscr{S}$ should relate the extensions $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$, we start by looking for a link between the projective representations ω_{χ_k} of $\mathrm{Sp}(\widetilde{W})$ and $\omega_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.

Remark 2.70. Proposition 2.65 provides a Weil representation of a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ (which is unique up to isomorphism if the rank $2n$ of $\widetilde{W}$ is such that $n \geqslant 3$). So, if the main aim of constructing the group homomorphism $\mathscr{S}$ is to extend the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to the group $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, then we do not really need to do the entire construction of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and its Weil representation, since we are able to define a Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ in a completely independent way. However, it is still interesting to know how the two constructions relate, so we pursue our search of $\mathscr{S}$.

Lemma 2.71. *Suppose that we made a choice for the sets $\{w_s$ satisfying* (100)*, $s \in \mathrm{ASp}(B_W)\}$ and $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying* (207)*, $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$. Let $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and $\sigma = \pi(\widetilde{\sigma}) \in \mathrm{Sp}(W)$, where π is defined in Proposition 2.7.iii. Recall that $\eta(\widetilde{\sigma}) = (\sigma, f_{\widetilde{\sigma}})$, where $f_{\widetilde{\sigma}}$ is defined by* (93)*.*

i. *Let $w_{\widetilde{\sigma}} = \widetilde{w}_{\widetilde{\sigma}} + 2\widetilde{W}$. Then $w_{\widetilde{\sigma}}$ satisfies* (100) *for $s = \eta(\widetilde{\sigma})$. In particular, $w_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})} \in \mathrm{I}(\sigma)$.*

ii. *Suppose that*

$$\begin{aligned} B_{\widetilde{W}} : \widetilde{W} &\to \mathbb{Z}/4\mathbb{Z} \\ (\widetilde{x}_1+\widetilde{x}_2, \widetilde{y}_1+\widetilde{y}_2) &\mapsto \langle \widetilde{x}_1, \widetilde{y}_2\rangle_{\widetilde{W}} \end{aligned} \quad \textit{and} \quad \begin{aligned} B = B_{\mathrm{Weil}} : W &\to \mathbb{F}_2 \\ (x_1+x_2, y_1+y_2) &\mapsto \langle x_1, y_2\rangle, \end{aligned} \tag{278}$$

and that the extension $\widetilde{\chi_k}$ *of* χ_k *is trivial on* X*. Then*

$$\begin{aligned} &\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\ &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1}\chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}}) - f(v_{\widetilde{\sigma}}) + \langle v_{\widetilde{\sigma}}, w_{\eta(\widetilde{\sigma})}\rangle_W + B_W(w_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})}, v_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})})) \\ &\quad \omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}), w_{\eta(\widetilde{\sigma})}), \end{aligned} \tag{279}$$

where $v_{\widetilde{\sigma}} \in W$ *is such that* $w_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})} = (1-\sigma)(v_{\widetilde{\sigma}})$*. In particular,* $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mu_4(\mathbb{C})\omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}), w_{\eta(\widetilde{\sigma})})$ *if and only if* $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1} \in \mu_4(\mathbb{C})$*.*

Proof. **i.** Let $w \in W$ and $\widetilde{w} \in \widetilde{W}$ be such that $\widetilde{w} + 2\widetilde{W} = w$. We already observed in the proof of (207) in Lemma 2.54 that $w \in \mathrm{K}(\sigma)$ if and only if $\widetilde{w} \in \mathrm{K}'(\widetilde{\sigma})$, i.e. $2\widetilde{w} \in \mathrm{K}(\widetilde{\sigma})$. Suppose that $w \in \mathrm{K}(\sigma)$. Then $(1-\widetilde{\sigma})(2\widetilde{w}) = 0$, i.e. there exists $\widetilde{a} \in \widetilde{W}$ such that $(1-\widetilde{\sigma})(\widetilde{w}) = 2\widetilde{a}$. Hence by definition of $B_{\widetilde{W}}$ and $f_{\widetilde{\sigma}}$ in (93),

$$\begin{aligned} \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w}\rangle_{\widetilde{W}} &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{w}) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{\sigma}(\widetilde{w})) \\ &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{\sigma}(\widetilde{w}) + 2\widetilde{a}) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w} - 2\widetilde{a}) \\ &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{\sigma}(\widetilde{w})) + B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), 2\widetilde{a}) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) + B_{\widetilde{W}}(\widetilde{w}, 2\widetilde{a}) \\ &= f_{\widetilde{\sigma}}(w) + B_{\widetilde{W}}(\widetilde{w} - 2\widetilde{a}, 2\widetilde{a}) + B_{\widetilde{W}}(\widetilde{w}, 2\widetilde{a}) \\ &= f_{\widetilde{\sigma}}(w). \end{aligned} \tag{280}$$

Thus, by definition of $w_{\eta(\widetilde{\sigma})}$ in (100) and $\widetilde{w}_{\widetilde{\sigma}}$ in (207),

$$\langle w_{\eta(\widetilde{\sigma})}, w\rangle_W = f_{\widetilde{\sigma}}(w) = \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w}\rangle_{\widetilde{W}} = 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}\rangle_{\widetilde{W}} = \langle w_{\widetilde{\sigma}}, w\rangle_W. \tag{281}$$

Since (281) is true for every $w \in \mathrm{K}(\sigma)$, this shows that $w_{\widetilde{\sigma}}$ satisfies (100) and $w_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})} \in \mathrm{K}(\sigma)^{\perp} = \mathrm{I}(\sigma)$.

ii. By hypothesis,

$$\begin{aligned} B_{\widetilde{W}} : \widetilde{W} &\to \mathbb{Z}/4\mathbb{Z} \\ (\widetilde{x}_1+\widetilde{x}_2, \widetilde{y}_1+\widetilde{y}_2) &\mapsto \langle \widetilde{x}_1, \widetilde{y}_2\rangle_{\widetilde{W}} \end{aligned} \quad \text{and} \quad \begin{aligned} B = B_{\mathrm{Weil}} : W &\to \mathbb{F}_2 \\ (x_1+x_2, y_1+y_2) &\mapsto \langle x_1, y_2\rangle. \end{aligned}$$

Therefore, $X \times \mathbb{Z}/4\mathbb{Z}$ is an abelian subgroup of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and we can set $\widetilde{\chi_k}(x,t) = \chi_k(t)$ for every $(x,t) \in X \times \mathbb{Z}/4\mathbb{Z}$. Let $\widetilde{w} = \widetilde{x} + \widetilde{y} \in \widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$ and let $w = x + y \in W = X \oplus Y$ such that $\widetilde{x} + 2\widetilde{X} = x$ and $\widetilde{y} + 2\widetilde{Y} = y$. Let $t \in \mathbb{Z}/4\mathbb{Z}$. By (96) and (201), for all $\varphi \in \mathrm{L}^2(Y)$ and $z \in Y$,

$$\begin{aligned} \rho_{\widetilde{\chi_k}}(x+y,t)\varphi(z) &= \widetilde{\chi_k}(x, t - Q_W(y) + \langle x, z\rangle_W - B_W(x+z, y))\varphi(z-y) \\ &= \chi_k(t + \langle x, z\rangle_W - \langle x, y\rangle_W)\varphi(z-y) \\ &= \chi_k(t + \langle x, z\rangle_W - 2\langle \widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})\varphi(z-y) \\ &= \chi_k(-\langle \widetilde{x}, \widetilde{y}\rangle_{\widetilde{W}})\rho_{\chi_k}(\widetilde{x}+\widetilde{y}, t)\varphi(z). \end{aligned}$$

In particular, this shows that, for every $w = \widetilde{w} + 2\widetilde{W}$,

$$\rho_{\chi_k}(\widetilde{w}) = \chi_k(\langle \widetilde{x}, \widetilde{y} \rangle_{\widetilde{W}}) \rho_{\widetilde{\chi_k}}(w) = \chi_k(B_{\widetilde{W}}(\widetilde{w}, \widetilde{w})) \rho_{\widetilde{\chi_k}}(w). \tag{282}$$

Thus

$$\begin{aligned}
&\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1} \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}) \\
&= \rho_{\chi_k}(-\widetilde{\sigma}(\widetilde{w})) \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}) \\
&= \chi_k(B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{\sigma}(\widetilde{w})) + B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}}) + B_{\widetilde{W}}(\widetilde{w}, \widetilde{w})) \rho_{\widetilde{\chi_k}}(\sigma(w)) \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w) \\
&= \chi_k(B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{\sigma}(\widetilde{w})) + B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}}) + B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) - B_W(\sigma(w), \sigma(w))) \\
&\qquad \rho_{\widetilde{\chi_k}}(\sigma(w))^{-1} \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w) \\
&= \chi_k(B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{\sigma}(\widetilde{w})) + B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}}) + B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) - 2B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \sigma(\widetilde{w}))) \\
&\qquad \rho_{\widetilde{\chi_k}}(\sigma(w))^{-1} \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w) \\
&= \chi_k(-f_{\widetilde{\sigma}}(w) + B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}})) \rho_{\widetilde{\chi_k}}(\sigma(w))^{-1} \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w).
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\rho_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma})(w))^{-1} = \rho_{\widetilde{\chi_k}}(\sigma(w), f_{\widetilde{\sigma}}(w))^{-1} &= \rho_{\widetilde{\chi_k}}((\sigma(w), 0)(0, f_{\widetilde{\sigma}}(w)))^{-1} \\
&= \chi_k(-f_{\widetilde{\sigma}}(w)) \rho_{\widetilde{\chi_k}}(\sigma(w))^{-1},
\end{aligned}$$

we obtain

$$\rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1} \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}) = \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}})) \rho_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma})(w))^{-1} \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w).$$

Now, by (103) and (215),

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\operatorname{K}(\widetilde{\sigma})|^{-1} |\operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w} \in \widetilde{W}} \rho_{\chi_k}(\widetilde{\sigma}(\widetilde{w}))^{-1} \rho_{\chi_k}(\widetilde{w}_{\widetilde{\sigma}}) \rho_{\chi_k}(\widetilde{w}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\operatorname{K}(\widetilde{\sigma})|^{-1} |\operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{w} \in \widetilde{W}} \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}})) \rho_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma})(w))^{-1} \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w) \\
&= \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}})) \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\operatorname{K}(\widetilde{\sigma})|^{-1} |\operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{w \in W} |2\widetilde{W}| \rho_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma})(w))^{-1} \rho_{\widetilde{\chi_k}}(w_{\widetilde{\sigma}}) \rho_{\widetilde{\chi_k}}(w) \\
&= \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w}_{\widetilde{\sigma}})) \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) |\operatorname{K}(\widetilde{\sigma})|^{-1} |\operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} |2\widetilde{W}| |\operatorname{K}(\sigma)| \gamma(\sigma)^{-1} \omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}), w_{\widetilde{\sigma}}).
\end{aligned}$$

where $|\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 = |2 \operatorname{I}(\widetilde{\sigma})|^{-1}$ and $|\gamma(\sigma)|^2 = 2^{-\dim \operatorname{I}(\sigma)}$.

Since the maps

$$\begin{array}{llll}
\widetilde{W}/\operatorname{K}(\widetilde{\sigma}) \to \operatorname{I}(\widetilde{\sigma}) & , & \operatorname{K}'(\widetilde{\sigma})/2\widetilde{W} \to \operatorname{K}(\sigma) & \text{and} \quad \operatorname{I}(\widetilde{\sigma})/(\operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W}) \to 2\operatorname{I}(\widetilde{\sigma}) \\
\widetilde{w} + \operatorname{K}(\widetilde{\sigma}) \mapsto (1 - \widetilde{\sigma})(\widetilde{w}) & & \widetilde{w} + 2\widetilde{W} \mapsto w & \widetilde{u} + \operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W} \mapsto 2\widetilde{u}
\end{array}$$

are group isomorphisms, and $\operatorname{K}'(\widetilde{\sigma})$ and $2\operatorname{I}(\widetilde{\sigma})$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_{\widetilde{W}}$, we have that

$$|\operatorname{K}(\widetilde{\sigma})|^{-1} |\operatorname{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} |2\widetilde{W}| |\operatorname{K}(\sigma)| = |\operatorname{I}(\widetilde{\sigma})| |\widetilde{W}|^{-1} |\operatorname{I}(\widetilde{\sigma})|^{-1} |2\operatorname{I}(\widetilde{\sigma})| |\operatorname{K}'(\widetilde{\sigma})| = 1.$$

Therefore,

$$\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1}\chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}},\widetilde{w}_{\widetilde{\sigma}}))\omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}),w_{\widetilde{\sigma}}). \tag{283}$$

Finally, since we proved in the first point above that there exists $v_{\widetilde{\sigma}} \in W$ such that $w_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})} = (1-\sigma)(v_{\widetilde{\sigma}})$, then by Proposition 2.21,

$$\begin{aligned}&\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\\ &= \gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1}\chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}},\widetilde{w}_{\widetilde{\sigma}}) - f(v_{\widetilde{\sigma}}) + \langle v_{\widetilde{\sigma}}, w_{\eta(\widetilde{\sigma})}\rangle_W + B_W(w_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})}, v_{\widetilde{\sigma}} - w_{\eta(\widetilde{\sigma})}))\\ &\quad \omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}),w_{\eta(\widetilde{\sigma})}).\end{aligned} \tag{284}$$

Since χ_k takes its values in $\mu_4(\mathbb{C})$, we obtain that $\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}}) \in \mu_4(\mathbb{C})\omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}),w_{\eta(\widetilde{\sigma})})$ if and only if $\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1} \in \mu_4(\mathbb{C})$.

□

Lemma 2.72 (Link between the cocycles $\widetilde{c}_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ and c_{χ_k} of $\mathrm{ASp}(B_W)$). *We use the notations and assumptions of Lemma 2.71, in particular* (278)*. Let* $\widetilde{\sigma}_1,\widetilde{\sigma}_2 \in \mathrm{Sp}(\widetilde{W})$ *and* $\sigma_1 = \pi(\widetilde{\sigma}_1)$, $\sigma_2 = \pi(\widetilde{\sigma}_2)$*. Let* $w_{\widetilde{\sigma}_1} = \widetilde{w}_{\widetilde{\sigma}_1} + 2\widetilde{W}$, $w_{\widetilde{\sigma}_2} = \widetilde{w}_{\widetilde{\sigma}_2} + 2\widetilde{W}$ *and* $w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} = \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2} + 2\widetilde{W}$ *in* W*. Then the cocycles* $\widetilde{c}_{\chi_k}$ *of* $\mathrm{Sp}(\widetilde{W})$ *and* c_{χ_k} *of* $\mathrm{ASp}(B_W)$ *are related by the following formula:*

$$\begin{aligned}&\widetilde{c}_{\chi_k}(\widetilde{\sigma}_1,\widetilde{\sigma}_2)c_{\chi_k}(\eta(\widetilde{\sigma}_1),\eta(\widetilde{\sigma}_2))^{-1}\\ &= \gamma(\widetilde{\sigma}_1,\widetilde{w}_{\widetilde{\sigma}_1})\gamma(\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_2})\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2,\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}(\gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1}\\ &\quad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1},\widetilde{w}_{\widetilde{\sigma}_1}) - f_{\widetilde{\sigma}_1}(v_{\widetilde{\sigma}_1}) + \langle v_{\widetilde{\sigma}_1}, w_{\eta(\widetilde{\sigma}_1)}\rangle_W + B_W(w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)}, v_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)}))\\ &\quad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_2},\widetilde{w}_{\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_2)}\rangle_W + B_W(w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)}))\\ &\quad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2},\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_1\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}\rangle_W\\ &\qquad + B_W(w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}))^{-1},\end{aligned} \tag{285}$$

where $v_{\widetilde{\sigma}_1}$, $v_{\widetilde{\sigma}_2}$ *and* $v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}$ *in* W *are such that* $w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)} = (1-\sigma_1)(v_{\widetilde{\sigma}_1})$, $w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)} = (1-\sigma_2)(v_{\widetilde{\sigma}_2})$ *and* $w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)} = (1-\sigma_1\sigma_2)(v_{\widetilde{\sigma}_1\widetilde{\sigma}_2})$.

Proof. Let $\widetilde{\sigma}_1,\widetilde{\sigma}_2 \in \mathrm{Sp}(\widetilde{W})$ and $\sigma_1 = \pi(\widetilde{\sigma}_1)$, $\sigma_2 = \pi(\widetilde{\sigma}_2)$. Set $w_{\widetilde{\sigma}_1} = \widetilde{w}_{\widetilde{\sigma}_1} + 2\widetilde{W}$, $w_{\widetilde{\sigma}_2} = \widetilde{w}_{\widetilde{\sigma}_2} + 2\widetilde{W}$ and $w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} = \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2} + 2\widetilde{W}$ in W. By Lemma 2.71, there exist $v_{\widetilde{\sigma}_1}$, $v_{\widetilde{\sigma}_2}$ and $v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}$ in W such that $w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)} = (1-\sigma_1)(v_{\widetilde{\sigma}_1})$, $w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)} = (1-\sigma_2)(v_{\widetilde{\sigma}_2})$ and $w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)} = (1-\sigma_1\sigma_2)(v_{\widetilde{\sigma}_1\widetilde{\sigma}_2})$. Then, by (279),

$$
\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\omega_{\chi_k}(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2}) \\
&= \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\sigma_1)^{-1}\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})\gamma(\sigma_2)^{-1} \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_{\widetilde{\sigma}_1}) - f_{\widetilde{\sigma}_1}(v_{\widetilde{\sigma}_1}) + \langle v_{\widetilde{\sigma}_1}, w_{\eta(\widetilde{\sigma}_1)}\rangle_W + B_W(w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)}, v_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)})) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_{\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_2)}\rangle_W + B_W(w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)})) \\
&\qquad \omega_{\widetilde{\chi}_k}(\eta(\widetilde{\sigma}_1), w_{\eta(\widetilde{\sigma}_1)})\omega_{\widetilde{\chi}_k}(\eta(\widetilde{\sigma}_2), w_{\eta(\widetilde{\sigma}_2)}) \\
&= \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\sigma_1)^{-1}\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})\gamma(\sigma_2)^{-1} \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_{\widetilde{\sigma}_1}) - f_{\widetilde{\sigma}_1}(v_{\widetilde{\sigma}_1}) + \langle v_{\widetilde{\sigma}_1}, w_{\eta(\widetilde{\sigma}_1)}\rangle_W + B_W(w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)}, v_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)})) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_{\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_2)}\rangle_W + B_W(w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)})) \\
&\qquad c_{\chi_k}(\eta(\widetilde{\sigma}_1), \eta(\widetilde{\sigma}_2))\omega_{\widetilde{\chi}_k}(\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2), w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}) \\
&= \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\sigma_1)^{-1}\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})\gamma(\sigma_2)^{-1}\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}\gamma(\sigma_1\sigma_2) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_{\widetilde{\sigma}_1}) - f_{\widetilde{\sigma}_1}(v_{\widetilde{\sigma}_1}) + \langle v_{\widetilde{\sigma}_1}, w_{\eta(\widetilde{\sigma}_1)}\rangle_W + B_W(w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)}, v_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)})) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_{\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_2)}\rangle_W + B_W(w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)})) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_1\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}\rangle_W \\
&\qquad\qquad + B_W(w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}))^{-1} c_{\chi_k}(\eta(\widetilde{\sigma}_1), \eta(\widetilde{\sigma}_2))\omega_{\chi_k}(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}).
\end{aligned}
$$

Thus, by definition of the cocycle $\widetilde{c}_{\chi_k}$, we obtain

$$
\begin{aligned}
&\widetilde{c}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{\sigma}_2) \qquad (286)\\
&= \gamma(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1})\gamma(\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_2})\gamma(\widetilde{\sigma}_1\widetilde{\sigma}_2, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2})^{-1}(\gamma(\sigma_1)\gamma(\sigma_2)\gamma(\sigma_1\sigma_2)^{-1})^{-1} \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1}, \widetilde{w}_{\widetilde{\sigma}_1}) - f_{\widetilde{\sigma}_1}(v_{\widetilde{\sigma}_1}) + \langle v_{\widetilde{\sigma}_1}, w_{\eta(\widetilde{\sigma}_1)}\rangle_W + B_W(w_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)}, v_{\widetilde{\sigma}_1} - w_{\eta(\widetilde{\sigma}_1)})) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_2}, \widetilde{w}_{\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_2)}\rangle_W + B_W(w_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_2)})) \\
&\qquad \chi_k(B_{\widetilde{W}}(\widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, \widetilde{w}_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) - f_{\widetilde{\sigma}_1\widetilde{\sigma}_2}(v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}) + \langle v_{\widetilde{\sigma}_1\widetilde{\sigma}_2}, w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}\rangle_W \\
&\qquad\qquad + B_W(w_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}, v_{\widetilde{\sigma}_1\widetilde{\sigma}_2} - w_{\eta(\widetilde{\sigma}_1\widetilde{\sigma}_2)}))^{-1} c_{\chi_k}(\eta(\widetilde{\sigma}_1), \eta(\widetilde{\sigma}_2)).
\end{aligned}
$$

□

The following result answers the question of the existence of the commutative diagram (D_{23}) for the following bilinear forms:

$$
\begin{aligned}
B_{\widetilde{W}} : \quad \widetilde{W} &\to \mathbb{Z}/4\mathbb{Z} \qquad \text{and} \quad B = B_{\mathrm{Weil}} : \quad W \to \mathbb{F}_2 \\
(\widetilde{x}_1 + \widetilde{x}_2, \widetilde{y}_1 + \widetilde{y}_2) &\mapsto \langle \widetilde{x}_1, \widetilde{y}_2\rangle_{\widetilde{W}} \qquad\qquad (x_1 + x_2, y_1 + y_2) \mapsto \langle x_1, y_2\rangle.
\end{aligned}
$$

In fact, the existence (and unicity) of a group homomorphism $\mathscr{S}$ which makes the diagram (D_{23}) commute is equivalent to the condition (288). The final diagram is (D_{25}). As in Remark 2.70, we notice that this diagram relates the two constructions of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$, hence the Weil representations $\widetilde{\omega}_{\chi_k}$ and $\widetilde{\omega}_{\widetilde{\chi}_k}$ of these two groups. However, it is only valid for a particular choice of $B_{\widetilde{W}}$ and B, whereas the construction of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ we made is completely independent of the choice of bilinear form $B_{\widetilde{W}}$, and already provides a (non-faithful) Weil representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$.

Proposition 2.73 (The homomorphism of central extensions $\mathscr{S}$). *We choose two sets $\{w_s$ satisfying* (100)*, $s \in \mathrm{ASp}(B_W)\}$ and $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying* (207)*, $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$. We choose also two*

sets $\{\gamma(\sigma),\, \sigma \in \mathrm{Sp}(W)\}$ *such that* (112) *and* (116) *are satisfied, and* $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\, \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *such that* (232) *is satisfied and the cocycle* $\widetilde{c}_{\chi_k}$ *takes its values in* $\mu_2(\mathbb{C})$. *We suppose that* (278) *is satisfied. If there exists a group homomorphism* $\mathscr{S}$ *such that the following diagram*

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \iota} & \overset{\mathscr{S}}{\nearrow} & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} & \overset{\eta}{\nearrow} & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Sp}(W) & \longrightarrow & 1 \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\chi_k}} & & \uparrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{Mp}(\widetilde{W})_{\chi_k} & \xrightarrow{\widetilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) & \longrightarrow & 1
\end{array}
\tag{D_{25}}
$$

commutes, then $\mathscr{S}$ *is defined by*

$$
\begin{aligned}
\mathscr{S} : \quad & \mathrm{Mp}(\widetilde{W})_{\chi_k} \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \\
& (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \mapsto (\eta(\widetilde{\sigma}), (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}))
\end{aligned}
\tag{287}
$$

where η *is defined in Proposition 2.7.vi. Moreover, the kernel of* $\mathscr{S}$ *is the kernel of* $\widetilde{\omega}_{\chi_k}$, *which is given in Proposition 2.65. The commutativity of the diagram* (D_{25}) *is equivalent to the following condition:*

$$
\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1} \in \mu_4(\mathbb{C}) \quad \textit{for all } \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W}), \textit{ where } \sigma = \pi(\widetilde{\sigma}),
\tag{288}
$$

and we recall that π *is defined in Proposition 2.7.iii.*

Proof. Let

$$
\begin{aligned}
\mathscr{S} : \quad & \mathrm{Mp}(\widetilde{W})_{\chi_k} \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \subseteq \mathrm{ASp}(B_W) \times \mathrm{U}(\mathrm{L}^2(Y)) \\
& \widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \mapsto (\mathscr{S}_1(\widetilde{s}), \mathscr{S}_2(\widetilde{s}))
\end{aligned}
$$

be a group homomorphism such that (D_{25}) commutes. Let $\widetilde{s} = (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \in \mathrm{Mp}(\widetilde{W})_{\chi_k}$. Then the commutativity of (D_{25}) implies:

- $\widetilde{\omega}_{\chi_k}(\widetilde{s}) = (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\widetilde{s}) = \widetilde{\omega}_{\widetilde{\chi_k}}(\mathscr{S}_1(\widetilde{s}), \mathscr{S}_2(\widetilde{s})) = \mathscr{S}_2(\widetilde{s})$, so $\mathscr{S}_2(\widetilde{s}) = (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$.
- $(\eta \circ \widetilde{\Psi})(\widetilde{s}) = (\Psi_{\widetilde{\chi_k}} \circ \mathscr{S})(\widetilde{s}) = \mathscr{S}_1(\widetilde{s})$, so $\mathscr{S}_1(\widetilde{s}) = \eta(\widetilde{\sigma})$.

Thus the commutativity of (D_{25}) determines entirely the expression of $\mathscr{S}$. Using the product laws in $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, and the fact that η is a group homomorphism, we prove that $\mathscr{S}$ is a group homomorphism, hence the expression (288) of $\mathscr{S}$ provides a commutative diagram (D_{25}). Therefore, the commutativity of (D_{25}) is equivalent to the fact that $\mathscr{S}$ is well-defined. By definition of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, the map $\mathscr{S}$ is well-defined if and only if the operator

$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ belongs to the set $\mu_4(\mathbb{C})\omega_{\widetilde{\chi_k}}(\eta(\widetilde{\sigma}), w_{\eta(\widetilde{\sigma})})$, for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$. By Lemma 2.71, this is equivalent to the condition $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})\gamma(\sigma)^{-1} \in \mu_4(\mathbb{C})$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$. Finally, since $\widetilde{\omega}_{\chi_k} = \widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ if (D_{25}) is commutative, and $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful by Proposition 2.32, we obtain that the kernel of $\mathscr{S}$ is exactly the kernel of $\widetilde{\omega}_{\chi_k}$, which is described in Proposition 2.65. □

Chapter 3

The lowest-dimensional example for $\mathbb{F} = \mathbb{F}_2$, i.e. $\widetilde{W} \simeq (\mathbb{Z}/4\mathbb{Z})^2$ and $W \simeq (\mathbb{F}_2)^2$

In this chapter, we apply all the results of the previous sections in order to obtain a complete and explicit description of the groups involved and the Weil representations when $\widetilde{W}$ is a $\mathbb{Z}/4\mathbb{Z}$-module of dimension two and W is a symplectic $\mathbb{F}_2$-vector space of dimension two. Most of the groups and their presentations are found explicitly thanks to the database "GroupNames" [GN]. The small size of the groups we have to consider in this case allows us to do all the computations by hand, and then we are able to verify with **GAP** some of the results we find. We keep the notations used in the preceding sections with $n = 1$. In particular:

- W is a $\mathbb{F}_2$-vector space of dimension 2 endowed with a nondegenerate and alternating bilinear form $\langle \cdot, \cdot \rangle$. Then W is isomorphic to $(\mathbb{F}_2)^2$ and we can set $W = X \oplus Y$ with $X = \mathrm{Span}\{(1,0)\}$ and $Y = \mathrm{Span}\{(0,1)\}$, i.e $W = \{(0,0),(1,0),(0,1),(1,1)\}$ where $0, 1 \in \mathbb{F}_2$. We will write an element $w \in W$ either as $w = x + y$ with $x \in X$ and $y \in Y$, or as $w = (x, y)$ or $w = \begin{pmatrix} x \\ y \end{pmatrix}$ if we consider that $x, y \in \mathbb{F}_2$.

- $\widetilde{W}$ is a free $\mathbb{Z}/4\mathbb{Z}$-module of rank 2 endowed with a nondegenerate and alternating bilinear form $\langle \cdot, \cdot \rangle_{\widetilde{W}}$. This means that we can choose an ordered basis $\{(1,0),(0,1)\}$ of $\widetilde{W}$ over $\mathbb{Z}/4\mathbb{Z}$ and two isotropic submodules $\widetilde{X}$ generated by $(1,0)$ and $\widetilde{Y}$ generated by $(0,1)$ such that $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$. We will write an element $\widetilde{w} \in \widetilde{W}$ indifferently as $\widetilde{w} = \widetilde{x} + \widetilde{y}$ considering $\widetilde{x} \in \widetilde{X}$, $\widetilde{y} \in \widetilde{Y}$, or as $\widetilde{w} = (\widetilde{x}, \widetilde{y})$ or $\widetilde{w} = \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix}$ if we consider that $\widetilde{x}, \widetilde{y} \in \mathbb{Z}/4\mathbb{Z}$. We identify $\widetilde{W}/2\widetilde{W} = \widetilde{X}/2\widetilde{X} \oplus \widetilde{Y}/2\widetilde{Y}$ with the $\mathbb{F}_2$-vector space $W = X \oplus Y$ defined in the previous point.

With respect to the fixed complete polarizations $W = X \oplus Y$ and $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$, the nondegenerate symplectic forms are given by

$$\langle x_1 + y_1, x_2 + y_2 \rangle = x_1 y_2 - y_1 x_2 \quad \text{and} \quad \langle \widetilde{x}_1 + \widetilde{y}_1, \widetilde{x}_2 + \widetilde{y}_2 \rangle_{\widetilde{W}} = \widetilde{x}_1 \widetilde{y}_2 - \widetilde{y}_1 \widetilde{x}_2, \tag{289}$$

for all $w_i = x_i + y_i \in W$ and $\widetilde{w}_i = \widetilde{x}_i + \widetilde{y}_i \in \widetilde{W}$. With respect to the ordered basis $\{(1,0),(0,1)\}$ fixed above, the matrices of these two bilinear forms are $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\widetilde{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

respectively. Since W is a two-dimensional vector space over $\mathbb{F}_2$, and hence the space Y is one-dimensional, $\dim_{\mathbb{C}}(\mathrm{L}^2(Y)) = 2$. Therefore all the operators we obtain in this section are two-by-two matrices.

For the three different constructions of the Weil representations we made in chapters 1 and 2, the first step is always to determine the Heisenberg group: $\mathrm{H}(B)$, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ or $\mathrm{H}(\widetilde{W})$. For the first two groups, this amounts to determine all the possible bilinear forms B and B_W such that for all $w_i = x_i + y_i \in W$ and $\widetilde{w}_i = \widetilde{x}_i + \widetilde{y}_i \in \widetilde{W}$, $i \in \{1, 2\}$,

$$\langle x_1{+}y_1, x_2{+}y_2\rangle = B(w_1, w_2){-}B(w_2, w_1) \text{ and } \langle x_1{+}y_1, x_2{+}y_2\rangle_W = B_W(w_1, w_2){-}B_W(w_2, w_1). \tag{290}$$

Recall that $B = \kappa \circ B_W$, where $\kappa : 2\mathbb{Z}/4\mathbb{Z} \to \mathbb{F}_2$ is the abelian group isomorphism which is defined in Lemma 2.1 and maps 0 to 0 and 2 to 1. Thus it suffices to determine all the possible bilinear forms B. In addition, recall that by Propositions 1.6 and 2.3, $\mathrm{H}(B)$ is an extraspecial 2-group and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is an almost extraspecial 2-group. Therefore, it is enough to find two bilinear forms B whose associated quadratic forms Q have different Witt index/Arf invariant in order to study the two possible different Heisenberg groups $\mathrm{H}(B)$, and these two bilinear forms will lead to the same group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. In terms of matrices, we are looking for $B = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{pmatrix}$ such that for all $x_1 + y_1 \in W$ and $x_2 + y_2 \in W$,

$$\begin{pmatrix} x_1 & y_1 \end{pmatrix} B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_2 & y_2 \end{pmatrix} B \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \Omega \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

This is equivalent to have $\beta_{1,2}(x_1y_2{-}y_1x_2){-}\beta_{2,1}(x_1y_2{-}y_1x_2) = x_1y_2{-}y_1x_2$, i.e. $\beta_{1,2}{-}\beta_{2,1} = 1$. This leads to eight possibilities for B:

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_6 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, B_7 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, B_8 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

These bilinear forms are given in coordinates by:

- $B_1(w_1, w_2) = x_1y_2$, i.e. $B_1 = B_{\text{Weil}}$
- $B_2(w_1, w_2) = x_1y_2 + x_1x_2$
- $B_3(w_1, w_2) = x_1y_2 + y_1y_2$
- $B_4(w_1, w_2) = x_1y_2 + x_1x_2 + y_1y_2$
- $B_5(w_1, w_2) = y_1x_2$
- $B_6(w_1, w_2) = y_1x_2 + x_1x_2$
- $B_7(w_1, w_2) = y_1x_2 + y_1y_2$
- $B_8(w_1, w_2) = y_1x_2 + x_1x_2 + y_1y_2$.

Thus the corresponding quadratic forms are written as follows:

- $Q_1(w) = xy$
- $Q_2(w) = xy + x^2$
- $Q_3(w) = xy + y^2$
- $Q_4(w) = xy + x^2 + y^2$
- $Q_5(w) = xy$
- $Q_6(w) = xy + x^2$
- $Q_7(w) = xy + y^2$
- $Q_8(w) = xy + x^2 + y^2$.

In particular, the quadratic forms of Witt index 1 (Arf invariant 0) are $Q_1 = Q_5$, $Q_2 = Q_6$ and $Q_3 = Q_7$, whereas the Witt index of $Q_4 = Q_8$ is 0 (Arf invariant 1). Therefore, in order to study all the possibilities for the Heisenberg groups $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, it is enough to consider for example $B_1 = B_{\mathrm{Weil}}$ and B_4.

We need to fix some additional notations. For $t \in \mathbb{F}_2$, we will denote $\kappa^{-1}(t)$ by $2t$, and define $t^2 \in \mathbb{Z}/4\mathbb{Z}$ by $t^2 = 0$ if $t = 0$ and $t^2 = 1$ if $t = 1$. In this way,

$$2t = 2\widetilde{t} \quad \text{and} \quad t^2 = \widetilde{t}^2 \quad \text{if } t = \widetilde{t} + 2\mathbb{Z}/4\mathbb{Z}. \tag{291}$$

We will also identify every $\mathbb{Z}/4\mathbb{Z}$-module homomorphism $\widetilde{\sigma} : \widetilde{W} \to \widetilde{W}$ with its matrix $\begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix}$ in the ordered basis $\widetilde{\mathscr{B}} = \{(1,0),(0,1)\}$ of $\widetilde{W}$, every linear map $\sigma : W \to W$ with its matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in the ordered basis $\mathscr{B} = \{(1,0),(0,1)\}$ of W and, similarly, every linear map $T : \mathrm{L}^2(Y) \to \mathrm{L}^2(Y)$ with its matrix in the basis $\mathscr{B}_Y = \{\delta_0, \delta_1\}$ of $\mathrm{L}^2(Y)$.

Before separating the two cases depending on the Witt index/Arf invariant of Q, we give some details about the additive maps and the symplectic groups involved in this example.

Proposition 3.1. ***i.*** *The set $\mathscr{Q}_a(W)$ of additive quadratic forms on W is given by the set of linear forms $(x,y) \mapsto 0$, $(x,y) \mapsto x$, $(x,y) \mapsto y$ and $(x,y) \mapsto x + y$.*

ii. *The set $\mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$ consists of the following maps: $(x,y) \mapsto 0$, $(x,y) \mapsto 2x$, $(x,y) \mapsto 2y$ and $(x,y) \mapsto 2x + 2y$.*

iii. *The symplectic group of W is*

$$\mathrm{Sp}(W) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

i.e. $\mathrm{Sp}(W) = \mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$*, and in particular*

$$\mathrm{Sp}(W) \simeq \mathrm{S}_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle,$$

where for example $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

iv. *The symplectic group of $\widetilde{W}$ is*

$$\mathrm{Sp}(\widetilde{W}) = \left\{ \begin{array}{l} \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&0\\2&1\end{pmatrix}, \pm\begin{pmatrix}1&2\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&2\\2&1\end{pmatrix}, \\ \pm\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}2&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}0&-1\\1&2\end{pmatrix}, \pm\begin{pmatrix}2&-1\\1&2\end{pmatrix}, \\ \pm\begin{pmatrix}0&1\\-1&1\end{pmatrix}, \pm\begin{pmatrix}2&1\\1&1\end{pmatrix}, \pm\begin{pmatrix}0&-1\\1&1\end{pmatrix}, \pm\begin{pmatrix}2&-1\\-1&1\end{pmatrix}, \\ \pm\begin{pmatrix}1&0\\1&1\end{pmatrix}, \pm\begin{pmatrix}-1&0\\1&-1\end{pmatrix}, \pm\begin{pmatrix}1&2\\1&-1\end{pmatrix}, \pm\begin{pmatrix}1&2\\-1&-1\end{pmatrix}, \\ \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&-1\\2&-1\end{pmatrix}, \pm\begin{pmatrix}-1&1\\0&-1\end{pmatrix}, \pm\begin{pmatrix}1&1\\2&-1\end{pmatrix}, \\ \pm\begin{pmatrix}1&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}1&1\\-1&0\end{pmatrix}, \pm\begin{pmatrix}1&1\\1&2\end{pmatrix}, \pm\begin{pmatrix}1&-1\\-1&2\end{pmatrix}, \end{array} \right\}$$

where the image under the epimorphism π (defined in Proposition 2.7.iii) of each element of the j-th row in the list is the j-th element of $\mathrm{Sp}(W)$ from the list in iii. In particular, $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is isomorphic to

$$\mathrm{A}_4 \rtimes \mathrm{C}_4 = \langle a, b, c, d \,|a^2 = b^2 = c^3 = d^4 = 1, cac^{-1} = dad^{-1} = ab = ba, cbc^{-1} = a, bd = db, \\ dcd^{-1} = c^{-1}\rangle,$$

an isomorphism being given by $a = \begin{pmatrix}-1&0\\2&-1\end{pmatrix}$, $b = \begin{pmatrix}-1&2\\0&-1\end{pmatrix}$, $c = \begin{pmatrix}2&1\\1&1\end{pmatrix}$ and $d = \begin{pmatrix}1&1\\2&-1\end{pmatrix}$.

Proof. **i.** The additive quadratic forms on W with values in $\mathbb{F}_2$ are just the linear maps $f : W \to \mathbb{F}_2$. Such a map f is entirely determined by $f(1,0)$ and $f(0,1)$. There are two possibilities for each of them, so this gives exactly four possibilities for f, which we can write as $f : (x,y) \mapsto xf(1,0) + yf(0,1)$.

ii. An additive map from W to $\mathbb{Z}/4\mathbb{Z}$ has to satisfy $2f(w) = f(2w) = f(0) = 0$, i.e. $f(w) \in 2\mathbb{Z}/4\mathbb{Z}$. Since f is entirely determined by $f(1,0) \in 2\mathbb{Z}/4\mathbb{Z}$ and $f(0,1) \in 2\mathbb{Z}/4\mathbb{Z}$, this gives four possibilities for f, which we can write as $f : (x,y) \mapsto xf(1,0)+yf(0,1)$.

iii. $\sigma = \begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$ is symplectic if and only if $\sigma^T \begin{pmatrix}0&1\\-1&0\end{pmatrix} \sigma = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$. This condition is equivalent to $\alpha\delta - \gamma\beta = 1$, i.e. $\sigma \in \mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$.

iv. $\widetilde{\sigma} = \begin{pmatrix}\widetilde{\alpha}&\widetilde{\beta}\\\widetilde{\gamma}&\widetilde{\delta}\end{pmatrix}$ is symplectic if and only if $\widetilde{\sigma}^T \begin{pmatrix}0&1\\-1&0\end{pmatrix} \widetilde{\sigma} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$. This condition is equivalent to $\widetilde{\alpha}\widetilde{\delta} - \widetilde{\beta}\widetilde{\gamma} = 1$, i.e. $\widetilde{\sigma} \in \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$.

□

3.1 The case $\mathrm{Arf}(Q) = 0$

In this section, we suppose that the bilinear form B which satisfies (290) is such that Q has Arf invariant 0, or equivalently Witt index 1. In particular, we can consider the bilinear form $B = B_{\mathrm{Weil}}$ which appears in Weil's article [Wei64], i.e.

$$B(w_1, w_2) = x_1 y_2 \quad \text{for all } w_1 = x_1 + y_1 \text{ and } w_2 = x_2 + y_2 \text{ in } W.$$

Hence the matrix of B in the ordered basis $\{(1,0),(0,1)\}$ of W is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $Q(w) = xy$ for all $w = x + y \in W$. In this case, $\mathrm{H}(B)$ is the set of elements $(x, y, t) \in (\mathbb{F}_2)^3$ with product

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 y_2). \tag{292}$$

In particular, $\mathrm{H}(B)$ is isomorphic to the dihedral group

$$\mathrm{D}_4 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle,$$

where for example $a = (1, 1, 0)$ and $b = (1, 0, 0)$.

We need to fix an extension $\widetilde{\chi}$ of the unique nontrivial character χ of $\mathbb{F}_2$ to the abelian subgroup $X \times \mathbb{F}_2$ of $\mathrm{H}(B)$. This extension is entirely determined by $\widetilde{\chi}(1,0,0)$, which can take the values ± 1 because X is Q-singular (cf. Lemma 1.15). Set

$$\widetilde{\varepsilon} = \widetilde{\chi}(1,0,0) \in \{\pm 1\}. \tag{293}$$

3.1.1 The pseudosymplectic group $\mathrm{Ps}(B)$ and its two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$

The first groups we determine explicitly are the orthogonal group $\mathrm{O}(Q)$ and the pseudosymplectic group $\mathrm{Ps}(B)$. Then, in order to obtain all the possible two-fold coverings

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\tilde{\chi}}} \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\Psi_{\tilde{\chi}}} \mathrm{Ps}(B) \longrightarrow 1$$

defined in Proposition 1.36.ii, we consider all the possible choices of $\{\gamma(\sigma), \sigma \in \mathrm{O}(Q)\}$ satisfying (46) and (48). We will prove in (294) that, up to isomorphism, there are only two different pseudosymplectic coverings.

Proposition 3.2. ***i.*** *The orthogonal group is* $\mathrm{O}(Q) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. *If we set* $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *then* $\mathrm{O}(Q) = \{1, \tau\} =< \tau >$.

ii. *The pseudosymplectic group is*

$$\mathrm{Ps}(B) = \left\{ \begin{array}{l} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, x + y \right), \\ \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + x \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + y \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + x + y \right) \end{array} \right\}.$$

We will denote these elements in the same order by $\{s_1, s_2, s_3, s_4, r_1, r_2, r_3, r_4\}$. In particular, $\mathrm{Ps}(B)$ is isomorphic to the dihedral group of order eight

$$\mathrm{D}_4 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle,$$

an isomorphism being for example given by $a = r_2$ and $b = s_2$.

iii. *The different choices of $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying* (46) *and* (48) *lead to two non-isomorphic pseudosymplectic extensions*

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\tilde{\chi}}} \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\Psi_{\tilde{\chi}}} \mathrm{Ps}(B) \longrightarrow 1.$$

More precisely,

- *If $\theta(\tau) \equiv 0 \mod \pi$ then the pseudosymplectic covering*

$$\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)),\ s \in \mathrm{Ps}(B),\ l \in \mathbb{Z}\}$$

is given by:

$$\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} = \left\{ \begin{array}{l} \left(s_1, (-1)^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(s_2, (-1)^l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \left(s_3, (-1)^l \widetilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \\ \left(s_4, (-1)^l \widetilde{\varepsilon} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \left(r_1, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & 1 \\ 1 & -\widetilde{\varepsilon} \end{pmatrix}\right), \left(r_2, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix}\right), \\ \left(r_3, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\varepsilon} \\ -\widetilde{\varepsilon} & 1 \end{pmatrix}\right), \left(r_4, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix}\right) \end{array} \right\}$$

where $\widetilde{\varepsilon} = \tilde{\chi}(1,0,0)$ is as in (293)*. In particular, $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ is isomorphic to the dihedral group*

$$\mathrm{D}_8 = \langle a, b \mid a^8 = b^2 = 1, bab = a^{-1} \rangle,$$

an isomorphism being given for example by $a = \left(r_2, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix}\right)$ and $b = \left(s_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$.

- *If $\theta(\tau) \equiv \frac{\pi}{2} \mod \pi$ then the pseudosymplectic covering*

$$\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)),\ s \in \mathrm{Ps}(B),\ l \in \mathbb{Z}\}$$

is given by:

$$\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} = \left\{ \begin{array}{l} \left(s_1, (-1)^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(s_2, (-1)^l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \left(s_3, (-1)^l \widetilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \\ \left(s_4, (-1)^l \widetilde{\varepsilon} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \left(r_1, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & 1 \\ 1 & -\widetilde{\varepsilon} \end{pmatrix}\right), \left(r_2, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix}\right), \\ \left(r_3, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\varepsilon} \\ -\widetilde{\varepsilon} & 1 \end{pmatrix}\right), \left(r_4, \frac{(-1)^l i}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix}\right) \end{array} \right\}$$

where $\widetilde{\varepsilon} = \widetilde{\chi}(1,0,0)$ *is as in* (293). *In particular,* $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ *is isomorphic to the semidihedral group*

$$\mathrm{SD}_{16} = \langle\, a, b \mid a^8 = b^2 = 1, bab = a^3 \rangle,$$

an isomorphism being given for example by $a = \left(r_2, \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix}\right)$ *and* $b = \left(s_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$.

Proof. **i.** σ is orthogonal if and only if $\begin{pmatrix} x & y \end{pmatrix} \sigma^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ for all $x, y \in \mathbb{F}_2$.

ii. According to Proposition 1.12, the elements of $\mathrm{Ps}(B)$ are given by $(\sigma, f_\sigma + f)$, where $\sigma \in \mathrm{O}(Q)$ and $f \in \mathscr{Q}_a(W)$. This gives $f_1 = 0$ and $f_\tau(x, y) = xy$. Proposition 3.1 then provides all the eight elements of $\mathrm{Ps}(B)$ listed above.

iii. First we make a choice of w_s satisfying (37) for all $s = (\sigma, f) \in \mathrm{Ps}(B)$. If $s = s_i$ $(1 \leqslant i \leqslant 4)$ then $\sigma = 1$ and $\mathrm{K}(\sigma) = W$. Hence there is only one choice of w_s satisfying $f_{|\,\mathrm{K}(\sigma)} = f = \langle w_s, \cdot \rangle$, namely $w_{s_1} = 0$, $w_{s_2} = (0,1)$, $w_{s_3} = (1,0)$ and $w_{s_4} = (1,1)$. If $s = r_i$ then $\sigma = \tau$ and $\mathrm{K}(\sigma) = \{0, (1,1)\}$. Hence there are two choices of w_s for each s, namely $w_{r_1} \in \{(1,0),(0,1)\}$, $w_{r_2} \in \{0,(1,1)\}$, $w_{r_3} \in \{0,(1,1)\}$ and $w_{r_4} \in \{(1,0),(0,1)\}$. Since the isomorphism class of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ (and the set $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$) does not depend on the choice of $\{w_s,\ s \in \mathrm{Ps}(B)\}$, we can choose for example $w_{r_1} = w_{r_4} = (1,0)$ and $w_{r_2} = w_{r_3} = 0$. This enables us to compute the operators $\omega_{\widetilde{\chi}}(s)$ for all $s \in \mathrm{Ps}(B)$ thanks to Proposition 1.28:

$$\omega_{\widetilde{\chi}}(s) = \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\widetilde{\chi}}(u + w_s),$$

where $w \in W$ is any element such that $u = (1 - \sigma)(w)$ and $\gamma(\sigma)$ satisfies (46) and (48). This formula together with (28) with $t = 0$ gives:

- $\omega_{\widetilde{\chi}}(s_1) = \rho_{\widetilde{\chi}}(w_{s_1}) = \rho_{\widetilde{\chi}}(0) = I_2$
- $\omega_{\widetilde{\chi}}(s_2) = \rho_{\widetilde{\chi}}(w_{s_2}) = \rho_{\widetilde{\chi}}(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\omega_{\widetilde{\chi}}(s_3) = \rho_{\widetilde{\chi}}(w_{s_3}) = \rho_{\widetilde{\chi}}(1,0) = \begin{pmatrix} \widetilde{\varepsilon} & 0 \\ 0 & -\widetilde{\varepsilon} \end{pmatrix}$
- $\omega_{\widetilde{\chi}}(s_4) = \rho_{\widetilde{\chi}}(w_{s_4}) = \rho_{\widetilde{\chi}}(1,1) = \begin{pmatrix} 0 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 0 \end{pmatrix}$
- $\omega_{\widetilde{\chi}}(r_1) = \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\widetilde{\chi}}(1,0) + \rho_{\widetilde{\chi}}(0,1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & 1 \\ 1 & -\widetilde{\varepsilon} \end{pmatrix}$
- $\omega_{\widetilde{\chi}}(r_2) = \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\widetilde{\chi}}(0) + \rho_{\widetilde{\chi}}(1,1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix}$

- $\omega_{\tilde{\chi}}(r_3) = \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\tilde{\chi}}(0) - \rho_{\tilde{\chi}}(1,1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\varepsilon} \\ -\widetilde{\varepsilon} & 1 \end{pmatrix}$
- $\omega_{\tilde{\chi}}(r_4) = \frac{e^{i\theta(\tau)}}{\sqrt{2}}(\rho_{\tilde{\chi}}(1,0) - \rho_{\tilde{\chi}}(0,1)) = \frac{e^{i\theta(\tau)}}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix}$.

Moreover, since $\tau^2 = 1$, (48) implies that

$$\theta(\tau) + \theta(\tau) - \theta(1) \equiv 0 \mod \pi.$$

Since $\theta(1) \equiv 0 \mod \pi$ by (48), we obtain that $\theta(\tau) \equiv 0 \mod \frac{\pi}{2}$. Hence there are four possibilities for $e^{i\theta(\tau)}$, but these choices lead to at most two non-isomorphic extensions by Proposition 1.36.v, corresponding to the choices $\theta(\tau) \equiv 0 \mod \pi$ and $\theta(\tau) \equiv \frac{\pi}{2} \mod \pi$, respectively. Then Proposition 1.36 leads to the list of all elements of $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ given in the statement above. Finally, we want to identify the two groups $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ defined in this way in the database [GN]. We know by ii. that $\mathrm{Ps}(B)$ is isomorphic to D_4, and we know by Proposition 1.36.ii that $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ is a non-split stem extension of $\mathrm{Ps}(B)$ by two elements. According to [GN], there are only three possibilities for $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$, namely the groups D_8, SD_{16} and Q_{16}. These groups have different numbers of elements of order two. Since we have already computed all the elements of the groups $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$, we can easily obtain the orders of these elements and then deduce the following group isomorphisms:

$$\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \begin{cases} \mathrm{D}_8 & \text{if } \theta(\tau) \equiv 0 \mod \pi \\ \mathrm{SD}_{16} & \text{if } \theta(\tau) \equiv \frac{\pi}{2} \mod \pi. \end{cases} \tag{294}$$

In particular, this shows that the two pseudosymplectic coverings we obtain are not isomorphic.

□

Remark 3.3 (Related to Remark 1.37)**.** The result obtained in (294) proves that in general, we cannot obtain all the non-split stem central extensions of $\mathrm{Ps}(B)$ by two elements with all possible choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ satisfying (46) and (48). In fact, the number of central extensions of $\mathrm{Ps}(B) \simeq \mathrm{D}_4$ by two elements can be found with the universal coefficient theorem, which provides the following short split exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{D}_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(\mathrm{D}_4, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{D}_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \to 0.$$

The Schur multiplier of D_4 is $\mathrm{H}_2(\mathrm{D}_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ (see [Kar87, p. 278]). Moreover, $\mathrm{H}_1(\mathrm{D}_4, \mathbb{Z}) \simeq \mathrm{D}_4/[\mathrm{D}_4, \mathrm{D}_4] \simeq (\mathbb{Z}/2\mathbb{Z})^2$, and thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{D}_4, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}((\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2$. Hence $|\mathrm{H}^2(\mathrm{D}_4, \mathbb{Z}/2\mathbb{Z})| = 8$, which means that there are eight non-isomorphic central extensions of $\mathrm{Ps}(B) \simeq \mathrm{D}_4$ by two elements. The database [GN] gives six possible groups associated to these extensions, three of them being associated to non-split stem extensions: D_8, SD_{16} and Q_{16}. However, by (294), there is no choice of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$ which enables us to obtain a group $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ which is isomorphic to the generalised quaternionic group Q_{16}. Nevertheless, by Corollary 1.38, $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ contains a copy of the Heisenberg group $\mathrm{H}(B)$, which is isomorphic to D_4 in this example, and Q_{16} does not contain D_4. Therefore, we obtain in fact all the

possible groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ associated to non-split stem extensions of $\mathrm{Ps}(B)$ by two elements which contain the Heisenberg group $\mathrm{H}(B)$. Moreover, (294) shows also that if (50) defines a coboundary, then the condition (51) is satisfied in this case (cf. Proposition 1.36.v).

Remark 3.4. We give a short proof of the fact that the extension defined by the pseudosymplectic group splits when $\dim_{\mathbb{F}}(W) = 2$ and $\mathbb{F} = \mathbb{F}_2$; see Proposition 1.11. Recall that $\mathrm{Ps}(B)$ defines a short exact sequence:

$$0 \to \mathscr{Q}_a(W) \xrightarrow{\Phi} \mathrm{Ps}(B) \xrightarrow{\Psi} \mathrm{O}(Q) \to 1 \tag{295}$$

where $\Phi : f_z = \langle z, \cdot \rangle \mapsto (1, f_z)$ and $\Psi : (\sigma, f) \mapsto \sigma$. Recall also that we defined a section $\sigma \mapsto f_\sigma$ of Ψ in Proposition 1.12. If we consider the ordered basis $\mathscr{B} = \{(1,0),(0,1)\}$ of W, then for all $\sigma \in \mathrm{O}(Q)$ and $(x,y) \in W$, by Proposition 3.2,

$$f_\sigma(x,y) = \begin{cases} 0 & \text{if} \quad \sigma = 1 \\ \\ xy & \text{if} \quad \sigma = \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{cases}$$

Hence

$$\begin{aligned} (f_1 + f_1 \circ 1)(x,y) &= 0 = f_{1\cdot 1} \\ (f_1 + f_\tau \circ 1)(x,y) &= xy = f_{\tau\cdot 1} \\ (f_\tau + f_1 \circ \sigma_1)(x,y) &= xy = f_{1\cdot\tau} \\ (f_\tau + f_\tau \circ \tau)(x,y) &= xy + yx = 0 = f_{\tau^2} = f_1. \end{aligned}$$

This shows that $\sigma \mapsto f_\sigma$ is a group homomorphism, which implies that the extension (295) splits. This proof can also be easily adapted to the case of a $\mathbb{F}$-vector space of dimension 2, where $\mathbb{F}$ is an arbitrary finite field of characteristic two.

3.1.2 The affine symplectic group $\mathrm{ASp}(B_W)$ and its four-fold covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$

We recall the expression of $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ and choose $B_{\widetilde{W}}$ so that it gives $B = \kappa \circ B_W = B_{\mathrm{Weil}}$:

$$\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} = \langle (\widetilde{x}_1, \widetilde{y}_1), (\widetilde{x}_2, \widetilde{y}_2) \rangle_{\widetilde{W}} = \widetilde{x}_1\widetilde{y}_2 - \widetilde{y}_1\widetilde{x}_2 \quad \text{and} \quad B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) = \langle \widetilde{x}_1, \widetilde{y}_2 \rangle_{\widetilde{W}} = \widetilde{x}_1\widetilde{y}_2, \tag{296}$$

for all $\widetilde{w}_1 = \widetilde{x}_1 + \widetilde{y}_1$ and $\widetilde{w}_2 = \widetilde{x}_2 + \widetilde{y}_2$ in $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$. In this case, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is the set of elements $(x,y,t) \in \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}/4\mathbb{Z}$ with the product

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2x_1y_2). \tag{297}$$

In particular, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is isomorphic to the Pauli group

$$\mathrm{C}_4 \circ \mathrm{D}_4 = \langle a, b, c \mid a^4 = c^2 = 1, b^2 = a^2, ab = ba, ac = ca, cbc = a^2b \rangle,$$

where for example $a = (0,0,1)$, $b = (1,1,0)$ and $c = (1,0,0)$.

We need to fix a faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ ($k = 1$ or $k = 3$ by Lemma 2.8) and an extension $\widetilde{\chi_k}$ of χ_k to $X \times \mathbb{Z}/4\mathbb{Z}$. This extension is entirely determined by $\widetilde{\chi_k}(1,0,0)$, which can take the values ± 1 because X is Q_W-singular (cf. Lemma 2.8). According to Corollary 2.28, the smallest non-split central extension of $\mathrm{ASp}(B_W)$ that we can construct to lift the projective Weil representation is an extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ by four elements, if we choose $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ such that (112) and (116) are satisfied, the latter condition being equivalent to (117) because W is two-dimensional. By Proposition 2.39, the central extensions associated to $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ are related, in the sense of the commutativity of the diagram (D_{22}), provided (192) is satisfied. Since we will be able to satisfy this condition, we will not consider in this section the extension $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$, whose construction was done in Proposition 2.29 in order to find a link between the two extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}_{\widetilde{\chi_k}}$, only in the case where there is no such link when $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. By (164), if we want to relate $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ using Proposition 2.39, we need $\widetilde{\chi_k}$ to coincide with $\widetilde{\chi}$ on $X \times \{0\}$. This is why we set

$$\widetilde{\chi_k}(1,0,0) = \widetilde{\chi}(1,0,0) = \widetilde{\varepsilon}, \tag{298}$$

as in (293). Now we can describe explicitly the group $\mathrm{ASp}(B_W)$ and the group homomorphism $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ defined in Proposition 2.7. Moreover, we can examine all the possible choices of $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ such that (112) and (117) are satisfied, in order to describe explicitly all the four-fold coverings

$$1 \longrightarrow \mu_4(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(B_W) \longrightarrow 1,$$

that we can obtain in this way by Proposition 2.29. Then we are able to decide in section 3.1.3 which of these four-fold coverings can be related to the pseudosymplectic covering, in the sense of the commutativity of the diagram (D_{22}).

Proposition 3.5. *Recall the notation introduced in* (291) *and the transvection of W defined in Proposition 2.7.iv by $\sigma_v : w \mapsto w + \langle w, v\rangle v$.*

i. *The group homomorphism $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ defined in Proposition 2.7 maps $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ listed in Proposition 3.1 to $\eta(\widetilde{\sigma}) \in \mathrm{ASp}(B_W)$ listed in the same position in the following set:*

$$\widetilde{\sigma} \mapsto (\sigma, f_{\widetilde{\sigma}})$$

$$\left\{\begin{array}{l}(1,0),(1,2x),(1,2y),(1,2x+2y),\\ \left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy+2x\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy+2y\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy+2x+2y\right),\\ \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},y^2+2xy\right),\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},y^2+2xy+2x\right),\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},y^2+2xy+2y\right),\\ \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},y^2+2xy+2x+2y\right)\\ \left(\begin{pmatrix}1&0\\1&1\end{pmatrix},x^2\right),\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},x^2+2x\right),\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},x^2+2y\right),\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},x^2+2x+2y\right),\\ \left(\begin{pmatrix}1&1\\0&1\end{pmatrix},y^2\right),\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},y^2+2x\right),\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},y^2+2y\right),\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},y^2+2x+2y\right),\\ \left(\begin{pmatrix}1&1\\1&0\end{pmatrix},x^2+2xy\right),\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},x^2+2xy+2x\right),\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},x^2+2xy+2y\right),\\ \left(\begin{pmatrix}1&1\\1&0\end{pmatrix},x^2+2xy+2x+2y\right)\end{array}\right\}.$$

ii. *The affine symplectic group* $\mathrm{ASp}(B_W)$ *is precisely the image of* η *given above. In particular,* $\mathrm{ASp}(B_W)$ *is isomorphic to*

$$\mathrm{S}_4 = \langle a,b,c,d \mid a^2=b^2=c^3=d^2=1,\, cac^{-1}=dad=ab=ba,\, cbc^{-1}=a,\, bd=db,\, dcd=c^{-1}\rangle,$$

an isomorphism being for example given by

$$a=(1,2x),\quad b=(1,2y),\quad c=\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},y^2+2xy\right)\quad \textit{and}\quad d=\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},y^2+2x+2y\right).$$

iii. *The choices of* $\{\delta(\sigma)=2^{-\frac{\dim \mathrm{I}(\sigma)}{2}}e^{i\nu(\sigma)},\ \sigma\in\mathrm{Sp}(W)\}$ *satisfying* (112) *and* (117) *lead to two non-isomorphic affine metaplectic coverings*

$$1\longrightarrow \mu_4(\mathbb{C})\xrightarrow{\Phi_{\widetilde{\chi_k}}}\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}\xrightarrow{\Psi_{\widetilde{\chi_k}}}\mathrm{ASp}(B_W)\longrightarrow 1.$$

More precisely,

- *If* $\nu(\sigma_{(1,0)})\equiv 0 \mod \frac{\pi}{2}$ *then the affine metaplectic covering is*

$$\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}=\{(s,i^l\omega_{\widetilde{\chi_k}}(s)),\ s\in\mathrm{ASp}(B_W),\ l\in\mathbb{Z}\},$$

where the set of operators $i^l\omega_{\widetilde{\chi_k}}(s)$ *is as follows (in the same order as the elements of* $\mathrm{ASp}(B_W)$ *above and where* $\widetilde{\varepsilon}$ *is as in* (298)*):*

$$\left\{\begin{array}{l} i^l\begin{pmatrix}1&0\\0&1\end{pmatrix}, i^l\begin{pmatrix}0&1\\1&0\end{pmatrix}, i^l\widetilde{\varepsilon}\begin{pmatrix}1&0\\0&-1\end{pmatrix}, i^l\widetilde{\varepsilon}\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \\ \frac{i^l}{\sqrt{2}}\begin{pmatrix}\widetilde{\varepsilon}&1\\1&-\widetilde{\varepsilon}\end{pmatrix}, \frac{i^l}{\sqrt{2}}\begin{pmatrix}1&\widetilde{\varepsilon}\\-\widetilde{\varepsilon}&1\end{pmatrix}, \frac{i^l}{\sqrt{2}}\begin{pmatrix}1&-\widetilde{\varepsilon}\\\widetilde{\varepsilon}&1\end{pmatrix}, \frac{i^l}{\sqrt{2}}\begin{pmatrix}-\widetilde{\varepsilon}&1\\1&\widetilde{\varepsilon}\end{pmatrix}, \\ \frac{i^l}{2}\begin{pmatrix}1-\widetilde{\varepsilon}\chi_k(1)&-\widetilde{\varepsilon}-\chi_k(1)\\\widetilde{\varepsilon}-\chi_k(1)&1+\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \frac{i^l}{2}\begin{pmatrix}1-\widetilde{\varepsilon}\chi_k(1)&\widetilde{\varepsilon}+\chi_k(1)\\-\widetilde{\varepsilon}+\chi_k(1)&1+\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \frac{i^l}{2}\begin{pmatrix}1+\widetilde{\varepsilon}\chi_k(1)&-\widetilde{\varepsilon}+\chi_k(1)\\\widetilde{\varepsilon}+\chi_k(1)&1-\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \\ \frac{i^l}{2}\begin{pmatrix}1+\widetilde{\varepsilon}\chi_k(1)&\widetilde{\varepsilon}-\chi_k(1)\\-\widetilde{\varepsilon}-\chi_k(1)&1-\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \\ \frac{i^l}{\sqrt{2}}\begin{pmatrix}1&\chi_k(1)\\\chi_k(1)&1\end{pmatrix}, \frac{i^l}{\sqrt{2}}\begin{pmatrix}\chi_k(1)&1\\1&\chi_k(1)\end{pmatrix}, \frac{i^l\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix}1&-\chi_k(1)\\\chi_k(1)&-1\end{pmatrix}, \frac{i^l\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix}1&\chi_k(1)\\-\chi_k(1)&-1\end{pmatrix}, \\ \frac{i^l}{\sqrt{2}}\begin{pmatrix}1-\widetilde{\varepsilon}\chi_k(1)&0\\0&1+\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \frac{i^l}{\sqrt{2}}\begin{pmatrix}0&1-\widetilde{\varepsilon}\chi_k(1)\\1+\widetilde{\varepsilon}\chi_k(1)&0\end{pmatrix}, \frac{i^l}{\sqrt{2}}\begin{pmatrix}1+\widetilde{\varepsilon}\chi_k(1)&0\\0&1-\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \\ \frac{i^l}{\sqrt{2}}\begin{pmatrix}0&1+\widetilde{\varepsilon}\chi_k(1)\\1-\widetilde{\varepsilon}\chi_k(1)&0\end{pmatrix}, \\ \frac{i^l}{2}\begin{pmatrix}1+\widetilde{\varepsilon}\chi_k(1)&\widetilde{\varepsilon}+\chi_k(1)\\-\widetilde{\varepsilon}+\chi_k(1)&1-\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \frac{i^l}{2}\begin{pmatrix}1-\widetilde{\varepsilon}\chi_k(1)&\widetilde{\varepsilon}-\chi_k(1)\\-\widetilde{\varepsilon}-\chi_k(1)&1+\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \frac{i^l}{2}\begin{pmatrix}1-\widetilde{\varepsilon}\chi_k(1)&-\widetilde{\varepsilon}+\chi_k(1)\\\widetilde{\varepsilon}+\chi_k(1)&1+\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \\ \frac{i^l}{2}\begin{pmatrix}1+\widetilde{\varepsilon}\chi_k(1)&-\widetilde{\varepsilon}-\chi_k(1)\\\widetilde{\varepsilon}-\chi_k(1)&1-\widetilde{\varepsilon}\chi_k(1)\end{pmatrix}, \end{array}\right\}.$$

Moreover, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *is isomorphic to*

$$\mathrm{C}_4 \cdot_6 \mathrm{S}_4 = \langle a,b,c,d,e \,|a^4 = d^3 = e^2 = 1, b^2 = c^2 = a^2, ab = ba, ac = ca, ad = da, ae = ea, \\ cbc^{-1} = a^2b, dbd^{-1} = a^2bc, ebe = bc, dcd^{-1} = b, ece = a^2c, ede = d^{-1}\rangle,$$

an isomorphism being given for example by:

$$a = \left((1,0), i\begin{pmatrix}1&0\\0&1\end{pmatrix}\right), \quad b = \left((1,2x), i\begin{pmatrix}0&1\\1&0\end{pmatrix}\right), \quad c = \left((1,2y), i\widetilde{\varepsilon}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),$$

$$d = \left(\left(\begin{pmatrix}0&1\\1&1\end{pmatrix}, y^2+2xy+2y\delta_1(k)\right), \frac{1}{2}\begin{pmatrix}-1-\widetilde{\varepsilon}i&\widetilde{\varepsilon}-i\\-\widetilde{\varepsilon}-i&-1+\widetilde{\varepsilon}i\end{pmatrix}\right),$$

$$e = \left(\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, y^2+2x+2y\delta_3(k)\right), \frac{1}{\sqrt{2}}\begin{pmatrix}0&1-\widetilde{\varepsilon}i\\1+\widetilde{\varepsilon}i&0\end{pmatrix}\right).$$

In particular, $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ *and* $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$ *are isomorphic groups.*

- *If* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ *then the affine metaplectic covering is* $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} = \{(s, i^l\omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}$*, where the set of operators* $i^l\omega_{\widetilde{\chi_k}}(s)$ *is exactly as the previous one, only replacing every factor* $\frac{i^l}{\sqrt{2}}$ *with* $\frac{(1-i)i^l}{2}$ *in the lines corresponding to the second, fourth and fifth line of the list of elements of* $\mathrm{ASp}(B_W)$*. Moreover,* $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *is isomorphic to*

$$\mathrm{U}_2(\mathbb{F}_3) = \langle a,b,c,d,e \,|a^4 = d^3 = 1, b^2 = c^2 = a^2, e^2 = a, ab = ba, ac = ca, ad = da, ae = ea, \\ cbc^{-1} = a^2b, dbd^{-1} = a^2bc, ebe^{-1} = bc, dcd^{-1} = b, ece^{-1} = a^2c, \\ ede^{-1} = d^{-1}\rangle,$$

an isomorphism being given for example by:

$$a = \left((1,0), i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right), \quad b = \left((1,2x), i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right), \quad c = \left((1,2y), i\widetilde{\varepsilon}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right),$$

$$d = \left(\left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, y^2 + 2xy + 2y\delta_1(k)\right), \frac{1}{2}\begin{pmatrix}-1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i\\ -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i\end{pmatrix}\right),$$

$$e = \left(\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, y^2 + 2x + 2y\delta_3(k)\right), \frac{1+i}{2}\begin{pmatrix}0 & 1-\widetilde{\varepsilon}i\\ 1+\widetilde{\varepsilon}i & 0\end{pmatrix}\right).$$

In particular, $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ *and* $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$ *are isomorphic groups.*

Remark 3.6. The fact that $\mathrm{ASp}(B_W)$ is isomorphic to S_4 implies that the extension defined in Proposition 2.7.ix splits, since S_4 is a semidirect product of C_2^2 and S_3.

Proof. **i.** We write the details only for the element $\eta\left(\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\right) = \left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, y^2 + 2xy\right)$, the other ones being obtained in a similar way. Essentially we use the definition of $f_{\widetilde{\sigma}}$ given by (93) in Proposition 2.7. For all $\widetilde{w}_1 = (\widetilde{x}_1, \widetilde{y}_1)$ and $\widetilde{w}_2 = (\widetilde{x}_2, \widetilde{y}_2)$ in $\widetilde{W}$,

$$\begin{aligned}\beta_{\widetilde{\sigma}}(\widetilde{w}_1, \widetilde{w}_2) &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) - B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)\\ &= B_{\widetilde{W}}((\widetilde{y}_1, -\widetilde{x}_1 + \widetilde{y}_1), (\widetilde{y}_2, -\widetilde{x}_2 + \widetilde{y}_2)) - \widetilde{x}_1\widetilde{y}_2\\ &= \widetilde{y}_1(\widetilde{y}_2 - \widetilde{x}_2) - \widetilde{x}_1\widetilde{y}_2.\end{aligned}$$

Hence, for all $w = x + y = \widetilde{w} + 2\widetilde{W} = \widetilde{x} + 2\widetilde{X} + \widetilde{y} + 2\widetilde{Y}$, by (291),

$$f_{\widetilde{\sigma}}(w) = \beta_{\widetilde{\sigma}}(\widetilde{w}, \widetilde{w}) = \widetilde{y}^2 + 2\widetilde{x}\widetilde{y} = y^2 + 2xy.$$

ii. According to Proposition 2.7, $\mathrm{ASp}(B_W)$ is the set of elements $(\sigma, f_{\widetilde{\sigma}} + f)$, where $\sigma \in \mathrm{Sp}(W)$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Proposition 3.1 and the computations of $f_{\widetilde{\sigma}}$ done above for η give precisely the image of η for the set $\mathrm{ASp}(B_W)$.

iii. Before computing the operators $\omega_{\widetilde{\chi_k}}(s, w_s)$ for all $s \in \mathrm{ASp}(B_W)$ with the formula (105), we study the different possibilities for the choices of $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (117). In fact, since we are interested in obtaining all the possible non-isomorphic non-split extensions $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, by Proposition 2.29.v, there is only a finite number of choices of $\delta(\sigma)$ to study. Recall the notation σ_v for the transvection defined in Proposition 2.7.iv. Then the symplectic group is given by

$$\mathrm{Sp}(W) = \left\{\sigma_{(0,0)} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \sigma_{(1,1)} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_{(1,0)}\sigma_{(0,1)} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, \sigma_{(0,1)} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\right.$$
$$\left.\sigma_{(1,0)} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \sigma_{(0,1)}\sigma_{(1,0)} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\right\}.$$

(We have listed the elements σ of $\mathrm{Sp}(W)$ in the order in which they appear as $(\sigma, \cdot)$ in i.) By (117), $\nu(\sigma_{(0,0)}) \equiv 0 \mod \frac{\pi}{2}$. Moreover, $\sigma_{(1,0)}^2 = \sigma_{(0,1)}^2 = \sigma_{(1,1)}^2 = 1$, so by (117), since $\nu(1) \equiv 0 \mod \frac{\pi}{2}$,

$$\nu(\sigma_v) \equiv 0 \mod \frac{\pi}{4} \quad \text{for every } v \neq 0. \tag{299}$$

Observe now that $\sigma_{(1,0)}\sigma_{(0,1)} = \sigma_{(0,1)}\sigma_{(1,1)} = \sigma_{(1,1)}\sigma_{(1,0)}$. By (117), this implies that

$$\nu(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv \begin{cases} \nu(\sigma_{(1,0)}) + \nu(\sigma_{(0,1)}) & \mod \frac{\pi}{2} \\ \nu(\sigma_{(0,1)}) + \nu(\sigma_{(1,1)}) & \mod \frac{\pi}{2} \\ \nu(\sigma_{(1,1)}) + \nu(\sigma_{(1,0)}) & \mod \frac{\pi}{2} \end{cases}$$

Hence

$$\nu(\sigma_{(1,0)}) \equiv \nu(\sigma_{(0,1)}) \equiv \nu(\sigma_{(1,1)}) \mod \frac{\pi}{2}. \tag{300}$$

Now by (299) and (300), there are two possibilities:

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then $\nu(\sigma_{(0,1)}) \equiv \nu(\sigma_{(1,1)}) \equiv 0 \mod \frac{\pi}{2}$. By (117), we also have $\nu(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \frac{\pi}{2}$ and $\nu(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. Finally, this proves that
$$\nu(\sigma) \equiv 0 \mod \frac{\pi}{2} \qquad \text{for all } \sigma \in \mathrm{Sp}(W). \tag{301}$$
In particular, by Proposition 2.29.v, any other choice $\{\delta'(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu'(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112),(117) and such that $\nu'(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, will lead to an extension which is isomorphic to this one.
- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ then $\nu(\sigma_{(0,1)}) \equiv \nu(\sigma_{(1,1)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. By (117), we have $\nu(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \frac{\pi}{2}$ and $\nu(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. In particular, by Proposition 2.29.v, any other choice $\{\delta'(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu'(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112),(117) and such that $\nu'(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, will lead to an extension which is isomorphic to this one.

This shows that, up to isomorphism, the different choices $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (117) lead to at most two affine metaplectic extensions. At this stage, we can in fact already affirm that these two extensions are non-isomorphic if we apply Proposition 2.39 (we give the details of this reasoning in the proof of Proposition 3.9 below).

Now we compute explicitly the operators $i^l\omega_{\widetilde{\chi_k}}(s)$ for $s \in \mathrm{ASp}(B_W)$. We provide the details of the computations only for the operators $i^l\omega_{\widetilde{\chi_k}}(s)$ for

$$s = (\sigma_{(1,0)}\sigma_{(0,1)}, f) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y^2 + 2xy \right),$$

the other ones being obtained in a similar way. First we make a choice of w_s satisfying (100) (the isomorphism class of the extension defined by $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ does not depend on this choice according to Proposition 2.29). Since $1 - \sigma_{(1,0)}\sigma_{(0,1)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, we see that $\mathrm{K}(\sigma_{(1,0)}\sigma_{(0,1)}) = \{0\}$ and $\mathrm{I}(\sigma_{(1,0)}\sigma_{(0,1)}) = W$. Hence every $w \in W$ can be chosen for w_s. We choose $w_s = 0$ to simplify the computations. This enables us to compute the operator $\omega_{\widetilde{\chi_k}}(s, 0)$ by (105):

$$\omega_{\widetilde{\chi_k}}(s, 0) = \delta(\sigma_{(1,0)}\sigma_{(0,1)}) \sum_{u \in \mathrm{I}(\sigma_{(1,0)}\sigma_{(0,1)})} \chi_k(f(w) + B_W(u, w))\rho_{\widetilde{\chi_k}}(u),$$

where $w \in W$ is any element such that $u = (1 - \sigma_{(1,0)}\sigma_{(0,1)})(w)$. Since we proved that $\nu(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \frac{\pi}{2}$ in any case, this formula gives

$$\begin{aligned}
&\omega_{\widetilde{\chi_k}}(s,0) \\
&= \frac{e^{i\nu(\sigma_{(1,0)}\sigma_{(0,1)})}}{2}\left(\rho_{\widetilde{\chi_k}}(0) + \chi_k(f(0,1) + B_W((1,0),(0,1)))\rho_{\widetilde{\chi_k}}(1,0)\right. \\
&\quad \left. + \chi_k(f(1,1) + B_W((0,1),(1,1)))\rho_{\widetilde{\chi_k}}(0,1) + \chi_k(f(1,0) + B_W((1,1),(1,0)))\rho_{\widetilde{\chi_k}}(1,1)\right) \\
&= \frac{1}{2}\left(\rho_{\widetilde{\chi_k}}(0) - \chi_k(1)\rho_{\widetilde{\chi_k}}(1,0) - \chi_k(1)\rho_{\widetilde{\chi_k}}(0,1) + \rho_{\widetilde{\chi_k}}(1,1)\right) \\
&= \frac{1}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \chi_k(1)\begin{pmatrix}\widetilde{\varepsilon} & 0\\ 0 & -\widetilde{\varepsilon}\end{pmatrix} - \chi_k(1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -\widetilde{\varepsilon}\\ \widetilde{\varepsilon} & 0\end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix}1 - \widetilde{\varepsilon}\chi_k(1) & -\widetilde{\varepsilon} - \chi_k(1)\\ \widetilde{\varepsilon} - \chi_k(1) & 1 + \widetilde{\varepsilon}\chi_k(1)\end{pmatrix}.
\end{aligned} \tag{302}$$

This allows us to establish the two lists of elements of the groups $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ given in the statement. As mentioned above, the extensions associated to the choices $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ and $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ are not isomorphic. In addition, even the groups $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ associated to these two extensions are not isomorphic, since they do not have the same number of elements of order two (this is an observation based directly on the lists of elements of these groups). Now to identify explicitly these two groups, we use again the database [GN]. There are three groups corresponding to non-split central extensions of $\mathrm{ASp}(B_W) \simeq \mathrm{S}_4$ by four elements, namely $\mathrm{A}_4 \rtimes \mathrm{C}_8$, $\mathrm{U}_2(\mathbb{F}_3)$ and $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$. If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ has 19 elements of order two, which implies immediately that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$. If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, the situation is less obvious since $\mathrm{A}_4 \rtimes \mathrm{C}_8$ and $\mathrm{U}_2(\mathbb{F}_3)$ both have 7 elements of order two. However, we know by Corollary 2.31 that $\mathrm{AMp}(B_W)$ contains a copy of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, which is isomorphic to the Pauli group $\mathrm{C}_4 \circ \mathrm{D}_4$ in this example. Since $\mathrm{U}_2(\mathbb{F}_3)$ is the only group containing $\mathrm{C}_4 \circ \mathrm{D}_4$, we can conclude that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3)$. Finally, we obtain

$$\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \begin{cases} \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2} \\ \mathrm{U}_2(\mathbb{F}_3) & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2} \end{cases}. \tag{303}$$

□

Remark 3.7 (Related to Remark 2.30)**.** The result obtained in (303) proves that in general, we cannot obtain all the non-split central extensions of $\mathrm{ASp}(B_W)$ by four elements with all possible choices of $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$. In fact, $\mathrm{ASp}(B_W)$ is isomorphic to S_4 and the universal coefficient theorem provides $|\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z})| = 4$ (the computation is done in Remark 3.8). The database [GN] gives exactly four possible associated groups, three of them being associated to non-split extensions: $\mathrm{A}_4 \rtimes \mathrm{C}_8$, $\mathrm{U}_2(\mathbb{F}_3)$ and $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$. However, by Proposition 3.5.iii, there is no choice of $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ which enables to obtain a group $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ which is isomorphic to $\mathrm{A}_4 \rtimes \mathrm{C}_8$. Nevertheless, by Corollary 2.31, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ contains a copy of the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, which is isomorphic to the Pauli group $\mathrm{C}_4 \circ \mathrm{D}_4$ in this example, and $\mathrm{A}_4 \rtimes \mathrm{C}_8$ does not contain $\mathrm{C}_4 \circ \mathrm{D}_4$. Therefore, we obtain in fact all the possible non-split

extensions of $\mathrm{ASp}(B_W)$ by four elements which contain the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Moreover, (303) shows also that if (143) defines a coboundary, then the condition (144) is satisfied in this case (cf. Proposition 2.29.v).

