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Input redundancy of parameter-dependent and switched systems

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Abstract

System outputs are not always uniquely determined by a single input, a phenomenon known as input redundancy, where different inputs produce the same output. This redundancy can be exploited to achieve objectives beyond the primary control task, leveraging inherent degrees of freedom to significantly improve performance. However, input redundancy has so far been studied only in the context of linear time-invariant systems.

This thesis extends the concept to parameter-dependent systems, where parameters vary over time, and switched systems, which combine continuous dynamics with discrete events. For both frameworks, we introduce new tailored notions of input redundancy that require only the existence and uniqueness of solutions. Focusing on systems with linear dependence on state and input (and polynomial parameter dependence in the parameter-dependent case) under unconstrained settings, we derive tractable conditions to verify these notions. Furthermore, we reveal the degrees of freedom associated with input redundancy, enabling their direct use to enhance system performance.

Keywords: Input redundancy, Parameter-dependent systems, Switched systems, Geometric control theory.

Résumé

Les sorties d'un système ne sont pas toujours uniquement déterminées par une seule entrée, un phénomène connu sous le nom de redondance d'entrée, où différentes entrées produisent la même sortie. Cette redondance peut être exploitée pour atteindre des objectifs au-delà de la tâche de commande principale, en tirant parti des degrés de liberté inhérents afin d'améliorer significativement les performances. Cependant, la redondance d'entrée n'a jusqu'à présent été étudiée que dans le cadre des systèmes linéaires invariants dans le temps.

Cette thèse étend le concept aux systèmes dépendant de paramètres, où ceux-ci évoluent au cours du temps, et aux systèmes à commutation, qui combinent dynamiques continues et événements discrets. Pour ces deux cadres, nous introduisons de nouvelles notions adaptées de redondance d'entrée, ne reposant que sur l'existence et l'unicité des solutions. En nous concentrant sur des systèmes à dépendance linéaire en l'état et en l'entrée (et à dépendance polynomiale en les paramètres dans le cas paramétré) dans un contexte non contraint, nous établissons des conditions exploitables permettant de vérifier ces notions. De plus, nous mettons en évidence les degrés de liberté associés à la redondance d'entrée, rendant possible leur utilisation directe pour améliorer les performances des systèmes.

Mots-clés: Redondance d'entrée, Systèmes dépendants de paramètres, Systèmes à commutation, Théorie géométrique du contrôle.

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Notation

In this notation section, x denotes a vector, A a linear application, $\mathcal{V}$ a vector space, and M a matrix.

$\mathbb{R}$	the set of the real numbers
$\mathbb{R}^+$	the set of real positive numbers including zero
$\mathbb{N}$	the set of natural numbers including zero
$\mathbb{N}^*$	the set of natural numbers not including zero
$\mathbb{Z}$	the set of integer numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^-$	the set of complex numbers with negative real part
$\mathbb{R}^n$	n-dimensional Euclidean space with $n \in \mathbb{N}^*$
$[\![a, b]\!]$	the set $[a, b] \cap \mathbb{Z}$ for any $a, b \in \mathbb{Z}$ satisfying $a \leq b$
$\in$	belongs to
$\emptyset$	empty set
$\exists$	exists
$\forall$	for all
$\subseteq$	non-strict set inclusion
$\cap$	set intersection
$\not\sim$	not-equivalent
$:=$	is defined as
$\otimes$	the Kronecker product
$\circ$	function composition $((f \circ g)(x) = f(g(x)))$
$x^\top$	transpose of $x \in \mathbb{R}^n$
x_i	the i-th component of x
I_n	identity matrix of dimension $n \times n$
$\mathbf{0}$	when no confusion can arise, the zero element of a space (vector, matrix, function)

$\mathbf{0}_n$	zero vector of size n
$\mathbf{0}_{n\times m}$	zero matrix with n rows and m columns
$\mathbf{1}_n$	vector of size n where each entry is 1
$M^\top$	transpose of $M \in \mathbb{R}^{n\times m}$
$\mathrm{diag}_{i=0}^{N} M_i$	the block diagonal matrix where matrix M_i is the i-th block.
$\underset{k\in\mathcal{K}}{\mathrm{cat_v}}\{M_k\}$	the vertical concatenation of matrices M_k for all k elements in the set of indexes $\mathcal{K}$
$\underset{k\in\mathcal{K}}{\mathrm{cat_h}}\{M_k\}$	the horizontal concatenation of matrices M_k for all k elements in the set of indexes $\mathcal{K}$
$\mathrm{Im}\,\{V\}$	the vector space spanned by the columns of matrix V
$\ker\{V\}$	the null space of matrix V
$\mathrm{rank}\,\{V\}$	the dimension of the column (or row) space of matrix V
$\lvert\cdot\rvert$	the cardinality of the set
$\times$	the Cartesian product
$\mathcal{V}^N$	the Cartesian product of $\mathcal{V}$, N times: $\mathcal{V}\times\mathcal{V}\times\cdots\times\mathcal{V}$
$A^{-1}\mathcal{V}$	the inverse map of set $\mathcal{V}$ by A: $\{u(t)\in\mathbb{R}^m;\ Au(t)\in\mathcal{V}\}$
$\mathcal{V}_\perp$	the orthogonal complement of $\mathcal{V}$
$\dim\{\mathcal{V}\}$	the dimension of $\mathcal{V}$
$\mathcal{W}\setminus\mathcal{M}$	the set of all elements that are in $\mathcal{W}$ but not in $\mathcal{M}$
$\mathcal{X}/\mathcal{V}$	the quotient of $\mathcal{X}$ by $\mathcal{V}$ ($\mathcal{X}$ a vector space of finite dimension, $\mathcal{V}$ a subspace of $\mathcal{X}$)
$\mathbf{F}(\mathcal{V})$	the set of friends of $\mathcal{V}$
$x_a \neq x_b$	$\int \lvert\lvert x_a(\tau)-x_b(\tau)\rvert\rvert \mathrm{d}\tau > 0$ for time functions x_a and x_b, defined on an appropriate time interval
$\mathcal{PC}(\mathbb{R}^+,\Theta)$	the set of piecewise continuous functions from $\mathbb{R}^+$ to Θ with a finite number of discontinuities in a finite length interval
$\beta_1 \mid \beta_2$	β_2 is divisible by β_1, for polynomial functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$
$\deg(\beta)$	the degree of the polynomial function β
$P_1\wedge P_2$	the property P_1 and P_2
$P_1\vee P_2$	the property P_1 or P_2
$\neg P$	the negative of proposition P

Chapter 1

General introduction

A dynamical system is a process that evolves over time according to natural laws. It is usually described by a mathematical model that specifies how the variables of the system, namely states, change in response to input functions. The output of the system typically depends on both its states and inputs. In control theory, the main objective is to design inputs that steer the system toward desired behaviors, such as stability or reference tracking. In many cases, a single input function uniquely determines the output trajectory. However, the input–output mapping is not always injective. Some systems exhibit input redundancy, where distinct inputs yield the same output. In practice, this provides degrees of freedom in selecting control inputs to achieve a desired output. Input redundancy is not confined to a narrow class of systems; it is in fact widespread in engineering applications. This stems from the fact that many systems are deliberately designed with more actuators than strictly necessary to achieve the primary control objectives. For instance, in aerospace, power electronics, and automotive systems, redundant actuators are introduced to improve performance by reducing unexpected harmonics, enabling faster responses or reducing overall consumption [51, 28, 60]. Redundancy is also exploited to extend actuator constraints, allowing operation in regimes that would not be feasible with a single actuator [18, 29]. Finally, the use of input redundancy is a cornerstone of fault tolerance, ensuring continued operation despite actuator failures by reallocating control effort among the remaining actuators [1, 63]. For these reasons, input redundancy is not merely an abstract property, but a deliberate design feature in many high-performance and safety-critical systems.

This property offers additional flexibility, enabling the pursuit of secondary objectives beyond the primary control task. This has motivated the development of a two-layer control architecture, where a primary controller computes the required total control effort, and a secondary allocation layer distributes this effort among redundant inputs to optimize auxiliary objectives. This approach, widely known as control allocation [11, 29, 41], relies on identifying which inputs lead to the same output. A rigorous characterization of input redundancy is therefore essential: it establishes whether a system possesses redundancy and reveals the associated degrees of freedom available for control. Without such characterization, part of the redundancy may remain inaccessible. While input redundancy has been characterized for Linear Time-Invariant (LTI) systems, the problem remains largely unresolved in more general settings.

1.1 Objective

The aim of this thesis is to extend the concept of input redundancy beyond the LTI framework. Achieving a complete characterization for general nonlinear systems would be ideal, but remains a formidable challenge. As a tractable yet significant step forward, we focus on two important classes: parameter-dependent linear systems and switched linear systems. These systems go beyond LTI systems by capturing a much richer set of dynamical behaviors, while retaining enough structure to enable rigorous analysis. They also arise naturally in many applications, making their study both practical and relevant.

Our contribution is twofold. First, we introduce new notions of input redundancy specifically tailored to parameter-dependent and switched systems, relying solely on the fundamental assumption of existence and uniqueness of solutions. This level of generality ensures that the notions remain valid even when the framework is later extended. Second, for the linear parameter-dependent and switched settings, we provide tractable conditions that make input redundancy verifiable in practice. Together, these results open the way for more systematic exploitation of input redundancy in control allocation strategies, ultimately enabling more efficient and robust control.

1.2 Organization of the Thesis

This thesis is structured as follows:

Chapter 2: This chapter provides an overview of the concept of input redundancy and the established literature on LTI systems. It begins by recalling the formal definition of input redundancy, illustrated with a practical example of a hybrid electric vehicle. A literature review then revisits the main characterizations of input redundancy for LTI systems. Following a reformulation of the definition in terms of signals, related theoretical developments are detailed and applied to the hybrid electric vehicle example. The chapter concludes with original contributions concerning specific elements of input redundancy in LTI systems.

Chapter 3: This chapter extends and characterizes input redundancy for parameter-dependent systems. It introduces the notions of *adaptive* and *robust input redundancy*, and presents the framework of linear parameter-dependent systems with system matrices depending polynomially on the parameters. Key elements for characterization are identified, including an invariant subspace confining state trajectories independently of the parameters, and a parameter-dependent subspace describing redundant inputs. The chapter then proposes tractable strategies for computing both subspaces. Finally, a formal characterization of robust and adaptive input redundancy for these systems is provided.

Chapter 4: This chapter addresses input redundancy in switched systems. Two notions are introduced: *continuous input redundancy*, when the switching signal is not a control input, and *switched input redundancy*, when it is. For continuous input redundancy, the chapter provides a full characterization for switched linear systems, along with constructive results to determine inputs producing identical outputs. For switched input redundancy, the analysis focuses on systems with two linear autonomous modes, presenting both the characterization and constructive methods to generate switching laws yielding the same output trajectory.

Chapter 5: This chapter summarizes the main results of the thesis and concludes with a discussion of open problems and potential directions for future research.

1.3 Contributions

The main contributions of this thesis concern the extension of input redundancy and the development of tractable characterizations for both parameter-dependent and switched linear systems. However, we also develop some technical results, related to the LTI case. They can be summarized as follows, and put by order of appearance in the thesis:

1. Linear Time-Invariant Systems

1.1. Extension of the controlled invariant and output invisible subspace algorithm to more general settings, accommodating larger matrix dimensions and higher-order subspaces (Section 2.4.1).

1.2. Analysis of *finite-time* trajectories and their relation to input redundancy (Section 2.4.2).

2. Parameter-Dependent Systems

2.1. Extension of input redundancy to parameter-dependent systems, relying only on the existence and uniqueness of solutions. Two new notions are introduced: *adaptive input redundancy* and *robust input redundancy*, depending on the availability of the time-varying parameter (Section 3.2).

2.2. Characterization tools for systems linear in state and input and polynomial in the parameter:

(a) Extension and tractable computation of the *weakly unobservable subspace* in the parameter-dependent context (Section 3.4).

(b) Extension and tractable computation of the *weakly unobservable input subspace* in the parameter-dependent context (Section 3.5).

2.3. Characterization of both adaptive and robust input redundancy based on the previous subspace extensions (Section 3.6).

3. Switched Linear Systems

3.1. Extension of input redundancy to switched systems, with no assumptions beyond existence and uniqueness of solutions. Two notions are defined: *continuous input redundancy* (switching signal not treated as a control input) and *switched input redundancy* (continuous input as a non-controlled variable) (Section 4.2).

3.2. Continuous input redundancy (Section 4.3):

(a) Characterization with respect to continuous inputs.

(b) Tools to construct distinct inputs producing identical outputs.

3.3. Switched input redundancy (Section 4.4):

(a) Focus on the simplified case of two autonomous modes.

(b) Characterization of switched input redundancy.

Contributions 1.1 and 2.2(a) are based on the journal paper [74], while contributions 2.2(b) and 2.3 build on [75]. Contribution 3.2 is derived from [73], and contribution 3.3 is based on [76]. For all of these contributions, the thesis includes additional original elements, making the work a substantial contribution in its own right.

The remaining contributions, namely 1.2, 2.1, and 3.1, are entirely original to this thesis and may form the basis for future publications.

1.4 Publications

In this section, a list summarizing the publications obtained from this thesis is provided.

- **Journal papers:**
  1. V. V. Viana, J. Kreiss, M. Jungers, On the computation of controlled invariant and output invisible subspaces for parameter-dependent systems, IEEE Transactions on Automatic Control, vol. 69, no. 7, pp. 4695-4701, 2024. [74]
  https://doi.org/10.1109/TAC.2024.3356170
  2. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched linear systems via polynomial parameter-dependent systems, IEEE Control Systems Letters, vol. 8, pp. 1066-1071, 2024. [73]
  https://dx.doi.org/10.1109/LCSYS.2024.3405380
  3. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched systems concerning the switching signal, IEEE Control Systems Letters, vol. 9, pp. 1520-1525, 2025. [76]
  https://doi.org/10.1109/LCSYS.2025.3582170
  4. V. V. Viana, J. Kreiss and M. Jungers, Adaptive input redundancy of polynomial parameter-dependent systems, in IEEE Transactions on Automatic Control, 2025. [75]
  https://dx.doi.org/10.1109/TAC.2025.3615034

- **Conference papers:**
  1. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched linear systems via polynomial parameter-dependent systems, IEEE Conference on Decision and Control (CDC), Milan, Italy, 2024. (Also in L-CSS).
  2. V. V. Viana, S. Galeani, M. Sassano, On the construction of a minimal order annihilator and its role in dynamic control allocation, IEEE Conference on Decision and Control (CDC), Rio de Janeiro, Brazil, 2025. [This contribution has been prepared during a 2-months visit of Valessa V. Viana to Università di Roma, Tor Vergata, co-funded by DREAM project from Université de Lorraine and by CRAN Laboratory.]
  3. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched systems concerning the switching signal, IEEE Conference on Decision and Control (CDC), Rio de Janeiro, Brazil, 2025. (Also in L-CSS).

Chapter 2

Background on input redundancy

2.1 Introduction

This chapter provides a comprehensive review of the established literature on input redundancy and its characterization within the framework of LTI systems. This review forms the foundation for the developments presented in Chapters 3 and 4. Using a guiding example, we aim to build intuitive insight into this concept and highlight its relevance in control theory. Section 2.3 revisits the main characterizations of input redundancy for LTI systems, following the recent perspectives introduced in [49], where the property is reformulated in terms of signals. This shift in viewpoint is crucial for extending the notion beyond LTI systems. Then, we return to the guiding example to illustrate the theoretical developments, offering a detailed explanation of how the degrees of freedom of the system associated with input redundancy can be systematically identified.

In addition to surveying existing results, two original contributions to the LTI framework are proposed in Section 2.4: (i) an extension of the algorithms associated with one of the core elements of input redundancy, the weakly unobservable subspace, to a specific dimension not directly linked to a dynamical system; and (ii) the unveiling of new structural properties related to the finite-time trajectories of input redundant systems. Although developed within the LTI setting, these contributions constitute essential building blocks for the broader developments proposed in Chapters 3 and 4.

2.2 The concept of input redundancy

2.2.1 Context and definition

In many applications, system architects choose to replicate actuators for various reasons, including fault tolerance, enhanced performance, and extension of actuator capabilities. The resulting over-actuated system contains more inputs than are strictly necessary to control its performance output, leading to what is known as an input redundant system.

Before proceeding further, we provide a precise and general definition of input redundancy, which serves as a foundation for the following discussions. To this end, consider a dynamical system Σ, represented by Figure 2.1 and defined below.

Definition 2.1 (System Σ)**.** *A system, denoted by Σ, is characterized by*

- *an input $u : \mathbb{R}^+ \to \mathcal{U}$, where $\mathcal{U}$ is the set of admissible input values,*

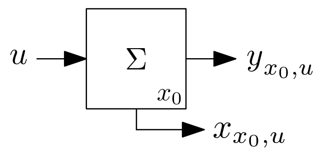

Figure 2.1: System Σ with input u, initial condition x_0, state $x = x_{x_0,u}$, and output $y = y_{x_0,u}$.

- *a state $x : \mathbb{R}^+ \to \mathcal{X}$ with initial condition $x(0) = x_0 \in \mathcal{X}$, where $\mathcal{X}$ is the set of admissible state values,*
- *and a performance output $y : \mathbb{R}^+ \to \mathcal{Y}$, where $\mathcal{Y}$ is the set of admissible output values. The output y reflects the control objective.*

Whenever, for a given initial condition x_0 and input u, the corresponding state and output trajectories are well defined and unique, we denote them by $x_{x_0,u}$ and $x_{x_0,u}$, respectively.

In what follows, we impose the following assumption to ensure the existence and uniqueness of the state and output trajectories of Σ.

Assumption 2.1 (Existence and Uniqueness)**.** *The dynamics of the system Σ is assumed to be sufficiently regular to guarantee well-posedness. Moreover, the admissible inputs of Σ belong to the set of piecewise continuous functions $\mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ from $\mathbb{R}^+$ to $\mathcal{U}$.*

We are now ready to formally define input redundancy.

Definition 2.2 ([49])**.** *System Σ is said to be Input Redundant (IR) if there exists an initial condition and two distinct input trajectories such that the resulting output trajectories are identical, i.e.:*

$$\exists x_0 \in \mathcal{X}, u_a, u_b \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U}) : \; u_a \neq u_b, \; y_{x_0,u_a} = y_{x_0,u_b}. \tag{2.1}$$

In this definition and throughout the rest of the thesis:

- The equality of output trajectories must be understood pointwise over time, i.e., $y_{x_0,u_a}(t) = y_{x_0,u_b}(t)$ for all $t \geq 0$;
- The difference between input trajectories must be understood in the sense of being different over a non-zero measure subset of time, i.e., $u_a \neq u_b$ means that $\int_0^{+\infty} ||u_a(\tau) - u_b(\tau)|| \mathrm{d}\tau > 0$ in the sense of Lebesgue integration.

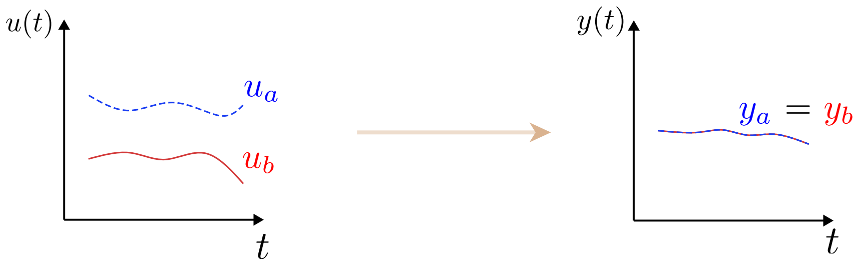

Figure 2.2: Input redundancy property.

This definition, which applies to a very general class of systems (no assumptions are made on the structure of Σ, except the ones for the existence and the uniqueness of the state and output trajectories), highlights the existence of degrees of freedom in IR systems. To see this, assume that y_{x_0,u_a} is a desired output trajectory. Since $y_{x_0,u_a} = y_{x_0,u_b}$, both inputs u_a and u_b achieve the same performance output from the same initial condition x_0, as described in Figure 2.2. The ability to choose between these two distinct inputs, without affecting the output, reflects a degree of freedom in the control input.

To make this notion more intuitive, we continue our discussion using a well-known physical system that exhibits this property and serves as a guiding example throughout the thesis: the Hybrid Electric Vehicle (HEV). The physical interpretation in this context is intended to help the reader grasp the core idea behind input redundancy in a concrete and meaningful way.