Remark 3.8 (Related to Remarks 2.22 and 2.36)**.** Proposition 3.5.iii shows that for a given choice of $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$, the groups $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$ are isomorphic, since they are both isomorphic to $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$ if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, or to $\mathrm{U}_2(\mathbb{F}_3)$ if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. In view of Proposition 2.35, we can wonder if the extensions defined by these groups are isomorphic. Since $\mathrm{ASp}(B_W) \simeq \mathrm{S}_4$ by Proposition 3.5.ii, we start by computing $\mathrm{H}^2(\mathrm{ASp}(B_W), \mathbb{Z}/4\mathbb{Z}) = \mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z})$. Applying the universal coefficient theorem, we obtain the following short split exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{S}_4, \mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) \to \mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{S}_4, \mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) \to 0.$$

By [Kar87, p. 279], the Schur multiplier of S_4 is $\mathrm{H}_2(\mathrm{S}_4, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Moreover, $\mathrm{H}_1(\mathrm{S}_4, \mathbb{Z}) \simeq \mathrm{S}_4/[\mathrm{S}_4, \mathrm{S}_4] \simeq \mathbb{Z}/2\mathbb{Z}$, and thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{S}_4, \mathbb{Z}), \mathbb{Z}/4\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. This implies that

$$\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{304}$$

In particular, there are four non-isomorphic central extensions of S_4 by $\mu_4(\mathbb{C})$. The database [GN] provides exactly four central extensions $1 \to \mu_4(\mathbb{C}) \to G \to \mathrm{S}_4 \to 1$, with a different group G for each of them. Therefore, the extensions defined by $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$, with the choices $\nu(\sigma)$ and $\nu'(\sigma)$, respectively, are isomorphic in this two-dimensional example if and only if $\nu(\sigma) + \nu'(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for all $\widetilde{\sigma} \in \mathrm{Sp}(W)$ (and in fact it is enough to verify this condition for $\sigma_{(1,0)}$). Nevertheless, this result raises a new question. In fact, since $\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by (304), and c_{χ_k} is not a coboundary by Corollary 2.26, then the cohomology class of c_{χ_k} in $\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z})$ has order two, which means that $c^2_{\chi_k}$ is a coboundary (for $k \in \{1, 3\}$ and every choice of $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$). This implies that we are able to prove the existence of a two-fold covering of $\mathrm{ASp}(B_W)$, using the same method as in Proposition 2.62.ii: we know that c_{χ_k} takes its values in $\mu_4(\mathbb{C})$ and $c^2_{\chi_k}$ is a coboundary, so we can "renormalize" the operators $\omega_{\widetilde{\chi_k}}(s, w_s)$ for all $s \in \mathrm{ASp}(B_W)$ in order to obtain a new associated cocycle c'_{χ_k} which takes its values in $\mu_2(\mathbb{C})$. At first reading, this may seem to be in contradiction with the fact that we proved in Corollary 2.28 and Proposition 2.29 that the smallest possible extension we can construct by means of c_{χ_k} is a four-fold covering. However, what we prove in this way is that considering the parameter $\delta(\sigma)$ as depending only on $\sigma \in \mathrm{Sp}(W)$ is not sufficient to obtain the best possible result. In fact, we would need to use the parameters $\delta(s, w_s)$ to construct a two-fold covering of $\mathrm{ASp}(B_W)$ and lift the projective Weil representation to a linear Weil representation (cf. Remarks 2.22 and 2.36). Unlike the situation where the parameters $\delta(\sigma)$ depend only on $\sigma \in \mathrm{Sp}(W)$, at present, we do not have necessary and sufficient conditions ensuring that the cocycles c_{χ_k} defined using general parameters $\delta(s, w_s)$ take values in $\mu_2(\mathbb{C})$. Nevertheless, if $\widetilde{\mathrm{G}}$ is a two-fold covering of $\mathrm{ASp}(B_W)$, obtained for example with the cocycle c'_{χ_k} defined above, then we can prove as in Proposition 1.36.ii that $\widetilde{\mathrm{G}}$ is a stem extension. Hence by [GN], $\widetilde{\mathrm{G}}$ is isomorphic to the conformal special unitary (or binary octahedral) group $\mathrm{CSU}_2(\mathbb{F}_3)$ or to the general linear group $\mathrm{GL}_2(\mathbb{F}_3)$, and we can exhibit two different choices of $\delta(s, w_s)$ that lead to these two-fold coverings.

- Consider the following choices for the two sets $\{w_s$ satisfying (100), $s \in \mathrm{ASp}(B_W)\}$ and $\{\delta(s, w_s)$ s.t. $|\gamma(s, w_s)|^2 = 2^{-\dim \mathrm{I}(\sigma)}$, $s \in \mathrm{ASp}(B_W)\}$, where the elements are given in the same order as the elements of $\mathrm{ASp}(B_W)$ in Proposition 3.2 above:

$$w_s \in \left\{ \begin{array}{l} (0,0),(0,1),(1,0),(1,1), \\ (0,1),(0,0),(0,0),(0,1), \\ (0,0),(0,0),(0,0),(0,0), \\ (0,0),(0,1),(1,0),(1,0), \\ (0,0),(0,1),(0,0),(1,1), \\ (0,0),(0,0),(0,0),(0,0) \end{array} \right\} \quad \text{and} \quad \delta(s, w_s) \in \left\{ \begin{array}{l} 1, i, i, 1, \\ \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \\ \frac{1}{2}, \frac{i}{2}, \frac{1}{2}, \frac{i}{2}, \\ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \\ \frac{1}{2}, \frac{1}{2}, \frac{i}{2}, \frac{i}{2} \end{array} \right\}. \tag{305}$$

  Then $\widetilde{\mathrm{G}} = \{(-1)^l \omega_{\widetilde{\chi_k}}(s, w_s),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}$ is a two-fold covering of $\mathrm{ASp}(B_W)$ which is isomorphic to

$$\mathrm{CSU}_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = 1, b^2 = d^2 = a^2, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = ab, \\ dad^{-1} = a^2 b, cbc^{-1} = a, dcd^{-1} = c^{-1} \rangle,$$

  an isomorphism being given for example by

$$a = \left((1, 2x), i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),\ b = \left((1, 2y), i\widetilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y^2 + 2xy + 2y\delta_1(k) \right), \frac{1}{2} \begin{pmatrix} -1 - \widetilde{\varepsilon} i & \widetilde{\varepsilon} - i \\ -\widetilde{\varepsilon} - i & -1 + \widetilde{\varepsilon} i \end{pmatrix} \right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2xy + 2x + 2y \right), \frac{-i}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix} \right).$$

- Consider the following choices for the two sets $\{w_s$ satisfying (100), $s \in \mathrm{ASp}(B_W)\}$ and $\{\delta(s, w_s)$ s.t. $|\gamma(s, w_s)|^2 = 2^{-\dim \mathrm{I}(\sigma)}$, $s \in \mathrm{ASp}(B_W)\}$, where the elements are given in the same order as the elements of $\mathrm{ASp}(B_W)$ in Proposition 3.2 above:

$$w_s \in \left\{ \begin{array}{l} (0,0),(0,1),(1,0),(1,1), \\ (0,1),(0,0),(0,0),(0,1), \\ (0,0),(0,0),(0,0),(0,0), \\ (0,0),(0,1),(1,0),(1,0), \\ (0,0),(0,1),(0,0),(1,1), \\ (0,0),(0,0),(0,0),(0,0) \end{array} \right\} \quad \text{and} \quad \delta(s, w_s) \in \left\{ \begin{array}{l} 1, i, i, 1, \\ \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \\ \frac{1}{2}, \frac{i}{2}, \frac{1}{2}, \frac{i}{2}, \\ \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \\ \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \\ \frac{1}{2}, \frac{1}{2}, \frac{i}{2}, \frac{i}{2} \end{array} \right\}. \tag{306}$$

Then $\widetilde{\mathrm{G}} = \{(-1)^l \omega_{\widetilde{\chi_k}}(s, w_s),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}$ is a two-fold covering of $\mathrm{ASp}(B_W)$ which is isomorphic to

$$\mathrm{GL}_2(\mathbb{F}_3) = \langle a, b, c, d \,|a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab, \\ dad = a^2 b, cbc^{-1} = a, dcd = c^{-1} \rangle,$$

an isomorphism being given for example by

$$a = \left((1, 2x), i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),\ b = \left((1, 2y), i\widetilde{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y^2 + 2xy + 2y\delta_1(k)\right), \frac{1}{2}\begin{pmatrix} -1 - \widetilde{\varepsilon}i & \widetilde{\varepsilon} - i \\ -\widetilde{\varepsilon} - i & -1 + \widetilde{\varepsilon}i \end{pmatrix}\right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2xy + 2x + 2y\right), \frac{1}{\sqrt{2}}\begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix}\right).$$

3.1.3 Relation between $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$

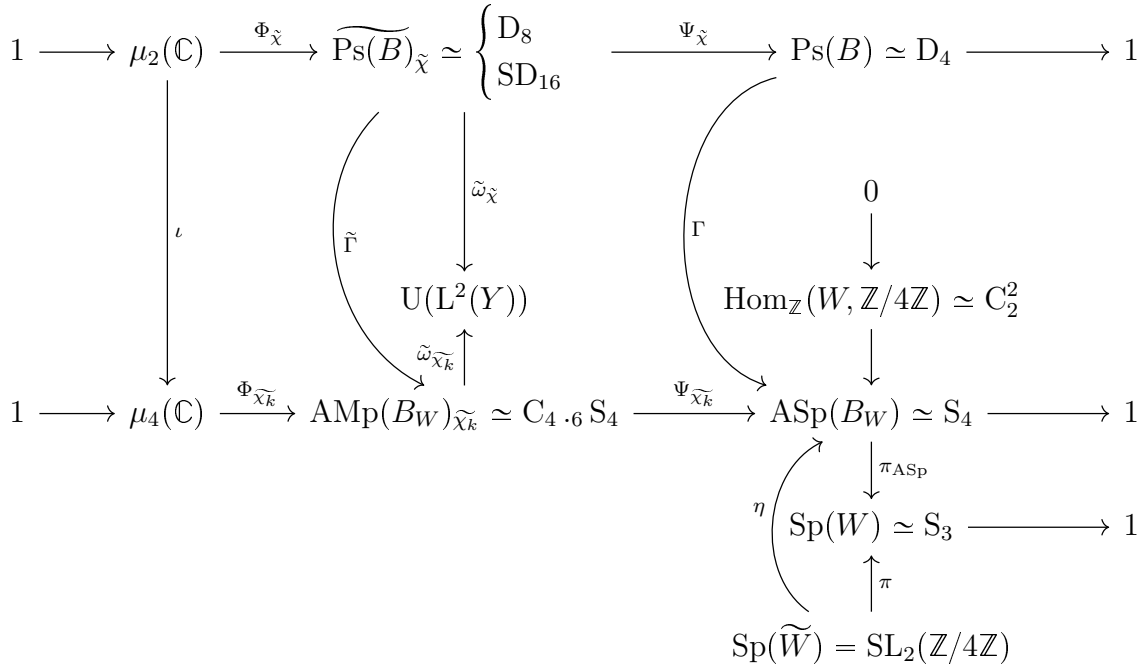
We say that the pseudosymplectic covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and the affine metaplectic covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ are compatible provided the diagram (D_{22}) commutes. Below, we use the results of sections 3.1.1 and 3.1.2 and Proposition 2.39 to see when it happens.

Proposition 3.9. *We fix choices $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying* (46) *and* (48) *to construct $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, and $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying* (112) *and* (117) *to construct $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Then the corresponding pseudosymplectic and affine metaplectic coverings are compatible if and only if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, i.e. if $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$.*

Proof. Observe that the element $\sigma_{(1,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which we have denoted by τ in Proposition 3.2, belongs to $\mathrm{O}(Q) \subseteq \mathrm{Sp}(W)$. By Proposition 2.39, the affine metaplectic covering and the pseudosymplectic covering are compatible if and only if (192) is satisfied. In this example, this means that $\nu(\sigma) \equiv \theta(\sigma) \mod \frac{\pi}{2}$ for $\sigma = 1$ and $\sigma = \sigma_{(1,1)} = \tau$. For $\sigma = 1$, this condition is always satisfied since $\theta(1) \equiv 0 \mod \pi$ by (48) and $\nu(1) \equiv 0 \mod \frac{\pi}{2}$ by (117). However, for $\sigma = \sigma_{(1,1)} = \tau$, since $\theta(\tau) \equiv 0 \mod \frac{\pi}{2}$ by Proposition 3.2.iii, this condition is satisfied if and only if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ (see the proof of Proposition 3.5.iii), which concludes the proof. □

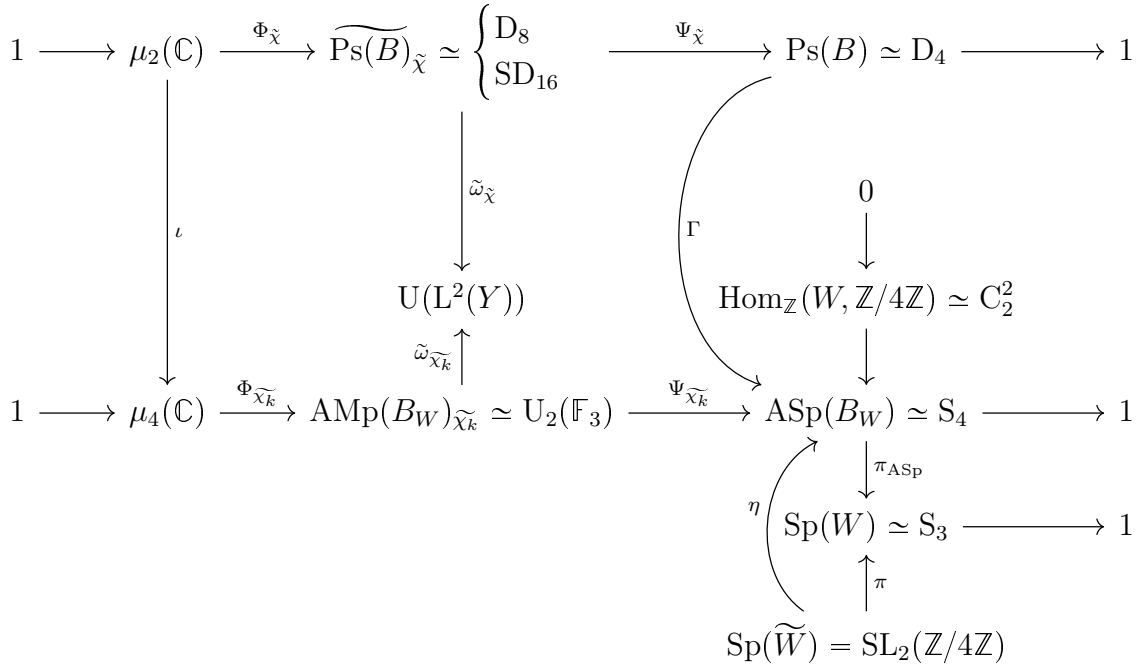
Combining all the results of sections 3.1.1, 3.1.2 and 3.1.3 proved so far with Proposition 3.9, we obtain two diagrams, depending on the choices made for $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying (46) and (48), and $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (117).

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then, regardless of the choice made for $\gamma(\tau)$, the diagram (D_{22}) commutes. It is given in this case by



(D_{26})

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ then, regardless of the choice of $\gamma(\tau)$, there is no group homomorphism $\widetilde{\Gamma}$ such that the diagram (D_{22}) commutes, i.e. we have the following diagram:



(D_{27})

Remark 3.10. The group $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is a central extension of $\mathrm{ASp}(B_W)$ by $\mu_4(\mathbb{C})$. We are not sure that the sheaf theoretical construction of Gurevich and Hadani in [GH12, Theorem 1.3] leads to one of the groups that we obtained. If the dimension of W over $\mathbb{F}_2$ is equal to two, then as mentioned in Proposition 3.5, there are only three possibilities for $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ by [GN], namely $\mathrm{A}_4 \rtimes \mathrm{C}_8$, $\mathrm{U}_2(\mathbb{F}_3)$ and $\mathrm{C}_4 \,._6\, \mathrm{S}_4$. Depending on the choice made for $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (117), our construction fixes $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ as isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ or $\mathrm{U}_2(\mathbb{F}_3)$. These groups are the only ones containing the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Moreover, the only possibility which allows to link the pseudosymplectic and the affine metaplectic coverings, in the sense of the commutativity of the diagram (D_{22}), is to choose $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ so that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \,._6\, \mathrm{S}_4$.

Now there are two different paths to follow:

- As in the article [GH12] of Gurevich and Hadani, we can try to define a Weil representation of $\mathrm{Mp}(\widetilde{W})$ as a pullback of the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.

- Since the groups are all explicitly identified, we can also try to define a Weil representation of a group which is "closer" to $\mathrm{Sp}(W)$ than $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ or $\mathrm{Mp}(\widetilde{W})$.

We investigate both directions in the following two sections.

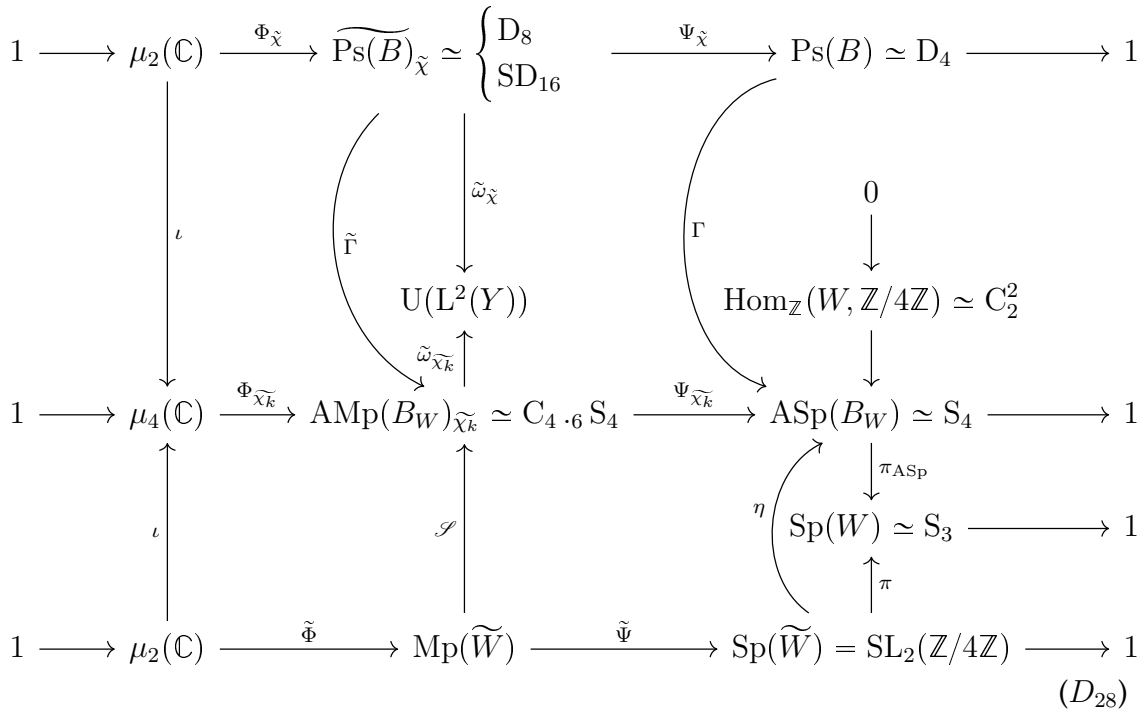
3.1.4 Following Gurevich-Hadani: the metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$

As mentioned in section 2.5, Gurevich and Hadani use the bilinear forms $B_{\widetilde{W}}$ and B_W corresponding to B_{Weil}. They construct a central extension

$$1 \to \mu_2(\mathbb{C}) \to \mathrm{Mp}(\widetilde{W}) \to \mathrm{Sp}(\widetilde{W}) \to 1,$$

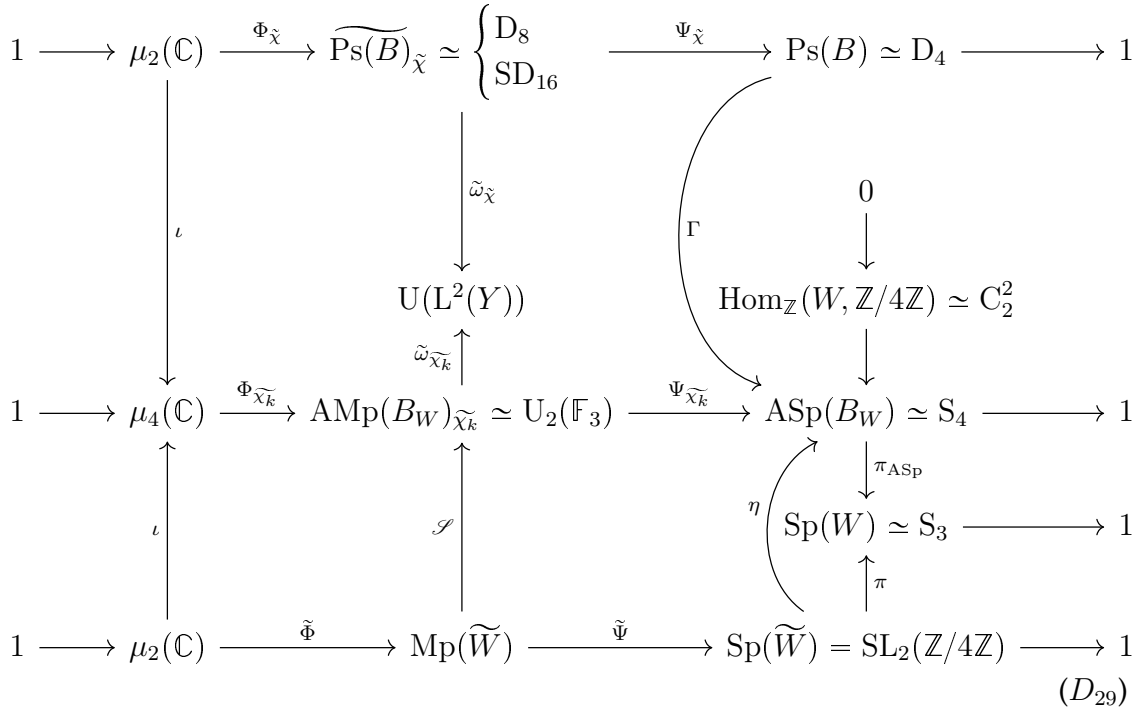
a homomorphism of central extensions $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and then a "splitting" of the Weil representation, i.e. a Weil representation of the metaplectic group $\mathrm{Mp}(\widetilde{W})$. This result appears as Theorem 1.4 in [GH12] and suggests additions to the commutative diagrams (D_{26}) and (D_{27}), as follows:

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then



(D_{28})

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then



(D_{29})

Remark 3.11. The construction of $\mathrm{Mp}(\widetilde{W})$ and the associated homomorphism of central extensions $\mathscr{S}$, starting from the Heisenberg group $\mathrm{H}(\widetilde{W})$ defined in section 2.5, is given

in section 3.3, where we give a specific choice of $\{\widetilde{w}_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and two choices of $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ which provide a cocycle $\widetilde{c}_{\chi_k}$ whose range is $\mu_2(\mathbb{C})$, and two associated two-fold coverings $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{ASp}(B_W)$. However, the results of the present section 3.1.4 were obtained before we made the construction of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ and they show that in this two-dimensional example, we are able to construct a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and its Weil representation, such that the diagram (D_{28}) or (D_{29}) commutes, using only the first two constructions and considerations on abstract presentations of the groups involved. Moreover, the methods used in this section can be reproduced also in section 3.2, whereas the construction of the homomorphism $\mathscr{S}$ of central extensions from section 2.5 is valid only for $B = B_{\mathrm{Weil}}$. Therefore, we first look for the construction of all possible two-fold coverings $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{ASp}(B_W)$ using only the results we have already obtained for this two-dimensional example.

To extend the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ to the metaplectic group $\mathrm{Mp}(\widetilde{W})$, we would expect that $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ stays unitary, irreducible and faithful, like $\widetilde{\omega}_{\widetilde{\chi_k}}$. However, while studying the case of dimension two, we will see that such a homomorphism of central extensions $\mathscr{S}$, satisfying that $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is injective, exists if and only if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. (We recall from Lemma 2.42 that the homomorphism $\mathscr{S}$ cannot be injective if the rank $2n$ of the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ is such that $n \geqslant 2$.)

We now study the expected commutative diagrams (D_{28}) and (D_{29}) above in more details, when the dimension of W over $\mathbb{F}_2$ is two. Recall the presentations of the different groups involved so far:

- $\mathrm{Sp}(W) \simeq \mathrm{S}_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$, where $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{A}_4 \rtimes \mathrm{C}_4 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^4 = 1, cac^{-1} = dad^{-1} = ab = ba, cbc^{-1} = a, bd = db, dcd^{-1} = c^{-1} \rangle$, where $a = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$, $b = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$, $c = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $d = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$.
- $\mathrm{ASp}(B_W) \simeq \mathrm{S}_4 = \langle\, a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = 1, cac^{-1} = dad = ab = ba, cbc^{-1} = a, bd = db, dcd = c^{-1} \rangle$, where $a = (1, 2x)$, $b = (1, 2y)$, $c = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y^2 + 2xy \right)$ and $d = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y^2 + 2x + 2y \right)$.
- $\mathrm{C}_4 \,._6\, \mathrm{S}_4 = \langle a, b, c, d, e \mid a^4 = d^3 = e^2 = 1, b^2 = c^2 = a^2, ab = ba, ac = ca, ad = da, ae = ea, cbc^{-1} = a^2b, dbd^{-1} = a^2bc, ebe = bc, dcd^{-1} = b, ece = a^2c, ede = d^{-1} \rangle$, where

$$a = \left((1,0), i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad b = \left((1, 2x), i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \quad c = \left((1, 2y), i\widetilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y^2 + 2xy + 2y\delta_1(k) \right), \frac{1}{2} \begin{pmatrix} -1 - \widetilde{\varepsilon}i & \widetilde{\varepsilon} - i \\ -\widetilde{\varepsilon} - i & -1 + \widetilde{\varepsilon}i \end{pmatrix} \right),$$

$$e = \left(\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y^2 + 2x + 2y\delta_3(k) \right), \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 - \widetilde{\varepsilon} i \\ 1 + \widetilde{\varepsilon} i & 0 \end{pmatrix} \right).$$

- $\mathrm{U}_2(\mathbb{F}_3) = \langle a, b, c, d, e \mid a^4 = d^3 = 1, b^2 = c^2 = a^2, e^2 = a, ab = ba, ac = ca, ad = da, ae = ea, cbc^{-1} = a^2 b, dbd^{-1} = a^2 bc, ebe^{-1} = bc, dcd^{-1} = b, ece^{-1} = a^2 c, ede^{-1} = d^{-1} \rangle$,

  where

$$a = \left((1,0), i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad b = \left((1, 2x), i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \quad c = \left((1, 2y), i\widetilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right),$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, y^2 + 2xy + 2y\delta_1(k) \right), \frac{1}{2} \begin{pmatrix} -1 - \widetilde{\varepsilon} i & \widetilde{\varepsilon} - i \\ -\widetilde{\varepsilon} - i & -1 + \widetilde{\varepsilon} i \end{pmatrix} \right),$$

$$e = \left(\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y^2 + 2x + 2y\delta_3(k) \right), \frac{1+i}{2} \begin{pmatrix} 0 & 1 - \widetilde{\varepsilon} i \\ 1 + \widetilde{\varepsilon} i & 0 \end{pmatrix} \right).$$

In terms of these generators, recall also the following homomorphisms, which are defined in Propositions 2.7.vi and 2.29:

$$\begin{aligned} \Phi_{\widetilde{\chi_k}} : \mu_4(\mathbb{C}) &\to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}, & \Psi_{\widetilde{\chi_k}} : \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} &\to \mathrm{ASp}(B_W) \\ i &\mapsto a & a &\mapsto 1 \\ & & b &\mapsto a \\ & & c &\mapsto b \\ & & d &\mapsto \begin{cases} ac & \text{if } \chi_k = \chi_1 \\ c & \text{if } \chi_k = \chi_3 \end{cases} \\ & & e &\mapsto \begin{cases} bd & \text{if } \chi_k = \chi_1 \\ d & \text{if } \chi_k = \chi_3 \end{cases} \end{aligned} \tag{307}$$

  and $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$.
  $a \mapsto a$
  $b \mapsto b$
  $c \mapsto ca$
  $d \mapsto d$

So far, we have explicitly constructed everything except the central extension

$$1 \to \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W}) \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1$$

and the homomorphism of central extensions $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. By [Bey86], the Schur multiplier of $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is $\mathrm{H}_2(\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence, applying the universal coefficient theorem, we get the following short split exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{SL}_2, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(\mathrm{SL}_2, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{SL}_2, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \to 0,$$

where $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$. Since $\mathrm{H}_1(\mathrm{SL}_2, \mathbb{Z}) \simeq \mathrm{SL}_2/[\mathrm{SL}_2, \mathrm{SL}_2] \simeq \mathbb{Z}/4\mathbb{Z}$, we deduce that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{SL}_2, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/gcd(2,4)\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$. Hence we obtain

$$\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{308}$$

Thus there are four non-isomorphic central extensions of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ by $\mu_2(\mathbb{C})$. By [GN], each of these four extensions $1 \to \mu_2(\mathbb{C}) \to G \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \to 1$ is associated to a different group G:

$$\begin{array}{ll} \bullet\ \mathrm{C}_2 \times \mathrm{A}_4 \rtimes \mathrm{C}_4 & \bullet\ \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \\ \bullet\ \mathrm{A}_4 \rtimes \mathrm{C}_8 & \bullet\ \mathrm{U}_2(\mathbb{F}_3) \end{array} \tag{309}$$

This implies that $\mathrm{Mp}(\widetilde{W})$ should be isomorphic to one of these four groups.

Observe now that in this two-dimensional example, $\mathrm{Mp}(\widetilde{W})$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ have the same number of elements. Hence, if in the diagrams (D_{28}) and (D_{29}) there exists a homomorphism of central extensions $\mathscr{S}$ which is injective, then automatically $\mathscr{S}$ is also surjective. Thus $\mathscr{S}$ is an isomorphism of central extensions. Recall that our construction leads to $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 ._6 \mathrm{S}_4$ if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3)$ otherwise, i.e. if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. Thus, if we want to find such an injective homomorphism $\mathscr{S}$, then we need to choose $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. In fact, for the alternative choice $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, since $\mathrm{C}_4 ._6 \mathrm{S}_4$ does not appear in the possibilities for $\mathrm{Mp}(\widetilde{W})$ in (309), there is no hope to find such an injective homomorphism $\mathscr{S}$, and therefore also no hope to find a faithful representation of $\mathrm{Mp}(\widetilde{W})$ of the form $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$. Of course, in this case, we can still look for a homomorphism $\mathscr{S}$ which is not injective, but then this provides a non-faithful representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$.

To keep as many properties of the Weil representation as possible while extending it to $\mathrm{Mp}(\widetilde{W})$, we first examine the existence of an injective homomorphism of central extensions $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. As mentioned above, this requires to choose $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ and to set $\mathrm{Mp}(\widetilde{W}) = \mathrm{U}_2(\mathbb{F}_3)$. So, we study the diagram (D_{29}).

Proposition 3.12. *Suppose that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ and set $\mathrm{Mp}(\widetilde{W}) = \mathrm{U}_2(\mathbb{F}_3)$. Then there exists an isomorphism of central extensions $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that the diagram (D_{29}) commutes. For example, we can set $\mathscr{S}$ equal to the identity. In this case, the group homomorphisms $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are of the following form:*

$$\begin{array}{lll} \widetilde{\Phi} : \mu_2(\mathbb{C}) \to \mathrm{Mp}(\widetilde{W}) & \textit{and} & \widetilde{\Psi} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{Sp}(\widetilde{W}) \\ -1 \mapsto a^2 & & a \mapsto d^2 \\ & & b \mapsto a \\ & & c \mapsto b \\ & & d \mapsto \begin{cases} b^{-1}c & \textit{if } \chi_k = \chi_1 \\ ca^{-1} & \textit{if } \chi_k = \chi_3 \end{cases} \\ & & e \mapsto \begin{cases} bd \textit{ or } bd^{-1} & \textit{if } \chi_k = \chi_1 \\ d \textit{ or } d^{-1} & \textit{if } \chi_k = \chi_3. \end{cases} \end{array} \tag{310}$$

Proof. The groups $\mathrm{Mp}(\widetilde{W})$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ are both isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$. So, if we set $\mathscr{S}$ equal to the identity (meaning that each generator a, b, c, d, e of $\mathrm{Mp}(\widetilde{W})$ is associated to the corresponding generator of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$), then $\mathscr{S}$ is a well-defined group isomorphism. Thus the diagram (D_{29}) commutes if and only if $\widetilde{\Phi} = \Phi_{\widetilde{\chi_k}} \circ \iota$ and $\eta \circ \widetilde{\Psi} = \Psi_{\widetilde{\chi_k}}$. We recall that the expressions of $\Phi_{\widetilde{\chi_k}}$, $\Psi_{\widetilde{\chi_k}}$ and η are given by (307). The condition $\widetilde{\Phi} = \Phi_{\widetilde{\chi_k}} \circ \iota$ is equivalent to define $\widetilde{\Phi}$ by $\widetilde{\Phi}(-1) = a^2$. The condition $\eta \circ \widetilde{\Psi} = \Psi_{\widetilde{\chi_k}}$ is equivalent to

$$\begin{cases} (\eta \circ \widetilde{\Psi})(a) &= 1 \\ (\eta \circ \widetilde{\Psi})(b) &= a \\ (\eta \circ \widetilde{\Psi})(c) &= b \\ (\eta \circ \widetilde{\Psi})(d) &= \begin{cases} ac & \text{if } \chi_k = \chi_1 \\ c & \text{if } \chi_k = \chi_3 \end{cases} \\ (\eta \circ \widetilde{\Psi})(e) &= \begin{cases} bd & \text{if } \chi_k = \chi_1 \\ d & \text{if } \chi_k = \chi_3 \end{cases} \end{cases} \overset{\mathrm{Ker}(\eta) = <d^2>}{\Longleftrightarrow} \begin{cases} \widetilde{\Psi}(a) \in \{1, d^2\} \\ \widetilde{\Psi}(b) \in \{a, ad^2\} \\ \widetilde{\Psi}(c) \in \{b, bd^2\} \\ \widetilde{\Psi}(d) \in \begin{cases} \{b^{-1}c, b^{-1}cd^2\} & \text{if } \chi_k = \chi_1 \\ \{ca^{-1}, ca^{-1}d^2\} & \text{if } \chi_k = \chi_3 \end{cases} \\ \widetilde{\Psi}(e) \in \begin{cases} \{bd, bd^{-1}\} & \text{if } \chi_k = \chi_1 \\ \{d, d^{-1}\} & \text{if } \chi_k = \chi_3. \end{cases} \end{cases} \tag{311}$$

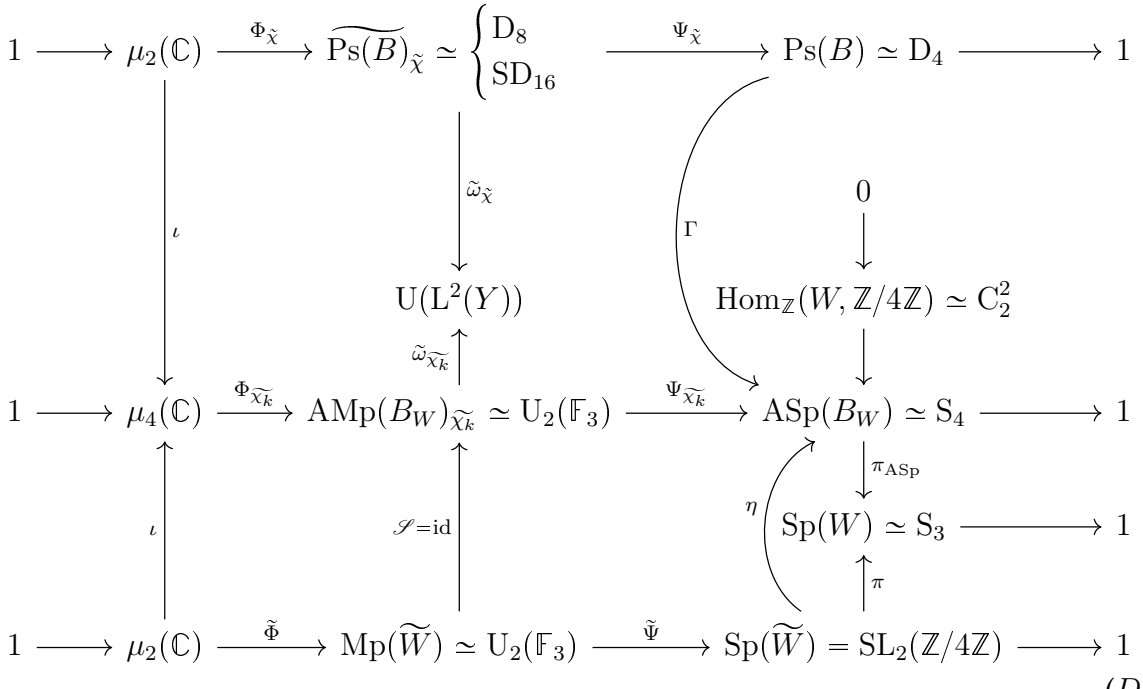
Suppose now that $\widetilde{\Psi}$ satisfies (311). To define an exact sequence, recall that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ have to satisfy that $\widetilde{\Phi}$ is injective (which is the case if we define $\widetilde{\Phi}$ by $\widetilde{\Phi}(-1) = a^2$), $\widetilde{\Psi}$ is surjective and $\mathrm{Ker}(\widetilde{\Psi}) = \mathrm{Im}(\widetilde{\Phi})$. Assuming (311), the last condition is equivalent to $\widetilde{\Psi}(a) = d^2$, since otherwise the kernel of $\widetilde{\Psi}$ would contain at least four elements (because a has order four in $\mathrm{U}_2(\mathbb{F}_3)$). Now $\widetilde{\Psi}$ is a group homomorphism if and only if the images $\widetilde{\Psi}(a) = d^2$, $\widetilde{\Psi}(b)$, $\widetilde{\Psi}(c)$, $\widetilde{\Psi}(d)$ and $\widetilde{\Psi}(e)$ satisfy the relations of the group $\mathrm{U}_2(\mathbb{F}_3)$. We examine these relations with (311).

- The relations $\widetilde{\Psi}(a)^4 = 1$, $\widetilde{\Psi}(b)^2 = \widetilde{\Psi}(c)^2 = \widetilde{\Psi}(a)^2$ and $\widetilde{\Psi}(e)^2 = \widetilde{\Psi}(a)$ are all satisfied because $a^2 = b^2 = d^4 = 1$ in $\mathrm{Sp}(\widetilde{W})$. Moreover, $\widetilde{\Psi}(a) = d^2$ is in the center of $\mathrm{Sp}(\widetilde{W})$.
- $\widetilde{\Psi}(d)^3 = 1$ is equivalent to $\widetilde{\Psi}(d) = \begin{cases} b^{-1}c & \text{if } \chi_k = \chi_1 \\ ca^{-1} & \text{if } \chi_k = \chi_3 \end{cases}$, since $b^{-1}cd^2$ and $ca^{-1}d^2$ have order six in $\mathrm{Sp}(\widetilde{W})$.
- $\widetilde{\Psi}(d)\widetilde{\Psi}(b)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b)\widetilde{\Psi}(c) \overset{d^2 \in \mathrm{Z}(\mathrm{Sp}(\widetilde{W}))}{\Longleftrightarrow} \begin{cases} b^{-1}cac^{-1}b = a\widetilde{\Psi}(c) & \text{if } \chi_k = \chi_1 \\ ca^{-1}aac^{-1} = a\widetilde{\Psi}(c) & \text{if } \chi_k = \chi_3 \end{cases}$
$$\Longleftrightarrow \widetilde{\Psi}(c) = \begin{cases} a^{-1}b^{-1}cac^{-1}b = b & \text{if } \chi_k = \chi_1 \\ a^{-1}ca^{-1}aac^{-1} = b & \text{if } \chi_k = \chi_3. \end{cases}$$
So in any case, $\widetilde{\Psi}(d)\widetilde{\Psi}(b)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b)\widetilde{\Psi}(c)$ is equivalent to $\widetilde{\Psi}(c) = b$.
- $\widetilde{\Psi}(c)\widetilde{\Psi}(b)\widetilde{\Psi}(c)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b) \overset{\widetilde{\Psi}(c)=b}{\Longleftrightarrow} b\widetilde{\Psi}(b)b^{-1} = \widetilde{\Psi}(b)$, which is satisfied because a and ad^2 commute with b.
- $\widetilde{\Psi}(d)\widetilde{\Psi}(c)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(b) \overset{\widetilde{\Psi}(c)=b}{\Longleftrightarrow} \widetilde{\Psi}(b) = \begin{cases} b^{-1}cbc^{-1}b = a & \text{if } \chi_k = \chi_1 \\ ca^{-1}bac^{-1} = a & \text{if } \chi_k = \chi_3. \end{cases}$
So in any case, $\widetilde{\Psi}(d)\widetilde{\Psi}(c)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(b)$ is equivalent to $\widetilde{\Psi}(b) = a$.

- $\widetilde{\Psi}(e)\widetilde{\Psi}(b)\widetilde{\Psi}(e)^{-1} = \widetilde{\Psi}(b)\widetilde{\Psi}(c) \overset{\widetilde{\Psi}(b)=a,\ \widetilde{\Psi}(c)=b}{\Longleftrightarrow} \widetilde{\Psi}(e)a\widetilde{\Psi}(e)^{-1} = ab$. This condition is satisfied because $bdad^{-1}b^{-1} = bbab = ab$ and $dad^{-1} = ab$. The condition $\widetilde{\Psi}(e)\widetilde{\Psi}(c)\widetilde{\Psi}(e)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(c)$ is equivalent to $\widetilde{\Psi}(e)b\widetilde{\Psi}(e)^{-1} = b$, which is also satisfied because b and d commute in $\mathrm{Sp}(\widetilde{W})$. Finally, the condition $\widetilde{\Psi}(e)\widetilde{\Psi}(d)\widetilde{\Psi}(e)^{-1} = \widetilde{\Psi}(d)^{-1}$ is satisfied because $bdb^{-1}cd^{-1}b^{-1} = dcd^{-1}b^{-1} = c^{-1}b^{-1} = (b^{-1}c)^{-1}$ and $dca^{-1}d^{-1} = c^{-1}da^{-1}d^{-1} = c^{-1}ca^{-1}c^{-1} = ac^{-1}$.

This shows that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are group homomorphisms such that the diagram (D_{29}) commutes if and only if their expressions are exactly as in (310). Finally, it only remains to prove that $\widetilde{\Psi}$ is surjective. This is the case because $a = \widetilde{\Psi}(b)$, $b = \widetilde{\Psi}(c)$, $c = \begin{cases} \widetilde{\Psi}(cd) & \text{if } \chi_k = \chi_1 \\ \widetilde{\Psi}(db) & \text{if } \chi_k = \chi_3 \end{cases}$ and $d = \begin{cases} \widetilde{\Psi}(c^{-1}e) \text{ or } \widetilde{\Psi}(ac^{-1}e) & \text{if } \chi_k = \chi_1 \\ \widetilde{\Psi}(e) \text{ or } \widetilde{\Psi}(ae) & \text{if } \chi_k = \chi_3 \end{cases}$. This concludes the proof. □

Proposition 3.12 shows that if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then there exists an isomorphism of central extensions $\mathscr{S}$ such that (D_{29}) commutes. In particular, we can choose $\mathscr{S}$ to be the identity, and in this case we obtain the following diagram, where all groups and homomorphisms are now defined explicitly:



(D_{30})

This concludes our study of the metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and the homomorphism of central extensions $\mathscr{S}$ in the two-dimensional case with B_{Weil} if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. Our work provides an explicit realization of the claims of [GH12] in this case. Specifically, we have constructed a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and an injective homomorphism $\mathscr{S}$ of central extensions (even an isomorphism, which we can choose to be the identity) such that (D_{29}) commutes. We obtain in this way a unitary, irreducible and faithful Weil representation

$\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$ which is a pullback of the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.

Now we examine the remaining case, where $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. We already mentioned that there is no hope to find a homomorphism of central extensions $\mathscr{S}$ such that at the same time the diagram (D_{28}) commutes and $\mathscr{S}$ is injective. So we only look for $\mathscr{S}$ such that (D_{28}) commutes. Recall that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$ and there are four possibilities for $\mathrm{Mp}(\widetilde{W})$ by (309). To make the appropriate selection, we examine the map $\mathscr{S}$ more closely.

Lemma 3.13. *Suppose that* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. *The commutativity of the diagram* (D_{28}) *implies that the image of* $\mathscr{S}$ *is a subgroup of index two of* $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$, *i.e.* $\mathrm{Im}(\mathscr{S})$ *is isomorphic to* $\mathrm{CSU}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ *or* $\mathrm{C}_4 \,.\, \mathrm{A}_4$.

Proof. Suppose that $\mathscr{S}$ is a group homomorphism which makes the diagram (D_{28}) commute. Then $\mathscr{S} \circ \widetilde{\Phi} = \Phi_{\widetilde{\chi_k}} \circ \iota$ and $\eta \circ \widetilde{\Psi} = \Psi_{\widetilde{\chi_k}} \circ \mathscr{S}$. In particular, $\Psi_{\widetilde{\chi_k}} \circ \mathscr{S}$ is surjective because so is $\eta \circ \widetilde{\Psi}$. Hence $\Psi_{\widetilde{\chi_k}|\,\mathrm{Im}(\mathscr{S})} : \mathrm{Im}(\mathscr{S}) \to \mathrm{ASp}(B_W)$ is surjective. This shows that $|\,\mathrm{ASp}(B_W)| = 24$ divides $|\,\mathrm{Im}(\mathscr{S})|$. We know that $\mathscr{S}$ is not injective, thus it cannot be surjective, since otherwise it would be bijective since $|\,\mathrm{Mp}(\widetilde{W})| = |\,\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}|$. So $\mathrm{Im}(\mathscr{S})$ is a proper subgroup of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$, i.e. $|\,\mathrm{Im}(\mathscr{S})| \in \{24, 48\}$. If $|\,\mathrm{Im}(\mathscr{S})| = 24$ then $\Psi_{\widetilde{\chi_k}|\,\mathrm{Im}(\mathscr{S})}$ is an isomorphism, which implies that $\mathrm{Im}(\mathscr{S})$ is a subgroup of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ isomorphic to S_4. This is impossible because $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$ does not contain S_4 as a subgroup. Thus $|\,\mathrm{Im}(\mathscr{S})| = 48$, i.e. $\mathrm{Im}(\mathscr{S})$ is a subgroup of index two in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Since $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$, we obtain by [GN] that $\mathrm{Im}(\mathscr{S})$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ or $\mathrm{C}_4 \,.\, \mathrm{A}_4$. □

Lemma 3.14. *Suppose that* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. *If the diagram* (D_{28}) *is commutative, then* $\mathrm{Mp}(\widetilde{W})$ *is isomorphic to* $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. *Moreover, the image of* $\mathscr{S}$ *is isomorphic to* $\mathrm{CSU}_2(\mathbb{F}_3)$ *or* $\mathrm{GL}_2(\mathbb{F}_3)$.

Proof. Lemma 3.13 implies that $\mathrm{Mp}(\widetilde{W})$ contains $\mathrm{CSU}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ or $\mathrm{C}_4 \,.\, \mathrm{A}_4$ as a quotient. Among the four possibilities $\mathrm{C}_2 \times \mathrm{A}_4 \rtimes \mathrm{C}_4$, $\mathrm{A}_4 \rtimes \mathrm{C}_8$, $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and $\mathrm{U}_2(\mathbb{F}_3)$ for $\mathrm{Mp}(\widetilde{W})$ in (309), $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ is the only group satisfying this condition. Indeed, it contains both $\mathrm{CSU}_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$ as quotients, but not $\mathrm{C}_4 \,.\, \mathrm{A}_4$. □

By Lemma 3.14, we can suppose from now on that $\mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. A presentation of this group is given by

$$\mathrm{Q}_8 \rtimes \mathrm{Dic}_3 = \langle a, b, c, d \,|\, a^4 = c^6 = 1, b^2 = a^2, d^2 = c^3, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = b, \\ dad^{-1} = a^2 b, cbc^{-1} = ab, dcd^{-1} = c^{-1} \rangle. \tag{312}$$

We already mentioned with the computation of $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ in (308) that there are exactly four central extensions of $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ by two elements, each of them corresponding to a different group. Thus all extensions $1 \to \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W}) \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1$ such that $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ are isomorphic. This allows us to fix the group homomorphisms $\tilde{\Phi}$ and $\tilde{\Psi}$ before continuing our investigation concerning the existence of $\mathscr{S}$. In fact, if we fix the choices of $\tilde{\Phi}$ and $\tilde{\Psi}$, and we consider any group homomorphisms $\tilde{\Phi}'$ and $\tilde{\Psi}'$ such that $1 \to \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}'} \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \xrightarrow{\tilde{\Psi}'} \mathrm{Sp}(\widetilde{W}) \to 1$ is also a central extension, then

these two extensions are isomorphic. So, by definition, there exists a group isomorphism $\mathscr{E}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W}) & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(W) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathscr{E}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}'} & \mathrm{Mp}(\widetilde{W}) & \xrightarrow{\tilde{\Psi}'} & \mathrm{Sp}(W) & \longrightarrow & 1 \end{array} \tag{313}$$

i.e. $\mathscr{E} \circ \widetilde{\Phi}' = \widetilde{\Phi}$ and $\widetilde{\Psi} \circ \mathscr{E} = \widetilde{\Psi}'$. Adding this part to the diagram (D_{28}), we obtain:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\hat{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \begin{cases} \mathrm{D}_8 \\ \mathrm{SD}_{16} \end{cases} & \xrightarrow{\Psi_{\hat{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{D}_4 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \iota} & \overset{\tilde{\Gamma}}{\curvearrowright} & \downarrow{\scriptstyle \tilde{\omega}_{\hat{\chi}}} & \overset{\Gamma}{\curvearrowright} & \downarrow{\scriptstyle 0} & & \\ & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\ & & & & \uparrow{\scriptstyle \tilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow & & \\ 1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \mathscr{S}} & \overset{\eta}{\curvearrowleft} & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\ & & & & & & \mathrm{Sp}(W) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\ & & & & & & \uparrow{\scriptstyle \pi} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathscr{E}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}'} & \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 & \xrightarrow{\tilde{\Psi}'} & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1 \end{array} \tag{D_{31}}$$

In particular, $\mathscr{S}$ is a group homomorphism making (D_{28}) commute if and only if $\mathscr{S} \circ \mathscr{E}$ is a group homomorphism such that $\mathscr{S} \circ \mathscr{E} \circ \tilde{\Phi}' = \mathscr{S} \circ \tilde{\Phi} = \Phi_{\widetilde{\chi_k}} \circ \iota$ and $\eta \circ \tilde{\Psi}' = \eta \circ \tilde{\Psi} \circ \mathscr{E} = \Psi_{\widetilde{\chi_k}} \circ \mathscr{S} \circ \mathscr{E}$. Consequently, we fix the group homomorphisms $\tilde{\Phi}$ and $\tilde{\Psi}$ in the following lemma.

Lemma 3.15. *Define* $\tilde{\Phi} : \mu_2(\mathbb{C}) \to \mathrm{Mp}(\widetilde{W})$ *and* $\tilde{\Psi} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{Sp}(\widetilde{W})$.

$$\begin{aligned} -1 &\mapsto a^2 & a &\mapsto b \\ & & b &\mapsto ab \\ & & c &\mapsto ac^{-1}d^2 \\ & & d &\mapsto cbcd \end{aligned}$$

Then $1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1$ *is a central extension.*

Proof. $\widetilde{\Phi}(-1) = a^2$ has order two in $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and $\widetilde{\Phi}$ is a group monomorphism by definition. Now we verify that the images $\widetilde{\Psi}(a)$, $\widetilde{\Psi}(b)$, $\widetilde{\Psi}(c)$ and $\widetilde{\Psi}(d)$ satisfy the relations of the group $\mathrm{Mp}(\widetilde{W})$ given by (312). In terms of matrices, $\widetilde{\Psi}(a) = b = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$, $\widetilde{\Psi}(b) = ab = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\widetilde{\Psi}(c) = ac^{-1}d^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\widetilde{\Psi}(d) = cbcd = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$. Then

- $\widetilde{\Psi}(a)^4 = b^4 = 1$ and $\widetilde{\Psi}(c)^6 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^6 = 1$.
- $\widetilde{\Psi}(b)^2 = (ab)^2 = abab = ab^2a = a^2 = 1$ and $\widetilde{\Psi}(a)^2 = b^2 = 1$, hence $\widetilde{\Psi}(b)^2 = \widetilde{\Psi}(a)^2$.
- $\widetilde{\Psi}(d)^2 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^2 = d^2 = \widetilde{\Psi}(c)^3$.
- $\widetilde{\Psi}(b)\widetilde{\Psi}(a)\widetilde{\Psi}(b)^{-1} = aba^{-1} = b = b^{-1} = \widetilde{\Psi}(a)^{-1}$ and $\widetilde{\Psi}(d)\widetilde{\Psi}(b)\widetilde{\Psi}(d)^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = b^{-1} = \widetilde{\Psi}(a)^{-1}$, hence $\widetilde{\Psi}(b)\widetilde{\Psi}(a)\widetilde{\Psi}(b)^{-1} = \widetilde{\Psi}(d)\widetilde{\Psi}(b)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(a)^{-1}$.
- $\widetilde{\Psi}(c)\widetilde{\Psi}(a)\widetilde{\Psi}(c)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = ab = \widetilde{\Psi}(b)$.
- $\widetilde{\Psi}(d)\widetilde{\Psi}(a)\widetilde{\Psi}(d)^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = ab$ and $\widetilde{\Psi}(a)^2\widetilde{\Psi}(b) = b^2ab = ab$, which implies that $\widetilde{\Psi}(d)\widetilde{\Psi}(a)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b)$.
- $\widetilde{\Psi}(c)\widetilde{\Psi}(b)\widetilde{\Psi}(c)^{-1} = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} = a$ and $\widetilde{\Psi}(a)\widetilde{\Psi}(b) = bab = b^2a = a$, which implies that $\widetilde{\Psi}(c)\widetilde{\Psi}(b)\widetilde{\Psi}(c)^{-1} = \widetilde{\Psi}(a)\widetilde{\Psi}(b)$.
- $\widetilde{\Psi}(d)\widetilde{\Psi}(c)\widetilde{\Psi}(d)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \widetilde{\Psi}(c)^{-1}$.

Hence the map $\widetilde{\Psi}$ defines a group homomorphism. Moreover, the map $\widetilde{\Psi}$ is surjective because $\widetilde{\Psi}(ba) = a$, $\widetilde{\Psi}(a) = b$, $\widetilde{\Psi}(bc^2) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = c$ and $\widetilde{\Psi}(d^{-1}c) = d$. In particular, since $|\mathrm{Mp}(\widetilde{W})| = 2|\mathrm{Sp}(\widetilde{W})|$, the group $\mathrm{Ker}(\widetilde{\Psi})$ has two elements. More precisely, $\mathrm{Ker}(\widetilde{\Psi}) = \{1, a^2\} = \mathrm{Im}(\widetilde{\Phi})$. Finally, a^2 is in the center of $\mathrm{Mp}(\widetilde{W})$ because $a^2b = b^3 = ba^2$, $a^2c = b^2c = bca = ca^2$ and $a^2d = (db^{-1}d^{-1})^2d = db^2d^{-1}d = da^2$. Thus

$$1 \to \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1$$

is a central extension. □

Proposition 3.16. *We use the group homomorphisms from Lemma 3.15 to define the central extension*

$$1 \to \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1.$$

Then there exists a group homomorphism $\mathscr{S}$ such that the diagram (D_{28}) is commutative. More precisely, every such homomorphism of central extensions $\mathscr{S}$ is of the following form:

$$(\mathscr{S}(a), \mathscr{S}(b)) \in \begin{cases} \{(c, bc), (c^{-1}, b^{-1}c)\} & \text{if } \chi_k = \chi_1 \\ \{(c^{-1}, bc), (c, b^{-1}c)\} & \text{if } \chi_k = \chi_3 \end{cases}$$

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}b, cebd), (d^{-1}b, c^{-1}ebd), (d^{-1}b^{-1}, acebd), (d^{-1}b^{-1}, a^{-1}cebd)\} & \text{if } \chi_k = \chi_1 \\ \{(d^{-1}, ed), (d^{-1}, a^2ed), (a^2d^{-1}, aed), (a^2d^{-1}, a^{-1}ed)\} & \text{if } \chi_k = \chi_3. \end{cases} \tag{314}$$

In particular, the image of $\mathscr{S}$ is isomorphic to the general linear group $\mathrm{GL}_2(\mathbb{F}_3)$ if

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}b, cebd), (d^{-1}b, c^{-1}ebd)\} & \text{if } \chi_k = \chi_1 \\ \{(d^{-1}, ed), (d^{-1}, a^2ed)\} & \text{if } \chi_k = \chi_3 \end{cases}, \tag{315}$$

and the image of $\mathscr{S}$ is isomorphic to the conformal special unitary (or binary octahedral) group $\mathrm{CSU}_2(\mathbb{F}_3)$ if

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}b^{-1}, acebd), (d^{-1}b^{-1}, a^{-1}cebd)\} & \text{if } \chi_k = \chi_1 \\ \{(a^2d^{-1}, aed), (a^2d^{-1}, a^{-1}ed)\} & \text{if } \chi_k = \chi_3 \end{cases}. \tag{316}$$

Proof. Suppose that $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is a group homomorphism. Then the diagram (D_{28}) commutes if and only if $\Phi_{\widetilde{\chi_k}} \circ \iota = \mathscr{S} \circ \widetilde{\Phi}$ and $\Psi_{\widetilde{\chi_k}} \circ \mathscr{S} = \eta \circ \widetilde{\Psi}$. Observe that $(\Phi_{\widetilde{\chi_k}} \circ \iota)(-1) = \Phi_{\widetilde{\chi_k}}(-1) = a^2$. Hence $\Phi_{\widetilde{\chi_k}} \circ \iota = \mathscr{S} \circ \widetilde{\Phi}$ is equivalent to $\mathscr{S}(a^2) = a^2$. Moreover,

$$\begin{cases} (\eta \circ \widetilde{\Psi})(a) = \eta(b) = b \\ (\eta \circ \widetilde{\Psi})(b) = \eta(ab) = ab \\ (\eta \circ \widetilde{\Psi})(c) = \eta(ac^{-1}d^2) = a(ca)^{-1}d^2 = c^{-1} \\ (\eta \circ \widetilde{\Psi})(d) = \eta(cbcd) = cabcad = cbacad = accbad = ac^{-1}bad = c^{-1}aad = c^{-1}d = dc. \end{cases}$$

Hence, the condition $\Psi_{\widetilde{\chi_k}} \circ \mathscr{S} = \eta \circ \widetilde{\Psi}$ is equivalent to

$$\begin{cases} \mathscr{S}(a) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{b\}) & = \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})c = \{c, ac, c^{-1}, a^{-1}c\} \\ \mathscr{S}(b) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{ab\}) & = \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})bc = \{bc, abc, b^{-1}c, a^{-1}bc\} \\ \mathscr{S}(c) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{c^{-1}\}) & = \begin{cases} \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})d^{-1}b = \{d^{-1}b, ad^{-1}b, d^{-1}b^{-1}, a^{-1}d^{-1}b\} & \text{if } \chi_k = \chi_1 \\ \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})d^{-1} = \{d^{-1}, ad^{-1}, a^2d^{-1}, a^{-1}d^{-1}\} & \text{if } \chi_k = \chi_3 \end{cases} \\ \mathscr{S}(d) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{dc\}) & = \begin{cases} \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})cebd = \{cebd, acebd, c^{-1}ebd, a^{-1}cebd\} & \text{if } \chi_k = \chi_1 \\ \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})ed = \{ed, aed, a^2ed, a^{-1}ed\} & \text{if } \chi_k = \chi_3. \end{cases} \end{cases} \tag{317}$$

Suppose that $\mathscr{S}$ satisfies (317). Since $\mathscr{S}$ is a group homomorphism, it has to satisfy the relations of $\mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ given by (312):

$$\mathscr{S}(a)^4 = \mathscr{S}(c)^6 = 1, \mathscr{S}(b)^2 = \mathscr{S}(a)^2, \mathscr{S}(d)^2 = \mathscr{S}(c)^3, \mathscr{S}(b)\mathscr{S}(a)\mathscr{S}(b)^{-1} = \mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}, \mathscr{S}(c)\mathscr{S}(a)\mathscr{S}(c)^{-1} = \mathscr{S}(b), \mathscr{S}(d)\mathscr{S}(a)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^2\mathscr{S}(b), \mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} = \mathscr{S}(a)\mathscr{S}(b), \mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}. \tag{318}$$

In particular,

- $\mathscr{S}(a)^4 = 1$ is satisfied for every possibility of $\mathscr{S}(a)$, but $\mathscr{S}(a)^2 = a^2$ if and only if $\mathscr{S}(a) \in \{c, c^{-1}\}$.
- $\mathscr{S}(b)^2 = \mathscr{S}(a)^2 = a^2$ if and only if $\mathscr{S}(b) \in \{bc, b^{-1}c\}$.
- If $\mathscr{S}(a) = c$ then $\mathscr{S}(b)\mathscr{S}(a)\mathscr{S}(b)^{-1} = bccc^{-1}b^{-1} = bcb^{-1} = c^{-1}bb^{-1} = c^{-1} = \mathscr{S}(a)^{-1}$. Similarly, if $\mathscr{S}(a) = c^{-1} = a^2c$, then $\mathscr{S}(b)\mathscr{S}(a)\mathscr{S}(b)^{-1} = \mathscr{S}(a)^{-1}$ (there is no need to compute again since a is in the center of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$).
- If $\mathscr{S}(b) = bc$ then

$$\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1}$$
$$= \begin{cases} cebdbc(cebd)^{-1} = cebdbcd^{-1}b^{-1}ec^{-1} = cebb^{-1}cdcd^{-1}b^{-1}ec^{-1} = cecdcd^{-1}b^{-1}ec^{-1} \\ = ebb^{-1}ec^{-1} = c^{-1} \qquad \text{if } \chi_k = \chi_1 \\ edbc(ed)^{-1} = edbcd^{-1}e = d^{-1}eebeed = d^{-1}bd = c \qquad \text{if } \chi_k = \chi_3. \end{cases}$$

  Hence $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}$ if and only if $\mathscr{S}(a) = \begin{cases} c & \text{if } \chi_k = \chi_1 \\ c^{-1} & \text{if } \chi_k = \chi_3 \end{cases}$. Similarly, if $\mathscr{S}(b) = b^{-1}c$, then $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}$ if and only if $\mathscr{S}(a) = \begin{cases} c^{-1} & \text{if } \chi_k = \chi_1 \\ c & \text{if } \chi_k = \chi_3 \end{cases}$ (there is no need to compute again since a is in the center). Thus $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}$ if and only if

$$(\mathscr{S}(a), \mathscr{S}(b)) \in \begin{cases} \{(c, bc), (c^{-1}, b^{-1}c)\} & \text{if } \chi_k = \chi_1 \\ \{(c^{-1}, bc), (c, b^{-1}c)\} & \text{if } \chi_k = \chi_3. \end{cases} \tag{319}$$

- If $\mathscr{S}(a) = c$ then

$$\mathscr{S}(c)\mathscr{S}(a)\mathscr{S}(c)^{-1}$$
$$= \begin{cases} d^{-1}bcb^{-1}d = d^{-1}c^{-1}d = d^{-1}d^{-1}b^{-1}dd = db^{-1}d^{-1} = cb^{-1} = bc & \text{if } \chi_k = \chi_1 \\ d^{-1}cd = d^{-1}d^{-1}bdd = dbd^{-1} = b^{-1}c & \text{if } \chi_k = \chi_3 \end{cases}$$
$$= \mathscr{S}(b).$$

  Similarly, if $\mathscr{S}(a) = c^{-1}$, then $\mathscr{S}(c)\mathscr{S}(a)\mathscr{S}(c)^{-1} = \mathscr{S}(b)$.

- If $\mathscr{S}(a) = c$ then

$$\mathscr{S}(d)\mathscr{S}(a)\mathscr{S}(d)^{-1}$$
$$= \begin{cases} cebdc(cebd)^{-1} = cebdcd^{-1}b^{-1}ec^{-1} = cebbb^{-1}ec^{-1} = cebec^{-1} = cbcc^{-1} = cb = b^{-1}c \\ \text{if } \chi_k = \chi_1 \\ edcd^{-1}e = ebe = bc \quad \text{if } \chi_k = \chi_3 \end{cases}$$
$$= \mathscr{S}(a)^2\mathscr{S}(b).$$

  Similarly, if $\mathscr{S}(a) = c^{-1}$, then $\mathscr{S}(d)\mathscr{S}(a)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^2\mathscr{S}(b)$.

- If $\mathscr{S}(b) = bc$ then

$$\begin{aligned}\mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} &= \begin{cases} d^{-1}bbcb^{-1}d = d^{-1}cbd = d^{-1}b^{-1}cd = d^{-1}db = b & \text{if } \chi_k = \chi_1 \\ d^{-1}bcd = d^{-1}db^{-1}d^{-1}d = b^{-1} & \text{if } \chi_k = \chi_3 \end{cases} \\ &= \mathscr{S}(a)\mathscr{S}(b).\end{aligned}$$

  Similarly, if $\mathscr{S}(b) = b^{-1}c$, then $\mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} = \mathscr{S}(a)\mathscr{S}(b)$.

- $\mathscr{S}(c)^6 = 1$ if and only if the order of $\mathscr{S}(c)$ in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ divides 6. Since a has order four, $(d^{-1}b)^3 = d^{-1}bd^{-1}bd^{-1}b = d^{-1}dcd^{-1}d^{-1}bd^{-1}b = cdbd^{-1}b = cbc^{-1}b = b^{-1}b = 1$ and $d^{-3} = 1$, the condition $\mathscr{S}(c)^6 = 1$ is satisfied if and only if

$$\mathscr{S}(c) \in \begin{cases} \{d^{-1}b, d^{-1}b^{-1}\} & \text{if } \chi_k = \chi_1 \\ \{d^{-1}, a^2d^{-1}\} & \text{if } \chi_k = \chi_3. \end{cases} \tag{320}$$

- $\mathscr{S}(d)^2 = \mathscr{S}(c)^3$ if and only if the order of $\mathscr{S}(d)$ divides 2 if $\mathscr{S}(c) = \begin{cases} d^{-1}b & \text{if } \chi_k = \chi_1 \\ d^{-1} & \text{if } \chi_k = \chi_3 \end{cases}$, and equals 4 if $\mathscr{S}(c) = \begin{cases} d^{-1}b^{-1} & \text{if } \chi_k = \chi_1 \\ a^2d^{-1} & \text{if } \chi_k = \chi_3 \end{cases}$. Since a has order four, $(cebd)^2 = cebdcebd = ceb^2debd = b^2cedebd = c^{-1}d^{-1}bd = c^{-1}c = 1$ and $(ed)^2 = eded = d^{-1}d = 1$, the condition $\mathscr{S}(d)^2 = \mathscr{S}(c)^3$ is satisfied if and only if

$$\mathscr{S}(d) \in \begin{cases} \{cebd, c^{-1}ebd\} & \text{if } \mathscr{S}(c) = d^{-1}b \text{ and } \chi_k = \chi_1 \\ \{acebd, a^{-1}cebd\} & \text{if } \mathscr{S}(c) = d^{-1}b^{-1} \text{ and } \chi_k = \chi_1 \\ \{ed, a^2ed\} & \text{if } \mathscr{S}(c) = d^{-1} \text{ and } \chi_k = \chi_3 \\ \{aed, a^{-1}ed\} & \text{if } \mathscr{S}(c) = a^2d^{-1} \text{ and } \chi_k = \chi_3. \end{cases} \tag{321}$$

- Now we examine the condition $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}$.

  - If $\mathscr{S}(c) = d^{-1}b$ and $\chi_k = \chi_1$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = cebdd^{-1}bd^{-1}b^{-1}ec^{-1} = ceb^2d^{-1}b^{-1}ec^{-1} = b^2ced^{-1}b^{-1}ec^{-1} = cdebec^{-1} = cdbcc^{-1} = cdb = cdbd^{-1}d = cbc^{-1}d = b^{-1}d = \mathscr{S}(c)^{-1}$.
  - If $\mathscr{S}(c) = d^{-1}b^{-1}$ and $\chi_k = \chi_1$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}$(there is no need to compute again since $d^{-1}b^{-1} = d^{-1}ba^2$ and a is in the center of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$).
  - If $\mathscr{S}(c) = d^{-1}$ and $\chi_k = \chi_3$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = edd^{-1}d^{-1}e = ed^{-1}e = d = \mathscr{S}(c)^{-1}$.
  - If $\mathscr{S}(c) = a^2d^{-1}$ and $\chi_k = \chi_3$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}$ (similarly, there is no need to compute again).

After studying all the relations in (318), combining (319), (320) and (321) together, we obtain that (318) is satisfied if and only if

$$(\mathscr{S}(a), \mathscr{S}(b)) \in \begin{cases} \{(c, bc), (c^{-1}, b^{-1}c)\} & \text{if } \chi_k = \chi_1 \\ \{(c^{-1}, bc), (c, b^{-1}c)\} & \text{if } \chi_k = \chi_3 \end{cases}$$
$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}b, cebd), (d^{-1}b, c^{-1}ebd), (d^{-1}b^{-1}, acebd), (d^{-1}b^{-1}, a^{-1}cebd)\} & \text{if } \chi_k = \chi_1 \\ \{(d^{-1}, ed), (d^{-1}, a^2ed), (a^2d^{-1}, aed), (a^2d^{-1}, a^{-1}ed)\} & \text{if } \chi_k = \chi_3. \end{cases} \tag{322}$$

Conversely, these computations also prove that every map $\mathscr{S}$, defined on the generators a, b, c, d of $\mathrm{Mp}(\widetilde{W})$ by a suitable choice in (322), defines a homomorphism of central extensions making the diagram (D_{28}) commute. According to Lemma 3.14, it only remains to identify the image of $\mathscr{S}$ between $\mathrm{CSU}_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$. In the database [GN], we notice that $\mathrm{GL}_2(\mathbb{F}_3)$ has 13 elements of order two, whereas $\mathrm{CSU}_2(\mathbb{F}_3)$ has only one element of order two. Observe that a^2, $cebd$ and ed are three distinct elements of order two in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Then we immediately obtain that the image of s is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$ if one of the following conditions is satisfied :

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}b, cebd), (d^{-1}b, c^{-1}ebd)\} & \text{if } \chi_k = \chi_1 \\ \{(d^{-1}, ed), (d^{-1}, a^2ed)\} & \text{if } \chi_k = \chi_3, \end{cases}$$

since then the image of $\mathscr{S}$ contains at least two different elements of order two. Suppose now that

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}b^{-1}, acebd), (d^{-1}b^{-1}, a^{-1}cebd)\} & \text{if } \chi_k = \chi_1 \\ \{(a^2d^{-1}, aed), (a^2d^{-1}, a^{-1}ed)\} & \text{if } \chi_k = \chi_3. \end{cases} \tag{323}$$

Observe that the group $\mathrm{CSU}_2(\mathbb{F}_3)$ contains $\mathrm{Q}_{16} =< a, b \mid a^8 = 1, b^2 = a^4, bab^{-1}{=}a^{-1} >$ as a subgroup, whereas $\mathrm{GL}_2(\mathbb{F}_3)$ does not contain Q_{16}. We are able to exhibit a subgroup of $\mathrm{Im}(\mathscr{S})$ which is isomorphic to Q_{16}. By assumption,

$$\begin{cases} \mathscr{S}(bdca) \in < a^2 > bcacebdd^{-1}b^{-1}c =< a^2 > bcacec =< a^2 > abce & \text{if } \chi_k = \chi_1 \\ \mathscr{S}(bdc) \in < a^2 > bcaeda^2d^{-1} =< a^2 > abce & \text{if } \chi_k = \chi_3. \end{cases}$$

This proves that $abce \in \mathrm{Im}(\mathscr{S})$. Since $(abce)^2 = a^2bcebce = a^2bcbe^2 = a^2c = c^{-1}$, the element $abce$ has order eight in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Moreover, by (322), the element bc of order four in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, belongs also to $\mathrm{Im}(\mathscr{S})$. Furthermore,

$$\begin{cases} (abce)^4 = c^2 \\ (bc)^2 = bcbc = b^2 = c^2 \\ (bc)abce(bc)^{-1} = bcabcec^{-1}b^{-1} = abcbcec^{-1}b^{-1} = ac^2ec^{-1}b^{-1} = ec^{-1}b^{-1}a^{-1} = (abce)^{-1}. \end{cases} \tag{324}$$

Consequently, (324) shows that the subgroup of $\mathrm{Im}(\mathscr{S})$ generated by $abce$ and bc is isomorphic to Q_{16}. In particular, this shows that $\mathrm{Im}(\mathscr{S})$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ if (323) is satisfied. This concludes the proof. □

Corollary 3.17. *The representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ obtained by Proposition 3.16 is unitary and irreducible but non-faithful.*

Proof. The representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is unitary since $\widetilde{\omega}_{\widetilde{\chi_k}}$ satisfies this property. By Proposition 3.16, the kernel of $\mathscr{S}$ has two elements. Since $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful, the kernel of $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is exactly the kernel of $\mathscr{S}$. More precisely,

$$\mathrm{Ker}(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}) = \begin{cases} < c^3 > & \text{if } \mathrm{Im}(\mathscr{S}) \simeq \mathrm{GL}_2(\mathbb{F}_3) \\ < a^2c^3 > & \text{if } \mathrm{Im}(\mathscr{S}) \simeq \mathrm{CSU}_2(\mathbb{F}_3) \end{cases}.$$

Now, let V be a vector subspace of $\mathrm{L}^2(Y)$ which is invariant under $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$. By (314), the elements $b = \left((1, 2x), i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ and $c = \left((1, 2y), i\widetilde{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ belong to $\mathrm{Im}(\mathscr{S})$. Hence V is invariant under the following operators:

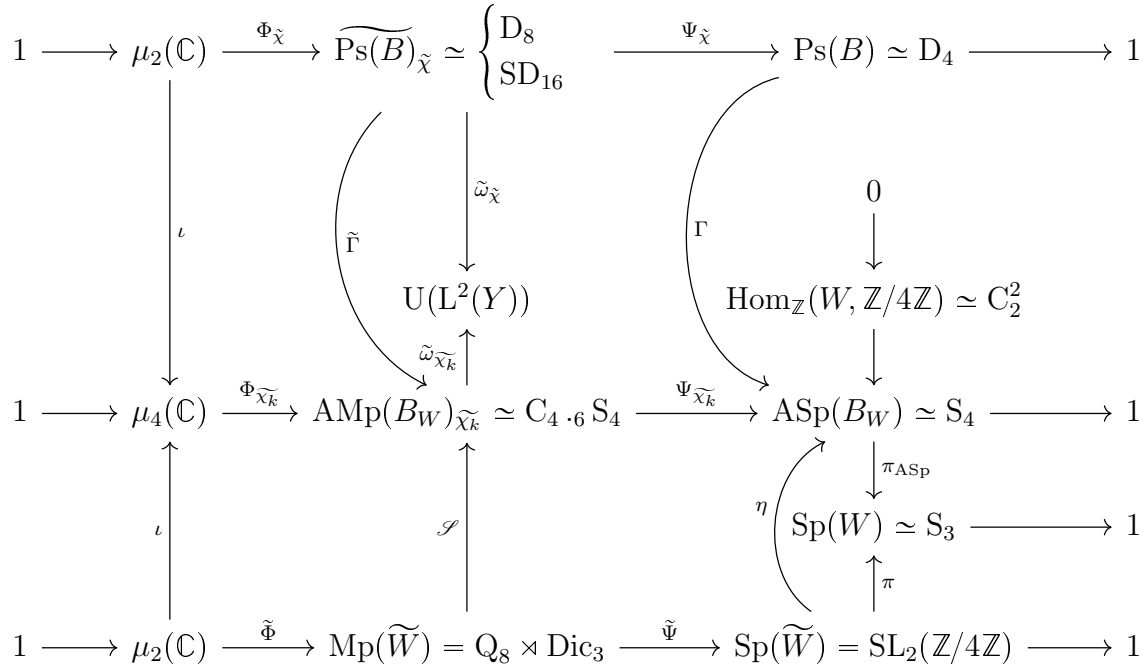
$$(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(b) = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(c) = i\widetilde{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(bc) = -\widetilde{\varepsilon}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since

$$\rho_{\widetilde{\chi_k}}(1,0) = \begin{pmatrix} \widetilde{\varepsilon} & 0 \\ 0 & -\widetilde{\varepsilon} \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho_{\widetilde{\chi_k}}(1,1) = \begin{pmatrix} 0 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 0 \end{pmatrix},$$

this implies that V is invariant under $\rho_{\widetilde{\chi_k}}(0)$, $i\rho_{\widetilde{\chi_k}}(0,1)$, $i\rho_{\widetilde{\chi_k}}(1,0)$ and $-\rho_{\widetilde{\chi_k}}(1,1)$. Since the set $\{\rho_{\widetilde{\chi_k}}(w),\ w \in W\}$ is a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by Lemma 2.15, and $W = \{0, (1,0), (0,1), (1,1)\}$, this shows that $V = \{0\}$ or $V = \mathrm{L}^2(Y)$. Therefore, $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is irreducible. □

Proposition 3.16 shows that if $\nu(\sigma_{(0,1)}) \equiv 0 \mod \frac{\pi}{2}$, then there exists a group homomorphism $\mathscr{S}$ such that (D_{28}) commutes. More precisely, this result gives the explicit formula for all possible such homomorphisms of central extensions $\mathscr{S}$, in the case where the maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are defined by Lemma 3.15. In this way, we obtain the following diagram, where all groups and homomorphisms are now defined explicitly:



(D_{32})

This concludes the study of the metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ in the two-dimensional case with $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. In fact, as in [GH12], we defined a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and a homomorphism of central extensions $\mathscr{S}$ such that (D_{32}) commutes. We obtain in this way a unitary and irreducible but non-faithful representation $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$ (its kernel has two elements).

Remark 3.18. The study of the two-dimensional case for $B = B_{\mathrm{Weil}}$ leads to the following results:

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then we obtain the commutative diagram (D_{32}). In this case, we really obtain an extension of Weil's work, in the sense of the existence of $\widetilde{\Gamma}$ (cf. Proposition 3.9 and 2.39). Moreover, there is only one possibility for the metaplectic covering $\mathrm{Mp}(\widetilde{W})$, and we obtain as claimed in [GH12] the existence of $\mathscr{S}$ and a unitary and irreducible representation $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$. However, the extension $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ does not preserve the injectivity of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}_k}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$.
- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then we obtain the commutative diagram (D_{30}). In this case, there is also only one possibility for the metaplectic covering $\mathrm{Mp}(\widetilde{W})$, and we obtain as announced in [GH12] the existence of $\mathscr{S}$ and a unitary and irreducible representation $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$. Moreover, this representation is faithful, as the Weil representation $\widetilde{\omega}_{\widetilde{\chi}_k}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$. However, we lose the relation to Weil's work, in the sense that the map $\widetilde{\Gamma}$ does not exist (cf. Propositions 3.9 and 2.39).

Remark 3.19. The conclusion of this study tells us that, while constructing $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$, we have

to choose between keeping a relation with Weil's work, in the sense of the commutativity of the diagram (D_{22}), or keeping the injectivity of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$. In higher dimension, recall that there is no hope to obtain the injectivity of the expected extension $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ to $\mathrm{Mp}(\widetilde{W})$ of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ (cf. Lemma 2.42).

3.1.5 Two metaplectic coverings $\mathrm{Mp}(W)$ and $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$

In this part, we consider again the diagrams (D_{26}) and (D_{27}). We try to extend the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ to a minimal extension of $\mathrm{Sp}(W)$, instead of $\mathrm{Sp}(\widetilde{W})$ as in section 3.1.4. While extending $\widetilde{\omega}_{\widetilde{\chi_k}}$, we try to keep its properties, namely its injectivity and irreducibility. The fact that in this example all groups are explicit and relatively small facilitates the task. We will use the presentations of the groups given in section 3.1.4. In particular, the maps $\Phi_{\widetilde{\chi_k}}$, $\Psi_{\widetilde{\chi_k}}$ and η are given by (307). Recall also the group homomorphism π_{ASp}:

$$\begin{aligned} \pi_{\mathrm{ASp}} : \mathrm{ASp}(B_W) &\to \mathrm{Sp}(W) \\ a &\mapsto 1 \\ b &\mapsto 1 \\ c &\mapsto a \\ d &\mapsto ba \end{aligned} \tag{325}$$

The first possibility would be to extend $\widetilde{\omega}_{\widetilde{\chi_k}}$ directly to $\mathrm{Sp}(W)$, that is to find a group homomorphism $\widehat{\mathscr{S}} : \mathrm{Sp}(W) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that (D_{26}) or (D_{27}) commutes. In particular, this would imply that $\Psi_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}} \circ \pi_{\mathrm{ASp}} = \mathrm{id}_{\mathrm{ASp}(B_W)}$, i.e. that the extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ splits, which is false by Proposition 2.29. Therefore, if we want a commutative diagram, then we need to define a central extension of $\mathrm{Sp}(W)$ by at least two elements. Recall that $\mathrm{Sp}(W) \simeq \mathrm{S}_3$ in this example. By [Kar87, p. 278], the Schur multiplier $\mathrm{H}_2(\mathrm{S}_3, \mathbb{Z})$ of S_3 is trivial. Thus, using the universal coefficient theorem as in Remark 3.3, we deduce that $\mathrm{H}^2(\mathrm{S}_3, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. By [GN], there are exactly two non-isomorphic central extensions $1 \to \mu_2(\mathbb{C}) \to G \to \mathrm{Sp}(W) \to 1$ and the corresponding groups G are not isomorphic: $G \simeq \mathrm{D}_6$ (trivial extension) or $G \simeq \mathrm{Dic}_3$ (non-split extension). The next proposition shows that there is only one possible choice: we have to set $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, and in this case $G \simeq \mathrm{Dic}_3$.

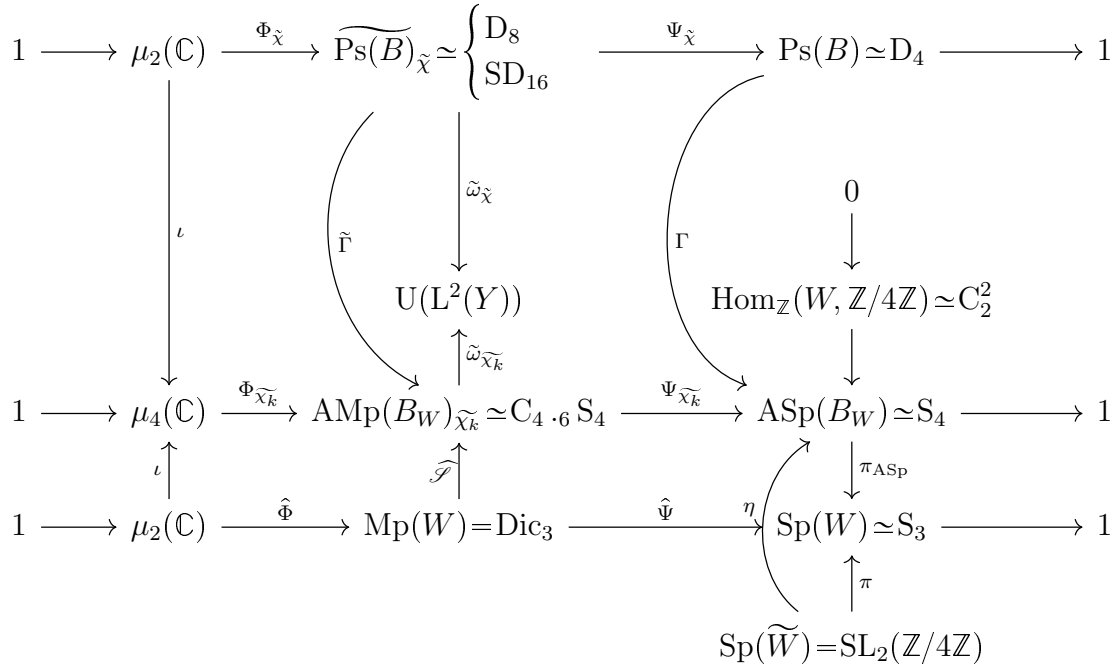
Proposition 3.20. • *Suppose that $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. Define*

$$\mathrm{Mp}(W) = \mathrm{Dic}_3 = \langle a, b \mid a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1} \rangle \tag{326}$$

and

$$\begin{aligned} \hat{\Phi} : \mu_2(\mathbb{C}) &\to \mathrm{Mp}(W) \\ -1 &\mapsto b^2 \end{aligned} \quad \textit{and} \quad \begin{aligned} \hat{\Psi} : \mathrm{Mp}(W) &\to \mathrm{Sp}(W). \\ a &\mapsto a \\ b &\mapsto b \end{aligned}$$

Then $1 \to \mu_2(\mathbb{C}) \xrightarrow{\hat{\Phi}} \mathrm{Mp}(W) \xrightarrow{\hat{\Psi}} \mathrm{Sp}(W) \to 1$ is a non-split central extension. Moreover, there exists an injective group homomorphism $\widehat{\mathscr{S}} : \mathrm{Mp}(W) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that the following diagram (D_{33}) is commutative:



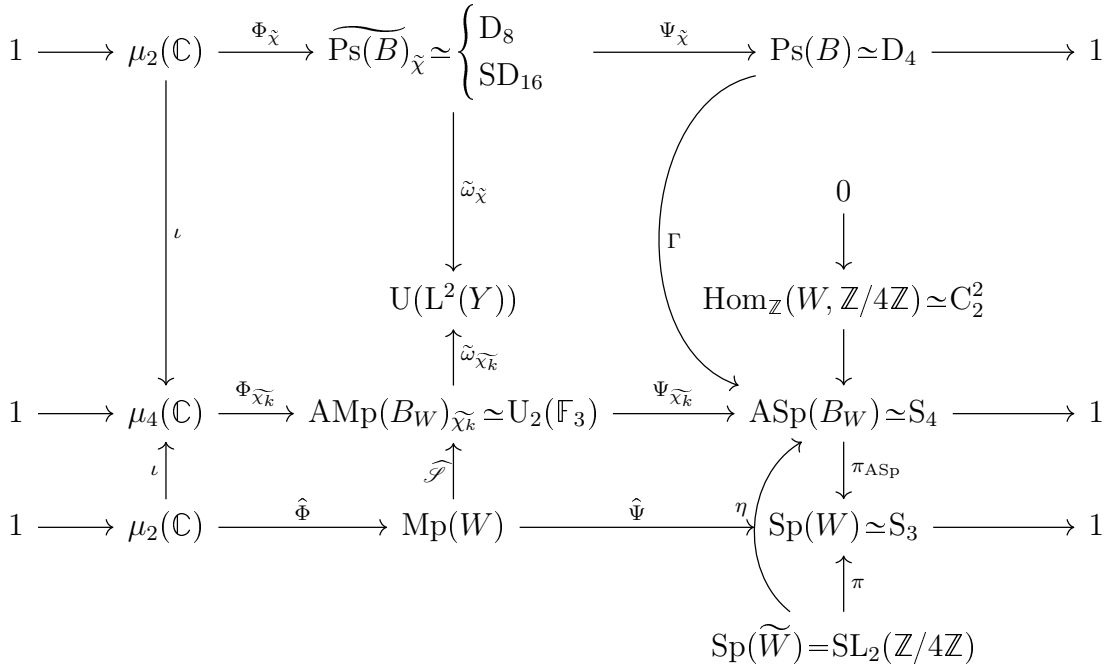
(D_{33})

More precisely, $\widehat{\mathscr{S}}$ is of the following form:

$$\begin{cases} \widehat{\mathscr{S}}(a) \in \{a^2d, bd\} \\ \widehat{\mathscr{S}}(b) \in \{ade, a^{-1}de\} \end{cases} \quad \textit{or} \quad \begin{cases} \widehat{\mathscr{S}}(a) \in \{cd, cbd\} \\ \widehat{\mathscr{S}}(b) \in \{abcde, a^{-1}bcde\}. \end{cases} \tag{327}$$

Moreover, Dic_3 is the only possibility as two-fold covering $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$ such that the diagram (D_{33}) is commutative.

- *Suppose that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. Then there is no group $\mathrm{Mp}(W)$ such that the following diagram commutes and $\widehat{\mathscr{S}}$ is injective:*



(D_{34})

Proof. • Suppose first that $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. Since $\mathrm{H}^2(\mathrm{S}_3, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, we already mentioned that all the extensions of the form $1 \to \mu_2(\mathbb{C}) \xrightarrow{\hat{\Phi}} \mathrm{Dic}_3 \xrightarrow{\hat{\Psi}} \mathrm{Sp}(W) \simeq \mathrm{S}_3 \to 1$ are isomorphic. This allows to fix the maps $\hat{\Phi}$ and $\hat{\Psi}$ (this is a similar reasoning to the one made to fix $\tilde{\Phi}$ and $\tilde{\Psi}$ just before Lemma 3.15). We only have to verify that $\hat{\Phi}$ and $\hat{\Psi}$ in fact define a central extension. $\hat{\Phi}$ is an injective group homomorphism because $b^2 \neq 1$ and $\hat{\Phi}(-1)^2 = b^4 = 1$. $\hat{\Psi}$ is a group homomorphism because its image satisfies the relations of Dic_3 given in (326). Indeed, $\hat{\Psi}(a)^6 = a^6 = 1$, $\hat{\Psi}(b)^2 = b^2 = 1 = a^3 = \Psi(a)^3$, and $\hat{\Psi}(b)\hat{\Psi}(a)\hat{\Psi}(b)^{-1} = bab^{-1} = a^{-1} = \hat{\Psi}(a)^{-1}$. $\hat{\Psi}$ is surjective by definition and $\mathrm{Ker}(\hat{\Psi}) = \{1, b^2\} = \mathrm{Im}(\hat{\Phi})$. Hence these maps define a central extension of $\mathrm{Sp}(W)$ by $\mu_2(\mathbb{C})$. This extension does not split, since otherwise Dic_3 would be isomorphic to the direct product of C_2 and S_3, i.e. D_6.
Now, if we consider a map $\widehat{\mathscr{S}} : \mathrm{Mp}(W) = \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$, then the diagram ($D_{33}$) commutes if and only if $\widehat{\mathscr{S}} \circ \hat{\Phi} = \Phi_{\widetilde{\chi}_k} \circ \iota$ and $\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k} \circ \widehat{\mathscr{S}} = \hat{\Psi}$. The condition $\widehat{\mathscr{S}} \circ \hat{\Phi} = \Phi_{\widetilde{\chi}_k} \circ \iota$ is equivalent to have $\widehat{\mathscr{S}}(\hat{\Phi}(-1)) = \Phi_{\widetilde{\chi}_k}(-1)$, i.e. $\widehat{\mathscr{S}}(b^2) = a^2$. Since $\hat{\Psi}(a) = a$, $\hat{\Psi}(b) = b$, $(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})(a) = (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})(b) = (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})(c) = 1$, $(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})(d) = a$ and $(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})(e) = ba$, we have the following equivalences:

$$\begin{cases} (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k} \circ \widehat{\mathscr{S}})(a) = \hat{\Psi}(a) \iff \widehat{\mathscr{S}}(a) \in (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})^{-1}(\{a\}) = \mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})d \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = < a, b, c > d \\ (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k} \circ \widehat{\mathscr{S}})(b) = \hat{\Psi}(b) \iff \widehat{\mathscr{S}}(b) \in (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})^{-1}(\{b\}) = \mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi}_k})de \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = < a, b, c > de. \end{cases}$$

Observe that

$$\mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}}) =< a, b, c >= \{1, a, a^2, a^{-1}, b, b^{-1}, c, c^{-1}, ab, ac, bc, ab^{-1}, ac^{-1}, cb, abc, a^{-1}bc\} \tag{328}$$

has 16 elements and is isomorphic to the Pauli group

$$\mathrm{C}_4 \circ \mathrm{D}_4 =< a, b, c \mid a^4 = c^2 = 1, b^2 = a^2, ab = ba, ac = ca, cbc = a^2 b >,$$

which is also isomorphic to the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. (For example, the map $f : \mathrm{C}_4 \circ \mathrm{D}_4 \to \mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})$ defined by $f(a) = a$, $f(b) = b$ and $f(c) = ac$ is a group isomorphism.)

Suppose now that $\widehat{\mathscr{S}} : \mathrm{Mp}(W) = \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is a map such that the diagram (D_{33}) commutes, i.e. $\widehat{\mathscr{S}}(b^2) = a^2$, $\widehat{\mathscr{S}}(a) \in < a, b, c > d$, and $\widehat{\mathscr{S}}(b) \in < a, b, c > de$. Then $\widehat{\mathscr{S}}$ is a group homomorphism if and only if its image satisfies the relations of Dic_3 given by (326), i.e. $\widehat{\mathscr{S}}(a)^6 = 1$, $\widehat{\mathscr{S}}(b)^2 = \widehat{\mathscr{S}}(a)^3 = a^2$ and $\widehat{\mathscr{S}}(b)\widehat{\mathscr{S}}(a)\widehat{\mathscr{S}}(b)^{-1} = \widehat{\mathscr{S}}(a)^{-1}$. If we examine all the possibilities for $\widehat{\mathscr{S}}(a) \in < a, b, c > d$ and $\widehat{\mathscr{S}}(b) \in < a, b, c > de$, we see that the conditions $\widehat{\mathscr{S}}(b)^2 = a^2$ and $\widehat{\mathscr{S}}(a)^3 = \widehat{\mathscr{S}}(b)^2 = a^2$ are satisfied if and only if $\widehat{\mathscr{S}}(a) \in \{a^2, b, c, cb\}d$ and $\widehat{\mathscr{S}}(b) \in \{a, a^{-1}, abc, a^{-1}bc\}de$. In this case, $\widehat{\mathscr{S}}(a)^6 = 1$ and $\widehat{\mathscr{S}}(b)^2 = \widehat{\mathscr{S}}(a)^3$. Finally, the condition $\widehat{\mathscr{S}}(b)\widehat{\mathscr{S}}(a)\widehat{\mathscr{S}}(b)^{-1} = \widehat{\mathscr{S}}(a)^{-1}$ provides all the possibilities for $\widehat{\mathscr{S}}$ given by (327). These computations prove that these possibilities define in fact group homomorphisms such that (D_{33}) commutes. So, it only remains to prove that $\widehat{\mathscr{S}}$ is injective. The image of $\widehat{\mathscr{S}}$ is generated by $\widehat{\mathscr{S}}(a)$ of order six and $\widehat{\mathscr{S}}(b)$ of order four. Moreover, $\widehat{\mathscr{S}}(a)$ and $\widehat{\mathscr{S}}(b)$ verify the relations of Dic_3 because $\widehat{\mathscr{S}}$ is a group homomorphism. Thus the image of $\widehat{\mathscr{S}}$ is isomorphic to Dic_3, and $\widehat{\mathscr{S}}$ is injective.

Finally, it remains to prove that Dic_3 is the only possible choice for $\mathrm{Mp}(W)$. We already mentioned that there are two possibilities for $\mathrm{Mp}(W)$, namely D_6 or Dic_3. If $\mathrm{Mp}(W) = D_6$ then the extension splits and there exists a group homomorphism $\hat{\tau} : \mathrm{Sp}(W) \to \mathrm{Mp}(W)$ such that $\hat{\Psi} \circ \hat{\tau} = \mathrm{id}_{\mathrm{Sp}(W)}$. Then, if the diagram (D_{33}) is commutative, $\Psi_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}} \circ \hat{\tau} \circ \pi_{\mathrm{ASp}} = \mathrm{id}_{\mathrm{ASp}(B_W)}$. In particular, the extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ splits, which is false according to Proposition 2.29.

- Assume that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. Since $\mathrm{Mp}(W)$ is isomorphic to D_6 or Dic_3, but $\mathrm{U}_2(\mathbb{F}_3)$ does not contain either D_6 or Dic_3 as a subgroup, there is no hope to find an injective group homomorphism $\widehat{\mathscr{S}} : \mathrm{Mp}(W) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$.

□

Corollary 3.21. *We keep the same notation as in Proposition 3.20 and assume that* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. *Then, for every homomorphism* $\widehat{\mathscr{S}}$ *defined by* (327), *the representation* $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ *of* $\mathrm{Mp}(W) = \mathrm{Dic}_3$ *is unitary, faithful and irreducible.*

Proof. $\widehat{\mathscr{S}}$ is an injective group homomorphism by Proposition 3.20, and $\widetilde{\omega}_{\widetilde{\chi_k}}$ is unitary and faithful by Proposition 2.32. Hence $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ is a unitary and faithful representation. Suppose now that $V \subseteq \mathrm{L}^2(Y)$ is a vector subspace which is invariant under $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$. Then V is

invariant under the operator $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(g)$ for every $g \in \mathrm{Mp}(W)$. In particular, V is invariant under $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a)$ and $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b)$, which are of the following form by (327):

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(a^2 d), \widetilde{\omega}_{\widetilde{\chi_k}}(bd)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(ade), \widetilde{\omega}_{\widetilde{\chi_k}}(a^{-1}de)\} \end{cases} \quad \text{or} \quad \begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(cd), \widetilde{\omega}_{\widetilde{\chi_k}}(cbd)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(abcde), \widetilde{\omega}_{\widetilde{\chi_k}}(a^{-1}bcde)\}, \end{cases}$$

i.e. by Proposition 3.5,

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \left\{ \frac{1}{2} \begin{pmatrix} 1+\widetilde{\varepsilon}i & -\widetilde{\varepsilon}+i \\ \widetilde{\varepsilon}+i & 1-\widetilde{\varepsilon}i \end{pmatrix}, \frac{i}{2} \begin{pmatrix} -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \\ -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \left\{ \frac{i}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix}, \frac{-i}{\sqrt{2}} \begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix} \right\} \end{cases}$$

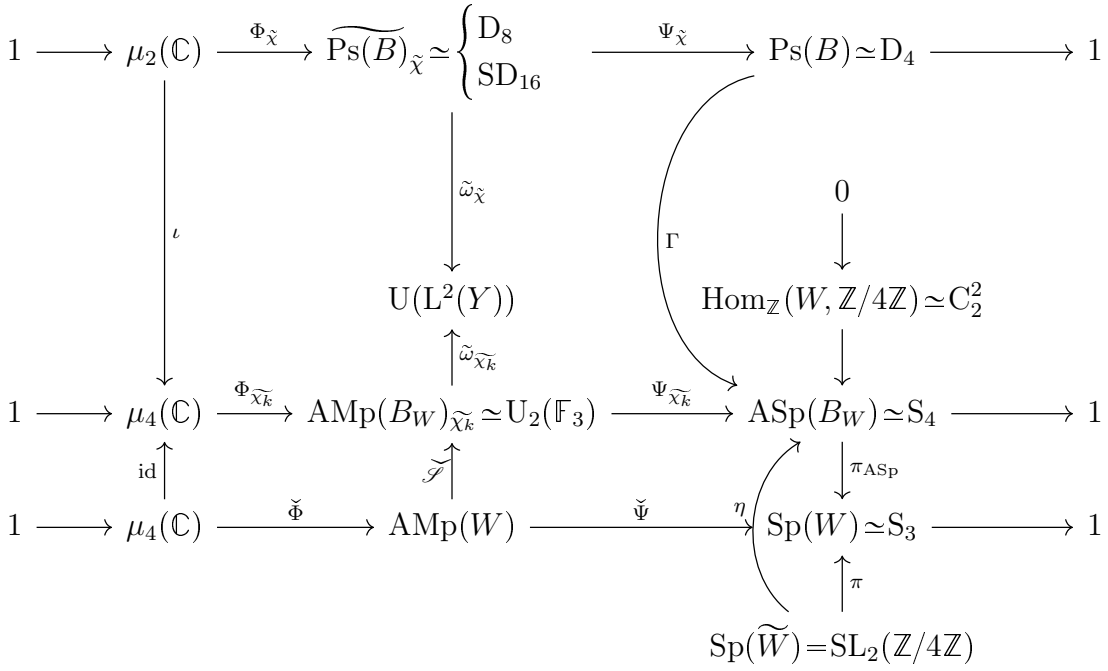
or

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \left\{ \frac{i\widetilde{\varepsilon}}{2} \begin{pmatrix} -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \\ \widetilde{\varepsilon}+i & 1-\widetilde{\varepsilon}i \end{pmatrix}, \frac{-\widetilde{\varepsilon}}{2} \begin{pmatrix} -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \\ 1+\widetilde{\varepsilon}i & -\widetilde{\varepsilon}+i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \left\{ \frac{-i\widetilde{\varepsilon}}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\varepsilon} \\ \widetilde{\varepsilon} & -1 \end{pmatrix}, \frac{i\widetilde{\varepsilon}}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\varepsilon} \\ \widetilde{\varepsilon} & -1 \end{pmatrix} \right\}. \end{cases}$$

In any case, the operators $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a)$ and $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b)$ do not have any common eigenvalue. Thus V cannot be a one-dimensional space. Since $\dim_{\mathbb{C}}(\mathrm{L}^2(Y)) = 2$, this implies that $V = \{0\}$ or $V = \mathrm{L}^2(Y)$. Therefore, $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ is irreducible. □

Remark 3.22. Proposition 3.20 and Corollary 3.21 provide the "best" result we could hope for if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, in the sense that the two-fold covering $\mathrm{Mp}(W) \simeq \mathrm{Dic}_3$ is the smallest possible central extension of $\mathrm{Sp}(W)$ which satisfies at the same time that the whole diagram (D_{33}) commutes and the resulting Weil representation of $\mathrm{Mp}(W)$ is a unitary, irreducible and faithful representation. Proposition 3.20 also shows that we cannot obtain such a two-fold covering if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$: if we want to obtain a faithful representation and a commutative diagram, then we need to define at least an extension of $\mathrm{Sp}(W)$ by four elements (this is what we will construct in Proposition 3.23).

Now, we suppose that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. We look for a four-fold covering $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$ such that the following diagram (D_{35}) commutes and $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}}$ is a unitary, irreducible and faithful representation.



(D_{35})

Proposition 3.23. *Suppose that* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. *Define*

$$\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8 = \langle a, b \mid a^3 = b^8 = 1, bab^{-1} = a^{-1} \rangle \tag{329}$$

and

$$\begin{aligned} \breve{\Phi} : \mu_4(\mathbb{C}) &\to \mathrm{AMp}(W) \\ i &\mapsto b^2 \end{aligned} \quad \textit{and} \quad \begin{aligned} \breve{\Psi} : \mathrm{AMp}(W) &\to \mathrm{Sp}(W). \\ a &\mapsto a \\ b &\mapsto b \end{aligned}$$

Then $1 \to \mu_4(\mathbb{C}) \xrightarrow{\breve{\Phi}} \mathrm{AMp}(W) \xrightarrow{\breve{\Psi}} \mathrm{Sp}(W) \to 1$ *is a non-split central extension. Moreover, there exists an injective group homomorphism* $\widetilde{\mathscr{S}} : \mathrm{AMp}(W) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *such that the diagram* (D_{35}) *is commutative. More precisely,* $\widetilde{\mathscr{S}}$ *is of the following form:*

$$\begin{cases} \widetilde{\mathscr{S}}(a) \in \{d, b^{-1}d\} \\ \widetilde{\mathscr{S}}(b) \in \{ed^{-1}, a^2ed^{-1}\}. \end{cases} \tag{330}$$

Moreover, $\mathrm{C}_3 \rtimes \mathrm{C}_8$ *is the only possibility as four-fold covering* $\mathrm{AMp}(W)$ *such that the diagram* (D_{35}) *is commutative.*

Proof. First of all, the commutativity of (D_{35}) implies that the group $\mathrm{AMp}(W)$ has to be contained in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ as a subgroup, and has to be associated to a central extension of $\mathrm{Sp}(W) \simeq \mathrm{S}_3$ by $\mu_4(\mathbb{C})$. Since $\mathrm{H}_2(\mathrm{S}_3, \mathbb{Z})$ is trivial by [Kar87, p.278], we compute in a similar way to Remark 3.3 with the universal coefficient theorem that $\mathrm{H}^2(\mathrm{S}_3, \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. According to [GN], $\mathrm{AMp}(W)$ has to be isomorphic to $\mathrm{C}_4 \times \mathrm{S}_3$ (trivial extension) or $\mathrm{C}_3 \rtimes \mathrm{C}_8$ (non-split

extension). In addition, the only subgroups of order 24 in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3)$ are $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{C}_3 \rtimes \mathrm{C}_8$. This implies that $\mathrm{AMp}(W) \simeq \mathrm{C}_3 \rtimes \mathrm{C}_8$. Since $|\,\mathrm{H}^2(\mathrm{S}_3, \mathbb{Z}/4\mathbb{Z})| = 2$, we can fix the maps $\breve{\Phi}$ and $\breve{\Psi}$ such that $1 \to \mu_4(\mathbb{C}) \xrightarrow{\breve{\Phi}} \mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8 \xrightarrow{\breve{\Psi}} \!\!\!\rightarrow \mathrm{Sp}(W) \to 1$ is a central extension (again, this reasoning is the same as before Lemma 3.15). Suppose that $\breve{\Phi}$ and $\breve{\Psi}$ are defined as in the statement. Then $\breve{\Phi}$ is an injective group homomorphism because b has order eight in $\mathrm{C}_3 \rtimes \mathrm{C}_8$, and $\breve{\Psi}$ is surjective by definition. Moreover, $a^3 = b^8 = 1$ and $bab^{-1} = a^{-1}$ in S_3, so $\breve{\Psi}$ is a group homomorphism since its image satisfies the relations of $\mathrm{C}_3 \rtimes \mathrm{C}_8$ given by (329). Since $\breve{\Psi}(b^2) = b^2 = 1$, we have that $\mathrm{Ker}(\breve{\Psi}) = \mathrm{Im}(\breve{\Phi})$, i.e. $\breve{\Phi}$ and $\breve{\Psi}$ define a central extension. Since $\mathrm{C}_3 \rtimes \mathrm{C}_8$ is not isomorphic to $\mathrm{C}_4 \times \mathrm{S}_3$, this extension does not split.

Now, the commutativity of (D_{35}) is equivalent to have $\Phi_{\widetilde{\chi_k}} = \breve{\mathscr{S}} \circ \breve{\Phi}$ and $\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}} = \breve{\Psi}$. The condition $\Phi_{\widetilde{\chi_k}} = \breve{\mathscr{S}} \circ \breve{\Phi}$ is equivalent to $a = \Phi_{\widetilde{\chi_k}}(i) = \breve{\mathscr{S}}(\breve{\Phi}(i))$, i.e. $\breve{\mathscr{S}}(b^2) = a$. Recall from (328) that $(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})(a) = (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})(b) = (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})(c) = 1$, $(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})(d) = a$, $(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})(e) = ba$ and

$$\mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}}) =< a, b, c >= \{1, a, a^2, a^{-1}, b, b^{-1}, c, c^{-1}, ab, ac, bc, ab^{-1}, ac^{-1}, cb, abc, a^{-1}bc\}.$$

Hence, since $\breve{\Psi}(a) = a$ and $\breve{\Psi}(b) = b$,

$$\begin{cases} (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(a) = \breve{\Psi}(a) \iff \breve{\mathscr{S}}(a) \in (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})^{-1}(\{a\}) = \mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})d \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =< a, b, c > d \\ (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(b) = \breve{\Psi}(b) \iff \breve{\mathscr{S}}(b) \in (\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})^{-1}(\{b\}) = \mathrm{Ker}(\pi_{\mathrm{ASp}} \circ \Psi_{\widetilde{\chi_k}})ed^{-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =< a, b, c > ed^{-1}. \end{cases}$$

Suppose now that $\breve{\mathscr{S}}$ is a map such that the diagram (D_{35}) is commutative, i.e. $\breve{\mathscr{S}}(b^2) = a$, $\breve{\mathscr{S}}(a) \in < a, b, c > d$, and $\breve{\mathscr{S}}(b) \in < a, b, c > ed^{-1}$. Then $\breve{\mathscr{S}}$ is a group homomorphism if and only if its image satisfies the relations of $\mathrm{C}_3 \rtimes \mathrm{C}_8$ given by (329), i.e. $\breve{\mathscr{S}}(b)^2 = a$, $\breve{\mathscr{S}}(a)^3 = \breve{\mathscr{S}}(b)^8 = 1$ and $\breve{\mathscr{S}}(b)\breve{\mathscr{S}}(a)\breve{\mathscr{S}}(b)^{-1} = \breve{\mathscr{S}}(a)^{-1}$. In particular, $\breve{\mathscr{S}}(b)^2 = a$ implies that $\breve{\mathscr{S}}(b)^8 = 1$. If we examine all the possibilities for $\breve{\mathscr{S}}(a) \in < a, b, c > d$ and $\breve{\mathscr{S}}(b) \in < a, b, c > ed^{-1}$, we see that the conditions $\breve{\mathscr{S}}(b)^2 = a$ and $\breve{\mathscr{S}}(a)^3 = 1$ are satisfied if and only if $\breve{\mathscr{S}}(a) \in \{1, b^{-1}, c^{-1}, bc\}d$ and $\breve{\mathscr{S}}(b) \in \{1, a^2, bc, cb\}ed^{-1}$. Finally, the condition $\breve{\mathscr{S}}(b)\breve{\mathscr{S}}(a)\breve{\mathscr{S}}(b)^{-1} = \breve{\mathscr{S}}(a)^{-1}$ provides all the possibilities for $\breve{\mathscr{S}}$ given by (330). These computations prove that these possibilities define in fact group homomorphisms such that (D_{35}) commutes. So, it only remains to prove that $\breve{\mathscr{S}}$ is injective. The image of $\breve{\mathscr{S}}$ is generated by $\breve{\mathscr{S}}(a)$ of order three and $\breve{\mathscr{S}}(b)$ of order eight. Moreover, $\breve{\mathscr{S}}(a)$ and $\breve{\mathscr{S}}(b)$ verify the relations of $\mathrm{C}_3 \rtimes \mathrm{C}_8$ because $\breve{\mathscr{S}}$ is a group homomorphism. Thus the image of $\breve{\mathscr{S}}$ is isomorphic to $\mathrm{C}_3 \rtimes \mathrm{C}_8$, and $\breve{\mathscr{S}}$ is injective. □

Corollary 3.24. *Suppose that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. We take the same notations as in Proposition 3.23. Then, for every homomorphism $\breve{\mathscr{S}}$ defined by* (330)*, the representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}}$ of $\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8$ is unitary, faithful and irreducible.*

Proof. $\breve{\mathscr{S}}$ is an injective group homomorphism by Proposition 3.23, and $\widetilde{\omega}_{\widetilde{\chi_k}}$ is unitary and faithful by Proposition 2.32. Hence $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}}$ is a unitary and faithful representation. Suppose now that $V \subseteq \mathrm{L}^2(Y)$ is a vector subspace which is invariant under $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}}$. Then V is invariant

under the operator $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(g)$ for every $g \in \mathrm{AMp}(W)$. In particular, V is invariant under $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(a)$ and $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(b)$, which are of the following form by (330):

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(d), \widetilde{\omega}_{\widetilde{\chi_k}}(b^{-1}d)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(ed^{-1}), \widetilde{\omega}_{\widetilde{\chi_k}}(a^2ed^{-1})\}, \end{cases}$$

i.e. by Proposition 3.5,

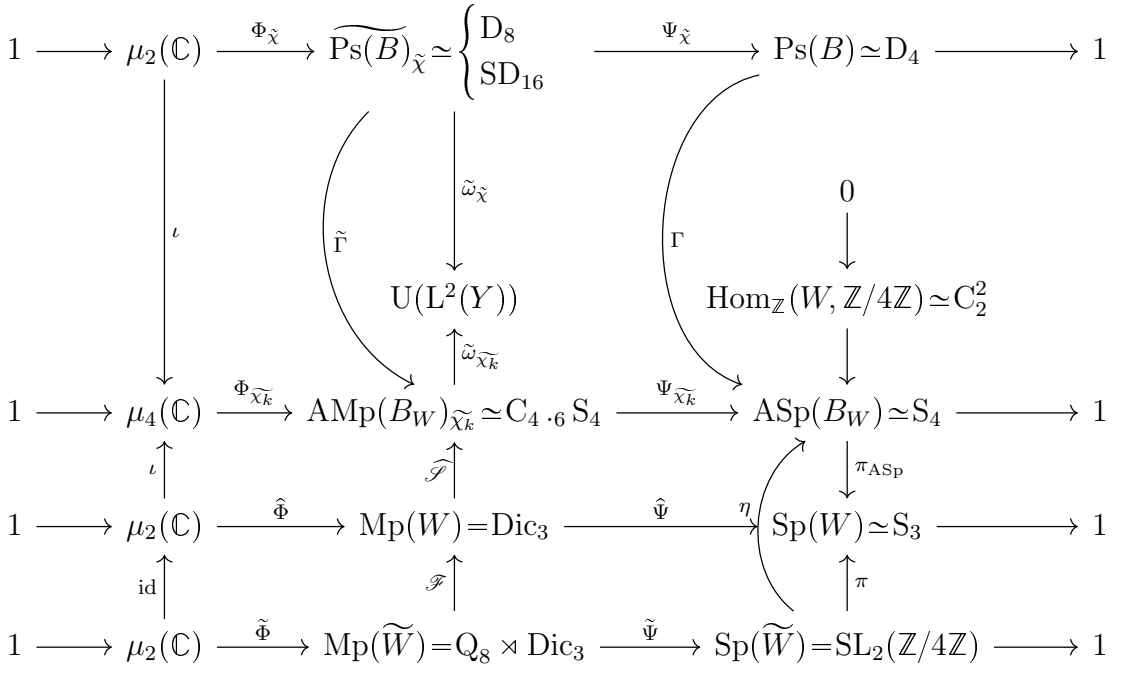
$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(a) \in \left\{ \frac{1}{2}\begin{pmatrix} -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \\ -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \end{pmatrix}, \frac{-i}{2}\begin{pmatrix} -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \\ -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(b) \in \left\{ \frac{1+i}{2}\begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix}, \frac{1+i}{2}\begin{pmatrix} -\widetilde{\varepsilon} & 1 \\ 1 & \widetilde{\varepsilon} \end{pmatrix} \right\}. \end{cases}$$

In any case, the operators $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(a)$ and $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(b)$ do not have any common eigenvalue. Thus V cannot be a one-dimensional space. Since $\dim_{\mathbb{C}}(\mathrm{L}^2(Y)) = 2$, this implies that $V = \{0\}$ or $V = \mathrm{L}^2(Y)$. Therefore, $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}}$ is irreducible. □

Remark 3.25. Recall from Proposition 3.20 that, if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then we cannot extend $\widetilde{\omega}_{\widetilde{\chi_k}}$ to a two-fold covering $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$, if we want a faithful representation and a commutative diagram. Hence, Proposition 3.23 and Corollary 3.24 provide the "best" result we could hope for in this case, in the sense that $\mathrm{AMp}(W) \simeq \mathrm{C}_3 \rtimes \mathrm{C}_8$ is the smallest possible central extension of $\mathrm{Sp}(W)$ which satisfies at the same time that the whole diagram (D_{35}) commutes and the resulting Weil representation of $\mathrm{AMp}(W)$ is a unitary, irreducible and faithful representation.

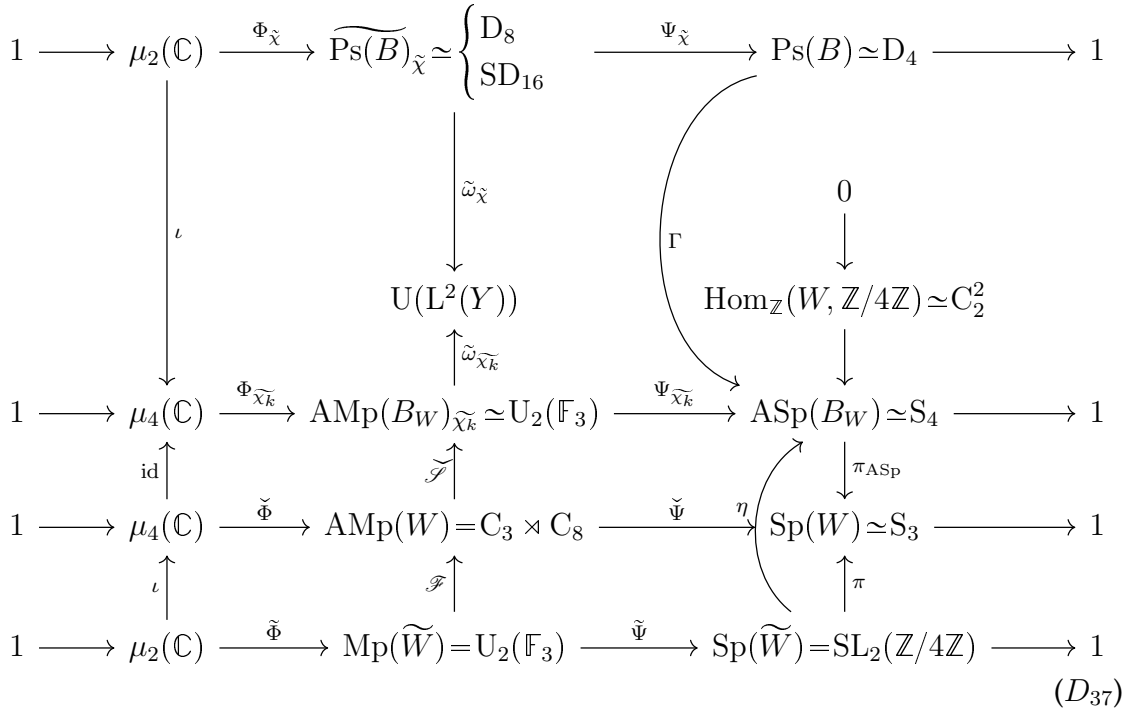
The last question we can ask to conclude the study of the two-dimensional case with B_{Weil} is whether we can insert the metaplectic two-fold covering $\mathrm{Mp}(\widetilde{W})$ obtained in section 3.1.4 into the diagrams (D_{33}) and (D_{35}), i.e. if we can obtain the following two commutative diagrams:

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then



(D_{36})

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then



(D_{37})

Unfortunately, the next proposition shows that it is impossible to "connect" the metaplectic coverings $\mathrm{Mp}(W)$ and $\mathrm{AMp}(W)$ to $\mathrm{Mp}(\widetilde{W})$ this way.

Proposition 3.26. *There are no group homomorphisms $\mathscr{F}$ such that any of the diagrams (D_{36}) or (D_{37}) commute.*

Proof. Recall that $\pi(a) = \pi(b) = 1$, $\pi(c) = a$ and $\pi(d) = ba$.

- Suppose that $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. Recall that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are defined in Lemma 3.15, and $\widehat{\Phi}$ and $\widehat{\Psi}$ in Proposition 3.20. Let $\mathscr{F} : \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \to \mathrm{Dic}_3$. Then the diagram (D_{36}) is commutative if and only if $\mathscr{F} \circ \widetilde{\Phi} = \widehat{\Phi}$ and $\pi \circ \widetilde{\Psi} = \widehat{\Psi} \circ \mathscr{F}$. The condition $\mathscr{F} \circ \widetilde{\Phi} = \widehat{\Phi}$ is equivalent to $\mathscr{F}(a^2) = b^2$. The condition $\pi \circ \widetilde{\Psi} = \widehat{\Psi} \circ \mathscr{F}$ is equivalent to

$$\begin{cases} (\widehat{\Psi} \circ \mathscr{F})(a) = (\pi \circ \widetilde{\Psi})(a) = \pi(b) = 1 \\ (\widehat{\Psi} \circ \mathscr{F})(b) = (\pi \circ \widetilde{\Psi})(b) = \pi(ab) = 1 \\ (\widehat{\Psi} \circ \mathscr{F})(c) = (\pi \circ \widetilde{\Psi})(c) = \pi(ac^{-1}d^2) = a^{-1} \\ (\widehat{\Psi} \circ \mathscr{F})(d) = (\pi \circ \widetilde{\Psi})(d) = \pi(cbcd) = a^2ba = ab \end{cases} \iff \begin{cases} \mathscr{F}(a) \in \mathrm{Ker}(\widehat{\Psi}) =< b^2 > \\ \mathscr{F}(b) \in \mathrm{Ker}(\widehat{\Psi}) =< b^2 > \\ \mathscr{F}(c) \in \widehat{\Psi}^{-1}(\{a^{-1}\}) = \mathrm{Ker}(\widehat{\Psi})a^{-1} \\ \mathscr{F}(d) \in \widehat{\Psi}^{-1}(\{ab\}) = \mathrm{Ker}(\widehat{\Psi})ab. \end{cases}$$

  In particular, if $\mathscr{F}$ is a group homomorphism such that (D_{36}) commutes, then $\mathscr{F}(a)^2 = b^2$ and $\mathscr{F}(a) \in < b^2 >$. This is impossible since b has order four in Dic_3.

- Suppose that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. Recall that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are given by (310), and $\breve{\Phi}$ and $\breve{\Psi}$ are defined in Proposition 3.23. Let $\mathscr{F} : \mathrm{Mp}(\widetilde{W}) = \mathrm{U}_2(\mathbb{F}_3) \to \mathrm{C}_3 \rtimes \mathrm{C}_8$. Then the diagram (D_{37}) is commutative if and only if $\mathscr{F} \circ \widetilde{\Phi} = \breve{\Phi} \circ \iota$ and $\pi \circ \widetilde{\Psi} = \breve{\Psi} \circ \mathscr{F}$. The condition $\mathscr{F} \circ \widetilde{\Phi} = \breve{\Phi} \circ \iota$ is equivalent to $\mathscr{F}(a^2) = b^4$. The condition $\pi \circ \widetilde{\Psi} = \breve{\Psi} \circ \mathscr{F}$ is equivalent to

$$\begin{cases} (\breve{\Psi} \circ \mathscr{F})(a) = (\pi \circ \widetilde{\Psi})(a) = \pi(d^2) = 1 \\ (\breve{\Psi} \circ \mathscr{F})(b) = (\pi \circ \widetilde{\Psi})(b) = \pi(a) = 1 \\ (\breve{\Psi} \circ \mathscr{F})(c) = (\pi \circ \widetilde{\Psi})(c) = \pi(b) = 1 \\ (\breve{\Psi} \circ \mathscr{F})(d) = (\pi \circ \widetilde{\Psi})(d) = a \\ (\breve{\Psi} \circ \mathscr{F})(e) = (\pi \circ \widetilde{\Psi})(e) = ba \end{cases} \iff \begin{cases} \mathscr{F}(a) \in \mathrm{Ker}(\breve{\Psi}) =< b^2 > \\ \mathscr{F}(b) \in \mathrm{Ker}(\breve{\Psi}) =< b^2 > \\ \mathscr{F}(c) \in \mathrm{Ker}(\breve{\Psi}) =< b^2 > \\ \mathscr{F}(d) \in \breve{\Psi}^{-1}(\{a\}) = \mathrm{Ker}(\breve{\Psi})a \\ \mathscr{F}(e) \in \breve{\Psi}^{-1}(\{ba\}) = \mathrm{Ker}(\breve{\Psi})ba. \end{cases}$$

  In particular, if $\mathscr{F}$ is a group homomorphism such that (D_{37}) commutes, then $\mathscr{F}(a)^2 = b^4$ and $\mathscr{F}(a)$, $\mathscr{F}(b)$ and $\mathscr{F}(c)$ all belong to $< b^2 >$, which is the center of $\mathrm{C}_3 \rtimes \mathrm{C}_8$. Since $\mathscr{F}$ is a group homomorphism, its image has to satisfy the relations of $\mathrm{U}_2(\mathbb{F}_3)$, in particular $\mathscr{F}(c)\mathscr{F}(b)\mathscr{F}(c)^{-1} = \mathscr{F}(a)^2\mathscr{F}(b)$. So the commutativity of (D_{37}) would imply that $\mathscr{F}(a)^2 = 1$, i.e. $b^4 = 1$, which is impossible since b has order eight in $\mathrm{C}_3 \rtimes \mathrm{C}_8$.

□

Remark 3.27. The present section was an attempt to find an extension of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to a group G which is "closer" to $\mathrm{Sp}(W)$ than $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\mathrm{Mp}(\widetilde{W})$ (96 elements). If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then we have obtained in Corollary 3.21 a unitary, irreducible and faithful Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ of a two-fold covering $\mathrm{Mp}(W) = \mathrm{Dic}_3$ (12 elements) of $\mathrm{Sp}(W)$. If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then Corollary 3.24 showed that the best we can obtain is a unitary, irreducible and faithful Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}}$

of a four-fold covering $\mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8$ (24 elements) of $\mathrm{Sp}(W)$. We were able to find such results because in this two-dimensional example $\mathrm{Sp}(W)$ is isomorphic to S_3, and hence $\mathrm{H}^2(\mathrm{Sp}(W), \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{H}^2(\mathrm{Sp}(W), \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ are not trivial. In order to know if similar results can be expected in higher dimension, we should first compute these cohomology groups. By [Ste81], if W is a $2n$-dimensional $\mathbb{F}_2$-vector space then the Schur multiplier of $\mathrm{Sp}(W)$ is

$$\mathrm{H}_2(\mathrm{Sp}(W), \mathbb{Z}) = \begin{cases} \{0\} & \text{if } n = 1 \text{ or } n \geqslant 4 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \in \{2, 3\}. \end{cases}$$

Hence, applying the universal coefficient theorem, we get the following short split exact sequence, where $N \in \{2, 4\}$:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{Sp}(W), \mathbb{Z}), \mathbb{Z}/N\mathbb{Z}) \to \mathrm{H}^2(\mathrm{Sp}(W), \mathbb{Z}/N\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(\mathrm{Sp}(W), \mathbb{Z}), \mathbb{Z}/N\mathbb{Z}) \to 0.$$

Since

$$\mathrm{H}_1(\mathrm{Sp}(W), \mathbb{Z}) \simeq \mathrm{Sp}(W)/[\mathrm{Sp}(W), \mathrm{Sp}(W)] \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \in \{1, 2\} \\ \{0\} & \text{if } n \geqslant 3 \end{cases},$$

by Proposition 2.7.v, we deduce that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(\mathrm{Sp}(W), \mathbb{Z}), \mathbb{Z}/N\mathbb{Z}) \simeq \begin{cases} \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) = \mathbb{Z}/gcd(2, N)\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} & \text{if } n \in \{1, 2\} \\ \{0\} & \text{if } n \geqslant 3. \end{cases}$$

Hence we obtain

$$\mathrm{H}^2(\mathrm{Sp}(W), \mathbb{Z}/N\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \in \{1, 3\} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2 \\ \{0\} & \text{if } n \geqslant 4. \end{cases} \tag{331}$$

Therefore, if $n \in \{2, 3\}$, it may be possible to obtain a unitary, irreducible and faithful Weil representation of either a two-fold covering $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$, or a four-fold covering $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$, as we just did in this section. However, such a result cannot be obtained if $n \geqslant 4$, i.e. if $\dim_{\mathbb{F}_2}(W) \geqslant 8$. On the other hand, since by [FP20],

$$\mathrm{H}_2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \quad \text{if } n \geqslant 3,$$

using again the universal coefficient theorem and Proposition 2.7.v, we obtain

$$\mathrm{H}^2(\mathrm{Sp}(\widetilde{W}), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \quad \text{if } n \geqslant 3. \tag{332}$$

Therefore, it makes sense to look for a Weil representation of a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ in this case. Unfortunately, as we proved in Lemma 2.42, there is no hope to obtain this way a faithful representation of $\mathrm{Mp}(\widetilde{W})$ if $n \geqslant 2$, even if the original Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is faithful (cf. section 2.5).

3.2 The case $\mathrm{Arf}(Q) = 1$

In this section, we suppose that the bilinear form B which satisfies (290) is such that Q has Arf invariant 1, or equivalently Witt index 0. In particular, we can consider the bilinear form B given by

$$B(w_1, w_2) = x_1y_2 + x_1x_2 + y_1y_2 \quad \text{for all } w_1 = x_1 + y_1 \text{ and } w_2 = x_2 + y_2 \text{ in } W.$$

Thus the matrix of B in the ordered basis $\{(1,0),(0,1)\}$ of W is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $Q(w) = xy + x^2 + y^2$ for all $w = x + y \in W$. In this case, $\mathrm{H}(B)$ is the set of elements $(x, y, t) \in (\mathbb{F}_2)^3$ with product

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1y_2 + x_1x_2 + y_1y_2). \tag{333}$$

In particular, $\mathrm{H}(B)$ is isomorphic to the quaternionic group

$$\mathrm{Q}_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} \rangle,$$

where for example $a = (1,0,0)$ and $b = (0,1,0)$.

We need to fix an extension $\widetilde{\chi}$ of χ to the abelian subgroup $X \times \mathbb{F}_2$ of $\mathrm{H}(B)$. This extension is entirely determined by $\widetilde{\chi}(1,0,0)$, which can take the values $\pm i$ since $B((1,0,0),(1,0,0)) = 1$ (cf. Lemma 1.15). Set

$$\widetilde{\xi} = \widetilde{\chi}(1,0,0) \in \{\pm i\}. \tag{334}$$

3.2.1 The pseudosymplectic group $\mathrm{Ps}(B)$ and its two-fold covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$

We determine first the groups $\mathrm{O}(Q)$ and $\mathrm{Ps}(B)$. Then, in order to obtain all the possible two-fold coverings

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi}}} \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \xrightarrow{\Psi_{\widetilde{\chi}}} \mathrm{Ps}(B) \longrightarrow 1$$

defined in Proposition 1.36.ii, we consider all the possible choices of $\{\gamma(\sigma), \sigma \in \mathrm{O}(Q)\}$ satisfying (46) and (48). As in Proposition 3.2, we will prove in (339) that there exist two different two-fold coverings up to isomorphism.

Proposition 3.28. *Recall the notation σ_v for the transvection of W defined in Proposition 2.7.iv by $\sigma_v : w \mapsto w + \langle w, v \rangle v$.*

i. *The orthogonal group is $\mathrm{O}(Q) = \mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$, i.e.*

$$\mathrm{O}(Q) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

In particular, this is the only case for which $\mathrm{O}(Q) = \mathrm{Sp}(W)$ (cf. Proposition 1.13).

ii. *The pseudosymplectic group is*

$$\mathrm{Ps}(B) = \left\{ \begin{array}{l} \left(\begin{pmatrix}1&0\\0&1\end{pmatrix},0\right), \left(\begin{pmatrix}1&0\\0&1\end{pmatrix},x\right), \left(\begin{pmatrix}1&0\\0&1\end{pmatrix},y\right), \left(\begin{pmatrix}1&0\\0&1\end{pmatrix},x+y\right), \\ \left(\begin{pmatrix}0&1\\1&0\end{pmatrix},xy\right), \left(\begin{pmatrix}0&1\\1&0\end{pmatrix},xy+x\right), \left(\begin{pmatrix}0&1\\1&0\end{pmatrix},xy+y\right), \left(\begin{pmatrix}0&1\\1&0\end{pmatrix},xy+x+y\right), \\ \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},0\right), \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},x\right), \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},y\right), \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},x+y\right), \\ \left(\begin{pmatrix}1&0\\1&1\end{pmatrix},xy\right), \left(\begin{pmatrix}1&0\\1&1\end{pmatrix},xy+x\right), \left(\begin{pmatrix}1&0\\1&1\end{pmatrix},xy+y\right), \left(\begin{pmatrix}1&0\\1&1\end{pmatrix},xy+x+y\right), \\ \left(\begin{pmatrix}1&1\\0&1\end{pmatrix},xy\right), \left(\begin{pmatrix}1&1\\0&1\end{pmatrix},xy+x\right), \left(\begin{pmatrix}1&1\\0&1\end{pmatrix},xy+y\right), \left(\begin{pmatrix}1&1\\0&1\end{pmatrix},xy+x+y\right), \\ \left(\begin{pmatrix}1&1\\1&0\end{pmatrix},0\right), \left(\begin{pmatrix}1&1\\1&0\end{pmatrix},x\right), \left(\begin{pmatrix}1&1\\1&0\end{pmatrix},y\right), \left(\begin{pmatrix}1&1\\1&0\end{pmatrix},x+y\right) \end{array} \right\}.$$

In particular, $\mathrm{Ps}(B)$ *is isomorphic to the symmetric group*

$$\mathrm{S}_4 = \langle\, a,b,c,d \mid a^2 = b^2 = c^3 = d^2 = 1,\, cac^{-1} = dad = ab = ba,\, cbc^{-1} = a,\, bd = db,\, dcd = c^{-1} \rangle,$$

an isomorphism being for example given by $a = (1,x)$, $b = (1,y)$, $c = \left(\begin{pmatrix}0&1\\1&1\end{pmatrix},0\right)$ *and* $d = \left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, xy+x+y\right)$.

iii. *The different choices of* $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ *satisfying* (46) *and* (48) *lead to two non-isomorphic pseudosymplectic extensions*

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\tilde{\chi}}} \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \xrightarrow{\Psi_{\tilde{\chi}}} \mathrm{Ps}(B) \longrightarrow 1.$$

More precisely,

- *If* $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$ *then the pseudosymplectic covering is*

$$\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} = \{(s, (-1)^l \omega_{\tilde{\chi}}(s)),\ s \in \mathrm{Ps}(B),\ l \in \mathbb{Z}\},$$

where the set of operators $(-1)^l \omega_{\tilde{\chi}}(s)$ *is given by (in the same order as the elements of* $\mathrm{Ps}(B)$ *above and where* $\widetilde{\xi}$ *is as in* (334)*):*

$$\left\{\begin{array}{l}
(-1)^l\begin{pmatrix}1&0\\0&1\end{pmatrix}, (-1)^l\begin{pmatrix}0&-1\\1&0\end{pmatrix}, (-1)^l\begin{pmatrix}\widetilde{\xi}&0\\0&-\widetilde{\xi}\end{pmatrix}, (-1)^l\begin{pmatrix}0&\widetilde{\xi}\\\widetilde{\xi}&0\end{pmatrix},\\
\frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}\widetilde{\xi}&-1\\1&-\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}1&\widetilde{\xi}\\\widetilde{\xi}&1\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}1&-\widetilde{\xi}\\-\widetilde{\xi}&1\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}\widetilde{\xi}&1\\-1&-\widetilde{\xi}\end{pmatrix},\\
\frac{(-1)^l}{2}\begin{pmatrix}1-\widetilde{\xi}&1-\widetilde{\xi}\\-1-\widetilde{\xi}&1+\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{2}\begin{pmatrix}1-\widetilde{\xi}&-1+\widetilde{\xi}\\1+\widetilde{\xi}&1+\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{2}\begin{pmatrix}1+\widetilde{\xi}&-1-\widetilde{\xi}\\1-\widetilde{\xi}&1-\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{2}\begin{pmatrix}1+\widetilde{\xi}&1+\widetilde{\xi}\\-1+\widetilde{\xi}&1-\widetilde{\xi}\end{pmatrix},\\
\frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}, \frac{(-1)^l\widetilde{\xi}}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}, \frac{(-1)^l\widetilde{\xi}}{\sqrt{2}}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix},\\
\frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}1-\widetilde{\xi}&0\\0&1+\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}0&-1-\widetilde{\xi}\\1-\widetilde{\xi}&0\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}1+\widetilde{\xi}&0\\0&1-\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{\sqrt{2}}\begin{pmatrix}0&-1+\widetilde{\xi}\\1+\widetilde{\xi}&0\end{pmatrix},\\
\frac{(-1)^l}{2}\begin{pmatrix}1+\widetilde{\xi}&-1+\widetilde{\xi}\\1+\widetilde{\xi}&1-\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{2}\begin{pmatrix}1-\widetilde{\xi}&1+\widetilde{\xi}\\-1+\widetilde{\xi}&1+\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{2}\begin{pmatrix}1-\widetilde{\xi}&-1-\widetilde{\xi}\\1-\widetilde{\xi}&1+\widetilde{\xi}\end{pmatrix}, \frac{(-1)^l}{2}\begin{pmatrix}1+\widetilde{\xi}&1-\widetilde{\xi}\\-1-\widetilde{\xi}&1-\widetilde{\xi}\end{pmatrix}
\end{array}\right\}.$$

In particular, $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is isomorphic to the conformal special unitary (or binary octahedral) group

$$\mathrm{CSU}_2(\mathbb{F}_3) = \langle a,b,c,d \mid a^4 = c^3 = 1, b^2 = d^2 = a^2, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = ab, dad^{-1} = a^2b, cbc^{-1} = a, dcd^{-1} = c^{-1}\rangle,$$

an isomorphism being given for example by

$$a = \left((1,x),\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right),\ b = \left((1,y),\begin{pmatrix}i&0\\0&-i\end{pmatrix}\right),$$

$$c = \left(\left(\begin{pmatrix}0&1\\1&1\end{pmatrix}, (x+y)\delta_{-i}(\widetilde{\xi})\right), -\frac{1}{2}\begin{pmatrix}1-i&1-i\\-1-i&1+i\end{pmatrix}\right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}, xy + (x+y)\delta_i(\widetilde{\xi})\right), \frac{1}{\sqrt{2}}\begin{pmatrix}i&1\\-1&-i\end{pmatrix}\right).$$

- *If $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$ then the pseudosymplectic covering is*

$$\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} = \{(s, (-1)^l\omega_{\widetilde{\chi}}(s)),\ s \in \mathrm{Ps}(B),\ l \in \mathbb{Z}\},$$

where the set of operators $(-1)^l\omega_{\widetilde{\chi}}(s)$ is exactly as the previous one, only replacing every factor $\frac{(-1)^l}{\sqrt{2}}$ with $\frac{(-1)^l i}{\sqrt{2}}$ in the second, fourth and fifth lines. In particular, $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is isomorphic to the general linear group

$$\mathrm{GL}_2(\mathbb{F}_3) = \langle a,b,c,d \mid a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab, dad = a^2b, cbc^{-1} = a, dcd = c^{-1}\rangle,$$

an isomorphism being given for example by

$$a = \left((1,x),\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right),\ b = \left((1,y),\begin{pmatrix}i&0\\0&-i\end{pmatrix}\right),$$

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (x+y)\delta_{-i}(\widetilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} \right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + (x+y)\delta_i(\widetilde{\xi}) \right), \frac{i}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right).$$

Proof. **i.** σ is orthogonal if and only if $\begin{pmatrix} x & y \end{pmatrix} \sigma^T \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ for all $x, y \in \mathbb{F}_2$.

ii. According to Proposition 1.12, the elements of $\mathrm{Ps}(B)$ are given by $(\sigma, f_\sigma + f)$, where $\sigma \in \mathrm{O}(Q)$ and $f \in \mathscr{Q}_a(W)$. This gives $f_\sigma = 0$ if $\sigma \in \left\{ I_2, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$, and $f_\sigma(x, y) = xy$ if $\sigma \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$. Proposition 3.1 then provides all the 24 elements of $\mathrm{Ps}(B)$ listed in the statement above.

iii. Before computing the operators $\omega_{\widetilde{\chi}}(s, w_s)$ for all $s \in \mathrm{Ps}(B)$ from formula (38), we are going to study the different possibilities for $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying (46) and (48). Recall the notation σ_v for the transvection defined in Proposition 2.7.iv. Then the group $\mathrm{O}(Q) = \mathrm{Sp}(W)$ is given by

$$\mathrm{Sp}(W) = \left\{ \sigma_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_{(1,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_{(1,0)}\sigma_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \sigma_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \right.$$
$$\left. \sigma_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \sigma_{(0,1)}\sigma_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

(We have listed the elements σ of $\mathrm{Sp}(W)$ in the order in which they appear as $(\sigma, \cdot)$ in i.) By (48), $\theta(\sigma_{(0,0)}) \equiv 0 \mod \pi$. Moreover, $\sigma_{(1,0)}^2 = \sigma_{(0,1)}^2 = \sigma_{(1,1)}^2 = 1$, so by (48),

$$\theta(\sigma_v) \equiv 0 \mod \frac{\pi}{2} \quad \text{for every } v \neq 0. \tag{335}$$

Observe now that $\sigma_{(1,0)}\sigma_{(0,1)} = \sigma_{(0,1)}\sigma_{(1,1)} = \sigma_{(1,1)}\sigma_{(1,0)}$. By (48), this implies that

$$\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv \begin{cases} \theta(\sigma_{(1,0)}) + \theta(\sigma_{(0,1)}) \mod \pi \\ \theta(\sigma_{(0,1)}) + \theta(\sigma_{(1,1)}) \mod \pi \\ \theta(\sigma_{(1,1)}) + \theta(\sigma_{(1,0)}) \mod \pi. \end{cases}$$

Hence

$$\theta(\sigma_{(1,0)}) \equiv \theta(\sigma_{(0,1)}) \equiv \theta(\sigma_{(1,1)}) \mod \pi. \tag{336}$$

Now by (335) and (336), there are two possibilities:

- If $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$ then by (336), $\theta(\sigma_{(0,1)}) \equiv \theta(\sigma_{(1,1)}) \equiv 0 \mod \pi$. By (48), we also have $\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \pi$ and $\theta(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \mod \pi$. Finally, this proves that

$$\theta(\sigma) \equiv 0 \mod \pi \qquad \text{for every } \sigma \in \mathrm{Sp}(W). \tag{337}$$

In particular, by Proposition 1.36.v, any other choice $\{\gamma'(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta'(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (46), (48) and such that $\theta'(\sigma_{(1,0)}) \equiv 0 \mod \pi$, will lead to an extension which is isomorphic to this one.

- If $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$ then by (336), $\theta(\sigma_{(0,1)}) \equiv \theta(\sigma_{(1,1)}) \equiv \frac{\pi}{2} \mod \pi$. By (48), we have $\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \pi$ and $\theta(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \mod \pi$. In particular, by Proposition 1.36.v, any other choice $\{\gamma'(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta'(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (46), (48) and such that $\theta'(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, will lead to an extension which is isomorphic to this one.

This shows that, up to isomorphism, the different choices $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (46) and (48) lead to at most two pseudosymplectic extensions. Now, we can compute explicitly the operators $\omega_{\widetilde{\chi}}(s)$ for all $s \in \mathrm{Ps}(B)$ using Proposition 1.28:

$$\omega_{\widetilde{\chi}}(s, w_s) = \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f(w) + \langle w_s, w\rangle + B(u, w + w_s))\rho_{\widetilde{\chi}}(u + w_s), \tag{338}$$

where $w \in W$ is any element such that $u = (1-\sigma)(w)$ and $\gamma(\sigma)$ satisfies (46) and (48). We give the details of the computations only for $s = \left(\sigma_{(1,0)}\sigma_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0\right)$, the other cases being obtained in a similar way. First we make a choice of w_s satisfying (37) (the isomorphism class of the extension defined by $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ does not depend on this choice according to Proposition 1.36). Here we have $\mathrm{K}(\sigma_{(1,0)}\sigma_{(0,1)}) = \mathrm{Ker} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \{0\}$, so every $w_s \in W$ satisfies (37), and we can choose for example $w_s = 0$. By (338) and (28),

$$\begin{aligned}
\omega_{\widetilde{\chi}}(s, 0) &= \gamma(\sigma_{(1,0)}\sigma_{(0,1)}) \sum_{u \in \mathrm{I}(\sigma)} \chi(B(u, w))\rho_{\widetilde{\chi}}(u) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} (\rho_{\widetilde{\chi}}(0) + \chi(B((1,0),(0,1)))\rho_{\widetilde{\chi}}(1,0) + \chi(B((0,1),(1,1)))\rho_{\widetilde{\chi}}(0,1) \\
&\qquad + \chi(B((1,1),(1,0)))\rho_{\widetilde{\chi}}(1,1)) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} (\rho_{\widetilde{\chi}}(0) - \rho_{\widetilde{\chi}}(1,0) - \rho_{\widetilde{\chi}}(0,1) - \rho_{\widetilde{\chi}}(1,1)) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \widetilde{\xi} & 0 \\ 0 & -\widetilde{\xi} \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \widetilde{\xi} \\ \widetilde{\xi} & 0 \end{pmatrix} \right) \\
&= \frac{e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})}}{2} \begin{pmatrix} 1 - \widetilde{\xi} & 1 - \widetilde{\xi} \\ -1 - \widetilde{\xi} & 1 + \widetilde{\xi} \end{pmatrix},
\end{aligned}$$

where $e^{i\theta(\sigma_{(1,0)}\sigma_{(0,1)})} = \pm 1$ since $\theta(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \pi$ for any choice of $\theta(\sigma_{(1,0)})$. This enables to establish the two lists of elements of the groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ given in the statement. Now, it only remains to identify the groups and extensions defined by the two possible choices of $\theta(\sigma_{(1,0)})$. We know that $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is a non-split stem central extension of $\mathrm{Ps}(B) \simeq \mathrm{S}_4$ by two elements. Hence, by [GN], $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is isomorphic to one of the following groups: $\mathrm{CSU}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_3)$. These groups having different numbers of

elements of order two, we can immediately deduce from the lists of elements given above the following group isomorphisms:

$$\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \begin{cases} \mathrm{CSU}_2(\mathbb{F}_3) & \text{if } \theta(\sigma_{(1,0)}) \equiv 0 \mod \pi \\ \mathrm{GL}_2(\mathbb{F}_3) & \text{if } \theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi. \end{cases} \tag{339}$$

□

Remark 3.29 (Related to Remarks 1.37 and 3.3)**.** The result obtained in (339) proves that in the case $\mathrm{Arf}(Q) = 1$, we obtain all the non-split stem central extensions of $\mathrm{Ps}(B)$ by two elements with all possible choices of $\{\gamma(\sigma),\ \sigma \in \mathrm{O}(Q)\}$. In fact, in the exact same way as we found (304), we compute with the universal coefficient theorem that

$$\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{340}$$

This means that there are four non-isomorphic central extensions of $\mathrm{Ps}(B) \simeq \mathrm{S}_4$ by two elements. The database [GN] gives exactly four possible associated groups, only two of them being associated to non-split stem extensions: $\mathrm{CSU}_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$. Moreover, by Corollary 1.38, $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ contains a copy of the Heisenberg group $\mathrm{H}(B)$, which is isomorphic to Q_8 in this example, and the groups $\mathrm{CSU}_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$ are also the only groups among the central extensions of S_4 by two elements which contain Q_8. Therefore, in this case, we obtain all the possible non-split stem central extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$ by two elements, and this set of extensions coincides with the central extensions of $\mathrm{Ps}(B)$ by two elements which contain the Heisenberg group $\mathrm{H}(B)$. Moreover, (339) shows also that if (50) defines a coboundary, then the condition (51) is satisfied in this case (cf. Proposition 1.36.v).

Recall from Proposition 1.13 that $\mathrm{O}(Q) \subseteq \mathrm{Sp}(W)$. At this stage, we already see that the situation is completely different from the case where $\mathrm{Arf}(Q) = 0$. In fact, the case for which $\dim_{\mathbb{F}_2}(W) = 2$ and $\mathrm{Arf}(Q) = 1$ is the only one which satisfies that $\mathrm{O}(Q) = \mathrm{Sp}(W)$. The fact that the orthogonal and symplectic groups coincide implies that the pseudosymplectic covering, hence the Weil representation, is here directly related to the entire symplectic group. This being said, there is not much interest in considering the affine symplectic group and the affine metaplectic group in this case, since in general these groups are a way to recover a missing link with the entire symplectic group when $\mathrm{O}(Q)$ is strictly contained in $\mathrm{Sp}(W)$. Notice that, in the present example, the groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ satisfy

$$|\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}| = 2|\,\mathrm{Ps}(B)| = 8|\,\mathrm{Sp}(W)| \quad \text{and} \quad |\,\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}| = 4|\,\mathrm{ASp}(B_W)| = 16|\,\mathrm{Sp}(W)|.$$

Therefore, $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is "twice closer" to $\mathrm{Sp}(W)$ than $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. For the sake of completeness, we still pursue the study of the groups $\mathrm{ASp}(B_W)$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, in order to obtain explicitly the entire diagram involving these groups and relate it to the one associated to B_{Weil} and to the work of Gurevich and Hadani.

3.2.2 The affine symplectic group $\mathrm{ASp}(B_W)$ and its coverings $\mathrm{G}^0_{\widetilde{\chi_k}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$

We recall the expression of $\langle \cdot, \cdot \rangle_{\widetilde{W}}$ and we choose the following bilinear form $B_{\widetilde{W}}$, so that it gives $B = \kappa \circ B_W = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

$$\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} = \langle (\widetilde{x}_1, \widetilde{y}_1), (\widetilde{x}_2, \widetilde{y}_2) \rangle_{\widetilde{W}} = \widetilde{x}_1\widetilde{y}_2 - \widetilde{y}_1\widetilde{x}_2 \quad \text{and} \quad B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) = \widetilde{x}_1\widetilde{y}_2 + \widetilde{x}_1\widetilde{x}_2 + \widetilde{y}_1\widetilde{y}_2,$$

for all $\widetilde{w}_1 = \widetilde{x}_1 + \widetilde{y}_1$ and $\widetilde{w}_2 = \widetilde{x}_2 + \widetilde{y}_2$ in $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$. In this case, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is the set of elements $(x, y, t) \in \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}/4\mathbb{Z}$ with the product:

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_1y_2 + x_1x_2 + y_1y_2)). \tag{341}$$

In particular, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is isomorphic to the Pauli group

$$\mathrm{C}_4 \circ \mathrm{D}_4 = \langle a, b, c \mid a^4 = c^2 = 1, b^2 = a^2, ab = ba, ac = ca, cbc = a^2b \rangle,$$

where for example $a = (0, 0, 1)$, $b = (1, 0, 0)$ and $c = (0, 1, 1)$.

We need to fix a faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$ ($k = 1$ or $k = 3$ by Lemma 2.8) and an extension $\widetilde{\chi_k}$ of χ_k to $X \times \mathbb{Z}/4\mathbb{Z}$. This extension is entirely determined by $\widetilde{\chi_k}(1, 0, 0)$, which can take the values $\pm i$ because $B_W((1, 0, 0), (1, 0, 0)) = 2$ (cf. Lemma 2.8). According to Corollary 2.28, the smallest non-split central extension of $\mathrm{ASp}(B_W)$ that we can construct to lift the projective Weil representation is a two-fold covering $\mathrm{G}^0_{\widetilde{\chi_k}}$ if $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ is chosen so that (112) and (118) are satisfied. We will also consider the extension $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ by four elements, by choosing $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$ such that (112) and (116) are satisfied, the latter condition being equivalent to (117) since W is two-dimensional. By Proposition 2.39, the extensions associated to $\mathrm{G}^0_{\widetilde{\chi_k}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ are compatible with $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, in the sense of the commutativity of the diagram (D_{22}), provided (190) and (192) are satisfied, respectively. Since we will be able to satisfy these conditions, we will not consider in this section the extension $\widetilde{\mathrm{ASp}(B_W)}_{\widetilde{\chi_k}}$, whose construction was done in Proposition 2.29 in order to find a link between the two extensions $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{G}_{\widetilde{\chi_k}}$, only in the case where there is no such link when $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{G}^0_{\widetilde{\chi_k}}$ or $\mathrm{G}_{\widetilde{\chi_k}} = \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. By (164), if we want to relate $\mathrm{G}_{\widetilde{\chi_k}}$ to $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ using Proposition 2.39, we need $\widetilde{\chi_k}$ to be equal to $\widetilde{\chi}$ on $X \times \{0\}$. This is why we set

$$\widetilde{\chi_k}(1, 0, 0) = \widetilde{\chi}(1, 0, 0) = \widetilde{\xi}, \tag{342}$$

as in (334). Now we can describe explicitly the group $\mathrm{ASp}(B_W)$ and the group homomorphism $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ defined in Proposition 2.7.vi. Moreover, we can examine all the possible choices of $\{\delta(\sigma)$ satisfying (112), $\sigma \in \mathrm{Sp}(W)\}$ such that (118) and (117) are satisfied, respectively, in order to describe explicitly all the two-fold coverings

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{G}^0_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(B_W) \longrightarrow 1$$

and four-fold coverings

$$1 \longrightarrow \mu_4(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(B_W) \longrightarrow 1$$

that we can obtain in this way by Proposition 2.29. Then we will be able to decide in section 3.2.3 which of these coverings is compatible with the pseudosymplectic covering, in the sense that the diagram (D_{22}) commutes.

Proposition 3.30. *Recall the notation introduced in* (291) *and the transvection of W defined in Proposition 2.7.iv by $\sigma_v : w \mapsto w + \langle w, v\rangle v$.*

i. *The group homomorphism $\eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W)$ defined in Proposition 2.7 is explicitly*

$$\widetilde{\sigma} \mapsto (\sigma, f_{\widetilde{\sigma}})$$

given by the following set (the images being given in the same order as the elements of $\mathrm{Sp}(\widetilde{W})$ listed in Proposition 3.1):

$$\left\{\begin{array}{l}
\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},0\right),\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},2x\right),\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},2y\right),\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},2x+2y\right),\\
\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy+2x\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy+2y\right),\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},2xy+2x+2y\right),\\
\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},2y\right),\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},2x+2y\right),\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},0\right),\left(\begin{pmatrix}0&1\\1&1\end{pmatrix},2x\right),\\
\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},2xy+2x\right),\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},2xy\right),\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},2xy+2x+2y\right),\left(\begin{pmatrix}1&0\\1&1\end{pmatrix},2xy+2y\right),\\
\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},2xy+2y\right),\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},2xy+2x+2y\right),\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},2xy\right),\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},2xy+2x\right),\\
\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},2x\right),\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},0\right),\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},2x+2y\right),\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},2y\right)
\end{array}\right\}.$$

ii. *The affine symplectic group is precisely the image of η given above. In particular, $\mathrm{ASp}(B_W)$ is isomorphic to*

$$\mathrm{S}_4 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = 1,\, cac^{-1} = dad = ab = ba,\, cbc^{-1} = a,\, bd = db,\, dcd = c^{-1}\rangle,$$

an isomorphism being for example given by $a = (1, 2x)$, $b = (1, 2y)$, $c = \left(\begin{pmatrix}0&1\\1&1\end{pmatrix}, 0\right)$ and $d = \left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, 2xy + 2x + 2y\right)$.

iii. *The choices of $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying* (112) *and* (118) *lead to two non-isomorphic two-fold coverings*

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{G}^0_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(B_W) \longrightarrow 1.$$

More precisely,

- *If* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \pi$ *then*

$$\mathrm{G}^0_{\widetilde{\chi_k}} = \{(s, (-1)^l \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\},$$

where the set of operators $(-1)^l \omega_{\widetilde{\chi_k}}(s)$ *is as follows (in the same order as the elements of* $\mathrm{ASp}(B_W)$ *above and where* $\widetilde{\xi}$ *is as in* (342)*),*

$$\left\{ \begin{array}{l}
(-1)^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (-1)^l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (-1)^l \begin{pmatrix} \widetilde{\xi} & 0 \\ 0 & -\widetilde{\xi} \end{pmatrix}, (-1)^l \begin{pmatrix} 0 & \widetilde{\xi} \\ \widetilde{\xi} & 0 \end{pmatrix}, \\
\frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \widetilde{\xi} & -1 \\ 1 & -\widetilde{\xi} \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\xi} \\ \widetilde{\xi} & 1 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\xi} \\ -\widetilde{\xi} & 1 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} \widetilde{\xi} & 1 \\ -1 & -\widetilde{\xi} \end{pmatrix}, \\
\frac{(-1)^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & -1-\widetilde{\xi} \\ 1-\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & 1+\widetilde{\xi} \\ -1+\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & 1-\widetilde{\xi} \\ -1-\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}, \\
\frac{(-1)^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & -1+\widetilde{\xi} \\ 1+\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}, \\
\frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{(-1)^l \widetilde{\xi}}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \frac{(-1)^l \widetilde{\xi}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
\frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1+\widetilde{\xi} & 0 \\ 0 & 1-\widetilde{\xi} \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 0 & -1+\widetilde{\xi} \\ 1+\widetilde{\xi} & 0 \end{pmatrix}, \frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1-\widetilde{\xi} & 0 \\ 0 & 1+\widetilde{\xi} \end{pmatrix}, \\
\frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 0 & -1-\widetilde{\xi} \\ 1-\widetilde{\xi} & 0 \end{pmatrix}, \\
\frac{(-1)^l}{\sqrt{2}} \begin{pmatrix} 1-\widetilde{\xi} & 1+\widetilde{\xi} \\ -1+\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & -1+\widetilde{\xi} \\ 1+\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{(-1)^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & 1-\widetilde{\xi} \\ -1-\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \\
\frac{(-1)^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & -1-\widetilde{\xi} \\ 1-\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}
\end{array} \right\}.$$

Moreover, $\mathrm{G}^0_{\widetilde{\chi_k}}$ *is isomorphic to the conformal special unitary (or binary octahedral) group*

$$\mathrm{CSU}_2(\mathbb{F}_3) = \langle a, b, c, d \,|\, a^4 = c^3 = 1,\, b^2 = d^2 = a^2,\, bab^{-1} = dbd^{-1} = a^{-1},\, cac^{-1} = ab,\, dad^{-1} = a^2 b,\, cbc^{-1} = a,\, dcd^{-1} = c^{-1} \rangle,$$

an isomorphism being given for example by

$$a = \left((1, 2x), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad b = \left((1, 2y), \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right),$$

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (2x + 2y)\delta_{-i}(\widetilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} \right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2xy + (2x + 2y)\delta_i(\widetilde{\xi}) \right), \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right).$$

In particular, $\mathrm{G}^0_{\widetilde{\chi_1}}$ *and* $\mathrm{G}^0_{\widetilde{\chi_3}}$ *are isomorphic groups.*

- *If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$ then $\mathrm{G}^0_{\widetilde{\chi_k}} = \{(s, (-1)^l \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}$, where the set of operators $(-1)^l \omega_{\widetilde{\chi_k}}(s)$ is exactly as the previous one, only replacing every factor $\frac{(-1)^l}{\sqrt{2}}$ with $\frac{(-1)^l i}{\sqrt{2}}$ in the lines corresponding to the second, fourth and fifth line of the list of elements of $\mathrm{ASp}(B_W)$. Moreover, $\mathrm{G}^0_{\widetilde{\chi_k}}$ is isomorphic to the general linear group*

$$\mathrm{GL}_2(\mathbb{F}_3) = \langle a, b, c, d \,|a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab,, \\ dad = a^2 b, cbc^{-1} = a, dcd = c^{-1} \rangle,$$

an isomorphism being given for example by

$$a = \left((1, 2x), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \quad b = \left((1, 2y), \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right),$$

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (2x + 2y)\delta_{-i}(\widetilde{\xi})\right), -\frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}\right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 2xy + (2x + 2y)\delta_i(\widetilde{\xi})\right), \frac{i}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}\right).$$

In particular, $\mathrm{G}^0_{\widetilde{\chi_1}}$ and $\mathrm{G}^0_{\widetilde{\chi_3}}$ are isomorphic groups.

iv. *The choices of $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying* (112) *and* (117) *lead to two non-isomorphic affine metaplectic coverings*

$$1 \longrightarrow \mu_4(\mathbb{C}) \xrightarrow{\Phi_{\widetilde{\chi_k}}} \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \xrightarrow{\Psi_{\widetilde{\chi_k}}} \mathrm{ASp}(W) \longrightarrow 1.$$

More precisely,

- *If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then the affine metaplectic covering is $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} = \{(s, i^l \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}$, where the set of operators $i^l \omega_{\widetilde{\chi_k}}(s)$ is as follows (in the same order as the elements of $\mathrm{ASp}(B_W)$ above and where $\widetilde{\xi}$ is as in* (342)*),*

$$\left\{ \begin{array}{l} i^l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i^l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, i^l \begin{pmatrix} \widetilde{\xi} & 0 \\ 0 & -\widetilde{\xi} \end{pmatrix}, i^l \begin{pmatrix} 0 & \widetilde{\xi} \\ \widetilde{\xi} & 0 \end{pmatrix}, \\ \frac{i^l}{\sqrt{2}} \begin{pmatrix} \widetilde{\xi} & -1 \\ 1 & -\widetilde{\xi} \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} 1 & \widetilde{\xi} \\ \widetilde{\xi} & 1 \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\xi} \\ -\widetilde{\xi} & 1 \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} \widetilde{\xi} & 1 \\ -1 & -\widetilde{\xi} \end{pmatrix}, \\ \frac{i^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & -1-\widetilde{\xi} \\ 1-\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{i^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & 1+\widetilde{\xi} \\ -1+\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{i^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & 1-\widetilde{\xi} \\ -1-\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}, \frac{i^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & -1+\widetilde{\xi} \\ 1+\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}, \\ \frac{i^l}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{i^l \widetilde{\xi}}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \frac{i^l \widetilde{\xi}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ \frac{i^l}{\sqrt{2}} \begin{pmatrix} 1+\widetilde{\xi} & 0 \\ 0 & 1-\widetilde{\xi} \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} 0 & -1+\widetilde{\xi} \\ 1+\widetilde{\xi} & 0 \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} 1-\widetilde{\xi} & 0 \\ 0 & 1+\widetilde{\xi} \end{pmatrix}, \frac{i^l}{\sqrt{2}} \begin{pmatrix} 0 & -1-\widetilde{\xi} \\ 1-\widetilde{\xi} & 0 \end{pmatrix}, \\ \frac{i^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & 1+\widetilde{\xi} \\ -1+\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix}, \frac{i^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & -1+\widetilde{\xi} \\ 1+\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{i^l}{2} \begin{pmatrix} 1+\widetilde{\xi} & 1-\widetilde{\xi} \\ -1-\widetilde{\xi} & 1-\widetilde{\xi} \end{pmatrix}, \frac{i^l}{2} \begin{pmatrix} 1-\widetilde{\xi} & -1-\widetilde{\xi} \\ 1-\widetilde{\xi} & 1+\widetilde{\xi} \end{pmatrix} \end{array} \right\}.$$

Moreover, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *is isomorphic to*

$$\mathrm{C}_4 \cdot_6 \mathrm{S}_4 = \langle a, b, c, d, e \,|a^4 = d^3 = e^2 = 1,\, b^2 = c^2 = a^2,\, ab = ba,\, ac = ca,\, ad = da,\, ae = ea,\\ cbc^{-1} = a^2 b,\, dbd^{-1} = a^2 bc,\, ebe = bc,\, dcd^{-1} = b,\, ece = a^2 c,\, ede = d^{-1}\rangle,$$

an isomorphism being given for example by:

$$a = \left((1,0), i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right), \quad b = \left((1, 2x+2y), -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right), \quad c = \left((1,2x), \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right),$$

$$d = \left(\left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, (2x+2y)\delta_{-i}(\widetilde{\xi})\right), -\frac{1}{2}\begin{pmatrix}1-i & 1-i\\ -1-i & 1+i\end{pmatrix}\right)$$

$$e = \left(\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, 2xy+2x+2y\right), \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\right).$$

In particular, $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ *and* $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$ *are isomorphic groups.*

- *If* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ *then the affine metaplectic covering is* $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} = \{(s, i^l \omega_{\widetilde{\chi_k}}(s)),\ s \in \mathrm{ASp}(B_W),\ l \in \mathbb{Z}\}$*, where the set of operators* $i^l\omega_{\widetilde{\chi_k}}(s)$ *is exactly as the previous one, only replacing every factor* $\frac{i^l}{\sqrt{2}}$ *with* $\frac{(1-i)i^l}{2}$ *in the lines corresponding to the second, fourth and fifth line of the list of elements of* $\mathrm{ASp}(B_W)$*. Moreover,* $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *is isomorphic to*

$$\mathrm{U}_2(\mathbb{F}_3) = \langle a, b, c, d, e \,|a^4 = d^3 = 1,\, b^2 = c^2 = a^2,\, e^2 = a,\, ab = ba,\, ac = ca,\, ad = da,\, ae = ea,\\ cbc^{-1} = a^2 b,\, dbd^{-1} = a^2 bc,\, ebe^{-1} = bc,\, dcd^{-1} = b,\, ece^{-1} = a^2 c,\\ ede^{-1} = d^{-1}\rangle,$$

an isomorphism being given for example by:

$$a = \left((1,0), i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right), \quad b = \left((1, 2x+2y), -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right), \quad c = \left((1,2x), \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right),$$

$$d = \left(\left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, (2x+2y)\delta_{-i}(\widetilde{\xi})\right), -\frac{1}{2}\begin{pmatrix}1-i & 1-i\\ -1-i & 1+i\end{pmatrix}\right)$$

$$e = \left(\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, 2xy+2x+2y\right), \frac{1+i}{2}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\right).$$

In particular, $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ *and* $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$ *are isomorphic groups.*

Remark 3.31. As in Remark 3.6, the fact that $\mathrm{ASp}(B_W)$ is isomorphic to S_4 implies that the extension defined in Proposition 2.7.ix splits, since S_4 is a semidirect product of C_2^2 and S_3.

Proof. **i.** We write the details only for the element $\eta\left(\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\right) = \left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, 2y\right)$, the other cases being obtained in a similar way. We use the definition of $f_{\widetilde{\sigma}}$ given by (93) in Proposition 2.7. For all $\widetilde{w}_1 = (\widetilde{x}_1, \widetilde{y}_1)$ and $\widetilde{w}_2 = (\widetilde{x}_2, \widetilde{y}_2)$ in $\widetilde{W}$,

$$\begin{aligned}\beta_{\widetilde{\sigma}}(\widetilde{w}_1, \widetilde{w}_2) &= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}_1), \widetilde{\sigma}(\widetilde{w}_2)) - B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)\\ &= B_{\widetilde{W}}((\widetilde{y}_1, -\widetilde{x}_1 + \widetilde{y}_1), (\widetilde{y}_2, -\widetilde{x}_2 + \widetilde{y}_2)) - \widetilde{x}_1\widetilde{y}_2 - \widetilde{x}_1\widetilde{x}_2 - \widetilde{y}_1\widetilde{y}_2\\ &= \widetilde{y}_1(\widetilde{y}_2 - \widetilde{x}_2) + \widetilde{y}_1\widetilde{y}_2 + (\widetilde{y}_1 - \widetilde{x}_1)(\widetilde{y}_2 - \widetilde{x}_2) - \widetilde{x}_1\widetilde{y}_2 - \widetilde{x}_1\widetilde{x}_2 - \widetilde{y}_1\widetilde{y}_2\\ &= 2\widetilde{y}_1\widetilde{y}_2 - 2\widetilde{x}_1\widetilde{y}_2 - 2\widetilde{y}_1\widetilde{x}_2.\end{aligned}$$

Hence by (291), for all $w = x + y = \widetilde{w} + 2\widetilde{W} = \widetilde{x} + 2\widetilde{X} + \widetilde{y} + 2\widetilde{Y}$,

$$f_{\widetilde{\sigma}}(w) = \beta_{\widetilde{\sigma}}(\widetilde{w}, \widetilde{w}) = 2\widetilde{y}^2 = 2y.$$

ii. According to Proposition 2.7, $\mathrm{ASp}(B_W)$ is the set of elements $(\sigma, f_{\widetilde{\sigma}} + f)$, where $\sigma \in \mathrm{Sp}(W)$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z})$. Proposition 3.1 and the computations of $f_{\widetilde{\sigma}}$ done above give the image of η inside $\mathrm{ASp}(B_W)$.

iii. There is nothing to compute here. In fact, we chose in (342) the character $\widetilde{\chi}_k$ such that it coincides with $\widetilde{\chi}$ on $X \times \{0\}$. Moreover, $f_{\widetilde{\sigma}}(w) \in 2\mathbb{Z}/4\mathbb{Z}$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and $w \in W$. Hence, by Propositions 1.28 and 2.21, the sets of operators $(-1)^l \omega_{\widetilde{\chi}}(s)$ and $(-1)^l \omega_{\widetilde{\chi}_k}(s)$ for all s in $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$, respectively, do not depend on the choice of w_s satisfying (37) and (100), respectively. By (165), we can choose $\{w_s \in W$ satisfying (37), $s \in \mathrm{Ps}(B)\}$ and $\{v_s \in W$ satisfying (100), $s \in \mathrm{ASp}(B_W)\}$, such that $v_{\Gamma(s)} = w_s$ for every $s \in \mathrm{Ps}(B)$. Then by Lemma 2.37, $\omega_{\widetilde{\chi}_k}(\Gamma(s), v_{\Gamma(s)}) = \delta(\sigma)\gamma(\sigma)^{-1}\omega_{\widetilde{\chi}}(s, w_s)$ for every $s \in \mathrm{Ps}(B)$. Since Γ is an isomorphism from $\mathrm{Ps}(B)$ to $\mathrm{ASp}(B_W)$ in this example, we obtain the different possibilities for $\mathrm{G}^0_{\widetilde{\chi}_k}$ from the two possibilities of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ given in Proposition 3.28. In fact, by the same reasoning as to obtain $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ in Proposition 3.28, we see that there are two possibilities for the choice of $\delta(\sigma)$ satisfying (112) and (118):

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \pi$ then $\nu(\sigma) \equiv 0 \mod \pi$ for every $\sigma \in \mathrm{Sp}(W)$. In this case, if we set also $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, then $\delta(\sigma) \in \{\pm\gamma(\sigma)\}$ for every $\sigma \in \mathrm{Sp}(W)$. Hence we directly obtain that $\omega_{\widetilde{\chi}_k}(s, v_s) = \delta(\sigma)\gamma(\sigma)^{-1}\omega_{\widetilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)})$ belongs to the set $\{\pm\omega_{\widetilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)})\}$ for every $s \in \mathrm{ASp}(B_W)$.
- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$ then $\nu(\sigma_v) \equiv \frac{\pi}{2} \mod \pi$ for all $v \neq 0$, $\nu(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \pi$ and $\nu(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \mod \pi$. In this case, if we set also $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, then $\delta(\sigma) \in \{\pm\gamma(\sigma)\}$ for every $\sigma \in \mathrm{Sp}(W)$. Hence we obtain directly that $\omega_{\widetilde{\chi}_k}(s, v_s)$ belongs to the set $\{\pm\omega_{\widetilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)})\}$ for every $s \in \mathrm{ASp}(B_W)$.