Guideline Example - HEV. *To enhance vehicle performance and efficiency, system architects explored the idea of integrating multiple energy sources within a single vehicle [62]. In this context, an HEV combines two power units: an internal combustion engine and an electric motor, both connected to a common driveline. Intuitively, to drive the vehicle at a desired velocity, the performance output in this case, the required power can be distributed arbitrarily between these two actuators. This power distribution constitutes a degree of freedom with respect to the speed control objective. It can be exploited to optimize energy consumption and reduce emissions.*

We illustrate this concept in more details using a simplified model of an HEV coming from [30, 62]. To construct the state-space model of an HEV, [30, 62] proceed in three steps: (*i*) *modeling the fuel engine,* (*ii*) *modeling the electric motor, and* (*iii*) *applying Newton's second law of motion.*

Fuel engine: *The fuel engine can be modeled in a simplified way as a combination of two components: a time delay, representing the latency between fuel injection and its arrival at the engine, and a first-order low-pass filter, capturing the combustion lag dynamics. Using the Padé approximation of order* $(1, 1)$ *[36] to represent the exponential related to the constant time delay, the fuel engine system* Σ_{FE} *can be expressed as:*

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -\frac{2}{t_d} & 0 \\ \frac{4}{t_d} & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_1(t), \\ T_{FE}(t) &= \frac{1}{\tau} x_2(t), \end{aligned} \tag{2.2}$$

where

- $u_1 : \mathbb{R}^+ \to \mathbb{R}$ *is the input of* Σ_{FE}*, representing the torque command;*
- x_1 *which represents the state associated with the Padé approximation of the time delay and* x_2 *which reflects the combustion dynamics are such that* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \mathbb{R}^+ \to \mathbb{R}^2$*;*
- *and* $T_{FE} : \mathbb{R}^+ \to \mathbb{R}$ *is the produced torque of the fuel engine (output).*

The parameters $t_d, \tau > 0$ *correspond to the fueling delay and the combustion time constant, respectively.*

Electric motor: *With an input voltage u_2 and a back Electromotive Force (EMF) that depends on the angular speed ω, the electric motor generates a current x_3 according to its electrical dynamics, which can be modeled as a first-order low-pass filter. The electric motor system Σ_{EM} dynamics are given by:*

$$\begin{aligned} \dot{x}_3(t) &= -\frac{R_a}{L_a}x_3(t) - K\omega(t) + u_2(t), \\ T_{EM}(t) &= \frac{K}{L_a}x_3(t), \end{aligned} \tag{2.3}$$

where

- $\begin{bmatrix} u_2 \\ \omega \end{bmatrix} : \mathbb{R}^+ \to \mathbb{R}^2$ *is the input of* Σ_{EM}*;*
- $x_3 : \mathbb{R}^+ \to \mathbb{R}$ *constitutes the* Σ_{EM} *state;*
- *and the produced torque by the electric motor* $T_{EM} : \mathbb{R}^+ \to \mathbb{R}$ *is the* Σ_{EM} *output.*

The parameters $R_a, L_a, K > 0$ denote the armature resistance, armature inductance, and back-EMF constant, respectively.

Newton's second law of motion: *Newton's second law states that the net torque acting on the vehicle's drivetrain is equal to the rotational inertia J multiplied by the angular acceleration $\dot{\omega}$. The net torque arises from the contribution of the engine torque T_B and the motor torque T_M, opposed by engine torque loss T_L due to friction torque, ancillary torque, and pumping loss. Additionally, the system experiences a speed-dependent resistive torque $b\omega$, where $b > 0$ is the rotational damping coefficient. The resulting rotational dynamics can be described by:*

$$J\dot{\omega}(t) + b\omega(t) = T_{FE}(t) + T_{EM}(t) - T_L(t), \tag{2.4}$$

where J denotes the rotational inertia. The torque loss is given by $T_L = m\omega$ where $m > 0$ is the loss coefficient. Introducing a new state variable $x_4 = J\omega$, the vehicle law of motion is described by the following system Σ_N:

$$\begin{aligned} \dot{x}_4(t) &= T_{FE}(t) + T_{EM}(t) - \frac{b+m}{J}x_4(t), \\ y(t) &= \frac{r}{J}x_4(t), \end{aligned} \tag{2.5}$$

where

- $\begin{bmatrix} T_{FE} \\ T_{EM} \end{bmatrix} : \mathbb{R}^+ \to \mathbb{R}^2$ *is the input of* Σ_N*;*
- $x_4 : \mathbb{R}^+ \to \mathbb{R}$ *is the* Σ_N *state;*
- $y : \mathbb{R}^+ \to \mathbb{R}$ *is the* Σ_N *output;*

and $r > 0$ is the wheel radius.

To build the HEV system Σ_{HEV}, we connect the three systems Σ_{FE}, Σ_{EM} and Σ_N together in the following way:

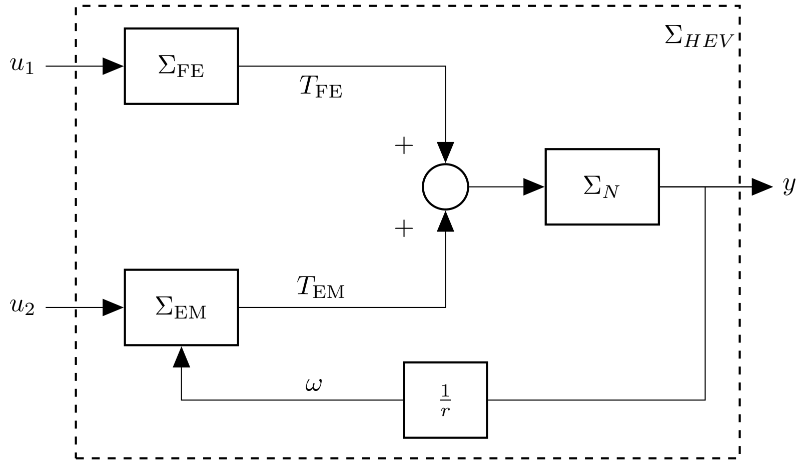

Figure 2.3: Block diagram of Σ_{HEV}.

- *The output of Σ_{FE}, namely T_{FE}, is connected to the first input of Σ_N;*
- *The output of Σ_{EM}, namely T_{EM}, is connected to the second input of Σ_N;*
- *the output of Σ_N divided by r, namely $\frac{y}{r}$ is connected to the second input of Σ_{EM}.*

Figure 2.3 illustrates this interconnection. We obtain the following description of Σ_{HEV}:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \frac{-2}{t_d} & 0 & 0 & 0 \\ \frac{4}{t_d} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & \frac{-K}{J} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & \frac{-(m+b)}{J} \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 0 & 0 & \frac{r}{J} \end{bmatrix} x(t), \end{aligned} \tag{2.6}$$

where

- $u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : \mathbb{R}^+ \to \mathbb{R}^2$ *is the input of* Σ_{HEV};
- $x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \mathbb{R}^+ \to \mathbb{R}^4$ *corresponds to its state;*
- y *is its output,*

Note that the performance output y corresponds to the linear velocity of the HEV. The parameter values from [30] that are considered for simulations are given in Table 2.1.

Parameter	**Value**	**Unit**
Fueling delay (t_d)	90	ms
Combustion lag (τ)	140	ms
Motor armature resistance (R_a)	1	Ω (Ohm)
Motor armature inductance (L_a)	0.3	H (Henry)
Motor back EMF constant (K)	0.25	Vsrad^{-1}
Effective hybrid rotational inertia (J)	0.6	kgm^{2}s^{-2}
Effective hybrid rotational damping (b)	0.125	Nms
Coefficient related to the torque loss (m)	0.12	-
Radius of the wheel (r)	0.3	m

Table 2.1: System Parameters and Values.

Suppose that we aim to regulate the vehicle's velocity to a reference value of $y_{ref} = 5$ m/s, *which corresponds to* 18 km/h. *One possible approach is to rely exclusively on the fuel engine by applying the following input trajectory:*

$$u_a(t) = \begin{bmatrix} \frac{m+b}{r} y_{ref} \\ \frac{K}{J} x_4(t) \end{bmatrix}, \tag{2.7}$$

where the term associated with[1] $u_{a,2}$ *compensates for the coupling between the vehicle speed and the electrical torque.*

With the notations $x_a = x_{x0,u_a}$ *and* $y_a = y_{x0,u_a}$, *Figure 2.4 shows that, although the electric motor input is nonzero, its generated torque remains zero throughout the entire time horizon since* $x_{a,3} = \mathbf{0}$. *This indicates that only the fuel engine contributes to the propulsion of the vehicle.*

Alternatively, we can achieve the exact same output trajectory by using only the electric motor. Consider the following input trajectory:

$$u_b(t) = \begin{bmatrix} 0 \\ \frac{R_a}{L_a} x_3(t) + \frac{K}{J} x_4(t) + \frac{L_a(m+b)}{rK(t_d-2\tau)} \left(4e^{-\frac{2t}{t_d}} - \left(2 + \frac{t_d}{\tau}\right) e^{-\frac{t}{\tau}}\right) y_{ref} \end{bmatrix}. \tag{2.8}$$

The results presented in Figure 2.5, $x_b = x_{x0,u_b}$ *and* $y_b = y_{x0,u_b}$, *confirm that the output remains identical. However, examining the state evolution reveals that, in this case, the torque applied to drive the vehicle is provided exclusively by the electric motor.*

We observe that two distinct input trajectories yield exactly the same output response. Therefore, we conclude that the HEV is IR.

This practical example illustrates that, even when physical intuition suggests that the system is IR (e.g., the required torque can be freely distributed between the two engines), it is not necessarily straightforward to determine an alternative input u_b that produces the same output as a given input u_a. The expression of the input trajectory u_b, given by Equation (2.8), are

[1]The i-th component of u_a is denoted $u_{a,i}$, as generally defined in the Notation section.

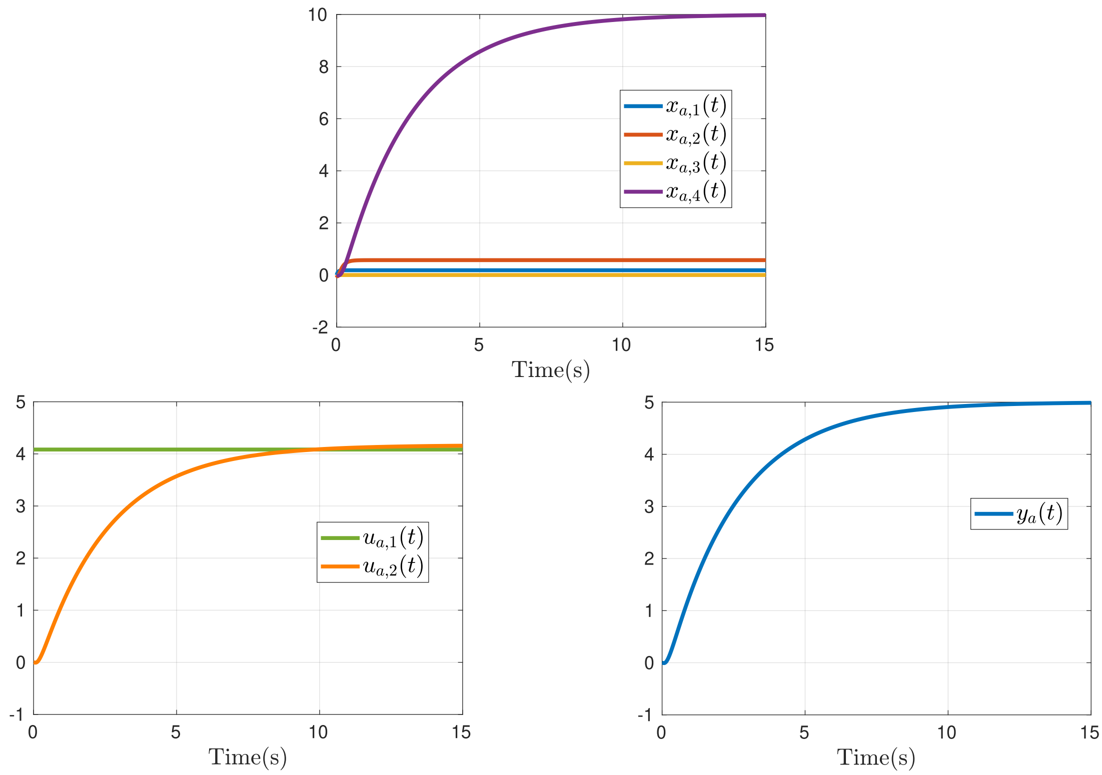

Figure 2.4: HEV trajectories for input u_a, given in (2.7), (Fuel Engine Only) and $x_0 = \mathbf{0}$.

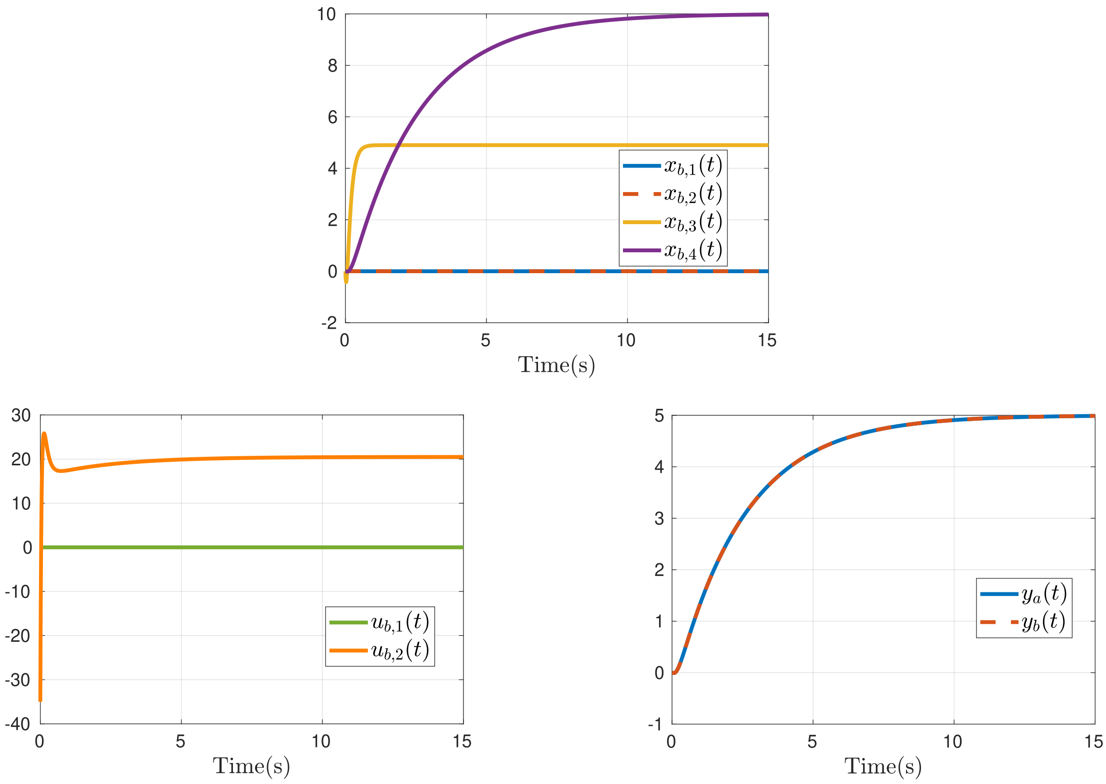

Figure 2.5: HEV trajectories for input u_b (Electric Motor Only) and $x_0 = \mathbf{0}$.

explained in the sequel of the chapter. Furthermore, it is worth noting that in the literature, this system is typically not classified as over-actuated, as discussed in [41], because the two inputs that produce the same output do not result in identical state trajectories. These two points lead us to the following fundamental questions:

- For a system, how can we systematically check if it is IR?
- For an IR system with initial state x_0, how can we design two different inputs that yield the same output?

To answer these questions, it is of foremost importance to deepen our understanding of input redundancy and develop a tractable characterization of the concept. The objective is not only to determine whether a system exhibits input redundancy, but also to provide the user with a complete description of the associated degrees of freedom.

2.2.2 Core characteristics of input redundancy

Based on Definition 2.2, input redundancy refers to the existence of two distinct input functions that yield the same output trajectory. A meaningful classification of input redundancy arises when we also consider the associated state trajectories, leading to different categories of input redundancy. The core question is thus: are the state trajectories the same or different? Before delving deeper into this classification, we first review the existing literature on input redundancy and highlight why the analysis of state trajectories is both relevant and necessary.

Traditionally, research has focused on the case where two different inputs producing the same output also generate the same state trajectory [14, 29]. To review the different conventions of input redundancy in the literature, let us focus on the framework commonly considered, namely the LTI case. System Σ, according to Definition 2.1, can be described by the following equations for all $t \geq 0$:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathcal{X}, \tag{2.9a}$$
$$y(t) = Cx(t) + Du(t), \tag{2.9b}$$

where $\mathcal{U} = \mathbb{R}^m$, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^p$, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are the system matrices.

The subclass of IR systems considered in the literature is defined by:

$$\dim\left(\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}\right) > 0, \tag{2.10}$$

and is usually referred to as *over-actuated systems* [8, 13]. These systems have been extensively studied [41, 59] to optimally manage the degrees of freedom of the actuators [39]. This field is known as *(static) control allocation*[2], and primarily focuses on optimization methods to distribute the total control effort among the actuators [11, 38].

Guideline Example - HEV. *System Σ_{HEV} does not belong to the class of over-actuated systems when considering the B matrix in Equation* (2.6)*. However, a common practice is to neglect the actuator dynamics (in this case, the motor dynamics) by assuming that their outputs can be*

[2]The term *static* is placed in parentheses because it does not explicitly appear in the cited literature. We include it for clarity in what follows.

treated as direct inputs to the system. Under this simplification, the HEV system reduces to Σ_N*, as given in Equation* (2.4)*, where* $B = \mathbf{1}_{1\times 2}$ *(the* 1×2 *matrix whose entries are all equal to one) and* $D = \mathbf{0}_{1\times 2}$ *(the* 1×2 *null matrix), which corresponds to an over-actuated system.*

Indeed, the difference between the torques $T_{FE} - T_{EM}$ *represents an additional degree of freedom that can be exploited to distribute the required torque between the two motors. Nevertheless, neglecting the motor dynamics may lead to inconsistencies in the modeling and control design process.*

In 2009, more than fifteen years after the notion of over-actuated systems emerged, Zaccarian [80] introduced the concept of *weak IR systems*, in contrast to *strong IR systems* (defined as in (2.10)) where the state trajectories are identical. Specifically, weak IR systems are defined as those satisfying

$$\dim\left(\ker\left\{\lim_{s\to 0} C(sI - A)^{-1}B + D\right\}\right) > 0, \tag{2.11}$$

where $s \in \mathbb{C}$ is the Laplace variable. This definition focuses on the steady-state behavior rather than the transient response and paved the way for *dynamic control allocation*, where allocation occurs primarily at steady-state.

A few years later, Serrani [64] revisited this class of systems with an alternative definition, under the assumptions that system Σ in (2.9) is strictly proper ($D = \mathbf{0}$) and right-invertible[3]. In this framework, weak IR systems are defined by $m = \operatorname{rank}\{B\} > p$. Serrani showed that, for such systems, multiple input trajectories can produce the same output trajectory, with the same initial condition, while generating different state trajectories. Several dynamic control allocation schemes have subsequently been proposed in the literature [22, 32, 33, 34, 35].

While these notions have been crucial for advancing the field of control allocation, they suffer from several limitations: (i) they are confined to LTI systems, often with additional restrictive assumptions (e.g., on stability in [80] and on strictly properness and right-invertibility in [35, 64]); (ii) multiple, non-equivalent definitions exist, particularly for weak input redundancy; and (iii) weak and strong input redundancy are not mutually exclusive. For instance, (2.10) implies (2.11) for a stable system. The signal-based definitions, as in Definition 2.2, proposed by [49], aim to address these issues. Further details on their connection with the literature can be found in that reference. We recall below the proposed taxonomy when state trajectories are taken into account.

Definition 2.3 ([49])**.** *An IR system* Σ *is said to be:*

- IR of Kind 1 *when for all initial conditions* x_0 *and for all input trajectories* u_a *and* u_b *producing* y_{x_0,u_a} *and* y_{x_0,u_b} *such that:*

$$u_a \neq u_b,\ y_{x_0,u_a} = y_{x_0,u_b},$$

the state trajectories are identical, i.e., $x_{x_0,u_a} = x_{x_0,u_b}$*.*

- IR of Kind 2 *when for all initial conditions* x_0 *and for all input trajectories* u_a *and* u_b *producing* y_{x_0,u_a} *and* y_{x_0,u_b} *such that:*

$$u_a \neq u_b,\ y_{x_0,u_a} = y_{x_0,u_b},$$

the state trajectories are distinct, i.e., $x_{x_0,u_a} \neq x_{x_0,u_b}$*.*

[3] System (2.9) is called right-invertible if for any arbitrarily assigned impulsive-smooth output y, there exists an impulsive-smooth input u yielding that output with zero initial condition, i.e. $y = y_{0,u}$ [70, Def. 8.11],[57, Def. 2].