Using the lists of elements of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ given in Proposition 3.28 and the expression of Γ, we obtain without further computations the lists of elements of $\mathrm{G}^0_{\widetilde{\chi}_k}$ given in the statement above. In particular,

$$\mathrm{G}^0_{\widetilde{\chi}_k} \simeq \begin{cases} \mathrm{CSU}_2(\mathbb{F}_3) & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \pi \\ \mathrm{GL}_2(\mathbb{F}_3) & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi. \end{cases} \tag{343}$$

iv. With the same reasoning as in the proof of Proposition 3.5.iii, we obtain two possibilities for the choice of $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (117), determined by the fact that $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ or $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$.

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then $\nu(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for every $\sigma \in \mathrm{Sp}(W)$. Then we follow the same proof as to find $\mathrm{G}^0_{\widetilde{\chi}_k}$ above. If we choose $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, then $\omega_{\widetilde{\chi}_k}(s, v_s) = \delta(\sigma)\gamma(\sigma)^{-1}\omega_{\widetilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)}) \in \{\pm 1, \pm i\}\omega_{\widetilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)})$ for every $s \in \mathrm{ASp}(B_W)$. This enables to establish the list of elements of $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$ given in the statement, using the first list of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ given in Proposition 3.28, only replacing the factor $(-1)^l$ with i^l in every line.

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ then $\nu(\sigma_v) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ for every $v \neq 0$, $\nu(\sigma_{(1,0)}\sigma_{(0,1)}) \equiv 0 \mod \frac{\pi}{2}$ and $\nu(\sigma_{(0,1)}\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. If we choose $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, then $\omega_{\widetilde{\chi_k}}(s, v_s) = \delta(\sigma)\gamma(\sigma)^{-1}\omega_{\tilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)}) \in \mu_8(\mathbb{C})\omega_{\tilde{\chi}}(\Gamma^{-1}(s), w_{\Gamma^{-1}(s)})$ for every $s \in \mathrm{ASp}(B_W)$. Thus, in order to obtain the list of elements of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ given in the statement, using the first list of $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ given in Proposition 3.28, we need some adjustments. In fact, the factor $\frac{(-1)^l}{\sqrt{2}}$ has to be replaced with $\frac{i^l}{\sqrt{2}}e^{\frac{i\pi}{4}} = \frac{i^l(1+i)}{2}$ in the second, fourth and fifth line, and every factor $(-1)^l$ has to be replaced by i^l in the other lines.

Recall from the proof of Proposition 3.5 that $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$ is the only non-split central extension of S_4 by C_4 having 19 elements of order two, and $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$ and $\mathrm{U}_2(\mathbb{F}_3)$ are the only non-split central extensions of S_4 by C_4 which contain $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$. Thus we deduce as in Proposition 3.5 that

$$\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \begin{cases} \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2} \\ \mathrm{U}_2(\mathbb{F}_3) & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \end{cases} \tag{344}$$

□

Remark 3.32. By Propositions 3.28 and 3.30, $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$ are isomorphic. This fact can also be observed using the group homomorphism Γ defined in Proposition 2.6. In fact, we proved that Γ is injective, and in this example $|\mathrm{ASp}(B_W)| = |W||\mathrm{Sp}(W)| = |W||\mathrm{O}(Q)| = |\mathrm{Ps}(B)|$, hence Γ is an isomorphism. This provides also an easy way to find explicitly the isomorphism between $\mathrm{ASp}(B_W)$ and S_4, by taking the images under Γ of the generators a, b, c and d of $\mathrm{Ps}(B)$ given in Proposition 3.28.

Remark 3.33 (Related to Remark 2.30)**.** Exactly as in Remark 3.7, observe that we do not obtain all the non-split central extensions of $\mathrm{ASp}(B_W)$ by four elements with all possible choices of $\{\delta(\sigma),\ \sigma \in \mathrm{Sp}(W)\}$, but we obtain all the possible non-split extensions of $\mathrm{ASp}(B_W)$ by four elements which contain the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. In addition, by Remark 3.29, the different possibilities for $\mathrm{G}^0_{\widetilde{\chi_k}}$ are precisely all the non-split stem central extensions of $\mathrm{ASp}(B_W)$ by two elements. Notice however that $\mathrm{G}^0_{\widetilde{\chi_k}}$ does not contain the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Moreover, (344) also shows that if (143) defines a coboundary, then the condition (144) is satisfied in this case (cf. Proposition 2.29.v).

Remark 3.34 (related to Remarks 2.22 and 2.36)**.** Proposition 3.30 shows that, with a fixed choice between $\nu(\sigma_{(1,0)}) \equiv 0 \mod \pi$ and $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, the groups defined by $\mathrm{G}^0_{\widetilde{\chi_1}}$ and $\mathrm{G}^0_{\widetilde{\chi_3}}$ are isomorphic. We computed in Remark 3.29 that $\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and we mentioned that there are exactly four central extensions of S_4 by $\mathbb{Z}/2\mathbb{Z}$, with a different group for each of them. Thus in fact also the extensions defined by $\mathrm{G}^0_{\widetilde{\chi_1}}$ and $\mathrm{G}^0_{\widetilde{\chi_3}}$ are isomorphic, provided they are defined with the same condition on $\nu(\sigma_{(1,0)})$. In particular, this answers one question in Remark 2.36: if the map defined by (161) is a coboundary, then the condition (162) is satisfied. In other words, $\mathrm{G}^0_{\widetilde{\chi_1}}$ and $\mathrm{G}^0_{\widetilde{\chi_3}}$ are isomorphic if and only if (162) is satisfied. Moreover, since $\mathrm{H}^2(\mathrm{S}_4, \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by (304), we can prove by the same method that the extensions defined by $\mathrm{AMp}(B_W)_{\widetilde{\chi_1}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_3}}$, with the choices $\nu(\sigma)$ and $\nu'(\sigma)$,

respectively, are isomorphic if and only if $\nu(\sigma) + \nu'(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for all $\sigma \in \mathrm{Sp}(W)$ (and in fact it is enough to verify this condition for $\sigma_{(1,0)}$). As in Remark 3.8 (where the reasoning is much more detailed), this proves that $c_{\chi_k}^2$ is a coboundary and that we could use this property to "renormalize" every operator $\omega_{\widetilde{\chi_k}}(s, w_s)$, using a parameter $\gamma(s, w_s)$ instead of $\gamma(\sigma)$, in order to obtain a central non-split stem extension which is a two-fold covering of $\mathrm{ASp}(B_W)$ instead of a four-fold covering. However, in contrast to the case of B_{Weil} in Remark 3.8, there is here no need to develop this process in order to obtain a two-fold covering of $\mathrm{ASp}(B_W)$. In fact, by Remark 3.33 the study of $\mathrm{G}^0_{\widetilde{\chi_k}}$ already provides all the possible non-split stem central extensions of $\mathrm{ASp}(B_W)$ by two elements. (This is the only case for which the parameter $\gamma(\sigma)$ can be chosen so that the cocycle c_{χ_k} takes its values in $\mu_2(\mathbb{C})$, so there would be no interest to consider the more complex parameter $\gamma(s, w_s)$.)

3.2.3 Relation between $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$

Using the results obtained in sections 3.2.1 and 3.2.2, and applying Proposition 2.39, we can now see under which conditions the pseudosymplectic covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is compatible with the two-fold covering $\mathrm{G}^0_{\widetilde{\chi_k}}$, or with the affine metaplectic covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Recall that their compatibility means that the diagram (D_{22}) commutes.

Proposition 3.35. ***i.*** *We fix the choices $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying* (46) *and* (48)*, and $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying* (112) *and* (118)*. Then the two-fold covering $\mathrm{G}^0_{\widetilde{\chi_k}}$ is compatible with the pseudosymplectic covering $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ if and only if $\theta(\sigma_{(1,0)}) \equiv \nu(\sigma_{(1,0)}) \mod \pi$. In this case, $\widetilde{\Gamma}$ is an isomorphism.*

ii. *We fix the choices $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying* (46) *and* (48)*, and $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying* (112) *and* (117)*. Then the pseudosymplectic and affine metaplectic coverings are compatible if and only if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, i.e. if $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is isomorphic to $\mathrm{C}_4 \cdot_6 \mathrm{S}_4$.*

Proof. **i.** By (190), the two-fold coverings $\mathrm{G}^0_{\widetilde{\chi_k}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ are compatible if and only if $\theta(\sigma) \equiv \nu(\sigma) \mod \pi$ for all $\sigma \in \mathrm{O}(Q) = \mathrm{Sp}(W)$. Thanks to our computations in Propositions 3.28 and 3.30, we can deduce that this happens if and only if $\theta(\sigma_{(1,0)}) \equiv \nu(\sigma_{(1,0)}) \mod \pi$. In this case, the groups $\mathrm{G}^0_{\widetilde{\chi_k}}$ and $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ are isomorphic. Since $\widetilde{\Gamma}$ is injective by Lemma 2.37, it is an isomorphism.

ii. By Proposition 3.28.iii, $\theta(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for every $\sigma \in \mathrm{O}(Q)$. Hence by (192), the diagram (D_{22}) commutes if and only if $\nu(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for every $\sigma \in \mathrm{Sp}(W)$, which is true by Proposition 3.30 if and only if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, i.e. $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$. □

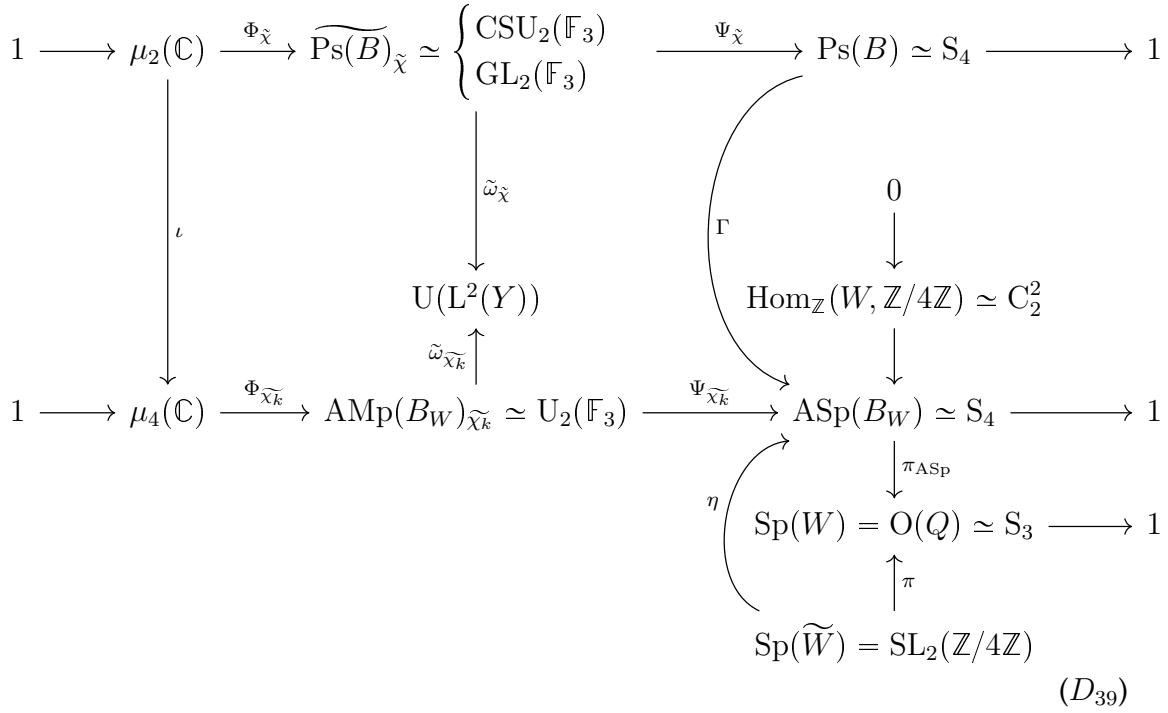
Comparing Propositions 3.28, 3.30 and 3.35, we realize that the construction of $\mathrm{G}^0_{\widetilde{\chi_k}}$ is just a repetition of the one of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, as expected since $\mathrm{O}(Q) = \mathrm{Sp}(W)$ and hence $\mathrm{ASp}(B_W) \simeq \mathrm{Ps}(B)$ in this case. Therefore, we will not consider $\mathrm{G}^0_{\widetilde{\chi_k}}$ anymore. Now, combining all the results of sections 3.2.1, 3.2.2 and 3.2.3 proved so far, we obtain two diagrams, depending on

the choice made for $\{\gamma(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\theta(\sigma)},\ \sigma \in \mathrm{O}(Q)\}$ satisfying (46) and (48), and $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (117).

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then, regardless of the choice made for $\gamma(\sigma_{(1,0)})$, the diagram (D_{22}) commutes. It is given in this case by

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \begin{cases} \mathrm{CSU}_2(\mathbb{F}_3) \\ \mathrm{GL}_2(\mathbb{F}_3) \end{cases} & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \tilde{\Gamma}} \ \downarrow{\scriptstyle \tilde{\omega}_{\tilde{\chi}}} & & \uparrow{\scriptstyle \Gamma} \ \downarrow{\scriptstyle 0} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & \uparrow{\scriptstyle \tilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & & & & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & & & & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & &
\end{array}
\qquad (D_{38})
$$

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ then, regardless of the choice of $\gamma(\sigma_{(1,0)})$, there is no group homomorphism $\widetilde{\Gamma}$ such that the diagram (D_{22}) commutes, i.e. we have the following diagram:



(D_{39})

Now, in the next two sections, we pursue our study with two different goals in mind. On one hand, in section 3.2.4, we look for a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and a homomorphism $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ such that $\tilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is unitary, irreducible and faithful, and the entire resulting diagram commutes, as in section 3.1.4. On the other hand, we look in section 3.2.5 for a way to define the Weil representation of an extension which is "closer" to $\mathrm{Sp}(W)$ than a two-fold covering of $\mathrm{Sp}(\widetilde{W})$, as in section 3.1.5.

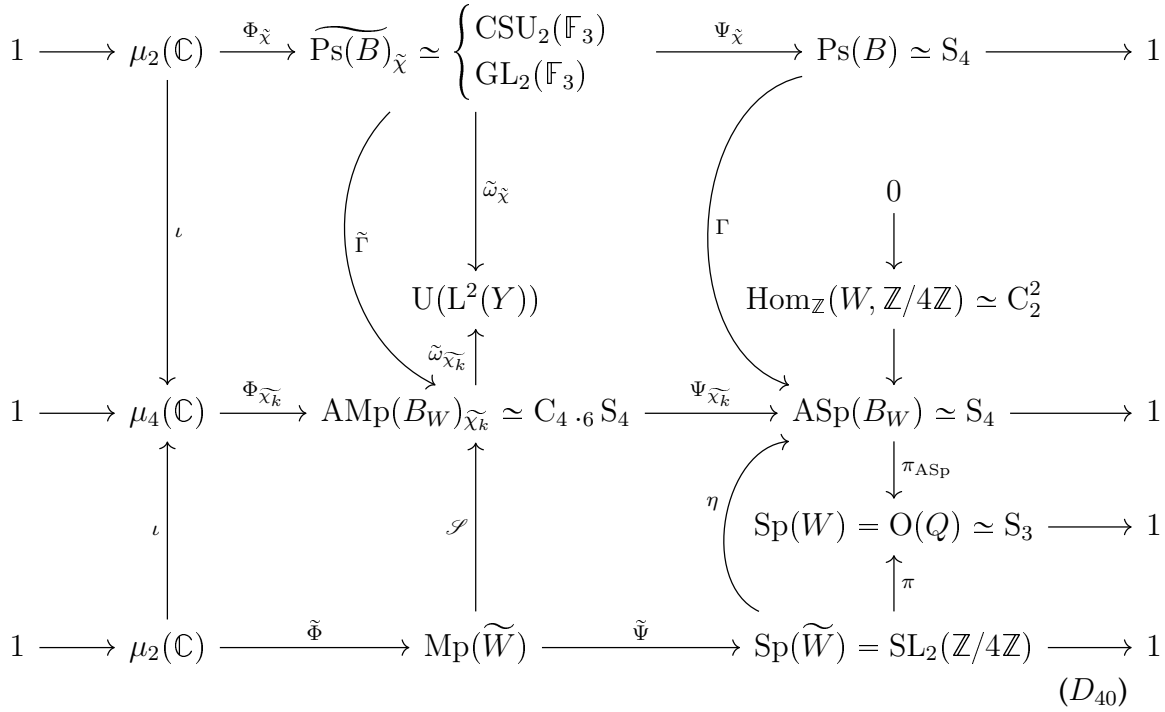
3.2.4 Following Gurevich-Hadani: the metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$

Recall that we proved in Proposition 2.3 that, up to isomorphism, there is only one Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. As a consequence, the bottom parts of the diagram (D_{26}) and (D_{38}) are isomorphic, and so are the bottom parts of the diagram (D_{27}) and (D_{39}). Therefore the central extensions

$$1 \to \mu_2(\mathbb{C}) \xrightarrow{\tilde{\Phi}} \mathrm{Mp}(\widetilde{W}) \xrightarrow{\tilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1$$

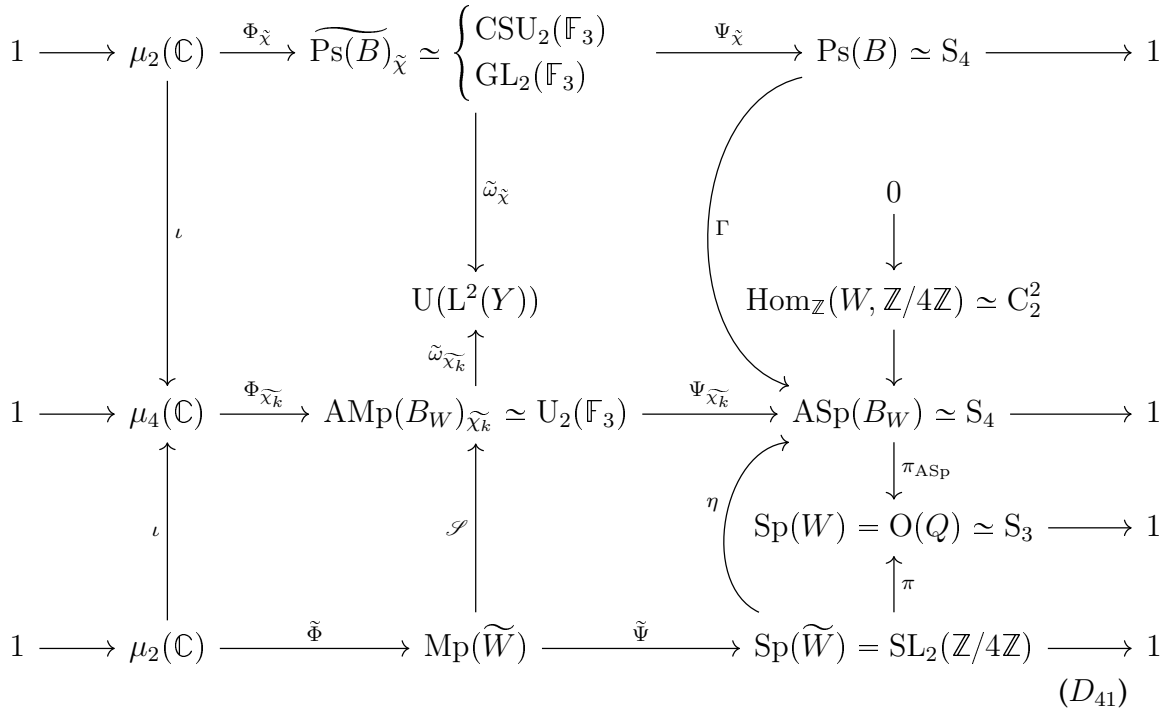
we are going to construct in both cases are going to be isomorphic. In both cases, we apply the same method as in section 3.1.4. The results and proofs are essentially the same, and the only adjustments to make are the explicit expressions of the group homomorphisms involved. More precisely, we look for a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ and a homomorphism of central extensions $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ pulling back the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to the metaplectic group $\mathrm{Mp}(\widetilde{W})$. We are therefore studying the following commutative diagrams (D_{38}) and (D_{39}):

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then



(D_{40})

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then



(D_{41})

In addition to the existence of $\mathrm{Mp}(\widetilde{W})$ and $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, we will also ask whether the expected lift of $\widetilde{\omega}_{\widetilde{\chi_k}}$ to $\mathrm{Mp}(\widetilde{W})$ stays unitary, irreducible and faithful, as in section 3.1.4. First, we recall the presentations of the different groups involved:

- $\mathrm{Sp}(W) \simeq \mathrm{S}_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$, where $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{A}_4 \rtimes \mathrm{C}_4 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^4 = 1, cac^{-1} = dad^{-1} = ab = ba, cbc^{-1} = a, bd = db, dcd^{-1} = c^{-1} \rangle$, where $a = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$, $b = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$, $c = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $d = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$.

- $\mathrm{ASp}(B_W) \simeq \mathrm{S}_4 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = 1, cac^{-1} = dad = ab = ba, cbc^{-1} = a, bd = db, dcd = c^{-1} \rangle$, where $a = (1, 2x)$, $b = (1, 2y)$, $c = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0 \right)$ and $d = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 2xy + 2x + 2y \right)$.

- $\mathrm{C}_4 \cdot_6 \mathrm{S}_4 = \langle a, b, c, d, e \mid a^4 = d^3 = e^2 = 1, b^2 = c^2 = a^2, ab = ba, ac = ca, ad = da, ae = ea, cbc^{-1} = a^2b, dbd^{-1} = a^2bc, ebe = bc, dcd^{-1} = b, ece = a^2c, ede = d^{-1} \rangle$, where

$$a = \left((1, 0), i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad b = \left((1, 2x + 2y), -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \quad c = \left((1, 2x), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right),$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (2x + 2y)\delta_{-i}(\widetilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1 - i & 1 - i \\ -1 - i & 1 + i \end{pmatrix} \right)$$

$$e = \left(\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 2xy + 2x + 2y \right), \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \right).$$

- $\mathrm{U}_2(\mathbb{F}_3) = \langle a, b, c, d, e \mid a^4 = d^3 = 1, b^2 = c^2 = a^2, e^2 = a, ab = ba, ac = ca, ad = da, ae = ea, cbc^{-1} = a^2b, dbd^{-1} = a^2bc, ebe^{-1} = bc, dcd^{-1} = b, ece^{-1} = a^2c, ede^{-1} = d^{-1} \rangle$, where

$$a = \left((1, 0), i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad b = \left((1, 2x + 2y), -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \quad c = \left((1, 2x), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right),$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (2x + 2y)\delta_{-i}(\widetilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1 - i & 1 - i \\ -1 - i & 1 + i \end{pmatrix} \right)$$

$$e = \left(\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 2xy + 2x + 2y \right), \frac{1 + i}{2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \right).$$

In terms of these generators, recall also the following homomorphisms, which are defined in Propositions 2.7.vi and 2.29:

$$\Phi_{\widetilde{\chi_k}} : \mu_4(\mathbb{C}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}, \qquad \Psi_{\widetilde{\chi_k}} : \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \to \mathrm{ASp}(B_W) \tag{345}$$

$$\begin{aligned} i &\mapsto a \end{aligned} \qquad \begin{aligned} a &\mapsto 1 \\ b &\mapsto ab \\ c &\mapsto a \\ d &\mapsto \begin{cases} c & \text{if } \widetilde{\xi} = i \\ bc & \text{if } \widetilde{\xi} = -i \end{cases} \\ e &\mapsto dc \end{aligned}$$

$$\text{and} \quad \eta : \mathrm{Sp}(\widetilde{W}) \to \mathrm{ASp}(B_W).$$

$$\begin{aligned} a &\mapsto a \\ b &\mapsto b \\ c &\mapsto bc \\ d &\mapsto db \end{aligned}$$

We first examine the possibility to obtain a faithful Weil representation of $\mathrm{Mp}(\widetilde{W})$ in this way, i.e. the existence of an injective group homomorphism $\mathscr{S}$. Using the results of the beginning of section 3.1.4, we can first affirm that this is only possible if we set $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, i.e. if $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3)$. We therefore restrict ourselves to the diagram (D_{41}). Since $\mathrm{Mp}(\widetilde{W})$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ have the same order, $\mathscr{S}$ has to be an isomorphism of central extensions in this case. This requires to set $\mathrm{Mp}(\widetilde{W}) = \mathrm{U}_2(\mathbb{F}_3)$.

Proposition 3.36. *Suppose that* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ *and set* $\mathrm{Mp}(\widetilde{W}) = \mathrm{U}_2(\mathbb{F}_3)$. *Then there exists an isomorphism of central extensions* $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ *such that the diagram* (D_{41}) *commutes. For example, we can set* $\mathscr{S}$ *equal to the identity. In this case, the group homomorphisms* $\widetilde{\Phi}$ *and* $\widetilde{\Psi}$ *are of the following form:*

$$\widetilde{\Phi} : \mu_2(\mathbb{C}) \to \mathrm{Mp}(\widetilde{W}) \qquad \textit{and} \qquad \widetilde{\Psi} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{Sp}(\widetilde{W}) \tag{346}$$

$$\begin{aligned} -1 &\mapsto a^2 \end{aligned} \qquad \begin{aligned} a &\mapsto d^2 \\ b &\mapsto ab \\ c &\mapsto a \\ d &\mapsto \begin{cases} bc & \textit{if } \widetilde{\xi} = i \\ c & \textit{if } \widetilde{\xi} = -i \end{cases} \\ e &\mapsto dc \textit{ or } d^{-1}c. \end{aligned}$$

Proof. The proof is identical to that of Proposition 3.12, except for the explicit computations with the generators. We indicate the modifications below. The diagram (D_{41}) commutes with $\mathscr{S} = \mathrm{id}$ (meaning that each generator a, b, c, d, e of $\mathrm{Mp}(\widetilde{W})$ is associated to the corresponding generator of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$) if and only if $\widetilde{\Phi} = \Phi_{\widetilde{\chi_k}} \circ \iota$ and $\eta \circ \widetilde{\Psi} = \Psi_{\widetilde{\chi_k}}$. The condition $\widetilde{\Phi} = \Phi_{\widetilde{\chi_k}} \circ \iota$

is equivalent to define $\widetilde{\Phi}$ by $\widetilde{\Phi}(-1) = a^2$. The condition $\eta \circ \widetilde{\Psi} = \Psi_{\widetilde{\chi_k}}$ is equivalent to

$$\begin{cases} (\eta \circ \widetilde{\Psi})(a) &= 1 \\ (\eta \circ \widetilde{\Psi})(b) &= ab \\ (\eta \circ \widetilde{\Psi})(c) &= a \\ (\eta \circ \widetilde{\Psi})(d) &= \begin{cases} \mathrm{c} & \text{if } \widetilde{\xi} = i \\ \mathrm{bc} & \text{if } \widetilde{\xi} = -i \end{cases} \\ (\eta \circ \widetilde{\Psi})(e) &= dc \end{cases} \overset{\mathrm{Ker}(\eta) = <d^2>}{\Longleftrightarrow} \begin{cases} \widetilde{\Psi}(a) \in \{1, d^2\} \\ \widetilde{\Psi}(b) \in \{ab, abd^2\} \\ \widetilde{\Psi}(c) \in \{a, ad^2\} \\ \widetilde{\Psi}(d) \in \begin{cases} \{bc, bcd^2\} & \text{if } \widetilde{\xi} = i \\ \{c, cd^2\} & \text{if } \widetilde{\xi} = -i \end{cases} \\ \widetilde{\Psi}(e) \in \{dc, d^{-1}c\}. \end{cases} \tag{347}$$

So (347) replaces (311). Suppose now that $\widetilde{\Psi}$ satisfies (347). Then, as in Proposition 3.12, $\widetilde{\Psi}(a) = d^2$ and we verify under which conditions the image of Ψ satisfies the relations of the group $\mathrm{U}_2(\mathbb{F}_3)$.

- The relations $\widetilde{\Psi}(a)^4 = 1$, $\widetilde{\Psi}(b)^2 = \widetilde{\Psi}(c)^2 = \widetilde{\Psi}(a)^2$ and $\widetilde{\Psi}(e)^2 = \widetilde{\Psi}(a)$ are all satisfied because $a^2 = b^2 = d^4 = 1$ in $\mathrm{Sp}(\widetilde{W})$. Moreover, $\widetilde{\Psi}(a) = d^2$ is in the center of $\mathrm{Sp}(\widetilde{W})$.

- $\widetilde{\Psi}(d)^3 = 1$ is equivalent to $\widetilde{\Psi}(d) = \begin{cases} bc & \text{if } \widetilde{\xi} = i \\ c & \text{if } \widetilde{\xi} = -i \end{cases}$, since bcd^2 and cd^2 have order six in $\mathrm{Sp}(\widetilde{W})$.

- $\widetilde{\Psi}(d)\widetilde{\Psi}(b)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b)\widetilde{\Psi}(c) \overset{d^2 \in \mathrm{Z}(\mathrm{Sp}(\widetilde{W}))}{\Longleftrightarrow} \begin{cases} bcabc^{-1}b^{-1} = ab\widetilde{\Psi}(c) & \text{if } \widetilde{\xi} = i \\ cabc^{-1} = a\widetilde{\Psi}(c) & \text{if } \widetilde{\xi} = -i \end{cases}$
$\Longleftrightarrow \widetilde{\Psi}(c) = a.$
So in any case, $\widetilde{\Psi}(d)\widetilde{\Psi}(b)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b)\widetilde{\Psi}(c)$ is equivalent to $\widetilde{\Psi}(c) = b$.

- $\widetilde{\Psi}(c)\widetilde{\Psi}(b)\widetilde{\Psi}(c)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(b) \overset{\widetilde{\Psi}(c)=a}{\Longleftrightarrow} a\widetilde{\Psi}(b)a^{-1} = \widetilde{\Psi}(b)$, which is satisfied because ab and abd^2 commute with a.

- $\widetilde{\Psi}(d)\widetilde{\Psi}(c)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(b) \overset{\widetilde{\Psi}(c)=a}{\Longleftrightarrow} \widetilde{\Psi}(b) = \begin{cases} bcac^{-1}b^{-1} = bbab = ab & \text{if } \widetilde{\xi} = i \\ cac^{-1} = ab & \text{if } \widetilde{\xi} = -i. \end{cases}$
So in any case, $\widetilde{\Psi}(d)\widetilde{\Psi}(c)\widetilde{\Psi}(d)^{-1} = \widetilde{\Psi}(b)$ is equivalent to $\widetilde{\Psi}(b) = ab$.

- $\widetilde{\Psi}(e)\widetilde{\Psi}(b)\widetilde{\Psi}(e)^{-1} = \widetilde{\Psi}(b)\widetilde{\Psi}(c) \overset{\widetilde{\Psi}(b)=ab,\ \widetilde{\Psi}(c)=a}{\Longleftrightarrow} \widetilde{\Psi}(e)ab\widetilde{\Psi}(e)^{-1} = aba = b$. This condition is satisfied because $dcabc^{-1}d^{-1} = dcbac^{-1}d^{-1} = dacac^{-1}d^{-1} = dbd^{-1} = b$. The condition $\widetilde{\Psi}(e)\widetilde{\Psi}(c)\widetilde{\Psi}(e)^{-1} = \widetilde{\Psi}(a)^2\widetilde{\Psi}(c)$ is equivalent to $\widetilde{\Psi}(e)a\widetilde{\Psi}(e)^{-1} = a$, which is also satisfied because $dcac^{-1}d^{-1} = dabd^{-1} = dad^{-1}b = abb = a$. Finally, the condition $\widetilde{\Psi}(e)\widetilde{\Psi}(d)\widetilde{\Psi}(e)^{-1} = \widetilde{\Psi}(d)^{-1}$ is satisfied because $dccc^{-1}d^{-1} = c^{-1}$ and $dcbcc^{-1}d^{-1} = dcbd^{-1} = dcd^{-1}b = c^{-1}b = (bc)^{-1}$.

This shows that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are group homomorphisms such that the diagram (D_{41}) commutes if and only if their expressions take the form shown in (346). Finally, it remains to prove

that $\widetilde{\Psi}$ is surjective. Indeed, $a = \widetilde{\Psi}(c)$, $b = \widetilde{\Psi}(cb)$, $c = \begin{cases} \widetilde{\Psi}(db) & \text{if } \widetilde{\xi} = i \\ \widetilde{\Psi}(d) & \text{if } \widetilde{\xi} = -i \end{cases}$ and $d = \begin{cases} \widetilde{\Psi}(ebd^{-1}) \text{ or } \widetilde{\Psi}(aebd^{-1}) & \text{if } \widetilde{\xi} = i \\ \widetilde{\Psi}(ed^{-1}) \text{ or } \widetilde{\Psi}(aed^{-1}) & \text{if } \widetilde{\xi} = -i \end{cases}$. This concludes the proof. □

Thus, Proposition 3.36 provides the same result as Proposition 3.12: if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then there exists an isomorphism of central extensions $\mathscr{S}$ such that (D_{41}) commutes. In particular, we can choose $\mathscr{S}$ to be the identity, and in this case we obtain the following diagram, where all groups and homomorphisms are now defined explicitly:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \begin{cases} \mathrm{CSU}_2(\mathbb{F}_3) \\ \mathrm{GL}_2(\mathbb{F}_3) \end{cases} & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \tilde{\omega}_{\tilde{\chi}}} & \overset{\Gamma}{\nearrow} & \downarrow{\scriptstyle 0} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & \uparrow{\scriptstyle \tilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \mathscr{S}=\mathrm{id}} & \overset{\eta}{\nearrow} & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
 & & & & & & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W}) \simeq \mathrm{U}_2(\mathbb{F}_3) & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1
\end{array}
\qquad (D_{42})
$$

We obtain in this way a unitary, irreducible and faithful Weil representation $\tilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$ which extends the Weil representation of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ (it is even exactly the same representation if $\mathscr{S}$ is the identity). This concludes the study of the two-dimensional case concerning the metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$.

Now we examine the remaining case, when $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. In this case, recall that there is no hope to find a homomorphism of central extensions $\mathscr{S}$ which is injective and makes the diagram (D_{40}) commute. So we only look for $\mathscr{S}$ such that (D_{40}) commutes. Since $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4$ in this case, applying Lemmas 3.13 and 3.14 from section 3.1.4, we know that the image of $\mathscr{S}$ has to be a subgroup of index two in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, which implies that $\mathrm{Mp}(\widetilde{W})$ has to be isomorphic to

$$
\mathrm{Q}_8 \rtimes \mathrm{Dic}_3 = \langle a, b, c, d \,|\, a^4 = c^6 = 1, b^2 = a^2, d^2 = c^3, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = b, \\ dad^{-1} = a^2 b, cbc^{-1} = ab, dcd^{-1} = c^{-1} \rangle. \qquad (348)
$$

Moreover, as shown with the diagram (D_{31}), we can fix the group homomorphisms $\widetilde{\Phi}$ and $\widetilde{\Psi}$ such that $1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\widetilde{\Phi}} \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \xrightarrow{\widetilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1$ is a central extension. Thus, we set $\mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and we define the maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$ as in Lemma 3.15:

$$\begin{aligned} \widetilde{\Phi} : \mu_2(\mathbb{C}) &\to \mathrm{Mp}(\widetilde{W}) \quad \text{and} \quad \widetilde{\Psi} : \mathrm{Mp}(\widetilde{W}) \to \mathrm{Sp}(\widetilde{W}). \\ -1 &\mapsto a^2 \qquad\qquad\qquad\quad a \mapsto b \\ & \qquad\qquad\qquad\quad\ b \mapsto ab \\ & \qquad\qquad\qquad\quad\ c \mapsto ac^{-1}d^2 \\ & \qquad\qquad\qquad\quad\ d \mapsto cbcd \end{aligned} \tag{349}$$

Proposition 3.37. *We use the maps given by* (349) *to define the central extension*

$$1 \to \mu_2(\mathbb{C}) \xrightarrow{\widetilde{\Phi}} \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \xrightarrow{\widetilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \to 1.$$

Then there exists a group homomorphism $\mathscr{S}$ such that the diagram (D_{40}) is commutative. More precisely, every such homomorphism of central extensions $\mathscr{S}$ is of the following form:

$$\begin{aligned} (\mathscr{S}(a), \mathscr{S}(b)) &= \begin{cases} (cb, b^{-1}) & \text{if } \widetilde{\xi} = i \\ (bc, b) & \text{if } \widetilde{\xi} = -i \end{cases} \\ (\mathscr{S}(c), \mathscr{S}(d)) &\in \begin{cases} \{(d^{-1}c, ec), (d^{-1}c, ec^{-1}), (d^{-1}c^{-1}, aec), (d^{-1}c^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = i \\ \{(d^{-1}b, ec), (d^{-1}b, ec^{-1}), (d^{-1}b^{-1}, aec), (d^{-1}b^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = -i. \end{cases} \end{aligned} \tag{350}$$

In particular, the image of $\mathscr{S}$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$ if

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}c, ec), (d^{-1}c, ec^{-1})\} & \text{if } \widetilde{\xi} = i \\ \{(d^{-1}b, ec), (d^{-1}b, ec^{-1})\} & \text{if } \widetilde{\xi} = -i, \end{cases} \tag{351}$$

and the image of $\mathscr{S}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ if

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}c^{-1}, aec), (d^{-1}c^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = i \\ \{(d^{-1}b^{-1}, aec), (d^{-1}b^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = -i. \end{cases} \tag{352}$$

Proof. Apart from explicit computations involving the generators, the proof is identical to that of Proposition 3.16. Suppose that $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ is a group homomorphism. Then the diagram (D_{40}) commutes if and only if $\Phi_{\widetilde{\chi_k}} \circ \iota = \mathscr{S} \circ \widetilde{\Phi}$ and $\Psi_{\widetilde{\chi_k}} \circ \mathscr{S} = \eta \circ \widetilde{\Psi}$. Observe that $(\Phi_{\widetilde{\chi_k}} \circ \iota)(-1) = \Phi_{\widetilde{\chi_k}}(-1) = a^2$. Hence $\Phi_{\widetilde{\chi_k}} \circ \iota = s \circ \widetilde{\Phi}$ is equivalent to $\mathscr{S}(a^2) = a^2$. Moreover,

$$\begin{cases} (\eta \circ \widetilde{\Psi})(a) = \eta(b) = b \\ (\eta \circ \widetilde{\Psi})(b) = \eta(ab) = ab \\ (\eta \circ \widetilde{\Psi})(c) = \eta(ac^{-1}d^2) = a(bc)^{-1}(db)^2 = ac^{-1}bd^2b^2 = ac^{-1}b = a^2c^{-1}a = c^{-1}a \\ (\eta \circ \widetilde{\Psi})(d) = \eta(cbcd) = bcbbcdb = bc^{-1}db = bdcb = dbac = dca. \end{cases}$$

Hence, the condition $\Psi_{\widetilde{\chi_k}} \circ \mathscr{S} = \eta \circ \widetilde{\Psi}$ is equivalent to

$$\begin{cases} \mathscr{S}(a) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{b\}) & = \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})cb = \{cb, acb, c^{-1}b, a^{-1}cb\} \\ \mathscr{S}(b) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{ab\}) & = \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})b = \{b, ab, b^{-1}, a^{-1}b\} \\ \mathscr{S}(c) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{c^{-1}a\}) & = \begin{cases} \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})d^{-1}c = \{d^{-1}c, ad^{-1}c, d^{-1}c^{-1}, a^{-1}d^{-1}c\} & \text{if } \widetilde{\xi} = i \\ \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})d^{-1}b = \{d^{-1}b, ad^{-1}b, d^{-1}b^{-1}, a^{-1}d^{-1}b\} & \text{if } \widetilde{\xi} = -i \end{cases} \\ \mathscr{S}(d) \in \Psi^{-1}_{\widetilde{\chi_k}}(\{dca\}) & = \mathrm{Ker}(\Psi_{\widetilde{\chi_k}})ec = \{ec, aec, a^2ec, a^{-1}ec\}. \end{cases} \tag{353}$$

Suppose that $\mathscr{S}$ satisfies (353). Since $\mathscr{S}$ is a group homomorphism, it has to satisfy the relations of $\mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ given by (348):

$$\begin{aligned} &\mathscr{S}(a)^4 = \mathscr{S}(c)^6 = 1, \mathscr{S}(b)^2 = \mathscr{S}(a)^2, \mathscr{S}(d)^2 = \mathscr{S}(c)^3, \mathscr{S}(b)\mathscr{S}(a)\mathscr{S}(b)^{-1} = \mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} \\ &= \mathscr{S}(a)^{-1}, \mathscr{S}(c)\mathscr{S}(a)\mathscr{S}(c)^{-1} = \mathscr{S}(b), \mathscr{S}(d)\mathscr{S}(a)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^2\mathscr{S}(b), \mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} \\ &= \mathscr{S}(a)\mathscr{S}(b), \mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}. \end{aligned} \tag{354}$$

In particular,

- $\mathscr{S}(a)^4 = 1$ is satisfied for every possibility of $\mathscr{S}(a)$, but $\mathscr{S}(a)^2 = a^2$ if and only if $\mathscr{S}(a) \in \{cb, c^{-1}b\} = \{cb, bc\}$.
- $\mathscr{S}(b)^2 = \mathscr{S}(a)^2 = a^2$ if and only if $\mathscr{S}(b) \in \{b, b^{-1}\}$.
- If $\mathscr{S}(a) = cb$ then $\mathscr{S}(b)\mathscr{S}(a)\mathscr{S}(b)^{-1} = bcbb^{-1} = bc = \mathscr{S}(a)^{-1}$. Similarly, if $\mathscr{S}(a) = c^{-1}b = a^2cb$, then $\mathscr{S}(b)\mathscr{S}(a)\mathscr{S}(b)^{-1} = \mathscr{S}(a)^{-1}$ (there is no need to compute again since a is in the center of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$).
- If $\mathscr{S}(b) = b$ then $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = ecbc^{-1}e = eb^{-1}e = (bc)^{-1}$. Hence we obtain that $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}$ if and only if $\mathscr{S}(a) = bc$. Similarly, if $\mathscr{S}(b) = b^{-1}$, then $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}$ if and only if $\mathscr{S}(a) = cb$ (there is no need to compute again since a is in the center). Thus $\mathscr{S}(d)\mathscr{S}(b)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^{-1}$ if and only if
$$(\mathscr{S}(a), \mathscr{S}(b)) \in \{(bc, b), (cb, b^{-1})\}. \tag{355}$$
- If $\mathscr{S}(a) = bc$ then
$$\mathscr{S}(c)\mathscr{S}(a)\mathscr{S}(c)^{-1} = \begin{cases} d^{-1}cbcc^{-1}d = d^{-1}cbd = d^{-1}a^2bcd = d^{-1}db = b & \text{if } \widetilde{\xi} = i \\ d^{-1}bbcb^{-1}d = d^{-1}cbd = b & \text{if } \widetilde{\xi} = -i \end{cases} = \mathscr{S}(b).$$
Similarly, if $\mathscr{S}(a) = cb$, then $\mathscr{S}(c)\mathscr{S}(a)\mathscr{S}(c)^{-1} = \mathscr{S}(b)$.
- If $\mathscr{S}(a) = bc$ then $\mathscr{S}(d)\mathscr{S}(a)\mathscr{S}(d)^{-1} = ecbcc^{-1}e = ecbe = c^{-1}ebe = c^{-1}bc = b^{-1} = \mathscr{S}(a)^2\mathscr{S}(b)$. Similarly, if $\mathscr{S}(a) = c^{-1}$, then $\mathscr{S}(d)\mathscr{S}(a)\mathscr{S}(d)^{-1} = \mathscr{S}(a)^2\mathscr{S}(b)$.
- If $\mathscr{S}(b) = b$ then
$$\mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} = \begin{cases} d^{-1}cbc^{-1}d = d^{-1}b^{-1}d = c^{-1} & \text{if } \widetilde{\xi} = i \\ d^{-1}bbb^{-1}d = d^{-1}bd = c & \text{if } \widetilde{\xi} = -i \end{cases},$$

hence $\mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} = \mathscr{S}(a)\mathscr{S}(b)$ if and only if $\widetilde{\xi} = -i$. Similarly, if $\mathscr{S}(b) = b^{-1}$, then $\mathscr{S}(c)\mathscr{S}(b)\mathscr{S}(c)^{-1} = \mathscr{S}(a)\mathscr{S}(b)$ if and only if $\widetilde{\xi} = i$. This implies that

$$(\mathscr{S}(a), \mathscr{S}(b)) = \begin{cases} (cb, b^{-1}) & \text{if } \widetilde{\xi} = i \\ (bc, b) & \text{if } \widetilde{\xi} = -i. \end{cases} \tag{356}$$

- $\mathscr{S}(c)^6 = 1$ if and only if the order of $\mathscr{S}(c)$ in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ divides 6. Since a has order four, $d^{-1}cd^{-1}cd^{-1}c = dbcd^{-1}c = d^{-1}b^{-1}dc = c^{-1}c = 1$ and $(d^{-1}b)^3 = d^{-1}bd^{-1}bd^{-1}b = d^{-1}dcd^{-1}d^{-1}bd^{-1}b = cdbd^{-1}b = cbc^{-1}b = b^{-1}b = 1$, the condition $\mathscr{S}(c)^6 = 1$ is satisfied if and only if

$$\mathscr{S}(c) \in \begin{cases} \{d^{-1}c, d^{-1}c^{-1}\} & \text{if } \widetilde{\xi} = i \\ \{d^{-1}b, d^{-1}b^{-1}\} & \text{if } \widetilde{\xi} = -i. \end{cases} \tag{357}$$

- $\mathscr{S}(d)^2 = \mathscr{S}(c)^3$ if and only if the order of $\mathscr{S}(d)$ divides 2 if $\mathscr{S}(c) = \begin{cases} d^{-1}c & \text{if } \widetilde{\xi} = i \\ d^{-1}b & \text{if } \widetilde{\xi} = -i \end{cases}$, and equals 4 if $\mathscr{S}(c) = \begin{cases} d^{-1}c^{-1} & \text{if } \widetilde{\xi} = i \\ d^{-1}b^{-1} & \text{if } \widetilde{\xi} = -i \end{cases}$. Since a has order four and $(ec)^2 = ecec = c^{-1}c = 1$, the condition $\mathscr{S}(d)^2 = \mathscr{S}(c)^3$ is satisfied if and only if

$$\mathscr{S}(d) \in \begin{cases} \{ec, ec^{-1}\} & \text{if } \mathscr{S}(c) = d^{-1}c \text{ and } \xi = i \\ \{aec, a^{-1}ec\} & \text{if } \mathscr{S}(c) = d^{-1}c^{-1} \text{ and } \xi = i \\ \{ec, ec^{-1}\} & \text{if } \mathscr{S}(c) = d^{-1}b \text{ and } \xi = -i \\ \{aec, a^{-1}ec\} & \text{if } \mathscr{S}(c) = d^{-1}b^{-1} \text{ and } \xi = -i. \end{cases} \tag{358}$$

- Now we examine the condition $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}$.
  - If $\mathscr{S}(c) = d^{-1}c$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = ecd^{-1}cc^{-1}e = ecd^{-1}e = eced = c^{-1}d$.
  - If $\mathscr{S}(c) = d^{-1}c^{-1}$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}$ (there is no need to compute again since $d^{-1}c^{-1} = d^{-1}ca^2$ and a is in the center of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$).
  - If $\mathscr{S}(c) = d^{-1}b$, then

$$\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = ecd^{-1}bc^{-1}e = ec^2d^{-1}c^{-1}e = a^2ed^{-1}c^{-1}e = a^2deec = a^2dc = a^2bd = b^{-1}d.$$

  - If $\mathscr{S}(c) = d^{-1}b^{-1}$, then $\mathscr{S}(d)\mathscr{S}(c)\mathscr{S}(d)^{-1} = \mathscr{S}(c)^{-1}$(there is no need to compute again).

After studying all the relations in (354) and combining (356), (357) and (358), we obtain that (354) is satisfied if and only if

$$\begin{aligned} (\mathscr{S}(a), \mathscr{S}(b)) &= \begin{cases} (cb, b^{-1}) & \text{if } \widetilde{\xi} = i \\ (bc, b) & \text{if } \widetilde{\xi} = -i \end{cases} \\ (\mathscr{S}(c), \mathscr{S}(d)) &\in \begin{cases} \{(d^{-1}c, ec), (d^{-1}c, ec^{-1}), (d^{-1}c^{-1}, aec), (d^{-1}c^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = i \\ \{(d^{-1}b, ec), (d^{-1}b, ec^{-1}), (d^{-1}b^{-1}, aec), (d^{-1}b^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = -i. \end{cases} \end{aligned} \tag{359}$$

Conversely, the above computations also prove that every map $\mathscr{S}$ defined on the generators a, b, c, d of $\mathrm{Mp}(\widetilde{W})$ by a suitable choice in (359), defines a homomorphism of central extensions making the diagram (D_{40}) commute. Now, by Lemma 3.14, it only remains to identify the image of $\mathscr{S}$ between $\mathrm{CSU}_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$. In the database [GN], we notice that $\mathrm{GL}_2(\mathbb{F}_3)$ has 13 elements of order two, whereas $\mathrm{CSU}_2(\mathbb{F}_3)$ has only one element of order two. Observe that a^2, ec and ec^{-1} are three distinct elements of order two in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Then we immediately obtain that the image of $\mathscr{S}$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$ if one of the following conditions is satisfied:

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}c, ec), (d^{-1}c, ec^{-1})\} & \text{if } \widetilde{\xi} = i \\ \{(d^{-1}b, ec), (d^{-1}b, ec^{-1})\} & \text{if } \widetilde{\xi} = -i, \end{cases}$$

since then the image of $\mathscr{S}$ contains at least two different elements of order two. Suppose now that

$$(\mathscr{S}(c), \mathscr{S}(d)) \in \begin{cases} \{(d^{-1}c^{-1}, aec), (d^{-1}c^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = i \\ \{(d^{-1}b^{-1}, aec), (d^{-1}b^{-1}, a^{-1}ec)\} & \text{if } \widetilde{\xi} = -i. \end{cases} \tag{360}$$

Observe that the group $\mathrm{CSU}_2(\mathbb{F}_3)$ contains $\mathrm{Q}_{16} =< a, b \mid a^8 = 1, b^2 = a^4, bab^{-1} = a^{-1} >$ as a subgroup, whereas $\mathrm{GL}_2(\mathbb{F}_3)$ does not contain Q_{16}. We now exhibit a subgroup of $\mathrm{Im}(\mathscr{S})$ which is isomorphic to Q_{16}. By assumption,

$$\begin{cases} \mathscr{S}(bdc) \in < a^2 > baecd^{-1}c^{-1} =< a^2 > abc^{-1}ed^{-1}c^{-1} =< a^2 > abc^{-1}dec^{-1} =< a^2 > adbec^{-1} \\ =< a^2 > adebcc^{-1} =< a^2 > adeb \qquad \text{if } \widetilde{\xi} = i \\ \mathscr{S}(bdca) \in < a^2 > baecd^{-1}b^{-1}bc =< a^2 > baecd^{-1}c =< a^2 > baecd^{-1}c^{-1} =< a^2 > adeb \\ \text{if } \widetilde{\xi} = -i. \end{cases}$$

This proves that $adeb \in \mathrm{Im}(\mathscr{S})$. Since $(adeb)^2 = a^2debdeb = a^2ed^{-1}bdeb = a^2eceb = cb$, and $(cb)^2 = cbcb = a^2bccb = a^2$, the element $adeb$ has order eight in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Moreover, by (359), the element b is of order four in $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and also belongs to $\mathrm{Im}(\mathscr{S})$. Furthermore,

$$\begin{cases} (adeb)^4 = (cb)^2 = a^2 \\ b^2 = a^2 \\ badebb^{-1} = bade = (adeb)^{-1}. \end{cases} \tag{361}$$

Consequently, (361) shows that the subgroup of $\mathrm{Im}(\mathscr{S})$ generated by $adeb$ and b is isomorphic to Q_{16}. Thus $\mathrm{Im}(\mathscr{S})$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ if (360) is satisfied. This concludes the proof. □

Corollary 3.38. *The representation* $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ *of* $\mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ *obtained by Proposition 3.37 is irreducible but non-faithful.*

Proof. The representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is unitary since $\widetilde{\omega}_{\widetilde{\chi_k}}$ satisfies this property. By Proposition 3.37, the kernel of $\mathscr{S}$ has two elements. Since $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful, the kernel of $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is exactly the kernel of $\mathscr{S}$. More precisely,

$$\mathrm{Ker}(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}) = \begin{cases} < c^3 > & \text{if } \mathrm{Im}(\mathscr{S}) \simeq \mathrm{GL}_2(\mathbb{F}_3) \\ < a^2c^3 > & \text{if } \mathrm{Im}(\mathscr{S}) \simeq \mathrm{CSU}_2(\mathbb{F}_3) \end{cases}.$$

Now, let V be a vector subspace of $\mathrm{L}^2(Y)$ which is invariant under $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$. By (350), the elements $b = \left((1, 2x+2y), -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ and $c = \left((1, 2x), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ belong to $\mathrm{Im}(\mathscr{S})$. Hence V is invariant under the following operators:

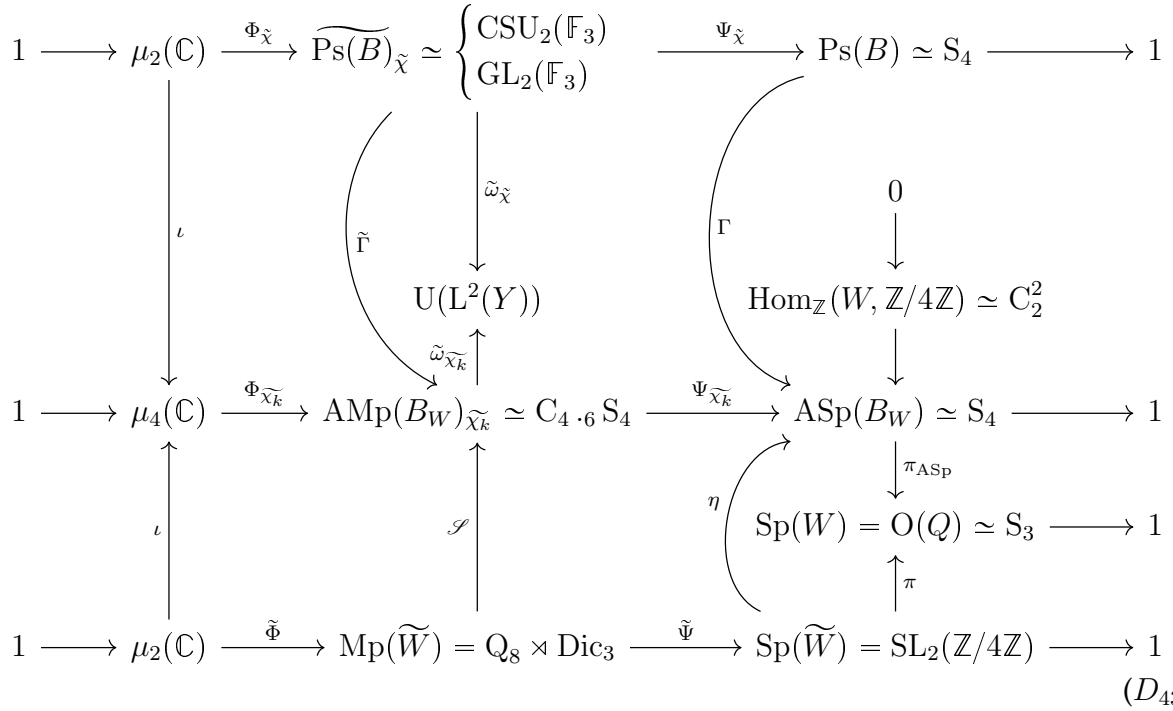
$$(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(b) = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(c) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(bc) = -i\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\rho_{\widetilde{\chi_k}}(1,0) = \begin{pmatrix} \widetilde{\xi} & 0 \\ 0 & -\widetilde{\xi} \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(0,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \rho_{\widetilde{\chi_k}}(1,1) = \begin{pmatrix} 0 & \widetilde{\xi} \\ \widetilde{\xi} & 0 \end{pmatrix},$$

and $W = \{0, (1,0), (0,1), (1,1)\}$, the space V is invariant under $\rho_{\widetilde{\chi_k}}(w)$ for every $w \in W$. Since $\{\rho_{\widetilde{\chi_k}}(w),\ w \in W\}$ is a basis of $\mathrm{End}_{\mathbb{C}}(\mathrm{L}^2(Y))$ by Lemma 2.15, this shows that $V = \{0\}$ or $V = \mathrm{L}^2(Y)$. Therefore, $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is irreducible. □

As Proposition 3.16 does, Proposition 3.37 establishes that if $\nu(\sigma_{(0,1)}) \equiv 0 \mod \frac{\pi}{2}$, then there exists a homomorphism $\mathscr{S}$ such that (D_{40}) commutes. It also provides explicit formulas for all possible such homomorphisms $\mathscr{S}$, when the maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are defined by (346). In this way, we obtain the following diagram, where all groups and homomorphisms are now explicit:



(D_{43})

We obtain in this way a unitary and irreducible but non-faithful representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$ (its kernel has two elements). This concludes the study of the two-dimensional case concerning the metaplectic covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$.

Remark 3.39. The study of the two-dimensional case for $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ leads to the following results (which are similar to those obtained for B_{Weil}):

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then the diagram (D_{43}) commutes. We obtain an extension of Weil's work, i.e. the existence of $\widetilde{\Gamma}$ (cf. Propositions 3.35.ii and 2.39). Moreover, there is only one possibility for the metaplectic covering $\mathrm{Mp}(\widetilde{W})$, and we prove the existence of the homomorphism $\mathscr{S}$ providing the unitary and irreducible representation $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$. However, unlike the Weil representation $\widetilde{\omega}_{\widetilde{\chi}_k}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$, the representation $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ is no longer faithful.
- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then the diagram (D_{42}) commutes. Also in this case, there is only one possibility for the metaplectic covering $\mathrm{Mp}(\widetilde{W})$, and we prove the existence of $\mathscr{S}$ and hence of the unitary and irreducible representation $\widetilde{\omega}_{\widetilde{\chi}_k} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$. Moreover, this representation is faithful, as the Weil representation $\widetilde{\omega}_{\widetilde{\chi}_k}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$. However, we lose the relation to Weil's work, in the sense that the homomorphism $\widetilde{\Gamma}$ does not exist (cf. Propositions 3.35.ii and 2.39).

Remark 3.40. Remark 3.19 can be reproduced here.

3.2.5 Two metaplectic coverings $\mathrm{Mp}(W)$ and $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$

In this part, as in section 3.1.5, we try to extend the Weil representation to a minimal extension of $\mathrm{Sp}(W)$, instead of $\mathrm{Sp}(\widetilde{W})$. Since $\mathrm{O}(Q) = \mathrm{Sp}(W)$ in this example, as already mentioned, the construction of $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$ was primarily intended to compare our construction with the work of Gurevich and Hadani, but it does not really bring anything new on the link with $\mathrm{Sp}(W)$. Therefore, we continue our study with a reduced diagram including just the pseudosymplectic covering:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi}}} & & \downarrow & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \begin{cases} \mathrm{CSU}_2(\mathbb{F}_3) \\ \mathrm{GL}_2(\mathbb{F}_3) \end{cases} & \xrightarrow{\Psi_{\widetilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & & & & \overset{\Gamma^{-1} \circ \eta}{\nwarrow} & \downarrow{\scriptstyle \pi_{\mathrm{Ps}}} & & \\
 & & & & & & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & &
\end{array}
\qquad (D_{44})
$$

First, we recall the presentations of the groups which appear in (D_{44}), and we give the explicit expressions of the group homomorphisms $\Phi_{\tilde{\chi}}$, $\Psi_{\tilde{\chi}}$ and π_{Ps}.

- $\mathrm{S}_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$, where $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
- $\mathrm{S}_4 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = 1, cac^{-1} = dad = ab = ba, cbc^{-1} = a, bd = db, dcd = c^{-1} \rangle$, where $a = (1, x), b = (1, y), c = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0 \right)$ and $d = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, xy + x + y \right)$,
- $\mathrm{CSU}_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = 1, b^2 = d^2 = a^2, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = ab, dad^{-1} = a^2 b, cbc^{-1} = a, dcd^{-1} = c^{-1} \rangle$, where

$$a = \left((1, x), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right), \quad b = \left((1, y), \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right),$$

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (x + y)\delta_{-i}(\widetilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1 - i & 1 - i \\ -1 - i & 1 + i \end{pmatrix} \right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + (x + y)\delta_i(\widetilde{\xi}) \right), \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right),$$

- $\mathrm{GL}_2(\mathbb{F}_3) = \langle a, b, c, d \mid a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab, dad = a^2 b, cbc^{-1} = a, dcd = c^{-1} \rangle$, where $a = \left((1, x), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$, $b = \left((1, y), \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right)$,

$$c = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, (x + y)\delta_{-i}(\widetilde{\xi}) \right), -\frac{1}{2} \begin{pmatrix} 1 - i & 1 - i \\ -1 - i & 1 + i \end{pmatrix} \right) \text{ and}$$

$$d = \left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, xy + (x + y)\delta_i(\widetilde{\xi}) \right), \frac{i}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right),$$

- $\Phi_{\tilde{\chi}} : \mu_2(\mathbb{C}) \to \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$, $\quad -1 \mapsto a^2$

$\Psi_{\tilde{\chi}} : \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \to \mathrm{Ps}(B)$

$$\begin{aligned} a &\mapsto a \\ b &\mapsto b \\ c &\mapsto \begin{cases} c & \text{if } \widetilde{\xi} = i \\ bc & \text{if } \widetilde{\xi} = -i \end{cases} \\ d &\mapsto \begin{cases} cd & \text{if } \widetilde{\xi} = i \\ abcd & \text{if } \widetilde{\xi} = -i \end{cases} \end{aligned}$$

and $\pi_{\mathrm{Ps}} : \mathrm{Ps}(B) \to \mathrm{O}(Q)$.

$$\begin{aligned} a &\mapsto 1 \\ b &\mapsto 1 \\ c &\mapsto a \\ d &\mapsto ba \end{aligned}$$

Since the extension $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ does not split, one cannot lift $\widetilde{\omega}_{\tilde{\chi}}$ directly to $\mathrm{Sp}(W)$, see section 3.1.5. Therefore we look for a covering of $\mathrm{Sp}(W)$ of degree at least 2. The following proposition uses results of section 3.1.5 to affirm that a two-fold covering can be found if only if $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, i.e. if $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{CSU}_2(\mathbb{F}_3)$ by Proposition 3.28.

Proposition 3.41. • *Suppose that* $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$. *Define*

$$\mathrm{Mp}(W) = \mathrm{Dic}_3 = \langle a, b \mid a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1} \rangle \tag{362}$$

and

$$\begin{array}{rcl} \hat{\Phi} : \mu_2(\mathbb{C}) \to \mathrm{Mp}(W) \\ -1 \mapsto b^2 \end{array} \quad \textit{and} \quad \begin{array}{rcl} \hat{\Psi} : \mathrm{Mp}(W) \to \mathrm{Sp}(W). \\ a \mapsto a \\ b \mapsto b \end{array}$$

Then $1 \to \mu_2(\mathbb{C}) \xrightarrow{\hat{\Phi}} \mathrm{Mp}(W) \xrightarrow{\hat{\Psi}} \mathrm{Sp}(W) \to 1$ *is a non-split central extension. Moreover, there exists an injective group homomorphism* $\widehat{\mathscr{S}} : \mathrm{Mp}(W) \to \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ *such that the following diagram* (D_{45}) *is commutative:*

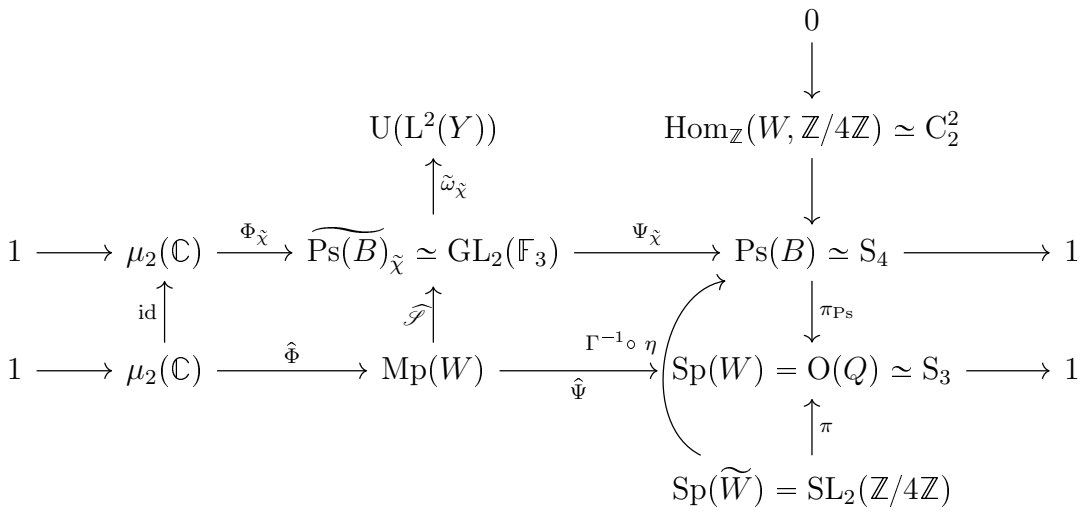
$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\ & & & & \uparrow{\scriptstyle \tilde{\omega}_{\tilde{\chi}}} & & \downarrow & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{CSU}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \widehat{\mathscr{S}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Ps}}} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\hat{\Phi}} & \mathrm{Mp}(W) = \mathrm{Dic}_3 & \xrightarrow[\hat{\Psi}]{\Gamma^{-1} \circ \eta} & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\ & & & & & & \uparrow{\scriptstyle \pi} & & \\ & & & & & & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & & \end{array} \tag{D_{45}}$$

More precisely, $\widehat{\mathscr{S}}$ *is of the following form:*

$$\begin{cases} \widehat{\mathscr{S}}(a) \in \{a^2c, a^{-1}c\} \\ \widehat{\mathscr{S}}(b) \in \{d, d^{-1}\} \end{cases} \quad \textit{or} \quad \begin{cases} \widehat{\mathscr{S}}(a) \in \{b^{-1}c, a^{-1}bc\} \\ \widehat{\mathscr{S}}(b) \in \{abd, a^{-1}bd\}. \end{cases} \tag{363}$$

Moreover, Dic_3 *is the only possibility as two-fold covering making diagram* (D_{45}) *commutative.*

• *Suppose that* $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$. *Then there is no group* $\mathrm{Mp}(W)$ *such that the following diagram commutes and* $\widehat{\mathscr{S}}$ *is injective:*



(D_{46})

Proof. • Suppose that $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$. The proof of the fact that $\hat{\Phi}$ and $\hat{\Psi}$ define a non-split central extension $1 \to \mu_2(\mathbb{C}) \xrightarrow{\hat{\Phi}} \mathrm{Dic}_3 \xrightarrow{\hat{\Psi}} \mathrm{Sp}(W) \simeq \mathrm{S}_3 \to 1$, as well as the fact that we can fix the maps $\hat{\Phi}$ and $\hat{\Psi}$, is identical to that of Proposition 3.20. Now, suppose that $\widehat{\mathscr{S}} : \mathrm{Mp}(W) = \mathrm{Dic}_3 \to \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ is a group homomorphism. Then the diagram (D_{45}) commutes if and only if $\widehat{\mathscr{S}} \circ \hat{\Phi} = \Phi_{\tilde{\chi}}$ and $\pi_{Ps} \circ \Psi_{\tilde{\chi}} \circ \widehat{\mathscr{S}} = \hat{\Psi}$. The condition $\widehat{\mathscr{S}} \circ \hat{\Phi} = \Phi_{\tilde{\chi}}$ is equivalent to $\widehat{\mathscr{S}}(\hat{\Phi}(-1)) = \Phi_{\tilde{\chi}}(-1)$, i.e. $\widehat{\mathscr{S}}(b^2) = a^2$. Since $\hat{\Psi}(a) = a$, $\hat{\Psi}(b) = b$, $(\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})(a) = (\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})(b) = 1$, $(\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})(c) = a$ and $(\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})(d) = aba = b$,

$$\begin{cases} (\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(a) = \hat{\Psi}(a) & \iff \widehat{\mathscr{S}}(a) \in (\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})^{-1}(\{a\}) = \mathrm{Ker}(\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})c =< a, b > c \\ (\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(b) = \hat{\Psi}(b) & \iff \widehat{\mathscr{S}}(b) \in (\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})^{-1}(\{b\}) = \mathrm{Ker}(\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}})d =< a, b > d. \end{cases}$$

Observe that

$$\mathrm{Ker}(\pi_{\mathrm{Ps}} \circ \Psi_{\tilde{\chi}}) =< a, b >= \{1, a, a^2, a^{-1}, b, b^{-1}, ab, a^{-1}b\} \tag{364}$$

has eight elements and is isomorphic to the group

$$\mathrm{Q}_8 =< a, b \mid a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} >,$$

which is also isomorphic to the Heisenberg group $\mathrm{H}(B)$.

Suppose now that $\widehat{\mathscr{S}}$ is a map such that the diagram (D_{45}) commutes, i.e. $\widehat{\mathscr{S}}(b^2) = a^2$, $\widehat{\mathscr{S}}(a) \in < a, b > c$, and $\widehat{\mathscr{S}}(b) \in < a, b > d$. Then $\widehat{\mathscr{S}}$ is a group homomorphism if and only if its image satisfies the relations of Dic_3 given by (362), i.e. $\widehat{\mathscr{S}}(a)^6 = 1$, $\widehat{\mathscr{S}}(b)^2 = \widehat{\mathscr{S}}(a)^3 = a^2$ and $\widehat{\mathscr{S}}(b)\widehat{\mathscr{S}}(a)\widehat{\mathscr{S}}(b)^{-1} = \widehat{\mathscr{S}}(a)^{-1}$. If we examine all the possibilities for $\widehat{\mathscr{S}}(a) \in < a, b > c$ and $\widehat{\mathscr{S}}(b) \in < a, b > d$, we see that the conditions $\widehat{\mathscr{S}}(b)^2 = a^2$ and $\widehat{\mathscr{S}}(a)^3 = \widehat{\mathscr{S}}(b)^2 = a^2$ are satisfied if and only if $\widehat{\mathscr{S}}(a) \in \{a^2, a^{-1}, b^{-1}, a^{-1}b\}c$ and $\widehat{\mathscr{S}}(b) \in \{1, a^2, ab, a^{-1}b\}d$. In this case, $\widehat{\mathscr{S}}(a)^6 = 1$ and $\widehat{\mathscr{S}}(b)^2 = \widehat{\mathscr{S}}(a)^3$. Finally, the

condition $\widehat{\mathscr{S}}(b)\widehat{\mathscr{S}}(a)\widehat{\mathscr{S}}(b)^{-1} = \widehat{\mathscr{S}}(a)^{-1}$ provides all the possibilities for $\widehat{\mathscr{S}}$ given by (363). These computations prove that these possibilities in fact define group homomorphisms such that (D_{45}) commutes. So, it only remains to prove that $\widehat{\mathscr{S}}$ is injective. The image of $\widehat{\mathscr{S}}$ is generated by $\widehat{\mathscr{S}}(a)$, of order six, and $\widehat{\mathscr{S}}(b)$, of order four. Moreover, $\widehat{\mathscr{S}}(a)$ and $\widehat{\mathscr{S}}(b)$ verify the relations of Dic_3 because $\widehat{\mathscr{S}}$ is a group homomorphism. Thus the image of $\widehat{\mathscr{S}}$ is isomorphic to Dic_3, and $\widehat{\mathscr{S}}$ is injective.

The fact that Dic_3 is the only possible choice for $\mathrm{Mp}(W)$ is due to the fact that, between the two possibilities D_6 and Dic_3, the group $\mathrm{CSU}_2(\mathbb{F}_3)$ contains only Dic_3 as a subgroup.

- Suppose that $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$. Then $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3)$ by Proposition 3.28. Since $\mathrm{GL}_2(\mathbb{F}_3)$ contains D_6 but not Dic_3 as a subgroup, there is no hope to find an injective group homomorphism $\widehat{\mathscr{S}} : \mathrm{Mp}(W) \to \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ such that the diagram commutes (otherwise the extension defined by $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}}$ would split, since D_6 defines a central split extension).

□

Corollary 3.42. *We keep the same notation as in Proposition 3.41. Suppose that* $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$. *Then, for every homomorphism* $\widehat{\mathscr{S}}$ *defined by* (363), *the representation* $\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}}$ *of* $\mathrm{Mp}(W)$ *is unitary, faithful and irreducible.*

Proof. $\widehat{\mathscr{S}}$ is an injective group homomorphism by Proposition 3.41, and $\widetilde{\omega}_{\tilde{\chi}}$ is faithful and irreducible by Proposition 1.39. Hence $\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}}$ is a unitary and faithful representation. Suppose now that $V \subseteq \mathrm{L}^2(Y)$ is a vector subspace which is invariant under $\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}}$. Then V is invariant under the operator $(\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(g)$ for every $g \in \mathrm{Mp}(W)$. In particular, V is invariant under $(\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(a)$ and $(\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(b)$, which are of the following form by (363):

$$\begin{cases} (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\tilde{\chi}}(a^2 c), \widetilde{\omega}_{\tilde{\chi}}(a^{-1} c)\} \\ (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\tilde{\chi}}(d), \widetilde{\omega}_{\tilde{\chi}}(d^{-1})\} \end{cases} \quad \text{or} \quad \begin{cases} (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\tilde{\chi}}(b^{-1} c), \widetilde{\omega}_{\tilde{\chi}}(a^{-1} b c)\} \\ (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\tilde{\chi}}(abd), \widetilde{\omega}_{\tilde{\chi}}(a^{-1} b d)\}, \end{cases}$$

i.e. by Proposition 3.28:

$$\begin{cases} (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(a) \in \left\{ \frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(b) \in \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}, -\frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \right\} \end{cases}$$

or

$$\begin{cases} (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(a) \in \left\{ \frac{1}{2} \begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\tilde{\chi}} \circ \widehat{\mathscr{S}})(b) \in \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}, -\frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \right\}. \end{cases}$$

In any case, the operators $(\widetilde{\omega}_{\widetilde{\chi}} \circ \widehat{\mathscr{S}})(a)$ and $(\widetilde{\omega}_{\widetilde{\chi}} \circ \widehat{\mathscr{S}})(b)$ do not have any common eigenvalue. Thus V cannot be a one-dimensional space. Since $\dim_{\mathbb{C}}(\mathrm{L}^2(Y)) = 2$, this implies that $V = \{0\}$ or $V = \mathrm{L}^2(Y)$. Therefore, $\widetilde{\omega}_{\widetilde{\chi}} \circ \widehat{\mathscr{S}}$ is irreducible. □

Remark 3.43 (Related to Remark 3.22)**.** Proposition 3.41 and Corollary 3.42 provide the "best" result we could hope for if $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, in the sense that the two-fold covering $\mathrm{Mp}(W) \simeq \mathrm{Dic}_3$ is the smallest possible central extension of $\mathrm{Sp}(W)$ for which the diagram (D_{45}) commutes and the resulting Weil representation of $\mathrm{Mp}(W)$ is unitary, irreducible and faithful. These statements also show that we cannot obtain such a result if $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$: if we want to obtain a faithful representation and a commutative diagram, then we need to define at least an extension of $\mathrm{Sp}(W)$ by four elements.

Now, we suppose that $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$. By Proposition 3.28, the group $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$. Thus we look for a four-fold covering $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$ such that the following diagram (D_{47}) commutes and $\widetilde{\omega}_{\widetilde{\chi}} \circ \widecheck{\mathscr{S}}$ is a unitary, irreducible and faithful representation.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi}}} & & \downarrow & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\widetilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \quad (D_{47}) \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \widecheck{\mathscr{S}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Ps}}} & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\widecheck{\Phi}} & \mathrm{AMp}(W) & \xrightarrow[\widecheck{\Psi}]{\Gamma^{-1} \circ \eta} & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & &
\end{array}
$$

Lemma 3.44. *There is no four-fold covering $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$ such that $\widecheck{\mathscr{S}}$ is an injective group homomorphism and the diagram (D_{47}) commutes.*

Proof. The group $\mathrm{AMp}(W)$ has to be a central extension of S_3 by four elements, and also a subgroup of index two in $\mathrm{GL}_2(\mathbb{F}_3)$. The only subgroup of index two in $\mathrm{GL}_2(\mathbb{F}_3)$ is $\mathrm{SL}_2(\mathbb{F}_3)$, and it does not appear in the list of central extensions of S_3 by four elements in [GN]. □

Lemma 3.44 shows that, if $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, then it is not possible to find a four-fold covering $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$ such that (D_{47}) commutes. If we want such a four-fold covering, then we have to consider again the affine metaplectic covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$, as for B_{Weil}. So, if we consider again the two possibilities for $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, we can resume the situation in the following way:

- If $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$ then $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{CSU}_2(\mathbb{F}_3)$ and there is no need to consider $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$: the commutative diagram (D_{45}) provides a unitary, faithful and irreducible Weil representation of a two-fold covering $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$, and maintains the link with Weil's work.
- If $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$ then $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3)$. In this case, we were not able to define a commutative diagram leading to a unitary, faithful and irreducible Weil representation of a two-fold or a four-fold covering of $\mathrm{Sp}(W)$. This is why we need to consider again $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, and this is what we will do in the rest of this section.

To complete our study, we now suppose that $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, i.e. $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3)$. Considering the possibilities for $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, we look for a unitary, irreducible and faithful Weil representation of the smallest possible extension of $\mathrm{Sp}(W)$.

Proposition 3.45. *Assume that* $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, *i.e.* $\widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3)$.

- *Suppose that* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. *We consider again the maps* $\hat{\Phi}$ *and* $\hat{\Psi}$ *defined in Proposition 3.41. Then the following diagram* (D_{48}) *commutes if and only if* $\widehat{\mathscr{S}}$ *is of the following form:*

$$\begin{cases} \widehat{\mathscr{S}}(a) \in \{a^2d, cd\} \\ \widehat{\mathscr{S}}(b) \in \{ad^{-1}e, a^{-1}d^{-1}e\} \end{cases} \quad \textit{or} \quad \begin{cases} \widehat{\mathscr{S}}(a) \in \{bd, cbd\} \\ \widehat{\mathscr{S}}(b) \in \{abd^{-1}e, a^{-1}bd^{-1}e\}. \end{cases} \tag{365}$$

  In this case, $\widehat{\mathscr{S}}$ *is injective. Moreover,* Dic_3 *is the only possible choice for the two-fold covering* $\mathrm{Mp}(W)$ *of* $\mathrm{Sp}(W)$ *such that* (D_{48}) *commutes.*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \tilde{\omega}_{\tilde{\chi}}} \ \ {\scriptstyle \tilde{\Gamma}}\uparrow & & \downarrow{\scriptstyle 0} \ \ {\scriptstyle \Gamma}\uparrow & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & \uparrow{\scriptstyle \tilde{\omega}_{\widetilde{\chi_k}}} & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \widehat{\mathscr{S}}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\hat{\Phi}} & \mathrm{Mp}(W) = \mathrm{Dic}_3 & \xrightarrow{\hat{\Psi}} \ \ {\scriptstyle \eta}\uparrow & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1
\end{array}$$

(D_{48})

- *If* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, *then there is no two-fold covering* $\mathrm{Mp}(W)$ *of* $\mathrm{Sp}(W)$ *which enables us to obtain a commutative diagram similar to* (D_{48}), *with* $\widehat{\mathscr{S}}$ *injective. However,*

if we consider again the maps $\breve{\Phi}$ and $\breve{\Psi}$ defined in Proposition 3.23, then the following diagram (D_{49}) commutes if and only if $\breve{\mathscr{S}}$ is of the following form:

$$\begin{cases} \breve{\mathscr{S}}(a) \in \{d, c^{-1}d\} \\ \breve{\mathscr{S}}(b) \in \{d^{-1}e, a^2d^{-1}e\} \end{cases} \quad \textit{or} \quad \begin{cases} \breve{\mathscr{S}}(a) \in \{b^{-1}d, bcd\} \\ \breve{\mathscr{S}}(b) \in \{bd^{-1}e, b^{-1}d^{-1}e\}. \end{cases} \tag{366}$$

In this case, $\breve{\mathscr{S}}$ is injective. Moreover, $\mathrm{C}_3 \rtimes \mathrm{C}_8$ is the only possible choice for the four-fold covering $\mathrm{AMp}(W)$ such that (D_{49}) commutes.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi_{\tilde{\chi}}} & \widetilde{\mathrm{Ps}(B)}_{\tilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\tilde{\chi}}} & \mathrm{Ps}(B) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \widetilde{\omega}_{\tilde{\chi}}} & & \overset{\Gamma}{\curvearrowright} \downarrow{\scriptstyle 0} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi}_k}} & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi}_k}} & \mathrm{AMp}(B_W)_{\widetilde{\chi}_k} \simeq \mathrm{U}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\widetilde{\chi}_k}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \breve{\mathscr{S}}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\breve{\Phi}} & \mathrm{AMp}(W) = \mathrm{C}_3 \rtimes \mathrm{C}_8 & \xrightarrow{\breve{\Psi} \quad \eta} & \mathrm{Sp}(W) = \mathrm{O}(Q) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & & & \uparrow{\scriptstyle \pi} & & \\
 & & & & & & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1
\end{array}$$

(D_{49})

Proof. The fact that Dic_3 and $\mathrm{C}_3 \rtimes \mathrm{C}_8$ are the only possibilities for $\mathrm{Mp}(W)$ and $\mathrm{AMp}(W)$, respectively, and that there is no suitable two-fold covering $\mathrm{Mp}(W)$ if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, is proven exactly as in Propositions 3.20 and 3.23.

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, the proof of the existence and explicit expression of $\widehat{\mathscr{S}}$ is exactly as in Proposition 3.20. Since only the expressions of $\Phi_{\widetilde{\chi}_k}$, $\Psi_{\widetilde{\chi}_k}$, η and π_{ASp} are different, we have to replace $(\pi_{\mathrm{ASp}} \circ \Psi)(e) = ba$ by ab. Then $\widehat{\mathscr{S}}(b) \in < a, b, c > d^{-1}e$ instead of $< a, b, c > de$, and the rest of the reasoning provides $\widehat{\mathscr{S}}(b) \in \{a, a^{-1}, ab, a^{-1}b\}d^{-1}e$ instead of $\widehat{\mathscr{S}}(b) \in \{a, a^{-1}, abc, a^{-1}bc\}de$. These modifications give the expression of $\widehat{\mathscr{S}}$ given in (365).
- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then the proof is analogous to that of Proposition 3.23. In fact, we only have to replace $(\pi_{\mathrm{ASp}} \circ \Psi)(e) = ba$ by ab, and then $\breve{\mathscr{S}}(b) \in < a, b, c > ed^{-1}$ by $< a, b, c > d^{-1}e$. Hence, $\widehat{\mathscr{S}}(b) \in \{1, a^2, b, b^{-1}\}d^{-1}e$, and the rest of the proof provides (366).

□

Corollary 3.46. *We keep the same notation as in Proposition 3.45.*

- *Suppose that* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. *Then, for every homomorphism* $\widehat{\mathscr{S}}$ *defined by* (365), *the representation* $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}}$ *of* $\mathrm{Mp}(W)$ *is unitary, faithful and irreducible.*

- *Suppose that* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. *Then, for every homomorphism* $\breve{\mathscr{S}}$ *defined by* (366), *the representation* $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}}$ *of* $\mathrm{AMp}(W)$ *is unitary, faithful and irreducible.*

Proof. The proof is identical to those of Corollaries 3.21 and 3.24. We only have to modify the operators $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a)$, $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b)$, $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(a)$and $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(b)$ according to (365) and (366):

- If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(a^2 d), \widetilde{\omega}_{\widetilde{\chi_k}}(cd)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(ad^{-1}e), \widetilde{\omega}_{\widetilde{\chi_k}}(a^{-1}d^{-1}e)\} \end{cases} \text{ or } \begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(bd), \widetilde{\omega}_{\widetilde{\chi_k}}(cbd)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(abd^{-1}e), \widetilde{\omega}_{\widetilde{\chi_k}}(a^{-1}bd^{-1}e)\}, \end{cases}$$

  i.e. by Proposition 3.30,

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \left\{ \frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \left\{ -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}, \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \right\} \end{cases}$$

  or

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(a) \in \left\{ \frac{i}{2} \begin{pmatrix} -1-i & 1+i \\ 1-i & 1-i \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 1-i & 1-i \\ 1+i & -1-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(b) \in \left\{ -\frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix} \right\}. \end{cases}$$

- If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(d), \widetilde{\omega}_{\widetilde{\chi_k}}(c^{-1}d)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}e), \widetilde{\omega}_{\widetilde{\chi_k}}(a^2 d^{-1}e)\} \end{cases} \text{ or } \begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(a) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(b^{-1}d), \widetilde{\omega}_{\widetilde{\chi_k}}(bcd)\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(b) \in \{\widetilde{\omega}_{\widetilde{\chi_k}}(bd^{-1}e), \widetilde{\omega}_{\widetilde{\chi_k}}(b^{-1}d^{-1}e)\}, \end{cases}$$

  i.e. by Proposition 3.30,

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(a) \in \left\{ -\frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}, -\frac{1}{2} \begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \breve{\mathscr{S}})(b) \in \left\{ -\frac{1+i}{2} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}, \frac{1+i}{2} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \right\} \end{cases}$$

or

$$\begin{cases} (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(a) \in \left\{ -\frac{i}{2} \begin{pmatrix} -1-i & 1+i \\ 1-i & 1-i \end{pmatrix}, -\frac{i}{2} \begin{pmatrix} 1-i & 1-i \\ 1+i & -1-i \end{pmatrix} \right\} \\ (\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(b) \in \left\{ \frac{-1+i}{2} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}, \frac{1-i}{2} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix} \right\}. \end{cases}$$

□

Now, by Proposition 3.45 and Corollary 3.46, if $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3)$, we have constructed a lift of the Weil representation to a two-fold or a four-fold covering of $\mathrm{Sp}(W)$. Unfortunately, arguing as in the proof of Proposition 3.26, we see that we cannot "connect" these coverings $\mathrm{Mp}(W)$ and $\mathrm{AMp}(W)$, respectively, to the two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ by a bigger commutative diagram, similar to (D_{36}) and (D_{37}), respectively.

Remark 3.47. The "best" results we obtain in this section for $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the following:

- If $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, i.e. $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \mathrm{CSU}_2(\mathbb{F}_3)$, then there is no need to consider $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. We obtain a unitary, irreducible and faithful Weil representation of a two-fold covering $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$ such that the diagram (D_{45}) commutes. In particular, we keep the link with Weil's work.
- If $\theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi$, i.e. $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \mathrm{GL}_2(\mathbb{F}_3)$, then the Weil representation does not lift to a two-fold or to a four-fold covering of $\mathrm{Sp}(W)$. Instead, we use the four-fold covering $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and its Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$. In this way, if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ then we obtain a unitary, irreducible and faithful Weil representation of a two-fold covering $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$ such that the diagram (D_{48}) commutes (in particular, we keep the link with Weil's work, in the sense of the commutativity of (D_{22})). If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ then we obtain a unitary, irreducible and faithful Weil representation of a four-fold covering $\mathrm{AMp}(W)$ of $\mathrm{Sp}(W)$ such that the diagram (D_{49}) commutes (but we lose the link with Weil's work).

Remark 3.48. As in Remark 3.27, we underline that no unitary, irreducible and faithful lift of the Weil representation to a two-fold or four-fold covering of $\mathrm{Sp}(W)$ exists if the dimension of W is $2n$ with $n \geqslant 4$. Moreover, we point out once more that the example considered in this section is really exceptional since it is the only one for which $\mathrm{O}(Q) = \mathrm{Sp}(W)$.

3.3 The two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ arising from the Heisenberg group $\mathrm{H}(\widetilde{W})$

In sections 3.1.4 and 3.2.4, we have exhibited all the possibilities for the construction of a two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ which are compatible with the pseudosymplectic and the affine metaplectic coverings, where "compatible" means that the diagrams (D_{30}), (D_{32}), (D_{42}) and (D_{43}) are commutative. Each of these diagrams provides also a homomorphism of central extensions $\mathscr{S}$ which enables us to extend the definition of the Weil representation to $\mathrm{Mp}(\widetilde{W})$, preserving its injectivity only for (D_{30}) and (D_{42}). A drawback of these constructions is that the possible groups $\mathrm{Mp}(\widetilde{W})$ are abstractly defined in terms of generators and relations, and no clear structure related to the symplectic group emerges from these results. In order to recover such a structure, we will follow in this section the path of section 2.5. More precisely, we will use the projective Weil representation $\tilde{\sigma} \mapsto \omega_{\chi_k}(\tilde{\sigma}, \tilde{w}_{\tilde{\sigma}})$ of $\mathrm{Sp}(\widetilde{W})$ defined in Proposition 2.56 to define a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. At the end of the section, we will compare these results with those obtained in sections 3.1.4 and 3.2.4.

We start our study with the Heisenberg group $\mathrm{H}(\widetilde{W})$ from Definition 2.47, associated to the symplectic module $\widetilde{W} \simeq (\mathbb{Z}/4\mathbb{Z})^2$. This group is the set $\widetilde{W} \times \mathbb{Z}/4\mathbb{Z}$ endowed with the following product:

$$(\tilde{w}_1, t_1)(\tilde{w}_2, t_2) = (\tilde{w}_1 + \tilde{w}_2, t_1 + t_2 + \langle \tilde{w}_1, \tilde{w}_2 \rangle_{\widetilde{W}}). \tag{367}$$

By Proposition 2.48, $\mathrm{H}(\widetilde{W})$ is a central, non-split, and non-stem extension of $\widetilde{W} \simeq (\mathbb{Z}/4\mathbb{Z})^2$ by $\mathbb{Z}/4\mathbb{Z}$. So this group has 64 elements and by [GN], it is isomorphic to $\mathrm{C}_{16} \rtimes_5 \mathrm{C}_4$, $\mathrm{C}_4^2 \rtimes_4 \mathrm{C}_4$ or $\mathrm{C}_2 \times \mathrm{C}_8 \rtimes \mathrm{C}_4$. Since for every $(\tilde{w}, t) \in \mathrm{H}(\widetilde{W})$, we have that $(\tilde{w}, t)^4 = (4\tilde{w}, 4t) = (0, 0)$, we know that the maximal order of an element in $\mathrm{H}(\widetilde{W})$ is four. This is enough to conclude that $\mathrm{H}(\widetilde{W})$ is isomorphic to

$$\mathrm{C}_4^2 \rtimes_4 \mathrm{C}_4 = \langle a, b, c, \mid a^4 = b^4 = c^4 = 1, ab = ba, cac^{-1} = ab^2, bc = cb \rangle,$$

an explicit isomorphism being for example given by $a = (1, 0, 0)$, $b = (0, 0, 1)$ and $c = (0, 1, 0)$. In fact, $\mathrm{C}_{16} \rtimes_5 \mathrm{C}_4$ contains C_{16} as a subgroup, and $\mathrm{C}_2 \times \mathrm{C}_8 \rtimes \mathrm{C}_4$ contains C_8 as a subgroup.

Recall from Proposition 3.1 that the symplectic group of W is

$$\mathrm{Sp}(W) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

and the symplectic group of $\widetilde{W}$ is

$$\mathrm{Sp}(\widetilde{W}) = \left\{ \begin{array}{l} \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&0\\2&1\end{pmatrix}, \pm\begin{pmatrix}1&2\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&2\\2&1\end{pmatrix}, \\ \pm\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}2&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}0&-1\\1&2\end{pmatrix}, \pm\begin{pmatrix}2&-1\\1&2\end{pmatrix}, \\ \pm\begin{pmatrix}0&1\\-1&1\end{pmatrix}, \pm\begin{pmatrix}2&1\\1&1\end{pmatrix}, \pm\begin{pmatrix}0&-1\\1&1\end{pmatrix}, \pm\begin{pmatrix}2&-1\\-1&1\end{pmatrix}, \\ \pm\begin{pmatrix}1&0\\1&1\end{pmatrix}, \pm\begin{pmatrix}-1&0\\1&-1\end{pmatrix}, \pm\begin{pmatrix}1&2\\1&-1\end{pmatrix}, \pm\begin{pmatrix}1&2\\-1&-1\end{pmatrix}, \\ \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&-1\\2&-1\end{pmatrix}, \pm\begin{pmatrix}-1&1\\0&-1\end{pmatrix}, \pm\begin{pmatrix}1&1\\2&-1\end{pmatrix}, \\ \pm\begin{pmatrix}1&-1\\1&0\end{pmatrix}, \pm\begin{pmatrix}1&1\\-1&0\end{pmatrix}, \pm\begin{pmatrix}1&1\\1&2\end{pmatrix}, \pm\begin{pmatrix}1&-1\\-1&2\end{pmatrix}, \end{array} \right\}$$

where the image under the epimorphism π (defined in Proposition 2.7.iii) of each element of the j-th row in the list of $\mathrm{Sp}(\widetilde{W})$ is the j-th element in the list of $\mathrm{Sp}(W)$. In particular, $\mathrm{Sp}(W) = \mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$ is isomorphic to $\mathrm{S}_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$, where for example $a = \begin{pmatrix}0&1\\1&1\end{pmatrix}$ and $b = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, and $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is isomorphic to

$$\begin{aligned} \mathrm{A}_4 \rtimes \mathrm{C}_4 = \langle a, b, c, d \mid & a^2 = b^2 = c^3 = d^4 = 1, cac^{-1} = dad^{-1} = ab = ba, cbc^{-1} = a, bd = db, \\ & dcd^{-1} = c^{-1} \rangle, \end{aligned}$$

an isomorphism being given by $a = \begin{pmatrix}-1&0\\2&-1\end{pmatrix}$, $b = \begin{pmatrix}-1&2\\0&-1\end{pmatrix}$, $c = \begin{pmatrix}2&1\\1&1\end{pmatrix}$ and $d = \begin{pmatrix}1&1\\2&-1\end{pmatrix}$.

We fix a faithful character χ_k of $\mathbb{Z}/4\mathbb{Z}$. Recall from Lemma 2.8 that $k \in \{1, 3\}$, where $\chi_1(1) = i$ and $\chi_3 = \overline{\chi_1}$. Then by Proposition 2.52, the representation ρ_{χ_k} of $\mathrm{H}(\widetilde{W})$ is given by the following operators:

$$\left\{ \begin{array}{ll} \rho_{\chi_k}(0,0) = \rho_{\chi_k}(2,0) = \rho_{\chi_k}(0,2) = \rho_{\chi_k}(2,2) & = \begin{pmatrix}1&0\\0&1\end{pmatrix} \\ \rho_{\chi_k}(1,0) = \rho_{\chi_k}(-1,0) = -\rho_{\chi_k}(1,2) = -\rho_{\chi_k}(-1,2) & = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \\ \rho_{\chi_k}(0,1) = \rho_{\chi_k}(0,-1) = -\rho_{\chi_k}(2,1) = -\rho_{\chi_k}(2,-1) & = \begin{pmatrix}0&1\\1&0\end{pmatrix} \\ \rho_{\chi_k}(1,1) = \rho_{\chi_k}(-1,-1) = -\rho_{\chi_k}(1,-1) = -\rho_{\chi_k}(-1,1) & = \begin{pmatrix}0&-\chi_k(1)\\\chi_k(1)&0\end{pmatrix}. \end{array} \right. \tag{368}$$

We first determine for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ all the possible choices of $\widetilde{w}_{\widetilde{\sigma}}$ satisfying (207). The set consisting of these choices is the following, in the same order as the elements of $\mathrm{Sp}(\widetilde{W})$ above, and where $\widetilde{x}, \widetilde{y} \in \mathbb{Z}/4\mathbb{Z}$:

$$\left\{\begin{array}{l}(2\widetilde{x}, 2\widetilde{y}), (2\widetilde{x}, 2\widetilde{y}+1), (2\widetilde{x}+1, 2\widetilde{y}), (2\widetilde{x}+1, 2\widetilde{y}+1),\\ (\widetilde{x}, \widetilde{x}+2\widetilde{y}+1), (\widetilde{x}, \widetilde{x}+2\widetilde{y}), (\widetilde{x}, \widetilde{x}+2\widetilde{y}), (\widetilde{x}, \widetilde{x}+2\widetilde{y}+1),\\ (\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{y}),\\ (2\widetilde{x}, \widetilde{y}), (2\widetilde{x}, \widetilde{y}), (2\widetilde{x}+1, \widetilde{y}), (2\widetilde{x}+1, \widetilde{y}),\\ (\widetilde{x}, 2\widetilde{y}), (\widetilde{x}, 2\widetilde{y}+1), (\widetilde{x}, 2\widetilde{y}), (\widetilde{x}, 2\widetilde{y}+1)\\ (\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{y}), (\widetilde{x}, \widetilde{y})\end{array}\right\}. \tag{369}$$

Then we compute the corresponding intertwining operators $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ defined in Proposition 2.56, using the expression (218). The set of operators $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})^{-1}\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}), \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ that we obtain in this way is as follows, in the same order as the elements of $\mathrm{Sp}(\widetilde{W})$ above and where $l \in \mathbb{Z}$ is such that $i^l \neq -1$:

$$\left\{\begin{array}{l}\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, \pm\begin{pmatrix}0&1\\1&0\end{pmatrix}, \pm\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \pm\begin{pmatrix}0&-\chi_k(1)\\\chi_k(1)&0\end{pmatrix},\\ \pm\begin{pmatrix}1&1\\1&-1\end{pmatrix}, i^l\begin{pmatrix}1&1\\-1&1\end{pmatrix}, i^l\begin{pmatrix}1&-1\\1&1\end{pmatrix}, \pm\begin{pmatrix}-1&1\\1&1\end{pmatrix},\\ i^l\begin{pmatrix}1-\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1+\chi_k(1)\end{pmatrix}, i^l\begin{pmatrix}1-\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1+\chi_k(1)\end{pmatrix}, i^l\begin{pmatrix}1+\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\ i^l\begin{pmatrix}1+\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\ i^l\begin{pmatrix}1&\chi_k(1)\\\chi_k(1)&1\end{pmatrix}, i^l\begin{pmatrix}1&-\chi_k(1)\\-\chi_k(1)&1\end{pmatrix}, \pm\begin{pmatrix}1&-\chi_k(1)\\\chi_k(1)&-1\end{pmatrix}, \pm\begin{pmatrix}1&\chi_k(1)\\-\chi_k(1)&-1\end{pmatrix},\\ i^l\begin{pmatrix}1-\chi_k(1)&0\\0&1+\chi_k(1)\end{pmatrix}, \pm\begin{pmatrix}0&1-\chi_k(1)\\1+\chi_k(1)&0\end{pmatrix}, i^l\begin{pmatrix}1+\chi_k(1)&0\\0&1-\chi_k(1)\end{pmatrix},\\ \pm\begin{pmatrix}0&1+\chi_k(1)\\1-\chi_k(1)&0\end{pmatrix},\\ i^l\begin{pmatrix}1+\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1-\chi_k(1)\end{pmatrix}, i^l\begin{pmatrix}1-\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1+\chi_k(1)\end{pmatrix}, i^l\begin{pmatrix}1-\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\ i^l\begin{pmatrix}1+\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1-\chi_k(1)\end{pmatrix}\end{array}\right\}. \tag{370}$$

Now, by definition of $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ in Proposition 2.56 and (232), we have that

$$\left\{\begin{array}{ll}\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})} & \text{if } \pi(\widetilde{\sigma}) = 1\\ \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \frac{1}{\sqrt{2}}e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})} & \text{if } \pi(\widetilde{\sigma}) \in \left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}\right\}\\ \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \frac{1}{2}e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})} & \text{if } \pi(\widetilde{\sigma}) \in \left\{\begin{pmatrix}0&1\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}\right\}.\end{array}\right. \tag{371}$$

Proposition 3.49. *Recall the notation $\widetilde{\sigma}_{\widetilde{v}}$ for the transvection of $\widetilde{W}$ defined in Proposition 2.7.iv by $\widetilde{\sigma}_{\widetilde{v}}(\widetilde{w}) = \widetilde{w} + \langle \widetilde{w}, \widetilde{v} \rangle_{\widetilde{W}} \widetilde{v}$. Suppose that the set $\{\widetilde{w}_{\widetilde{\sigma}}$ satisfying* (207)*, $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *is fixed as follows:*

$$\left\{ \begin{array}{l} (0,0),(2,1),(1,2),(1,1), \\ (0,1),(0,0),(0,0),(0,1), \\ (0,0),(0,0),(0,0),(0,0), \\ (0,0),(0,0),(1,0),(1,0), \\ (0,0),(0,1),(0,0),(0,1), \\ (0,0),(0,0),(0,0),(0,0) \end{array} \right\}. \tag{372}$$

i. *We choose the following set $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying* (232)*, $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$, where the elements are given in the same order as in the list of $\mathrm{Sp}(\widetilde{W})$ above:*

$$\left\{ \begin{array}{l} 1, i, 1, i, 1, i, i, 1, \\ \frac{1-i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \\ \frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \frac{i}{2}, \frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \frac{i}{2}, \\ \frac{1-i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \\ \frac{1+i}{2}, \frac{1-i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \frac{1+i}{2}, \frac{1-i}{2}, \\ \frac{i}{2}, \frac{1}{2}, \frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \frac{i}{2}, \frac{1}{2}, \frac{i}{2} \end{array} \right\}. \tag{373}$$

Then the set of operators $\{\pm\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is the following, again in the same order as the elements of $\mathrm{Sp}(\widetilde{W})$ given above:

$$\left\{\begin{array}{l}
\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm i\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm\begin{pmatrix}0&1\\1&0\end{pmatrix},\pm i\begin{pmatrix}0&1\\1&0\end{pmatrix},\pm\begin{pmatrix}1&0\\0&-1\end{pmatrix},\pm i\begin{pmatrix}1&0\\0&-1\end{pmatrix},\\
\pm i\begin{pmatrix}0&-\chi_k(1)\\\chi_k(1)&0\end{pmatrix},\pm\begin{pmatrix}0&-\chi_k(1)\\\chi_k(1)&0\end{pmatrix},\\
\pm\frac{1-i}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix},\pm\frac{1-i}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix},\\
\pm\frac{1-i}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix},\pm\frac{1-i}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}-1&1\\1&1\end{pmatrix},\\
\pm\frac{i}{2}\begin{pmatrix}1-\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{i}{2}\begin{pmatrix}1-\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{i}{2}\begin{pmatrix}1+\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{i}{2}\begin{pmatrix}1+\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{1-i}{2}\begin{pmatrix}1&\chi_k(1)\\\chi_k(1)&1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}1&\chi_k(1)\\\chi_k(1)&1\end{pmatrix},\\
\pm\frac{1-i}{2}\begin{pmatrix}1&-\chi_k(1)\\-\chi_k(1)&1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}1&-\chi_k(1)\\-\chi_k(1)&1\end{pmatrix},\\
\pm\frac{1-i}{2}\begin{pmatrix}1&-\chi_k(1)\\\chi_k(1)&-1\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}1&-\chi_k(1)\\\chi_k(1)&-1\end{pmatrix},\\
\pm\frac{1+i}{2}\begin{pmatrix}1&\chi_k(1)\\-\chi_k(1)&-1\end{pmatrix},\pm\frac{1-i}{2}\begin{pmatrix}1&\chi_k(1)\\-\chi_k(1)&-1\end{pmatrix}\\
\pm\frac{1+i}{2}\begin{pmatrix}1-\chi_k(1)&0\\0&1+\chi_k(1)\end{pmatrix},\pm\frac{1-i}{2}\begin{pmatrix}1-\chi_k(1)&0\\0&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1-i}{2}\begin{pmatrix}0&1-\chi_k(1)\\1+\chi_k(1)&0\end{pmatrix},\pm\frac{1+i}{2}\begin{pmatrix}0&1-\chi_k(1)\\1+\chi_k(1)&0\end{pmatrix},\\
\pm\frac{1+i}{2}\begin{pmatrix}1+\chi_k(1)&0\\0&1-\chi_k(1)\end{pmatrix},\pm\frac{1-i}{2}\begin{pmatrix}1+\chi_k(1)&0\\0&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{1+i}{2}\begin{pmatrix}0&1+\chi_k(1)\\1-\chi_k(1)&0\end{pmatrix},\pm\frac{1-i}{2}\begin{pmatrix}0&1+\chi_k(1)\\1-\chi_k(1)&0\end{pmatrix},\\
\pm\frac{i}{2}\begin{pmatrix}1+\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{i}{2}\begin{pmatrix}1-\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{i}{2}\begin{pmatrix}1-\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{i}{2}\begin{pmatrix}1+\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1-\chi_k(1)\end{pmatrix}
\end{array}\right\}.$$

Moreover, the range of the cocycle $\widetilde{c}_{\chi_k}$ is $\mu_2(\mathbb{C}) = \{\pm 1\}$. Hence by Proposition 2.63, the set

$$\mathrm{Mp}(\widetilde{W})_{\chi_k} = \{(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W}),\ l \in \mathbb{Z}\},$$

endowed with the direct product, defines a non-split stem central extension

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\widetilde{\Phi}} \mathrm{Mp}(\widetilde{W})_{\chi_k} \xrightarrow{\widetilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \longrightarrow 1,$$

i.e. a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. The group $\mathrm{Mp}(\widetilde{W})$ is isomorphic to

$$\mathrm{U}_2(\mathbb{F}_3) = \langle a, b, c, d, e \mid a^4 = d^3 = 1,\ b^2 = c^2 = a^2,\ e^2 = a,\ ab = ba,\ ac = ca,\ ad = da,\ ae = ea,\ cbc^{-1} = a^2 b,\ dbd^{-1} = a^2 bc,\ ebe^{-1} = bc,\ dcd^{-1} = b,\ ece^{-1} = a^2 c,\ ede^{-1} = d^{-1} \rangle,$$

where

$$a = \left(-1, i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad b = \left(\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad c = \left(\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

$$d = \begin{cases} \left(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix}\right) & \text{if } \chi_k = \chi_1 \\ \left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1-i & 1-i \\ -1-i & -1+i \end{pmatrix}\right) & \text{if } \chi_k = \chi_3 \end{cases} \quad \text{and}$$

$$e = \begin{cases} \left(\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \frac{1+i}{2}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}\right) & \text{if } \chi_k = \chi_1 \\ \left(\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \frac{1+i}{2}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}\right) & \text{if } \chi_k = \chi_3. \end{cases}$$

In particular, the groups $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ are isomorphic.

ii. *We choose the following set $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ satisfying (232), $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$, where the elements are given in the same order as in the list of $\mathrm{Sp}(\widetilde{W})$ above,*

$$\left\{\begin{array}{l} 1, 1, i, i, i, i, i, i, \\ \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \\ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}, \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{array}\right\} \tag{374}$$

Then the set of operators $\{\pm\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is the following, again in the same order as the elements of $\mathrm{Sp}(\widetilde{W})$ given above:

$$
\left\{
\begin{array}{l}
\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm i\begin{pmatrix}0&1\\1&0\end{pmatrix},\pm i\begin{pmatrix}0&1\\1&0\end{pmatrix},\pm i\begin{pmatrix}1&0\\0&-1\end{pmatrix},\pm i\begin{pmatrix}1&0\\0&-1\end{pmatrix},\\
\pm i\begin{pmatrix}0&-\chi_k(1)\\\chi_k(1)&0\end{pmatrix},\pm i\begin{pmatrix}0&-\chi_k(1)\\\chi_k(1)&0\end{pmatrix},\\
\pm\frac{i}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix},\\
\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}-1&1\\1&1\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}-1&1\\1&1\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&\chi_k(1)\\\chi_k(1)&1\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&\chi_k(1)\\\chi_k(1)&1\end{pmatrix},\\
\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&-\chi_k(1)\\-\chi_k(1)&1\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1&-\chi_k(1)\\-\chi_k(1)&1\end{pmatrix},\\
\pm\frac{i}{\sqrt{2}}\begin{pmatrix}1&-\chi_k(1)\\\chi_k(1)&-1\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}1&-\chi_k(1)\\\chi_k(1)&-1\end{pmatrix},\\
\pm\frac{i}{\sqrt{2}}\begin{pmatrix}1&\chi_k(1)\\-\chi_k(1)&-1\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}1&\chi_k(1)\\-\chi_k(1)&-1\end{pmatrix}\\
\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1-\chi_k(1)&0\\0&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1-\chi_k(1)&0\\0&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{i}{\sqrt{2}}\begin{pmatrix}0&1-\chi_k(1)\\1+\chi_k(1)&0\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}0&1-\chi_k(1)\\1+\chi_k(1)&0\end{pmatrix},\\
\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1+\chi_k(1)&0\\0&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1+\chi_k(1)&0\\0&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{i}{\sqrt{2}}\begin{pmatrix}0&1+\chi_k(1)\\1-\chi_k(1)&0\end{pmatrix},\pm\frac{i}{\sqrt{2}}\begin{pmatrix}0&1+\chi_k(1)\\1-\chi_k(1)&0\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&1+\chi_k(1)\\-1+\chi_k(1)&1-\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&1-\chi_k(1)\\-1-\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1-\chi_k(1)&-1+\chi_k(1)\\1+\chi_k(1)&1+\chi_k(1)\end{pmatrix},\\
\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1-\chi_k(1)\end{pmatrix},\pm\frac{1}{2}\begin{pmatrix}1+\chi_k(1)&-1-\chi_k(1)\\1-\chi_k(1)&1-\chi_k(1)\end{pmatrix}
\end{array}
\right\}.
$$

Moreover, the range of the cocycle $\widetilde{c}_{\chi_k}$ is $\mu_2(\mathbb{C}) = \{\pm 1\}$. Hence by Proposition 2.63, the set

$$\mathrm{Mp}(\widetilde{W})_{\chi_k} = \{(\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W}),\ l \in \mathbb{Z}\},$$

endowed with the direct product, defines a non-split stem central extension

$$1 \longrightarrow \mu_2(\mathbb{C}) \xrightarrow{\widetilde{\Phi}} \mathrm{Mp}(\widetilde{W})_{\chi_k} \xrightarrow{\widetilde{\Psi}} \mathrm{Sp}(\widetilde{W}) \longrightarrow 1,$$

i.e. a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. Furthermore, the group $\mathrm{Mp}(\widetilde{W})$ is isomorphic to

$$\mathrm{Q}_8 \rtimes \mathrm{Dic}_3 = \langle a, b, c, d \,|a^4 = c^6 = 1,\, b^2 = a^2,\, d^2 = c^3,\, bab^{-1} = dbd^{-1} = a^{-1},\, cac^{-1} = b,\\ dad^{-1} = a^2 b,\, cbc^{-1} = ab,\, dcd^{-1} = c^{-1}\rangle.$$

where

$$a = \left(\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \chi_k(1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad b = \left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \chi_k(1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right),$$

$$c = \left(\begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-\chi_k(1) & 1+\chi_k(1) \\ -1+\chi_k(1) & 1+\chi_k(1) \end{pmatrix}\right) \quad \textit{and}$$

$$d = \left(\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1+\chi_k(1) \\ 1-\chi_k(1) & 0 \end{pmatrix}\right).$$

In particular, the groups $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ are isomorphic.

Proof. The choices of $\widetilde{w}_{\widetilde{\sigma}}$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ given by (372), (373) and (374) are valid because they satisfy (369) and (371). We obtain the conditions (369) and (371) by studying explicitly (207) and (232) for every $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$. The list of operators $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ for all $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ can be obtained directly with the formula (218). We give the details only for $\widetilde{\sigma} = \widetilde{\sigma}_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, the reasoning being similar for the other elements of $\mathrm{Sp}(\widetilde{W})$. In this case, $1 - \widetilde{\sigma} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. Then $\mathrm{K}(\widetilde{\sigma}) = \{(0, \widetilde{y}),\ \widetilde{y} \in \mathbb{Z}/4\mathbb{Z}\}$, $\mathrm{I}(\widetilde{\sigma}) = \mathrm{K}(\widetilde{\sigma})$, $\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W} = \{(0, 2\widetilde{y}),\ \widetilde{y} \in \mathbb{Z}/4\mathbb{Z}\} = 2\,\mathrm{I}(\widetilde{\sigma})$ and $\mathrm{K}'(\widetilde{\sigma}) = (2\,\mathrm{I}(\widetilde{\sigma}))^{\perp} = \{(2\widetilde{x}, \widetilde{y}),\ \widetilde{x}, \widetilde{y} \in \mathbb{Z}/4\mathbb{Z}\}$. Then $\widetilde{w}_{\widetilde{\sigma}} = (\widetilde{a}, \widetilde{b}) \in \widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$ satisfies (207) if and only if, for every $(2\widetilde{x}, \widetilde{y}) \in \mathrm{K}'(\widetilde{\sigma})$,

$$2\langle(\widetilde{a}, \widetilde{b}), (2\widetilde{x}, \widetilde{y})\rangle_{\widetilde{W}} = \langle\widetilde{\sigma}(2\widetilde{x}, \widetilde{y}), (2\widetilde{x}, \widetilde{y})\rangle_{\widetilde{W}} = \langle(2\widetilde{x}, 2\widetilde{x} + \widetilde{y}), (2\widetilde{x}, \widetilde{y})\rangle_{\widetilde{W}},$$

i.e.

$$2\widetilde{a}\widetilde{y} = 2\widetilde{x}\widetilde{y} - 2\widetilde{x}\widetilde{y} = 0.$$

Hence, $\widetilde{w}_{\widetilde{\sigma}} = (\widetilde{a}, \widetilde{b})$ satisfies (207) if and only if $2\widetilde{a} = 0$, i.e. $\widetilde{w}_{\widetilde{\sigma}}$ is of the form $(2\widetilde{c}, \widetilde{b})$, where $\widetilde{b},\ \widetilde{c} \in \mathbb{Z}/4\mathbb{Z}$, as announced in (369). Now, since $|2\,\mathrm{I}(\widetilde{\sigma})| = 2$, we have that $|\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|^2 = 2^{-1}$, so we can write $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \frac{1}{\sqrt{2}} e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})}$, where $\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \in \mathbb{R}$. Moreover, we can choose $S_{\widetilde{\sigma}} = \{(0,0), (0,1)\}$ as system of representatives of $\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})$. Then, by (218),

$$\omega_{\chi_k}(\widetilde{\sigma}, (2\widetilde{x}, \widetilde{y})) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\widetilde{u} \in S_{\widetilde{\sigma}}} \chi_k(\langle\widetilde{u}, \widetilde{w} + (2\widetilde{x}, \widetilde{y})\rangle_{\widetilde{W}} + 2\langle(2\widetilde{x}, \widetilde{y}), \widetilde{w}\rangle_{\widetilde{W}}) \rho_{\chi_k}(\widetilde{u} + (2\widetilde{x}, \widetilde{y})),$$

where $(1-\widetilde{\sigma})(\widetilde{w})=\widetilde{u}$. Hence if we set $\widetilde{w}_{\widetilde{\sigma}}=(2\widetilde{x},\widetilde{y})=(0,0)$ and use (368),

$$\begin{aligned}\omega_{\chi_k}(\widetilde{\sigma},(0,0)) &= \gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})\left(\rho_{\chi_k}(0,0)+\chi_k(\langle(0,1),(-1,0)\rangle_{\widetilde{W}})\rho_{\chi_k}(0,1)\right)\\ &= \frac{1}{\sqrt{2}}e^{i\theta(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})}\left(\rho_{\chi_k}(0,0)+\chi_k(1)\rho_{\chi_k}(0,1)\right)\\ &= \begin{cases}\frac{1-i}{2}\begin{pmatrix}1 & \chi_k(1)\\ \chi_k(1) & 1\end{pmatrix} & \text{if } \theta(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})=-\frac{\pi}{4}\\ \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \chi_k(1)\\ \chi_k(1) & 1\end{pmatrix} & \text{if } \theta(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})=0.\end{cases}\end{aligned}$$

Using the choices (372), and (373) and (374) respectively, and doing similar computations for every other element in $\mathrm{Sp}(\widetilde{W})$, we obtain the two lists of operators given in the statement.

Now we want to prove that, with the choices (372), and (373) and (374) respectively, the cocycle $\widetilde{c}_{\chi_k}$ has a range equal to $\{\pm1\}$. Since $\widetilde{W}$ is a free symplectic $\mathbb{Z}/4\mathbb{Z}$-module of rank two, we can apply Theorem A in [Ish89] to give a smaller generating set of $\mathrm{Sp}(\widetilde{W})$ than $\{\widetilde{\sigma}_{\widetilde{v}},\ \widetilde{v}\in\widetilde{W}\}$. In fact, since $\{(1,0),(0,1)\}$ is a generating set of $\widetilde{W}$ as $\mathbb{Z}/4\mathbb{Z}$-module and $\langle(1,0),(0,1)\rangle_{\widetilde{W}}=1$, we have that $\{(1,0),(0,1)\}$ is a connected spanning set of $\widetilde{W}$. Thus, $\mathrm{Sp}(\widetilde{W})$ is generated by the set of transvections $\left\{\widetilde{\sigma}_{(1,0)}=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix},\widetilde{\sigma}_{(0,1)}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\}$. Therefore, by (245), $\widetilde{c}_{\chi_k}$ takes its values in $\{\pm1\}$ if and only if $\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{(1,0)},\widetilde{\tau})\in\{\pm1\}$ and $\widetilde{c}_{\chi_k}(\widetilde{\sigma}_{(0,1)},\widetilde{\tau})\in\{\pm1\}$ for every $\widetilde{\tau}\in\mathrm{Sp}(\widetilde{W})$. Since we already have the explicit lists of operators $\omega_{\chi_k}(\widetilde{\tau},\widetilde{w}_{\widetilde{\tau}})$ for all $\widetilde{\tau}\in\mathrm{Sp}(\widetilde{W})$, we can verify this property by hand. In fact, we compare $\omega_{\chi_k}(\widetilde{\sigma}_{(1,0)},\widetilde{w}_{\widetilde{\sigma}_{(1,0)}})\omega_{\chi_k}(\widetilde{\tau},\widetilde{w}_{\widetilde{\tau}})$ with $\omega_{\chi_k}(\widetilde{\sigma}_{(1,0)}\widetilde{\tau},\widetilde{w}_{\widetilde{\sigma}_{(1,0)}\widetilde{\tau}})$ for every $\widetilde{\tau}\in\mathrm{Sp}(\widetilde{W})$, and similarly for $\widetilde{\sigma}_{(0,1)}$. This proves that $\widetilde{c}_{\chi_k}$ takes its values in $\{\pm1\}$. Moreover, since $\widetilde{c}_{\chi_k}$ is not a coboundary by Corollary 2.59, its range is at least $\{\pm1\}$. Thus the range of $\widetilde{c}_{\chi_k}$ is exactly $\{\pm1\}$. Consequently, the fact that the set $\mathrm{Mp}(\widetilde{W})_{\chi_k}=\{(\widetilde{\sigma},(-1)^l\omega_{\chi_k}(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})),\ \widetilde{\sigma}\in\mathrm{Sp}(\widetilde{W}),\ l\in\mathbb{Z}\}$ defines a non-split stem central extension

$$1\longrightarrow\mu_2(\mathbb{C})\xrightarrow{\widetilde{\Phi}}\mathrm{Mp}(\widetilde{W})_{\chi_k}\xrightarrow{\widetilde{\Psi}}\mathrm{Sp}(\widetilde{W})\longrightarrow 1$$

follows directly from Proposition 2.63.

It remains to identify the group $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ when it is defined by the choice (373) or (374), respectively. Since this group is associated to a non-split stem central extension of $\mathrm{Sp}(\widetilde{W})\simeq\mathrm{A}_4\rtimes\mathrm{C}_4$ by two elements, we know by [GN] that $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is either isomorphic to $\mathrm{Q}_8\rtimes\mathrm{Dic}_3$ or to $\mathrm{U}_2(\mathbb{F}_3)$. Since $\mathrm{Q}_8\rtimes\mathrm{Dic}_3$ has three elements of order two and $\mathrm{U}_2(\mathbb{F}_3)$ has seven such elements, we deduce with the explicit expressions of the 96 elements of the group $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ that

$$\mathrm{Mp}(\widetilde{W})_{\chi_k}\simeq\begin{cases}\mathrm{U}_2(\mathbb{F}_3) & \text{if (373) is satisfied}\\ \mathrm{Q}_8\rtimes\mathrm{Dic}_3 & \text{if (374) is satisfied.}\end{cases}$$

□

Remark 3.50 (related to Remark 2.69)**.** Recall that we were able to find in Corollary 2.62 a necessary and sufficient condition on $\gamma(\widetilde{\sigma},\widetilde{w}_{\widetilde{\sigma}})$ so that $\widetilde{c}_{\chi_k}$ takes its values in $\mu_4(\mathbb{C})$, but no such

condition for $\mu_2(\mathbb{C})$. Proposition 3.49 provides a choice (372) of $\{\widetilde{w}_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ and two choices (373) and (374) of $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ such that the cocycle $\widetilde{c}_{\chi_k}$ has a range equal to $\mu_2(\mathbb{C})$. In spite of the fact that these specific choices do not describe all the possible choices, they still provide the only two possibilities for the two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$. In fact, by Proposition 2.63, $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is a non-split stem central extension of $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{A}_4 \rtimes \mathrm{C}_4$ by two elements. Since $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by (308), there are exactly four central extensions of $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ by two elements. By [GN], each of these extensions corresponds to a different group, and only two of them are associated to non-split stem extensions, namely $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and $\mathrm{U}_2(\mathbb{F}_3)$. Furthermore, this implies also that with the same choice (373) or (374), respectively, not only the groups $\mathrm{Mp}(\widetilde{W})_{\chi_1}$ and $\mathrm{Mp}(\widetilde{W})_{\chi_3}$ are isomorphic, but also the associated extensions are isomorphic.

Now we have recovered the two possibilities for the two-fold covering $\mathrm{Mp}(\widetilde{W})$ of $\mathrm{Sp}(\widetilde{W})$ that we obtained in sections 3.1.4 and 3.2.4, but in a more natural way. In fact, $\mathrm{Mp}(\widetilde{W})$ was only defined with an abstract presentation in these sections. In contrast, Proposition 3.49 provides the construction of (the two possibilities for) a two-fold covering $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ of $\mathrm{Sp}(\widetilde{W})$ as a subgroup of $\mathrm{Sp}(\widetilde{W}) \times \mathrm{U}(\mathrm{L}^2(Y))$, arising from intertwining operators of the projective Weil representation of $\mathrm{Sp}(\widetilde{W})$ defined in Proposition 2.56. Now, we study some properties of the Weil representation $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ defined in Proposition 2.65, and exhibit the homomorphism of central extensions $\mathscr{S}$ defined in Proposition 2.73.

Proposition 3.51. *Suppose that the set $\{\widetilde{w}_{\widetilde{\sigma}},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is fixed by* (372).

- ***i.*** *If the set $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is fixed by* (373) *then the Weil representation $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{U}_2(\mathbb{F}_3)$ is unitary, irreducible and faithful.*

- ***ii.*** *If the set $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}),\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is fixed by* (374) *then the Weil representation $\widetilde{\omega}_{\chi_k}$ of $\mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ is unitary and irreducible but not faithful. More precisely, its kernel is*

$$\mathrm{Ker}(\widetilde{\omega}_{\chi_k}) = \{(1, I_2), (-1, I_2)\}\,, \tag{375}$$

  and its image is isomorphic to the conformal special unitary (or binary octahedral) group $\mathrm{CSU}_2(\mathbb{F}_3)$.

Proof. The fact that $\widetilde{\omega}_{\chi_k}$ is a unitary and irreducible representation of $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ follows directly from Proposition 2.65. The kernel of $\widetilde{\omega}_{\chi_k}$ can be found either with (264) in Proposition 2.65, or directly with the sets of operators $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ we computed in Proposition 3.49. Then, if $\mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$, the image of $\widetilde{\omega}_{\chi_k}$ is a group of 48 elements which is also isomorphic to a quotient of $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$. By [GN], the image of $\widetilde{\omega}_{\chi_k}$ is thus isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$, $\mathrm{GL}_2(\mathbb{F}_3)$ or $\mathrm{A}_4 \rtimes \mathrm{C}_4$, which have respectively one, thirteen and seven elements of order two. Looking at the list of operators that we find in Proposition 3.49 with the choice (374), we observe that the image of $\widetilde{\omega}_{\chi_k}$ contains only one element of order two. Therefore, the image of $\widetilde{\omega}_{\chi_k}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$. □

Remark 3.52. If we define $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ with the choices (372) and (374), then by Propositions 3.49 and 3.51, $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is isomorphic to $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and the image of the Weil representation $\widetilde{\omega}_{\chi_k}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$. Recall that we obtained in Propositions 3.16 and 3.37 a Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ such that its image is isomorphic to either $\mathrm{CSU}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_3)$. Since we only provided one explicit choice of parameters $\widetilde{w}_{\widetilde{\sigma}}$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ which leads to $\mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$, it is very likely that another choice could provide a Weil representation $\widetilde{\omega}_{\chi_k}$ whose image is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$.

Proposition 3.53. *Assume that the set* $\{\widetilde{w}_{\widetilde{\sigma}}$ *satisfying* (207)*,* $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *is fixed by* (372)*. Suppose also that* $B = B_{\mathrm{Weil}}$ *and* $B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) = \langle \widetilde{x}_1, \widetilde{y}_2 \rangle_{\widetilde{W}} = \widetilde{x}_1 \widetilde{y}_2$ *for all* $\widetilde{w}_1 = \widetilde{x}_1 + \widetilde{y}_1$ *and* $\widetilde{w}_2 = \widetilde{x}_2 + \widetilde{y}_2$ *in* $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$*. Let* $\{w_s,\ s \in \mathrm{ASp}(B_W)\}$ *be a set such that* w_s *satisfies* (100) *for all* $s \in \mathrm{ASp}(B_W)$*. Let* $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ *be a set satisfying* (112) *and* (117)*.*

i. *Suppose that the set* $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = |2\,\mathrm{I}(\widetilde{\sigma})|^{-\frac{1}{2}} e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *is fixed by* (373)*. If* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ *then there is no group homomorphism* $\mathscr{S}$ *such that the diagram* (D_{25}) *commutes. If* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ *then the diagram* (D_{25}) *commutes if and only if* $\mathscr{S}$ *is defined by* (287)*, i.e.*

$$\begin{aligned} \mathscr{S} :\ \mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{U}_2(\mathbb{F}_3) &\to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3) \\ (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) &\mapsto (\eta(\widetilde{\sigma}), (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \end{aligned} \tag{376}$$

where η *is defined in Proposition 2.7.vi and explicitly given in Proposition 3.5.i. In this case,* $\mathscr{S}$ *is an isomorphism of central extensions and we can write* (D_{25}) *as follows, where all the groups and maps are explicitly identified:*

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\ & & & & & & \downarrow & & \\ 1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3) & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \widetilde{\omega}_{\widetilde{\chi_k}}} \ \uparrow{\scriptstyle \mathscr{S}} & & \downarrow{\scriptstyle \pi_{\mathrm{ASp}}} \ \uparrow{\scriptstyle \eta} & & \\ & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Sp}(W) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\ & & & & \uparrow{\scriptstyle \widetilde{\omega}_{\chi_k}} & & \uparrow{\scriptstyle \pi} & & \\ 1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{U}_2(\mathbb{F}_3) & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1 \end{array} \tag{D_{50}}$$

ii. *Suppose that the set* $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = |2\,\mathrm{I}(\widetilde{\sigma})|^{-\frac{1}{2}} e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ *is fixed by* (374)*. If* $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$ *then there is no group homomorphism* $\mathscr{S}$ *such that the diagram* (D_{25}) *commutes. If* $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$ *then the diagram* (D_{25}) *commutes if and only if* $\mathscr{S}$ *is defined by* (287)*, i.e.*

$$\begin{aligned} \mathscr{S} : \mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 &\to \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4 \\ (\widetilde{\sigma}, (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) &\mapsto (\eta(\widetilde{\sigma}), (-1)^l \omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})) \end{aligned} \tag{377}$$

where η is defined in Proposition 2.7.vi and explicitly given in Proposition 3.5.i. In this case, the kernel of $\mathscr{S}$ is $\mathrm{Ker}(\mathscr{S}) = \{(1, I_2), (-1, I_2)\}$ and its image is isomorphic to the conformal special unitary (or binary octahedral) group $\mathrm{CSU}_2(\mathbb{F}_3)$. We can write ($D_{25}$) as follows, where all the groups and maps are explicitly identified:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathrm{Hom}_{\mathbb{Z}}(W, \mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{C}_2^2 & & \\
 & & & & & & \downarrow & & \\
1 & \longrightarrow & \mu_4(\mathbb{C}) & \xrightarrow{\Phi_{\widetilde{\chi_k}}} & \mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \xrightarrow{\Psi_{\widetilde{\chi_k}}} & \mathrm{ASp}(B_W) \simeq \mathrm{S}_4 & \longrightarrow & 1 \\
 & & \uparrow \iota & \overset{\mathscr{S}}{\nwarrow} & \downarrow \widetilde{\omega}_{\widetilde{\chi_k}} & \overset{\eta}{\nwarrow} & \downarrow \pi_{\mathrm{ASp}} & & \\
 & & & & \mathrm{U}(\mathrm{L}^2(Y)) & & \mathrm{Sp}(W) \simeq \mathrm{S}_3 & \longrightarrow & 1 \\
 & & & & \uparrow \widetilde{\omega}_{\chi_k} & & \uparrow \pi & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\tilde{\Phi}} & \mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 & \xrightarrow{\tilde{\Psi}} & \mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1
\end{array}
\tag{D_{51}}
$$

Proof. This is a direct application of Proposition 2.73.

i. Suppose that $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = |2\,\mathrm{I}(\widetilde{\sigma})|^{-\frac{1}{2}} e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is fixed by (373). By Proposition 2.73, the diagram (D_{25}) commutes, and uniquely determines the expression of $\mathscr{S}$, if and only if (288) is satisfied, i.e. if and only if

$$\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) - \nu(\sigma) \equiv 0 \mod \frac{\pi}{2} \quad \text{for all } \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W}), \text{ where } \sigma = \pi(\widetilde{\sigma}), \tag{378}$$

since $|\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})| = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} = |\,\mathrm{I}(\sigma)|^{-\frac{1}{2}} = |\,\mathrm{I}(\widetilde{\sigma})/(\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W})|^{-\frac{1}{2}} = |2\,\mathrm{I}(\widetilde{\sigma})|^{-\frac{1}{2}} = |\delta(\sigma)|$ by definition of π (see Proposition 2.7.iii) and Lemma 2.54. By the definition of π and (373),

$$\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \equiv \begin{cases} 0 \mod \frac{\pi}{2} & \text{if } \pi(\widetilde{\sigma}) \in \{\sigma_{(0,0)}, \sigma_{(1,0)}\sigma_{(0,1)}, \sigma_{(0,1)}\sigma_{(1,0)}\} \\ \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } \pi(\widetilde{\sigma}) \in \{\sigma_{(1,0)}, \sigma_{(0,1)}, \sigma_{(1,1)}\}. \end{cases} \tag{379}$$

If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, we proved in Proposition 3.5 that $\nu(\sigma) \equiv 0 \mod \frac{\pi}{2}$ for all $\sigma \in \mathrm{Sp}(W)$. So by (379), the condition (378) is not satisfied. Hence there is no group homomorphism $\mathscr{S}$ such that the diagram (D_{25}) commutes. Now, if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then by Proposition 3.5,

$$\nu(\sigma) \equiv \begin{cases} 0 \mod \frac{\pi}{2} & \text{if } \sigma \in \{\sigma_{(0,0)}, \sigma_{(1,0)}\sigma_{(0,1)}, \sigma_{(0,1)}\sigma_{(1,0)}\} \\ \frac{\pi}{4} \mod \frac{\pi}{2} & \text{if } \sigma \in \{\sigma_{(1,0)}, \sigma_{(0,1)}, \sigma_{(1,1)}\}. \end{cases} \tag{380}$$

Hence (379) and (380) imply that (378) is satisfied. Thus the diagram (D_{25}) commutes and $\mathscr{S}$ is defined by (287). Moreover, $\mathscr{S}$ and $\widetilde{\omega}_{\chi_k}$ have the same kernel. Since $\widetilde{\omega}_{\chi_k}$ is faithful by Proposition 3.51, this implies that $\mathscr{S}$ is injective. Since $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ are both isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$ in this case, this is enough to conclude that $\mathscr{S}$ is an isomorphism of central extensions.

ii. Suppose that $\{\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = |2\,\mathrm{I}(\widetilde{\sigma})|^{-\frac{1}{2}} e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})},\ \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})\}$ is fixed by (374). Then

$$\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \equiv 0 \mod \frac{\pi}{2} \quad \text{for all } \widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W}). \tag{381}$$

Using the same arguments as for the preceding point, we deduce that the condition (378) is not satisfied if $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$. Hence there is no group homomorphism $\mathscr{S}$ such that the diagram (D_{25}) commutes. However, (378) is satisfied if $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$. In this case, the diagram (D_{25}) commutes and $\mathscr{S}$ is defined by (287). Moreover, $\mathscr{S}$ and $\widetilde{\omega}_{\chi_k}$ have the same kernel, namely $\{(1, I_2), (-1, I_2)\}$ by Proposition 3.51. Since $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S} = \widetilde{\omega}_{\chi_k}$, the representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful by Proposition 2.32, and the image of $\widetilde{\omega}_{\chi_k}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ by Proposition 3.51, we conclude that the image of $\mathscr{S}$ is also isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$.

□

Remark 3.54. Thanks to Proposition 3.53, we recover almost all the results of Propositions 3.12 3.16, 3.36 and 3.37. A difference is that if $\mathrm{Mp}(\widetilde{W})_{\chi_k}$ is isomorphic to $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ then the image of $\mathscr{S}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$, whereas in Propositions 3.16 and 3.37 the image of $\mathscr{S}$ can be isomorphic to either $\mathrm{CSU}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_3)$. As observed in Remark 3.52, we mention that it is very likely that another choice of $\widetilde{w}_{\widetilde{\sigma}}$ and $\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ such that $\mathrm{Mp}(\widetilde{W})_{\chi_k} \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ could provide a group homomorphism $\mathscr{S}$ whose image is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$. Another difference is that, even if we fix the isomorphism class of the image of $\mathscr{S}$, there are several possibilities for the expression of $\mathscr{S}$ in Propositions 3.12 3.16, 3.36 and 3.37, whereas the homomorphism of central extensions $\mathscr{S}$ provided by Proposition 3.53 is unique. This is also due to the specific choices we made for constructing $\mathrm{Mp}(\widetilde{W})_{\chi_k}$.

3.4 A link with an article of Savin [Sav12]

For a commutative ring with unit A, let $\mathrm{G}(A)$ denote the Chevalley group over A corresponding to a fixed simply laced root system Φ. Let R be the ring of integers of the p-adic field $\mathbb{Q}_{2^r}$, where $r \geqslant 1$. In [Sav12], Savin constructs a two-fold covering $\mathrm{G}'(R/4R)$ of the group $\mathrm{G}(R/4R)$ and a one-to-one correspondence between genuine representations of $\mathrm{G}'(R/4R)$ and representations of $\mathrm{G}(R/2R)$. In particular, Savin provides the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathrm{G}(R/2R) & & \mathrm{G}(R/2R) & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \longrightarrow & \mathrm{G}'(R/4R) & \longrightarrow & \mathrm{G}(R/4R) & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \longrightarrow & \mathfrak{g}'(R/2R) & \longrightarrow & \mathfrak{g}(R/2R) & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\qquad (D_{52})
$$

where $\mathfrak{g}(R/2R)$ is the Lie algebra of $\mathrm{G}(R/2R)$. If we consider $R = \mathbb{Z}_2$, the ring of integers of $\mathbb{Q}_2$, and the root system $\Phi = A_1 = \{\alpha, -\alpha\}$ in [Sav12], then we obtain the groups $\mathrm{G}(R/2R) = \mathrm{Sp}(W)$ and $\mathrm{G}(R/4R) = \mathrm{Sp}(\widetilde{W})$ which correspond to the groups studied in sections 3.1 and 3.2, i.e. for the case in which $\dim_{\mathbb{F}_2}(W) = 2$. Hereafter, we write Savin's results for this example explicitly, and we relate them to our diagrams and representations obtained in sections 3.1 and 3.2.

If we take $\Phi = A_{n-1}$ with $n \geqslant 2$, then the Chevalley group corresponding to Φ is SL_n (cf. [Ste16]). If $R = \mathbb{Z}_2$, the ring of integers of $\mathbb{Q}_2$, then $R/2R \simeq \mathbb{Z}/2\mathbb{Z}$ and $R/4R \simeq \mathbb{Z}/4\mathbb{Z}$. For $n = 2$ and a local ring A, the group $\mathrm{G}(A)$ is generated by the matrices $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ for all $u \in A^*$. Hence, we can prove that $\mathrm{G}(R/2R) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Sp}(W)$ and $\mathrm{G}(R/4R) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) = \mathrm{Sp}(\widetilde{W})$, where the rank of the $\mathbb{Z}/4\mathbb{Z}$-module $\widetilde{W}$ is two and $\dim_{\mathbb{F}_2}(W) = 2$. Then the associated Lie algebra $\mathfrak{g}(R/2R)$ over $\mathbb{Z}/2\mathbb{Z}$ can be written

$$\mathfrak{g}(R/2R) = (\mathbb{Z}H_\alpha \oplus \mathbb{Z}E_\alpha \oplus \mathbb{Z}E_{-\alpha}) \otimes \mathbb{Z}/2\mathbb{Z}, \tag{382}$$

and it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ as a vector space.

Now, the group in the diagram (D_{52}) we are first interested in is $\mathrm{SL}_2'(R/4R)$. To get it explicitly, according to the definition of $\mathrm{SL}_2'(R/4R)$ in [Sav12], we need to understand $\mathrm{SL}_n'(R/4R)$ for a larger value of n. In general, i.e. not in the low rank exceptions, $\mathrm{G}'(R/4R)$ is defined from the Steinberg group $\mathrm{G}''(R/4R)$, which is the universal central extension of $\mathrm{G}(R/4R)$ if the rank of the root system is greater or equal to 4, i.e. $n \geqslant 5$ according to Theorem 5.3 in [Ste71b]. Since we do not use the fact that the Steinberg group is the universal central extension of $\mathrm{G}(R/4R)$ in what follows (only that it is a central extension), we suppose that $n \geqslant 3$. In fact, we will need to suppose that $n = 3$ to use **GAP** in order to obtain $\mathrm{SL}_2'(R/4R)$ explicitly, see section B.1. We begin by showing that for $n \geqslant 3$, $\mathrm{SL}_n'(R/4R) = \mathrm{SL}_n''(R/4R)$. The definition of the Steinberg group can be found for example in [DS75]: for a commutative ring with unit A and $n \geqslant 3$, $\mathrm{G}''(A)$ is denoted there by $\mathrm{St}(n, A)$. The group $\mathrm{St}(n, A)$ is generated by the elements $x_{ij}(u)$, where $1 \leqslant i \neq j \leqslant n$ and $u \in A$, subject to the Steinberg

relations:

$$\begin{cases} x_{ij}(u)x_{ij}(v) & = x_{ij}(u+v) \\ [x_{ij}(u), x_{kl}(v)] & = \begin{cases} 1 & \text{if } i \neq l,\ j \neq k \\ x_{il}(uv) & \text{if } i \neq l,\ j = k. \end{cases} \end{cases} \tag{383}$$

Let φ_n be the map which sends every generator $x_{ij}(u)$ in $\mathrm{St}(n, A)$ to the elementary matrix $e_{ij}(u) = I_n + uE_{ij}$ in $\mathrm{GL}(n, A)$. Then φ_n is a group homomorphism whose image is the elementary subgroup $\mathrm{E}(n, A)$ of $\mathrm{GL}(n, A)$, i.e. the subgroup generated by all the matrices of the form $I_n + uE_{ij}$. Using [PV] and the references given there, we can deduce that $\mathrm{E}(n, A) = \mathrm{SL}(n, A)$ if A is a semi-local ring, since $\mathrm{SL}(n, \mathbb{C})$ is simply connected. Thus, if A is semi-local, then there is a short exact sequence

$$1 \to \mathrm{K}_2(n, A) \xrightarrow{\iota} \mathrm{SL}''_n(A) = \mathrm{St}(n, A) \xrightarrow{\varphi_n} \mathrm{SL}_n(A) \to 1, \tag{384}$$

where $\mathrm{K}_2(n, A) = \mathrm{Ker}(\varphi_n)$. We define also $\mathrm{K}_2(A)$ as the direct limit of $\mathrm{K}_2(n, A)$ for $n \to +\infty$. Dennis and Stein proved in [DS75] that $\mathrm{K}_2(n, A) \simeq \mathrm{K}_2(A)$ for all $n \geqslant 3$, and if $A = \mathbb{Z}/4\mathbb{Z}$ then $\mathrm{K}_2(A) \simeq \mathbb{Z}/2\mathbb{Z}$. In other words, for all $n \geqslant 3$, this means that the Steinberg group $\mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z})$ is a central extension by two elements of the Chevalley group $\mathrm{SL}_n(\mathbb{Z}/4\mathbb{Z})$. This extension can be related to the Steinberg symbol $(\cdot, \cdot)_S$ used by Savin. For all $i \neq j$ and $u, v \in A^*$, we define

$$\begin{cases} w_{ij}(u) & = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u) \\ h_{ij}(u) & = w_{ij}(u)w_{ij}(-1) \end{cases} \quad \text{and} \quad (u, v)_S = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}. \tag{385}$$

It turns out that $(u, v)_S$ does not depend on the choice of i and j. Moreover, if A is a local ring, then $\mathrm{K}_2(n, A)$ is generated by the symbols $(u, v)_S$ for all $u, v \in A^*$ (cf. [Ste73, Theorem 2.13]). In particular, $\mathrm{K}_2(n, \mathbb{Z}/4\mathbb{Z})$ is generated by the symbol $(-1, -1)_S = h_{ij}(-1)^2 = w_{ij}(-1)^4$, which is therefore of order two because $\mathrm{K}_2(n, \mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. Now, the Hilbert symbol on $\mathbb{Q}_2$ induces a nondegenerate bilinear form on $R/2R \simeq \mathbb{Z}/2\mathbb{Z}$, which is given by

$$(u, v)_2 = (-1)^{uv}. \tag{386}$$

This provides a nontrivial group homomorphism

$$\begin{aligned} \phi : \mathrm{K}_2(n, \mathbb{Z}/4\mathbb{Z}) &\to \mu_2(\mathbb{C}) = \{\pm 1\}, \\ (u, v)_S &\mapsto (u, v)_2 \end{aligned}$$

which can be used to "push out" the central extension defined by (384). This means that we define $\mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z})$ as the free product $\mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z}) \star \{\pm 1\}$ with the relation $\iota(k)\phi(k)^{-1} = 1$ for all $k \in \mathrm{K}_2(n, \mathbb{Z}/4\mathbb{Z})$. These relations amount to $\iota((-1, -1)_S) = \phi((-1, -1)_S)$, i.e. $(-1, -1)_S = (-1, -1)_2 = -1$ in $\mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z})$. If we define furthermore

$$\begin{aligned} \psi_n : \mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z}) &\to \mathrm{SL}_n(\mathbb{Z}/4\mathbb{Z}), \\ x_{ij}(u) &\mapsto e_{ij}(u) \end{aligned}$$

then we obtain a central extension:

$$1 \to \{\pm 1\} \xrightarrow{\iota} \mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z}) \xrightarrow{\psi_n} \mathrm{SL}_n(\mathbb{Z}/4\mathbb{Z}) \to 1. \tag{387}$$

Now, if we define the group homomorphism $\mathscr{E} : \mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z}) \to \mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z})$, then

$$x_{ij}(u) \mapsto x_{ij}(u)$$

$$\begin{cases} (\mathscr{E} \circ \iota)((-1,-1)_S) & = \mathscr{E}((-1,-1)_S) = (-1,-1)_S \\ (\iota \circ \phi)((-1,-1)_S) & = \iota((-1,-1)_2) = \iota(-1) = -1. \end{cases}$$

Since $(-1,-1)_S = -1$ in $\mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z})$, this implies that $\mathscr{E} \circ \iota = \iota \circ \phi$. In addition, for all $i \neq j$ and $u \in \mathbb{Z}/4\mathbb{Z}$,

$$\begin{cases} \varphi_n(x_{ij}(u)) & = e_{ij}(u) \\ (\psi_n \circ \mathscr{E})(x_{ij}(u)) & = \psi_n(x_{ij}(u)) = e_{ij}(u), \end{cases}$$

i.e. $\varphi_n = \psi_n \circ \mathscr{E}$. This shows that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{K}_2(n, \mathbb{Z}/4\mathbb{Z}) & \xrightarrow{\iota} & \mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z}) & \xrightarrow{\varphi_n} & \mathrm{SL}_n(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathscr{E}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \{\pm 1\} & \xrightarrow{\iota} & \mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z}) & \xrightarrow{\psi_n} & \mathrm{SL}_n(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & 1 \end{array} \qquad (D_{53})$$

In particular, the extensions (384) and (387), and a fortiori the groups $\mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z})$ and $\mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z})$, are isomorphic. Finally, this enables us to define $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$. In fact, we can assume that $\mathrm{SL}'_n(\mathbb{Z}/4\mathbb{Z}) = \mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z})$ and then choose any $i \neq j$ to define $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ as the subgroup of $\mathrm{SL}''_n(\mathbb{Z}/4\mathbb{Z})$ generated by $x_{ij}(u)$ and $x_{ji}(u)$ for all $u \in \mathbb{Z}/4\mathbb{Z}$. We are grateful to Savin for providing us with some explanations which were useful to exhibit the construction of $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$.

The next step is to identify explicitly the group $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ in the database [GN]. Recall that $\mathrm{Sp}(\widetilde{W}) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{A}_4 \rtimes \mathrm{C}_4$ and $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by (308). The database [GN] provides four possible groups corresponding to central extensions of $\mathrm{A}_4 \rtimes \mathrm{C}_4$ by two elements, i.e. four possibilities for $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$, namely $\mathrm{C}_2 \times \mathrm{A}_4 \rtimes \mathrm{C}_4$, $\mathrm{A}_4 \rtimes \mathrm{C}_8$, $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and $\mathrm{U}_2(\mathbb{F}_3)$. The group $\mathrm{C}_2 \times \mathrm{A}_4 \rtimes \mathrm{C}_4$ is ruled out because it is a trivial extension. The center of $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ is isomorphic to C_2^2, and $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ has only three elements of order 2. Hence these three elements generate its center, whereas $x_{ij}(2)$ and $x_{ji}(2)$ are two elements of order 2 in $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ which do not belong to the center of $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$. In fact, if they were in the center, then their images under φ_n should be in the center of $\mathrm{SL}_n(\mathbb{Z}/4\mathbb{Z})$, which is false. Thus $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ is isomorphic to either $\mathrm{A}_4 \rtimes \mathrm{C}_8$ or $\mathrm{U}_2(\mathbb{F}_3)$. Using **GAP** with $n = 3$, we prove below that $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{U}_2(\mathbb{F}_3)$. More precisely, in terms of **GAP**'s generators, a presentation of the group $\mathrm{U}_2(\mathbb{F}_3)$ is

$$\begin{aligned} \mathrm{U}_2(\mathbb{F}_3) = \langle f_1, \dots, f_6 \,|\, & f_1^2 = f_2,\, f_2^2 = f_4^2 = f_5^2 = f_6,\, f_3^3 = 1,\, f_6^2 = 1,\, [f_2, f_i] = [f_i, f_6] = 1, \\ & [f_3^{-1}, f_1^{-1}] = f_3,\, [f_4^{-1}, f_1^{-1}] = f_4 f_5 f_6,\, [f_5^{-1}, f_1^{-1}] = f_4 f_5, \\ & [f_4^{-1}, f_3^{-1}] = f_4 f_5,\, [f_5^{-1}, f_3^{-1}] = f_4 f_6,\, [f_5^{-1}, f_4^{-1}] = f_6 \rangle. \end{aligned}$$

Then $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ is the subgroup of $\mathrm{U}_2(\mathbb{F}_3)$ generated by f_1 and f_4, and an explicit isomorphism from $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ to $\mathrm{U}_2(\mathbb{F}_3)$ is given by $f_1 \mapsto f_1 f_2 f_3 f_4$ and $f_4 \mapsto f_1 f_3^2 f_4 f_5 f_6$ (see appendix B.1).

We have identified $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z})$, so we can determine the last group missing in the diagram (D_{52}), which is $\mathfrak{g}'(R/2R)$. This group has to be isomorphic to a subgroup of $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{U}_2(\mathbb{F}_3)$ and it must also define a central extension of $\mathfrak{g}(R/2R) \simeq \mathrm{C}_2^3$ by two elements. According to [GN], the only possibility is therefore the Pauli group $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$.

Now, we use **GAP** to make all the homomorphisms explicit in the commutative diagram (D_{52}). This will provide the diagram (D_{54}). Recall the following **GAP**'s presentations:

- $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) = \langle f_1, \dots, f_5 \mid f_1^2 = f_2,\, f_2^2 = f_3^3 = f_4^2 = f_5^2 = 1,\, [f_3^{-1}, f_1^{-1}] = f_3,$
$[f_2, f_i] = [f_4, f_5] = 1,\, [f_4^{-1}, f_1^{-1}] = [f_5^{-1}, f_1^{-1}] = [f_4^{-1}, f_3^{-1}] = f_4 f_5,$
$[f_5^{-1}, f_3^{-1}] = f_4 \rangle$
- $\mathrm{C}_4 \circ \mathrm{D}_4 = \langle f_1, f_2, f_3, f_4 \mid f_1^2 = f_2^2 = f_4^2 = 1,\, f_3^2 = f_4,\, [f_2^{-1}, f_1^{-1}] = f_4,$
$[f_3, f_i] = [f_4, f_i] = 1 \rangle$
- $\mathrm{C}_2^3 = \langle f_1, f_2, f_3 \mid f_1^2 = f_2^2 = f_3^2 = 1,\, [f_1, f_2] = [f_2, f_3] = [f_1, f_3] = 1 \rangle$.

Proposition 3.55. *In terms of **GAP**'s presentations, the following maps on generators define group homomorphisms:*

- $\Psi' : \mathrm{U}_2(\mathbb{F}_3) \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ *by* $\Psi'(f_1) = f_1 f_2 f_3$, $\Psi'(f_2) = f_2$, $\Psi'(f_3) = f_3^2$, $\Psi'(f_4) = f_4$, $\Psi'(f_5) = f_4 f_5$ *and* $\Psi'(f_6) = 1$.
- $\mathscr{F}' : \mathrm{C}_4 \circ \mathrm{D}_4 \to \mathrm{U}_2(\mathbb{F}_3)$ *by* $\mathscr{F}'(f_1) = f_1 f_4 f_1$, $\mathscr{F}'(f_2) = f_1^2 f_4$, $\mathscr{F}'(f_3) = f_1^2$ *and* $\mathscr{F}'(f_4) = f_1^4 = f_6$.
- $\psi' : \mathrm{C}_4 \circ \mathrm{D}_4 \to \mathrm{C}_2^3$ *by* $\psi'(f_1) = f_1 f_2$, $\psi'(f_2) = f_2 f_3$, $\psi'(f_3) = f_1 f_2 f_3$ *and* $\psi'(f_4) = 1$.
- $\iota' : \mathrm{C}_2^3 \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ *by* $\iota'(f_1) = f_4$, $\iota'(f_2) = f_2 f_5$ *and* $\iota'(f_3) = f_5 f_4$.

We define also the group homomorphisms $\varphi' : \mu_2(\mathbb{C}) \to \mathrm{C}_4 \circ \mathrm{D}_4$ *by* $\varphi'(-1) = f_4$ *and* $\Phi' : \mu_2(\mathbb{C}) \to \mathrm{U}_2(\mathbb{F}_3)$ *by* $\Phi'(-1) = f_6$. *Then the following diagram* (D_{54}) *is commutative:*

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathrm{Sp}(W)=\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})\simeq \mathrm{S}_3 & & \mathrm{Sp}(W)=\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})\simeq \mathrm{S}_3 & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\Phi'} & \mathrm{G}'(R/4R)\simeq \mathrm{U}_2(\mathbb{F}_3) & \xrightarrow{\Psi'} & \mathrm{Sp}(\widetilde{W})=\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})\simeq \mathrm{A}_4 \rtimes \mathrm{C}_4 & \longrightarrow & 1 \\
 & & \mathrm{id}\uparrow & & \mathscr{F}'\uparrow & & \iota'\uparrow & & \\
1 & \longrightarrow & \mu_2(\mathbb{C}) & \xrightarrow{\varphi'} & \mathfrak{g}'(R/2R)\simeq \mathrm{C}_4 \circ \mathrm{D}_4 & \xrightarrow{\psi'} & \mathfrak{g}(R/2R)\simeq \mathrm{C}_2^3 & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\qquad (D_{54})
$$

Proof. The fact that the maps defined on generators are group homomorphisms is checked using **GAP** (see appendix B.1). Next, we check that the diagram (D_{54}) commutes. The conditions $\varphi'(-1) = f_4$ and $\Phi'(-1) = f_6$ guarantee that the group homomorphisms φ' and ψ', and Φ' and Ψ', respectively, define two central extensions. Since $(\mathscr{F}' \circ \varphi')(-1) = \mathscr{F}'(f_4) = f_6 = \Phi'(-1)$, we have that $\mathscr{F}' \circ \varphi' = \Phi'$. Moreover, we verify with **GAP** that

$$\begin{cases} (\iota' \circ \psi')(f_1) &= \iota'(f_1 f_2) = f_4 f_2 f_5 \\ (\iota' \circ \psi')(f_2) &= \iota'(f_2 f_3) = f_2 f_4 \\ (\iota' \circ \psi')(f_3) &= \iota'(f_1 f_2 f_3) = f_2 \\ (\iota' \circ \psi')(f_4) &= \iota'(1) = 1 \end{cases} \text{and} \quad \begin{cases} (\Psi' \circ \mathscr{F}')(f_1) &= \Psi'(f_1 f_4 f_1) = f_1 f_3 f_4 f_1 f_3 \\ (\Psi' \circ \mathscr{F}')(f_2) &= \Psi'(f_1^2 f_4) = f_1 f_3 f_1 f_3 f_4 \\ (\Psi' \circ \mathscr{F}')(f_3) &= \Psi'(f_1^2) = f_1 f_3 f_1 f_3 \\ (\Psi' \circ \mathscr{F}')(f_4) &= \Psi'(f_6) = 1. \end{cases}$$

Using the presentation of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ given above, we obtain that $\iota' \circ \psi' = \Psi' \circ \mathscr{F}'$, which is enough to conclude that the diagram (D_{54}) is commutative. □

The explicit homomorphisms provided in Proposition 3.55 allow us also to compute the bilinear form on $\mathfrak{g}(R/2R) \simeq \mathrm{C}_2^3$ defined by $\omega(x, y) = [\overline{x}, \overline{y}]$, where $\overline{x}$ denotes any element of $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$ such that $\psi'(\overline{x}) = x$, for all $x \in \mathfrak{g}(R/2R) \simeq \mathrm{C}_2^3$. In fact, by definition of ψ', we have that $f_1 = \psi'(f_2 f_3)$, $f_2 = \psi'(f_1 f_2 f_3)$ and $f_3 = \psi'(f_1 f_3)$. Hence, for example in the ordered basis $\{f_1 f_2 f_3, f_1, f_2\}$ of C_2^3 considered as a $\mathbb{Z}/2\mathbb{Z}$-vector space, and with the isomorphism between $\langle f_4 \rangle$ in $\mathrm{C}_4 \circ \mathrm{D}_4$ and $\{\pm 1\}$, the matrix of ω is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

In particular, the kernel of ω is $Z = \langle f_1 f_2 f_3 \rangle$. Moreover, if we denote by β the bilinear form defined over $\mathfrak{g}(R/2R) = (\mathbb{Z}H_\alpha \oplus \mathbb{Z}E_\alpha \oplus \mathbb{Z}E_{-\alpha}) \otimes \mathbb{Z}/2\mathbb{Z}$ by

$$\beta(X \otimes u, Y \otimes v) = (1 + 2u, 1 + 2v)_2^{\kappa(X,Y)} = (-1)^{uv\kappa(X,Y)}, \tag{388}$$

where the matrix of κ in the ordered basis $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ is given by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then the matrix of β in the ordered basis $\{H_\alpha \otimes 1, E_\alpha \otimes 1, E_{-\alpha} \otimes 1\}$ of $\mathfrak{g}(R/2R)$ is given by $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. This is exactly the matrix of ω, and this verifies Proposition 3.1 in [Sav12].

Now, we are going to describe explicitly the irreducible representation ρ_χ of $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$ given in [Sav12]. Observe that $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$ is the Pauli group we already encountered in sections 3.1.2 and 3.2.2 as the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Let χ be a genuine character of the center of $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$, i.e. of $Z' = \langle f_3 \rangle \simeq \mathbb{Z}/4\mathbb{Z}$. There are two such characters χ_1 and χ_3, where $\chi_1(1) = i$ and $\chi_3 = \overline{\chi_1}$ (cf. Lemma 2.8). So, if we identify the generator f_3 of Z' with the generator 1 of $\mathbb{Z}/4\mathbb{Z}$, then we can write $\chi = \chi_k$ and $\chi(f_3) = \chi_k(1)$. A maximal subspace S of C_2^3 such that $\omega_{|S} = 1$ is for example $S = \langle f_1 f_2 f_3, f_1 \rangle$ (and then S is isomorphic to $\langle H_\alpha \otimes 1, E_\alpha \otimes 1 \rangle$ in $\mathfrak{g}(R/2R)$). Then $S' = \psi'^{-1}(S) = \langle f_3, f_2 f_3 \rangle \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Notice that $\chi = \chi_k$ can be extended to S' by $\chi_S(f_3) = \chi_k(1)$ and $\chi_S(f_2f_3) = \chi_k(1)$ because χ_S has only to satisfy $\chi_{S|Z'} = \chi_k$. This gives:

x	1	f_3	$f_3^2 = (f_2f_3)^2 = f_4$	f_3^{-1}	f_2f_3	$(f_2f_3)^{-1} = f_2f_3^{-1}$	f_2f_4	f_2
$\chi_S(x)$	1	$\chi_k(1)$	-1	$-\chi_k(1)$	$\chi_k(1)$	$-\chi_k(1)$	-1	1

Now the representation ρ_χ is defined by $\rho_\chi = \mathrm{Ind}_{S'}^{\mathfrak{g}'(R/2R)}(\chi_S)$, i.e. if we define

$$\mathscr{H} = \{f : \mathfrak{g}'(R/2R) \to \mathbb{C} | f(s'g) = \chi_S(s')f(g)\ \forall s' \in S',\ \forall g \in \mathfrak{g}'(R/2R)\},$$

then for all $g, g' \in \mathfrak{g}'(R/2R)$ and $f \in \mathscr{H}$:

$$\rho_\chi(g)(f)(g') = f(g'g).$$

Every complex-valued function on $\mathfrak{g}'(R/2R)$ is given by a sum of characteristic functions of the form $f = \sum_{h \in \mathfrak{g}'(R/2R)} \lambda_h \delta_h$, where $\lambda_h \in \mathbb{C}$. Using the definition of $\mathscr{H}$, we can prove that $\{H_1, H_2\}$ is a $\mathbb{C}$-basis of $\mathscr{H}$, where

$$\begin{cases} H_1 = \delta_1 + \chi_k(1)\delta_{f_3} + \chi_k(1)\delta_{f_2f_3} - \chi_k(1)\delta_{f_3^{-1}} - \chi_k(1)\delta_{f_2f_3^{-1}} - \delta_{f_4} - \delta_{f_2f_4} + \delta_{f_2} \\ H_2 = \delta_{f_1} + \chi_k(1)\delta_{f_1f_3} + \chi_k(1)\delta_{f_2f_1f_3} - \chi_k(1)\delta_{f_1f_3^{-1}} - \chi_k(1)\delta_{f_1f_2f_3} - \delta_{f_1f_4} - \delta_{f_1f_2} + \delta_{f_2f_1}. \end{cases}$$

Then we compute a matrix representation of ρ_χ in the basis $\{H_1, H_2\}$:

$$\rho_\chi(f_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \rho_\chi(f_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \rho_\chi(f_3) = \begin{pmatrix} \chi_k(1) & 0 \\ 0 & \chi_k(1) \end{pmatrix} \text{ and } \rho_\chi(f_4) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now relate Savin's work to the preceding sections 3.1 and 3.2. We know from Stone-von Neumann theorem 2.16 that the irreducible representation ρ_χ is isomorphic to the Schrödinger model of the Heisenberg representation $\rho_{\widetilde{\chi_k}}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ defined in Lemma 2.11. If we follow Savin's article, we see that the extension ρ'_χ (not unique by Proposition 4.1 in [Sav12]) of ρ_χ to $\mathrm{G}'(R/4R) \simeq \mathrm{U}_2(\mathbb{F}_3)$ and our Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ of $\mathrm{Mp}(\widetilde{W})$ both are representations of a two-fold covering of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ which are constructed from the same (unique) irreducible representation of $\mathrm{C}_4 \circ \mathrm{D}_4$ which acts as $\chi = \chi_k$ on its center. Therefore, if we want to relate the two representations ρ'_χ and $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$, then we need $\mathrm{G}'(R/4R)$ and $\mathrm{Mp}(\widetilde{W})$ to be the same group, i.e. $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{U}_2(\mathbb{F}_3)$. By sections 3.1 and 3.2, whether it is for $B_{\text{Weil}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or for $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this is true if and only if we have that $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}} \simeq \mathrm{U}_2(\mathbb{F}_3)$. By Propositions 3.5 and 3.30, this is equivalent to define the projective representation $s \mapsto \omega_{\widetilde{\chi_k}}(s, w_s)$ with a choice of $\{\delta(\sigma) = 2^{-\frac{\dim \mathrm{I}(\sigma)}{2}} e^{i\nu(\sigma)},\ \sigma \in \mathrm{Sp}(W)\}$ satisfying (112) and (116) such that $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, where $\sigma_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In this case, by Propositions 3.12 and 3.36, the map $\mathscr{S}$ is an isomorphism of central extensions, the representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$ is a unitary, irreducible

and faithful representation of $\mathrm{U}_2(\mathbb{F}_3)$, and we obtain in this way the commutative diagrams (D_{30}) and (D_{42}), respectively. Observe again that this is an exceptional case, since it is the only example, in any dimension, for which we can define a lift of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to $\mathrm{Mp}(\widetilde{W})$ such that this representation remains faithful, cf. Lemma 2.42. Now, the representation ρ'_χ is an irreducible 2-dimensional representation of the group $\mathrm{G}'(R/4R) \simeq \mathrm{U}_2(\mathbb{F}_3)$ such that $Z' = Z(\mathfrak{g}'(R/2R)) \simeq \mathbb{Z}/4\mathbb{Z}$ acts by the character $\chi = \chi_1$ or $\chi = \chi_3$. Notice that $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$ and hence $Z' = Z(\mathrm{U}_2(\mathbb{F}_3))$. We verify this explicitly in terms of **GAP**'s generators. The center of $\mathfrak{g}'(R/2R) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$ is $Z' = \langle f_3 \rangle$ and the center of $\mathrm{G}'(R/4R) \simeq \mathrm{U}_2(\mathbb{F}_3)$ is $Z(\mathrm{U}_2(\mathbb{F}_3)) = \langle f_2 \rangle$. Since $\mathscr{F}'(f_3) = f_1^2 = f_2$, the homomorphism $\mathscr{F}$ maps Z' bijectively onto $Z(\mathrm{U}_2(\mathbb{F}_3))$. According to [GN], $\mathrm{U}_2(\mathbb{F}_3)$ has six two-dimensional irreducible representations. Among them, two are such that Z' acts by χ_1, and two others are such that Z' acts by χ_3. Thus there are two possibilities for the extension ρ'_χ, and one of them has to coincide with our Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S}$.