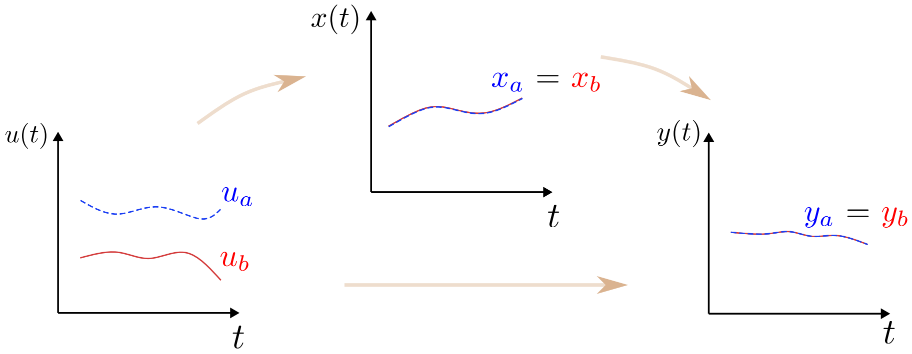

Figure 2.6: Input redundancy of Kind 1.

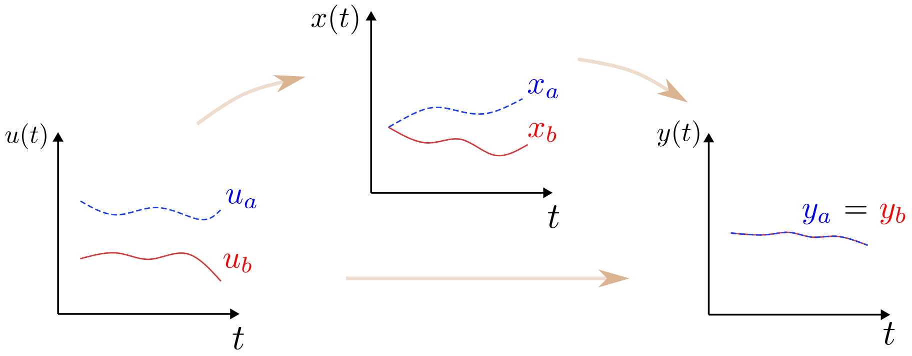

Figure 2.7: Input redundancy of Kind 2.

- IR of Kind 3 *when it is not of kind 1 or kind 2, i.e., there exists* x_0 *such that there exist* u_a, u_b, u_c *such that:*

$$\begin{aligned} &u_a \neq u_b,\;\; x_{x0,u_a} = x_{x0,u_b},\;\; y_{x0,u_a} = y_{x0,u_b},\\ &u_a \neq u_c,\;\; x_{x0,u_a} \neq x_{x0,u_c},\;\; y_{x0,u_a} = y_{x0,u_c}. \end{aligned}$$

Trajectories of an IR system of the first (resp. second and third) kind are illustrated in Figure 2.6 (resp. Figures 2.7 and 2.8).

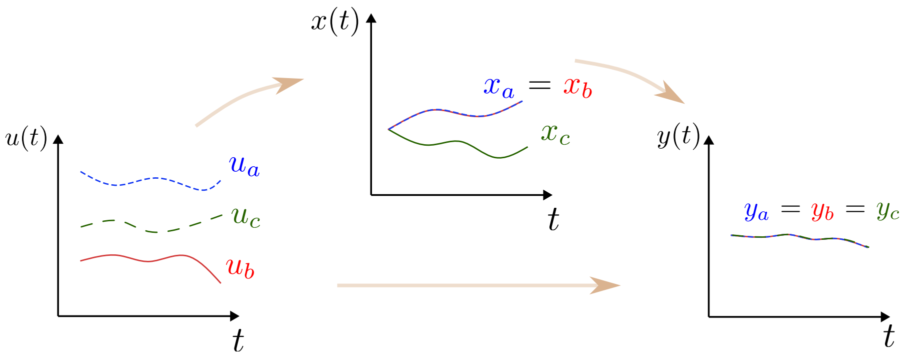

Figure 2.8: Input redundancy of Kind 3.

Guideline Example - HEV. *Figures 2.4 and 2.5 show the state trajectories generated by inputs* u_a *and* u_b*. We verify that such state trajectories differ. Consequently, according to Definition 2.3,* Σ_{HEV} *is not IR of Kind 1, it is either Kind 2 or 3.*

2.3 Characterization of input redundancy for LTI systems

Input redundancy has been fully characterized in the context of LTI systems, as represented by (2.9) in [49]. We focus on this framework to recall and explain this characterization.

2.3.1 Linearity implies uniformity

This fundamental feature is what allows a complete characterization of input redundancy in LTI systems. Recall that the input redundancy property is based on the *existence of an initial condition* x_0 for which two different inputs produce the same output. One of the main advantages of LTI systems is that, due to linearity, this property then holds for *all* initial conditions. This is formally stated in the following proposition:

Proposition 2.1. *[49, Prop. 2.1] If system* (2.9) *is IR, then it holds that for all* x_0*, there exist* u_a *and* u_b *such that* $u_a \neq u_b$ *and* $y_{x_0,u_a} = y_{x_0,u_b}$*.*

2.3.2 Characterization of input redundancy

The characterization relies on an incremental analysis based on the system's linearity. For an IR system, consider an initial condition $x_0 \in \mathbb{R}^n$ and two inputs u_a and u_b such that

$$u_a \neq u_b, \quad y_{x_0,u_a} = y_{x_0,u_b}.$$

By linearity, defining $\tilde{u} := u_a - u_b$ and having that $x_0 - x_0 = \mathbf{0}$, we then have

$$\tilde{u} \neq \mathbf{0}, \quad y_{\mathbf{0},\tilde{u}} = y_{x_0,u_a} - y_{x_0,u_b} = \mathbf{0}.$$

This shows that the system exhibits *invisible inputs*. The notion of invisible inputs is well characterized in geometric control theory [5, 70, 78], recalled in Lemma 2.1, through the so-called weakly unobservable subspace $\mathcal{V}^\star$.

Definition 2.4. *[70, Section 7.3] The weakly unobservable subspace* $\mathcal{V}^\star$ *of System* (2.9) *is defined as the set of points* $x_0 \in \mathbb{R}^n$ *for which there exists an input* u *such that the corresponding output* $y_{x_0,u}$ *is identically zero.*

Relations of a weakly unobservable subspace: The weakly unobservable subspace $\mathcal{V}^\star$ is the largest subspace $\mathcal{V} \subset \mathbb{R}^n$ such that:

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subset \mathcal{V} \times \{\mathbf{0}_p\} + \operatorname{Im}\left\{ \begin{bmatrix} B \\ D \end{bmatrix} \right\}. \tag{2.12}$$

For such a subspace $\mathcal{V}$, the first n lines of inclusion (2.12) corresponds to the property of *controlled invariance*, meaning that there exists an input capable of keeping $\mathcal{V}$ invariant. The remaining p lines of inclusion (2.12) express the property of *output invisibility*: the input that maintains $\mathcal{V}$ invariant also ensures that the output remains identically zero.

Such an input can always be expressed as a state feedback of the form $u = Fx$, where the linear map F is referred to as a *friend* of $\mathcal{V}$. Such a map might not be unique. The set of friends of a subspace $\mathcal{V}$ is denoted by $\mathbf{F}(\mathcal{V})$.

To fully characterize the set of inputs that produce zero output, we introduce, for convenience, the following definition of the *weakly unobservable input subspace* $\mathcal{N} \subset \mathbb{R}^m$, now formulated for the input space.

Definition 2.5. *The weakly unobservable input subspace $\mathcal{N}$ of System* (2.9) *is defined as the set of input directions*[4] *that produce a zero output when the state is confined to the weakly unobservable subspace $\mathcal{V}^\star$. The weakly unobservable input subspace reads*

$$\mathcal{N} := B^{-1}\mathcal{V}^\star \cap \ker\{D\}. \tag{2.13}$$

Here $B^{-1}\mathcal{V}^\star$ refers to the inverse map of $\mathcal{V}^\star$ by B, i.e.

$$B^{-1}\mathcal{V}^\star := \{u \in \mathbb{R}^m : \ Bu \in \mathcal{V}^\star\}. \tag{2.14}$$

Lemma 2.1. *[70, Theorem 7.11] Let $\mathcal{V}^\star$ and $\mathcal{N}$ be the weakly unobservable subspace and weakly unobservable input subspace of System* (2.9)*, respectively. Let $x_0 \in \mathcal{V}^\star$ and let u be an input function. Then the state trajectory resulting from x_0 and u remains in $\mathcal{V}^\star$ and $y_{x_0,u} = \mathbf{0}$ if and only if u has the form*

$$u(t) = Fx_{x_0,u}(t) + Lw(t), \tag{2.15}$$

with $F \in \mathbf{F}(\mathcal{V}^\star)$, L such that $\operatorname{Im}\{L\} = \mathcal{N}$, and for any $w : \mathbb{R}^+ \to \mathbb{R}^{\dim \mathcal{N}}$.

From Lemma (2.1), the necessary and sufficient conditions for an input trajectory u to remain invisible, given an initial condition x_0, are as follows:

$$\begin{cases} x_0 \in \mathcal{V}^\star, \\ u = Fx_{x_0,u} + Lw, \end{cases}$$

for some $w : \mathbb{R}^+ \to \mathbb{R}^{\dim(\mathcal{N})}$.

In the context of input redundancy, recall that both state trajectories x_{x_0,u_a} and x_{x_0,u_b} share the same initial condition. Consequently, the incremental trajectory starts from the zero initial condition, i.e. $x_{\mathbf{0},\tilde{u}} = x_{x_0,u_a} - x_{x_0,u_b}$. Since $\mathcal{V}^\star$ is a vector subspace of $\mathbb{R}^n$, it follows that $\mathbf{0} \in \mathcal{V}^\star$ which satisfies the first condition of Lemma 2.1. Therefore, the only way to satisfy $\tilde{u} \neq \mathbf{0}$ (i.e., to have two distinct inputs u_a and u_b) is to ensure that $Lw \neq \mathbf{0}$. As w is arbitrary, the sole requirement is that $L \neq \mathbf{0}$, which is equivalent to $\dim \mathcal{N} > 0$. This reasoning forms the basis of the proof of the following theorem (omitted here for briefness but available in [49]) that characterizes input redundancy.

Theorem 2.1. *[49, Theorem 3.1] Let $\mathcal{V}^\star \subset \mathbb{R}^n$ and $\mathcal{N} \subset \mathbb{R}^m$ be the weakly unobservable subspace and the weakly unobservable input subspace* (2.13) *of system* (2.9)*, respectively. The following statements are equivalent:*

(i) System (2.9) *is IR.*

(ii) $\dim(\mathcal{N}) > 0$.

[4] Here, “directions” refer to the orientations of the vectors spanning the subspace.

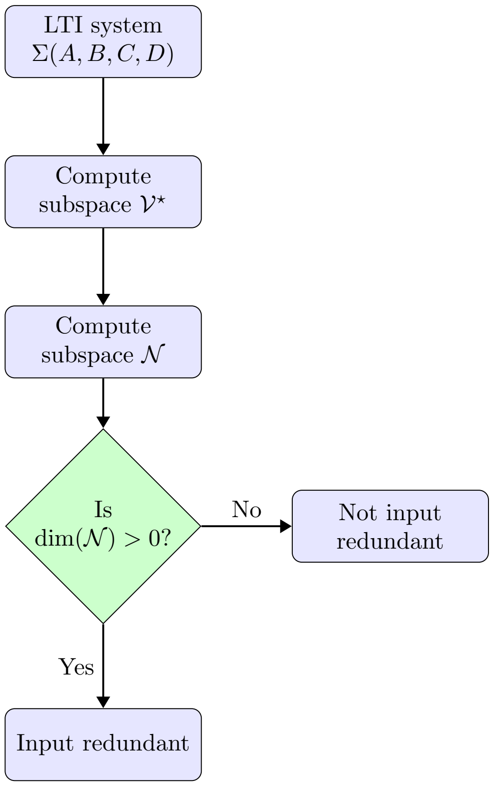

Figure 2.9: Algorithm to check input redundancy.

The main advantage of this theorem lies in its ability to provide a tractable characterization of input redundancy. Instead of exhaustively testing every initial condition with all possible input trajectories to verify whether two of them yield the same output, it suffices to compute the dimension of the subspace $\mathcal{N}$. Furthermore, all the components required for this computation can be obtained algorithmically. The overall procedure is illustrated in Figure 2.9. In particular, the subspace $\mathcal{V}^\star$ can be systematically determined using the finite-step Algorithm 2.1 ([70, Section 7.3]), usually called the *invariant and invisible subspace algorithm*. Algorithm 2.1 starts with the largest possible subspace, i.e., $\mathbb{R}^n$, and returns $\mathcal{V}^\star$ in at most n steps. All the steps shown in Figure 2.9 can be implemented in MATLAB through the geometric control toolbox described

Algorithm 2.1: Invariant and invisible subspace algorithm.

input : A, B, C, D
output : $\mathcal{V}^\star$
$\mathcal{V}_0 \leftarrow \mathbb{R}^n$, $k = 0$;
repeat

$$\mathcal{V}_{k+1} \leftarrow \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\mathcal{V}_k \times \{\mathbf{0}_p\} + \operatorname{Im}\left\{ \begin{bmatrix} B \\ D \end{bmatrix} \right\} \right); \tag{2.16}$$

$k = k + 1$;
until $\mathcal{V}_{k+1} = \mathcal{V}_k$;
return $\mathcal{V}^\star = \mathcal{V}_k$;

in [55].

2.3.3 Characterization of the kinds of input redundancy

Let us now examine in more detail the core component for characterizing input redundancy, namely the weakly unobservable input subspace $\mathcal{N}$. From the definition of $B^{-1}\mathcal{V}^{\star}$ (see (2.14)), it is evident that $\ker\{B\} \subset B^{-1}\mathcal{V}^{\star}$, which in turn implies

$$\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\} \subset \mathcal{N}.$$

This subset of $\mathcal{N}$ gathers all input directions that produce identical state and output trajectories. Interestingly, this shows that all over-actuated systems (see (2.10)) are included as a particular case of IR systems.

As a consequence, the subspace $\mathcal{N}$ can be decomposed into two components, according to the following relation:

$$\dim \mathcal{N} = \dim\left(\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}\right) + \dim\left(\mathcal{N}/\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}\right), \tag{2.17}$$

where $\mathcal{N}/\ker\begin{bmatrix} B \\ D \end{bmatrix}$ denotes the quotient space[5] of $\mathcal{N}$ by $\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}$. This quotient space is related to the input directions that generate different state trajectories while leading to the same output. This decomposition provides a basis for the characterization of the taxonomy introduced in Definition 2.3.

Proposition 2.2. *Consider an IR system* (2.9) *and define* $\rho := \dim\left(\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}\right)$ *and* $\eta := \dim\left(\mathcal{N}/\ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}\right)$. *System* (2.9) *is IR:*

- *of the first Kind when* $\rho > 0$ *and* $\eta = 0$*;*
- *of the second Kind when* $\rho = 0$ *and* $\eta > 0$*;*
- *of the third Kind when* $\rho > 0$ *and* $\eta > 0$.

Guideline Example - HEV. *The HEV model described by Equation* (2.6) *is an LTI system, which falls within the scope of this section. Consequently, the previously established results can be directly applied. To illustrate the process, let us follow step by step the procedure outlined in Figure 2.9.*

[5] See Appendix A.

The first step is to compute the weakly unobservable subspace $\mathcal{V}^\star$. Using Algorithm 2.1, we find that after three iterations:

$$\mathcal{V}^\star = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & \frac{K}{L_a} \\ 0 & -\frac{1}{\tau} \\ 0 & 0 \end{bmatrix}\right\}.$$

Next, we determine the set of inputs whose image through B lies within $\mathcal{V}^\star$: $B^{-1}\mathcal{V}^\star = \mathrm{Im}\{L\}$ where

$$L := \begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} \end{bmatrix}. \tag{2.18}$$

Since in this example $D = \mathbf{0}_{1\times 2}$, we have $\ker\{D\} = \mathbb{R}^2$, which implies $\mathcal{N} = B^{-1}\mathcal{V}^\star = \mathrm{Im}\{L\}$.

$$\mathcal{N} = B^{-1}\mathcal{V}^\star = \mathrm{Im}\left\{\begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} \end{bmatrix}\right\}.$$

As a result, we obtain an IR system with degree $(\rho = 0, \eta = 1)$. Therefore, the system is classified as IR of kind 2 according to Definition 2.3.

Previously in this chapter, we showed that the two distinct inputs u_a in (2.7) and u_b in (2.8) lead to the same output trajectory. Actually, it can be shown that such inputs admit an alternative representation. For all $t \geq 0$:

$$u_a(t) = Fx_{\mathbf{0},u_a}(t) + \left(L + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)\right)\frac{m+b}{r} y_{ref}, \tag{2.19}$$

$$u_b(t) = Fx_{\mathbf{0},u_b}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)\frac{m+b}{r} y_{ref}, \tag{2.20}$$

where

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{4L_a}{K\tau t_d} & \frac{L_a}{K\tau^2} & \frac{R_a}{L_a} & \frac{K}{J} \end{bmatrix} \in \mathbf{F}(\mathcal{V}^\star),$$

and

$$v : t \mapsto \frac{L_a}{K(t_d - 2\tau)}\left(4e^{-\frac{2t}{t_d}} - \left(2 + \frac{t_d}{\tau}\right)e^{-\frac{t}{\tau}}\right). \tag{2.21}$$

As a consequence, we can write $u_a = u_b + \tilde{u}$ with

$$\tilde{u} = Fx_{\mathbf{0},\tilde{u}} + L\frac{m+b}{r} y_{ref} \tag{2.22}$$

which is consistent with the structure of an invisible input (see Lemma 2.1). It is important to emphasize that appending any term of the form Lw to the inputs does not affect the output. In other words, w can be chosen arbitrarily, as it represents the intrinsic degrees of freedom of the system.

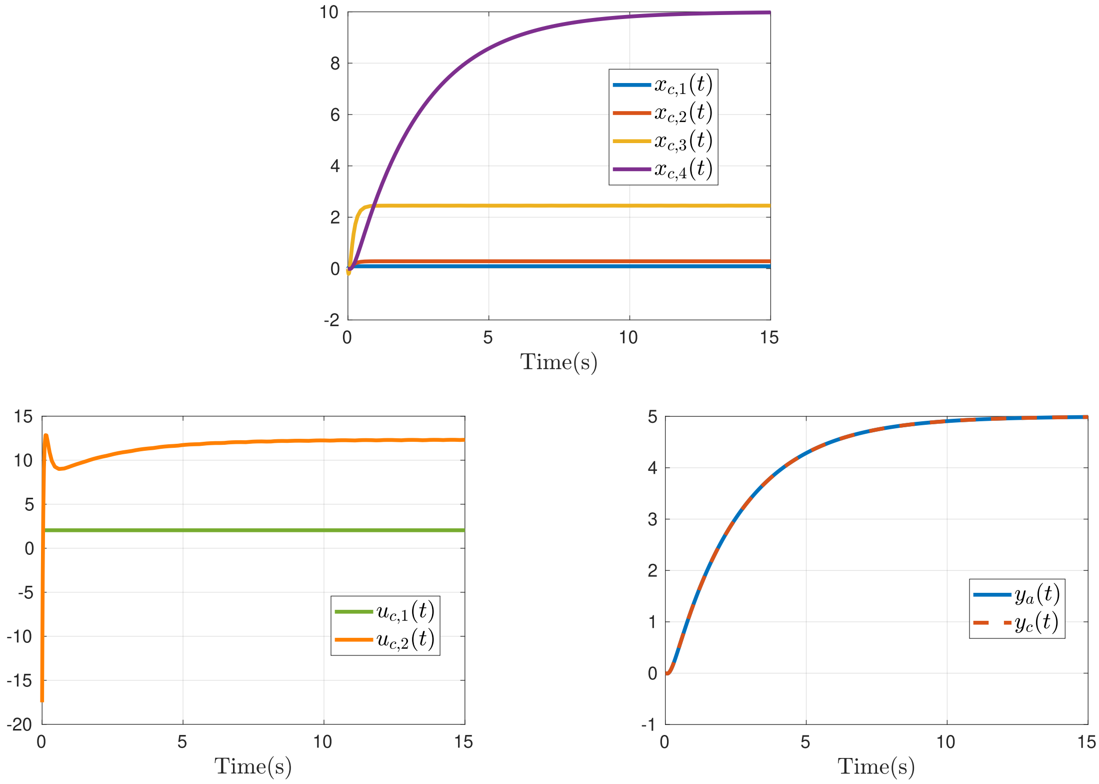

Figure 2.10: HEV trajectories for input u_c (Combination of Fuel Engine and Electric Motor) and $x_0 = \mathbf{0}$.