Remark 3.56. A one-to-one correspondence between isomorphism classes of irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Sp}(W) \simeq \mathrm{S}_3$ and isomorphism classes of irreducible representations of $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z}) = \mathrm{U}_2(\mathbb{F}_3)$ such that Z' acts by the character χ_k is not possible because S_3 has three isomorphism classes of irreducible representations, whereas $\mathrm{U}_2(\mathbb{F}_3)$ has precisely two irreducible representations such that Z' acts by the character χ_k. Observe that this only depends on the fact that $\mathrm{SL}'_2(\mathbb{Z}/4\mathbb{Z})$ is a two-fold covering of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$. In fact, recall from (309) that there are four non-isomorphic central extensions of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ by two elements, namely $\mathrm{C}_2 \times \mathrm{A}_4 \rtimes \mathrm{C}_4$, $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$, $\mathrm{A}_4 \rtimes \mathrm{C}_8$ and $\mathrm{U}_2(\mathbb{F}_3)$. The groups $\mathrm{C}_2 \times \mathrm{A}_4 \rtimes \mathrm{C}_4$ (which defines a split extension) and $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ do not have any two-dimensional irreducible representation such that Z' acts by the character χ_k, whereas $\mathrm{A}_4 \rtimes \mathrm{C}_8$ has only one such representation. Therefore, neither of these groups could provide the aforementioned one-to-one correspondence. This confirms the fact that Theorem 4.1 in [Sav12] does not apply to the exceptional case of $G(R/4R) = \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$.

Chapter 4

A link with Quantum Information Theory

> "Quantum information theory is the study of the achievable limits of information processing within quantum mechanics. Many different types of information can be accomodated within quantum mechanics, including *classical information, coherent quantum information*, and *entanglement*. Exploring the rich variety of capabilities allowed by these types of information is the subject of quantum information theory [...]"
>
> – Michael A. Nielsen [Nie00]

This section is intended to provide some insights on the links between the mathematical results we obtain in this thesis, and the formalism of Quantum Information Theory. It is certainly not exhaustive, and it could deserve more attention in the future, for example concerning potential applications of our results for people working in quantum error-correction. Therefore, we will only touch some deep concepts of Quantum Information Theory in order to reach directly the results that interest us. We refer to [NC11] for a complete introduction to this subject, with precise references and historical backgrounds. We also mention that Heinrich analyzes in detail in [Hei21] the links between the stabilizer formalism (which is described below) and the construction of the Weil representation for finite fields. However, he points to difficulties of the construction in characteristic two and refers to Gurevich-Hadani [GH12] to treat this case. Our construction of the Weil representations clarifies the case of the field $\mathbb{F}_2$ and provides explicit formulas that we will use to recognize Clifford gates and stabilizer codes.

The basic building blocks of classical information are called "bits". Each bit can take the value 0 or 1. Hence the state of a bit can be physically stored as for example the ON/OFF state of a transistor, or the orientation of a field in a magnetic domain. A basic unit of quantum information is fundamentally different: it consists of a quantum particle, whose behaviour is then governed by the laws of quantum mechanics. In fact, a "quantum bit", also called "qubit", is not limited to only two possible states: it can exist in any superposition of the two basis states denoted by $|0\rangle$ and $|1\rangle$, respectively. Formally, a single qubit is a vector

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{389}$$

parametrized by two complex numbers α and β which satisfy $|\alpha|^2+|\beta|^2=1$. In other words, a qubit is described by a unit vector in a two-dimensional complex Hilbert space. This suggests the potential advantage of using qubits instead of bits: if $n \geqslant 1$, there are 2^n possible states if we manipulate n bits, but infinitely many possible states lying in a 2^n-dimensional complex Hilbert space if we manipulate n qubits instead. However, the peculiarity of the quantum behaviour of the particles is that this powerful superposition of states is hidden in the system, in the sense that measuring a qubit always leads to only one state. We say that the qubit $|\psi\rangle$ collapses to either the state $|0\rangle$, with probability $|\alpha|^2$, or to the state $|1\rangle$, with probability $|\beta|^2$, and we lose the initial state of the qubit after the measurement. The question of the physical implementation of qubits is a huge challenge, which is primarily due to two phenomena. First, quantum particles are very fragile since they are strongly affected by their environment, thus it is difficult to store them reliably for a long period; this is called "decoherence". Secondly, quantum particles may be correlated in the sense that any action of one particle may affect all the related particles, no matter the distance between them; this is called "entanglement". The race to find the physical implementation that will permit to build large-scale quantum computers has begun around 1982, when Feyman raised the need of quantum computers to simulate quantum systems efficiently, and it is now gaining momentum. To cite the most promising examples, the research is oriented towards superconducting, trapped ion, photonic, spin and topological qubits. In this context, the power of the mathematical abstract model of qubits lies in the fact that it allows to build a theory which is independent of any physical implementation.

Classical information is transmitted through circuits, and bit manipulation is performed using logic gates. The analog in quantum computation consists in operating on qubits by quantum gates through quantum circuits. More precisely, an operation on a single qubit is described by a linear operator U of the two-dimensional complex Hilbert space in which the qubit lives, i.e. a two-by-two matrix. Since U sends a quantum state to another quantum state, it has to preserve its norm. In fact, any unitary two-by-two matrix defines a valid operation on a single qubit. More generally, an operation on a system of n qubits is described by a complex unitary matrix of size 2^n. The first quantum gates we mention are the Pauli matrices

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad X=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \text{and}\quad Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{390}$$

where X is also called a bit flip (since it exchanges $|0\rangle$ and $|1\rangle$) and Z is also called a phase flip (since $|0\rangle$ is invariant and $|1\rangle$ becomes $-|1\rangle$). The three matrices X, Y and Z generate the Pauli group $\mathscr{P}_1$ on one qubit:

$$\mathscr{P}_1=\{\pm I,\pm iI,\pm X,\pm iX,\pm Y,\pm iY,\pm Z,\pm iZ\}. \tag{391}$$

The presentation of this group which appears in [GN] is

$$\mathrm{C}_4\circ\mathrm{D}_4=\langle a,b,c\mid a^4=c^2=1,b^2=a^2,ab=ba,ac=ca,cbc=a^2b\rangle,$$

and we have already encountered it as the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, where B_W is any bilinear form which satisfies (77) in the two-dimensional example treated in chapter 3. More precisely, by Propositions 3.5 and 3.30, the matrices in the ordered basis $\{\delta_0,\delta_1\}$ of $\mathrm{L}^2(Y)$ of

the operators $\rho_{\widetilde{\chi_k}}(w)$, for all $w \in W$, of the Heisenberg representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, are the following:

$$\rho_{\widetilde{\chi_k}}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(1,0) = \begin{pmatrix} \widetilde{\varepsilon} & 0 \\ 0 & -\widetilde{\varepsilon} \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(1,1) = \begin{pmatrix} 0 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 0 \end{pmatrix} \tag{392}$$

if $B = B_{\mathrm{Weil}}$, and

$$\rho_{\widetilde{\chi_k}}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(0,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(1,0) = \begin{pmatrix} \widetilde{\xi} & 0 \\ 0 & -\widetilde{\xi} \end{pmatrix}, \quad \rho_{\widetilde{\chi_k}}(1,1) = \begin{pmatrix} 0 & \widetilde{\xi} \\ \widetilde{\xi} & 0 \end{pmatrix} \tag{393}$$

if $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where $\widetilde{\varepsilon} \in \{\pm 1\}$ and $\widetilde{\xi} \in \{\pm i\}$ are defined by (298) and (342), respectively.

We define also the Pauli group $\mathscr{P}_n$ on n qubits as the set of n-fold tensor products of matrices in $\mathscr{P}_1$. We can prove that the Pauli group $\mathscr{P}_n$ and the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ of order 2^{2n+2} are isomorphic.

Proposition 4.1. *If W is a $2n$-dimensional symplectic vector space over $\mathbb{F}_2$, then the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ defined by* (79)*, and the n-qubit Pauli group $\mathscr{P}_n$, are isomorphic.*

Proof. Since two elements in $\mathscr{P}_1$ either commute or anti-commute (i.e. either $xyx^{-1}y^{-1} = 1$ or $xyx^{-1}y^{-1} = -1$), this is the same for any two elements in $\mathscr{P}_n$. Hence the commutator subgroup of $\mathscr{P}_n$ is exactly $\{\pm I\}$. Since every element of $\mathscr{P}_n$ has an order which divides four, and its center is $\{\pm I, \pm iI\}$, we prove exactly as in Proposition A.17 for $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ that $\mathscr{P}_n$ is an almost extraspecial 2-group of order 2^{2n+2}. Since there exists a unique such almost extraspecial 2-group by Proposition A.14, we deduce that $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\mathscr{P}_n$ are isomorphic groups. □

A fundamental property of Pauli matrices is that every complex square matrix of size 2^n can be written as a linear combination of elements in $\mathscr{P}_n$ (cf. Lemma 2.15 for a proof of this property for $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$). To understand why this is important, we need to recall that quantum particles are very fragile, and thus quantum computations are highly susceptible to errors. Therefore, correcting these errors is a main challenge to deal with in order to perform reliable computations. This is exactly the goal of quantum error-correction. However, the task of protecting qubits against errors is made difficult by the fact that we cannot measure a qubit to detect an error, since measuring it would destroy its superposition of states. Moreover, in constrast with classical information, the "no-cloning theorem" implies also that we cannot copy a qubit without destroying it. Therefore, quantum error-correction has to find ways to detect and correct errors without ever knowing the state of the qubits which are transmitted. To this extent, quantum error-correcting codes use several physical qubits in order to encode one logical qubit (the qubit we want to transmit), operate on and measure the physical qubits without altering the state of the logical qubit in order to detect the "error syndrome", and then perform a "recovery operation" to restore the original state of the qubit. Formally, suppose that we have n qubits, living in a 2^n-dimensional complex Hilbert space H. A quantum code which encodes k qubits with n physical qubits is a vector subspace C of H of dimension 2^k. It is called the "codespace" and its elements are called "codewords". Let C be such quantum

code and let P be the projector onto the codespace C. Let $\{E_i,\ i \in J\}$ be a set of operators on H, representing the errors that we want to be able to correct. By the quantum error-correction conditions, provided for example by Knill-Laflamme [KL97], which we write as in [NC11, Theorem 10.1], a necessary and sufficient condition for the existence of a recovery operation which corrects the set of errors $\{E_i,\ i \in J\}$ on C is that there exists a hermitian matrix $A = (a_{i,j})_{i,j \in J}$ such that

$$PE_i^* E_j P = a_{i,j} P \tag{394}$$

for all $i, j \in J$, where $*$ denotes the adjoint. Pauli matrices come into play in the process called "digitization of errors". Since they form a basis of operators on H, using (394), we can prove that they form a basis of errors. This means that correcting any error which occurs on k qubits as a Pauli gate on C is enough to correct an arbitrary error on k qubits on C. In other words, correcting bit flip and phase flip errors on C is enough to be able to correct an arbitrary error in $\{E_i,\ i \in J\}$ on C.

Pauli matrices are also especially relevant in the "stabilizer formalism" of quantum error-correction, developed by Gottesman in [Got97]. The philosophy of this field is to use group theory in order to describe quantum codes with operators instead of states. More precisely, we suppose that S is a commutative subgroup of $\mathscr{P}_n$ which does not contain $-I$. Then the elements of S are simultaneously diagonalizable, and we can define the codespace C as the joint eigenspace associated to the eigenvalue 1 of the operators in S. The space S is the "stabilizer" of all the codewords in C. In this context in which we are studying Pauli operators rather than quantum states, we are also concerned with the following question: how does the action of a unitary operator U on a codeword $|\psi\rangle \in C$ affect the corresponding set of Pauli operators in S? Since for every element M in S,

$$U|\psi\rangle = UM|\psi\rangle = UMU^*U|\psi\rangle, \tag{395}$$

studying the effect of U on $|\psi\rangle$ is equivalent to studying the effect of conjugation by U on M. To allow reliable computations, it would therefore be convenient to work only with operators U that preserve the stabilizer S. This is why we can choose to work only with unitary operators which belong to the normalizer of $\mathscr{P}_n$ in the unitary group $\mathrm{U}(2^n, \mathbb{C})$. This group is denoted by $\mathscr{C}_n$, called the "Clifford group", and its elements are called "Clifford gates". Notice that we use here the term Clifford group for the infinite group of operators which normalize the Pauli group. In order to obtain a smaller group, we may quotient it by the unit circle (which means that we ignore the global phase of every Clifford gate). Some examples of widely used Clifford gates are the Hadamard gate H and the phase gate S on one qubit:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \tag{396}$$

and the CNOT gate on two qubits, which is written in the ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{397}$$

Beyond the beauty of relating quantum error-correcting codes with group theory, the stabilizer formalism has powerful features. We cite two examples. First, a codespace C of dimension 2^k encoding k logical qubits in a Hilbert space of dimension 2^n can be described using only $n-k$ generators of the stabilizer group S of C, instead of 2^k codewords for a basis of C [Got97]. This allows us to considerably reduce the amount of information needed to physically handle the whole code on a computer. Secondly, the stabilizer formalism allows to simulate efficiently quantum computations on classical computers. This result is known as the Gottesman-Knill theorem, which appears in [Got98]:

> "Any quantum computer performing only: a) Clifford group gates, b) measurements of Pauli group operators, and c) Clifford group operations conditioned on classical bits, which may be the results of earlier measurements, can be perfectly simulated in polynomial time on a probabilistic classical computer."

The stabilizer formalism is strongly related to our construction of the Weil representations. In fact, we already proved in Proposition 4.1 that, if $\dim_{\mathbb{F}_2}(W) = 2n$, then the Pauli group $\mathscr{P}_n$ is isomorphic to the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Moreover, recall from (104) in Proposition 2.21 that the operators $\omega_{\widetilde{\chi_k}}(s, w_s)$ satisfy for all $s \in \mathrm{ASp}(B_W)$ and $h \in \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ the equality

$$\omega_{\widetilde{\chi_k}}(s, w_s) \circ \rho_{\widetilde{\chi_k}}(h) \circ \omega_{\widetilde{\chi_k}}(s, w_s)^{-1} = \rho_{\widetilde{\chi_k}}(s(h)).$$

Since the elements of $\mathrm{ASp}(B_W)$ are automorphisms of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, and the representation $\rho_{\widetilde{\chi_k}}$ is faithful, this means exactly that the operators $\omega_{\widetilde{\chi_k}}(s, w_s)$ normalize the group $\rho_{\widetilde{\chi_k}}(\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W))$, which is the Pauli group $\mathscr{P}_n$. Therefore, every operator $\omega_{\widetilde{\chi_k}}(s, w_s)$ may be thought as a Clifford gate, and more precisely the image of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ defined in Proposition 2.32 defines a relevant finite subgroup of the Clifford group $\mathscr{C}_n$.

Remark 4.2. So far we have only mentioned the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, because Proposition 4.1 states that the Pauli group $\mathscr{P}_n$ is the group $\rho_{\widetilde{\chi_k}}(\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W))$ if the dimension of the $\mathbb{F}_2$-vector space W is $2n$. Nevertheless, this is not the only case of interest. In fact, recall from Remark 2.4 that $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ contains the two non-isomorphic Heisenberg groups $\mathrm{H}(B')$ as subgroups, where the quadratic form Q' associated to B' has Witt index n or $n-1$. So, via $\rho_{\widetilde{\chi_k}}$, the group $\mathrm{H}(B')$ is a proper subgroup of the Pauli group $\mathscr{P}_n$, no matter the Witt index of Q'. In particular, if Q' has Witt index n, and we identify $\mathrm{H}(B')$ with the image of the Heisenberg representation $\rho_{\widetilde{\chi}}$ defined in Lemma 1.17, then we see by Lemma 1.15.ii and (28) that the matrix of $\rho_{\widetilde{\chi}}(h)$ is real for all h in $\mathrm{H}(B')$. Thus $\mathrm{H}(B')$ can be identified with the "real Pauli group" $\mathscr{P}_n^{\mathbb{R}}$. The elements of $\mathscr{P}_n^{\mathbb{R}}$ are exactly all the n-fold tensor products of elements of $\mathscr{P}_1^{\mathbb{R}}$, which is the subgroup of $\mathscr{P}_1$ generated by X and Z. The real Pauli group is useful for example to simplify some quantum computing models, since it uses less gates and no complex numbers. Moreover, even if the definition of $\mathrm{H}(\widetilde{W})$ in 2.47 is quite different, it eventually provides a representation ρ_{χ_k} and a Weil representation $\widetilde{\omega}_{\chi_k}$ whose operators are Clifford gates too. Therefore, we will look for links with Quantum Information Theory in a uniform way, studying the three constructions of the Weil representations we have provided in chapters 1 and 2.

In the following, we will provide explicit relations between the Weil representations we construct in this thesis and the Pauli groups, stabilizer codes and Clifford gates. To this extent, we start by mathematical preliminaries on the tensor product of representations in section 4.1, in order to be able to recognize some quantum gates on n qubits for general n. In section 4.2.1, we start by identifying the Pauli groups $\mathscr{P}_n^{\mathbb{R}}$ inside $\widetilde{\mathrm{Ps}(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$ and $\mathscr{P}_n$ inside $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ thanks to the Weil representations $\widetilde{\omega}_{\widetilde{\chi}}$ and $\widetilde{\omega}_{\widetilde{\chi_k}}$, and then we express widely used stabilizer error-correcting codes in terms of elements of $\widetilde{\mathrm{Ps}(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. Section 4.2.2 is devoted to the Hadamard(-Rademacher-Walsh) transform. Finally, we exhibit in section 4.2.3 all the finite subgroups of one-qubit Clifford gates defined with the constructions of the Weil representations from chapter 3. We will always assume that $\mathbb{F} = \mathbb{F}_2$ in this section.

4.1 Tensor product of representations

Recall from the second chapter that $(\widetilde{W}, \langle\cdot,\cdot\rangle_{\widetilde{W}})$ is a symplectic $\mathbb{Z}/4\mathbb{Z}$-module of rank $2n$, where $n \geqslant 1$, and $B_{\widetilde{W}}$ is a bilinear form on $\widetilde{W}$ such that for all $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{W}$,

$$\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} = B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) - B_{\widetilde{W}}(\widetilde{w}_2, \widetilde{w}_1). \tag{398}$$

Recall from Lemma 2.1 that the abelian group isomorphism $\kappa : 2R \to \mathbb{F}_2$ enables us to identify $\widetilde{W}/2\widetilde{W}$ with a $2n$-dimensional symplectic vector space $(W, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle = \kappa(\langle\cdot,\cdot\rangle_W)$ and

$$\langle w_1, w_2 \rangle_W = \langle \widetilde{w}_1 + 2\widetilde{W}, \widetilde{w}_2 + 2\widetilde{W} \rangle_W = 2\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} \tag{399}$$

for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$. Then the bilinear form B_W, defined by $B_W(w_1, w_2) = 2B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)$ for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$, satisfies the following equality:

$$\langle w_1, w_2 \rangle_W = 2\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}} = 2(B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2) - B_{\widetilde{W}}(\widetilde{w}_2, \widetilde{w}_1)) = B_W(w_1, w_2) - B_W(w_2, w_1). \tag{400}$$

Furthermore, the bilinear form $B = \kappa \circ B_W$ is such that

$$\langle w_1, w_2 \rangle = \kappa(\langle w_1, w_2 \rangle_W) = \kappa(B_W(w_1, w_2)) - \kappa(B_W(w_2, w_1)) = B(w_1, w_2) - B(w_2, w_1). \tag{401}$$

Finally, recall that we can fix complete polarizations $\widetilde{W} = \widetilde{X} \oplus \widetilde{Y}$ and $W = X \oplus Y$. Moreover, $\widetilde{W}$ and W can be endowed with symplectic bases $\widetilde{\mathscr{B}} = (\widetilde{e_1}, \ldots, \widetilde{e_n}, \widetilde{f_1}, \ldots, \widetilde{f_n})$ and $\mathscr{B} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$, respectively, where $e_i = \widetilde{e_i} + 2\widetilde{W}$, $f_j = \widetilde{f_j} + 2\widetilde{W}$, $\langle \widetilde{e_i}, \widetilde{e_j} \rangle_{\widetilde{W}} = 0$, $\langle \widetilde{f_i}, \widetilde{f_j} \rangle_{\widetilde{W}} = 0$ and $\langle \widetilde{e_i}, \widetilde{f_j} \rangle_{\widetilde{W}} = \delta_{i,j}$ for all $1 \leqslant i, j \leqslant n$. We can choose $\widetilde{\mathscr{B}}$ and $\mathscr{B}$ so that $\{\widetilde{e_1}, \ldots, \widetilde{e_n}\}$ is a basis of $\widetilde{X}$, $\{\widetilde{f_1}, \ldots, \widetilde{f_n}\}$ is a basis of $\widetilde{Y}$, $\{e_1, \ldots, e_n\}$ is a basis of X and $\{f_1, \ldots, f_n\}$ is a basis of Y. If for every $1 \leqslant i \leqslant n$ we define $\widetilde{W}_i = \mathbb{Z}/4\mathbb{Z}\widetilde{e_i} \oplus \mathbb{Z}/4\mathbb{Z}\widetilde{f_i}$ and $W_i = \mathrm{Span}_{\mathbb{F}_2}\{e_i, f_i\}$, then $\widetilde{W}_i$ is a symplectic module of rank two and W_i is a two-dimensional symplectic vector space (which is also called a hyperbolic plane). Defining $\widetilde{X}_i = \mathbb{Z}/4\mathbb{Z}\widetilde{e_i}$, $\widetilde{Y}_i = \mathbb{Z}/4\mathbb{Z}\widetilde{f_i}$, $X_i = \mathrm{Span}_{\mathbb{F}_2}\{e_i\}$ and $Y_i = \mathrm{Span}_{\mathbb{F}_2}\{f_i\}$, we have the complete polarizations $\widetilde{W}_i = \widetilde{X}_i \oplus \widetilde{Y}_i$ and $W_i = X_i \oplus Y_i$. Moreover, we have the following decompositions: $\widetilde{W} = \widetilde{W}_1 \oplus \cdots \oplus \widetilde{W}_n$, $W = W_1 \oplus \cdots \oplus W_n$, $\widetilde{X} = \widetilde{X}_1 \oplus \cdots \oplus \widetilde{X}_n$, $X = X_1 \oplus \cdots \oplus X_n$, $\widetilde{Y} = \widetilde{Y}_1 \oplus \cdots \oplus \widetilde{Y}_n$ and $Y = Y_1 \oplus \cdots \oplus Y_n$.

Lemma 4.3. *The map*

$$\begin{aligned} T :\ \mathrm{L}^2(Y) &\to \mathrm{L}^2(Y_1) \otimes \cdots \otimes \mathrm{L}^2(Y_n) \\ \delta_y = \delta_{y_1+\cdots+y_n} &\mapsto \delta_{y_1} \otimes \cdots \otimes \delta_{y_n} \end{aligned} \tag{402}$$

defines a vector space isomorphism.

Proof. Since the set Y_i is finite for every $1 \leqslant i \leqslant n$, the space $\mathrm{L}^2(Y_i)$ is just the space of all functions from Y_i to $\mathbb{C}$. Hence, a basis of $\mathrm{L}^2(Y_i)$ is given by $\{\delta_{y_i},\ y_i \in Y_i\} = \{\delta_0, \delta_{f_i}\}$, and a basis of $\mathrm{L}^2(Y)$ is given by $\{\delta_y,\ y \in Y\}$, where $|Y| = |\mathbb{F}_2|^n = 2^n$. Thus, the spaces $\mathrm{L}^2(Y)$ and $\mathrm{L}^2(Y_1) \otimes \cdots \otimes \mathrm{L}^2(Y_n)$ have the same dimension n over $\mathbb{F}_2$. Now, for every $y = \sum_{i=1}^{n} y_i$ in $Y = Y_1 \oplus \cdots Y_n$, if we map δ_y to $\delta_{y_1} \otimes \cdots \otimes \delta_{y_n}$, and we extend this definition to $\mathrm{L}^2(Y)$ by linearity, then we define a linear map which sends a basis of $\mathrm{L}^2(Y)$ to a basis of $\mathrm{L}^2(Y_1) \otimes \cdots \otimes \mathrm{L}^2(Y_n)$, hence an isomorphism of vector spaces. □

Let $i \in \{1, \ldots, n\}$. We denote by $\langle \cdot, \cdot \rangle_{\widetilde{W}_i}$, $\langle \cdot, \cdot \rangle_{W_i}$ and $\langle \cdot, \cdot \rangle_i$ the restrictions of $\langle \cdot, \cdot \rangle_{\widetilde{W}}$, $\langle \cdot, \cdot \rangle_W$ and $\langle \cdot, \cdot \rangle$ to $\widetilde{W}_i$, W_i and W_i, respectively. We also denote by $B_{\widetilde{W}_i}$, B_{W_i}, Q_{W_i}, B_i and Q_i the restrictions of $B_{\widetilde{W}}$, B_W, Q_W, $B = \kappa \circ B_W$ and Q to $\widetilde{W}_i$ for the first map and W_i for the other ones (recall that κ is defined in Lemma 2.1). Hence these bilinear forms satisfy the following equalities:

$$\begin{cases} B_{W_i}(w_1, w_2) &= 2B_{\widetilde{W}_i}(\widetilde{w}_1, \widetilde{w}_2) \\ B_i(w_1, w_2) &= \kappa(B_{W_i}(w_1, w_2)) \\ \langle w_1, w_2 \rangle_i &= \kappa(\langle w_1, w_2 \rangle_{W_i}) \end{cases} \quad \text{and} \quad \begin{cases} \langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}_i} &= B_{\widetilde{W}_i}(\widetilde{w}_1, \widetilde{w}_2) - B_{\widetilde{W}_i}(\widetilde{w}_2, \widetilde{w}_1) \\ \langle w_1, w_2 \rangle_{W_i} &= B_{W_i}(w_1, w_2) - B_{W_i}(w_2, w_1) \\ \langle w_1, w_2 \rangle_i &= B_i(w_1, w_2) - B_i(w_2, w_1), \end{cases} \tag{403}$$

for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}_i$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}_i$ in W_i. In this way, we can consider the subgroups of $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ generated by the elements in $W_i \times \mathbb{F}_2$ and $W_i \times \mathbb{Z}/4\mathbb{Z}$, respectively. These subgroups are isomorphic to the Heisenberg groups $\mathrm{H}(B_i)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_{W_i})$ of order eight and sixteen, respectively. More precisely, by Propositions 1.6 and 2.3, we have that $\mathrm{H}(B_i) \simeq \mathrm{D}_4$ or Q_8 and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_{W_i}) \simeq \mathrm{C}_4 \circ \mathrm{D}_4$. Moreover, the restrictions $\widetilde{\chi}^{(i)}$ of $\widetilde{\chi}$ to $X_i \times \mathbb{F}_2$ and $\widetilde{\chi_k}^{(i)}$ of $\widetilde{\chi_k}$ to $X_i \times \mathbb{Z}/4\mathbb{Z}$ are extensions of χ and χ_k to the abelian subgroups $X_i \times \mathbb{F}_2$ of $\mathrm{H}(B_i)$ and $X_i \times \mathbb{Z}/4\mathbb{Z}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_{W_i})$, respectively. Then, according to Lemmas 1.17 and 2.11, the Schrödinger models $\rho_{\widetilde{\chi}^{(i)}}$ and $\rho_{\widetilde{\chi_k}^{(i)}}$ of the Heisenberg representations of $\mathrm{H}(B_i)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_{W_i})$, respectively, are well-defined on $\mathrm{L}^2(Y_i)$. In the same way, the subgroup of $\mathrm{H}(\widetilde{W})$ (defined in 2.47) generated by the elements in $\widetilde{W}_i \times \mathbb{Z}/4\mathbb{Z}$ is isomorphic to the Heisenberg group $\mathrm{H}(\widetilde{W}_i) \simeq \mathrm{C}_4^2 \rtimes_4 \mathrm{C}_4$ of order 64 (see the beginning of section 3.3). Then we denote by $\rho_{\chi_k}^{(i)}$ the representation of $\mathrm{H}(\widetilde{W}_i)$ defined in Proposition 2.52.

In this section, we will suppose that the bilinear form B satisfies the following property:

$$B(w_i, w_j) = 0 \quad \text{for all } w_i \in W_i \text{ and } w_j \in W_j \text{ such that } i \neq j. \tag{404}$$

For example, this property is satisfied if B is one of the two following forms:

$$\begin{cases} B(x+y,x'+y') = B_{\text{Weil}}(x+y,x'+y') = x^T y' & \text{(in this case the Witt index of } Q \text{ is } n) \\ B(x+y,x'+y') = x^T y' + x^T x' + y^T y' & \text{(in this case the Witt index of } Q \text{ is } n-1). \end{cases} \tag{405}$$

Lemma 4.4 (The Schrödinger model of the Heisenberg representation, as a tensor product). *We take the same notations as in Lemmas 1.17 and 2.11, and Proposition 2.52.*

i. *We can realize the Heisenberg representation $\rho_{\widetilde{\chi}}$ of $\mathrm{H}(B)$ given in Lemma 1.17 as a tensor product. In fact, if we set*

$$\rho_{\widetilde{\chi}}^{\otimes}(w,t) = \chi(t)\rho_{\widetilde{\chi}^{(1)}}(w_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n,0) \quad \textit{for all } (w,t)\in\mathrm{H}(B), \tag{406}$$

then $\rho_{\widetilde{\chi}}^{\otimes}$ defines a representation of $\mathrm{H}(B)$ on the space $\mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n)$. Moreover,

$$\rho_{\widetilde{\chi}}^{\otimes}(h) = T\circ\rho_{\widetilde{\chi}}(h)\circ T^{-1} \quad \textit{for all } h\in\mathrm{H}(B), \tag{407}$$

where T is as in (402)*, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{L}^2(Y) & \xrightarrow{\quad\rho_{\widetilde{\chi}}(h)\quad} & \mathrm{L}^2(Y) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ \mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n) & \xrightarrow{\rho_{\widetilde{\chi}}^{\otimes}(h)} & \mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n) \end{array} \tag{D_{55}}$$

ii. *We can realize the Heisenberg representation $\rho_{\widetilde{\chi_k}}$ of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ given in Lemma 2.11 as a tensor product. In fact, if we set*

$$\rho_{\widetilde{\chi_k}}^{\otimes}(w,t) = \chi_k(t)\rho_{\widetilde{\chi_k}^{(1)}}(w_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi_k}^{(n)}}(w_n,0) \quad \textit{for all } (w,t)\in\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W), \tag{408}$$

then $\rho_{\widetilde{\chi_k}}^{\otimes}$ defines a representation of $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ on the space $\mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n)$. Moreover,

$$\rho_{\widetilde{\chi_k}}^{\otimes}(h) = T\circ\rho_{\widetilde{\chi_k}}(h)\circ T^{-1} \quad \textit{for all } h\in\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W), \tag{409}$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{L}^2(Y) & \xrightarrow{\quad\rho_{\widetilde{\chi_k}}(h)\quad} & \mathrm{L}^2(Y) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ \mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n) & \xrightarrow{\rho_{\widetilde{\chi_k}}^{\otimes}(h)} & \mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n) \end{array} \tag{D_{56}}$$

iii. *We can realize the representation ρ_{χ_k} of $\mathrm{H}(\widetilde{W})$ given in Proposition 2.52 as a tensor product. In fact, if we set*

$$\rho_{\chi_k}^{\otimes}(\widetilde{w},t) = \chi_k(t)\rho_{\chi_k}^{(1)}(\widetilde{w}_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}_n,0) \quad \textit{for all } (\widetilde{w},t)\in\mathrm{H}(\widetilde{W}), \tag{410}$$

then $\rho^{\otimes}_{\chi_k}$ *defines a representation of* $\mathrm{H}(\widetilde{W})$ *on the space* $\mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n)$. *Moreover,*

$$\rho^{\otimes}_{\chi_k}(h) = T\circ\rho_{\chi_k}(h)\circ T^{-1} \quad \textit{for all } h\in\mathrm{H}(\widetilde{W}), \tag{411}$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{L}^2(Y) & \xrightarrow{\quad\rho_{\chi_k}(h)\quad} & \mathrm{L}^2(Y) \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle T} \\ \mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n) & \xrightarrow{\rho^{\otimes}_{\chi_k}(h)} & \mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n) \end{array} \tag{D_{57}}$$

Proof. **i.** For every $(w,t)\in\mathrm{H}(B)$ such that $w=x+y\in X\oplus Y$, we write more precisely $w=w_1+\cdots+w_n\in W_1\oplus\cdots\oplus W_n$ and $w_i=x_i+y_i\in X_i\oplus Y_i$ for all $1\leqslant i\leqslant n$. For all $(w,t)\in\mathrm{H}(B)$, we define

$$\rho^{\otimes}_{\widetilde{\chi}}(w,t) = \chi(t)\rho_{\widetilde{\chi}^{(1)}}(w_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n,0). \tag{412}$$

This formula defines a representation of $\mathrm{H}(B)$ on $\mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n)$. In fact, for all (w,t) and (w',t') in $\mathrm{H}(B)$,

$$\begin{aligned}
&\rho^{\otimes}_{\widetilde{\chi}}(w,t)\rho^{\otimes}_{\widetilde{\chi}}(w',t')\\
&= (\chi(t)\rho_{\widetilde{\chi}^{(1)}}(w_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n,0))(\chi(t')\rho_{\widetilde{\chi}^{(1)}}(w'_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w'_n,0))\\
&= \chi(t+t')\rho_{\widetilde{\chi}^{(1)}}(w_1,0)\rho_{\widetilde{\chi}^{(1)}}(w'_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n,0)\rho_{\widetilde{\chi}^{(n)}}(w'_n,0)\\
&= \chi(t+t')\chi(B_1(w_1,w'_1)+\cdots+B_n(w_n,w'_n))\rho_{\widetilde{\chi}^{(1)}}(w_1+w'_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n+w'_n,0)\\
&= \chi(t+t'+B(w,w'))\rho_{\widetilde{\chi}^{(1)}}(w_1+w'_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n+w'_n,0)\\
&= \rho^{\otimes}_{\widetilde{\chi}}(w+w',t+t'+B(w,w'))\\
&= \rho^{\otimes}_{\widetilde{\chi}}((w,t)(w',t')),
\end{aligned}$$

where the fourth equality is due to the hypothesis we made for the choice of B. In fact, (404) implies that

$$B(w_i,w'_j)=0 \quad \text{for all } w_i\in W_i \text{ and } w'_j\in W_j \text{ such that } i\neq j. \tag{413}$$

Moreover by (33), for all $i\in\{1,\dots,n\}$ and $z_i\in Y_i$,

$$\rho_{\widetilde{\chi}^{(i)}}(w_i,0)\delta_{z_i} = \widetilde{\chi}^{(i)}(x_i,\langle x_i,z_i\rangle_i - B_i(y_i,x_i+z_i))\delta_{z_i+y_i} = \widetilde{\chi}(x_i,\langle x_i,z_i\rangle - B(y_i,x_i+z_i))\delta_{z_i+y_i}.$$

Hence

$$\begin{aligned}
&(\rho_{\widetilde{\chi}^{(1)}}(w_1,0)\otimes\cdots\otimes\rho_{\widetilde{\chi}^{(n)}}(w_n,0))(\delta_{z_1}\otimes\cdots\otimes\delta_{z_n})\\
&= (\rho_{\widetilde{\chi}^{(1)}}(w_1,0)\delta_{z_1})\otimes\cdots\otimes(\rho_{\widetilde{\chi}^{(n)}}(w_n,0)\delta_{z_n})\\
&= (\widetilde{\chi}(x_1,\langle x_1,z_1\rangle - B(y_1,x_1+z_1))\delta_{z_1+y_1})\otimes\cdots\otimes(\widetilde{\chi}(x_n,\langle x_n,z_n\rangle - B(y_n,x_n+z_n))\delta_{z_n+y_n})\\
&= \widetilde{\chi}(x_1,\langle x_1,z_1\rangle - B(y_1,x_1+z_1))\cdots\widetilde{\chi}(x_n,\langle x_n,z_n\rangle - B(y_n,x_n+z_n))\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n}.
\end{aligned}$$

Using the product law in $\mathrm{H}(B)$, we obtain that

$$\begin{aligned}&\widetilde{\chi}(x_1,\langle x_1,z_1\rangle - B(y_1,x_1+z_1))\cdots\widetilde{\chi}(x_n,\langle x_n,z_n\rangle - B(y_n,x_n+z_n))\\&= \widetilde{\chi}(x_1+\cdots+x_n)\chi\left(\sum_{i=1}^n(\langle x_i,z_i\rangle - B(y_i,x_i+z_i)) + \sum_{1\leqslant i<j\leqslant n} B(x_i,x_j)\right).\end{aligned}$$

Thus by (413),

$$\rho_{\widetilde{\chi}}^{\otimes}(w,t) = \chi(t)\widetilde{\chi}\left(x,\sum_{i=1}^n(\langle x_i,z_i\rangle - B(y_i,x_i+z_i))\right)\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n}. \tag{414}$$

On the other hand,

$$\begin{aligned}&(T\circ\rho_{\widetilde{\chi}}(w,t)\circ T^{-1})(\delta_{z_1}\otimes\cdots\otimes\delta_{z_n})\\&= (T\circ\rho_{\widetilde{\chi}}(w,t))\delta_{z_1+\cdots+z_n}\\&= T\left(\widetilde{\chi}(x,t+\langle x,z_1+\cdots+z_n\rangle - B(y,x+(z_1+\cdots+z_n)))\delta_{(z_1+\cdots+z_n)+y}\right)\\&= \widetilde{\chi}(x,t+\langle x,z_1+\cdots+z_n\rangle - B(y,x+(z_1+\cdots+z_n)))\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n}\\&= \widetilde{\chi}\left(x,t+\sum_{i=1}^n\langle x_i,z_i\rangle - \sum_{i=1}^n B(y_i,x_i+z_i)\right)\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n},\end{aligned} \tag{415}$$

where the last equality comes from (413) and the fact that $\langle w_i,w_j\rangle = 0$ for $i\neq j$. Comparing the expressions given by (414) and (415), we finally obtain that $\rho_{\widetilde{\chi}}^{\otimes}(w,t) = T\circ\rho_{\widetilde{\chi}}(w,t)\circ T^{-1}$.

ii. This is the same proof as for the preceding point, only replacing B, $\langle\cdot,\cdot\rangle$, χ, $\mathrm{H}(B)$ and $\rho_{\widetilde{\chi}}$ by B_W, $\langle\cdot,\cdot\rangle_W$, χ_k, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ and $\rho_{\widetilde{\chi}_k}$, respectively.

iii. For every $(\widetilde{w},t)\in\mathrm{H}(\widetilde{W})$, where $\widetilde{w}=\widetilde{w}_1+\cdots+\widetilde{w}_n\in\widetilde{W}_1\oplus\cdots\oplus\widetilde{W}_n$, we define

$$\rho_{\chi_k}^{\otimes}(\widetilde{w},t) = \chi_k(t)\rho_{\chi_k}^{(1)}(\widetilde{w}_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}_n,0). \tag{416}$$

This formula defines a representation of $\mathrm{H}(\widetilde{W})$ on $\mathrm{L}^2(Y_1)\otimes\cdots\otimes\mathrm{L}^2(Y_n)$ since for all $(\widetilde{w},t)$ and $(\widetilde{w}',t')$ in $\mathrm{H}(\widetilde{W})$,

$$\begin{aligned}&\rho_{\chi_k}^{\otimes}(\widetilde{w},t)\rho_{\chi_k}^{\otimes}(\widetilde{w}',t')\\&= (\chi_k(t)\rho_{\chi_k}^{(1)}(\widetilde{w}_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}_n,0))(\chi_k(t')\rho_{\chi_k}^{(1)}(\widetilde{w}'_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}'_n,0))\\&= \chi_k(t+t')\rho_{\chi_k}^{(1)}(\widetilde{w}_1,0)\rho_{\chi_k}^{(1)}(\widetilde{w}'_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}_n,0)\rho_{\chi_k}^{(n)}(\widetilde{w}'_n,0)\\&= \chi_k(t+t')\chi_k(\langle\widetilde{w}_1,\widetilde{w}'_1\rangle_{\widetilde{W}_1}+\cdots+\langle\widetilde{w}_n,\widetilde{w}'_n\rangle_{\widetilde{W}_n})\rho_{\chi_k}^{(1)}(\widetilde{w}_1+\widetilde{w}'_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}_n+\widetilde{w}'_n,0)\\&= \chi_k(t+t'+\langle\widetilde{w},\widetilde{w}'\rangle_{\widetilde{W}})\rho_{\chi_k}^{(1)}(\widetilde{w}_1+\widetilde{w}'_1,0)\otimes\cdots\otimes\rho_{\chi_k}^{(n)}(\widetilde{w}_n+\widetilde{w}'_n,0)\quad\text{because }\langle\widetilde{w}_i,\widetilde{w}_j\rangle_{\widetilde{W}}=0\text{ if }i\neq j\\&= \rho_{\chi_k}^{\otimes}(\widetilde{w}+\widetilde{w}',t+t'+\langle\widetilde{w},\widetilde{w}'\rangle_{\widetilde{W}})\\&= \rho_{\chi_k}^{\otimes}((\widetilde{w},t)(\widetilde{w}',t')).\end{aligned}$$

Moreover by (203), for all $i\in\{1,\ldots,n\}$ and $z_i\in Y_i$,

$$\rho_{\chi_k}^{(i)}(\widetilde{w}_i,0)\delta_{z_i} = \chi_k(\langle x_i,z_i\rangle_{W_i}+\langle\widetilde{x}_i,\widetilde{y}_i\rangle_{\widetilde{W}_i})\delta_{z_i+y_i} = \chi_k(\langle x_i,z_i\rangle_W+\langle\widetilde{x}_i,\widetilde{y}_i\rangle_{\widetilde{W}})\delta_{z_i+y_i},$$

where $x_i = \widetilde{x}_i + 2\widetilde{X}_i$ and $y_i = \widetilde{y}_i + 2\widetilde{Y}_i$. This implies that

$$\begin{aligned}
&\rho^{\otimes}_{\chi_k}(\widetilde{w},t)(\delta_{z_1}\otimes\cdots\otimes\delta_{z_n})\\
&= \chi_k(t)(\rho^{(1)}_{\chi_k}(\widetilde{w}_1,0)\otimes\cdots\otimes\rho^{(n)}_{\chi_k}(\widetilde{w}_n,0))(\delta_{z_1}\otimes\cdots\otimes\delta_{z_n})\\
&= \chi_k(t)(\rho^{(1)}_{\chi_k}(\widetilde{w}_1,0)\delta_{z_1})\otimes\cdots\otimes(\rho^{(n)}_{\chi_k}(\widetilde{w}_n,0)\delta_{z_n})\\
&= \chi_k(t)(\chi_k(\langle x_1,z_1\rangle_W+\langle\widetilde{x}_1,\widetilde{y}_1\rangle_{\widetilde{W}})\delta_{z_1+y_1})\otimes\cdots\otimes(\chi_k(\langle x_n,z_n\rangle_W+\langle\widetilde{x}_n,\widetilde{y}_n\rangle_{\widetilde{W}})\delta_{z_n+y_n})\\
&= \chi_k(t)\chi_k\left(\sum_{i=1}^{n}(\langle x_i,z_i\rangle_W+\langle\widetilde{x}_i,\widetilde{y}_i\rangle_{\widetilde{W}})\right)\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n}\\
&= \chi_k(t+\langle x,z\rangle_W+\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n},
\end{aligned} \tag{417}$$

where the last equality comes from the fact $\langle w_i,w_j\rangle_W = 0$ and $\langle\widetilde{w}_i,\widetilde{w}_j\rangle_{\widetilde{W}} = 0$ if $i\neq j$. On the other hand,

$$\begin{aligned}
&(T\circ\rho_{\chi_k}(\widetilde{w},t)\circ T^{-1})(\delta_{z_1}\otimes\cdots\otimes\delta_{z_n})\\
&= (T\circ\rho_{\chi_k}(\widetilde{w},t))\delta_{z_1+\cdots+z_n}\\
&= T\left(\chi_k(t+\langle x,z_1+\cdots+z_n\rangle_W+\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\delta_{(z_1+\cdots+z_n)+y}\right)\\
&= \chi_k(t+\langle x,z_1+\cdots+z_n\rangle_W+\langle\widetilde{x},\widetilde{y}\rangle_{\widetilde{W}})\delta_{z_1+y_1}\otimes\cdots\otimes\delta_{z_n+y_n}.
\end{aligned} \tag{418}$$

By (417) and (418), we finally obtain that $\rho^{\otimes}_{\chi_k}(\widetilde{w},t) = T\circ\rho_{\chi_k}(\widetilde{w},t)\circ T^{-1}$.

□

The next step is to look for elements $s\in\mathrm{Ps}(B)$, $s'\in\mathrm{ASp}(B_W)$ and $\widetilde{\sigma}\in\mathrm{Sp}(\widetilde{W})$ such that we can express the operators $\omega_{\widetilde{\chi}}(s)$, $\omega_{\widetilde{\chi_k}}(s')$ and $\omega_{\chi_k}(\widetilde{\sigma})$ of $\mathrm{L}^2(Y)$, respectively, as tensor products of operators of $\mathrm{L}^2(Y_i)$ for all $1\leqslant i\leqslant n$.

For every $1\leqslant i\leqslant n$, an element $\sigma_i\in\mathrm{Sp}(W_i)$ can be considered as an element of $\mathrm{Sp}(W)$ if we set for all $w = w_1+\cdots+w_n\in W = W_1\oplus\cdots\oplus W_n$:

$$\sigma_i(w) = w_1+\cdots+w_{i-1}+\sigma_i(w_i)+w_{i+1}+\cdots+w_n. \tag{419}$$

In this case, we will denote this element of $\mathrm{Sp}(W)$ by $(1,\ldots,1,\sigma_i,1,\ldots,1)$, or simply by σ_i. We can do the same with $\mathrm{Sp}(\widetilde{W}_i)$ and $\mathrm{Sp}(\widetilde{W})$. In this way, we define two (proper if $n\geqslant 1$) subgroups $\mathrm{Sp}^{\oplus}(W) = \mathrm{Sp}(W_1)\times\cdots\times\mathrm{Sp}(W_n)$ of $\mathrm{Sp}(W)$ and $\mathrm{Sp}^{\oplus}(\widetilde{W}) = \mathrm{Sp}(\widetilde{W}_1)\times\cdots\times\mathrm{Sp}(\widetilde{W}_n)$ of $\mathrm{Sp}(\widetilde{W})$. An element $(\sigma_1,\ldots,\sigma_n)\in\mathrm{Sp}^{\oplus}(W)$ acts on W by

$$(\sigma_1,\ldots,\sigma_n)(w_1+\cdots+w_n) = \sigma_1\cdots\sigma_n(w_1+\cdots+w_n) = \sigma_1(w_1)+\cdots+\sigma_n(w_n) \tag{420}$$

for all $w = w_1+\cdots+w_n\in W$. The product of two elements in $\mathrm{Sp}^{\oplus}(W)$ is given by $(\sigma_1,\ldots,\sigma_n)(\tau_1,\ldots,\tau_n) = (\sigma_1\tau_1,\ldots,\sigma_n\tau_n)$. We obtain similar expressions for $\mathrm{Sp}^{\oplus}(\widetilde{W})$. Recall that $\mathrm{O}(Q)\subseteq\mathrm{Sp}(W)$. Since (404) is satisfied, we can also consider the subgroup $\mathrm{O}^{\oplus}(Q) = \mathrm{O}(Q_1)\times\cdots\times\mathrm{O}(Q_n)$ of $\mathrm{O}(Q)$.

Furthermore, we define $\mathrm{Ps}^{\oplus}(B)$ and $\mathrm{ASp}^{\oplus}(B_W)$ as the subgroups of $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$ consisting of the elements of the form (σ,f) where $\sigma\in\mathrm{O}^{\oplus}(Q)$ and $\sigma\in\mathrm{Sp}^{\oplus}(W)$, respectively.

More precisely, by Propositions 1.12 and 2.7, every element of $\mathrm{Ps}^{\oplus}(B)$ can be written under the form $(\sigma, f_\sigma + \langle a, \cdot \rangle)$, where $\sigma \in \mathrm{O}^{\oplus}(Q)$ and $a \in W$, and every element of $\mathrm{ASp}^{\oplus}(B_W)$ can be written under the form $(\sigma, f_{\tilde{\sigma}} + \langle a, \cdot \rangle_W)$, where $\sigma \in \mathrm{Sp}^{\oplus}(W)$, $\pi(\tilde{\sigma}) = \sigma$ and $a \in W$ (π is defined in Proposition 2.7.iii).

Lemma 4.5. ***i.*** *Let $(\sigma, f_\sigma + \langle a, \cdot \rangle) \in \mathrm{Ps}^{\oplus}(B)$ be such that $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathrm{O}^{\oplus}(Q)$, f_σ is defined by* (18) *and expressed in the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of W, and $a = a_1 + \cdots + a_n \in W$. Then*

$$\prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle) = (\sigma, f_\sigma + \langle a, \cdot \rangle). \tag{421}$$

ii. *Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathrm{Sp}^{\oplus}(W)$. For every $1 \leqslant i \leqslant n$, let $\tilde{\sigma}_i \in \mathrm{Sp}(\widetilde{W}_i)$ be such that $\pi(\tilde{\sigma}_i) = \sigma_i$, and set $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n) \in \mathrm{Sp}^{\oplus}(\widetilde{W})$. Let $(\sigma, f_{\tilde{\sigma}} + \langle a, \cdot \rangle_W) \in \mathrm{ASp}^{\oplus}(B_W)$, where $a = a_1 + \cdots + a_n \in W$. Then*

$$\prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\tilde{\sigma}_i} + \langle a_i, \cdot \rangle_W) = (\sigma, f_{\tilde{\sigma}} + \langle a, \cdot \rangle_W). \tag{422}$$

Proof. **i.** Let $(\sigma, f_\sigma + \langle a, \cdot \rangle) \in \mathrm{Ps}^{\oplus}(B)$ be such that $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathrm{O}^{\oplus}(Q)$ and $a = a_1 + \cdots + a_n \in W$. By (18), if we express f_σ in the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of W, we have for every $w = \sum_{1 \leqslant i \leqslant n} (\lambda_i e_i + \mu_i f_i)$,

$$\begin{aligned} &f_\sigma \left(w = \sum_{1 \leqslant i \leqslant n} (\lambda_i e_i + \mu_i f_i) \right) \\ &= \sum_{1 \leqslant i < j \leqslant n} \lambda_i \lambda_j \beta_\sigma(e_i, e_j) + \sum_{1 \leqslant i, j \leqslant n} \lambda_i \mu_j \beta_\sigma(e_i, f_j) + \sum_{1 \leqslant i < j \leqslant n} \mu_i \mu_j \beta_\sigma(f_i, f_j), \end{aligned} \tag{423}$$

where for all $1 \leqslant i, j \leqslant n$, by (404),

$$\begin{cases} \beta_\sigma(e_i, e_j) = B(\sigma(e_i), \sigma(e_j)) - B(e_i, e_j) = B(\sigma_i(e_i), \sigma_j(e_j)) - B(e_i, e_j) \\ \qquad\qquad = (B(\sigma_i(e_i), \sigma_i(e_i)) - B(e_i, e_i))\delta_i(j) \\ \beta_\sigma(e_i, f_j) = B(\sigma(e_i), \sigma(f_j)) - B(e_i, f_j) = B(\sigma_i(e_i), \sigma_j(f_j)) - B(e_i, f_j) \\ \qquad\qquad = (B(\sigma_i(e_i), \sigma_i(f_i)) - B(e_i, f_i))\delta_i(j) \\ \beta_\sigma(f_i, f_j) = B(\sigma(f_i), \sigma(f_j)) - B(f_i, f_j) = B(\sigma_i(f_i), \sigma_j(f_j)) - B(f_i, f_j) \\ \qquad\qquad = (B(\sigma_i(f_i), \sigma_i(f_i)) - B(f_i, f_i))\delta_i(j). \end{cases} \tag{424}$$

Hence

$$f_\sigma \left(w = \sum_{1 \leqslant i \leqslant n} (\lambda_i e_i + \mu_i f_i) \right) = \sum_{1 \leqslant i \leqslant n} \lambda_i \mu_i (B(\sigma_i(e_i), \sigma_i(f_i)) - B(e_i, f_i)). \tag{425}$$

Now, replacing σ by σ_l in (424) for every $1 \leqslant l \leqslant n$ (where $\sigma_l = (1, \ldots, 1, \sigma_l, 1, \ldots, 1)$ is considered as an element of $\mathrm{O}^{\oplus}(Q) \subseteq \mathrm{O}(Q)$, see (419)), we have

$$\begin{cases} \beta_{\sigma_l}(e_i, e_j) = (B(\sigma_i(e_i), \sigma_i(e_i)) - B(e_i, e_i))\delta_i(j)\delta_i(l) \\ \beta_{\sigma_l}(e_i, f_j) = (B(\sigma_i(e_i), \sigma_i(f_i)) - B(e_i, f_i))\delta_i(j)\delta_i(l) \\ \beta_{\sigma_l}(f_i, f_j) = (B(\sigma_i(f_i), \sigma_i(f_i)) - B(f_i, f_i))\delta_i(j)\delta_i(l). \end{cases}$$

Then, replacing σ by σ_l in (423), we obtain

$$f_{\sigma_l}(w) = \lambda_l \mu_l (B(\sigma_l(e_l), \sigma_l(f_l)) - B(e_l, f_l)). \tag{426}$$

Thus, by (425),

$$f_\sigma(w) = \sum_{i=1}^{n} f_{\sigma_i}(w). \tag{427}$$

Furthermore, for all $i \neq j$ and for all $w \in W$, since the components of w and $\sigma_j(w)$ on W_i are the same, by (426),

$$\begin{aligned}
&(\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle)(\sigma_j, f_{\sigma_j} + \langle a_j, \cdot \rangle)(w) \\
&= (\sigma_i \sigma_j, f_{\sigma_j} + \langle a_j, \cdot \rangle + (f_{\sigma_i} + \langle a_i, \cdot \rangle) \circ \sigma_j)(w) \\
&= (\sigma_i \sigma_j(w), f_{\sigma_j}(w) + \langle a_j, w \rangle + f_{\sigma_i}(\sigma_j(w)) + \langle a_i, \sigma_j(w) \rangle) \\
&= (\sigma_i \sigma_j(w), f_{\sigma_j}(w) + \langle a_j, w \rangle + f_{\sigma_i}(w) + \langle a_i, w \rangle).
\end{aligned}$$

By (427), this proves that

$$\prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle) = \left(\sigma_1 \cdots \sigma_n, \sum_{1 \leqslant i \leqslant n} (f_{\sigma_i} + \langle a_i, \cdot \rangle) \right) = (\sigma, f_\sigma + \langle a, \cdot \rangle). \tag{428}$$

ii. Let $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathrm{Sp}^{\oplus}(W)$. For every $1 \leqslant i \leqslant n$, there exists $\widetilde{\sigma}_i \in \mathrm{Sp}(\widetilde{W}_i)$ such that $\pi(\widetilde{\sigma}_i) = \sigma_i$. If we set $\widetilde{\sigma} = (\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_n) \in \mathrm{Sp}^{\oplus}(\widetilde{W})$, then $\pi(\widetilde{\sigma}) = \sigma$. Let $(\sigma, f_{\widetilde{\sigma}} + \langle a, \cdot \rangle_W) \in \mathrm{ASp}^{\oplus}(B_W)$, where $a = a_1 + \cdots + a_n \in W$. Then for all $w = w_1 + \cdots + w_n \in W$, and $\widetilde{w} = \widetilde{w}_1 + \cdots + \widetilde{w}_n \in \widetilde{W}$ such that $\widetilde{w}_i + 2\widetilde{W}_i = w_i$ for all $1 \leqslant i \leqslant n$, by (93),

$$\begin{aligned}
&f_{\widetilde{\sigma}}(w) \\
&= B_{\widetilde{W}}(\widetilde{\sigma}(\widetilde{w}), \widetilde{\sigma}(\widetilde{w})) - B_{\widetilde{W}}(\widetilde{w}, \widetilde{w}) \\
&= B_{\widetilde{W}}(\widetilde{\sigma}_1(\widetilde{w}_1) + \cdots + \widetilde{\sigma}_n(\widetilde{w}_n), \widetilde{\sigma}_1(\widetilde{w}_1) + \cdots + \widetilde{\sigma}_n(\widetilde{w}_n)) - B_{\widetilde{W}}(\widetilde{w}_1 + \cdots + \widetilde{w}_n, \widetilde{w}_1 + \cdots + \widetilde{w}_n) \\
&= \sum_{1 \leqslant i,j \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_j(\widetilde{w}_j)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_j)) \\
&= \sum_{1 \leqslant i \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_i)) + \sum_{1 \leqslant i < j \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_j(\widetilde{w}_j)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_j)) \\
&\quad + \sum_{1 \leqslant j < i \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_j(\widetilde{w}_j)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_j)) \\
&= \sum_{1 \leqslant i \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_i)) \\
&\quad + \sum_{1 \leqslant i < j \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_j(\widetilde{w}_j)) + B_{\widetilde{W}}(\widetilde{\sigma}_j(\widetilde{w}_j), \widetilde{\sigma}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_j) - B_{\widetilde{W}}(\widetilde{w}_j, \widetilde{w}_j)) \\
&= \sum_{1 \leqslant i \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_i)) \\
&\quad + \sum_{1 \leqslant i < j \leqslant n} (\langle \widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_j(\widetilde{w}_j) \rangle_{\widetilde{W}} + 2B_{\widetilde{W}}(\widetilde{\sigma}_j(\widetilde{w}_j), \widetilde{\sigma}_i(\widetilde{w}_i)) - \langle \widetilde{w}_i, \widetilde{w}_j \rangle_{\widetilde{W}} - 2B_{\widetilde{W}}(\widetilde{w}_j, \widetilde{w}_j))
\end{aligned}$$

Since $\widetilde{W}_i$ and $\widetilde{W}_j$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\widetilde{W}}$ and $2B_{\widetilde{W}}(\widetilde{w}_i,\widetilde{w}_j) = B_W(w_i,w_j) = 0$ for $i \neq j$ by (404), we obtain that

$$f_{\widetilde{\sigma}}(w) = \sum_{1\leqslant i\leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i),\widetilde{\sigma}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i,\widetilde{w}_i)). \tag{429}$$

Now for every $1 \leqslant l \leqslant n$, we find in the same way:

$$f_{\widetilde{\sigma}_l}(w) = B_{\widetilde{W}}(\widetilde{\sigma}_l(\widetilde{w}_l),\widetilde{\sigma}_l(\widetilde{w}_l)) - B_{\widetilde{W}}(\widetilde{w}_l,\widetilde{w}_l). \tag{430}$$

Thus by (429) and (430),

$$f_{\widetilde{\sigma}}(w) = \sum_{1\leqslant i\leqslant n} f_{\widetilde{\sigma}_i}(w). \tag{431}$$

Finally, we prove exactly as in the preceding point that

$$\prod_{1\leqslant i\leqslant n} (\sigma_i, f_{\widetilde{\sigma}_i}+\langle a_i,\cdot\rangle_W) = \left(\sigma_1\cdots\sigma_n, \sum_{1\leqslant i\leqslant n} (f_{\widetilde{\sigma}_i} + \langle a_i,\cdot\rangle_W)\right) = (\sigma, f_{\widetilde{\sigma}}+\langle a,\cdot\rangle_W). \tag{432}$$

□

Recall that for every $1 \leqslant i \leqslant n$, we can consider the subgroups of $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ generated by the elements in $W_i \times \mathbb{F}_2$ and $W_i \times \mathbb{Z}/4\mathbb{Z}$, respectively, and that these subgroups are isomorphic to the Heisenberg groups $\mathrm{H}(B_i)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_{W_i})$ of order eight and sixteen, respectively. Similarly the subgroup of $\mathrm{H}(\widetilde{W})$ generated by the elements in $\widetilde{W}_i \times \mathbb{Z}/4\mathbb{Z}$ is isomorphic to the Heisenberg group $\mathrm{H}(\widetilde{W}_i) \simeq \mathrm{C}_4^2 \rtimes_4 \mathrm{C}_4$ of order 64. In the same way, the subgroups of $\mathrm{Ps}(B)$ and $\mathrm{ASp}(B_W)$ generated by the elements $s = (\sigma, f)$ such that σ belongs to $\mathrm{O}(Q_i)$ and $\mathrm{Sp}(W_i)$ are isomorphic to $\mathrm{Ps}(B_i)$ and $\mathrm{ASp}(B_{W_i})$, respectively. In particular, every element $s_i = (\sigma_i, f_{\sigma_i} + \langle a_i,\cdot\rangle)$ such that $\sigma_i \in \mathrm{O}(Q_i)$ can be considered as:

- either an element of $\mathrm{Ps}(B_i)$, where in this case $s_i(w_i,t) = (\sigma_i(w_i), t+f_{\sigma_i}(w_i)+\langle a_i,w_i\rangle_i)$ for all $(w_i,t) \in \mathrm{H}(B_i)$. We denote by $w_{s_i}^{(i)}$ any element of W_i satisfying (37), i.e. such that
$$\langle w_{s_i}^{(i)}, w_i\rangle_i = f_{\sigma_i}(w_i) + \langle a_i,w_i\rangle_i \qquad \text{for all } w_i \in \mathrm{K}(\sigma_i) \subseteq W_i. \tag{433}$$
- either an element of $\mathrm{Ps}(B)$, where in this case by (419) applied to $(1,\dots,1,\sigma_i,\dots,1)$, $s_i(w,t) = (w_1+\cdots+w_{i-1}+\sigma_i(w_i)+w_{i+1}+\cdots+w_n, f_{\sigma_i}(w)+\langle a_i,w\rangle)$ for all $(w,t) \in \mathrm{H}(B)$. We denote by w_{s_i} any element of W satisfying (37), i.e. such that
$$\langle w_{s_i}, w\rangle = f_{\sigma_i}(w) + \langle a_i,w\rangle \quad \text{for all } w \in \mathrm{K}(1,\dots,1,\sigma_i,1,\dots,1) \subseteq W, \tag{434}$$
where $\mathrm{K}(1,\dots,1,\sigma_i,1,\dots,1) = W_1 \oplus \cdots \oplus W_{i-1} \oplus \mathrm{K}(\sigma_i) \oplus W_{i+1} \oplus \cdots \oplus W_n$.

In particular, we will prove in the next lemma that $w_{s_i} - w_{s_i}^{(i)} \in \mathrm{I}(\sigma_i)$. In the same way, every element $s_i = (\sigma_i, f_{\widetilde{\sigma}_i}+\langle a_i,\cdot\rangle_W)$ such that $\sigma_i \in \mathrm{Sp}(W_i)$ can be considered either as an element of $\mathrm{ASp}(B_{W_i})$ or an element of $\mathrm{ASp}(B_W)$, and we denote also by $w_{s_i}^{(i)}$ and w_{s_i} the corresponding elements of W_i and W, respectively, which satisfy (100). Finally for every $\widetilde{\sigma}_i \in \mathrm{Sp}(\widetilde{W}_i)$, we denote by $\widetilde{w}_{\widetilde{\sigma}_i}^{(i)}$ and $\widetilde{w}_{\widetilde{\sigma}_i}$ any elements of $\widetilde{W}_i$ and $\widetilde{W}$ satisfying (207) when we consider that $\widetilde{\sigma}_i \in \mathrm{Sp}(\widetilde{W}_i)$ and $\widetilde{\sigma}_i \in \mathrm{Sp}(\widetilde{W})$, respectively.

Lemma 4.6. *We keep the notations and assumptions of Lemma 4.5.*

i. *Let $s = s_1 \cdots s_n \in \mathrm{Ps}^{\oplus}(B)$ be as in* (421)*, where $s_i = (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle)$ for all $1 \leqslant i \leqslant n$. Then $w_s \in W$ is an element satisfying* (37) *if and only if $w_s - (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ belongs to $\mathrm{I}(\sigma) = \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n)$.*

ii. *Let $s = s_1 \cdots s_n \in \mathrm{ASp}^{\oplus}(B_W)$ be as in* (422)*, where $s_i = (\sigma_i, f_{\widetilde{\sigma}_i} + \langle a_i, \cdot \rangle_W)$ for all $1 \leqslant i \leqslant n$. Then $w_s \in W$ is an element satisfying* (100) *if and only if $w_s - (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ belongs to $\mathrm{I}(\sigma) = \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n)$.*

iii. *Let $\widetilde{\sigma} = (\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n) \in \mathrm{Sp}^{\oplus}(\widetilde{W})$. Then $\widetilde{w}_{\widetilde{\sigma}} \in \widetilde{W}$ is an element satisfying* (207) *if and only if $2\widetilde{w}_{\widetilde{\sigma}} - 2(\widetilde{w}_{\widetilde{\sigma}_1}^{(1)}, \ldots, \widetilde{w}_{\widetilde{\sigma}_n}^{(n)})$ belongs to $2\,\mathrm{I}(\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}_1) \oplus \cdots \oplus 2\,\mathrm{I}(\widetilde{\sigma}_n)$.*

Proof. **i.** Let $s = (\sigma, f) \in \mathrm{Ps}^{\oplus}(B)$ be such that

$$s = (\sigma, f_\sigma + \langle a, \cdot \rangle) = \prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle) = s_1 \cdots s_n,$$

where $\sigma_i \in \mathrm{O}(Q_i)$ and $s_i = (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle)$ for all $1 \leqslant i \leqslant n$. If $w_s = \sum_{1 \leqslant i \leqslant n} w_{s,i} \in W_1 \oplus \cdots \oplus W_n$, then w_s satisfies (37) if and only if we have for all $w = w_1 + \cdots + w_n \in \mathrm{K}(\sigma) = \mathrm{K}(\sigma_1) \oplus \cdots \oplus \mathrm{K}(\sigma_n)$:

$$\begin{aligned}
&\langle w_s, w \rangle = (f_\sigma + \langle a, \cdot \rangle)(w) \\
&\overset{(427)}{\iff} \sum_{1 \leqslant i \leqslant n} \langle w_{s,i}, w_i \rangle = \sum_{1 \leqslant i \leqslant n} (f_{\sigma_i}(w) + \langle a_i, w \rangle) \\
&\overset{(426)}{\iff} \sum_{1 \leqslant i \leqslant n} \langle w_{s,i}, w_i \rangle_i = \sum_{1 \leqslant i \leqslant n} (f_{\sigma_i}(w_i) + \langle a_i, w_i \rangle_i) \\
&\iff \sum_{1 \leqslant i \leqslant n} \langle w_{s,i}, w_i \rangle_i = \sum_{1 \leqslant i \leqslant n} \langle w_{s_i}^{(i)}, w_i \rangle_i \qquad \text{since } w_i \in \mathrm{K}(\sigma_i) \text{ for all } i.
\end{aligned}$$

This shows that w_s satisfies (37) if and only if $w_{s,i} - w_{s_i}^{(i)}$ belongs to $\mathrm{I}(\sigma_i)$ for all i, or equivalently $w_s - (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ belongs to $\mathrm{I}(\sigma) = \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n)$.

ii. This is the exact same proof as the one of the preceding point, only using the second part of Lemma 4.5 instead of the first one.

iii. Let $\widetilde{\sigma} = (\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n) = \widetilde{\sigma}_1 \cdots \widetilde{\sigma}_n \in \mathrm{Sp}^{\oplus}(\widetilde{W})$. If $\widetilde{w}_{\widetilde{\sigma}} = \sum_{1 \leqslant i \leqslant n} \widetilde{w}_{\widetilde{\sigma},i} \in \widetilde{W}_1 \oplus \cdots \widetilde{W}_n$ then $\widetilde{w}_{\widetilde{\sigma}}$ satisfies (207) if and only if for every $w \in \mathrm{K}'(\widetilde{\sigma}) = \mathrm{K}'(\widetilde{\sigma}_1) \oplus \cdots \oplus \mathrm{K}'(\widetilde{\sigma}_n)$,

$$\begin{aligned}
\langle 2\widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}} = \langle \widetilde{\sigma}(\widetilde{w}), \widetilde{w} \rangle_{\widetilde{W}} &\iff \sum_{1 \leqslant i \leqslant n} \langle 2\widetilde{w}_{\widetilde{\sigma},i}, \widetilde{w}_i \rangle_{\widetilde{W}_i} = \sum_{1 \leqslant i \leqslant n} \langle \widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{w}_i \rangle_{\widetilde{W}_i} \\
&\iff \sum_{1 \leqslant i \leqslant n} \langle 2\widetilde{w}_{\widetilde{\sigma},i}, \widetilde{w}_i \rangle_{\widetilde{W}_i} = \sum_{1 \leqslant i \leqslant n} \langle 2\widetilde{w}_{\widetilde{\sigma}_i}^{(i)}, \widetilde{w}_i \rangle_{\widetilde{W}_i}.
\end{aligned}$$

Hence $\widetilde{w}_{\widetilde{\sigma}}$ satisfies (207) if and only if $2\widetilde{w}_{\widetilde{\sigma},i} - 2\widetilde{w}_{\widetilde{\sigma}_i}^{(i)}$ belongs to the submodule of $\widetilde{W}_i$ which is orthogonal to $\mathrm{K}'(\widetilde{\sigma}_i)$ with respect to $\langle \cdot, \cdot \rangle_{\widetilde{W}_i}$, i.e. $2\,\mathrm{I}(\widetilde{\sigma}_i)$. This is equivalent to the fact that $2\widetilde{w}_{\widetilde{\sigma}} - 2(\widetilde{w}_{\widetilde{\sigma}_1}^{(1)}, \ldots, \widetilde{w}_{\widetilde{\sigma}_n}^{(n)})$ belongs to $2\,\mathrm{I}(\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}_1) \oplus \cdots \oplus 2\,\mathrm{I}(\widetilde{\sigma}_n)$.

□

For all $s \in \mathrm{Ps}^{\oplus}(B)$, $s \in \mathrm{ASp}^{\oplus}(B_W)$ and $\widetilde{\sigma} \in \mathrm{Sp}^{\oplus}(\widetilde{W})$, we are now ready to express the operators $\omega_{\widetilde{\chi}}(s, w_s)$, $\omega_{\widetilde{\chi_k}}(s, w_s)$ and $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ of $\mathrm{L}^2(Y)$, respectively, as tensor products of operators of $\mathrm{L}^2(Y_i)$ for all $1 \leqslant i \leqslant n$. We denote by $\omega_{\widetilde{\chi}^{(i)}}$, $\omega_{\widetilde{\chi_k}^{(i)}}$ and $\omega_{\chi_k}^{(i)}$ the projective Weil representations of $\mathrm{Ps}(B_i)$, $\mathrm{ASp}(B_{W_i})$ and $\mathrm{Sp}(\widetilde{W}_i)$ on $\mathrm{L}^2(Y_i)$ defined in Propositions 1.28, 2.21 and 2.56, respectively.

Proposition 4.7. *We keep the notations from all the previous results of this section.*

i. *Let* $s = (\sigma, f_\sigma + \langle a, \cdot \rangle) = \prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle) = s_1 \cdots s_n$ *be an element of* $\mathrm{Ps}^{\oplus}(B)$ *as in* (421)*, where* $\sigma_i \in \mathrm{O}(Q_i)$ *and* $s_i = (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle)$ *for all* $1 \leqslant i \leqslant n$*. If* $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ *then we can write the operator* $\omega_{\widetilde{\chi}}(s, w_s)$ *as a tensor product, i.e.*

$$\omega_{\widetilde{\chi}}(s, w_s) = \gamma(\sigma)(\gamma^{(1)}(\sigma_1) \cdots \gamma^{(n)}(\sigma_n))^{-1} T^{-1} \circ \big(\omega_{\widetilde{\chi}^{(1)}}(s_1, w_{s_1}^{(1)}) \otimes \cdots \otimes \omega_{\widetilde{\chi}^{(n)}}(s_n, w_{s_n}^{(n)})\big) \circ T, \tag{435}$$

where $\omega_{\widetilde{\chi}^{(i)}}(s_i, w_{s_i}^{(i)})$ *is defined with* $\gamma^{(i)}(\sigma_i)$ *satisfying* (46) *for every* $1 \leqslant i \leqslant n$*.*

ii. *Let* $s = (\sigma, f_{\widetilde{\sigma}} + \langle a, \cdot \rangle_W) = \prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\widetilde{\sigma}_i} + \langle a_i, \cdot \rangle_W) = s_1 \cdots s_n$ *be an element of* $\mathrm{ASp}^{\oplus}(B_W)$ *as in* (422)*, where* $\widetilde{\sigma}_i \in \mathrm{Sp}(\widetilde{W}_i)$*,* $\sigma_i \in \mathrm{Sp}(W_i)$*,* $\pi(\widetilde{\sigma}_i) = \sigma_i$ *and* $s_i = (\sigma_i, f_{\widetilde{\sigma}_i} + \langle a_i, \cdot \rangle_W)$ *for all* $1 \leqslant i \leqslant n$*. If* $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ *then we can write the operator* $\omega_{\widetilde{\chi_k}}(s, w_s)$ *as a tensor product, i.e.*

$$\omega_{\widetilde{\chi_k}}(s, w_s) = \gamma(\sigma)(\gamma^{(1)}(\sigma_1) \cdots \gamma^{(n)}(\sigma_n))^{-1} T^{-1} \circ \Big(\omega_{\widetilde{\chi_k}^{(1)}}(s_1, w_{s_1}^{(1)}) \otimes \cdots \otimes \omega_{\widetilde{\chi_k}^{(n)}}(s_n, w_{s_n}^{(n)})\Big) \circ T, \tag{436}$$

where $\omega_{\widetilde{\chi_k}^{(i)}}(s_i, w_{s_i}^{(i)})$ *is defined with* $\gamma^{(i)}(\sigma_i)$ *satisfying* (112) *for every* $1 \leqslant i \leqslant n$*.*

iii. *Let* $\widetilde{\sigma} = (\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n) \in \mathrm{Sp}^{\oplus}(\widetilde{W})$*. If* $\widetilde{w}_{\widetilde{\sigma}} = (\widetilde{w}_{\widetilde{\sigma}_1}^{(1)}, \ldots, \widetilde{w}_{\widetilde{\sigma}_n}^{(n)})$ *then we can write the operator* $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ *as a tensor product, i.e.*

$$\begin{aligned}&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\ &= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})(\gamma^{(1)}(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}^{(1)}) \cdots \gamma^{(n)}(\widetilde{\sigma}_n, \widetilde{w}_{\widetilde{\sigma}_n}^{(n)}))^{-1} T^{-1} \circ \Big(\omega_{\chi_k}^{(1)}(\widetilde{\sigma}_1, \widetilde{w}_{\widetilde{\sigma}_1}^{(1)}) \otimes \cdots \otimes \omega_{\chi_k}^{(n)}(\widetilde{\sigma}_n, \widetilde{w}_{\widetilde{\sigma}_n}^{(n)})\Big) \circ T,\end{aligned} \tag{437}$$

where $\omega_{\chi_k}^{(i)}(\widetilde{\sigma}_i, \widetilde{w}_{\widetilde{\sigma}_i}^{(i)})$ *is defined with* $\gamma^{(i)}(\widetilde{\sigma}_i)$ *satisfying* (232)*, for every* $1 \leqslant i \leqslant n$*.*

Proof. **i.** Let

$$s = (\sigma, f_\sigma + \langle a, \cdot \rangle) = \prod_{1 \leqslant i \leqslant n} (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle) = s_1 \cdots s_n$$

be an element of $\mathrm{Ps}^{\oplus}(B)$, where $\sigma_i \in \mathrm{O}(Q_i)$ and $s_i = (\sigma_i, f_{\sigma_i} + \langle a_i, \cdot \rangle)$ for all $1 \leqslant i \leqslant n$. By Lemma 4.6, we can choose $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$. By (40),

$$\omega_{\widetilde{\chi}}(s, w_s) = \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f_\sigma(w) + \langle a, w \rangle + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\widetilde{\chi}}(u + w_s),$$

where $w \in W$ is such that $u = (1 - \sigma)(w)$. We use the following facts:

- Since $\mathrm{I}(\sigma) = \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n)$, every $u \in \mathrm{I}(\sigma)$ can be written as $u = u_1 + \cdots + u_n$, where $u_i \in \mathrm{I}(\sigma_i)$ for all i. Moreover, for every $w = w_1 + \cdots + w_n \in W$ such that $u = (1 - \sigma)(w)$, we have that $u_i = (1 - \sigma_i)(w_i)$ for all i.
- By (427) and since W_i and W_j are orthogonal with respect to $\langle \cdot, \cdot \rangle$ for $i \neq j$, we have that $f_\sigma(w) + \langle a, w \rangle = \sum_{1 \leqslant i \leqslant n} (f_{\sigma_i}(w_i) + \langle a_i, w_i \rangle_i)$.
- By (404) and with our choice of w_s, $B(u, w + w_s) = \sum_{1 \leqslant i \leqslant n} B_i(u_i, w_i + w_{s_i}^{(i)})$.
- By Lemma 4.4,

$$\begin{aligned}
\rho_{\widetilde{\chi}}(u + w_s) &= \rho_{\widetilde{\chi}} \left(\sum_{1 \leqslant i \leqslant n} (u_i + w_{s_i}^{(i)}) \right) \\
&= T^{-1} \circ \rho_{\widetilde{\chi}}^{\otimes} \left(\sum_{1 \leqslant i \leqslant n} (u_i + w_{s_i}^{(i)}) \right) \circ T \\
&= T^{-1} \circ (\rho_{\widetilde{\chi}^{(1)}}(u_1 + w_{s_1}^{(1)}) \otimes \cdots \otimes \rho_{\widetilde{\chi}^{(n)}}(u_n + w_{s_n}^{(n)})) \circ T.
\end{aligned}$$

Then

$$\begin{aligned}
&\omega_{\widetilde{\chi}}(s, w_s) \\
&= \gamma(\sigma) \sum_{u \in \mathrm{I}(\sigma)} \chi(f_\sigma(w) + \langle a, w \rangle + \langle w_s, w \rangle + B(u, w + w_s)) \rho_{\widetilde{\chi}}(u + w_s) \\
&= \gamma(\sigma) \sum_{(u_1, \ldots, u_n) \in \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n)} \chi \left(\sum_{1 \leqslant i \leqslant n} (f_{\sigma_i}(w_i) + \langle a_i, w_i \rangle_i + \langle w_{s_i}^{(i)}, w_i \rangle_i + B_i(u_i, w_i + w_{s_i}^{(i)})) \right) \\
&\qquad\qquad T^{-1} \circ (\rho_{\widetilde{\chi}^{(1)}}(u_1 + w_{s_1}^{(1)}) \otimes \cdots \otimes \rho_{\widetilde{\chi}^{(n)}}(u_n + w_{s_n}^{(n)})) \circ T \\
&= \gamma(\sigma) \sum_{(u_1, \ldots, u_n) \in \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n)} \prod_{1 \leqslant i \leqslant n} \chi(f_{\sigma_i}(w_i) + \langle a_i, w_i \rangle_i + \langle w_{s_i}^{(i)}, w_i \rangle_i + B_i(u_i, w_i + w_{s_i}^{(i)})) \\
&\qquad\qquad T^{-1} \circ (\rho_{\widetilde{\chi}^{(1)}}(u_1 + w_{s_1}^{(1)}) \otimes \cdots \otimes \rho_{\widetilde{\chi}^{(n)}}(u_n + w_{s_n}^{(n)})) \circ T \\
&= \gamma(\sigma) T^{-1} \\
&\circ \left(\left(\sum_{u_1 \in \mathrm{I}(\sigma_1)} \chi(f_{\sigma_1}(w_1) + \langle a_1, w_1 \rangle_1 + \langle w_{s_1}^{(1)}, w_1 \rangle_1 + B_1(u_1, w_1 + w_{s_1}^{(1)})) \rho_{\widetilde{\chi}^{(1)}}(u_1 + w_{s_1}^{(1)}) \right) \right. \\
&\qquad \otimes \cdots \otimes \\
&\qquad \left. \left(\sum_{u_n \in \mathrm{I}(\sigma_n)} \chi(f_{\sigma_n}(w_n) + \langle a_n, w_n \rangle_n + \langle w_{s_n}^{(n)}, w_n \rangle_n + B_n(u_n, w_n + w_{s_n}^{(n)})) \rho_{\widetilde{\chi}^{(n)}}(u_n + w_{s_n}^{(n)}) \right) \right) \\
&\circ T \\
&= \gamma(\sigma)(\gamma^{(1)}(\sigma_1) \cdots \gamma^{(n)}(\sigma_n))^{-1} T^{-1} \circ \left(\omega_{\widetilde{\chi}^{(1)}}(s_1, w_{s_1}^{(1)}) \otimes \cdots \otimes \omega_{\widetilde{\chi}^{(n)}}(s_n, w_{s_n}^{(n)}) \right) \circ T.
\end{aligned}$$

ii. This part of the proof is identical to that of the preceding point, only using the formulas and results corresponding to $\mathrm{ASp}(B_W)$ instead of $\mathrm{Ps}(B)$.

iii. Let $\widetilde{\sigma} = (\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_n) \in \mathrm{Sp}^{\oplus}(\widetilde{W})$. By Lemma 4.6, we can choose $\widetilde{w}_{\widetilde{\sigma}} = (\widetilde{w}^{(1)}_{\widetilde{\sigma}_1}, \dots, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n})$. Then by (217),

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{\widetilde{u} \in \mathrm{I}(\widetilde{\sigma})} \chi_k(\langle \widetilde{u}, \widetilde{w} + \widetilde{w}_{\widetilde{\sigma}} \rangle_{\widetilde{W}} + 2\langle \widetilde{w}_{\widetilde{\sigma}}, \widetilde{w} \rangle_{\widetilde{W}}) \rho_{\chi_k}(\widetilde{u} + \widetilde{w}_{\widetilde{\sigma}}),$$

where $\widetilde{w} \in \widetilde{W}$ is any element such that $(1 - \widetilde{\sigma})(\widetilde{w}) = \widetilde{u}$. We know that $\mathrm{I}(\widetilde{\sigma}) = \mathrm{I}(\widetilde{\sigma}_1) \oplus \cdots \oplus \mathrm{I}(\widetilde{\sigma}_n)$, and $\widetilde{W}_i$ and $\widetilde{W}_j$ are orthogonal with respect to the symplectic form of $\widetilde{W}$ for $i \neq j$. As in the first part of the proof above, we can write

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \sum_{(\widetilde{u}_1, \dots, \widetilde{u}_n) \in \mathrm{I}(\widetilde{\sigma}_1) \oplus \cdots \oplus \mathrm{I}(\widetilde{\sigma}_n)} \chi_k \left(\sum_{1 \leqslant i \leqslant n} (\langle \widetilde{u}_i, \widetilde{w}_i + \widetilde{w}^{(i)}_{\widetilde{\sigma}_i} \rangle_{\widetilde{W}_i} + 2\langle \widetilde{w}^{(i)}_{\widetilde{\sigma}_i}, \widetilde{w}_i \rangle_{\widetilde{W}_i}) \right) \\
&\qquad\qquad \rho_{\chi_k} \left(\sum_{1 \leqslant i \leqslant n} (\widetilde{u}_i + \widetilde{w}^{(i)}_{\widetilde{\sigma}_i}) \right) \\
&\overset{(411)}{=} \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \\
&\qquad \sum_{(\widetilde{u}_1, \dots, \widetilde{u}_n) \in \mathrm{I}(\widetilde{\sigma}_1) \oplus \cdots \oplus \mathrm{I}(\widetilde{\sigma}_n)} \prod_{1 \leqslant i \leqslant n} \chi_k(\langle \widetilde{u}_i, \widetilde{w}_i + \widetilde{w}^{(i)}_{\widetilde{\sigma}_i} \rangle_{\widetilde{W}_i} + 2\langle \widetilde{w}^{(i)}_{\widetilde{\sigma}_i}, \widetilde{w}_i \rangle_{\widetilde{W}_i}) \\
&\qquad\qquad T^{-1} \circ \left(\rho^{(1)}_{\chi_k}(\widetilde{u}_1 + \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \otimes \cdots \otimes \rho^{(n)}_{\chi_k}(\widetilde{u}_n + \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}) \right) \circ T \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})|\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} \\
&\qquad T^{-1} \circ \Bigg(\sum_{\widetilde{u}_1 \in \mathrm{I}(\widetilde{\sigma}_1)} \chi_k(\langle \widetilde{u}_1, \widetilde{w}_1 + \widetilde{w}^{(1)}_{\widetilde{\sigma}_1} \rangle_{\widetilde{W}_1} + 2\langle \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}, \widetilde{w}_1 \rangle_{\widetilde{W}_1}) \rho^{(1)}_{\chi_k}(\widetilde{u}_1 + \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \\
&\qquad\qquad \otimes \cdots \otimes \\
&\qquad\qquad \sum_{\widetilde{u}_n \in \mathrm{I}(\widetilde{\sigma}_n)} \chi_k(\langle \widetilde{u}_n, \widetilde{w}_n + \widetilde{w}^{(n)}_{\widetilde{\sigma}_n} \rangle_{\widetilde{W}_n} + 2\langle \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}, \widetilde{w}_n \rangle_{\widetilde{W}_n}) \rho^{(n)}_{\chi_k}(\widetilde{u}_n + \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}) \Bigg) \circ T \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})(\gamma^{(1)}(\widetilde{\sigma}_1, \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \cdots \gamma^{(n)}(\widetilde{\sigma}_n, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}))^{-1} |\,\mathrm{I}(\widetilde{\sigma}) \cap 2\widetilde{W}|^{-1} |\,\mathrm{I}(\widetilde{\sigma}_1) \cap 2\widetilde{W}_1| \cdots |\,\mathrm{I}(\widetilde{\sigma}_n) \cap 2\widetilde{W}_n| \\
&\qquad T^{-1} \circ \left(\omega^{(1)}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \otimes \cdots \otimes \omega^{(n)}_{\chi_k}(\widetilde{\sigma}_n, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}) \right) \circ T \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})(\gamma^{(1)}(\widetilde{\sigma}_1, \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \cdots \gamma^{(n)}(\widetilde{\sigma}_n, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}))^{-1} T^{-1} \circ \left(\omega^{(1)}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \otimes \cdots \otimes \omega^{(n)}_{\chi_k}(\widetilde{\sigma}_n, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}) \right) \circ T.
\end{aligned}$$

□

4.2 Quantum gates and stabilizer codes in the language of Weil representations

We keep the notation of section 4.1. In the following, we suppose that $\dim_{\mathbb{F}_2}(W) = 2n$, where $n \geqslant 1$, and either

$$B = B_{\mathrm{Weil}} = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_{\widetilde{W}} = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} \tag{438}$$

or

$$B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad B_{\widetilde{W}} = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \tag{439}$$

in the symplectic bases $(e_1, \dots, e_n, f_1, \dots, f_n)$ of W and $(\widetilde{e}_1, \dots, \widetilde{e}_n, \widetilde{f}_1, \dots, \widetilde{f}_n)$ of $\widetilde{W}$, respectively. This section is intended to provide explicit links between our constructions of the Weil representations and the setting of Quantum Information Theory. First, we use the Weil representations $\widetilde{\omega}_{\widetilde{\chi}}$ of $\widetilde{\mathrm{Ps}(B_{\mathrm{Weil}})_{\widetilde{\chi}}}$ and $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ in order to identify the Pauli groups $\mathscr{P}_n^{\mathbb{R}}$ and $\mathscr{P}_n$ with subgroups of $\widetilde{\mathrm{Ps}(B_{\mathrm{Weil}})_{\widetilde{\chi}}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, respectively. Then we apply this result to describe a correspondence between generators of the stabilizer groups of widely used stabilizer codes and generators of the corresponding subgroups of $\widetilde{\mathrm{Ps}(B_{\mathrm{Weil}})_{\widetilde{\chi}}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$. In section 4.2.2, we identify the elements of $\widetilde{\mathrm{Ps}(B)_{\widetilde{\chi}}}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ and $\mathrm{Sp}(\widetilde{W})$ which provide the Hadamard(-Rademacher-Walsh) transform via the corresponding projective Weil representations. Finally, we exhibit in section 4.2.3 the finite subgroups of the Clifford group $\mathscr{C}_1$ that we obtain thanks to our constructions of the Weil representations from chapter 3.