To illustrate this property, starting from the same initial condition $x_0 = \mathbf{0}$, consider the input

$$u_c(t) = Fx_{\mathbf{0},u_c}(t) + \left(L + \begin{bmatrix} 0 \\ 2 \end{bmatrix} v(t)\right) \frac{m+b}{2r} y_{ref}, \tag{2.23}$$

for all $t \geq 0$ where v is defined by (2.21), $y_{ref} = 5$ m/s, and the parameters considered for simulation are given in Table 2.1. The corresponding state and output trajectories $x_c = x_{\mathbf{0},u_c}$ and $y_c = y_{\mathbf{0},u_c}$ respectively, are shown in Figure 2.10, where we observe that the output trajectory remains unchanged. However, in steady state, the two motors now share the required torque equally, as illustrated in Figure 2.11, to maintain the same velocity. In this example, the degree of freedom w can be interpreted as a parameter controlling torque distribution between the two motors. This flexibility can be exploited to optimize the allocation of torque and improve performance without altering the output. Even though the vehicle's internal behavior changes, these adjustments are completely imperceptible to the passagers, as they do not affect the delivered output.

2.4 Extensions to the classical LTI input redundancy framework

Finally in this chapter, we propose original developments in the LTI framework that provide a foundation for extending the notion of input redundancy to the context of parameter-dependent systems in Chapter 3 and of switched systems in Chapter 4.

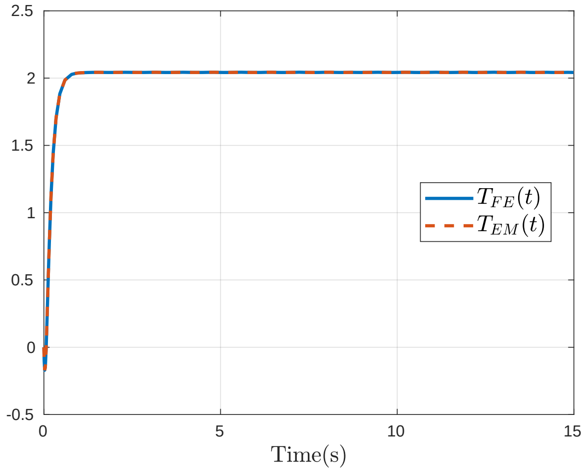

Figure 2.11: HEV torques for input u_c.

2.4.1 Extension of the invariant and invisible subspace algorithm

In this subsection, we present an extension of the invariant and invisible subspace algorithm (Algorithm 2.1) to deal with matrices of extended dimension. We first establish subspace inclusion properties involving extended matrix dimensions and powers of subspaces, as formalized in Lemma 2.2.

Lemma 2.2. *Consider given matrices* $\mathbf{A} \in \mathbb{R}^{nZ_1 \times n}$, $\mathbf{B} \in \mathbb{R}^{nZ_1 \times r}$, $\mathbf{C} \in \mathbb{R}^{pZ_2 \times n}$, $\mathbf{D} \in \mathbb{R}^{pZ_2 \times r}$, *where* $n,\ p,\ r,\ Z_1,\ Z_2 \in \mathbb{N}^*$, *and a subspace* $\mathcal{V} \subset \mathbb{R}^n$. *The following statements are equivalent:*

(i) $\mathcal{V} \subset \mathbb{R}^n$ *satisfies*[6]

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \mathcal{V} \subset (\mathcal{V}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2}) + \mathrm{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\}, \tag{2.24}$$

(ii) There exists a matrix $\mathbf{F} \in \mathbb{R}^{r \times n}$, *such that*

$$\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \mathbf{F} \right) \mathcal{V} \subset \mathcal{V}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2}. \tag{2.25}$$

Proof. $(i) \Rightarrow (ii)$ Let us assume that inclusion (2.24) holds, and let $V \in \mathbb{R}^{n \times q}$ be a basis[7] of $\mathcal{V}$ where $q \le n$ corresponds to the dimension of $\mathcal{V}$. Then, for each column v of V there exist Z_1 vectors $\tilde{v}_i \in \mathcal{V}$, $i = 1, \dots, Z_1$, and a vector $\tilde{u} \in \mathbb{R}^r$ such that

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} v = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \tilde{u} + (\tilde{v}_1^\top \cdots \ \tilde{v}_{Z_1}^\top \quad \mathbf{0}_{1\times p} \cdots \mathbf{0}_{1\times p})^\top. \tag{2.26}$$

[6] $\mathcal{V}^N$ denotes the Cartesian product of $\mathcal{V}$, N times, as defined in the Notation section.

[7] Note that when $\mathcal{V} = \{\mathbf{0}\}$, its dimension q is zero such that the basis V is empty. In this case, $\mathbf{F} = \mathbf{0}$ holds by definition. This trivial situation is omitted in our algorithm.

$\tilde{v}_i \in \mathcal{V}$ can be expressed as $\tilde{v}_i = V\tilde{y}_i$, where $\tilde{y}_i \in \mathbb{R}^q$. Thus, by selecting each column of V for v in (2.26), it is clear that there exist a matrix $U \in \mathbb{R}^{r\times q}$ and a matrix $Y \in \mathbb{R}^{qZ_1\times q}$ such that

$$\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} V = \begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix} U + \begin{bmatrix}\mathrm{diag}_{i=1}^{Z_1} V\\ \mathbf{0}_{pZ_2\times qZ_1}\end{bmatrix} Y. \tag{2.27}$$

Finally, since $V^\top V$ is invertible, we can define the following gain satisfying inclusion (2.25)

$$\mathbf{F} = -U(V^\top V)^{-1}V^\top. \tag{2.28}$$

$(ii) \Rightarrow (i)$ Let us assume that there exists $\mathbf{F}$ such that inclusion (2.25) holds. Then, it is straightforward that inclusion (2.24) also holds. □

Interestingly, the set of subspaces $\mathcal{V}$ satisfying (2.24), denoted by $\mathcal{V}_\mathcal{S}$, forms a *lattice*, that is, an ordered set under the inclusion relation between subspaces. In particular, any two elements of $\mathcal{V}_\mathcal{S}$ admit both a largest lower bound (their intersection) and a smallest upper bound (their sum). The following lemma establishes that $\mathcal{V}_\mathcal{S}$ admits a maximal element.

Lemma 2.3. *The set $\mathcal{V}_\mathcal{S}$ admits a unique maximal element, which is denoted by $\mathcal{V}_{k^\star}$. It means that if $\mathcal{V} \in \mathcal{V}_\mathcal{S}$, then $\mathcal{V} \subset \mathcal{V}_{k^\star}$.*

Proof. The proof is inspired by the developments in [79] and [48]. By definition, the singleton vector is an element of $\mathcal{V}_\mathcal{S}$, i.e., $\{\mathbf{0}_{n\times 1}\} \in \mathcal{V}_\mathcal{S}$. Moreover, $\mathcal{V}_\mathcal{S}$ is closed by addition, i.e. if $\mathcal{V}_1 \in \mathcal{V}_\mathcal{S}$ and $\mathcal{V}_2 \in \mathcal{V}_\mathcal{S}$, $\mathcal{V}_1 + \mathcal{V}_2 \in \mathcal{V}_\mathcal{S}$. From [78, Lemma 4.4], since $\mathcal{V}_\mathcal{S}$ is a nonempty subspace and closed under addition, then $\mathcal{V}_\mathcal{S}$ contains a supremal element denoted by $\mathcal{V}_{k^\star}$. Thus, if $\mathcal{V} \in \mathcal{V}_\mathcal{S}$, $\dim(\mathcal{V} + \mathcal{V}_{k^\star}) \leq \dim(\mathcal{V}_{k^\star})$, which implies $\mathcal{V} \subset \mathcal{V}_{k^\star}$. On the other hand, assume that there exist two distinct maximal elements, these subspaces are included in each other by the definition of maximality. Then, they are identical, proving that the maximal element $\mathcal{V}_{k^\star} \in \mathcal{V}_\mathcal{S}$ is unique. □

An iterative algorithm to compute the maximal subspace $\mathcal{V}_{k^\star}$ is presented in Algorithm 2.2. Its convergence properties are established in Proposition 2.3.

Algorithm 2.2: Extended invariant and invisible subspace algorithm

input : $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$
output : $\mathcal{V}_{k^\star}$
$\mathcal{V}_0 \leftarrow \mathbb{R}^n$, $k = 0$;
repeat
$$\mathcal{V}_{k+1} \leftarrow \begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}^{-1}\left(\mathcal{V}_k^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \mathrm{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\}\right); \tag{2.29}$$
$k = k + 1$;
until $\mathcal{V}_{k+1} = \mathcal{V}_k$;
return $\mathcal{V}_{k^\star} = \mathcal{V}_k$;

Proposition 2.3. *Algorithm 2.2 converges in at most n steps and returns the largest subspace $\mathcal{V}_{k^\star}$ that satisfies* (2.25).

Proof. Let us prove, by recurrence, that the given algorithm returns the largest subspace such that (2.24) or (2.25) holds. First, we prove that the sequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ is not increasing in the sense of inclusion. We have $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathcal{V}_1$, then considering that $\mathcal{V}_{k-1} \supseteq \mathcal{V}_k$ and applying some manipulations, it yields

$$\mathcal{V}_{k-1}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \operatorname{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\} \supseteq \mathcal{V}_{k}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \operatorname{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\},$$

and finally

$$\mathcal{V}_k = \begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}^{-1}\left(\mathcal{V}_{k-1}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \operatorname{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\}\right) \supseteq \begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}^{-1}\left(\mathcal{V}_{k}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \operatorname{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\}\right) = \mathcal{V}_{k+1}.$$

Furthermore, we are also able to prove by recurrence that $\mathcal{V}_k \supseteq \mathcal{V}_{k^\star}$. We also have $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathcal{V}_{k^\star}$. Consider $\mathcal{V}_k \supseteq \mathcal{V}_{k^\star}$, and let us show that $\mathcal{V}_{k+1} \supseteq \mathcal{V}_{k^\star}$,

$$\mathcal{V}_{k+1} = \begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}^{-1}\left(\mathcal{V}_{k}^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \operatorname{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\}\right) \supseteq \begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}^{-1}\left((\mathcal{V}_{k^\star})^{Z_1} \times \{\mathbf{0}_p\}^{Z_2} + \operatorname{Im}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\right\}\right) = \mathcal{V}_{k^\star}.$$

Thus, $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ is a non-increasing sequence, and there exists one integer $K \leq n-1$ such that Algorithm 2.2 converges in finite number of steps and $\mathcal{V}_{K+1} = \mathcal{V}_K$, then it follows that $\mathcal{V}_K \in \mathcal{V}$. Since there exists a maximal element $\mathcal{V}_{k^\star}$, we have $\mathcal{V}_K \supseteq \mathcal{V}_{k^\star} \supseteq \mathcal{V}_K$, which is equivalent to the equality $\mathcal{V}_K = \mathcal{V}_{k^\star}$, completing the proof. □

Remark 2.1. *Note that Algorithm 2.2 reduces, as a special case, to the classical invariant and invisible subspace algorithm when setting*

$$Z_1 = Z_2 = 1, \quad \mathbf{A} = A, \quad \mathbf{B} = B, \quad \mathbf{C} = C, \quad \textit{and} \quad \mathbf{D} = D,$$

thereby allowing the computation of the weakly unobservable subspace $\mathcal{V}^\star$.

2.4.2 On trajectories in finite time for input redundancy

From Definition 2.2, it follows that the property of input redundancy is manifested through input-output trajectories defined on $\mathbb{R}^+$. An important question naturally arises: what can be said about *finite-time trajectories*? In what follows, we establish in Lemma 2.4 that an IR system can be fully characterized by specific finite-time trajectories. To this end, we denote by $u|_{[0,T]}$ a trajectory u defined specifically over the interval $[0,T]$, i.e. $u|_{[0,T]} : [0,T] \to \mathcal{U}$. We also use this notation when restricting a trajectory $u : \mathbb{R}^+ \to \mathcal{U}$ to the time interval $[0,T]$.

Lemma 2.4. *Consider a system Σ of the form* (2.9). *The two following statements are equivalent:*

(S1) $\exists x_0 \in \mathbb{R}^n, T > 0, u_a|_{[0,T]}, u_b|_{[0,T]}: \; u_a|_{[0,T]} \neq u_b|_{[0,T]},\; y_{x_0,u_a|_{[0,T]}}|_{[0,T]} = y_{x_0,u_b|_{[0,T]}}|_{[0,T]}$;

(S2) System (2.9) *is IR.*

Proof. (S2)⇒(S1): Suppose that System (2.9) is IR. Then there exist $\tilde{x}_0 \in \mathbb{R}^n$, $\tilde{u}_a$, $\tilde{u}_b$ such that $\tilde{u}_a \neq \tilde{u}_b$ and $y_{\tilde{x}_0,\tilde{u}_a} = y_{\tilde{x}_0,\tilde{u}_b}$. Considering $x_0 = \tilde{x}_0$, and $u_a|_{[0,T]}(t) = \tilde{u}_a(t)$, and $u_b|_{[0,T]}(t) = \tilde{u}_b(t)$, $\forall t \in [0, T]$, where $T > \min_t(\|\tilde{u}_a(t) - \tilde{u}_b(t)\| > 0)$ show the implication.

(S1)⇒(S2): Let us proceed by contraposition. Suppose that System (2.9) is not IR. By virtue of Theorem 2.1, this is equivalent to $\dim(\mathcal{N}) = 0$ where we recall that $\mathcal{N} = B^{-1}\mathcal{V}^\star \cap \ker\{D\}$ and $\mathcal{V}^\star$ is the weakly unobservable subspace of System (2.9). Using now [70, Lemma 7.9] as well as Lemma 2.1, we conclude that the following equivalence holds:

$$y_{x_0,u} = \mathbf{0} \;\Leftrightarrow\; x_0 \in \mathcal{V}^\star,\; u = F x_{x_0,u} \tag{2.30}$$

where $F \in \mathbf{F}(\mathcal{V}^\star)$ is a friend of $\mathcal{V}^\star$. Bering in mind that the system is linear allows us to say that $\forall x_0 \in \mathbb{R}^n$, and $\forall u_a, u_b$, the difference between trajectories can be rewritten as $(\tilde{u}, x_{\tilde{x}_0,\tilde{u}}, y_{\tilde{x}_0,\tilde{u}})$ where $\tilde{u} = u_a - u_b$ and $\tilde{x}_0 = x_{x_0,u_a}(0) - x_{x_0,u_b}(0) = \mathbf{0}$. Since $\mathbf{0} \in \mathcal{V}^\star$, a direct consequence of (2.30) is that $y_{\tilde{x}_0=\mathbf{0},\tilde{u}} = \mathbf{0}$ implies that $\exists F \in \mathbf{F}(\mathcal{V}^\star)$ such that $\tilde{u} = F x_{\tilde{x}_0=\mathbf{0},\tilde{u}} = 0$. As a consequence, for any $T > 0$, restricting the time horizon to $[0, T]$, we have that $\tilde{y}_{\tilde{x}_0=\mathbf{0},\tilde{u}|_{[0,T]}}|_{[0,T]} = \mathbf{0}$, $\forall t \in [0, T]$ implies $\tilde{u}|_{[0,T]} = \mathbf{0}$. This is equivalent to the negation of statement (S1) and concludes the proof. □

After establishing that it is possible to construct two distinct input trajectories over a finite time interval that produce the same output on this interval, the following lemma goes further by showing that such inputs can always be constructed so that the system states coincide exactly at the end of the interval.

Lemma 2.5. *Consider an IR System* (2.9)*. For any* $T > 0$ *and any initial condition, there exist two distinct inputs defined on the interval* $[0, T]$ *such that the corresponding outputs on the same interval* $[0, T]$ *are identical so as the corresponding states at time* T*, i.e.,* $\forall x_0 \in \mathbb{R}^n$, $\forall T > 0$, $\exists u_a|_{[0,T]} \neq u_b|_{[0,T]}$:

$$y_{x_0,u_a|_{[0,T]}}|_{[0,T]} = y_{x_0,u_b|_{[0,T]}}|_{[0,T]},\; x_{x_0,u_a|_{[0,T]}}(T) = x_{x_0,u_b|_{[0,T]}}(T).$$

Proof. Assume that System (2.9) is IR. As stated in [49, Theorem 3.1], this is equivalent to $\rho > 0$ (ρ defined in Proposition 2.2) or $\dim(\mathcal{R}^\star) > 0$, where $\mathcal{R}^\star$ is the controllable weakly unobservable subspace of system (2.9) (see [70, Section 7.4] for its definition). We consider separately the two cases where ρ is strictly positive or equal to zero.

- $\rho > 0$: Let $u_a|_{[0,T]}$ be any input function on $[0, T]$ and let $\tilde{u}|_{[0,T]}$ be a nontrivial function satisfying $\tilde{u}|_{[0,T]}(t) \in \ker\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\}$ for all $t \in [0, T]$. Define $u_b|_{[0,T]} = u_a|_{[0,T]} + \tilde{u}|_{[0,T]}$. Then we have

$$\begin{cases} u_a|_{[0,T]} \neq u_b|_{[0,T]}, \\ y_{x_0,u_b|_{[0,T]}} = y_{x_0,u_a|_{[0,T]}+\tilde{u}|_{[0,T]}} = y_{x_0,u_a|_{[0,T]}}, \\ x_{x_0,u_b|_{[0,T]}} = x_{x_0,u_a|_{[0,T]}+\tilde{u}|_{[0,T]}} = x_{x_0,u_a|_{[0,T]}}. \end{cases}$$

  In particular, evaluating the states at $t = T$ gives $x_{x_0,u_a|_{[0,T]}}(T) = x_{x_0,u_b|_{[0,T]}}(T)$.

- $\rho = 0$: Then necessarily $\dim(\mathcal{R}^\star) > 0$. Pick $x_1 \in \mathcal{R}^\star$, $x_1 \neq \mathbf{0}$. Since the system is unconstrained ($\mathcal{U} = \mathbb{R}^m$, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^p$), $\mathcal{R}^\star$ is the reachable subset of the controllable weakly unobservable subspace (see [70, Theorem 7.14]). Consequently, there exists a nontrivial control $\tilde{u}|_{[0,T/2]}$ steering $\mathbf{0}$ to x_1 in time $T/2$, i.e., $x_{\mathbf{0},\tilde{u}|_{[0,T/2]}}(T/2) = x_1$, while keeping the output zero, $y_{\mathbf{0},\tilde{u}|_{[0,T/2]}}(t) = \mathbf{0}$ for all $t \in [0, T/2]$. Moreover, since $\mathcal{R}^\star$ is controllable and weakly unobservable, there exists a control $\tilde{u}|_{[T/2,T]}$ steering x_1 to $\mathbf{0}$ in time $T/2$, i.e., $x_{x_1,u|_{[T/2,T]}}(T) = \mathbf{0}$, while keeping the output zero, $y_{x_1,\tilde{u}|_{[T/2,T]}} = \mathbf{0}$. Concatenating $\tilde{u}|_{[0,T/2]}$ and $\tilde{u}|_{[T/2,T]}$ and applying the resulting function to the difference system concludes the proof.

$\square$

2.5 Conclusion

In this chapter, we introduced the concept of input redundancy for a generic system, defined as the existence of two distinct input trajectories that produce the same output. After illustrating this concept using the HEV example, which highlighted its practical relevance in control theory, we reviewed the existing literature on input redundancy, including key definitions, properties, and characterizations. Our discussion then focused on a recent reformulation of this concept in terms of signal-based representations. This reformulation provides the foundation for extending the concept to other classes of systems, and its characterization is grounded in tools from geometric control theory. Then, in the HEV example, we demonstrate how these theoretical tools can be applied to identify a system's redundancy and degrees of freedom. Finally, new technical results are presented in the LTI context. One contribution concerns the invariant and unobservable subspace algorithm, which we extend to non-standard dimensions and powers. The other contributions focus on the structural properties of finite-time trajectories of an IR system.

Chapter 3

Input redundancy for parameter-dependent systems

3.1 Introduction

In practice, many systems exhibiting input redundancy are also subject to (possibly unknown) parameter variations; see, for example, [27, 30, 47, 54]. A typical case is the mass of an HEV, which depends on the number of passengers and is therefore uncertain. Similar situations arise in other domains, such as component values varying with temperature in power electronics, or changes in the operating point for which a linear model is derived in aeronautics.

To address such variations and uncertainties, it is common to employ a *parameter-dependent* model, for which a wide range of tools already exist to analyze stability [19, 23, 26], and to controller design [3, 10, 50]. However, as far as we are aware, the problem of input redundancy in this parameter-dependent context remains largely unexplored.

The purpose of this chapter is to address this gap. Building on the signal–based definitions of input redundancy, we first introduce in Section 3.2 two new notions tailored to the parameter-dependent setting: *adaptive* and *robust input redundancy.* Because we believe these definitions should not depend on any simplifying assumptions, we formulate them without imposing restrictions on the parameter dependence beyond the existence and uniqueness of solutions of the system. We then focus on a simplified yet practically relevant framework: linear parameter-dependent systems with polynomial parameter dependence. Within this framework, we derive tractable conditions for determining whether a parameter-dependent system is (adaptively or robustly) input redundant. Our approach extends the reasoning developed in the LTI context. As recalled in Chapter 2, input redundancy in LTI systems is characterized through an incremental analysis involving *invisible inputs*, which consist of two components: (i) a term ensuring that the state remains within a designated state subspace, the weakly unobservable subspace, and (ii) a term that may be chosen freely within a particular input subspace, the weakly unobservable input subspace. Section 3.4 extends the notion of the weakly unobservable subspace to the parameter-dependent setting and discusses its tractable construction. Section 3.5.2 develops a corresponding extension for the weakly unobservable input subspace. These extensions inevitably introduce a degree of conservatism, which is discussed in detail throughout the subsequent sections. Finally, Section 3.5 combines the previous results to provide tractable conditions for both adaptive and robust input redundancy. Throughout the chapter, several examples, including the running HEV example, are provided to make the developments more accessible.

3.2 The concept of input redundancy in the parameter-dependent context

3.2.1 System description

In this chapter, we consider systems represented in Figure 3.1 described by the following definition.

Definition 3.1 (Parameter-dependent system Σ_θ)**.** *A parameter-dependent system, denoted by Σ_θ, is characterized by:*

- *an input $u : \mathbb{R}^+ \to \mathcal{U}$, where $\mathcal{U}$ is the set of admissible input values,*
- *a state $x : \mathbb{R}^+ \to \mathcal{X}$ with initial condition $x(0) = x_0 \in \mathcal{X}$, where $\mathcal{X}$ is the set of admissible state values,*
- *a parameter $\theta : \mathbb{R}^+ \to \Theta$ where Θ is the set of admissible parameter values,*
- *and a performance output $y : \mathbb{R}^+ \to \mathcal{Y}$, where $\mathcal{Y}$ is the set of admissible output values. The output y reflects the control objective.*

For a given input u, parameter function θ, and initial condition x_0, leading to existence and uniqueness of state and output trajectories, we denote by $x_{x_0,u,\theta}$ (resp. $y_{x_0,u,\theta}$) the state (resp. output) trajectory generated by the system.

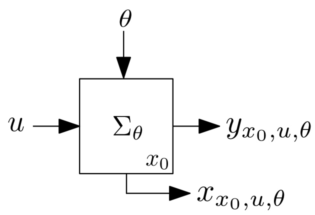

Figure 3.1: System Σ_θ with input u, state x, output y, and parameter θ.

We adopt the following assumption to ensure the existence and uniqueness of solutions for Σ_θ throughout this chapter.

Assumption 3.1 (Existence and Uniqueness)**.** *The dynamics of the system Σ_θ are assumed to be sufficiently regular to guarantee well-posedness. Moreover, the input and parameter trajectories u and θ are assumed to belong to $\mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ and $\mathcal{PC}(\mathbb{R}^+, \Theta)$, the sets of piecewise continuous functions from the time domain $\mathbb{R}^+$ to the set of input values $\mathcal{U}$ and of parameter values Θ, respectively.*

3.2.2 Input redundancy definitions

In the context of parameter-dependent systems, an important question is how to adapt the general definition of input redundancy (Definition 2.2) to this specific setting. Since input redundancy is fundamentally linked to the *existence* of two distinct input trajectories that produce the same output, it is natural to extend this concept by also considering the *existence* of a parameter trajectory under which this property holds. Clearly, this property depends on the evolution of θ, which motivates introducing a dedicated notion, referred to as *adaptive input redundancy*. The formal definition is provided below.

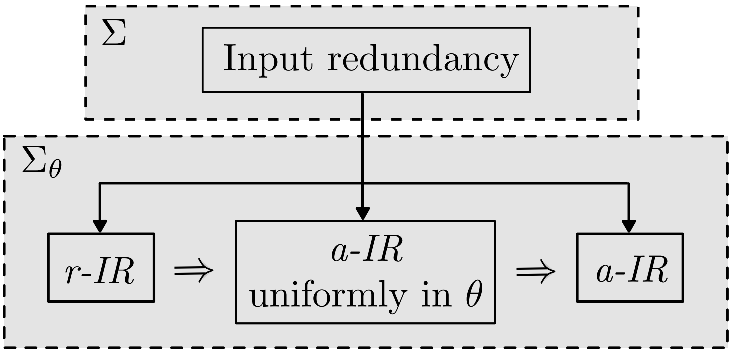

Figure 3.2: Three frameworks of input redundancy of parameter-dependent systems.

Definition 3.2 (Adaptive input redundancy). *System Σ_θ is said to be Adaptively Input Redundant (a-IR) if there exist an initial condition, a parameter trajectory, and two distinct input trajectories such that the resulting output trajectories are identical, i.e.:*

$$\exists x_0 \in \mathcal{X}, \theta \in \mathcal{PC}(\mathbb{R}^+, \Theta), \text{ and } u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U}): \; u_a \neq u_b, \; y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta}.$$

It is clear from the definition that, for a given parameter trajectory θ, the system Σ_θ may admit different input trajectories producing the same output, while this property might not hold for other trajectories of θ. When the goal is to ensure this property *regardless of the parameter trajectory*, a uniformity with respect to θ must be imposed, as formalized in the following definition.

Definition 3.3 (Adaptive input redundancy uniformly in θ). *System Σ_θ is said to be a-IR, uniformly in θ if, for all parameter trajectories, there exist an initial condition and two distinct input trajectories such that the resulting output trajectories are identical, i.e.:*

$$\forall \theta \in \mathcal{PC}(\mathbb{R}^+, \Theta), \; \exists x_0 \in \mathcal{X}, u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U}): \; u_a \neq u_b, \; y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta}.$$

In this previous definition, although it is uniform in θ, both the initial condition and the input trajectories are possibly depending in θ. In contrast, we may be interested in another *uniform version* of adaptive input redundancy, where the initial condition and input trajectories must be independent of θ. This consideration is particularly relevant when the parameter is either unknown or cannot be measured. To address this situation, we introduce the notion of *robust input redundancy*, defined as follows.

Definition 3.4 (Robust input redundancy). *System Σ_θ is said to be Robustly Input Redundant (r-IR) if there exist an initial condition and two distinct input trajectories such that, for all parameter trajectories, the resulting output trajectories are identical, i.e.:*

$$\exists x_0 \in \mathcal{X}, u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U}): \; u_a \neq u_b \text{ and } \forall \theta \in \mathcal{PC}(\mathbb{R}^+, \Theta), y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta}.$$

Note that the distinction among these three definitions lies in the quantifiers associated with θ, specifically, whether "∃" or "∀" is used and where the quantifier appears in the statement.

Fact 3.1. *The following chain of implications holds:*

$$\Sigma_\theta \text{ is r-IR } \Rightarrow \Sigma_\theta \text{ is a-IR, uniformly in } \theta \; \Rightarrow \Sigma_\theta \text{ is a-IR.}$$

Proof. Assume Σ_θ is r-IR. By definition there exist $x_0 \in \mathcal{X}$ and $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ such that for every admissible parameter trajectory $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$ the outputs coincide $y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta}$. Fix any particular $\bar{\theta} \in \mathcal{PC}(\mathbb{R}^+, \Theta)$. The above equality evaluated at $\theta = \bar{\theta}$ shows that, for the same x_0, u_a, u_b, the system $\Sigma_{\bar{\theta}}$ produces identical outputs under u_a and u_b. Hence $\Sigma_{\bar{\theta}}$ is a-IR. As $\bar{\theta}$ was arbitrary, Σ_θ is a-IR uniformly in θ. □

Figure 3.2 highlights the distinct frameworks that appear when dealing with input redundancy of parameter-dependent systems, and their relation.

Similar to the LTI case, we can also classify different types of input redundancy of parameter-dependent systems based on the property of distinct inputs producing not only identical output but also identical state trajectory in both contexts of robust and adaptive IR.

Definition 3.5. *A system Σ_θ is said to be a-IR (uniformly in θ) of:*

- Kind 1 *when it is a-IR (uniformly in θ) and for all initial conditions $x_0 \in \mathcal{X}$, all parameter dependent trajectories $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$ and all input trajectories $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ such that:*
$$u_a \neq u_b \text{ and } y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta},$$
the state trajectories are identical, i.e., $x_{x_0,u_a,\theta} = x_{x_0,u_b,\theta}$.
- Kind 2 *when is a-IR (uniformly in θ) and for all initial conditions x_0, all parameter dependent trajectories $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$ and all input trajectories $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ such that:*
$$u_a \neq u_b \text{ and } y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta},$$
the state trajectories are not identical, i.e., $x_{x_0,u_a,\theta} \neq x_{x_0,u_b,\theta}$.
- Kind 3 *when it is a-IR (uniformly in θ) but not of Kind 1 or Kind 2.*

Definition 3.6. *A system Σ_θ is r-IR of:*

- Kind 1 *when it is r-IR and for all initial conditions $x_0 \in \mathcal{X}$, and all input trajectories $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ satisfying:*
$$u_a \neq u_b \text{ and } y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta},$$
for all parameter functions $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$, the state trajectories are identical, i.e., $x_{x_0,u_a,\theta} = x_{x_0,u_b,\theta}$.
- Kind 2 *when it is r-IR and for all initial conditions $x_0 \in \mathcal{X}$, and all input trajectories $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U})$ satisfying:*
$$u_a \neq u_b \text{ and } y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta},$$
for all parameter functions $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$, the state trajectories are not identical, i.e., $x_{x_0,u_a,\theta} \neq x_{x_0,u_b,\theta}$, $\forall \theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$.
- Kind 3 *when it is r-IR but not of Kind 1 or Kind 2.*

To make these notions more intuitive, let us continue our discussion on the guiding example of the HEV.

Guideline Example - HEV. *Recall the model* (2.6) *of the HEV driveline Σ_{HEV}.*

Robust input redundancy analysis of the HEV model. *In practice, the inertia J of the vehicle varies with its total mass, which in turn depends on the number of passengers and the amount of luggage carried. Since this is not measured, it should be treated as an uncertain parameter. Other coefficients, such as damping b, resistance R_a, or inductance L_a may also vary, but here we focus on one varying parameter to illustrate our framework. The key question is: does input redundancy persist despite parameter variations, and if so, does it hold robustly or only adaptively?*

To analyze robust input redundancy on the HEV, we model the effective inertia as uncertain with $\pm 50\%$ variation, and define $\theta : t \mapsto \theta(t) \in [\underline{\theta}, \bar{\theta}]$ where $\underline{\theta} = \frac{2}{J}$ and $\bar{\theta} = \frac{2}{3J}$. The resulting uncertain HEV model is given by the following equation:

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} \frac{-2}{t_d} & 0 & 0 & 0 \\ \frac{4}{t_d} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & -K\theta(t) \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & -(m+b)\theta(t) \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & 0 & 0 & r\theta(t) \end{bmatrix} x(t).
\end{aligned}
\tag{3.1}
$$

Ideally, we would reproduce the input u_a from (2.7), which produces an output of 18 km/h. *However, this input explicitly depends on the inertia. Since our objective here is solely to analyze the redundancy property, we therefore adopt a simplified input $u_a = \mathbf{0}$. For $x_0 = \mathbf{0}$ and $\theta : t \mapsto 3$, this yields $x_{\mathbf{0},u_a,\theta} = \mathbf{0}$, and $y_{\mathbf{0},u_a,\theta} = \mathbf{0}$.*

Next, we consider the input

$$
u_b(t) = \begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} - \frac{4L_a}{K\tau t_d} x_{b,1}(t) + \frac{L_a - R_a\tau}{K\tau^2} x_{b,2}(t) \end{bmatrix}. \tag{3.2}
$$

The resulting trajectories, $x_b = x_{\mathbf{0},u_b,\theta}$ and $y_b = y_{\mathbf{0},u_b,\theta}$, are shown in Fig. 3.3. We observe that the output trajectory remains identical to the output generated by u_a.

We repeat the simulation for a different parameter value within the admissible range, namely $\theta : t \mapsto 1.5$. The corresponding trajectories are reported in Fig. 3.4, again showing that the output remains unchanged despite the variation in θ.

These results confirm that the chosen input structure achieves the same output trajectory for two distinct values of the uncertain parameter. This led us to think that the system may be r-IR. However, to confirm that, we must check for all piecewise continuous parameter functions $\theta : t \mapsto \theta(t) \in [\underline{\theta}, \bar{\theta}]$, leading to an infinite number of simulations.

Adaptive input redundancy analysis of the HEV. *As reported in [30], the variables t_d and τ depend on the rotational speed ω of the driveshaft and the fuel engine torque T_{FE} (see Equation (2.2)), with their corresponding values listed in Table 3.1.*

We note that t_d exhibits significant variation, which can strongly affect the system dynamics. It is therefore natural to treat t_d as a time-varying parameter. Assuming it can be estimated, we define

$$
\theta(t) = \frac{1}{t_d(\omega(t), T_{FE}(t))} \in [\underline{\theta}, \bar{\theta}] = [10, 13.88] \ \textit{(s)},
$$

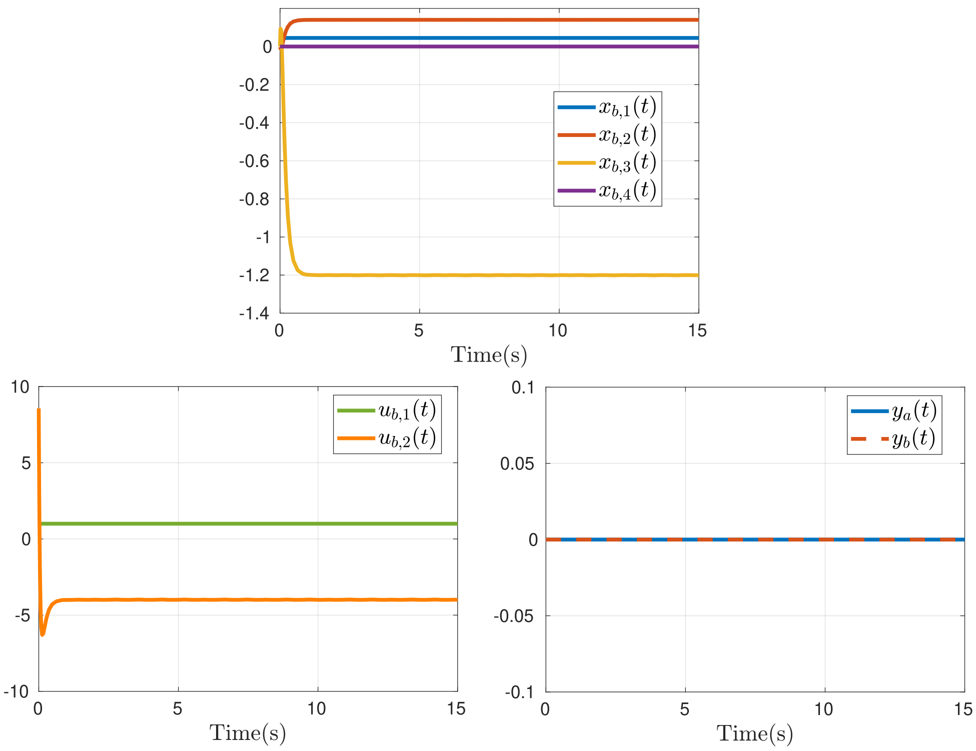

Figure 3.3: Trajectories of system (3.1) for input u_b in (3.2) and $\theta : t \mapsto 3$.

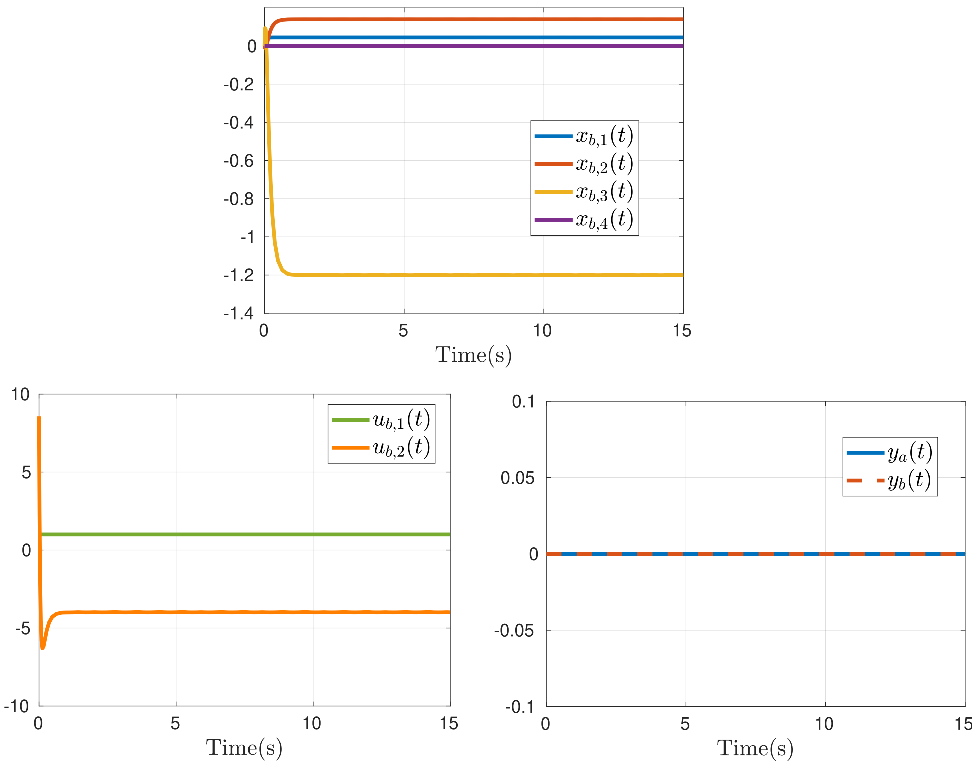

Figure 3.4: Trajectories of system (3.1) for input u_b in (3.2) and $\theta : t \mapsto 1.5$.

ω (tr.min^{-1})	1200	1200	1600	1600	2000	2000
T_{FE} (N.m)	50	100	50	100	50	100
t_d (ms)	100	140	84	96	80	72
τ (ms)	144	142	140	137	140	134

Table 3.1: Delay and combustion lag vs. driveshaft speed and fuel engine torque.

according to Table 2.1, and rewrite the HEV as a parameter-dependent system:

$$\dot{x}(t) = \begin{bmatrix} -2\theta(t) & 0 & 0 & 0 \\ 4\theta(t) & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & \frac{-K}{J} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & \frac{-(m+b)}{J} \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t), \tag{3.3}$$
$$y(t) = \begin{bmatrix} 0 & 0 & 0 & \frac{r}{J} \end{bmatrix} x(t).$$

Here, we consider the inputs u_a from (2.7) *and u_b from* (2.8)*, which yield the desired output of* 18 km/h. *In this case, the input u_b depends on the parameter θ through its dependence on t_d. For simplicity, we disregard the variation of θ as a function of ω and, instead, prescribe the following trajectory:*

$$\theta(t) = \begin{cases} 10, & 0 \le t < 2, \\ 0.388\,(t-2) + 10, & 2 \le t < 12, \\ 13.88, & t \ge 12. \end{cases} \tag{3.4}$$

The resulting state and output trajectories, $x_b = x_{\mathbf{0},u_b,\theta}$ and $y_b = y_{\mathbf{0},u_b,\theta}$ are shown in Figure 3.5. We observe that the output remains unchanged, even as the parameter $\theta(t)$ varies over time. Consequently, the parameter-dependent system (3.3) *of the HEV preserves input redundancy. More specifically, it seems to be* a-IR *according to Definition 3.2.*

This practical example shows that when certain parameters become time-varying and/or unknown, an IR system may effectively behave as an a-IR or r-IR parameter-dependent system. Specifically, the system exhibits r-IR characteristics when inertia is treated as a time-varying parameter and a-IR characteristics when the fueling delay is considered instead. Yet, to date there are no established methods in the literature for rigorously verifying input redundancy in such parameter-dependent systems. The remainder of this chapter addresses this gap by developing an appropriate parameter-dependent framework.

3.3 Problem statement

The considered problem of this chapter is the following:

Problem 3.1. *Provide tractable conditions under which a parameter-dependent system Σ_θ can be certified as either a-IR or r-IR, and, whenever this holds, exhibit its inherent degrees of freedom.*

Tackling this problem within the fully general framework of Definition 3.1 is highly challenging. To make the analysis feasible, we impose additional structure, as detailed below.