4.2.1 Pauli gates and stabilizer codes

Recall that the Pauli group $\mathscr{P}_n$ is the group of matrices of size 2^n whose elements are n-fold tensor products of elements of $\mathscr{P}_1$, where $\mathscr{P}_1$ is the group generated by the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For all $1 \leqslant i \leqslant n$, we will denote by $X_{(i)}$ the matrix $I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots \otimes I$, where X operates on the i-th qubit, and similarly for $Y_{(i)}$ and $Z_{(i)}$. Recall from Remark 4.2 that the real Pauli group $\mathscr{P}_n^{\mathbb{R}}$ is the subgroup of $\mathscr{P}_n$ whose elements are the n-fold tensor products of elements of $\mathscr{P}_1^{\mathbb{R}}$, which is the subgroup of $\mathscr{P}_1$ generated by X and Z. In other words, $\mathscr{P}_n^{\mathbb{R}}$ is the set of all matrices of $\mathscr{P}_n$ with real coefficients. Recall from Proposition 1.12.vi that $\mathrm{Ps}_a(B)$ is the following abelian subgroup of $\mathrm{Ps}(B)$:

$$\mathrm{Ps}_a(B) = \{(1, \langle w, \cdot \rangle), w \in W\}.$$

We define $\widetilde{\mathrm{Ps}_a(B)_{\widetilde{\chi}}}$ as the subgroup of $\widetilde{\mathrm{Ps}(B)_{\widetilde{\chi}}}$ which consists of the elements of the form $\{(s, (-1)^l \omega_{\widetilde{\chi}}(s, w_s)),\ s \in \mathrm{Ps}_a(B),\ l \in \mathbb{Z}\}$. Similarly, we recall the abelian subgroup

$$\mathrm{ASp}_a(B_W) = \{(1, \langle w, \cdot \rangle_W), w \in W\}$$

of $\mathrm{ASp}(B_W)$ from Proposition 2.7.viii, and we define $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ as the subgroup of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ which consists of the elements $\{(s, i^l \omega_{\widetilde{\chi}}(s, w_s)),\ s \in \mathrm{ASp}_a(B_W),\ l \in \mathbb{Z}\}$.

Remark 4.8. By definition of $\mathrm{Ps}_a(B)$, every element of $\mathrm{Ps}_a(B)$ is uniquely written as $s_w = (1, \langle w, \cdot \rangle)$, where $w \in W$. By Lemma 1.27.iv, the element w_{s_w} which satisfies (37) is unique and is equal to w. Hence we can denote the operator $\omega_{\widetilde{\chi}}(s_w, w)$ of the projective Weil representation of $\mathrm{Ps}_a(B)$ by $\omega_{\widetilde{\chi}}(s_w)$. Since $\gamma(1) \in \{\pm 1\}$ by (48), every element of $\widetilde{\mathrm{Ps}_a(B)_{\widetilde{\chi}}}$

can uniquely be written under the form $(s_w, (-1)^l\gamma(1)\omega_{\widetilde{\chi}}(s_w))$. In the same way, since every element of $\mathrm{ASp}_a(B_W)$ is uniquely written as $s_w = (1, \langle w, \cdot\rangle_W)$, where $w \in W$, and the element w_{s_w} which satisfies (100) is unique and equal to w by Lemma 2.20.iii, then every element of $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ can uniquely be written under the form $(s_w, i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w))$, where $\gamma(1) \in \{\pm 1, \pm i\}$ by (116), and $\omega_{\widetilde{\chi_k}}(s_w)$ denotes the operator $\omega_{\widetilde{\chi_k}}(s_w, w)$ of the projective Weil representation of $\mathrm{ASp}_a(B_W)$.

We will first prove that the Weil representations $\widetilde{\omega}_{\widetilde{\chi}}$ of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ defined in Proposition 1.39, and $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ defined in Proposition 2.32, allow us to identify the Pauli groups $\mathscr{P}_n^{\mathbb{R}}$ inside $\widetilde{\mathrm{Ps}(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$, and $\mathscr{P}_n$ inside $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, where B_W corresponds to either B_{Weil} or $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$.

Proposition 4.9. ***i.*** *Suppose that $B = B_{\mathrm{Weil}}$. Then the restriction of the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ of $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$ to $\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}$ defines a group isomorphism between $\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}$ and the real Pauli group $\mathscr{P}_n^{\mathbb{R}}$. More precisely, write the elements of $\mathrm{Ps}_a(B)$ as $s_w = (1, \langle w, \cdot\rangle)$ with $w \in W$. Then this isomorphism is explicitly given by*

$$\begin{aligned}&\widetilde{\omega}_{\widetilde{\chi}|\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}} : \widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}} \to \mathscr{P}_n^{\mathbb{R}}\\ &(s_w, (-1)^l\gamma(1)\omega_{\widetilde{\chi}}(s_w)) \mapsto (-1)^l\widetilde{\chi}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0)X_{(1)}^{\langle e_1,w\rangle}Z_{(1)}^{\langle w,f_1\rangle}\cdots X_{(n)}^{\langle e_n,w\rangle}Z_{(n)}^{\langle w,f_n\rangle}.\end{aligned} \tag{440}$$

ii. *The restriction of the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ of $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$ to $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ defines a group isomorphism between $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ and the Pauli group $\mathscr{P}_n$. More precisely, write the elements of $\mathrm{ASp}_a(B_W)$ as $s_w = (1, \langle w, \cdot\rangle_W)$ with $w \in W$. If $B = B_{\mathrm{Weil}}$, this isomorphism is explicitly given by*

$$\begin{aligned}\widetilde{\omega}_{\widetilde{\chi_k}|\,\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}} &: \mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}} \to \mathscr{P}_n\\ (s_w, i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w)) &\mapsto i^l\widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0)\\ &\qquad X_{(1)}^{\langle e_1,w\rangle}Z_{(1)}^{\langle w,f_1\rangle}\cdots X_{(n)}^{\langle e_n,w\rangle}Z_{(n)}^{\langle w,f_n\rangle},\end{aligned} \tag{441}$$

and if $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ it is given by

$$\begin{aligned}\widetilde{\omega}_{\widetilde{\chi_k}|\,\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}} &: \mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}} \to \mathscr{P}_n\\ (s_w, i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w)) &\mapsto i^l\widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0)\\ &\qquad X_{(1)}^{\langle e_1,w\rangle}Z_{(1)}^{\langle e_1-f_1,w\rangle}\cdots X_{(n)}^{\langle e_n,w\rangle}Z_{(n)}^{\langle e_n-f_n,w\rangle}.\end{aligned} \tag{442}$$

Proof. **i.** Suppose that $B = B_{\mathrm{Weil}}$. By Remark 4.8 and since $\gamma(1) \in \{\pm 1\}$ by (48), every element of $\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}$ can uniquely be written under the form $(s_w, (-1)^l\gamma(1)\omega_{\widetilde{\chi}}(s_w))$. Then, by Proposition 1.39, for every $l \in \mathbb{Z}$,

$$\widetilde{\omega}_{\widetilde{\chi}}(s_w, (-1)^l\gamma(1)\omega_{\widetilde{\chi}}(s_w)) = (-1)^l\gamma(1)\omega_{\widetilde{\chi}}(s_w) = (-1)^l\rho_{\widetilde{\chi}}(w, 0). \tag{443}$$

Now, we decompose $w = w_1 + \cdots + w_n$ in the direct sum $W = W_1 \oplus \cdots \oplus W_n$, and we consider the symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of W, where $\{e_i, f_i\}$ is a basis the symplectic space W_i. Let $1 \leqslant i \leqslant n$. Recall that by (28), for every $z_i \in Y_i$, if we write $w_i = x_i + y_i$, where $x_i \in X_i$ and $y_i \in Y_i$, then

$$\rho_{\widetilde{\chi}^{(i)}}(w_i, 0)\delta_{z_i} = \widetilde{\chi}^{(i)}(x_i, \langle x_i, z_i\rangle_i - B_i(y_i, x_i + z_i))\delta_{z_i+y_i}. \tag{444}$$

In particular, this implies that in the ordered basis $\{\delta_0, \delta_{f_i}\}$ of Y_i,

$$\begin{cases} \rho_{\widetilde{\chi}^{(i)}}(e_i, 0) & = \widetilde{\chi}^{(i)}(e_i, 0)Z \\ \rho_{\widetilde{\chi}^{(i)}}(f_i, 0) & = X, \end{cases} \tag{445}$$

where $\widetilde{\chi}^{(i)}(e_i, 0) \in \{\pm 1\}$ by Lemma 1.15.ii. Since

$$\begin{aligned} \rho_{\widetilde{\chi}^{(i)}}(f_i, 0)\rho_{\widetilde{\chi}^{(i)}}(e_i, 0) = \rho_{\widetilde{\chi}^{(i)}}(e_i + f_i, B_{\text{Weil}}(f_i, e_i)) &= \chi(B_{\text{Weil}}(f_i, e_i))\rho_{\widetilde{\chi}^{(i)}}(e_i + f_i, 0) \\ &= \rho_{\widetilde{\chi}^{(i)}}(e_i + f_i, 0), \end{aligned}$$

we obtain that

$$\begin{aligned} \rho_{\widetilde{\chi}^{(i)}}(w_i, 0) = \rho_{\widetilde{\chi}^{(i)}}(\langle w_i, f_i\rangle_i e_i + \langle e_i, w_i\rangle_i f_i, 0) &= \rho_{\widetilde{\chi}^{(i)}}(\langle e_i, w_i\rangle_i f_i, 0)\rho_{\widetilde{\chi}^{(i)}}(\langle w_i, f_i\rangle_i e_i, 0) \\ &= \widetilde{\chi}^{(i)}(\langle w_i, f_i\rangle_i e_i, 0)X^{\langle e_i, w_i\rangle_i} Z^{\langle w_i, f_i\rangle_i}. \end{aligned} \tag{446}$$

Thus, by Lemma 4.4 and since the restriction of B_{Weil} to $X \times X$ vanishes,

$$\begin{aligned} \rho_{\widetilde{\chi}}(w, 0) &= \rho_{\widetilde{\chi}}(w_1 + \cdots + w_n, 0) \\ &= T^{-1} \circ \left(\rho_{\widetilde{\chi}^{(1)}}(w_1, 0) \otimes \cdots \otimes \rho_{\widetilde{\chi}^{(n)}}(w_n, 0)\right) \circ T \\ &= \widetilde{\chi}^{(1)}(\langle w_1, f_1\rangle_1 e_1, 0) \cdots \widetilde{\chi}^{(n)}(\langle w_n, f_n\rangle_n e_n, 0) X^{\langle e_1, w_1\rangle_1} Z^{\langle w_1, f_1\rangle_1} \otimes \cdots \otimes X^{\langle e_n, w_n\rangle_n} Z^{\langle w_n, f_n\rangle_n} \\ &= \widetilde{\chi}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) X^{\langle e_1, w\rangle} Z^{\langle w, f_1\rangle} \otimes \cdots \otimes X^{\langle e_n, w\rangle} Z^{\langle w, f_n\rangle} \\ &= \widetilde{\chi}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle w, f_1\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle w, f_n\rangle}, \end{aligned} \tag{447}$$

where $\widetilde{\chi}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) \in \{\pm 1\}$. Hence by (443),

$$\begin{aligned} &\widetilde{\omega}_{\widetilde{\chi}}(s_w, (-1)^l \gamma(1)\omega_{\widetilde{\chi}}(s_w)) \\ &= (-1)^l \widetilde{\chi}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle w, f_1\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle w, f_n\rangle} \in \mathscr{P}_n^{\mathbb{R}}. \end{aligned} \tag{448}$$

Since the Weil representation $\widetilde{\omega}_{\widetilde{\chi}}$ is faithful by Proposition 1.39, and the groups $\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}$ and $\mathscr{P}_n^{\mathbb{R}}$ have the same order 2^{2n+1}, this is enough to conclude that the restriction of $\widetilde{\omega}_{\widetilde{\chi}}$ to $\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}$ defines a group isomorphism between $\widetilde{\mathrm{Ps}_a(B)}_{\widetilde{\chi}}$ and $\mathscr{P}_n^{\mathbb{R}}$.

ii. This proof is very similar to the one of the previous point, except that we treat the two possibilities $B = B_{\text{Weil}}$ and $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$. In fact, as mentioned in Remark 4.8, every element of $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ can uniquely be written under the form $(s_w, i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w))$, where $s_w = (1, \langle w, \cdot\rangle_W)$ and $\gamma(1) \in \{\pm 1, \pm i\}$. Then, by Proposition 2.32, for every $l \in \mathbb{Z}$,

$$\widetilde{\omega}_{\widetilde{\chi_k}}(s_w, i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w)) = i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w) = i^l\rho_{\widetilde{\chi_k}}(w, 0). \tag{449}$$

Now by (96), for every $1 \leqslant i \leqslant n$, and for every $z_i \in Y_i$, if we write $w = w_1 + \cdots + w_n \in W_1 \oplus \cdots \oplus W_n$ and $w_i = x_i + y_i$, where $x_i \in X_i$ and $y_i \in Y_i$, then

$$\rho_{\widetilde{\chi_k}^{(i)}}(w_i, 0)\delta_{z_i} = \widetilde{\chi_k}^{(i)}(x_i, \langle x_i, z_i\rangle_{W_i} - B_{W_i}(y_i, x_i + z_i))\delta_{z_i+y_i}. \tag{450}$$

In particular, this implies that in the ordered basis $\{\delta_0, \delta_{f_i}\}$ of Y_i,

$$\begin{cases} \rho_{\widetilde{\chi_k}^{(i)}}(e_i, 0) & = \widetilde{\chi_k}^{(i)}(e_i, 0)Z \\ \rho_{\widetilde{\chi_k}^{(i)}}(f_i, 0) & = \begin{cases} X & \text{if } B = B_{\text{Weil}} \\ XZ & \text{if } B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}, \end{cases} \end{cases} \tag{451}$$

where $\widetilde{\chi_k}^{(i)}(e_i, 0) \in \{\pm 1\}$ if $B = B_{\text{Weil}}$ and $\widetilde{\chi_k}^{(i)}(e_i, 0) \in \{\pm i\}$ if $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$ by the paragraph which follows (95). Then we prove as in (446) that

$$\begin{aligned} \rho_{\widetilde{\chi_k}^{(i)}}(w_i, 0) &= \rho_{\widetilde{\chi_k}^{(i)}}(\langle w_i, f_i\rangle_i e_i + \langle e_i, w_i\rangle_i f_i, 0) \\ &= \rho_{\widetilde{\chi_k}^{(i)}}(\langle e_i, w_i\rangle_i f_i, 0)\rho_{\widetilde{\chi_k}^{(i)}}(\langle w_i, f_i\rangle_i e_i, 0) \\ &= \begin{cases} \widetilde{\chi_k}^{(i)}(\langle w_i, f_i\rangle_i e_i, 0)X^{\langle e_i, w_i\rangle_i} Z^{\langle w_i, f_i\rangle_i} & \text{if } B = B_{\text{Weil}} \\ \widetilde{\chi_k}^{(i)}(\langle w_i, f_i\rangle_i e_i, 0)(XZ)^{\langle e_i, w_i\rangle_i} Z^{\langle w_i, f_i\rangle_i} & \text{if } B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \end{cases} \\ &= \begin{cases} \widetilde{\chi_k}^{(i)}(\langle w_i, f_i\rangle_i e_i, 0)X^{\langle e_i, w_i\rangle_i} Z^{\langle w_i, f_i\rangle_i} & \text{if } B = B_{\text{Weil}} \\ \widetilde{\chi_k}^{(i)}(\langle w_i, f_i\rangle_i e_i, 0)X^{\langle e_i, w_i\rangle_i} Z^{\langle e_i - f_i, w_i\rangle_i} & \text{if } B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}. \end{cases} \end{aligned} \tag{452}$$

Then we prove exactly as in (447) that for $B = B_{\text{Weil}}$,

$$\rho_{\widetilde{\chi_k}}(w, 0) = \widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0)X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle w, f_1\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle w, f_n\rangle}, \tag{453}$$

where $\widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) \in \{\pm 1\}$.
For $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$, since $B_W(e_i, e_j) = 0$ for $i \neq j$, we obtain that

$$\begin{aligned} &\rho_{\widetilde{\chi_k}}(w, 0) \\ &= \widetilde{\chi_k}^{(1)}(\langle w_1, f_1\rangle_1 e_1, 0) \cdots \widetilde{\chi_k}^{(n)}(\langle w_n, f_n\rangle_n e_n, 0)X^{\langle e_1, w_1\rangle_1} Z^{\langle e_1 - f_1, w_1\rangle_1} \otimes \cdots \otimes X^{\langle e_n, w_n\rangle_n} Z^{\langle e_n - f_n, w_n\rangle_n} \\ &= \widetilde{\chi_k}(\langle w, f_1\rangle e_1, 0) \cdots \widetilde{\chi_k}(\langle w, f_n\rangle e_n, 0)X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle e_1 - f_1, w\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle e_n - f_n, w\rangle} \\ &= \widetilde{\chi_k}\left(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, \sum_{1 \leqslant i < j \leqslant n} B_W(\langle w, f_i\rangle e_i, \langle w, f_j\rangle e_j)\right) \\ &\quad X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle e_1 - f_1, w\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle e_n - f_n, w\rangle} \\ &= \widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0)X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle e_1 - f_1, w\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle e_n - f_n, w\rangle}, \end{aligned} \tag{454}$$

where $\widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) \in \{\pm 1, \pm i\}$. Hence by (449),

$$\begin{aligned}
&\widetilde{\omega}_{\widetilde{\chi_k}}(s_w, i^l\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_w)) \\
&= \begin{cases} i^l\widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle w, f_1\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle w, f_n\rangle} \\ \text{if } B = B_{\text{Weil}} \\ i^l\widetilde{\chi_k}(\langle w, f_1\rangle e_1 + \cdots + \langle w, f_n\rangle e_n, 0) X_{(1)}^{\langle e_1, w\rangle} Z_{(1)}^{\langle e_1 - f_1, w\rangle} \cdots X_{(n)}^{\langle e_n, w\rangle} Z_{(n)}^{\langle e_n - f_n, w\rangle} \\ \text{if } B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \end{cases} \\
&\in \mathscr{P}_n.
\end{aligned} \tag{455}$$

Since the Weil representation $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful by Proposition 2.32, and the groups $\mathscr{P}_n$ and $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ have the same order 2^{2n+2}, we conclude that the restriction of $\widetilde{\omega}_{\widetilde{\chi_k}}$ to $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ defines a group isomorphism between $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ and $\mathscr{P}_n$.

□

Proposition 4.9 enables us to recognize the Pauli groups $\mathscr{P}_n^{\mathbb{R}}$ and $\mathscr{P}_n$ as subgroups of $\widetilde{\mathrm{Ps}(B_{\text{Weil}})}_{\widetilde{\chi}}$ and $\mathrm{AMp}(B_W)_{\widetilde{\chi_k}}$, respectively. Recall from the introduction of chapter 4 that in the stabilizer formalism, the codespace C of a quantum error-correcting code is the set of quantum states which are fixed by a stabilizer group S, where S is an abelian subgroup of $\mathscr{P}_n$ which does not contain $-I$. Therefore, we can use Proposition 4.9 to express stabilizer groups in terms of elements of $\widetilde{\mathrm{Ps}_a(B_{\text{Weil}})}_{\widetilde{\chi}}$ and $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$. We will now translate the generators of some widely used stabilizer codes in the language of Weil representations. Stabilizer generators are the operators we measure in order to detect the errors. In each of the following tables, the first column provides the list of the stabilizer generators in the real Pauli group $\mathscr{P}_n^{\mathbb{R}}$, the second column gives the associated elements in $\widetilde{\mathrm{Ps}_a(B_{\text{Weil}})}_{\widetilde{\chi}}$ and $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ for $B = B_{\text{Weil}}$, and the last column gives the associated elements in $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ for $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$.

i. The three-qubit bit flip code. This code allows to correct a bit flip (X) error on a single qubit, by encoding a logical qubit into three physical qubits.

$\mathscr{P}_n^{\mathbb{R}}$	$\widetilde{\mathrm{Ps}_a(B_{\text{Weil}})}_{\widetilde{\chi}}$ for $B = B_{\text{Weil}}$ $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$	$\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ for $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$
$Z_{(1)}Z_{(2)}$	$(s_{e_1+e_2}, \widetilde{\chi}(e_1+e_2)\gamma(1)\omega_{\widetilde{\chi}}(s_{e_1+e_2}))$ $(s_{e_1+e_2}, \widetilde{\chi_k}(e_1+e_2)\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_{e_1+e_2}))$	$(s_{e_1+e_2}, \widetilde{\chi_k}(e_1+e_2)^{-1}\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_{e_1+e_2}))$
$Z_{(2)}Z_{(3)}$	$(s_{e_2+e_3}, \widetilde{\chi}(e_2+e_3)\gamma(1)\omega_{\widetilde{\chi}}(s_{e_2+e_3}))$ $(s_{e_2+e_3}, \widetilde{\chi_k}(e_2+e_3)\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_{e_2+e_3}))$	$(s_{e_2+e_3}, \widetilde{\chi_k}(e_2+e_3)^{-1}\gamma(1)^{-1}\omega_{\widetilde{\chi_k}}(s_{e_2+e_3}))$

ii. The three-qubit phase flip code. This code allows to correct a phase flip (Z) error on a single qubit, by encoding a logical qubit into three physical qubits.

$\mathscr{P}_n^{\mathbb{R}}$	$\widetilde{\mathrm{Ps}_a(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$ for $B = B_{\mathrm{Weil}}$ $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$	$\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ for $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$
$X_{(1)}X_{(2)}$	$(s_{f_1+f_2}, \gamma(1)\omega_{\widetilde{\chi}}(s_{f_1+f_2}))$ $(s_{f_1+f_2}, \gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{f_1+f_2}))$	$(s_{e_1+e_2+f_1+f_2}, \widetilde{\chi_k}(e_1+e_2)^{-1}\gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{e_1+e_2+f_1+f_2}))$
$X_{(2)}X_{(3)}$	$(s_{f_2+f_3}, \gamma(1)\omega_{\widetilde{\chi}}(s_{f_2+f_3}))$ $(s_{f_2+f_3}, \gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{f_2+f_3}))$	$(s_{e_2+e_3+f_2+f_3}, \widetilde{\chi_k}(e_2+e_3)^{-1}\gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{e_2+e_3+f_2+f_3}))$

iii. The five-qubit code. This is the smallest possible code that corrects an arbitrary single-qubit error, in the sense that it uses the smallest number of physical qubits required to encode one logical qubit.

$\mathscr{P}_n^{\mathbb{R}}$	$\widetilde{\mathrm{Ps}_a(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$ for $B = B_{\mathrm{Weil}}$ $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$	$\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ for $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$
$X_{(1)}Z_{(2)}Z_{(3)}X_{(4)}$	$(s_{e_2+e_3+f_1+f_4}, \widetilde{\chi}(e_2+e_3)\gamma(1)$ $\omega_{\widetilde{\chi}}(s_{e_2+e_3+f_1+f_4}))$ $(s_{e_2+e_3+f_1+f_4}, \widetilde{\chi_k}(e_2+e_3)\gamma(1)^{-1}$ $\omega_{\widetilde{\chi}_k}(s_{e_2+e_3+f_1+f_4}))$	$(s_{e_1+e_2+e_3+e_4+f_1+f_4}, \widetilde{\chi_k}(e_1+e_2+e_3+e_4)^{-1}$ $\gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{e_1+e_2+e_3+e_4+f_1+f_4}))$
$X_{(2)}Z_{(3)}Z_{(4)}X_{(5)}$	$(s_{e_3+e_4+f_2+f_5}, \widetilde{\chi}(e_3+e_4)\gamma(1)$ $\omega_{\widetilde{\chi}}(s_{e_3+e_4+f_2+f_5}))$ $(s_{e_3+e_4+f_2+f_5}, \widetilde{\chi_k}(e_3+e_4)\gamma(1)^{-1}$ $\omega_{\widetilde{\chi}_k}(s_{e_3+e_4+f_2+f_5}))$	$(s_{e_2+e_3+e_4+e_5+f_2+f_5}, \widetilde{\chi_k}(e_2+e_3+e_4+e_5)^{-1}$ $\gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{e_2+e_3+e_4+e_5+f_2+f_5}))$
$X_{(1)}X_{(3)}Z_{(4)}Z_{(5)}$	$(s_{e_4+e_5+f_1+f_3}, \widetilde{\chi}(e_4+e_5)\gamma(1)$ $\omega_{\widetilde{\chi}}(s_{e_4+e_5+f_1+f_3}))$ $(s_{e_4+e_5+f_1+f_3}, \widetilde{\chi_k}(e_4+e_5)\gamma(1)^{-1}$ $\omega_{\widetilde{\chi}_k}(s_{e_4+e_5+f_1+f_3}))$	$(s_{e_1+e_3+e_4+e_5+f_1+f_3}, \widetilde{\chi_k}(e_1+e_3+e_4+e_5)^{-1}$ $\gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{e_1+e_3+e_4+e_5+f_1+f_3}))$
$Z_{(1)}X_{(2)}X_{(4)}Z_{(5)}$	$(s_{e_1+e_5+f_2+f_4}, \widetilde{\chi}(e_1+e_5)\gamma(1)$ $\omega_{\widetilde{\chi}}(s_{e_1+e_5+f_2+f_4}))$ $(s_{e_1+e_5+f_2+f_4}, \widetilde{\chi_k}(e_1+e_5)\gamma(1)^{-1}$ $\omega_{\widetilde{\chi}_k}(s_{e_1+e_5+f_2+f_4}))$	$(s_{e_1+e_2+e_4+e_5+f_2+f_4}, \widetilde{\chi_k}(e_1+e_2+e_4+e_5)^{-1}$ $\gamma(1)^{-1}\omega_{\widetilde{\chi}_k}(s_{e_1+e_2+e_4+e_5+f_2+f_4}))$

We may think that we could stop here with the five-qubit code, since it enables us to correct any error on a single logical qubit with the minimal number of physical qubits. However, we mention also the two following other examples of stabilizer codes which allow us to correct an arbitrary error on a single qubit.

iv. The seven-qubit Steane code. This code protects against arbitrary single-qubit errors, using seven physical qubits to encode one logical qubit. It belongs to the class of Calderbank-Shor-Steane (CSS) codes. Despite requiring more physical qubits than the

five-qubit code, this code is easier to implement and it is of better use for large-scale computations. The six generators of the stabilizer group of the seven qubit Steane code are $X_{(4)}X_{(5)}X_{(6)}X_{(7)}$, $X_{(2)}X_{(3)}X_{(6)}X_{(7)}$, $X_{(1)}X_{(3)}X_{(5)}X_{(7)}$, $Z_{(4)}Z_{(5)}Z_{(6)}Z_{(7)}$, $Z_{(2)}Z_{(3)}Z_{(6)}Z_{(7)}$ and $Z_{(1)}Z_{(3)}Z_{(5)}Z_{(7)}$. To save some space, we will not write explicitly the correspondence between these generators and the elements of $\widetilde{\mathrm{Ps}_a(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$ and $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$, but the pattern follows the same structure as for the previous tables.

v. **The nine-qubit Shor code.** This code is historically important because it is the first example of a quantum error-correcting code which allows us to correct an arbitrary error on a single qubit. It was introduced by Shor in 1995. The stabilizer group of this code consists of the eight following generators: $Z_{(1)}Z_{(2)}$, $Z_{(2)}Z_{(3)}$, $Z_{(4)}Z_{(5)}$, $Z_{(5)}Z_{(6)}$, $Z_{(7)}Z_{(8)}$, $Z_{(8)}Z_{(9)}$, $X_{(1)}X_{(2)}X_{(3)}X_{(4)}X_{(5)}X_{(6)}$ and $X_{(4)}X_{(5)}X_{(6)}X_{(7)}X_{(8)}X_{(9)}$. The correspondence with the elements $\widetilde{\mathrm{Ps}_a(B_{\mathrm{Weil}})}_{\widetilde{\chi}}$ and $\mathrm{AMp}_a(B_W)_{\widetilde{\chi_k}}$ can also be obtained in the same way as for the tables above.

4.2.2 The Hadamard(-Rademacher-Walsh) transform

For every $1 \leqslant i \leqslant n$, all the operators $\omega_{\widetilde{\chi}^{(i)}}(s, w_s)$, $\omega_{\widetilde{\chi_k}^{(i)}}(s, w_s)$ and $\omega^{(i)}_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ of the projective Weil representations of $\mathrm{Ps}(B_i)$, $\mathrm{ASp}(B_{W_i})$ and $\mathrm{Sp}(\widetilde{W}_i)$ on $\mathrm{L}^2(Y_i)$ are computed in chapter 3. They correspond to one-qubit Clifford gates. Hence, by Proposition 4.7, we can compute and express in terms of operators of the projective Weil representations $\omega_{\widetilde{\chi}}$ of $\mathrm{Ps}(B)$, $\omega_{\widetilde{\chi_k}}$ of $\mathrm{ASp}(B_W)$ or ω_{χ_k} of $\mathrm{Sp}(\widetilde{W})$, every n-qubit Clifford gate which is an n-fold tensor product of these single-qubit Clifford gates. In particular, as shown below, we recover in this way the Hadamard(-Rademacher-Walsh) transform. The Hadamard(-Rademacher-Walsh) transform is a unitary operator on the Hilbert space $\mathrm{L}^2(Y)$ of dimension 2^n, where Y stands for the set of basis quantum states of a system of n qubits. In terms of matrices, it is defined by

$$H = \underbrace{H_1 \otimes \cdots \otimes H_1}_{n \text{ times}}, \quad \text{where each } H_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ acts on the } i\text{-th qubit.} \tag{456}$$

This operator plays a prominent role in Quantum Information Theory. If we look at its definition, we see that H_1 sends the basis states $|0\rangle$ to $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ to $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Hence we observe first that H_1 performs a change of bases between $(|0\rangle, |1\rangle)$ and $(|+\rangle, |-\rangle)$, which may be useful for example to easier detect some errors in quantum error-correcting codes. Secondly, the Hadamard gate creates superposition of basis states with equal weights, starting from basis states. We already mentioned that superposition of states is a key principle since it should allow quantum computers to explore different paths at the same time. Finally, we observe that the Hadamard gate belongs to the Clifford group, i.e. it normalizes the Pauli group $\mathscr{P}_n$. These properties make the Hadamard gate a prime ingredient for many quantum algorithms, such as Deutsch-Josza's, Grover's, Shor's, and the Quantum Fourier Transform algorithms.

Remark 4.10. Let χ be the unique nontrivial additive character of $\mathbb{F}_2$. The Fourier transform

$\mathscr{F}_1 f$ of a function $f \in \mathrm{L}^2(\mathbb{F}_2)$ is the function on $\mathbb{F}_2$ defined by

$$(\mathscr{F}_1 f)(y) = \frac{1}{\sqrt{2}} \sum_{x \in \mathbb{F}_2} f(x)\chi(xy) \tag{457}$$

for all $y \in \mathbb{F}_2$. Hence the matrix of $\mathscr{F}_1$ with respect to the ordered basis $\{\delta_0, \delta_1\}$ is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Consider the abelian group $\otimes^n \mathbb{F}_2 = \underbrace{\mathbb{F}_2 \otimes \cdots \otimes \mathbb{F}_2}_{n \text{ times}}$. For $x = x_1 \otimes \cdots \otimes x_n \in \otimes^n \mathbb{F}_2$ and $y = y_1 \otimes \cdots \otimes y_n \in \otimes^n \mathbb{F}_2$, set

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n \in \mathbb{F}_2. \tag{458}$$

Then every character of $\otimes^n \mathbb{F}_2$ is of the form χ_y for $y = (y_1, \ldots, y_n) \in \otimes^n \mathbb{F}_2$, where

$$\chi_y(x) = \chi(x \cdot y) \tag{459}$$

for $x = (x_1, \ldots, x_n) \in \otimes^n \mathbb{F}_2$. Hence the Fourier transform $\mathscr{F} f$ of a function $f \in \mathrm{L}^2(\otimes^n \mathbb{F}_2)$ is the function on $\otimes^n \mathbb{F}_2$ given by

$$(\mathscr{F} f)(y) = \frac{1}{2^{\frac{n}{2}}} \sum_{x \in \otimes^n \mathbb{F}_2} f(x)\chi(x \cdot y) \tag{460}$$

for all $y \in \otimes^n \mathbb{F}_2$. In particular, if $f = f_1 \otimes \cdots \otimes f_n$, then

$$\mathscr{F} f = \mathscr{F}_1 f_1 \otimes \cdots \otimes \mathscr{F}_1 f_n. \tag{461}$$

In other words, $\mathscr{F} = \underbrace{\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_1}_{n \text{ times}}$. Furthermore, H is the matrix of $\mathscr{F}$ with respect to the basis $\delta_{i_1} \otimes \cdots \otimes \delta_{i_n}$, where $i_k \in \{0, 1\}$ for all $k \in \{1, \ldots, n\}$.

Proposition 4.11. *We keep the notations from section 4.1. Let $\widetilde{\sigma} \in \mathrm{Sp}(\widetilde{W})$ and $\widetilde{\tau} \in \mathrm{Sp}(\widetilde{W})$ be such that*

$$\begin{cases} \widetilde{\sigma}(\widetilde{x}, \widetilde{y}) = (\widetilde{y}, -\widetilde{x}) & \text{for all } \widetilde{w} = (\widetilde{x}, \widetilde{y}) \in \widetilde{W} = \widetilde{X} \oplus \widetilde{Y}, \quad \text{i.e.} \quad \widetilde{\sigma} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \\ \widetilde{\tau}(\widetilde{x}, \widetilde{y}) = (\widetilde{x}, \widetilde{x} + \widetilde{y}) & \text{for all } \widetilde{w} = (\widetilde{x}, \widetilde{y}) \in \widetilde{W} = \widetilde{X} \oplus \widetilde{Y}, \quad \text{i.e.} \quad \widetilde{\tau} = \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix}. \end{cases} \tag{462}$$

Then $\widetilde{\sigma}$ and $\widetilde{\tau}$ belong to $\mathrm{Sp}^{\oplus}(\widetilde{W})$. More precisely, $\widetilde{\sigma} = (\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ and $\widetilde{\tau} = (\widetilde{\tau}_1, \ldots, \widetilde{\tau}_n)$, where $\widetilde{\sigma}_i, \widetilde{\tau}_i \in \mathrm{Sp}(\widetilde{W}_i)$ are given by $\widetilde{\sigma}_i : (\widetilde{x}_i, \widetilde{y}_i) \mapsto (\widetilde{y}_i, -\widetilde{x}_i)$ and $\widetilde{\tau}_i : (\widetilde{x}_i, \widetilde{y}_i) \mapsto (\widetilde{x}_i, \widetilde{x}_i + \widetilde{y}_i)$ for all $(\widetilde{x}_i, \widetilde{y}_i) \in \widetilde{W}_i$, i.e. $\widetilde{\sigma}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\widetilde{\tau}_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $1 \leqslant i \leqslant n$, respectively. We set $\sigma_i = \pi(\widetilde{\sigma}_i)$ and $\tau_i = \pi(\widetilde{\tau}_i)$ for all i, $\sigma = \pi(\widetilde{\sigma})$ and $\tau = \pi(\widetilde{\tau})$, where π is defined in Proposition 2.7.iii.

i. *We define the following elements* $s_i \in \mathrm{Ps}(B)$ *if* (438) *is satisfied and* $r_i \in \mathrm{Ps}(B)$ *if* (439) *is satisfied:*

$$\begin{cases} s_i & = \begin{cases} (\sigma_i, f_{\sigma_i}) & \text{if } \widetilde{\chi}^{(i)}(e_i, 0) = 1 \\ (\sigma_i, w \mapsto f_{\sigma_i}(w) + x_i + y_i) & \text{if } \widetilde{\chi}^{(i)}(e_i, 0) = -1 \end{cases} \\ r_i & = (\tau_i, w \mapsto f_{\tau_i}(w) + y_i), \end{cases} \tag{463}$$

where $w = \sum_{1 \leqslant i \leqslant n} (x_i e_i + y_i f_i) \in W = W_1 \oplus \cdots \oplus W_n$ *and by* (419), σ_i *and* τ_i *are considered as elements of* $\mathrm{Ps}(B)$. *We define also* $s = s_1 \cdots s_n$ *and* $r = r_1 \cdots r_n$. *Then* s *belongs to* $\mathrm{Ps}^{\oplus}(B)$ *for* $B = B_{\mathrm{Weil}}$ *and* r *belongs to* $\mathrm{Ps}^{\oplus}(B)$ *for* $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$. *If* $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ *and* $w_r = (w_{r_1}^{(1)}, \ldots, w_{r_n}^{(n)})$, *then the operators* $\omega_{\widetilde{\chi}}(s, w_s)$ *and* $\omega_{\widetilde{\chi}}(r, w_r)$ *correspond, up to a unitary factor, to the Hadamard(-Rademacher-Walsh) transform* H.

ii. *We define the following elements* $s_i \in \mathrm{ASp}(B_W)$ *if* (438) *is satisfied and* $r_i \in \mathrm{ASp}(B_W)$ *if* (439) *is satisfied:*

$$\begin{cases} s_i & = \begin{cases} (\sigma_i, f_{\widetilde{\sigma}_i}) & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = 1 \\ (\sigma_i, w \mapsto f_{\widetilde{\sigma}_i}(w) + 2x_i + 2y_i) & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = -1 \end{cases} \\ r_i & = (\tau_i, w \mapsto f_{\widetilde{\tau}_i}(w) + 2x_i + 2y_i), \end{cases} \tag{464}$$

where $w = \sum_{1 \leqslant i \leqslant n} (x_i e_i + y_i f_i) \in W = W_1 \oplus \cdots \oplus W_n$ *and by* (419), σ_i *and* τ_i *are considered as elements of* $\mathrm{ASp}(B_W)$. *We define also* $s = s_1 \cdots s_n$ *and* $r = r_1 \cdots r_n$. *Then* s *belongs to* $\mathrm{ASp}^{\oplus}(B_W)$ *for* $B = B_{\mathrm{Weil}}$ *and* r *belongs to* $\mathrm{ASp}^{\oplus}(B_W)$ *for* $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$. *If* $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ *and* $w_r = (w_{r_1}^{(1)}, \ldots, w_{r_n}^{(n)})$, *then the operators* $\omega_{\widetilde{\chi_k}}(s, w_s)$ *and* $\omega_{\widetilde{\chi_k}}(r, w_r)$ *correspond, up to a unitary factor, to the Hadamard(-Rademacher-Walsh) transform* H.

iii. *If* $\widetilde{w}_{\widetilde{\sigma}} = (\widetilde{w}_{\widetilde{\sigma}_1}^{(1)}, \ldots, \widetilde{w}_{\widetilde{\sigma}_n}^{(n)})$ *then the operator* $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ *corresponds, up to a unitary factor, to the Hadamard(-Rademacher-Walsh) transform* H.

Proof. **i.** Let $\widetilde{\sigma}$ and $\widetilde{\tau}$ be defined as in (462). Then $\widetilde{\sigma} = (\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ and $\widetilde{\tau} = (\widetilde{\tau}_1, \ldots, \widetilde{\tau}_n)$ belong to $\mathrm{Sp}(\widetilde{W})$ by (90). Moreover, $\sigma = (\sigma_1, \ldots, \sigma_n)$ belongs to $\mathrm{O}^{\oplus}(Q)$ if $B = B_{\mathrm{Weil}}$ and $\tau = (\tau_1, \ldots, \tau_n)$ belongs to $\mathrm{O}^{\oplus}(Q)$ if $B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$. In fact, for all $1 \leqslant i \leqslant n$, for all $w_i = (x_i, y_i) \in W_i$, since B takes its values in $\mathbb{F}_2$,

$$\begin{cases} Q_i(\sigma_i(w_i)) = B_i(\sigma_i(w_i), \sigma_i(w_i)) = B_i((y_i, -x_i), (y_i, -x_i)) = -x_i y_i = x_i y_i = Q_i(w_i) \\ \text{if } B = B_{\mathrm{Weil}} \\ Q_i(\tau_i(w_i)) = B_i(\tau_i(w_i), \tau_i(w_i)) = B_i((x_i, x_i + y_i), (x_i, x_i + y_i)) = x_i y_i + x_i + y_i = Q_i(w_i) \\ \text{if } B = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}. \end{cases}$$

Then we have by (426) and (427) that

$$f_\sigma\left(w=\sum_{1\leqslant i\leqslant n}(x_ie_i+y_if_i)\right)=\sum_{1\leqslant i\leqslant n}f_{\sigma_i}(w)=\sum_{1\leqslant i\leqslant n}x_iy_i(B(\sigma_i(e_i),\sigma_i(f_i))-B(e_i,f_i))$$
$$=\sum_{1\leqslant i\leqslant n}x_iy_i,$$

and

$$f_\tau\left(w=\sum_{1\leqslant i\leqslant n}(x_ie_i+y_if_i)\right)=\sum_{1\leqslant i\leqslant n}f_{\tau_i}(w)=\sum_{1\leqslant i\leqslant n}x_iy_i(B(\tau_i(e_i),\tau_i(f_i))-B(e_i,f_i))$$
$$=\sum_{1\leqslant i\leqslant n}x_iy_i$$

where in particular $f_{\sigma_i}(w)=f_{\tau_i}(w)=x_iy_i$. Now, for every $1\leqslant i\leqslant n$, we define the following elements $s_i\in\mathrm{Ps}(B)$ if (438) is satisfied and $r_i\in\mathrm{Ps}(B)$ if (439) is satisfied:

$$\begin{cases}s_i &= \begin{cases}(\sigma_i,f_{\sigma_i}) & \text{if } \widetilde{\chi}^{(i)}(e_i,0)=1\\ (\sigma_i,w\mapsto f_{\sigma_i}(w)+x_i+y_i) & \text{if } \widetilde{\chi}^{(i)}(e_i,0)=-1\end{cases}\\ r_i &= (\tau_i,w\mapsto f_{\tau_i}(w)+y_i),\end{cases}\tag{465}$$

Then $s=s_1\cdots s_n$ and $r=r_1\cdots r_n$ belong to $\mathrm{Ps}^{\oplus}(B)$ by definition, where B is B_{Weil} and $\begin{pmatrix}I_n & I_n\\ 0 & I_n\end{pmatrix}$, respectively. We suppose that $w_s=(w_{s_1}^{(1)},\ldots,w_{s_n}^{(n)})$ and $w_r=(w_{r_1}^{(1)},\ldots,w_{r_n}^{(n)})$. Then by (435),

$$\begin{cases}\omega_{\widetilde{\chi}}(s,w_s) &= \gamma(\sigma)(\gamma^{(1)}(\sigma_1)\cdots\gamma^{(n)}(\sigma_n))^{-1}T^{-1}\circ\left(\omega_{\widetilde{\chi}^{(1)}}(s_1,w_{s_1}^{(1)})\otimes\cdots\otimes\omega_{\widetilde{\chi}^{(n)}}(s_n,w_{s_n}^{(n)})\right)\circ T\\ \omega_{\widetilde{\chi}}(r,w_r) &= \gamma(\tau)(\gamma^{(1)}(\tau_1)\cdots\gamma^{(n)}(\tau_n))^{-1}T^{-1}\circ\left(\omega_{\widetilde{\chi}^{(1)}}(r_1,w_{r_1}^{(1)})\otimes\cdots\otimes\omega_{\widetilde{\chi}^{(n)}}(r_n,w_{r_n}^{(n)})\right)\circ T.\end{cases}\tag{466}$$

Let $i\in\{1,\ldots,n\}$. By (37), since $\mathrm{K}(\sigma_i)=\mathrm{Span}_{\mathbb{F}_2}\{e_i+f_i\}\subseteq W_i$ and $\mathrm{K}(\tau_i)=\mathrm{Span}_{\mathbb{F}_2}\{f_i\}\subseteq W_i$, we have that $w_{s_i}^{(i)}\in\{e_i,f_i\}$, $w_{r_i}^{(i)}\in\{e_i,e_i+f_i\}$. By (40), for any choice of such elements $w_{s_i}^{(i)}$ and $w_{r_i}^{(i)}$, we have that

$$\omega_{\widetilde{\chi}^{(i)}}(s_i,w_{s_i}^{(i)})$$
$$=\begin{cases}\gamma^{(i)}(\sigma_i)(\rho_{\widetilde{\chi}^{(i)}}(e_i)+\rho_{\widetilde{\chi}^{(i)}}(f_i))=\frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}}(\rho_{\widetilde{\chi}^{(i)}}(e_i)+\rho_{\widetilde{\chi}^{(i)}}(f_i)) & \text{if } \widetilde{\chi}^{(i)}(e_i,0)=1\\ \pm\gamma^{(i)}(\sigma_i)(\rho_{\widetilde{\chi}^{(i)}}(e_i)-\rho_{\widetilde{\chi}^{(i)}}(f_i))=\pm\frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}}(\rho_{\widetilde{\chi}^{(i)}}(e_i)-\rho_{\widetilde{\chi}^{(i)}}(f_i)) & \text{if } \widetilde{\chi}^{(i)}(e_i,0)=-1\end{cases}\tag{467}$$

and

$$\omega_{\widetilde{\chi}^{(i)}}(r_i,w_{r_i}^{(i)})=\gamma^{(i)}(\tau_i)(\rho_{\widetilde{\chi}^{(i)}}(e_i)+\rho_{\widetilde{\chi}^{(i)}}(e_i+f_i))=\frac{e^{i\theta^{(i)}(\tau_i)}}{\sqrt{2}}(\rho_{\widetilde{\chi}^{(i)}}(e_i)+\rho_{\widetilde{\chi}^{(i)}}(e_i+f_i)).\tag{468}$$

By (466), this allows us to compute explicitly the expression of the operators $\omega_{\widetilde{\chi}}(s, w_s)$ and $\omega_{\widetilde{\chi}}(r, w_r)$, for any choice of $w_s = (w_{s_1}^{(1)}, \dots, w_{s_n}^{(n)})$ and $w_r = (w_{r_1}^{(1)}, \dots, w_{r_n}^{(n)})$ such that $w_{s_i}^{(i)} \in \{e_i, f_i\}$ and $w_{r_i}^{(i)} \in \{e_i, e_i + f_i\}$ for all $1 \leqslant i \leqslant n$. To simplify the notation for s, we do the computations only in the case in which $\widetilde{\chi}^{(i)}(e_i, 0) = 1$ for all $1 \leqslant i \leqslant n$, but the same method applies in the general case. We obtain

$$\begin{aligned}
\omega_{\widetilde{\chi}}(s, w_s) &= \gamma(\sigma) T^{-1} \circ \left((\rho_{\widetilde{\chi}^{(1)}}(e_1) + \rho_{\widetilde{\chi}^{(1)}}(f_1)) \otimes \cdots \otimes (\rho_{\widetilde{\chi}^{(n)}}(e_n) + \rho_{\widetilde{\chi}^{(n)}}(f_n)) \right) \circ T \\
&= \gamma(\sigma) T^{-1} \circ \left(\sum_{\substack{a_i \in \{e_i, f_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\widetilde{\chi}^{(1)}}(a_1) \otimes \cdots \otimes \rho_{\widetilde{\chi}^{(n)}}(a_n) \right) \circ T \\
&= \gamma(\sigma) \sum_{\substack{a_i \in \{e_i, f_i\} \\ 1 \leqslant i \leqslant n}} T^{-1} \circ \left(\rho_{\widetilde{\chi}^{(1)}}(a_1) \otimes \cdots \otimes \rho_{\widetilde{\chi}^{(n)}}(a_n) \right) \circ T \\
&= \gamma(\sigma) \sum_{\substack{a_i \in \{e_i, f_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\widetilde{\chi}}(a_1 + \cdots + a_n) \qquad \text{by Lemma 4.4} \\
&= \frac{e^{i\theta(\sigma)}}{2^{\frac{n}{2}}} \sum_{\substack{a_i \in \{e_i, f_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\widetilde{\chi}}(a_1 + \cdots + a_n),
\end{aligned} \tag{469}$$

since $\mathrm{I}(\sigma) = \mathrm{I}(\sigma_1) \oplus \cdots \oplus \mathrm{I}(\sigma_n) = \mathrm{Span}_{\mathbb{F}_2}\{e_1 + f_1, \dots, e_n + f_n\}$, and similarly

$$\omega_{\widetilde{\chi}}(r, w_r) = \frac{e^{i\theta(\tau)}}{2^{\frac{n}{2}}} \sum_{\substack{a_i \in \{e_i, e_i + f_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\widetilde{\chi}}(a_1 + \cdots + a_n), \tag{470}$$

since $\mathrm{I}(\tau) = \mathrm{I}(\tau_1) \oplus \cdots \oplus \mathrm{I}(\tau_n) = \mathrm{Span}_{\mathbb{F}_2}\{f_1, \dots, f_n\}$. In terms of matrices, by (467) and (468), $\omega_{\widetilde{\chi}^{(i)}}(s_i, w_{s_i}^{(i)})$ and $\omega_{\widetilde{\chi}^{(i)}}(r_i, w_{r_i}^{(i)})$ are written in the ordered basis $\{\delta_0, \delta_{f_i}\}$ of $\mathrm{L}^2(Y_i)$ as

$$\omega_{\widetilde{\chi}^{(i)}}(s_i, w_{s_i}^{(i)}) = \begin{cases} \frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\theta^{(i)}(\sigma_i)} H_1 & \text{if } \widetilde{\chi}^{(i)}(e_i, 0) = 1 \\ \pm \frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \pm e^{i\theta^{(i)}(\sigma_i)} H_1 & \text{if } \widetilde{\chi}^{(i)}(e_i, 0) = -1, \end{cases} \tag{471}$$

and

$$\omega_{\widetilde{\chi}^{(i)}}(r_i, w_{r_i}^{(i)}) = \frac{e^{i\theta^{(i)}(\tau_i)}}{\sqrt{2}} \widetilde{\chi}^{(i)}(e_i, 0) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\theta^{(i)}(\tau_i)} \widetilde{\chi}^{(i)}(e_i, 0) H_1, \tag{472}$$

where H_1 denotes here the two-dimensional Hadamard matrix, or equivalently the one-qubit Hadamard quantum gate. In particular, by (466), (469) and (470) (since by (471) the result is the same up to a sign if we do not suppose that $\widetilde{\chi}^{(i)}(e_i, 0) = 1$ for all $1 \leqslant i \leqslant n$), this implies that the operators $\omega_{\widetilde{\chi}}(s, w_s)$ and $\omega_{\widetilde{\chi}}(r, w_r)$ are up to a unitary factor the

n-fold tensor product of the one-qubit Hadamard gate:

$$\begin{cases} \omega_{\widetilde{\chi}}(s, w_s) & = \pm e^{i\theta(\sigma)} \underbrace{H_1 \otimes \cdots \otimes H_1}_{n \text{ times}} = \pm e^{i\theta(\sigma)} H \\ \omega_{\widetilde{\chi}}(r, w_r) & = e^{i\theta(\tau)} \prod_{1 \leqslant i \leqslant n} \widetilde{\chi}^{(i)}(e_i, 0) \underbrace{H_1 \otimes \cdots \otimes H_1}_{n \text{ times}} = \widetilde{\chi}(e_1 + \cdots + e_n, 0) e^{i\theta(\tau)} H, \end{cases} \tag{473}$$

i.e. they correspond (up to a unitary factor) to the Hadamard(-Rademacher-Walsh) transform (456).

ii. This proof is very similar to that of the first point above. In fact, we only have to modify the definition of s_i, r_i, s and r as follows. For every $1 \leqslant i \leqslant n$, we have by (429) and (430) that

$$\begin{aligned} f_{\widetilde{\sigma}}\left(w = \sum_{1 \leqslant i \leqslant n} (x_i e_i + y_i f_i)\right) = \sum_{1 \leqslant i \leqslant n} f_{\widetilde{\sigma}_i}(w) &= \sum_{1 \leqslant i \leqslant n} (B_{\widetilde{W}}(\widetilde{\sigma}_i(\widetilde{w}_i), \widetilde{\sigma}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_i)) \\ &= \sum_{1 \leqslant i \leqslant n} 2x_i y_i \quad \text{by (438)}, \end{aligned}$$

where $f_{\widetilde{\sigma}_i}(w) = 2x_i y_i$, and

$$\begin{aligned} f_{\widetilde{\tau}}\left(w = \sum_{1 \leqslant i \leqslant n} (x_i e_i + y_i f_i)\right) = \sum_{1 \leqslant i \leqslant n} f_{\widetilde{\tau}_i}(w) &= \sum_{1 \leqslant i \leqslant n} (B_{\widetilde{W}}(\widetilde{\tau}_i(\widetilde{w}_i), \widetilde{\tau}_i(\widetilde{w}_i)) - B_{\widetilde{W}}(\widetilde{w}_i, \widetilde{w}_i)) \\ &= \sum_{1 \leqslant i \leqslant n} (2x_i y_i + 2x_i) \quad \text{by (439)}, \end{aligned}$$

where $f_{\widetilde{\tau}_i}(w) = 2x_i y_i + 2x_i$. Then we set for every $1 \leqslant i \leqslant n$:

$$\begin{cases} s_i & = \begin{cases} (\sigma_i, f_{\widetilde{\sigma}_i}) & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = 1 \\ (\sigma_i, w \mapsto f_{\widetilde{\sigma}_i}(w) + 2x_i + 2y_i) & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = -1 \end{cases} \\ r_i & = (\tau_i, w \mapsto f_{\widetilde{\tau}_i}(w) + 2x_i + 2y_i), \end{cases} \tag{474}$$

$s = s_1 \cdots s_n$ and $r = r_1 \cdots r_n$. Then s and r belong to $\mathrm{ASp}^{\oplus}(B_W)$ by definition, where B_W corresponds to B_{Weil} and $\begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$, respectively. We suppose that $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ and $w_r = (w_{r_1}^{(1)}, \ldots, w_{r_n}^{(n)})$. Then by (436),

$$\begin{cases} \omega_{\widetilde{\chi_k}}(s, w_s) & = \gamma(\sigma)(\gamma^{(1)}(\sigma_1) \cdots \gamma^{(n)}(\sigma_n))^{-1} T^{-1} \circ \left(\omega_{\widetilde{\chi_k}^{(1)}}(s_1, w_{s_1}^{(1)}) \otimes \cdots \otimes \omega_{\widetilde{\chi_k}^{(n)}}(s_n, w_{s_n}^{(n)})\right) \circ T \\ \omega_{\widetilde{\chi_k}}(r, w_r) & = \gamma(\tau)(\gamma^{(1)}(\tau_1) \cdots \gamma^{(n)}(\tau_n))^{-1} T^{-1} \circ \left(\omega_{\widetilde{\chi_k}^{(1)}}(r_1, w_{r_1}^{(1)}) \otimes \cdots \otimes \omega_{\widetilde{\chi_k}^{(n)}}(r_n, w_{r_n}^{(n)})\right) \circ T, \end{cases} \tag{475}$$

where by (100) and (105), for every $1 \leqslant i \leqslant n$, for every choice of $w_{s_i}^{(i)} \in \{e_i, f_i\}$ and $w_{r_i}^{(i)} \in \{e_i, e_i + f_i\}$,

$$\omega_{\widetilde{\chi_k}^{(i)}}(s_i, w_{s_i}^{(i)}) = \begin{cases} \gamma^{(i)}(\sigma_i)(\rho_{\widetilde{\chi_k}^{(i)}}(e_i) + \rho_{\widetilde{\chi_k}^{(i)}}(f_i)) = \frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}}(\rho_{\widetilde{\chi_k}^{(i)}}(e_i) + \rho_{\widetilde{\chi_k}^{(i)}}(f_i)) & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = 1 \\ \pm\gamma^{(i)}(\sigma_i)(\rho_{\widetilde{\chi_k}^{(i)}}(e_i) - \rho_{\widetilde{\chi_k}^{(i)}}(f_i)) = \pm\frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}}(\rho_{\widetilde{\chi_k}^{(i)}}(e_i) - \rho_{\widetilde{\chi_k}^{(i)}}(f_i)) & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = -1 \end{cases} \tag{476}$$

and

$$\omega_{\widetilde{\chi_k}^{(i)}}(r_i, w_{r_i}^{(i)}) = \gamma^{(i)}(\tau_i)(\rho_{\widetilde{\chi_k}^{(i)}}(e_i) + \rho_{\widetilde{\chi_k}^{(i)}}(e_i + f_i)) = \frac{e^{i\theta^{(i)}(\tau_i)}}{\sqrt{2}}(\rho_{\widetilde{\chi_k}^{(i)}}(e_i) + \rho_{\widetilde{\chi_k}^{(i)}}(e_i + f_i)). \tag{477}$$

Then (475) allows us to compute explicitly the expression of the operators $\omega_{\widetilde{\chi_k}}(s, w_s)$ and $\omega_{\widetilde{\chi_k}}(r, w_r)$, for any choice of $w_s = (w_{s_1}^{(1)}, \ldots, w_{s_n}^{(n)})$ and $w_r = (w_{r_1}^{(1)}, \ldots, w_{r_n}^{(n)})$ such that $w_{s_i}^{(i)} \in \{e_i, f_i\}$ and $w_{r_i}^{(i)} \in \{e_i, e_i + f_i\}$ for all $1 \leqslant i \leqslant n$. We obtain as in (469) and (470) that

$$\omega_{\widetilde{\chi_k}}(s, w_s) = \frac{e^{i\theta(\sigma)}}{2^{\frac{n}{2}}} \sum_{\substack{a_i \in \{e_i, f_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\widetilde{\chi_k}}(a_1 + \cdots + a_n) \tag{478}$$

if $\widetilde{\chi_k}^{(i)}(e_i, 0) = 1$ for all $1 \leqslant i \leqslant n$ (again, this hypothesis is made to simplify the notation but the method is the same in the general case), and

$$\omega_{\widetilde{\chi_k}}(r, w_r) = \frac{e^{i\theta(\tau)}}{2^{\frac{n}{2}}} \sum_{\substack{a_i \in \{e_i, e_i + f_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\widetilde{\chi_k}}(a_1 + \cdots + a_n). \tag{479}$$

In terms of matrices, by (476) and (477), $\omega_{\widetilde{\chi_k}^{(i)}}(s_i, w_{s_i}^{(i)})$ and $\omega_{\widetilde{\chi_k}^{(i)}}(r_i, w_{r_i}^{(i)})$ are written in the ordered basis $\{\delta_0, \delta_{f_i}\}$ of $\mathrm{L}^2(Y_i)$ as

$$\omega_{\widetilde{\chi_k}^{(i)}}(s_i, w_{s_i}^{(i)}) = \begin{cases} \frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\theta^{(i)}(\sigma_i)} H_1 & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = 1 \\ \pm\frac{e^{i\theta^{(i)}(\sigma_i)}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \pm e^{i\theta^{(i)}(\sigma_i)} H_1 & \text{if } \widetilde{\chi_k}^{(i)}(e_i, 0) = -1, \end{cases} \tag{480}$$

and

$$\omega_{\widetilde{\chi_k}^{(i)}}(r_i, w_{r_i}^{(i)}) = \frac{e^{i\theta^{(i)}(\tau_i)}}{\sqrt{2}} \widetilde{\chi_k}^{(i)}(e_i, 0) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\theta^{(i)}(\sigma_i)} \widetilde{\chi_k}^{(i)}(e_i, 0) H_1. \tag{481}$$

Thus we obtain as in the preceding point that $\omega_{\widetilde{\chi_k}}(s, w_s)$ and $\omega_{\widetilde{\chi_k}}(r, w_r)$ are up to a unitary factor the n-fold tensor product of the one-qubit Hadamard gate:

$$\begin{cases} \omega_{\widetilde{\chi_k}}(s, w_s) & = \pm e^{i\theta(\sigma)} \underbrace{H_1 \otimes \cdots \otimes H_1}_{n \text{ times}} = \pm e^{i\theta(\sigma)} H \\ \omega_{\widetilde{\chi_k}}(r, w_r) & = e^{i\theta(\tau)} \prod_{1 \leqslant i \leqslant n} \widetilde{\chi_k}^{(i)}(e_i, 0) \underbrace{H_1 \otimes \cdots \otimes H_1}_{n \text{ times}} = \widetilde{\chi_k}(e_1 + \cdots + e_n, 0) e^{i\theta(\tau)} H. \end{cases} \tag{482}$$

iii. The elements $\widetilde{\sigma} = (\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_n)$ and $\widetilde{\tau} = (\widetilde{\tau}_1, \dots, \widetilde{\tau}_n)$ belong to $\mathrm{Sp}^{\oplus}(\widetilde{W})$ by definition. Let $i \in \{1, \dots, n\}$. Since $\mathrm{K}(\widetilde{\sigma}_i) = 2\mathbb{Z}/4\mathbb{Z}(\widetilde{e}_i + \widetilde{f}_i) \subseteq \widetilde{W}_i$, we have that $\mathrm{K}'(\widetilde{\sigma}_i) = \mathbb{Z}/4\mathbb{Z}(\widetilde{e}_i + \widetilde{f}_i) \oplus 2\mathbb{Z}/4\mathbb{Z}\widetilde{f}_i \subseteq \widetilde{W}_i$. This implies that $\widetilde{w}^{(i)}_{\widetilde{\sigma}_i} \in \widetilde{W}_i$ satisfies (207) if and only if $\widetilde{w}^{(i)}_{\widetilde{\sigma}_i} \in \mathbb{Z}/4\mathbb{Z}(\widetilde{e}_i + \widetilde{f}_i) \oplus ((\mathbb{Z}/4\mathbb{Z})\backslash(2\mathbb{Z}/4\mathbb{Z}))\widetilde{f}_i$. Since $\mathrm{I}(\widetilde{\sigma}_i) = \mathbb{Z}/4\mathbb{Z}(\widetilde{e}_i + \widetilde{f}_i) \oplus 2\mathbb{Z}/4\mathbb{Z}\widetilde{f}_i \subseteq \widetilde{W}_i$, a system of representatives of $\mathrm{I}(\widetilde{\sigma}_i)/(\mathrm{I}(\widetilde{\sigma}_i) \cap 2\widetilde{W}_i)$ is for example $\{0, \widetilde{e}_i + \widetilde{f}_i\}$. Then by (218), using (368), we obtain that for very choice of $\widetilde{w}^{(i)}_{\widetilde{\sigma}_i}$,

$$\omega^{(i)}_{\chi_k}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i}) = \pm\gamma^{(i)}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})(\rho^{(i)}_{\chi_k}(\widetilde{e}_i) + \rho^{(i)}_{\chi_k}(\widetilde{f}_i)) = \pm\frac{e^{i\theta^{(i)}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})}}{\sqrt{2}}(\rho^{(i)}_{\chi_k}(\widetilde{e}_i) + \rho^{(i)}_{\chi_k}(\widetilde{f}_i)), \tag{483}$$

since $|\gamma^{(i)}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})| = |2\,\mathrm{I}(\widetilde{\sigma}_i))|^{-\frac{1}{2}} = 2^{-\frac{1}{2}}$. By (437), if $\widetilde{w}_{\widetilde{\sigma}} = (\widetilde{w}^{(1)}_{\widetilde{\sigma}_1}, \dots, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n})$, then we obtain (using the same method as to prove (469))

$$\begin{aligned}
&\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \\
&= \gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})(\gamma^{(1)}(\widetilde{\sigma}_1, \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \cdots \gamma^{(n)}(\widetilde{\sigma}_n, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n}))^{-1} T^{-1} \circ \left(\omega^{(1)}_{\chi_k}(\widetilde{\sigma}_1, \widetilde{w}^{(1)}_{\widetilde{\sigma}_1}) \otimes \cdots \otimes \omega^{(n)}_{\chi_k}(\widetilde{\sigma}_n, \widetilde{w}^{(n)}_{\widetilde{\sigma}_n})\right) \circ T \\
&= \pm\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) T^{-1} \circ \left((\rho^{(1)}_{\chi_k}(\widetilde{e}_1) + \rho^{(1)}_{\chi_k}(\widetilde{f}_1)) \otimes \cdots \otimes (\rho^{(n)}_{\chi_k}(\widetilde{e}_n) + \rho^{(n)}_{\chi_k}(\widetilde{f}_n))\right) \circ T \\
&= \pm\gamma(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) \sum_{\substack{\widetilde{a}_i \in \{\widetilde{e}_i, \widetilde{f}_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\chi_k}(\widetilde{a}_1 + \cdots + \widetilde{a}_n) \qquad \text{by Lemma 4.4} \\
&= \frac{e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})}}{2^{\frac{n}{2}}} \sum_{\substack{\widetilde{a}_i \in \{\widetilde{e}_i, \widetilde{f}_i\} \\ 1 \leqslant i \leqslant n}} \rho_{\chi_k}(\widetilde{a}_1 + \cdots + \widetilde{a}_n),
\end{aligned} \tag{484}$$

since $2\,\mathrm{I}(\widetilde{\sigma}) = 2\,\mathrm{I}(\widetilde{\sigma}_1) \oplus \cdots \oplus 2\,\mathrm{I}(\widetilde{\sigma}_n) = \bigoplus_{1 \leqslant i \leqslant n} 2\mathbb{Z}/4\mathbb{Z}(\widetilde{e}_i + \widetilde{f}_i)$. In terms of matrices, by (483), the operator $\omega^{(i)}_{\chi_k}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})$ is written in the ordered basis $\{\delta_0, \delta_{f_i}\}$ of $\mathrm{L}^2(Y_i)$ as

$$\omega^{(i)}_{\chi_k}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i}) = \pm\frac{e^{i\theta^{(i)}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})}}{\sqrt{2}}(\rho^{(i)}_{\chi_k}(\widetilde{e}_i) + \rho^{(i)}_{\chi_k}(\widetilde{f}_i)) = \pm\frac{e^{i\theta^{(i)}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})}}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \pm e^{i\theta^{(i)}(\widetilde{\sigma}_i, \widetilde{w}^{(i)}_{\widetilde{\sigma}_i})} H_1, \tag{485}$$

which implies by (485) that, up to a unitary factor, the operator $\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})$ corresponds precisely to the Hadamard(-Rademacher-Walsh) transform (456):

$$\omega_{\chi_k}(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}}) = \pm e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})} \underbrace{H_1 \otimes \cdots \otimes H_1}_{n \text{ times}} = \pm e^{i\theta(\widetilde{\sigma}, \widetilde{w}_{\widetilde{\sigma}})} H. \tag{486}$$

□

4.2.3 Finite subgroups of single-qubit Clifford gates

In this section, we suppose that $\widetilde{W}$ is a $\mathbb{Z}/4\mathbb{Z}$-module of rank two. Hence W is a two-dimensional $\mathbb{F}_2$-vector space. In chapter 3, we computed explicitly all the operators of the different Weil representations we constructed. In particular, all these operators correspond to one-qubit Clifford

gates, and more precisely the images of the (linear) Weil representations we define correspond to finite subgroups of Clifford gates (see the beginning of chapter 4). As already mentioned, Clifford gates are of prime interest in the stabilizer formalism, since they preserve the Pauli group, hence the stabilizer groups in stabilizer codes. Moreover, defining finite subgroups of Clifford gates may be useful to allow reliable computations, and perhaps to reveal hidden structures that could be of interest. In this section, we will write the correspondence between the generators of the images of the Weil representations of the groups $\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}$, $\mathrm{G}^0_{\widetilde{\chi}_k}$, $\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}$, $\mathrm{Mp}(\widetilde{W})$, $\mathrm{Mp}(\widetilde{W})_{\chi_k}$, $\mathrm{Mp}(W)$ and $\mathrm{AMp}(W)$ from chapter 3, and the one-qubit Clifford gates that arise in this setting. These generators therefore define some finite subgroups of one-qubit Clifford gates, i.e. finite subgroups of the Clifford group $\mathscr{C}_1$. We use the notation from chapter 3. In particular, $\widetilde{\varepsilon} \in \{\pm 1\}$ is defined by (298), $\widetilde{\xi} \in \{\pm i\}$ is defined by (342), and the generators a, b, c, d, e of the different groups involved refer to the presentations given in appendix B.2.

First, we consider $B_{\mathrm{Weil}} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

i. $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathbf{Ps}(\boldsymbol{B})}_{\widetilde{\boldsymbol{\chi}}}) \simeq \mathbf{D_8}$ **or** $\mathbf{SD_{16}}$. By Proposition 3.2 and since $\widetilde{\omega}_{\widetilde{\chi}}$ is faithful, $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ is a subgroup of $\mathscr{C}_1$ of order 16 such that

$$\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}) \simeq \begin{cases} \mathrm{D}_8 & \text{if } \theta(\tau) \equiv 0 \mod \pi \\ \mathrm{SD}_{16} & \text{if } \theta(\tau) \equiv \frac{\pi}{2} \mod \pi. \end{cases} \tag{487}$$

More precisely, $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ is generated by the following gates:

- $\widetilde{\omega}_{\widetilde{\chi}}(a) = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix} = \begin{cases} XH & \text{if } \widetilde{\varepsilon} = 1 \\ HX & \text{if } \widetilde{\varepsilon} = -1 \end{cases} & \text{if } \theta(\tau) \equiv 0 \mod \pi \\ \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ \widetilde{\varepsilon} & 1 \end{pmatrix} = \begin{cases} iXH & \text{if } \widetilde{\varepsilon} = 1 \\ iHX & \text{if } \widetilde{\varepsilon} = -1 \end{cases} & \text{if } \theta(\tau) \equiv \frac{\pi}{2} \mod \pi \end{cases}$
- $\widetilde{\omega}_{\widetilde{\chi}}(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X.$

ii. $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathbf{AMp}(\boldsymbol{B_W})_{\widetilde{\boldsymbol{\chi}_k}}) \simeq \mathbf{C_4 \cdot_6 S_4}$ **or** $\mathbf{U_2}(\mathbb{F}_3)$. By Proposition 3.5 and since $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful, $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k})$ is a subgroup of $\mathscr{C}_1$ of order 96 such that

$$\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}) \simeq \begin{cases} \mathrm{C}_4 \cdot_6 \mathrm{S}_4 & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2} \\ \mathrm{U}_2(\mathbb{F}_3) & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \end{cases} \tag{488}$$

More precisely, $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k})$ is generated by the following gates:

- $\widetilde{\omega}_{\widetilde{\chi_k}}(a) = iI$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(b) = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = iX$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(c) = i\widetilde{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\widetilde{\varepsilon}Z$

- $\widetilde{\omega}_{\widetilde{\chi_k}}(d) = \frac{1}{2}\begin{pmatrix} -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \\ -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \end{pmatrix} = \begin{cases} -SHSHZ & \text{if } \widetilde{\varepsilon}=1 \\ -SHS^{-1}H & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(e) = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-\widetilde{\varepsilon}i \\ 1+\widetilde{\varepsilon}i & 0 \end{pmatrix} = \begin{cases} e^{-i\frac{\pi}{4}}SX & \text{if } \widetilde{\varepsilon}=1 \\ e^{-i\frac{\pi}{4}}XS & \text{if } \widetilde{\varepsilon}=-1 \end{cases} & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2} \\ \frac{1+i}{2}\begin{pmatrix} 0 & 1-\widetilde{\varepsilon}i \\ 1+\widetilde{\varepsilon}i & 0 \end{pmatrix} = \begin{cases} SX & \text{if } \widetilde{\varepsilon}=1 \\ XS & \text{if } \widetilde{\varepsilon}=-1 \end{cases} & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \end{cases}$

iii. $\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi_k}}}(\mathbf{Mp}(\widetilde{\boldsymbol{W}})) \simeq \mathbf{U_2}(\mathbb{F}_3)$, $\mathbf{GL_2}(\mathbb{F}_3)$ **or** $\mathbf{CSU_2}(\mathbb{F}_3)$. If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{AMp}(B_W)_{\widetilde{\chi}_k} \simeq \mathrm{U}_2(\mathbb{F}_3)$ by Proposition 3.12, and the group $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is precisely the group $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k})$ of order 96 given in the previous point. If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and the subgroup $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ obtained by Proposition 3.16 and Corollary 3.17 has order 48. If $\mathscr{S}$ satisfies (314) and (315) then $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$, and generated by the following gates (here the generators a, b, c, d, e are those of $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ since $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\widetilde{\chi}_k} \simeq \mathrm{C}_4 \,._6\, \mathrm{S}_4$, so we can directly use the computations made in the previous point):

- $\widetilde{\omega}_{\widetilde{\chi_k}}(c) = i\widetilde{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\widetilde{\varepsilon}Z$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(bc) = -\widetilde{\varepsilon}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\widetilde{\varepsilon}XZ$
- $\begin{cases} \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}b) = \frac{i}{2}\begin{pmatrix} -\widetilde{\varepsilon}+i & -1+\widetilde{\varepsilon}i \\ -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}+i \end{pmatrix} = \begin{cases} -ZHSHS & \text{if } \widetilde{\varepsilon}=1 \\ -HS^{-1}HS & \text{if } \widetilde{\varepsilon}=-1 \end{cases} & \text{if } \chi_k = \chi_1 \\ \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}) = \frac{1}{2}\begin{pmatrix} -1+\widetilde{\varepsilon}i & -\widetilde{\varepsilon}+i \\ \widetilde{\varepsilon}+i & -1-\widetilde{\varepsilon}i \end{pmatrix} = \begin{cases} iHSHS & \text{if } \widetilde{\varepsilon}=1 \\ -iZHS^{-1}HS & \text{if } \widetilde{\varepsilon}=-1 \end{cases} & \text{if } \chi_k = \chi_3 \end{cases}$
- $\begin{cases} \widetilde{\omega}_{\widetilde{\chi_k}}(cebd) = \frac{\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} = \widetilde{\varepsilon}ZSHS & \text{if } \chi_k = \chi_1 \\ \widetilde{\omega}_{\widetilde{\chi_k}}(ed) = \frac{\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} = -\widetilde{\varepsilon}SHSZ & \text{if } \chi_k = \chi_3. \end{cases}$

If $\mathscr{S}$ satisfies (314) and (316) then $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ and generated by

- $\widetilde{\omega}_{\widetilde{\chi_k}}(c) = i\widetilde{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\widetilde{\varepsilon}Z$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(bc) = -\widetilde{\varepsilon}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\varepsilon XZ$

- $\begin{cases} \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}b^{-1}) = -\frac{i}{2}\begin{pmatrix} -\widetilde{\varepsilon}+i & -1+\widetilde{\varepsilon}i \\ -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}+i \end{pmatrix} = \begin{cases} ZHSHS & \text{if } \widetilde{\varepsilon}=1 \\ HS^{-1}HS & \text{if } \widetilde{\varepsilon}=-1 \end{cases} & \text{if } \chi_k=\chi_1 \\ \widetilde{\omega}_{\widetilde{\chi_k}}(a^2d^{-1}) = -\frac{1}{2}\begin{pmatrix} -1+\widetilde{\varepsilon}i & -\widetilde{\varepsilon}+i \\ \widetilde{\varepsilon}+i & -1-\widetilde{\varepsilon}i \end{pmatrix} = \begin{cases} -iHSHS & \text{if } \widetilde{\varepsilon}=1 \\ iZHS^{-1}HS & \text{if } \widetilde{\varepsilon}=-1 \end{cases} & \text{if } \chi_k=\chi_3 \end{cases}$
- $\begin{cases} \widetilde{\omega}_{\widetilde{\chi_k}}(acebd) = \frac{i\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} = i\widetilde{\varepsilon}ZSHS & \text{if } \chi_k=\chi_1 \\ \widetilde{\omega}_{\widetilde{\chi_k}}(aed) = \frac{i\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} = -i\widetilde{\varepsilon}SHSZ & \text{if } \chi_k=\chi_3. \end{cases}$

iv. $(\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi_k}}} \circ \widehat{\mathscr{S}})(\mathbf{Mp}(\boldsymbol{W})) \simeq \mathbf{Dic_3}$. If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then we can define a two-fold covering $\mathrm{Mp}(W) \simeq \mathrm{Dic}_3$ of $\mathrm{Sp}(W) \simeq \mathrm{S}_3$ such that the group $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widehat{\mathscr{S}})(\mathrm{Mp}(W))$ obtained by Proposition 3.20 and Corollary 3.21 is of order 12, is isomorphic to Dic_3, and is generated by the following gates, depending on the definition of $\widehat{\mathscr{S}}$ given by (327). (Observe again that the generators a, b, c, d, e are those of $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ since $\widehat{\mathscr{S}} : \mathrm{Mp}(W) \simeq \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\widetilde{\chi}_k} \simeq \mathrm{C}_4 \,._6\, \mathrm{S}_4$.)