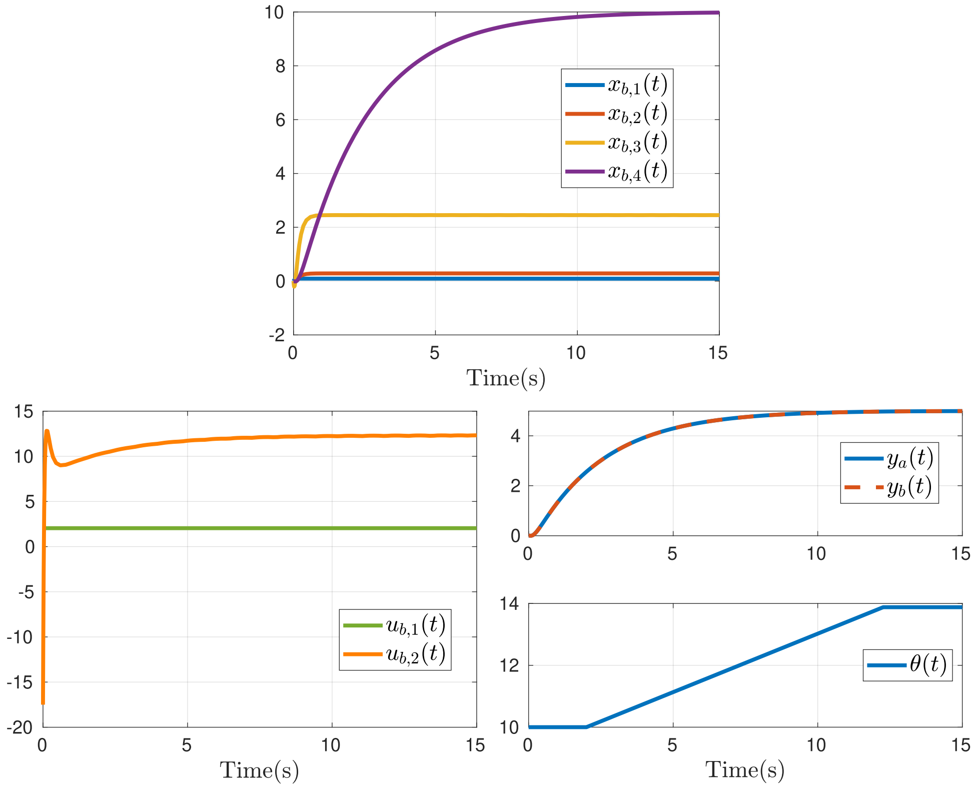

Figure 3.5: Trajectories of system (3.3) for input u_b in (2.8) and parameter θ in (3.4).

3.3.1 Considered framework

We begin by introducing the following assumptions.

Assumption 3.2 (Admissible sets)**.** *The input u, state x, and output y are assumed to be unconstrained, i.e., $\mathcal{U} = \mathbb{R}^m$, $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^p$. Moreover, the parameter set Θ is assumed to be a compact subset of $\mathbb{R}^l$, for $l \in \mathbb{N}^*$.*

Assumption 3.3 (Dependence on u, x and θ)**.** *The system Σ_θ is assumed to be* linear *in the state x and the input u, and polynomial in the parameter θ.*

Under these assumptions, System Σ_θ can be represented, for all $t \geq 0$, as follows:

$$\begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t), \\ y(t) = C(\theta(t))x(t) + D(\theta(t))u(t), \end{cases} \tag{3.5}$$

where $A : \Theta \to \mathbb{R}^{n\times n}$, $B : \Theta \to \mathbb{R}^{n\times m}$, $C : \Theta \to \mathbb{R}^{p\times n}$, and $D : \Theta \to \mathbb{R}^{p\times m}$ are matri functions polynomial on θ.

Remark 3.1. *Note that restricting the dependence on θ to be polynomial still allows one to capture more complex dependencies, sometimes in an equivalent manner. To illustrate this, consider the system Σ_θ described by (3.5), where $A : \Theta \subset \mathbb{R} \to \mathbb{R}$ is defined by $\theta \mapsto \sin(\theta)$, while B, C, and D are independent of θ. For any $\Theta \subset \mathbb{R}$, one can define $\hat{\Theta} = [-1, 1]$ and $\hat{\theta} = \sin(\theta) \in \hat{\Theta}$, which transforms the system into a polynomially dependent system $\hat{\Sigma}_{\hat{\theta}}$ with state matrix $\hat{A} : \hat{\Theta} \to \mathbb{R}$ given by $\hat{A}(\hat{\theta}) = \hat{\theta}$. Moreover, when Θ is such that $\sin(\Theta) = \hat{\Theta}$, the two representations Σ_θ and $\hat{\Sigma}_{\hat{\theta}}$ are equivalent.*

About the existence and uniqueness of system trajectories. Since the matrix functions A, B, C, D are polynomial on the compact set Θ, they are uniformly bounded. Moreover, the parameter trajectory $t \mapsto \theta(t)$ is piecewise continuous. As a consequence, when the input $t \mapsto u(t)$ is assumed to be piecewise continuous, the vector field $(t, x) \mapsto A(\theta(t))x + B(\theta(t))u(t) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t and globally Lipschitz in x. Hence, by the global existence and uniqueness theorem [46, Theorem 3.2], System (3.5) admits a unique solution on $\mathbb{R}^+$ for each initial condition $x_0 \in \mathbb{R}^n$.

Notations for the parameter dependence. Since θ is a time-varying function and the system matrix functions A, B, C, and D depend on this parameter, to avoid confusion in the notation, along this chapter, we consider the following three notations:

- $\theta(t) \in \Theta$ denotes the value at time t of the function θ.
- $\theta\ (= \theta(\cdot))$ is the signal or function: $\left\{ \begin{array}{lcl} \mathbb{R}^+ & \to & \Theta \\ t & \mapsto & \theta(t). \end{array} \right.$
- $\xi \in \Theta$ corresponds to the indeterminate of a function.

Remark 3.2. *From System Σ_θ given by Equation* (3.5)*, the expressions at a given time t of the dynamics can be obtained by composing the matrix functions A, B, C, D depending on an indeterminate ξ with the time-dependent signal $t \mapsto \theta(t)$, as illustrated in Equation* (3.6) *for the matrix function A.*

$$\left(\left\{ \begin{array}{lcl} \mathbb{R}^+ & \to & \mathbb{R}^{n\times n} \\ t & \mapsto & A(\theta(t)) \end{array} \right. \right) = \left(\left\{ \begin{array}{lcl} \Theta & \to & \mathbb{R}^{n\times n} \\ \xi & \mapsto & A(\xi) \end{array} \right. \right) \circ \left(\left\{ \begin{array}{lcl} \mathbb{R}^+ & \to & \Theta \\ t & \mapsto & \theta(t) \end{array} \right. \right). \tag{3.6}$$

3.3.2 Existing works related to Problem 3.1

Several contributions have studied the interplay between input redundancy and parameter dependency [35, 66, 67, 72]. In [35], a parameter-dependent allocation scheme is proposed, although the plant itself remains LTI. By contrast, [66, 67, 72] focus directly on parameter-dependent systems, developing control methods that estimate time-varying parameters to exploit the degrees of freedom provided by input redundancy. However, these works consider only adaptive input redundancy of kind 1, where two different inputs depending on the parameter produce the same output and also result in the same state trajectory. This concept was introduced in [72] under the term *strong input redundancy* for parameter-dependent systems and is defined in terms of kernel spaces rather than signal properties.

Similarly, in the context of output regulation problems, [66] studied adaptive input redundancy of Kind 1 using a steady-state optimizing controller to handle over-actuation. Following the same perspective, [67] also relied on adaptive input redundancy of Kind 1 for linear systems subject to faults modeled as parameter uncertainties.

More recently, adopting a different perspective, robust input redundancy, as defined in Definition 3.4, was introduced in [48] and fully characterized for linear parameter-dependent systems with system matrices that depend affinely on the parameters. This represents a first step toward the complete characterization of input redundancy in parameter-dependent systems. Nevertheless, the characterization of adaptive input redundancy remains an open and challenging problem. Furthermore, extending the results of [48] on robust input redundancy to more general parameter dependencies, such as the polynomial dependence considered in Assumption 3.3, also remains an open issue.

3.3.3 Outline of the proposed approach and fundamental challenges

Since Σ_θ is linear in x and u (Assumption 3.3), it is natural to adopt an *incremental analysis* similar to the LTI case (Section 2.3.2). For two inputs u_a and u_b producing the same output, their difference $\tilde{u} = u_a - u_b$ satisfies

$$\tilde{u} = F x_{x_0,\tilde{u}} + Lw,$$

where F is a friend of the weakly unobservable subspace $\mathcal{V}^\star$, L satisfies $\operatorname{im} L = \mathcal{N}$ where $\mathcal{N}$ is the weakly unobservable input subspace, and w is arbitrary. This structure relies on two core ingredients:

(i) the weakly unobservable subspace $\mathcal{V}^\star$ together with a friend $F \in \mathbf{F}(\mathcal{V}^\star)$;

(ii) the weakly unobservable input subspace $\mathcal{N}$, which captures the input degrees of freedom.

Extending this incremental framework to linear parameter-dependent systems requires generalizing these two notions. For LTI systems, $\mathcal{V}^\star$ is the largest subspace that is both controlled invariant and output invisible. Therefore, a first step is to re-examine these two properties, controlled invariance and output invisibility, in the parameter-dependent context.

Two complementary settings must be distinguished:

- **Robust case:** the inputs that render a subspace invariant and output-invisible must be *independent of the parameters*.
- **Adaptive case:** these inputs are allowed to *depend on the parameters*, so both the subspace and its friends may vary with θ.

Determining such subspaces is highly challenging. In particular, in the adaptive case the directions of the subspace may change over time as the parameters evolve. To avoid this difficulty in the present work, we restrict our analysis to *parameter-independent* (later called *generalized*) controlled-invariant and output-invisible subspaces, while still allowing their associated friends to depend on the parameters.

Even under this simplification, three fundamental questions remain:

- How can we define and characterize parameter-independent controlled invariant and output invisible subspaces in the parameter-dependent setting?
- How can we compute the *largest* such subspaces together with their associated friends?
- Given a subspace $\mathcal{V} \subset \mathbb{R}^n$, how can we compute a basis for the parameter-dependent set of input directions producing zero output,

$$B^{-1}(\xi)\,\mathcal{V} \cap \ker\{D(\xi)\}?$$

The remainder of this section addresses these three questions in turn. Section 3.4 characterizes controlled invariance and output invisibility in the parameter-dependent context. Section 3.5 presents a method to compute the parameter-dependent set of invisible input directions. Finally, Section 3.6 establishes tractable conditions for adaptive and robust input redundancy.

3.4 Controlled invariance and output invisibility in the parameter-dependent context

3.4.1 Generalized (adaptive) subspaces

For the system Σ_θ described in (3.5), the notion of (controlled) invariance is strongly influenced by the time dependence of θ. In particular, under Assumption 3.3, θ may be discontinuous at isolated time instants. At such points, (controlled) invariance may fail to hold, since the state trajectory itself cannot exhibit discontinuities in time. To exclude this undesirable situation, we restrict our attention to invariance properties that are *independent of* θ, even in the case of adaptive input redundancy. We emphasize that this perspective already goes beyond the scope of the existing literature. This motivates the following definitions.

Definition 3.7 ([48, Def. 2])**.** *A subspace* $\mathcal{V}_G \subset \mathbb{R}^n$ *is called a* generalized controlled invariant and output invisible subspace *for System* (3.5) *if for any* $x_0 \in \mathcal{V}_G$*, there exists an input function* $u \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ *such that, for all parameter trajectories* $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$*,*

$$x_{x_0,u,\theta}(t) \in \mathcal{V}_G, \ \forall t \geq 0 \quad \textit{and} \quad y_{x_0,u,\theta} = \mathbf{0}.$$

Definition 3.8. *A subspace* $\mathcal{V}_{GA} \subset \mathbb{R}^n$ *is called a* generalized adaptively controlled invariant and output invisible subspace *for System* (3.5) *if for all* $x_0 \in \mathcal{V}_{GA}$ *and all parameter trajectory* $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$*, there exists an input function* $u \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ *such that*

$$x_{x_0,u,\theta}(t) \in \mathcal{V}_{GA}, \ \forall t \geq 0 \quad \textit{and} \quad y_{x_0,u,\theta} = \mathbf{0}.$$

The distinction between these two notions lies, once again, in the placement of the quantifier "for all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$". In Definition 3.7, its position implies that the input functions ensuring that $\mathcal{V}_G$ is invariant and invisible are *independent* of θ. In contrast, in Definition 3.8, the input is allowed to *depend* on θ.

3.4.2 Existing works on generalized (adaptive) subspaces

The computation of subspaces $\mathcal{V}_G$ and $\mathcal{V}_{GA}$ is a difficult task that has attracted considerable attention. Most existing methods rely on specific structural assumptions on the system to preserve tractability.

For $\mathcal{V}_G$, a tractable approach was first proposed in [9] under the following assumptions: (i) the system is strictly proper, i.e., $D(\xi) = \mathbf{0}$ for all $\xi \in \Theta$; (ii) the system matrices depend *affinely* on the parameter; and (iii) the maps A, B, and C depend on *distinct* parameters (no parameter sharing). This framework was later extended in [20] by allowing A, B, and C to depend polynomially on the parameter, albeit still restricted to a *scalar* parameter. More recently, [48] further generalized the approach by considering maps A, B, C, and D depending affinely on *multiple* parameters, possibly shared between matrices, and by admitting non–strictly proper systems.

For $\mathcal{V}_{GA}$, [6] addressed the case of a *finite* parameter set. Subsequent works [4, 12, 68] considered strictly proper systems where A and C depend affinely on the parameter while B remains parameter–independent, with the parameter vector constrained to a polytopic set.

Although these approaches represent significant progress, they are not directly applicable to the more general settings considered here, e.g., *polynomial dependence on multiple parameters varying over a continuous set*, which is the focus of this chapter. Overcoming the limitations linked to affine dependence, scalar parameters, or finite parameter sets motivates the alternative approach proposed here.

3.4.3 Intractable characterization of generalized (adaptive) subspaces

Let us propose a characterization of the *generalized* controlled invariant and output invisible subspaces in Lemma 3.1, and extend the characterization given by [4] of *generalized adaptively* controlled invariant and output invisible subspaces to non-strictly proper parameter-dependent systems in Lemma 3.2.

Lemma 3.1. *Consider System* (3.5) *and let* $\mathcal{V}_G \subset \mathbb{R}^n$*. The two following statements are equivalent:*

(i) $\mathcal{V}_G$ *is a generalized controlled invariant and output invisible subspace,*

(ii) There exists a matrix $F \in \mathbb{R}^{m\times n}$*, such that, for all* $\xi \in \Theta$

$$\begin{aligned} &(A(\xi) + B(\xi)F)\mathcal{V}_G \subset \mathcal{V}_G, \\ &(C(\xi) + D(\xi)F)\mathcal{V}_G = \mathbf{0}. \end{aligned} \tag{3.7}$$

Proof. $(i) \Rightarrow (ii)$ Let $\mathcal{V}_G$ be a generalized controlled invariant and output invisible subspace of System (3.5). Then, by definition, for every $x_0 \in \mathcal{V}_G$, there exists an input function $u \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ such that, for any function $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$, $x_{x_0,u,\theta}(t) \in \mathcal{V}_G$ for all $t \geq 0$ and $y_{x_0,u,\theta} = \mathbf{0}$. It follows that $\dot{x}(0^+) = \lim_{t\to 0^+} \frac{1}{t}(x_{x_0,u,\theta}(t) - x_0) \in \mathcal{V}_G$, and there exists $u_0 \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A(\xi) \\ C(\xi) \end{bmatrix} x_0 + \begin{bmatrix} B(\xi) \\ D(\xi) \end{bmatrix} u_0 \in \mathcal{V}_G \times \{0_{p\times 1}\}, \tag{3.8}$$

for all $\xi \in \Theta$. Then, consider that $V \in \mathbb{R}^{n\times q}$ is a basis of $\mathcal{V}_G$ where $q \leq n$ corresponds to its dimension. For each column v of V there exists a vector-valued function $\tilde{v} : \Theta \to \mathcal{V}_G$, and a vector $\tilde{u} \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A(\xi) \\ C(\xi) \end{bmatrix} v = \begin{bmatrix} B(\xi) \\ D(\xi) \end{bmatrix} \tilde{u} + \begin{bmatrix} \tilde{v}(\xi) \\ \mathbf{0}_p \end{bmatrix}. \tag{3.9}$$

Since matrix V is a basis of $\mathcal{V}_G$, then $\tilde{v}(\xi)$ can be expressed as $\tilde{v}(\xi) = V\tilde{y}(\xi)$, where $\tilde{y} : \Theta \to \mathbb{R}^q$. Thus, it is clear that there exist a matrix $U = [\tilde{u}_1, \ldots, \tilde{u}_q] \in \mathbb{R}^{m\times q}$ and a mapping $Y : \Theta \to \mathbb{R}^{q\times q}$ such that

$$\begin{bmatrix} A(\xi) \\ C(\xi) \end{bmatrix} V = \begin{bmatrix} B(\xi) \\ D(\xi) \end{bmatrix} U + \begin{bmatrix} V \\ \mathbf{0}_{p\times q} \end{bmatrix} Y(\xi). \tag{3.10}$$

Finally, since V is full column rank, we have that $V^\top V$ is invertible. Then, we can define the gain $F = -U(V^\top V)^{-1}V^\top$, which is a friend of $\mathcal{V}_G$, leading to the inclusion (3.7).
$(ii) \Rightarrow (i)$ For any $x_0 \in \mathcal{V}_G$, consider $u(t) = Fx(t)$, with F verifying (3.7), then $x_{x_0,u,\theta}(t) \in \mathcal{V}_G$ for all $t \geq 0$ and $y_{x_0,u,\theta} = \mathbf{0}$, both for all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$. □

Lemma 3.2. *Consider System* (3.5) *and let* $\mathcal{V}_{GA} \subset \mathbb{R}^n$*. The following statements are equivalent:*

(i) $\mathcal{V}_{GA}$ *is a generalized adaptively controlled invariant and output invisible subspace,*

(ii) There exists a mapping $F : \Theta \to \mathbb{R}^{m\times n}$*, such that, for all* $\xi \in \Theta$

$$\begin{aligned} &(A(\xi) + B(\xi)F(\xi))\mathcal{V}_{GA} \subset \mathcal{V}_{GA}, \\ &(C(\xi) + D(\xi)F(\xi))\mathcal{V}_{GA} = \mathbf{0}. \end{aligned} \tag{3.11}$$

(iii) for all $\xi \in \Theta$, $\mathcal{V}_{GA}$ satisfies

$$\begin{bmatrix} A(\xi) \\ C(\xi) \end{bmatrix} \mathcal{V}_{GA} \subset \mathcal{V}_{GA} \times \{\mathbf{0}_p\} + \mathrm{Im}\left\{ \begin{bmatrix} B(\xi) \\ D(\xi) \end{bmatrix} \right\}. \tag{3.12}$$

Proof. $(i) \Rightarrow (iii)$ Following the developments of the proof of Lemma 3.1, we obtain

$$\begin{bmatrix} A(\xi) \\ C(\xi) \end{bmatrix} x_0 + \begin{bmatrix} B(\xi) \\ D(\xi) \end{bmatrix} u_0(\xi) \in \mathcal{V}_{GA} \times \{\mathbf{0}_p\}, \tag{3.13}$$

where, in this case, u_0 depends on ξ, since u may depend on it. Equation (3.13) leads directly to the inclusion (3.12).
$(iii) \Rightarrow (ii)$ Let us assume that inclusion (3.12) holds. By following the same proof of item $(i) \Rightarrow (ii)$ of Lemma 3.1, with a matrix $U(\xi)$ depending on ξ in this case, we have

$$\begin{bmatrix} A(\xi) \\ C(\xi) \end{bmatrix} V = \begin{bmatrix} B(\xi) \\ D(\xi) \end{bmatrix} U(\xi) + \begin{bmatrix} V \\ \mathbf{0}_{p\times q} \end{bmatrix} Y(\xi). \tag{3.14}$$

Then, the gain $F(\xi) = -U(\xi)(V^\top V)^{-1}V^\top$ is a friend of $\mathcal{V}_{GA}$ and satisfies relations (3.11).
$(ii) \Rightarrow (i)$ For any $x_0 \in \mathcal{V}_{GA}$, applying the control $u(t) = F(\theta(t))x(t)$, with F satisfying (3.11), leads $x_{x_0,u,\theta}(t) \in \mathcal{V}_{GA}$ for all $t \geq 0$ and $y_{x_0,u,\theta} = \mathbf{0}$, both for all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$. □

Remark 3.3. *We would like to highlight that such characterizations are pointwise with respect to ξ and that they are not using neither the assumptions that the parameter lies in a compact set, nor that the dependency is polynomial.*

Although Lemmas 3.1 and 3.2 provide insightful expressions for these subspaces, their conditions are not directly verifiable in practice. Indeed, they must hold for all possible values of $\xi \in \Theta$, resulting in an infinite number of conditions. Accordingly, we focus on the following objectives:

- Developing tractable conditions for establishing the controlled invariance and output invisibility of the subspaces $\mathcal{V}_G$ and $\mathcal{V}_{GA}$.

- Computing one of the corresponding friends of $\mathcal{V}_G$ and $\mathcal{V}_{GA}$.

- Demonstrating that the set of $\mathcal{V}_G$ (and likewise the set of $\mathcal{V}_{GA}$) admits a maximal element, and providing a method for computing this maximal element.

Remark 3.4. *Identifying the maximal element of the sets of generalized and generalized adaptively controlled invariant and output invisible subspaces is an essential step, as the analysis of robust and adaptive input redundancy fundamentally relies on these maximal elements. This role is analogous to that of the weakly unobservable subspace in the study of input redundancy for LTI systems, as recalled in Chapter 2.*

3.4.4 Homogeneous polynomial representation of parameter-dependent systems

Before providing tractable characterizations of the sets $\mathcal{V}_G$ and $\mathcal{V}_{GA}$, we first need to obtain a *homogeneous polynomial representation* of System (3.5), which underpins our subsequent results. Although parameter-dependent systems are often modeled with linear dependence on the parameters, particularly in stability and robustness analysis [2, 19, 23], such an approach typically introduces conservatism into the results. To address this limitation, and even to reduce conservatism compared with a generic (non-homogeneous) polynomial representation, we adopt a *homogeneous representation* for the system mappings A, B, C, and D. This approach has already proven to be effective in mitigating conservatism in stability analysis of parameter-dependent systems (see, e.g., [43, 58]). More details on the construction of homogeneous polynomials are provided in Appendix B. The definition of a homogeneous polynomial is given here after.

Definition 3.9. *Given $n_P, n_1, n_2 \in \mathbb{N}^*$, a homogeneous polynomial matrix-valued function*[8] *$\hat{P} : \Theta \subset \mathbb{R}^l \to \mathbb{R}^{n_1 \times n_2}$ of degree n_P can be expressed*

$$\hat{P}(\xi) = \sum_{k \in \mathcal{K}_{n_P}} \xi^k \hat{P}_k \tag{3.15}$$

where $k = (k_1, k_2, \cdots, k_l)$, $\xi^k = \prod_{i=1}^{l} \xi_i^{k_i}$ with $k_i \in \mathbb{N}$ for all $i \in [\![1, l]\!]$ where $[\![1, l]\!] = [1, l] \cap \mathbb{Z}$ and

$$\mathcal{K}_{n_P} \coloneqq \left\{ k \in \mathbb{N}^l : \sum_{i=1}^{l} k_i = n_P \right\}$$

is the set of l-tuples with all possible combinations of k such that the sum of all k_i, $i \in [\![1, l]\!]$, is equal to n_P. Moreover, $\prod_{i=1}^{l} \xi_i^{k_i}$ are monomials in $\xi \in \Theta$, and $\hat{P}_k \in \mathbb{R}^{n_1 \times n_2}$, for all $k \in \mathcal{K}_{n_P}$.

For the system Σ_θ described in (3.5), we adopt the following assumption.

Assumption 3.4. *Let $(A, B, C, D) : \Theta \to (\mathbb{R}^{n \times n}, \mathbb{R}^{n \times m}, \mathbb{R}^{p \times n}, \mathbb{R}^{p \times m})$ be a quadruple of polynomial matrix-valued functions, associated with System* (3.5)*. We assume that there exist an integer $H \in \mathbb{N}^*$, a vector-valued mapping $\alpha : \Theta \to \Lambda_H$ with*

$$\Lambda_H \coloneqq \Big\{ \alpha \in \mathbb{R}^H : \ \alpha_i \geq 0, \ \forall i \in [\![1, H]\!], \ \sum_{i=1}^{H} \alpha_i = 1 \Big\},$$

and four polynomial matrix-valued functions $\hat{A} : \Lambda_H \to \mathbb{R}^{n \times n}$, $\hat{B} : \Lambda_H \to \mathbb{R}^{n \times m}$, $\hat{C} : \Lambda_H \to \mathbb{R}^{p \times n}$, and $\hat{D} : \Lambda_H \to \mathbb{R}^{p \times m}$ such that

$$(A, B, C, D) = (\hat{A} \circ \alpha, \ \hat{B} \circ \alpha, \ \hat{C} \circ \alpha, \ \hat{D} \circ \alpha).$$

This assumption allows (A, B, C, D) to be *equivalently expressed* as a quadruple of homogeneous polynomial matrix-valued functions $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. In Assumption 3.4, Λ_H denotes the H-dimensional unit simplex. From Remark B.1, whenever Θ is polytopic and A, B, C, D are polynomial matrix-valued functions, Assumption 3.4 is automatically satisfied.

Throughout the remainder of this chapter, we shall work under this assumption. Moreover, representation (3.5), with the notations θ, Θ, A, B, C, and D, is used when referring to a *general*

[8] Throughout this chapter, homogeneous polynomials and their (matrix) coefficients are systematically denoted with a hat.

parameter-dependent representation. In contrast, the following form is employed when dealing with a *homogeneous polynomial parameter-dependent representation*:

$$\begin{cases} \dot{x}(t) = \hat{A}(\alpha(t))x(t) + \hat{B}(\alpha(t))u(t), \\ y(t) = \hat{C}(\alpha(t))x(t) + \hat{D}(\alpha(t))u(t), \end{cases} \tag{3.16}$$

where $\alpha : \mathbb{R}^+ \to \Lambda_H$ denotes, with a slight abuse of notation, the composition $\alpha \circ \theta$, and $\hat{A} : \Lambda_H \to \mathbb{R}^{n\times n}$, $\hat{B} : \Lambda_H \to \mathbb{R}^{n\times m}$, $\hat{C} : \Lambda_H \to \mathbb{R}^{p\times n}$, and $\hat{D} : \Lambda_H \to \mathbb{R}^{p\times m}$ are homogeneous polynomial functions in α of respective degrees n_A, n_B, n_C, and n_D. In accordance with Definition 3.9, they can be written as

$$\hat{A} : \alpha \mapsto \sum_{k\in\mathcal{K}_{n_A}} \alpha^k \hat{A}_k, \quad \hat{B} : \alpha \mapsto \sum_{k\in\mathcal{K}_{n_B}} \alpha^k \hat{B}_k, \quad \hat{C} : \alpha \mapsto \sum_{k\in\mathcal{K}_{n_C}} \alpha^k \hat{C}_k, \quad \hat{D} : \alpha \mapsto \sum_{k\in\mathcal{K}_{n_D}} \alpha^k \hat{D}_k, \tag{3.17}$$

with matrices $\hat{A}_k$ for $k \in \mathcal{K}_{n_A}$, $\hat{B}_k$ for $k \in \mathcal{K}_{n_B}$, $\hat{C}_k$ for $k \in \mathcal{K}_{n_C}$, and $\hat{D}_k$ for $k \in \mathcal{K}_{n_D}$, all of appropriate dimensions.

The following example illustrates how a system Σ_θ, which is not originally in homogeneous polynomial form, can be transformed into such a representation.

Guideline Example - HEV. *Consider again the uncertain model of the HEV* (3.1) *developed previously in this chapter, where the uncertain parameter is the vehicle inertia.*

Computation of homogeneous polynomial representation of the HEV model (3.1)**:** *In order to obtain the homogeneous polynomial representation of this system, we define* $\alpha : [\underline{\theta}, \bar{\theta}] \to \Lambda_2$, *such that*

$$\alpha_1 = \frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}}, \quad \alpha_2 = 1 - \alpha_1, \tag{3.18}$$

and $\theta = (\bar{\theta} - \underline{\theta})\alpha_1 + \underline{\theta}$. *Here, we have* $n_A = 1$, $\mathcal{K}_{n_A} = \{(1,0),(0,1)\}$, $|\mathcal{K}_{n_A}| = 2$, $n_B = 1$, $\mathcal{K}_{n_B} = \{(1,0),(0,1)\}$, $|\mathcal{K}_{n_B}| = 2$, *leading to applications of the form*

$$\hat{A}(\alpha) = \alpha_1\hat{A}_{10} + \alpha_2\hat{A}_{01}, \ \hat{B}(\alpha) = \alpha_1\hat{B}_{10} + \alpha_2\hat{B}_{01}, \ \hat{C}(\alpha) = \alpha_1\hat{C}_{10} + \alpha_2\hat{C}_{01}, \tag{3.19}$$

where

$$\hat{A}_{10} = \begin{bmatrix} \frac{-2}{t_d} & 0 & 0 & 0 \\ \frac{4}{t_d} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & -K\bar{\theta} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & -(m+b)\bar{\theta} \end{bmatrix}, \ \hat{A}_{01} = \begin{bmatrix} \frac{-2}{t_d} & 0 & 0 & 0 \\ \frac{4}{t_d} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & -K\underline{\theta} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & -(m+b)\underline{\theta} \end{bmatrix},$$

$$\hat{B}_{10} = \hat{B}_{01} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \hat{C}_{10} = \begin{bmatrix} 0 & 0 & 0 & r\bar{\theta} \end{bmatrix}, \ \hat{C}_{01} = \begin{bmatrix} 0 & 0 & 0 & r\underline{\theta} \end{bmatrix}.$$

Now, let us consider the second parameter-dependent model (3.3)*, where the time-varying parameter is* $\frac{1}{t_d}$.

Computation of homogeneous polynomial representation of the HEV model (3.3)**:** *Constructing* $\alpha \in \Lambda_2$ *as in* (3.18) *gives*

$$\theta(t) = (\bar{\theta} - \underline{\theta})\alpha_1(t) + \underline{\theta},$$

and the system matrices in homogeneous polynomial form, as in (3.19)*, with*

$$\hat{A}_{10} = \begin{bmatrix} -2\bar{\theta} & 0 & 0 & 0 \\ 4\bar{\theta} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & -\frac{K}{J} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & -\frac{m+b}{J} \end{bmatrix}, \ \hat{A}_{01} = \begin{bmatrix} -2\underline{\theta} & 0 & 0 & 0 \\ 4\underline{\theta} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & -\frac{K}{J} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & -\frac{m+b}{J} \end{bmatrix},$$

$$\hat{B}_{10} = \hat{B}_{01} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \hat{C}_{10} = \hat{C}_{01} = \begin{bmatrix} 0 & 0 & 0 & \frac{r}{J} \end{bmatrix}.$$

3.4.5 Tractable conditions of generalized (adaptive) subspaces

In Lemmas 3.1 and 3.2, the mapping F is allowed to be very general. The following example illustrates that, even for a polynomially parameter-dependent system, it may be necessary to consider *rational* functions of the parameters in order for a subspace to exhibit controlled invariance and output invisibility.

Example 3.1. *Consider the linear parameter-dependent system* Σ_θ *represented by* (3.5) *with system matrices*

$$A : \xi \mapsto \begin{bmatrix} 1 & 1 \\ \xi_1 & 1 \end{bmatrix}, \quad B : \xi \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \xi_2 \end{bmatrix}, \quad C : \xi \mapsto \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D : \xi \mapsto \mathbf{0}_{1\times 2}, \tag{3.20}$$

where ξ_1 *and* ξ_2 *are independent parameters ranging in* $[0, 1]$*.*

In this case, one can verify that the subspace

$$\mathcal{V}_{GA} = \mathrm{Im}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

is invariant and output invisible for all $\xi \in [0, 1]^2$ *such that* $\xi_2 \neq 0$ *with the friend*

$$F : \xi \mapsto \begin{bmatrix} 0 & 0 \\ -\frac{\xi_1}{\xi_2} & 0 \end{bmatrix} \in \mathbf{F}(\mathcal{V}_{GA}),$$

since

$$(A(\xi) + B(\xi)F(\xi))\mathcal{V}_{GA} = \mathrm{Im}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subset \mathcal{V}_{GA}, \quad \textit{and} \quad C(\xi)\mathcal{V}_{GA} = 0.$$

Notice that F *is a rational function in* ξ*, and* F *is not defined on all the compact set* $\Theta = [0, 1]^2$*.*

This example illustrates the difficulty of working with fully general mappings F: even for polynomially parameter-dependent systems, rational dependencies may naturally appear. To ensure tractability, we restrict our attention to friends $\hat{F}$ that belong to the class of homogeneous polynomial mappings of a fixed degree n_F.

Remark 3.5. *Two sources of conservatism are inherent in this restriction. The first arises from limiting our attention to polynomial friends only (since any polynomial can be transformed into a homogeneous one according to Lemma B.1). The second source of conservatism stems from fixing the degree n_F a priori. To mitigate the latter, one may envision an iterative procedure where n_F is tested over multiple values, thereby reducing conservatism.*

To move from the infinite number of conditions required in Lemma 3.2 to the finite set of conditions stated in Theorem 3.1, we build upon the parameter-independent conditions introduced in the previous chapter (see Section 2.4.1). In particular, Lemma 2.2 extended the classical controlled invariance and output invisibility conditions of LTI systems to a broader setting by allowing (i) enlarged matrix dimensions and (ii) higher powers of subspaces.

Theorem 3.1 exploits this result by showing that the parameter-dependent conditions formulated in Lemma 3.2 can be reformulated as parameter-independent conditions involving enlarged dimensions and higher-order subspaces.

Before presenting these results, we introduce the following auxiliary notations.

Auxiliary notations. Let $\hat{P}$ be a homogeneous polynomial matrix-valued function (see Definition 3.9) with matrices $\hat{P}_k \in \mathbb{R}^{n_1 \times n_2}$, indexed by $k \in \mathcal{K}_{n_P}$. We define their vertical concatenation and horizontal concatenation, respectively, as

$$\operatorname*{cat_v}_{k \in \mathcal{K}_{n_P}} \{\hat{P}_k\}, \qquad \operatorname*{cat_h}_{k \in \mathcal{K}_{n_P}} \{\hat{P}_k\}.$$

We also define the indicator function $\delta : \mathcal{K}_{n_P} \times \mathcal{K}_{n_P} \to \{0, 1\}$ by

$$\delta(k, k') = \begin{cases} 1, & \text{if } k = k', \\ 0, & \text{otherwise,} \end{cases}$$

so that, for example,

$$\operatorname*{cat_h}_{k \in \mathcal{K}_{n_P}} \delta_{kj} = \begin{bmatrix} \delta(k_1, j) & \delta(k_2, j) & \cdots & \delta(k_{N_P}, j) \end{bmatrix},$$

where $N_P = |\mathcal{K}_{n_P}|$ is the cardinality of $\mathcal{K}_{n_P}$. For instance, if $j = k_1$, then

$$\operatorname*{cat_h}_{k \in \mathcal{K}_{n_P}} \delta_{kj} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Theorem 3.1. *Consider System Σ_θ in the homogeneous polynomial representation* (3.16) *and let $\mathcal{V}_{GA} \subset \mathbb{R}^n$ and $n_F \in \mathbb{N}$. Consider*

$$\mathbf{A} = \operatorname*{cat_v}_{j \in \mathcal{K}_{n_B + n_F}} \left\{ (n_B + n_F - n_A)! \sum_{\begin{cases} \tau \in \mathcal{K}_{(n_B + n_F - n_A)} \\ j - \tau \in \mathcal{K}_{n_A} \end{cases}} \frac{\hat{A}_{j-\tau}}{\pi(\tau)} \right\},$$

$$\mathbf{B} = \operatorname*{cat_v}_{j\in\mathcal{K}_{n_B+n_F}} \left\{ \sum_{\begin{cases} \ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_B} \end{cases}} \left(\operatorname*{cat_h}_{i\in\mathcal{K}_{n_F}} \delta_{i\ell} \otimes \hat{B}_{j-\ell} \right) \right\},$$

$$\mathbf{C} = \operatorname*{cat_v}_{j\in\mathcal{K}_{n_D+n_F}} \left\{ (n_D+n_F-n_C)! \sum_{\begin{cases} \tau \in \mathcal{K}_{(n_D+n_F-n_C)} \\ j-\tau \in \mathcal{K}_{n_C} \end{cases}} \frac{\hat{C}_{j-\tau}}{\pi(\tau)} \right\},$$

$$\mathbf{D} = \operatorname*{cat_v}_{j\in\mathcal{K}_{n_D+n_F}} \left\{ \sum_{\begin{cases} \ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_D} \end{cases}} \left(\operatorname*{cat_h}_{i\in\mathcal{K}_{n_F}} \delta_{i\ell} \otimes \hat{D}_{j-\ell} \right) \right\},$$

where $\pi(\tau) = (\tau_1!)(\tau_2!)\cdots(\tau_H!)$. Without loss of generality, we assume that $n_B + n_F \geq n_A$ and $n_D + n_F \geq n_C$. Then, the following statements are equivalent:

(i) There exists a homogeneous polynomial $\hat{F} : \Lambda_H \to \mathbb{R}^{m\times n}$ of degree n_F with respect to α such that, for all $\alpha \in \Lambda_H$,

$$(\hat{A}(\alpha) + \hat{B}(\alpha)\hat{F}(\alpha))\mathcal{V}_{GA} \subset \mathcal{V}_{GA}, \tag{3.21}$$

$$(\hat{C}(\alpha) + \hat{D}(\alpha)\hat{F}(\alpha))\mathcal{V}_{GA} = \mathbf{0}. \tag{3.22}$$

(ii) There exists a matrix $\mathbf{F} \in \mathbb{R}^{|\mathcal{K}_{n_F}|m\times n}$, such that

$$\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \mathbf{F} \right) \mathcal{V}_{GA} \subset \mathcal{V}_{GA}^{|\mathcal{K}_{n_B+n_F}|} \times \{\mathbf{0}_p\}^{|\mathcal{K}_{n_D+n_F}|}. \tag{3.23}$$

Proof. (i) $\Rightarrow$ (ii): There exists an homogeneous polynomial $\hat{F} : \Lambda_H \to \mathbb{R}^{m\times n}$ that is called a friend of $\mathcal{V}_{GA}$, such that (3.21)-(3.22) hold. The multiplication of polynomials $\hat{B}(\alpha)$ and $\hat{F}(\alpha)$ is

$$\hat{B}(\alpha)\hat{F}(\alpha) = \left(\sum_{k\in\mathcal{K}_{n_B}} \alpha^k \hat{B}_k \right) \left(\sum_{j\in\mathcal{K}_{n_F}} \alpha^j \hat{F}_j \right).$$

Thanks to $\alpha^k \hat{B}_k \alpha^j \hat{F}_j = \alpha^{k+j} \hat{B}_k \hat{F}_j$ and by applying a change of variables $k + j = \ell \in \mathcal{K}_{n_B+n_F}$, we can represent $\hat{B}(\alpha)\hat{F}(\alpha)$ in the following form

$$\hat{B}(\alpha)\hat{F}(\alpha) = \sum_{\begin{cases} j \in \mathcal{K}_{n_B+n_F} \\ \ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_B} \end{cases}} \alpha^j \hat{B}_{j-\ell}\hat{F}_\ell. \tag{3.24}$$

Moreover, as $\sum_{i=1}^{H} \alpha_i = 1$ and[9] $n_B + n_F \geq n_A$, the polynomial matrix valued application $\hat{A}(\alpha)$ can also be rewritten as

$$\hat{A}(\alpha) = \left(\sum_{i=1}^{H} \alpha_i \right)^{n_T} \hat{A}(\alpha),$$

where $n_T = (n_B + n_F - n_A)$, to have the same degree of the polynomial $\hat{B}(\alpha)\hat{F}(\alpha)$. By following the technique presented by [61] and [37, Section 2.24], we obtain the representation

$$\hat{A}(\alpha) = n_T! \sum_{\begin{cases} j \in \mathcal{K}_{n_B+n_F} \\ \tau \in \mathcal{K}_{n_T} \\ j - \tau \in \mathcal{K}_{n_A} \end{cases}} \alpha^j \frac{\hat{A}_{j-\tau}}{\pi(\tau)}. \tag{3.25}$$

Since the polynomials in (3.24) and (3.25) have the same degree n_B+n_F, the sum $\hat{A}(\alpha)+\hat{B}(\alpha)\hat{F}(\alpha)$ can be rearranged by gathering together the monomials α^j for all $j \in \mathcal{K}_{n_B+n_F}$. That implies that condition (3.21) is equivalent to

$$\left[n_T! \sum_{\begin{cases} \tau \in \mathcal{K}_{n_T} \\ j - \tau \in \mathcal{K}_{n_A} \end{cases}} \frac{\hat{A}_{j-\tau}}{\pi(\tau)} + \sum_{\begin{cases} \ell \in \mathcal{K}_{n_F} \\ j - \ell \in \mathcal{K}_{n_B} \end{cases}} \hat{B}_{j-\ell}\hat{F}_\ell \right] \mathcal{V}_{GA} \subset \mathcal{V}_{GA}, \tag{3.26}$$

for all $j \in \mathcal{K}_{n_B+n_F}$. The same procedure applies for $\hat{C}(\alpha) + \hat{D}(\alpha)\hat{F}(\alpha)$, such that condition (3.22) can be represented as

$$\left[n_P! \sum_{\begin{cases} \tau \in \mathcal{K}_{n_P} \\ j - \tau \in \mathcal{K}_{n_C} \end{cases}} \frac{\hat{C}_{j-\tau}}{\pi(\tau)} + \sum_{\begin{cases} \ell \in \mathcal{K}_{n_F} \\ j - \ell \in \mathcal{K}_{n_D} \end{cases}} \hat{D}_{j-\ell}\hat{F}_\ell \right] \mathcal{V}_{GA} \subset \mathcal{V}_{GA}, \tag{3.27}$$

for all $j \in \mathcal{K}_{n_D+n_F}$, where $n_P = n_D + n_F - n_C$. Moreover, we can rewrite (3.26) and (3.27) as condition (3.23), where $\mathbf{F} = \underset{\ell \in \mathcal{K}_{n_F}}{\mathrm{cat_v}} \{\hat{F}_\ell\} \in \mathbb{R}^{N_F m \times n}$ is the vertical concatenation of $\hat{F}_\ell$ for all $\ell \in \mathcal{K}_{n_F}$.
(ii) $\Rightarrow$ (i) For any $x_0 \in \mathcal{V}_{GA}$, the control $u(t) = \sum_{\ell \in \mathcal{K}_{n_F}} \alpha(\theta(t))^\ell \hat{F}_\ell x(t)$ ensures that $x_{x_0,u,\theta}(t) \in \mathcal{V}_{GA}$ for all $t \geq 0$ and $y_{x_0,u,\theta} = \mathbf{0}$. □

From Theorem 3.1, the conditions related to the *generalized* controlled invariant and output invisible subspaces are obtained by setting $n_F = 0$, indicating that the friend is independent of the parameter. The equivalence between condition (3.7) in Lemma 3.1 and condition (2.25) in Lemma 2.2 is established in Corollary 3.1, which is a specialization of Theorem 3.1 with $n_F = 0$. Consequently, Algorithm 2.2 can also be used to compute the largest generalized subspace.