- $\widetilde{\omega}_{\widetilde{\chi_k}}(a^2d) = -\frac{1}{2}\begin{pmatrix} -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \\ -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \end{pmatrix} = \begin{cases} SHSHZ & \text{if } \widetilde{\varepsilon}=1 \\ SHS^{-1}H & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$

  or $\widetilde{\omega}_{\widetilde{\chi_k}}(bd) = \frac{i}{2}\begin{pmatrix} -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \\ -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \end{pmatrix} = \begin{cases} -iSHSH & \text{if } \widetilde{\varepsilon}=1 \\ iSHS^{-1}HZ & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(ade) = \frac{i}{\sqrt{2}}\begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix} = \begin{cases} -iXHX & \text{if } \widetilde{\varepsilon}=1 \\ -iH & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$

or

- $\widetilde{\omega}_{\widetilde{\chi_k}}(cd) = \frac{i\widetilde{\varepsilon}}{2}\begin{pmatrix} -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \\ \widetilde{\varepsilon}+i & 1-\widetilde{\varepsilon}i \end{pmatrix} = \begin{cases} SHS^{-1}H & \text{if } \widetilde{\varepsilon}=1 \\ SHSHZ & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$

  or $\widetilde{\omega}_{\widetilde{\chi_k}}(cbd) = -\frac{\widetilde{\varepsilon}}{2}\begin{pmatrix} -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \\ 1+\widetilde{\varepsilon}i & -\widetilde{\varepsilon}+i \end{pmatrix} = \begin{cases} -iSHS^{-1}HZ & \text{if } \widetilde{\varepsilon}=1 \\ iSHSH & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(abcde) = -\frac{i\widetilde{\varepsilon}}{\sqrt{2}}\begin{pmatrix} 1 & \widetilde{\varepsilon} \\ \widetilde{\varepsilon} & -1 \end{pmatrix} = \begin{cases} -iH & \text{if } \widetilde{\varepsilon}=1 \\ -iXHX & \text{if } \widetilde{\varepsilon}=-1. \end{cases}$

v. $(\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi_k}}} \circ \widetilde{\mathscr{S}})(\mathbf{AMp}(\boldsymbol{W})) \simeq \mathbf{C_3} \rtimes \mathbf{C_8}$. If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then we can define a four-fold covering $\mathrm{AMp}(W) \simeq \mathrm{C}_3 \rtimes \mathrm{C}_8$ of $\mathrm{Sp}(W) \simeq \mathrm{S}_3$ such that the group $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(\mathrm{AMp}(W))$ obtained by Proposition 3.23 and Corollary 3.24 is of order 24, is isomorphic to $\mathrm{C}_3 \rtimes \mathrm{C}_8$, and is generated by the following gates, depending on the definition of $\widetilde{\mathscr{S}}$ given by (330). (Observe here that the generators a, b, c, d, e are those of $\mathrm{U}_2(\mathbb{F}_3)$ since $\widetilde{\mathscr{S}} : \mathrm{AMp}(W) \simeq \mathrm{C}_3 \rtimes \mathrm{C}_8 \to \mathrm{AMp}(B_W)_{\widetilde{\chi}_k} \simeq \mathrm{U}_2(\mathbb{F}_3)$.)

- $\widetilde{\omega}_{\widetilde{\chi_k}}(d) = \frac{1}{2}\begin{pmatrix} -1-\widetilde{\varepsilon}i & \widetilde{\varepsilon}-i \\ -\widetilde{\varepsilon}-i & -1+\widetilde{\varepsilon}i \end{pmatrix} = \begin{cases} -SHSHZ & \text{if } \widetilde{\varepsilon}=1 \\ -SHS^{-1}H & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$

  or $\widetilde{\omega}_{\widetilde{\chi_k}}(b^{-1}d) = \frac{i}{2}\begin{pmatrix} \widetilde{\varepsilon}+i & 1-\widetilde{\varepsilon}i \\ 1+\widetilde{\varepsilon}i & -\widetilde{\varepsilon}+i \end{pmatrix} = \begin{cases} iSHSH & \text{if } \widetilde{\varepsilon}=1 \\ -iSHS^{-1}HZ & \text{if } \widetilde{\varepsilon}=-1 \end{cases}$

- $\widetilde{\omega}_{\widetilde{\chi_k}}(ed^{-1}) = \frac{1+i}{2}\begin{pmatrix} \widetilde{\varepsilon} & -1 \\ -1 & -\widetilde{\varepsilon} \end{pmatrix} = \begin{cases} -e^{\frac{i\pi}{4}}XHX & \text{if } \widetilde{\varepsilon}=1 \\ -e^{\frac{i\pi}{4}}H & \text{if } \widetilde{\varepsilon}=-1. \end{cases}$

Now, we suppose that $\boldsymbol{B} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$.

i. $\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi}}}(\widetilde{\mathbf{Ps}(\boldsymbol{B})}_{\widetilde{\boldsymbol{\chi}}}) \simeq \mathbf{CSU_2}(\mathbb{F}_\mathbf{3})$ or $\mathbf{GL_2}(\mathbb{F}_\mathbf{3})$. By Proposition 3.28 and since $\widetilde{\omega}_{\widetilde{\chi}}$ is faithful, $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ is a subgroup of $\mathscr{C}_1$ of order 48 such that

$$\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}}) \simeq \begin{cases} \mathrm{CSU}_2(\mathbb{F}_3) & \text{if } \theta(\sigma_{(1,0)}) \equiv 0 \mod \pi \\ \mathrm{GL}_2(\mathbb{F}_3) & \text{if } \theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi. \end{cases} \tag{489}$$

More precisely, $\widetilde{\omega}_{\widetilde{\chi}}(\widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}})$ is generated by the following gates:

- $\widetilde{\omega}_{\widetilde{\chi}}(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = XZ$
- $\widetilde{\omega}_{\widetilde{\chi}}(b) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ$
- $\widetilde{\omega}_{\widetilde{\chi}}(c) = -\frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = iHSHS$
- $\widetilde{\omega}_{\widetilde{\chi}}(d) = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} = iSHSZ & \text{if } \theta(\sigma_{(1,0)}) \equiv 0 \mod \pi \\ \frac{i}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} = -SHSZ & \text{if } \theta(\sigma_{(1,0)}) \equiv \frac{\pi}{2} \mod \pi. \end{cases}$

ii. $\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi_k}}}(\mathbf{AMp}(\boldsymbol{B_W})_{\widetilde{\boldsymbol{\chi_k}}}) \simeq \mathbf{C_4 \,._6\, S_4}$ or $\mathbf{U_2}(\mathbb{F}_\mathbf{3})$. By Proposition 3.30 and since $\widetilde{\omega}_{\widetilde{\chi_k}}$ is faithful, $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k})$ is a subgroup of $\mathscr{C}_1$ of order 96 such that

$$\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k}) \simeq \begin{cases} \mathrm{C}_4 \,._6\, \mathrm{S}_4 & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2} \\ \mathrm{U}_2(\mathbb{F}_3) & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \end{cases} \tag{490}$$

More precisely, $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\widetilde{\chi}_k})$ is generated by the following gates:

- $\widetilde{\omega}_{\widetilde{\chi_k}}(a) = i\,\mathrm{I}_2$,
- $\widetilde{\omega}_{\widetilde{\chi_k}}(b) = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -iX$

- $\widetilde{\omega}_{\widetilde{\chi_k}}(c) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = ZX$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(d) = -\frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = -iHSHS$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(e) = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = -XHX & \text{if } \nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2} \\ \frac{1+i}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = -e^{\frac{i\pi}{4}}XHX & \text{if } \nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}. \end{cases}$

iii. $\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi_k}}}(\mathbf{Mp}(\widetilde{\boldsymbol{W}})) \simeq \mathbf{U_2}(\mathbb{F}_3)$**,** $\mathbf{GL_2}(\mathbb{F}_3)$ **or** $\mathbf{CSU_2}(\mathbb{F}_3)$. If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{AMp}(B_W)_{\tilde{\chi}_k} \simeq \mathrm{U}_2(\mathbb{F}_3)$ by Proposition 3.36, and the group $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is precisely the group $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{AMp}(B_W)_{\tilde{\chi}_k})$ of order 96 given in the previous point. If $\nu(\sigma_{(1,0)}) \equiv 0 \mod \frac{\pi}{2}$, then $\mathrm{Mp}(\widetilde{W}) \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and the subgroup $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ obtained by Proposition 3.37 and Corollary 3.38 has order 48. If $\mathscr{S}$ satisfies (350) and (351) then $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$, and generated by the following gates (here the generators a, b, c, d, e are those of $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ since $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) \simeq \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \to \mathrm{AMp}(B_W)_{\tilde{\chi}_k} \simeq \mathrm{C}_4 \,._6\, \mathrm{S}_4$, so we can directly use the computations made in the previous point):

- $\widetilde{\omega}_{\widetilde{\chi_k}}(bc) = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iZ$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(b) = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -iX$
- $\begin{cases} \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}c) = -\frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix} = -SHS^{-1}H & \text{if } \widetilde{\xi} = i \\ \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}b) = \frac{i}{2}\begin{pmatrix} -1+i & 1+i \\ 1-i & 1+i \end{pmatrix} = -iSHS^{-1}HZ & \text{if } \widetilde{\xi} = -i \end{cases}$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(ec) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H.$

If $\mathscr{S}$ satisfies (350) and (352) then $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \mathscr{S})(\mathrm{Mp}(\widetilde{W}))$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ and generated by

- $\widetilde{\omega}_{\widetilde{\chi_k}}(bc) = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iZ$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(b) = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -iX$

- $\begin{cases} \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}c^{-1}) = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix} = SHS^{-1}H & \text{if } \widetilde{\xi} = i \\ \widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}b^{-1}) = -\frac{i}{2}\begin{pmatrix} -1+i & 1+i \\ 1-i & 1+i \end{pmatrix} = iSHS^{-1}HZ & \text{if } \widetilde{\xi} = -i \end{cases}$
- $\widetilde{\omega}_{\widetilde{\chi_k}}(aec) = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = iH.$

iv. $(\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi}}} \circ \widehat{\mathscr{S}})(\mathbf{Mp}(\boldsymbol{W})) \simeq \mathbf{Dic_3}$. If $\theta(\sigma_{(1,0)}) \equiv 0 \mod \pi$, then we can define a two-fold covering $\mathrm{Mp}(W) \simeq \mathrm{Dic}_3$ of $\mathrm{Sp}(W) \simeq \mathrm{S}_3$ such that the group $(\widetilde{\omega}_{\widetilde{\chi}} \circ \widehat{\mathscr{S}})(\mathrm{Mp}(W))$ obtained by Proposition 3.41 and Corollary 3.42 is of order 12, is isomorphic to Dic_3, and is generated by the following gates, depending on the definition of $\widehat{\mathscr{S}}$ given by (363). (Observe that the generators a, b, c, d are those of $\mathrm{CSU}_2(\mathbb{F}_3)$ since $\widehat{\mathscr{S}} : \mathrm{Mp}(W) \simeq \mathrm{Dic}_3 \to \widetilde{\mathrm{Ps}(B)}_{\widetilde{\chi}} \simeq \mathrm{CSU}_2(\mathbb{F}_3)$.)

- $\widetilde{\omega}_{\widetilde{\chi}}(a^2c) = \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = -iHSHS$

  or $\widetilde{\omega}_{\widetilde{\chi}}(a^{-1}c) = \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} = iZHS^{-1}HS$
- $\widetilde{\omega}_{\widetilde{\chi}}(d) = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} = iSHSZ,$

or

- $\widetilde{\omega}_{\widetilde{\chi}}(b^{-1}c) = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} = ZHSHS$

  or $\widetilde{\omega}_{\widetilde{\chi}}(a^{-1}bc) = \frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix} = HS^{-1}HS$
- $\widetilde{\omega}_{\widetilde{\chi}}(abd) = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} = -iZSHS.$

v. $(\widetilde{\boldsymbol{\omega}}_{\widetilde{\boldsymbol{\chi}_k}} \circ \widetilde{\mathscr{S}})(\mathbf{AMp}(\boldsymbol{W})) \simeq \mathbf{C_3} \rtimes \mathbf{C_8}$. If $\nu(\sigma_{(1,0)}) \equiv \frac{\pi}{4} \mod \frac{\pi}{2}$, then we can define a four-fold covering $\mathrm{AMp}(W) \simeq \mathrm{C}_3 \rtimes \mathrm{C}_8$ of $\mathrm{Sp}(W) \simeq \mathrm{S}_3$ such that the group $(\widetilde{\omega}_{\widetilde{\chi_k}} \circ \widetilde{\mathscr{S}})(\mathrm{AMp}(W))$ obtained by Proposition 3.45 and Corollary 3.46 is of order 24, is isomorphic to $\mathrm{C}_3 \rtimes \mathrm{C}_8$, and is generated by the following gates, depending on the definition of $\widetilde{\mathscr{S}}$ given by (366). (Observe here that the generators a, b, c, d, e are those of $\mathrm{U}_2(\mathbb{F}_3)$ since $\widetilde{\mathscr{S}} : \mathrm{AMp}(W) \simeq \mathrm{C}_3 \rtimes \mathrm{C}_8 \to \mathrm{AMp}(B_W)_{\widetilde{\chi}_k} \simeq \mathrm{U}_2(\mathbb{F}_3)$.)

- $\widetilde{\omega}_{\widetilde{\chi_k}}(d) = -\frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = iHSHS$

  or $\widetilde{\omega}_{\widetilde{\chi_k}}(c^{-1}d) = -\frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix} = -iZHS^{-1}HS$

- $\widetilde{\omega}_{\widetilde{\chi_k}}(d^{-1}e) = -\frac{1+i}{2}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix} = -e^{\frac{i\pi}{4}}SHSZ,$

or

- $\widetilde{\omega}_{\widetilde{\chi_k}}(b^{-1}d) = \frac{i}{2}\begin{pmatrix}1+i & -1-i\\ -1+i & -1+i\end{pmatrix} = -HS^{-1}HS$

  or $\widetilde{\omega}_{\widetilde{\chi_k}}(bcd) = -\frac{i}{2}\begin{pmatrix}1-i & 1-i\\ 1+i & -1-i\end{pmatrix} = -ZHSHS$

- $\widetilde{\omega}_{\widetilde{\chi_k}}(bd^{-1}e) = \frac{i(1+i)}{2}\begin{pmatrix}i & -1\\ 1 & -i\end{pmatrix} = -e^{\frac{i\pi}{4}}ZSHS.$

Finally, we do not exhibit the generating sets of gates of the subgroups $\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\chi_k}}(\mathbf{Mp}(\widetilde{\boldsymbol{W}})_{\boldsymbol{\chi_k}})$ defined and explicitly computed in Proposition 3.49 and 3.51. In fact, this would be redundant with the aforementioned groups $\widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{Mp}(\widetilde{W}))$ for the case in which $B = B_{\mathrm{Weil}}$, since the path followed in section 3.3 is only another way to construct the same groups. In fact, if (373) is satisfied then $\widetilde{\omega}_{\chi_k}(\mathrm{Mp}(\widetilde{W})_{\chi_k}) \simeq \widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{Mp}(\widetilde{W})) \simeq \mathrm{U}_2(\mathbb{F}_3)$, and if (374) is satisfied then $\widetilde{\omega}_{\chi_k}(\mathrm{Mp}(\widetilde{W})_{\chi_k}) \simeq \widetilde{\omega}_{\widetilde{\chi_k}}(\mathrm{Mp}(\widetilde{W})) \simeq \mathrm{CSU}_2(\mathbb{F}_3)$ or $\mathrm{GL}_2(\mathbb{F}_3)$.

Appendix A

Extraspecial and almost extraspecial p-groups

Let p be a prime number. Unless otherwise stated, all the groups in the following are supposed to be finite. We denote by $\Phi(G)$ the Frattini subgroup of a finite group G, which is the intersection of all (proper) maximal subgroups of G. The aim of this section is to describe the structure of extraspecial and almost extraspecial 2-groups in order to prove Propositions 1.6 and 2.3.

Definition A.1. Let G be a p-group.

- **i.** G is said to be extraspecial if $\Phi(G)$ has order p and $\Phi(G) = [G, G] = Z(G)$.
- **ii.** G is said to be almost extraspecial if $\Phi(G)$ has order p, $\Phi(G) = [G, G]$ and $Z(G)$ is cyclic of order p^2.

Remark A.2. This definition of extraspecial group is given for instance in [Gor68], [Hup67] and [LM02]. Glasby uses in [Gla95] another definition: G is extraspecial if it is a p-group such that $[G, G] = Z(G)$ has order p and $G/Z(G)$ is elementary abelian (i.e. all elements different from the trivial one have the same order). These definitions are equivalent. In fact, let G be a p-group. Suppose that $[G, G] = Z(G)$ has order p and $G/Z(G)$ is elementary abelian. Then $Z(G) = [G, G] \cap Z(G) \subseteq \Phi(G)$ by [Hup67, III.3.12] and since $\Phi(G)$ is the smallest normal subgroup N of G such that G/N is elementary abelian (cf. [Hup67, III.3.14.a)]), one has $\Phi(G) \subseteq Z(G)$ and thus $\Phi(G) = Z(G)$. Conversely, if G satisfies A.1.i, then $G/Z(G) = G/\Phi(G)$ and $G/\Phi(G)$ is elementary abelian, again by [Hup67, III.3.14.a)].

We point out that p-groups play a substantial role in the study of finite groups: first among all finite groups since the number of groups of order p^n grows as $p^{\frac{2}{27}n^3+O(n^{\frac{8}{3}})}$ by Higman-Sims's asymptotic formula, but also for the internal structure of a group if we think for example of Sylow's theorems. The structure of classes of p-groups satisfying certain properties has been widely studied and has also been relevant for the classification of finite simple groups. In our study, we focus on extraspecial and almost extraspecial groups since they are related to the Heisenberg groups we define in sections 1.1 and 2.1. This appendix is intended to be self-contained or at least give precise references of the results used without proof. In particular, we prove in detail the results for $p = 2$ but only cite some of the results concerning p odd. The

reader interested in p odd can find the results and proofs for extraspecial groups for general p in [Hup67, III], [Gor68, Chapter 5], [Suz86, Chapter 4, 4] and [LM02, 2.2]. On extraspecial p-groups, their representations and the link to Heisenberg groups, see also [Aub00]. For almost extraspecial groups, the results are cited in [CT00].

A.1 Central product of groups

In order to understand the structure of extraspecial and almost extraspecial groups, we first need to recall the central product of two groups.

Definition A.3.
- A group G is the internal central product of two subgroups H and K if $[H, K] = 1$, $G = HK$ and $H \cap K \subseteq Z(G)$.
- A group G is the external central product of two groups H and K if there exist two subgroups $H_1 \subseteq Z(H)$ and $K_1 \subseteq Z(K)$, and an isomorphism $\theta : H_1 \to K_1$ such that $G = (H \times K)/N$, where

$$N = \{(h, k) \in H_1 \times K_1 \mid \theta(h) = k^{-1}\}.$$

  We shall denote G by $H \circ_\theta K$.

Remark A.4. There are some variations in the definition of central product of two groups. For example, in [LM02, 2.2.6], there are stronger requirements : $H \cap K$ has to be equal to $Z(G)$ for the internal central product, and $H_1 = Z(H)$, $K_1 = Z(K)$ for the external central product. All the following results concerning extraspecial groups remain true if we had used this definition of central product in [LM02, 2.2.6], but we need Definition A.3 under this form to obtain the results concerning almost extraspecial groups.

Notation A.5. The structure of a central product $H \circ_\theta K$ defined in A.3 depends on the isomorphism θ, i.e. on the way to identify the elements of the centers of H and K. If θ is implicit for the groups H and K which are studied, like in the case of internal central products, where θ is the identity map, or if θ is unique, then we simply denote the central product of H and K by $G = H \circ K$. We point out that constructing the central product of a finite number of groups is a commutative and associative operation.

We include in this section a crucial result for the study of extraspecial and almost extraspecial p-groups, which shows in particular that the decomposition of a central product of two groups is not unique. We recall the presentations of the dihedral and quaternionic groups of order eight:

$$\mathrm{D}_4 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$$
$$\mathrm{Q}_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle,$$

and we define for p odd two groups of order p^3:

$$\mathrm{M}(p) = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle$$
$$\mathrm{N}(p) = \langle x, y \mid x^{p^2} = y^p = 1, yxy^{-1} = x^{p+1} \rangle.$$

Lemma A.6. ***i.*** *The central products* $\mathrm{D}_4 \circ \mathrm{D}_4$ *and* $\mathrm{Q}_8 \circ \mathrm{Q}_8$ *are isomorphic.*

ii. *The central products* $\mathrm{N}(p) \circ \mathrm{N}(p)$ *and* $\mathrm{N}(p) \circ \mathrm{M}(p)$ *are isomorphic.*

Proof. **i.** Since the centers of D_4 and Q_8 contain only two elements, there is only one way to identify them and Notation A.5 applies. In particular there is no need to mention θ. Define two copies of Q_8 as $G_1 = \langle x_1, y_1 \mid x_1^4 = 1, y_1^2 = x_1^2, y_1 x_1 y_1^{-1} = x_1^{-1} \rangle$ and $G_2 = \langle x_2, y_2 \mid x_2^4 = 1, y_2^2 = x_2^2, y_2 x_2 y_2^{-1} = x_2^{-1} \rangle$. The central product of these groups is written

$$G_1 \circ G_2 = \langle x_1, y_1, x_2, y_2 \mid y_1^2 = x_1^2 = y_2^2 = x_2^2 = z,\, z^2 = 1,\, y_1 x_1 y_1^{-1} = x_1^{-1},\, y_2 x_2 y_2^{-1} = x_2^{-1},\\ x_1 x_2 x_1^{-1} = x_2,\, x_1 y_2 x_1^{-1} = y_2,\, y_1 x_2 y_1^{-1} = x_2,\, y_1 y_2 y_1^{-1} = y_2 \rangle,$$

where we identify the centers $Z(G_i) = \langle x_i^2 = y_i^2 \rangle$, and the last four equalities come from the fact that G_1 centralizes G_2 in the central product. In a similar way, if we set $A_1 = \langle a_1, b_1 \mid a_1^4 = b_1^2 = 1, b_1 a_1 b_1^{-1} = a_1^{-1} \rangle$ and $A_2 = \langle a_2, b_2 \mid a_2^4 = b_2^2 = 1, b_2 a_2 b_2^{-1} = a_2^{-1} \rangle$, then A_1 and A_2 are two copies of D_4 and their central product is written

$$A_1 \circ A_2 = \langle a_1, b_1, a_2, b_2 \mid a_1^2 = a_2^2 = u,\, u^2 = b_1^2 = b_2^2 = 1,\, b_1 a_1 b_1^{-1} = a_1^{-1},\, b_2 a_2 b_2^{-1} = a_2^{-1},\\ a_1 a_2 a_1^{-1} = a_2,\, a_1 b_2 a_1^{-1} = b_2,\, b_1 a_2 b_1^{-1} = a_2,\, b_1 b_2 b_1^{-1} = b_2 \rangle.$$

Now set $H_1 = \langle x_1, y_1 x_2 \rangle \subseteq G_1 \circ G_2$ and $H_2 = \langle x_2, y_2 x_1 \rangle \subseteq G_1 \circ G_2$. Then $(y_1 x_2)^2 = y_1^2 x_2^2 = y_1^4 = 1$ and $(y_1 x_2) x_1 (y_1 x_2)^{-1} = y_1 x_2 x_1 y_1 x_2 = y_1 x_1 y_1 x_2^2 = x_1^{-1} y_1^2 x_2^2 = x_1^{-1}$. This shows that $H_1 \simeq \mathrm{D}_4$, and similar computations show that $H_2 \simeq \mathrm{D}_4$. Moreover, H_1 centralizes H_2 because G_1 centralizes G_2. Let $\phi : A_1 \circ A_2 \to G_1 \circ G_2$ be defined by $\phi(a_1) = x_1$, $\phi(b_1) = y_1 x_2$, $\phi(a_2) = x_2$ and $\phi(b_2) = y_2 x_1$. Then ϕ can be extended to a group homomorphism in the natural way. Moreover, ϕ is surjective because $x_1 = \phi(a_1)$, $y_1 = \phi(b_1 a_2^{-1})$, $x_2 = \phi(a_2)$ and $y_2 = \phi(b_2 a_1^{-1})$. Since $A_1 \circ A_2$ and $G_1 \circ G_2$ have the same number of elements, this is enough to conclude that these two groups are isomorphic.

ii. See [Gor68, Chapter 5, Theorem 5.2].

□

A.2 The structure of extraspecial and almost extraspecial p-groups

The essential tool we need to describe precisely the structure of extraspecial and almost extraspecial p-groups is the central product of two groups, which we have defined in the previous section. The next technical result will be needed to give a symplectic structure to extraspecial groups and relate them to the Heisenberg groups defined in 1.1 and 2.2.

Lemma A.7. *Let* G *be an extraspecial* p*-group. Then:*

i. G *is nilpotent of step* 2*.*

ii. *The exponent of* G *is either* p *or* p^2*, and more precisely it equals* 4 *if* $p = 2$*.*

iii. *$G/Z(G)$ can be identified with a finite-dimensional $\mathbb{F}_p$-vector space. More precisely, $G/Z(G)$ can be endowed with a symplectic vector space structure by defining the non-degenerate symplectic form $\langle xZ(G), yZ(G)\rangle = l$, where $Z(G) = \langle z\rangle$ and $[x,y] = z^l$, $0 \leqslant l \leqslant p-1$.*

iv. *Let $n \geqslant 1$ such that $|G| = p^{2n+1}$. Then every maximal abelian normal subgroup of G has order p^{n+1}.*

Proof. **i.** The lower central series of G is $G \rhd [G,G] \rhd 1$ because G is non-abelian and $[[G,G],G] = [Z(G),G] = 1$.

ii. For all x and y in G, $1 = [x,y]^p = [x^p, y]$ by [Hup67, III.1.3] and the fact that $[G,G] = Z(G)$ has order p. Hence $x^p \in Z(G)$ and thus $x^{p^2} = 1$ because $Z(G)$ has order p. This shows that the exponent of G is either p or p^2. If we suppose that $p = 2$ and G has exponent 2 then G should be abelian since every element has order two. This is impossible by definition of an extraspecial group, hence G has exponent $p^2 = 4$.

iii. Remark A.2 implies that $G/Z(G)$ is elementary abelian. Since $G/Z(G)$ is also a p-group, it follows from the fundamental theorem of abelian groups that there exists $m \geqslant 1$ such that $G/Z(G) \simeq (\mathbb{Z}/p\mathbb{Z})^m$. (Here we use the additive notation for $G/Z(G)$.) Thus there exists a generating set $\{x_1, \dots, x_m\}$ of $G/Z(G)$, i.e. for all $x \in G/Z(G)$ there exists $\lambda_i \in \{0, \dots, p-1\}$ such that $x = \sum_{1 \leqslant i \leqslant m} \lambda_i x_i$. Since every element of $G/Z(G)$ is of order p, we can consider the λ_i as elements of $\mathbb{F}_p = \{0, \dots, p-1\}$. For $\lambda \in \mathbb{F}_p$, define $\lambda \cdot x = \sum_{1 \leqslant i \leqslant m} \lambda\lambda_i x_i$. In this way, $G/Z(G)$ becomes an $\mathbb{F}_p$-vector space of dimension m with basis $\{x_1, \dots, x_m\}$.

Now let x and y be in G. Since $[G,G] = Z(G)$ is cyclic of order p, there exists $z \in G$ such that $[G,G] = Z(G) = \langle z\rangle$. Hence there exists a unique element $0 \leqslant l \leqslant p-1$ such that $[x,y] = z^l$, i.e. $\langle \cdot, \cdot \rangle$ is well defined. Now let $0 \leqslant k \leqslant p-1$ and x, y, u be in G such that $[x,y] = z^{l_1}$ and $[u,y] = z^{l_2}$. Then:

- $[x,x] = 1$ and $[y,x] = [x,y]^{-1}$. Hence $\langle xZ(G), xZ(G)\rangle = 0$ and $\langle yZ(G), xZ(G)\rangle = -l_1 = -\langle xZ(G), yZ(G)\rangle$.
- $[xu, y] = xuyu^{-1}x^{-1}y^{-1} = x[u,y]yx^{-1}y^{-1} = [x,y][u,y]$ because $[G,G] \subseteq Z(G)$. This implies that $[xu^k, y] = [x,y][u,y]^k$. Hence $\langle xZ(G)(uZ(G))^k, yZ(G)\rangle = \langle xu^kZ(G), yZ(G)\rangle = l_1 + kl_2 = \langle xZ(G), yZ(G)\rangle + k\langle uZ(G), yZ(G)\rangle$.
- Suppose that $\langle xZ(G), vZ(G)\rangle = 0$ for all $v \in G$, then $[x,v] = 1$ for all $v \in G$, i.e. $x \in Z(G)$.

This shows that $\langle \cdot, \cdot \rangle$ is an anti-symmetric, alternating and non-degenerate bilinear form on $G/Z(G)$, i.e. $(G/Z(G), \langle \cdot, \cdot \rangle)$ is a symplectic vector space. In particular, the dimension of $G/Z(G)$ is even.

iv. The existence of $n \geqslant 1$ such that $|G| = p^{2n+1}$ has been established in iii. Let A be a maximal abelian normal subgroup of G. Then A contains the center of G. There is a bijective correspondence between subgroups of G containing $Z(G)$ and subgroups of $G/Z(G)$. Moreover, $G/Z(G)$ is identified with a symplectic vector space. Therefore $A/Z(G)$ is a

vector subspace of $G/Z(G)$ of maximal dimension that satisfies $\langle a_1 Z(G), a_2 Z(G)\rangle = 0$ for all $a_1, a_2 \in A$. In other terms, $A/Z(G)$ is a maximal isotropic subspace of $G/Z(G)$ and hence of dimension n. Then $|A| = |A/Z(G)||Z(G)| = p^n p = p^{n+1}$.

□

If G is a group and H is a proper normal subgroup of G, then G acts on H by conjugation: the map

$$\begin{aligned} &G \to \mathrm{Aut}(H), \quad \text{where } \phi_g(h) = ghg^{-1} \text{ for all } h \in H, \\ &g \mapsto \phi_g \end{aligned} \tag{491}$$

defines a group homomorphism which kernel is the centralizer $C_G(H)$.

Lemma A.8. *(cf.* [Gor68, Chapter 5, Lemma 4.6]*) Let G be a p-group and H be an extraspecial p-group.*

i. *Every automorphism of H acting trivially on $H/Z(H)$ is an inner automorphism of H.*

ii. *If H is an extraspecial subgroup of G such that $[G, H] \subseteq Z(H)$ then $G = H \circ C_G(H)$.*

Proof. **i.** Let $\varphi \in \mathrm{Aut}(H)$ such that φ acts trivially on $H/Z(H)$. By Lemma A.7.iii, there exists $\{x_1, \ldots, x_m\}$ a basis of $H/Z(H)$ as a vector space over $\mathbb{F}_p$. Let $Z(H) = \langle z\rangle$, so that z has order p and H has order p^{m+1}. By hypothesis, there exists $0 \leqslant n_i \leqslant p-1$ such that $\varphi(x_i) = x_i z^{n_i}$ for all $1 \leqslant i \leqslant m$. This shows that there are at most p^m distinct automorphisms of H acting trivially on $H/Z(H)$. Now let $\{y_1, \ldots, y_N\}$ be a system of cosets representatives of $H/Z(H)$, where $N = p^m$. Consider the inner automorphisms $\phi_{y_1}, \ldots, \phi_{y_N}$ of H defined by (491). Since for all $1 \leqslant i \neq j \leqslant N$, y_i and y_j belong to distinct cosets by hypothesis, then $\phi_{y_i} \neq \phi_{y_j}$. Moreover, for every $h \in H$, $\phi_{y_i}(h)$ and h belong to the same coset of $H/Z(H)$ because $\phi_{y_i}(h)h^{-1} = y_i h y_i^{-1} h^{-1} = [y_i, h] \in [H, H] = Z(H)$ since H is extraspecial. This shows that ϕ_{y_i} acts trivially on $H/Z(H)$ for all $1 \leqslant i \leqslant N$. This leads to $N = p^m$ distinct automorphisms of H acting trivially on $H/Z(H)$. Therefore, the inner automorphisms $\phi_{y_1}, \ldots, \phi_{y_N}$ are precisely all the inner automorphisms of H acting trivially on $H/Z(H)$.

ii. Since the result is true if $H = G$, we can suppose that H is a proper subgroup of G. First we mention that $H \trianglelefteq G$ since $[G, H] \subseteq Z(H)$. This implies in particular that every inner automorphism ϕ_g of G can be restricted to an automorphim of H. Let $x \in G$. In order to prove that $G = HC_G(H)$, it is enough to find $y \in H$ such that $y^{-1}x$ centralizes H, since in this case $x = yu$ with $y \in H$ and $u = y^{-1}x \in C_G(H)$. Moreover, finding $y \in H$ such that $y^{-1}x$ centralizes H is equivalent to finding $y \in H$ such that the inner automorphisms ϕ_x and ϕ_y of G coincide on H. Since $[G, H] \subseteq Z(H)$, the restriction of ϕ_x to H is an automorphism of H which acts trivially on $H/Z(H)$. In fact, for every $h \in H$, $\phi_x(h)h^{-1} = xhx^{-1}h^{-1} = [x, h] \in [G, H] \subseteq Z(H)$. By i, we deduce that ϕ_x is an inner automorphism of H. This provides the existence of the desired $y \in H$. Thus $G = HC_G(H)$. Moreover, $H \cap C_G(H) = Z(H)$, and $Z(H) \subseteq Z(G)$ since $G = HC_G(H)$. Finally, $[H, C_G(H)] = 1$ by definition of the centralizer. Therefore, we proved that $G = HC_G(H)$, $[H, C_G(H)] = 1$ and $H \cap C_G(H) \subseteq Z(G)$, i.e. G is the central product $H \circ C_G(H)$.

□

Lemma A.9. *(cf.* [Gor68, Chapter 5, Theorem 5.1]*) Let G be a non-abelian group of order p^3. Then G is extraspecial. More precisely, if $p = 2$ then G is isomorphic to either D_4 or Q_8, and if p is odd then G is isomorphic to either $\mathrm{M}(p)$ or $\mathrm{N}(p)$.*

Proof. Since the center of a p-group is non-trivial and G is non-abelian, we have $|Z(G)| \in \{p, p^2\}$. If $|Z(G)| = p^2$ then $|G/Z(G)| = p$, i.e. $G/Z(G)$ is cyclic. Then G is abelian, which is false by hypothesis. Hence $|Z(G)| = p$ and $|G/Z(G)| = p^2$. This implies that the group $G/Z(G)$ is abelian, and thus $[G, G] \subseteq Z(G)$. We can deduce that $[G, G] = Z(G)$ because $[G, G] \neq 1$. In addition, $G/Z(G)$ is elementary abelian. In fact, otherwise it would contain an element of order p^2 and thus $G/Z(G)$ would be cyclic, which contradicts again the fact that G is non-abelian. By Remark A.2, this shows that G is extraspecial. For the rest of the proof, we can refer to [Gor68, Chapter 5, Theorem 5.1]. For $p = 2$, one can also say that the result comes from the fact that there are only two non-abelian groups of order eight: D_4 and Q_8. These two groups are in fact extraspecial because they satisfy $Z(G) \simeq \mathbb{Z}/2\mathbb{Z}$ (order 2) and $G/Z(G) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ (elementary abelian). □

Now we come to the central result of this section: up to isomorphism, there are exactly two distinct extraspecial p-groups of the same order and they can be described as central products of extraspecial groups of order p^3.

Proposition A.10. *(cf.* [Gor68, Chapter 5, Theorem 5.2]*) Let G be an extraspecial p-group. Then there exists $n \geqslant 1$ such that $|G| = p^{2n+1}$. Moreover:*

- *if $p = 2$ then G is isomorphic to either the central product of n copies of the dihedral group D_4 (which we denote by $\mathrm{D}_4^{\circ n}$ or 2_+^{1+2n}) or the central product of $n-1$ copies of D_4 and one copy of the quaternionic group Q_8 (which we denote by $\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)}$ or 2_-^{1+2n}).*
- *If p is odd then G is isomorphic to either the central product of n copies of the group $\mathrm{M}(p)$ (which we denote by $\mathrm{M}(p)^{\circ n}$ or p_+^{1+2n}) or the central product of $n-1$ copies of $\mathrm{M}(p)$ and one copy of the group $\mathrm{N}(p)$ (which we denote by $\mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-1)}$ or p_-^{1+2n}).*

Remark A.11. The claim of Proposition A.10 regarding the order of G is a direct consequence of the induction in the following proof, but we have already established it in Lemma A.7.iv.

Proof. We reason by induction on the order N of G. All p-groups of order p^k with $k \leqslant 2$ being abelian, we have $N \geqslant p^3$. If $N = p^3$ then the conclusions follow from Lemma A.9. Suppose now that $N > p^3$ and that every extraspecial group of order $p^k < N$ is of the form given in the statement. Let x be an element of $G \backslash Z(G)$. Since $[G, G] = Z(G)$, there exist $y \in G$ and $z \in Z(G) \backslash \{1\}$ such that $[x, y] = z$. Since the center of G has order p, $Z(G) = \langle z \rangle$. Let $H_1 = \langle x, y, z \rangle$ be the subgroup of G generated by the elements x, y and z. By Lemma A.7.ii, x^p and y^p belong to $Z(G)$. Hence $H_1/Z(G) \simeq \langle a, b \mid a^p = b^p = 1, [a, b] = 1 \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^2$, where we can set $a = xZ(G)$ and $b = yZ(G)$. Then H_1 is a non-abelian group of order p^3 and thus an extraspecial group isomorphic to D_4, Q_8, $\mathrm{M}(p)$ or $\mathrm{N}(p)$ by Lemma A.9. Moreover, $[G, H_1] \subseteq [G, G] = Z(G) = Z(H_1)$, so we can apply Lemma A.8 to affirm that $G = H_1 \cdot C_G(H_1)$, where $H_1 \cap C_G(H_1) = Z(H_1) = Z(G)$.

Now, we prove that $C_G(H_1)$ is extraspecial. Every element of $Z(C_G(H_1))$ centralizes $C_G(H_1)$, but also H_1 by definition of the centralizer. Hence $Z(C_G(H_1)) \subseteq Z(G)$ since $G =$

$H_1C_G(H_1)$. Since $Z(G) \subseteq C_G(H_1)$, we have that $Z(G) \subseteq Z(C_G(H_1))$, so we can conclude that $Z(C_G(H_1)) = Z(G)$ has order p. Moreover, $[C_G(H_1), C_G(H_1)] \subseteq [G,G] = Z(G)$. If $C_G(H_1)$ were abelian then we would have $C_G(H_1) = Z(C_G(H_1)) = Z(G)$. Hence $G = H_1 \circ C_G(H_1) = H_1 \circ Z(G)$ would have order $|H_1| = p^3$, which we excluded by hypothesis. Thus $[C_G(H_1), C_G(H_1)] = Z(G) = Z(C_G(H_1))$ has order p. Since the group $C_G(H_1)/Z(C_G(H_1)) = C_G(H_1)/Z(G)$ is a subgroup of the elementary abelian group $G/Z(G)$, it is elementary abelian too. Then $C_G(H_1)$ is extraspecial and we obtain by induction the existence of $m \geqslant 1$ such that $|C_G(H_1)| = p^{2m+1}$ and $C_G(H_1)$ is isomorphic to one of the central products $\mathrm{D}_4^{\circ m}$, $\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(m-1)}$, $\mathrm{M}(p)^{\circ m}$ or $\mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(m-1)}$. Thus, if we set $n = m+1$, then $|G| = p^3 p^{2m+1} p^{-1} = p^{2m+3} = p^{2(m+1)+1} = p^{2n+1}$ and G is isomorphic to one of the following central products: $H_1 \circ \mathrm{D}_4^{\circ(n-1)}$, $H_1 \circ (\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-2)})$, $H_1 \circ \mathrm{M}(p)^{\circ(n-1)}$ or $H_1 \circ (\mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-2)})$.

Now we separate the different possibilities for H_1 to obtain the final decomposition of G:

- if $H_1 \simeq \mathrm{D}_4$ then $G \simeq \mathrm{D}_4 \circ \mathrm{D}_4^{\circ(n-1)} = \mathrm{D}_4^{\circ n}$ or $G \simeq \mathrm{D}_4 \circ (\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-2)}) = \mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)}$.
- If $H_1 \simeq \mathrm{Q}_8$ then $G \simeq \mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)}$ or $G \simeq \mathrm{Q}_8 \circ (\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-2)}) = \mathrm{D}_4^{\circ n}$, since $\mathrm{Q}_8 \circ \mathrm{Q}_8$ and $\mathrm{D}_4 \circ \mathrm{D}_4$ are isomorphic by Lemma A.6.
- If $H_1 \simeq \mathrm{M}(p)$ then $G \simeq \mathrm{M}(p) \circ \mathrm{M}(p)^{\circ(n-1)} = \mathrm{M}(p)^{\circ n}$ or $G \simeq \mathrm{M}(p) \circ (\mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-2)}) = \mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-1)}$.
- If $H_1 \simeq \mathrm{N}(p)$ then $G \simeq \mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-1)}$ or $G \simeq \mathrm{N}(p) \circ (\mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-2)}) = \mathrm{N}(p) \circ \mathrm{M}(p)^{\circ(n-1)}$, since $\mathrm{N}(p) \circ \mathrm{M}(p)$ and $\mathrm{N}(p) \circ \mathrm{N}(p)$ are isomorphic by Lemma A.6.

To complete the proof, it only remains to show that the two possibilities for G for each p are not isomorphic. For p odd, we can refer to [Gor68, Chapter 5, Theorem 5.2], where the argument is combinatorial, based on a counting of subgroups of order four in G. For $p = 2$, the same combinatorial argument appearing in [Gor68] for p odd can be reproduced. Instead, we will follow the more structural proof given in [Hup67, III.13.8]. First we show that G possesses a maximal abelian normal (m.a.n.) subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{n-1}$. Then we show that every m.a.n. subgroup of 2_-^{1+2n} is of this type, whereas 2_+^{1+2n} contains a m.a.n. subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n+1}$. Since G is of exponent 4 (see Lemma A.7.i), G possesses a cyclic subgroup $\langle a \rangle$ of order four. By Lemma A.7.i, $1 \neq a^2 \in Z(G)$. Then $[G,G] = Z(G) = \langle a^2 \rangle$ because $Z(G)$ contains two elements. Moreover, $\langle a \rangle \trianglelefteq G$ since for all $g \in G$, $gag^{-1} = [g,a]a \in \{a, a^{-1}\}$. Now, let A be a m.a.n. subgroup of G containing a. By the fundamental theorem of finitely generated abelian groups, there exists $k_1 \geqslant 0$ and $k_2 \geqslant 1$ such that $A \simeq (\mathbb{Z}/2\mathbb{Z})^{k_1} \times (\mathbb{Z}/4\mathbb{Z})^{k_2}$. Recall that $G/Z(G)$ is elementary abelian and thus of exponent 2. Since $A/Z(G)$ is a subgroup of $G/Z(G)$, this implies that $k_2 < 2$. Hence A is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{n-1}$. This shows that every extraspecial group of order p^{2n+1} has a m.a.n. subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{n-1}$.

Suppose now that $G = 2_+^{1+2n} = \mathrm{D}_4^{\circ n}$. Then a presentation of G is

$$G = \langle a_1, \dots, a_n, b_1, \dots, b_n \mid a_i^4 = b_i^2 = 1,\, b_i a_i b_i^{-1} = a_i^{-1},\, a_i^2 = a_j^2,\\ [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \text{ for } i \neq j \rangle.$$

The subgroup $\langle a_1^2, b_1, \dots, b_n \rangle$ is abelian, normal in G and isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n+1}$, hence maximal by Lemma A.7.iv. Finally, suppose that $G = 2_-^{1+2n} = \mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)}$ and let A be

a m.a.n. subgroup of G. Then every element x of A can be written in the form x_1x_2, with $x_1 \in \mathrm{Q}_8$ and $x_2 \in \mathrm{D}_4^{\circ(n-1)}$, where this decomposition is unique up to multiplication by $Z(G)$. If we set A_1 and A_2 the subgroups of A generated by the elements $x = x_1x_2$ such that x_2 (respectively x_1) is in the center of G, then $A \subseteq A_1A_2$. Since A is a m.a.n. subgroup of G, then A_1 and A_2 are abelian normal subgroups of the extraspecial groups Q_8 and $\mathrm{D}_4^{\circ(n-1)}$, respectively. By Lemma A.7.iv and the fact that $A_1 \cap A_2 = Z(G)$,

$$|A_1A_2| = \frac{|A_1||A_2|}{|A_1 \cap A_2|} = \frac{|A_1||A_2|}{2} \leqslant \frac{2^2 2^n}{2} = 2^{n+1} = |A|.$$

Thus we can conclude that $A = A_1A_2$, and in particular A has exponent four because A_1 is a m.a.n. subgroup of Q_8, hence isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Thus every m.a.n subgroup of 2_-^{1+2n} is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{n-1}$, which concludes the proof. □

We now study the almost extraspecial p-groups. Proposition A.14 shows that the structure of these groups is strongly related to that of extraspecial groups.

Lemma A.12. *Let G be a non-trivial p-group. Then:*

i. *G is nilpotent.*

ii. *Every (proper) maximal subgroup of G is of index p and normal in G.*

Proof. **i.** The upper central series of G is $1 = Z_0(G) \subseteq Z(G) = Z_1(G) \subseteq Z_2(G) \subseteq \cdots$, where $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ for all $i \geqslant 0$. Let i be such that $G/Z_i(G) \neq 1$, then $Z(G/Z_i(G)) \neq 1$ since $G/Z_i(G)$ is a p-group. Hence $Z_{i+1}(G)$ strictly contains $Z_i(G)$. Since G is finite, it must be an index i such that $G/Z_i(G) = 1$, i.e. $Z_i(G) = G$, which implies that G is nilpotent.

ii. Since G is nilpotent, every (proper) maximal subgroup of G is normal by [Hup67, III.2.3]. Let M be a maximal subgroup of G. There is a bijective correspondence between subgroups N of G such that $M \subseteq N \subseteq G$ and subgroups of G/M. Since M is maximal, every such subgroup N satisfies $N = M$ or $N = G$. Hence G/M does not have any non-trivial subgroup, which implies that the order of G/M is prime. Therefore, the index of M in G equals p.

□

Lemma A.13. *The central products $\mathrm{D}_4 \circ (\mathbb{Z}/4\mathbb{Z})$ and $\mathrm{Q}_8 \circ (\mathbb{Z}/4\mathbb{Z})$ are isomorphic.*

Proof. Since D_4 and Q_8 have only two elements in their centers, there is only one way to identify the elements of these centers with elements of $\mathbb{Z}/4\mathbb{Z}$, and there is no need to mention the isomorphism θ (cf. Notation A.5). Presentations of $\mathrm{D}_4 \circ (\mathbb{Z}/4\mathbb{Z})$ and $\mathrm{Q}_8 \circ (\mathbb{Z}/4\mathbb{Z})$ can then be given by:

$$\mathrm{D}_4 \circ (\mathbb{Z}/4\mathbb{Z}) = \langle a, b, c \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}, a^2 = c^2, cac^{-1} = a, cbc^{-1} = b \rangle$$
$$\mathrm{Q}_8 \circ (\mathbb{Z}/4\mathbb{Z}) = \langle x, y, d \mid x^4 = 1, x^2 = y^2 = d^2, yxy^{-1} = x^{-1}, dxd^{-1} = x, dyd^{-1} = y \rangle.$$

Define $\phi(a) = x$, $\phi(b) = dy$ and $\phi(c) = d$. Then $\phi(a)^4 = x^4 = 1$, $\phi(b)^2 = dydy = d^2y^2 = d^4 = 1$, $\phi(b)\phi(a)\phi(b)^{-1} = dyxy^{-1}d^{-1} = dx^{-1}d^{-1} = x^{-1} = \phi(a)^{-1}$, $\phi(a)^2 = x^2 = d^2 = \phi(c)^2$,

$\phi(c)\phi(a)\phi(c)^{-1} = dxd^{-1} = x = \phi(a)$ and $\phi(c)\phi(b)\phi(c)^{-1} = ddyd^{-1} = dy = \phi(b)$. This proves that ϕ is a group homomorphism. Moreover, $x = \phi(a)$, $y = \phi(c^{-1}b)$ and $d = \phi(c)$, so ϕ is surjective. Since the groups $\mathrm{D}_4 \circ (\mathbb{Z}/4\mathbb{Z})$ and $\mathrm{Q}_8 \circ (\mathbb{Z}/4\mathbb{Z})$ have the same order, ϕ is an isomorphism. □

Proposition A.14. *If G is almost extraspecial then there exists $n \geqslant 1$ such that G has order p^{2n+2}. There exists a normal subgroup H of index p in G such that H does not contain the center $Z(G)$. Then H is extraspecial and $G = H \circ Z(G)$. In particular, up to isomorphism, there is only one almost extraspecial group of order p^{2n+2} which is isomorphic to $\mathrm{D}_4^{\circ n} \circ (\mathbb{Z}/4\mathbb{Z})$.*

Proof. Let G be an almost extraspecial p-group. Then G is a non-trivial p-group, so every maximal (proper) subgroup of G is normal and of index p. Since the center of G has order p^2 and the Frattini subgroup $\Phi(G)$ has order p, $\Phi(G)$ does not contain $Z(G)$. Since $\Phi(G)$ is the intersection of all (proper) maximal subgroups of G, there exists a (proper) maximal subgroup H of G which does not contain $Z(G)$. Now, since H has index p in G, the subgroup $HZ(G)$ of G verifies

$$|HZ(G)| = \frac{|H||Z(G)|}{|H \cap Z(G)|} = \frac{|G|p^2}{p|H \cap Z(G)|} = \frac{|G|p}{|H \cap Z(G)|}. \tag{492}$$

This implies that $|H \cap Z(G)| \geqslant p$, i.e. $|H \cap Z(G)| \in \{p, p^2\}$. Since $Z(G)$ is not contained in H, we deduce that $|H \cap Z(G)| = p$, whence $G = HZ(G)$ by (492). Then $Z(H)$ is contained in $Z(G)$, which proves that $Z(H) = H \cap Z(G)$ has order p. Moreover, since G is nilpotent by Lemma A.12, and $[G, G]$ is normal in G, we have that $[G, G] \cap Z(G) \neq \{1\}$ by [Hup67, II.2.6]. Hence $[G, G] \subseteq Z(G)$ because $|[G, G]| = p$ and $|Z(G)| = p^2$. Thus

$$[H, H] \subseteq [G, G] \cap H \subseteq Z(G) \cap H = Z(H),$$

so the commutator subgroup $[H, H]$ is either trivial or equal to $Z(H)$. Since $G = HZ(G)$ is non-abelian, H cannot be abelian, and thus $[H, H] = Z(H)$. In addition, $H \lhd G$ so $\Phi(H) \subseteq \Phi(G)$ by [Hup67, II.3.3]. Since $\Phi(G)$ has order p and $Z(H) = [H, H] \cap Z(H) \subseteq \Phi(H)$ by [Hup67, II.3.12], we obtain that $\Phi(H) = \Phi(G) = [G, G] = [H, H]$. This enables us to conclude that H is extraspecial. Now, $G = HZ(G)$, $[H, Z(G)] = 1$ and $H \cap Z(G) = Z(H) \subseteq Z(G)$, so G is the central product of H and $Z(G)$ by Definition A.3. Since H is extraspecial and $Z(G) \simeq \mathbb{Z}/4\mathbb{Z}$, by Proposition A.10, there exists $n \geqslant 1$ such that

$$|G| = \frac{|H||Z(G)|}{|H \cap Z(G)|} = \frac{p^{2n+1}p^2}{p} = p^{2n+2}, \tag{493}$$

and G is isomorphic to either $\mathrm{D}_4^{\circ n} \circ (\mathbb{Z}/4\mathbb{Z})$ or $\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)} \circ (\mathbb{Z}/4\mathbb{Z})$. By Lemma A.13, $\mathrm{D}_4^{\circ n} \circ (\mathbb{Z}/4\mathbb{Z})$ and $\mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)} \circ (\mathbb{Z}/4\mathbb{Z})$ are isomorphic. This implies that G is unique up to isomorphism, which concludes the proof. □

A.3 The link with the Heisenberg groups $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ in the case $p = 2$

In section 1.1, we introduce the Heisenberg group $\mathrm{H}(B)$, where the form

$$\langle \cdot, \cdot \rangle : \ (w_1, w_2) \mapsto B(w_1, w_2) - B(w_2, w_1)$$

is non-degenerate and $\mathbb{F} = \mathbb{F}_2$. Recall that this endows W with a symplectic vector space structure $(W, \langle \cdot, \cdot \rangle)$ of dimension $2n$ over $\mathbb{F}_2$, where $n \geqslant 1$. In section 2.1, starting from a symplectic $\mathbb{Z}/4\mathbb{Z}$-module $(\widetilde{W}, \langle \cdot, \cdot \rangle_{\widetilde{W}})$ of rank $2n$, we consider $W = \widetilde{W}/2\widetilde{W}$ and we introduce the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$, where $\langle w_1, w_2 \rangle_W = 2\langle \widetilde{w}_1, \widetilde{w}_2 \rangle_{\widetilde{W}}$ and $B_W(w_1, w_2) = 2B_{\widetilde{W}}(\widetilde{w}_1, \widetilde{w}_2)$ for all $w_1 = \widetilde{w}_1 + 2\widetilde{W}$ and $w_2 = \widetilde{w}_2 + 2\widetilde{W}$, and the form

$$\langle \cdot, \cdot \rangle_W : \ (w_1, w_2) \mapsto B_W(w_1, w_2) - B_W(w_2, w_1)$$

is non-degenerate. Using the abelian group homomorphism κ defined in Lemma 2.1, this allows us to identify $(\widetilde{W}/2\widetilde{W}, \langle \cdot, \cdot \rangle_W)$ with a symplectic $\mathbb{F}_2$- vector space $(W, \langle \cdot, \cdot \rangle)$ of dimension $2n$, where $\langle \cdot, \cdot \rangle = \kappa \circ \langle \cdot, \cdot \rangle_W$. The last step before proving Propositions 1.6 and 2.3 is to exhibit the link between:

- extraspecial 2-groups and the Heisenberg group $\mathrm{H}(B)$,
- almost extraspecial 2-groups and the Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$.

A priori the groups $\mathrm{H}(B)$ and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ depend on the bilinear forms B and B_W chosen to define them, and there are many such bilinear forms. However, as we are going to prove in Propositions A.16 and A.17, these groups are very particular: $\mathrm{H}(B)$ is in fact an extraspecial group of order 2^{2n+1} and $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is almost extraspecial of order 2^{2n+2}. This implies that, in spite of the multiple possibilities for B and B_W, there are up to isomorphism only two different groups $\mathrm{H}(B)$, corresponding to the two non-equivalent quadratic forms from W to $\mathbb{F}_2$, and only one Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Recall from the paragraph after Remark 1.4 the index $\nu(Q)$ and the Arf invariant $\mathrm{Arf}(Q)$ of a quadratic form Q on W. Then $\nu(Q) = n$ corresponds to $\mathrm{Arf}(Q) = 0$, and $\nu(Q) = n - 1$ corresponds to $\mathrm{Arf}(Q) = 1$.

Lemma A.15. *To every extraspecial group G, we can associate a symplectic structure over $G/Z(G)$. More precisely, if $Z(G) = \langle z \rangle$ then we can define a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ and a quadratic form Q associated to $\langle \cdot, \cdot \rangle$ by*

$$\langle xZ(G), yZ(G) \rangle = k, \quad \textit{where } [x, y] = z^k, \quad \textit{and} \quad Q(xZ(G)) = l, \quad \textit{where } x^2 = z^l.$$

Moreover, if $G \simeq 2_+^{1+2n}$ then $\nu(Q) = n$, whereas if $G \simeq 2_-^{1+2n}$ then $\nu(Q) = n - 1$.

Proof. The fact that $(G/Z(G), \langle \cdot, \cdot \rangle)$ is a symplectic vector space has already been proven in Lemma A.7.iii. Let $x, y \in G$ be such that $[x, y] = z^k$. By Lemma A.7.ii, we can also write $x^2 = z^{l_1}$ and $y^2 = z^{l_2}$ since $x^2,\ y^2 \in Z(G)$. Hence Q is well defined. In addition,

$$(xy)^2 = xyxy = [x, y]yx^2y = x^2y^2[x, y] = z^{l_1 + l_2 + k},$$

which shows that $Q(xZ(G)yZ(G)) - Q(xZ(G)) - Q(yZ(G)) = k = \langle xZ(G), yZ(G) \rangle$, i.e. $(xZ(G), yZ(G)) \mapsto Q(xZ(G)yZ(G)) - Q(xZ(G)) - Q(yZ(G))$ is bilinear. Thus Q is quadratic.
Suppose now that $G \simeq 2_+^{1+2n} = \mathrm{D}_4^{\circ n}$, with presentation

$$G = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid a_i^4 = b_i^2 = 1, b_i a_i b_i^{-1} = a_i^{-1}, a_i^2 = a_j^2, \\ [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \text{ for } i \neq j \rangle.$$

Then every element x of G can be written as a product $x = a_1^{k_1} b_1^{l_1} \cdots a_n^{k_n} b_n^{l_n}$, where $0 \leqslant k_i \leqslant 3$ and $0 \leqslant l_i \leqslant 1$. This enables us to compute the expression of $Q(xZ(G))$. First, let $1 \leqslant i \leqslant n$. If we do not distinguish between $0 \leqslant k_i \leqslant 3$ and its projection in $\mathbb{Z}/2\mathbb{Z}$, and if we use that $[a_i^{k_i}, b_i^{l_i}] = [a_i, b_i]^{k_i l_i}$ (see Lemma A.7.iii), then:

$$Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = Q(a_i^{k_i} Z(G)) + Q(b_i^{l_i} Z(G)) + \langle a_i^{k_i} Z(G), b_i^{l_i} Z(G) \rangle = k_i + k_i l_i.$$

Hence, since two copies of D_4 commute in the central product, we have by induction that

$$Q(xZ(G)) = \sum_{i=1}^{n} Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = \sum_{i=1}^{n} (k_i + k_i l_i).$$

Applying the transformation $(k_i, l_i) \mapsto (k_i, k_i + l_i)$, we modify $Q(xZ(G))$ into $\sum_{i=1}^{n} k_i l_i$, which implies that Q has index n.
Finally we suppose that $G \simeq 2_-^{1+2n} = \mathrm{Q}_8 \circ \mathrm{D}_4^{\circ(n-1)}$, with presentation

$$G = \langle x_1, y_1, a_2 \ldots, a_n, b_2 \ldots, b_n \,|\, x_1^4 = 1,\, y_1 x_1 y_1^{-1} = x^{-1},\, a_i^4 = b_i^2 = 1,\, b_i a_i b_i^{-1} = a_i^{-1},\\ x_1^2 = y_1^2 = a_i^2 = a_j^2,\, [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \text{ for } i \neq j \rangle.$$

Then every element x of G can be written as a product $x = x_1^{k_1} y_1^{l_1} a_2^{k_2} b_2^{l_2} \cdots a_n^{k_n} b_n^{l_n}$, where $0 \leqslant k_i \leqslant 3$ for all i, $0 \leqslant l_1 \leqslant 3$, and $0 \leqslant l_i \leqslant 1$ for $i \geqslant 2$. This enables us to compute the expression of $Q(xZ(G))$. First, as above, for every $2 \leqslant i \leqslant n$:

$$Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = Q(a_i^{k_i} Z(G)) + Q(b_i^{k_i} Z(G)) + \langle a_i^{k_i} Z(G), b_i^{l_i} Z(G) \rangle = k_i + k_i l_i,$$

and similarly,

$$Q(x_1^{k_1} Z(G) y_1^{l_1} Z(G)) = Q(x_1^{k_1} Z(G)) + Q(y_1^{k_1} Z(G)) + \langle x_1^{k_1} Z(G), y_1^{l_1} Z(G) \rangle = k_1 + l_1 + k_1 l_1.$$

Hence

$$Q(xZ(G)) = Q(x_1^{k_1} Z(G) y_1^{l_1} Z(G)) + \sum_{i=2}^{n} Q(a_i^{k_i} Z(G) b_i^{l_i} Z(G)) = k_1 + l_1 + k_1 l_1 + \sum_{i=2}^{n} (k_i + k_i l_i).$$

Applying the transformation $(k_i, l_i) \mapsto (k_i, k_i + l_i)$, we modify $Q(xZ(G))$ into $k_1 + l_1 + \sum_{i=1}^{n} k_i l_i$, which implies that Q has index $n - 1$. □

Proposition A.16. *The Heisenberg group* $\mathrm{H}(B)$ *is extraspecial. In particular,* $\mathrm{H}(B)$ *is isomorphic to* $2_+^{1+2n} = \mathrm{D}_4^{\circ n}$ *if* $\nu(Q) = n$, *or to* $2_-^{1+2n} = \mathrm{D}_4^{\circ(n-1)} \circ \mathrm{Q}_8$ *if* $\nu(Q) = n - 1$.

Proof. Let $h_1 = (w_1, t_1)$ and $h_2 = (w_2, t_2)$ be in $\mathrm{H}(B)$. Then

$$\begin{aligned}[h_1, h_2] &= (w_1 + w_2, t_1 + t_2 + B(w_1, w_2))(-w_1 - w_2, -t_1 - t_2 + Q(w_1) + Q(w_2) + B(w_1, w_2)) \\ &= (0, -Q(w_1 + w_2) + Q(w_1) + Q(w_2)) = (0, -\langle w_1, w_2 \rangle).\end{aligned}$$

This shows that $[\mathrm{H}(B), \mathrm{H}(B)] \simeq \mathbb{F}_2$, i.e. $[\mathrm{H}(B), \mathrm{H}(B)]$ has order two and $[\mathrm{H}(B), \mathrm{H}(B)] = Z(\mathrm{H}(B))$. Moreover $\mathrm{H}(B)/Z(\mathrm{H}(B)) \simeq W$ is elementary abelian, since it is isomorphic to

$\mathbb{F}_2^{2n}$. Therefore we can conclude that $\mathrm{H}(B)$ is extraspecial by Remark A.2. In addition, $h^2 = (0, Q(w))$ for every $h = (w, t) \in \mathrm{H}(B)$. This implies that the quadratic form associated to $\mathrm{H}(B)$ as an extraspecial group by Lemma A.15 is exactly the quadratic form Q associated to B. In this way, Lemma A.15 implies that $\mathrm{H}(B)$ is isomorphic to 2_+^{1+2n} if $\nu(Q) = n$, or to 2_-^{1+2n} if $\nu(Q) = n - 1$. □

Proposition A.17. *The Heisenberg group $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is almost extraspecial. In particular, $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is unique up to isomorphism and isomorphic to $\mathrm{H}(B) \circ \mathbb{Z}/4\mathbb{Z}$ for any choice of B.*

Proof. Let $h_1 = (w_1, t_1)$ and $h_2 = (w_2, t_2)$ be in $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$. Then

$$\begin{aligned}[h_1, h_2] &= (w_1 + w_2, t_1 + t_2 + B_W(w_1, w_2)) \\ &\quad \cdot (-w_1 - w_2, -t_1 - t_2 + Q_W(w_1) + Q_W(w_2) + B_W(w_1, w_2)) \\ &= (0, -Q_W(w_1 + w_2) + Q_W(w_1) + Q_W(w_2)) \\ &= (0, -\langle w_1, w_2 \rangle_W).\end{aligned}$$

This shows that $D := [\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W), \mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)] \simeq 2\mathbb{Z}/4\mathbb{Z}$, i.e. D has order 2. Now, since $D \subseteq Z(\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W))$, we have that

$$h_1 D h_2 D = h_1 h_2 D = h_2 h_1 (0, -\langle w_1, w_2 \rangle_W) D = h_2 h_1 D = h_2 D h_1 D,$$

and $(h_1 D)^2 = h_1^2 D = (0, 2t_1 + Q_W(w_1))D = D$. Thus $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)/D$ is elementary abelian. Since for a p-group G, $\Phi(G)$ is the smallest normal subgroup N of G such that G/N is elementary abelian (cf. [Hup67, III.3.14.a)]), we have that $\Phi(G) \simeq \mathbb{Z}/2\mathbb{Z}$. Since $D = D \cap Z(\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)) \subseteq \Phi(G)$ by [Hup67, III.3.12], we can deduce that $\Phi(G) = D$. Moreover, $Z(\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)) \simeq \mathbb{Z}/4\mathbb{Z}$ is cyclic of order $4 = 2^2$. So we can finally conclude that $\mathrm{H}_{\mathbb{Z}/4\mathbb{Z}}(B_W)$ is almost extraspecial, and thus unique up to isomorphism by Proposition A.14. The isomorphism with $\mathrm{H}(B) \circ \mathbb{Z}/4\mathbb{Z}$ is a consequence of Propositions A.14 and A.16. □

Appendix B

GAP codes and glossary of finite groups

In this section, we give the **GAP** code we use in section 3.4 in order to identify explicitly the group $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z})$, and other **GAP** commands which can be used to verify the expression of the group homomorphisms that we find by hand in section 3. Moreover, we write a list of all the finite groups we use in this thesis, with their identification in the SmallGroup library and a presentation for each of them.

B.1 GAP codes

In section 3.4, we identify explicitly the group $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z})$ thanks to **GAP**. In fact, we prove that $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{U}_2(\mathbb{F}_3)$ using $\mathrm{SL}_n'(\mathbb{Z}/4\mathbb{Z})$ with $n = 3$ and the following code:

```
f:=FreeGroup(6);

g:=f/[f.1^4,f.2^4,f.3^4,f.4^4,f.5^4,f.6^4,
f.1*f.5*f.1^3*f.5^3,f.1*f.3*f.1^3*f.3^3,
f.3*f.2*f.3^3*f.2^3,f.4*f.2*f.4^3*f.2^3,
f.4*f.6*f.4^3*f.6^3,f.6*f.5*f.6^3*f.5^3,
f.1*f.2*f.1^3*f.2^3*f.3^3,f.1^2*f.2*f.1^2*f.2^3*f.3^2,
f.1^3*f.2*f.1*f.2^3*f.3,f.1*f.2^2*f.1^3*f.2^2*f.3^2,
f.1^2*f.2^2*f.1^2*f.2^2,f.1^3*f.2^2*f.1*f.2^2*f.3^2,
f.1*f.2^3*f.1^3*f.2*f.3,f.1^2*f.2^3*f.1^2*f.2*f.3^2,
f.1^3*f.2^3*f.1*f.2*f.3^3,f.3*f.5*f.3^3*f.5^3*f.1^3,
f.3^2*f.5*f.3^2*f.5^3*f.1^2,f.3^3*f.5*f.3*f.5^3*f.1,
f.3*f.5^2*f.3^3*f.5^2*f.1^2,f.3^2*f.5^2*f.3^2*f.5^2,
f.3^3*f.5^2*f.3*f.5^2*f.1^2,f.3*f.5^3*f.3^3*f.5*f.1,
f.3^2*f.5^3*f.3^2*f.5*f.1^2,f.3^3*f.5^3*f.3*f.5*f.1^3,
f.2*f.6*f.2^3*f.6^3*f.4^3,f.2^2*f.6*f.2^2*f.6^3*f.4^2,
f.2^3*f.6*f.2*f.6^3*f.4,f.2*f.6^2*f.2^3*f.6^2*f.4^2,
f.2^2*f.6^2*f.2^2*f.6^2,f.2^3*f.6^2*f.2*f.6^2*f.4^2,
f.2*f.6^3*f.2^3*f.6*f.4,f.2^2*f.6^3*f.2^2*f.6*f.4^2,
f.2^3*f.6^3*f.2*f.6*f.4^3,f.4*f.3*f.4^3*f.3^3*f.2^3,
```

```
f.4^2*f.3*f.4^2*f.3^3*f.2^2,f.4^3*f.3*f.4*f.3^3*f.2,
f.4*f.3^2*f.4^3*f.3^2*f.2^2,f.4^2*f.3^2*f.4^2*f.3^2,
f.4^3*f.3^2*f.4*f.3^2*f.2^2,f.4*f.3^3*f.4^3*f.3*f.2,
f.4^2*f.3^3*f.4^2*f.3*f.2^2,f.4^3*f.3^3*f.4*f.3*f.2^3,
f.6*f.1*f.6^3*f.1^3*f.5^3,f.6^2*f.1*f.6^2*f.1^3*f.5^2,
f.6^3*f.1*f.6*f.1^3*f.5,f.6*f.1^2*f.6^3*f.1^2*f.5^2,
f.6^2*f.1^2*f.6^2*f.1^2,f.6^3*f.1^2*f.6*f.1^2*f.5^2,
f.6*f.1^3*f.6^3*f.1*f.5,f.6^2*f.1^3*f.6^2*f.1*f.5^2,
f.6^3*f.1^3*f.6*f.1*f.5^3,f.5*f.4*f.5^3*f.4^3*f.6^3,
f.5^2*f.4*f.5^2*f.4^3*f.6^2,f.5^3*f.4*f.5*f.4^3*f.6,
f.5*f.4^2*f.5^3*f.4^2*f.6^2,f.5^2*f.4^2*f.5^2*f.4^2,
f.5^3*f.4^2*f.5*f.4^2*f.6^2,f.5*f.4^3*f.5^3*f.4*f.6,
f.5^2*f.4^3*f.5^2*f.4*f.6^2,f.5^3*f.4^3*f.5*f.4*f.6^3];

h:=Subgroup(g,[g.1,g.4]);

k:=SmallGroup(96,67);

IsomorphismGroups(h,k);
```

Explanation: We generate a free group f with 6 generators, and then a finitely presented group g corresponding to $\mathrm{SL}_3''(\mathbb{Z}/4\mathbb{Z})$, by imposing the Steinberg relations (383) on the generators of the free group f. Then h is defined as the subgroup of g corresponding to $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z})$, where we choose the generators g.1=$x_{12}(1)$ and g.4=$x_{21}(1)$. The group k is the one which is isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$ and available in the SmallGroup library (the identification in the SmallGroup library is also mentioned in [GN]). The last command returns a group isomorphism from h to k or "fail" if the groups h and k are not isomorphic. Since this code returns such group isomorphism, this implies that $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z})$ is in fact isomorphic to $\mathrm{U}_2(\mathbb{F}_3)$.

As mentioned in section 3.4, we could verify that $\mathrm{SL}_2'(\mathbb{Z}/4\mathbb{Z}) \simeq \mathrm{U}_2(\mathbb{F}_3)$ starting from $\mathrm{SL}_n'(\mathbb{Z}/4\mathbb{Z})$ with $n = 5$. To this extent, we need to generate a free group f with 20 generators and the corresponding finitely presented group $\mathrm{SL}_5''(\mathbb{Z}/4\mathbb{Z})$ with the Steinberg relations (383) imposed on the generators of f. However, there are more than 680 such relations, and unfortunately the code seems to execute without end.

Other useful **GAP** commands are:

- `IsomorphicSubgroups(G,H)`

  It returns all injective homomorphisms from H to G up to G-conjugation. This is used to find the homomorphism $\mathscr{F}'$ in section 3.4.

- `GQuotients(G,H)`

  It returns all surjective homomorphisms from G to H up to H-conjugation. This is used to find the homomorphisms Ψ and ψ' in section 3.4.

- `IsomorphismGroups(G,H)`

  It returns a group isomorphism from G to H or "fail" if the groups G and H are not isomorphic. To cite only one example, this can be used to verify that the image of the map $\mathscr{S} : \mathrm{Mp}(\widetilde{W}) = \mathrm{Q}_8 \rtimes \mathrm{Dic}_3 \to \mathrm{AMp}(\widetilde{W}) \simeq \mathrm{C}_4 \,._6\, \mathrm{S}_4$, which is defined in Proposition 3.16 by $\mathscr{S}(a) = c$, $\mathscr{S}(b) = bc$, $\mathscr{S}(c) = d^{-1}b^{-1}$ and $\mathscr{S}(d) = acebd$ (using the generators of $\mathrm{Q}_8 \rtimes \mathrm{Dic}_3$ and $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ corresponding to the presentations given in section B.2), generates a subgroup of $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ which is isomorphic to the group $\mathrm{CSU}_2(\mathbb{F}_3)$. The code would be the following. We first generate a free group f with five generators, and then a finitely presented group g whose relations are the one in the presentation of $\mathrm{C}_4 \,._6\, \mathrm{S}_4$ given in section B.2. Then we define the subgroup h of g generated by the elements $g.3$, $g.2{*}g.3$, $g.4^{-1}{*}g.2^{-1}$ and $g.1{*}g.3{*}g.5{*}g.2{*}g.4$, which correspond to c, bc, $d^{-1}b^{-1}$ and $acebd$ in $\mathrm{C}_4 \,._6\, \mathrm{S}_4$, respectively. Finally, the group k is the one which is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$ in the SmallGroup library. Then the last command must return an isomorphism if the image of $\mathscr{S}$ is isomorphic to $\mathrm{CSU}_2(\mathbb{F}_3)$.

```
f:=FreeGroup(5);

g:=f/[f.1^4,f.4^2,f.5^2,f.2^2*f.1^2,f.3^2*f.1^2,
f.1*f.2*f.1^3*f.2^3,f.1*f.3*f.1^3*f.3^3,
f.1*f.4*f.1^3*f.4^2,f.1*f.5*f.1^3*f.5,
f.3*f.2*f.3^3*f.2,f.4*f.2*f.4^2*f.3^3*f.2,
f.5*f.2*f.1*f.3^3*f.2^3,f.4*f.3*f.4^2*f.2^3,
f.5*f.3*f.5*f.3,f.5*f.4*f.5*f.4];

h:=Subgroup(g,[g.3,g.2*g.3,g.4^2*g.2^3,g.1*g.3*g.5*g.2*g.4]);

k:=SmallGroup(48,28);

IsomorphismGroups(h,k);
```

B.2 Glossary of finite groups

Below is the list of all the explicit finite groups we mention in this thesis. For each of them, we provide the name, notation (the one we use, but they may be many aliases, see [GN]), order, SmallGroup identification and a presentation. This information is collected thanks to the database “GroupNames” [GN].

- The cyclic group of order 2 is denoted by $\mathbf{C_2}$, $\mathbb{Z}/\mathbf{2}\mathbb{Z}$ or $\boldsymbol{\mu_2}(\mathbb{C})$.

$$\mathrm{SmallGroup}(2,1) = \langle a | a^2 = 1 \rangle$$

- The cyclic group of order 4 is either denoted by $\mathbf{C_4}$, $\mathbb{Z}/\mathbf{4}\mathbb{Z}$ or $\boldsymbol{\mu_4}(\mathbb{C})$.

$$\mathrm{SmallGroup}(4,1) = \langle a | a^4 = 1 \rangle$$

- The Klein 4-group of order 4 is either denoted by $\mathbf{C_2^2}$ or $(\mathbb{Z}/\mathbf{2}\mathbb{Z})^{\mathbf{2}}$.

$$\mathrm{SmallGroup}(4,2) = \langle a, b | a^2 = b^2 = 1, ab = ba \rangle$$

- The symmetric group of order 6 is denoted by $\mathbf{S_3}$.

$$\mathrm{SmallGroup}(6,1) = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$$

- The cyclic group of order 8 is either denoted by $\mathbf{C_8}$, $\mathbb{Z}/\mathbf{8}\mathbb{Z}$ or $\boldsymbol{\mu_8}(\mathbb{C})$.

$$\mathrm{SmallGroup}(8,1) = \langle a | a^8 = 1 \rangle$$

- The dihedral group of order 8 is denoted by $\mathbf{D_4}$.

$$\mathrm{SmallGroup}(8,3) = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$$

- The quaternion group of order 8 is denoted by $\mathbf{Q_8}$.

$$\mathrm{SmallGroup}(8,4) = \langle a, b \mid a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1} \rangle$$

- The elementary abelian group of type [2,2,2] of order 8 is denoted by $\mathbf{C_2^3}$ or $(\mathbb{Z}/\mathbf{2}\mathbb{Z})^{\mathbf{3}}$.

$$\mathrm{SmallGroup}(8,5) = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$$

- The dicyclic group of order 12 is denoted by $\mathbf{Dic_3}$.

$$\mathrm{SmallGroup}(12,1) = \langle a, b \mid a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1} \rangle$$

- The dihedral group of order 12 is denoted by $\mathbf{D_6}$.

$$\mathrm{SmallGroup}(12,4) = \langle a, b \mid a^6 = b^2 = 1, bab = a^{-1} \rangle$$

- The dihedral group of order 16 is denoted by $\mathbf{D_8}$.

$$\mathrm{SmallGroup}(16,7) = \langle a, b \mid a^6 = b^2 = 1, bab = a^{-1} \rangle$$

- The semidihedral group of order 16 is denoted by $\mathbf{SD_{16}}$.

$$\mathrm{SmallGroup}(16,8) = \langle a, b \mid a^8 = b^2 = 1, bab = a^3 \rangle$$

- The generalised quaternion group of order 16 is denoted by $\mathbf{Q_{16}}$.

$$\mathrm{SmallGroup}(16,9) = \langle a, b \mid a^8 = 1, b^2 = a^4, bab^{-1} = a^{-1} \rangle$$

- The Pauli group of order 16 is denoted by $\mathbf{C_4} \circ \mathbf{D_4}$.

$$\mathrm{SmallGroup}(16,13) = \langle a, b, c \mid a^4 = c^2 = 1, b^2 = a^2, ab = ba, ac = ca, cbc = a^2b \rangle$$

- The group $\mathbf{C_3} \rtimes \mathbf{C_8}$ of order 24.

$$\mathrm{SmallGroup}(24,1) = \langle a, b \mid a^3 = b^8 = 1, bab^{-1} = a^{-1} \rangle$$

- The group $\mathbf{C_4} \times \mathbf{S_3}$ of order 24.

$$\mathrm{SmallGroup}(24,5) = \langle a, b, c \mid a^4 = b^3 = c^2 = 1, ab = ba, ac = ca, cbc = b^{-1} \rangle$$

- The symmetric group of order 24 is denoted by $\mathbf{S_4}$.

$$\mathrm{SmallGroup}(24,12) = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^2 = 1, cac^{-1} = dad = ab = ba, cbc^{-1} = a, bd = db, dcd = c^{-1} \rangle$$

- The conformal special unitary (or binary octahedral) group of order 48 is denoted by $\mathbf{CSU_2}(\mathbb{F}_3)$.

$$\mathrm{SmallGroup}(48,28) = \langle a, b, c, d \mid a^4 = c^3 = 1, b^2 = d^2 = a^2, bab^{-1} = dbd^{-1} = a^{-1}, cac^{-1} = ab, dad^{-1} = a^2b, cbc^{-1} = a, dcd^{-1} = c^{-1} \rangle$$

- The general linear group of order 48 is denoted by $\mathbf{GL_2}(\mathbb{F}_3)$.

$$\mathrm{SmallGroup}(48,29) = \langle a, b, c, d \mid a^4 = c^3 = d^2 = 1, b^2 = a^2, bab^{-1} = dbd = a^{-1}, cac^{-1} = ab, dad = a^2b, cbc^{-1} = a, dcd = c^{-1} \rangle$$

- The group $\mathbf{SL_2}(\mathbb{Z}/4\mathbb{Z}) \simeq \mathbf{A_4} \rtimes \mathbf{C_4}$ of order 48.

$$\mathrm{SmallGroup}(48,30) = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^4 = 1, cac^{-1} = dad^{-1} = ab = ba, cbc^{-1} = a, bd = db, dcd^{-1} = c^{-1} \rangle$$

- The group $\mathbf{C_4} \mathbf{.} \mathbf{A_4}$ of order 48.

$$\mathrm{SmallGroup}(48,33) = \langle a, b, c, d \mid a^4 = d^3 = 1, b^2 = c^2 = a^2, ab = ba, ac = ca, ad = da, cbc^{-1} = a^2b, dbd^{-1} = a^2bc, dcd^{-1} = b \rangle$$

- The group $\mathbf{C_{16}} \rtimes_{\mathbf{5}} \mathbf{C_4}$ of order 64.

$$\mathrm{SmallGroup}(64,27) = \langle a, b \mid a^{16} = b^4 = 1, bab^{-1} = a^9 \rangle$$

- The group $\mathbf{C_4^2} \rtimes_{\mathbf{4}} \mathbf{C_4}$ of order 64.

$$\mathrm{SmallGroup}(64,57) = \langle a, b, c \mid a^4 = b^4 = c^4 = 1, ab = ba, cac^{-1} = ab^2, bc = cb \rangle$$

- The group $\mathbf{C_2} \times \mathbf{C_8} \rtimes \mathbf{C_4}$ of order 64.

$$\mathrm{SmallGroup}(64,84) = \langle a, b, c \mid a^2 = b^8 = c^4 = 1, ab = ba, ac = ca, cbc^{-1} = b^5 \rangle$$

- The group $\mathbf{A_4} \rtimes \mathbf{C_8}$ of order 96.

$$\begin{aligned}\mathrm{SmallGroup}(96,65) = \langle a,b,c,d \,|& a^2 = b^2 = c^3 = d^8 = 1,\, cac^{-1} = dad^{-1} = ab = ba,\\ & cbc^{-1} = a,\, bd = db,\, dcd^{-1} = c^{-1}\rangle\end{aligned}$$

- The group $\mathbf{Q_8} \rtimes \mathbf{Dic_3}$ of order 96.

$$\begin{aligned}\mathrm{SmallGroup}(96,66) = \langle a,b,c,d \,|& a^4 = c^6 = 1,\, b^2 = a^2,\, d^2 = c^3,\, bab^{-1} = dbd^{-1} = a^{-1},\\ & cac^{-1} = b,\, dad^{-1} = a^2b,\, cbc^{-1} = ab,\, dcd^{-1} = c^{-1}\rangle\end{aligned}$$

- The unitary group $\mathbf{U_2}(\mathbb{F}_3)$ of order 96.

$$\begin{aligned}\mathrm{SmallGroup}(96,67) = \langle a,b,c,d,e \,|& a^4 = d^3 = 1, b^2 = c^2 = a^2,\, e^2 = a,\, ab = ba,\, ac = ca,\\ & ad = da,\, ae = ea,\, cbc^{-1} = a^2b,\, dbd^{-1} = a^2bc,\, ebe^{-1} = bc,\\ & dcd^{-1} = b,\, ece^{-1} = a^2c,\, ede^{-1} = d^{-1}\rangle\end{aligned}$$

- The group $\mathbf{C_4} \centerdot_{\mathbf{6}} \mathbf{S_4}$ of order 96.

$$\begin{aligned}\mathrm{SmallGroup}(96,192) = \langle a,b,c,d,e \,|& a^4 = d^3 = e^2 = 1,\, b^2 = c^2 = a^2,\, ab = ba,\, ac = ca,\\ & ad = da,\, ae = ea,\, cbc^{-1} = a^2b,\, dbd^{-1} = a^2bc,\, ebe = bc,\\ & dcd^{-1} = b,\, ece = a^2c,\, ede = d^{-1}\rangle\end{aligned}$$

- The group $\mathbf{C_2} \times \mathbf{A_4} \rtimes \mathbf{C_4}$ of order 96.

$$\begin{aligned}\mathrm{SmallGroup}(96,194) = \langle a,b,c,d,e \,|& a^2 = b^2 = c^2 = d^3 = e^4 = 1,\, ab = ba,\, ac = ca,\, ad = da,\\ & ae = ea,\, dbd^{-1} = ebe^{-1} = bc = cb,\, dcd^{-1} = b,\, ce = ec,\\ & ede^{-1} = d^{-1}\rangle\end{aligned}$$

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