[9] If $n_B + n_F < n_A$, we can use the same technique to increase the degree of $\hat{B}(\alpha)\hat{F}(\alpha)$ to obtain the same degree of $\hat{A}(\alpha)$.

Corollary 3.1. *Consider System Σ_θ in the homogeneous polynomial representation* (3.16) *and let $\mathcal{V}_G \subset \mathbb{R}^n$. Consider*

$$\mathbf{A} = \underset{j\in\mathcal{K}_{n_B}}{\mathrm{cat_v}}\left\{(n_B-n_A)!\sum_{\begin{cases}\tau\in\mathcal{K}_{n_B-n_A}\\ j-\tau\in\mathcal{K}_{n_A}\end{cases}}\frac{\hat{A}_{j-\tau}}{\pi(\tau)}\right\},\ \mathbf{B} = \underset{k\in\mathcal{K}_{n_B}}{\mathrm{cat_v}}\left\{\hat{B}_k\right\},$$

$$\mathbf{C} = \underset{j\in\mathcal{K}_{n_D}}{\mathrm{cat_v}}\left\{(n_D-n_C)!\sum_{\begin{cases}\tau\in\mathcal{K}_{n_D-n_C}\\ j-\tau\in\mathcal{K}_{n_C}\end{cases}}\frac{\hat{C}_{j-\tau}}{\pi(\tau)}\right\},\ \mathbf{D} = \underset{k\in\mathcal{K}_{n_D}}{\mathrm{cat_v}}\left\{\hat{D}_k\right\},$$

and, without loss of generality, that $n_B \geq n_A$ and $n_D \geq n_C$. Then the two following statements are equivalent.

(i) There exists a matrix $F \in \mathbb{R}^{m\times n}$ such that for all $\alpha \in \Lambda_H$,

$$(\hat{A}(\alpha) + \hat{B}(\alpha)F)\mathcal{V}_G \subset \mathcal{V}_G, \tag{3.28}$$

$$(\hat{C}(\alpha) + \hat{D}(\alpha)F)\mathcal{V}_G = \mathbf{0}. \tag{3.29}$$

(ii) There exists a matrix $\mathbf{F} \in \mathbb{R}^{m\times n}$, such that

$$\left(\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}\mathbf{F}\right)\mathcal{V}_G \subset \mathcal{V}_G^{|\mathcal{K}_{n_B}|} \times \{\mathbf{0}_p\}^{|\mathcal{K}_{n_D}|}. \tag{3.30}$$

Proof. The proof follows from Theorem 3.1 with $\mathcal{V}_{GA} = \mathcal{V}_G$, $n_F = 0$ (leading to $N_F = 1$), and $\hat{F} : \xi \mapsto F = \mathbf{F}$. □

Computation of a friend of the subspaces: Relying again on the extension of LTI framework provided in Section 2.4.1, the procedure for computing a friend follows from Lemma 2.2. Given a subspace $\mathcal{V}_{GA}$ (or $\mathcal{V}_G$), we first compute a basis, denoted by V. Then, by solving the linear system (2.27) for $[U^\top \ \ Y^\top]^\top$, we obtain the matrix U, from which the friend $\mathbf{F}$ is determined using relation (2.28).

Note that, unlike the conditions in Lemmas 3.1 and 3.2, which must hold for all parameter values $\alpha \in \Lambda_H$, the reformulated condition (3.23) is *independent of* α. This independence makes it directly verifiable through an algorithmic procedure.

Particular cases: The strategy presented in this chapter is a generalized framework dealing both with polynomial systems considering polynomial or constant friends and with the cases already covered in the literature, that are presented now.

1. System (3.16) with linear dependence and with constant friends [48]: The system matrices present a linear dependence on the parameter, then $n_A = n_B = n_C = n_D = 1$ and

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D})(\alpha) = \sum_{i=1}^{H} \alpha_i(\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i).$$

The bold matrices are

$$\mathbf{A} = [\hat{A}_1^\top, \dots, \hat{A}_H^\top]^\top,\ \mathbf{B} = [\hat{B}_1^\top, \dots, \hat{B}_H^T]^\top,$$
$$\mathbf{C} = [\hat{C}_1^\top, \dots, \hat{C}_H^\top]^\top,\ \mathbf{D} = [\hat{D}_1^\top, \dots, \hat{D}_H^T]^\top,$$

We obtain a largest generalized controlled invariant and output invisible subspace by applying Algorithm 2.2 with $Z_1 = Z_2 = H$, since the friend is constant.

2. System (3.16) with linear dependence and with friends depending linearly on the parameter [12]: The system matrices present a linear dependence on the parameter, then $n_A = n_B = n_C = n_D = 1$. Moreover, the friend also presents a linear dependence on the parameter, i.e., $n_F = 1$. Then, the expressions of the bold matrices are given by

$$\mathbf{A} = f\left(\{\hat{A}_i\}_{i=1}^H\right), \mathbf{B} = g\left(\{\hat{B}_i\}_{i=1}^H\right), \mathbf{C} = f\left(\{\hat{C}_i\}_{i=1}^H\right), \mathbf{D} = g\left(\{\hat{D}_i\}_{i=1}^H\right),$$

where

$$f\left(\{\hat{A}_i\}_{i=1}^H\right) = \left[\begin{array}{c} \hat{A}_1 \\ \vdots \\ \hat{A}_H \\ \hline\hline \hat{A}_1 + \hat{A}_2 \\ \vdots \\ \hat{A}_1 + \hat{A}_H \\ \hline \hat{A}_2 + \hat{A}_3 \\ \vdots \\ \hat{A}_2 + \hat{A}_H \\ \hline \vdots \\ \hline \hat{A}_{H-1} + \hat{A}_H \end{array}\right], \quad g\left(\{\hat{B}_i\}_{i=1}^H\right) = \left[\begin{array}{ccccc} \hat{B}_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \hat{B}_H \\ \hline\hline \hat{B}_2 & \hat{B}_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \hat{B}_H & 0 & \cdots & \cdots & \hat{B}_1 \\ \hline 0 & \hat{B}_3 & \hat{B}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{B}_H & 0 & \cdots & \hat{B}_2 \\ \hline & & \vdots & & \\ \hline 0 & 0 & \cdots & \hat{B}_H & \hat{B}_{H-1} \end{array}\right],$$

$\mathbf{F} = [\hat{F}_1^\top, \dots, \hat{F}_H^\top]^\top$. We obtain a generalized adaptively controlled invariance and output invisible subspace by setting $Z_1 = Z_2 = \frac{H(H+1)}{2}$, since the friend is parameter-dependent.

3.4.6 The largest generalized (adaptive) subspaces

Observe that condition (3.23) of Theorem 3.1, applied to $\mathcal{V}_{GA}$ with a friend of degree n_F (or to $\mathcal{V}_G$ via Corollary 3.1), directly matches the form of (2.25) in the proposed extension of the invariant and invisible subspace algorithm presented in the previous chapter, namely in Lemma 2.2.

In other words, $\mathcal{V}_{GA}$ (and similarly $\mathcal{V}_G$) can be regarded as a particular instance of the subspace $\mathcal{V}$ considered in Lemma 2.2. Lemma 2.3 then implies that the set of subspaces $\mathcal{V}_{GA}$ with a homogeneous polynomial friend of degree n_F (and the set of $\mathcal{V}_G$) admits a maximal element, which we denote by $\mathcal{V}^\star_{GA_{n_F}}$ and $\mathcal{V}^\star_G$, respectively. These subspaces are computed using the extended invariant and invisible subspace algorithm (Algorithm 2.2).

By analogy with the weakly unobservable subspace $\mathcal{V}$ in the LTI setting, we adopt the following terminology in the parameter-dependent case:

- *Generalized adaptively weakly unobservable subspace* for $\mathcal{V}^\star_{GA}$;
- n_F*-generalized adaptively weakly unobservable subspace* for $\mathcal{V}^\star_{GA_{n_F}}$;
- *Generalized weakly unobservable subspace* for $\mathcal{V}^\star_{G}$.

Note that $\mathcal{V}^\star_{GA_{n_F}}$ is not strictly equivalent to the generalized adaptively weakly unobservable subspace $\mathcal{V}^\star_{GA}$. Nevertheless, this degree dependence shapes the ordered hierarchy of such subspaces as established in the following proposition.

Proposition 3.1. *Let* $n_{F,1}, n_{F,2} \in \mathbb{N}$ *with* $0 \leq n_{F,1} \leq n_{F,2}$*. Then the following inclusions hold:*

$$\mathcal{V}^\star_{G} \subset \mathcal{V}^\star_{GA_{n_{F,1}}} \subset \mathcal{V}^\star_{GA_{n_{F,2}}} \subset \mathcal{V}^\star_{GA}.$$

Proof. Let $n_F \in \mathbb{N}$ and consider the n_F-generalized adaptively weakly unobservable subspace with an associated degree-n_F friend $\hat{F}_{n_F}(\alpha)$ satisfying conditions (3.21)–(3.22). Define

$$\hat{F}_{n_F+1}(\alpha) := \Big(\sum_{i=1}^{H} \alpha_i\Big) \hat{F}_{n_F}(\alpha).$$

Then $\hat{F}_{n_F+1}(\alpha)$ is a degree-(n_F+1) friend associated with the same subspace $\mathcal{V}^\star_{GA_{n_F}}$ and it satisfies

$$\big(\hat{A}(\alpha) + \hat{B}(\alpha)\hat{F}_{n_F+1}(\alpha)\big)\mathcal{V}^\star_{GA_{n_F}} \subset \mathcal{V}^\star_{GA_{n_F}}, \qquad \big(\hat{C}(\alpha) + \hat{D}(\alpha)\hat{F}_{n_F+1}(\alpha)\big)\mathcal{V}^\star_{GA_{n_F}} = \mathbf{0}.$$

By definition of the largest subspace for degree n_F+1, this implies $\mathcal{V}^\star_{GA_{n_F}} \subset \mathcal{V}^\star_{GA_{n_F+1}}$. Since $\mathcal{V}^\star_{G} = \mathcal{V}^\star_{GA_{n_F=0}}$, we obtain for any $0 \leq n_{F,1} \leq n_{F,2}$:

$$\mathcal{V}^\star_{G} \subset \mathcal{V}^\star_{GA_{n_{F,1}}} \subset \mathcal{V}^\star_{GA_{n_{F,2}}}.$$

Finally, because $\mathcal{V}^\star_{GA}$ is, by definition, the largest subspace admitting some friend F without any restriction on its dependence on α, it follows directly that $\mathcal{V}^\star_{GA_{n_F}} \subset \mathcal{V}^\star_{GA}$ for all n_F, which completes the proof. □

Remark 3.6. *It is important to note that, although we do not provide conditions ensuring that* $\mathcal{V}^\star_{GA_{n_F}}$ *is nontrivial, the algorithm is constructive and always yields the largest subspace satisfying the imposed invariance and invisibility conditions, for the given* n_F*. Thus, if* $\mathcal{V}^\star_{GA_{n_F}}$ *is trivial, our method correctly returns this singleton.*

Now, let us apply the developed procedure to compute generalized and generalized adaptively subspaces of the parameter-dependent models of the HEV, intending to use them to verify input redundancy later.

Guideline Example - HEV. *Consider again the uncertain model of the HEV* (3.1)*. Since the parameter* J *(inertia) is uncertain, it cannot be used in the inputs to render a subspace invariant and output invisible. Consequently, we verify if we can compute the generalized weakly unobservable subspace.*

Computation of $\mathcal{V}_G^\star$ of the uncertain HEV model (3.1)**:** *In order to compute this subspace, we consider the homogeneous polynomial form obtained in* (3.19) *and compute the bold matrices as particular case 1, where $H = 2$, $\hat{A}_1 = \hat{A}_{10}, \hat{A}_2 = \hat{A}_{01}$, and the same applies for $\hat{B}_i$ and $\hat{C}_i$. Applying Algorithm 2.2 yields the nontrivial invariant subspace*

$$\mathcal{V}_G^\star = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & \frac{K}{L_a} \\ 0 & -\frac{1}{\tau} \\ 0 & 0 \end{bmatrix}\right\},$$

as in the LTI case.

Now, let us consider the second parameter-dependent model in (3.3)*, where the time-varying parameter is $\frac{1}{t_d}$.*

Computation of $\mathcal{V}_G^\star$ and $\mathcal{V}_{GA_{n_F}}^\star$ of the parameter-dependent HEV model (3.3)**:** *First, we want to compute $\mathcal{V}_G^\star$ for this system. Using the homogeneous polynomial form* (3.19)*, with a constant friend, we construct the bold matrices also as in particular case 1, with the corresponding weighting matrices for system* (3.3)*. Applying Algorithm 2.2 yields $\mathcal{V}_G^\star = \{\mathbf{0}\}$. This means that there are no invariant subspaces that are independent of the parameters.*

Nevertheless, in this case, $\theta(t)$ is available, thus we allow the friend to depend on the parameter. Considering a friend of degree one,

$$F(\alpha) = \alpha_1 F_{10} + \alpha_2 F_{01},$$

we construct the bold matrices as in particular case 2, where $H = 2$, $\hat{A}_1 = \hat{A}_{10}, \hat{A}_2 = \hat{A}_{01}$, and the same applies for $\hat{B}_i, \hat{C}_i$ and $\hat{F}_i$.

Using Algorithm 2.2, we obtain a nontrivial invariant subspace:

$$\mathcal{V}_{GA_1}^\star = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & \frac{K}{L_a} \\ 0 & -\frac{1}{\tau} \\ 0 & 0 \end{bmatrix}\right\}.$$

Example to illustrate the advantages of the homogeneous polynomial representation

Now, we highlight the advantages of considering a polynomial dependence instead of an affine one, as considered in previous works from the literature.

Example 3.2. *Consider a system of the form* (3.5)*, where*

$$A(\xi) = \begin{bmatrix} 1 & 1 \\ \xi^2 & 1 \end{bmatrix},\ B(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D = \mathbf{0}_{1\times 2}, \tag{3.31}$$

with $\xi \in \Theta = [0, 1]$.

We want to compute a generalized adaptively subspace of this system. The simplest way to deal with such a system and to recover the results of the literature that mostly deal with affine *dependence on the parameters is to take an affine representation of the system by selecting new parameters* $\rho_2 = \xi^2$ *and* $\rho_1 = \xi$*, such that representation* (3.20) *from Example 3.1 is obtained. This representation is not equivalent and could be conservative since the link between* ρ_1 *and* ρ_2 *is omitted to obtain it.*

To compute the n_F*-generalized adaptively weakly unobservable subspace for the affine system* (3.20) *using the strategy developed in this section, we select a friend of degree* $n_F = 1$ *and we build matrices* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$*, as in particular case 2, to apply Algorithm 2.2. Then, we find* $\mathcal{V}^{\star}_{GA_{n_F}} = \{\mathbf{0}\}$*. Note that increasing* n_F *does not increase* $\mathcal{V}^{\star}_{GA_{n_F}}$ *in this case. Since the link between the parameters* ρ_2 *and* ρ_1 *is not considered, the friends are rational functions, as shown in Example 3.1, and our strategy, as any strategy in the literature, cannot handle rational functions.*

On the other hand, we can also represent system (3.31) *by a homogeneous polynomial representation. It is observed that matrices* $A(\xi)$ *and* $B(\xi)$ *are polynomials in* ξ*, but in each one, there are monomials of distinct degrees. To obtain a homogeneous polynomial representation, we need to compute* $\alpha = \alpha(\xi) \in \Lambda_H$ *and increase the degree of some terms thanks to the relation* $\sum_{i=1}^{H} \alpha_i = 1$*. We can proceed as follows to rewrite these matrices as homogeneous polynomials in* α*. First, we compute* $\alpha_1 : \xi \mapsto \xi$*,* $\alpha_2 : \xi \mapsto 1 - \alpha_1(\xi)$*, and we observe that* $n_A = 2$*,* $n_B = 1$ *and* $n_C = 0$*. Consequently, we obtain the homogeneous representation* (3.16) *where:*

$$\hat{A}(\alpha) = \begin{bmatrix} (\alpha_1+\alpha_2)^2 & (\alpha_1+\alpha_2)^2 \\ \alpha_1^2 & (\alpha_1+\alpha_2)^2 \end{bmatrix},\ \hat{B}(\alpha) = \begin{bmatrix} \alpha_1+\alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix},\ \hat{C} = \begin{bmatrix} 0 & 1 \end{bmatrix},\ \hat{D} = \mathbf{0}_{1\times 2}.$$

Such re-parameterization is not unique, however, the chosen parameterization is equivalent to the original system since $\Theta = \Lambda_2$*. Here, we have* $n_A = 2$*,* $\mathcal{K}_{n_A} = \{(2,0),(1,1),(0,2)\}$*,* $|\mathcal{K}_{n_A}| = 3$*,* $n_B = 1$*,* $\mathcal{K}_{n_B} = \{(1,0),(0,1)\}$*,* $|\mathcal{K}_{n_B}| = 2$*, leading to*

$$\hat{A}(\alpha) = \alpha_1^2 \hat{A}_{20} + \alpha_1\alpha_2 \hat{A}_{11} + \alpha_2^2 \hat{A}_{02}$$

and

$$\hat{B}(\alpha) = \alpha_1 \hat{B}_{10} + \alpha_2 \hat{B}_{01},$$

with

$$\hat{A}_{20} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ \hat{A}_{11} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix},\ \hat{A}_{02} = \frac{1}{2}\hat{A}_{11},\ \hat{B}_{01} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \hat{B}_{10} = I_2.$$

In this case, we have an equivalent representation of the original system. Since our strategy also deals with this type of system, we can apply the same approach to compute $\mathcal{V}^{\star}_{GA_{n_F}}$*. Then, choosing a degree* $n_F = 1$ *for the friend, we build matrices*

$$\mathbf{A} = \begin{bmatrix} \hat{A}_{20} \\ \hat{A}_{11} \\ \hat{A}_{02} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \hat{B}_{10} & 0 \\ \hat{B}_{01} & \hat{B}_{10} \\ 0 & \hat{B}_{01} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} C \\ C \end{bmatrix}, \mathbf{D} = \mathbf{0},$$

with $Z_1 = 3$*,* $Z_2 = 2$*, and* $r = m|\mathcal{K}_{n_F}| = 4$ *in Lemma 2.2. By applying Algorithm 2.2 with such bold matrices, we find:*

$$\mathcal{V}^{\star}_{GA_{n_F}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{R},$$

which has a larger dimension than the subspace found with the affine-dependent system. Moreover, our strategy also provides one of the friends, which is

$$F(\xi) = \begin{bmatrix} 0 & 0 \\ -\xi & 0 \end{bmatrix}. \tag{3.32}$$

By considering the homogeneous polynomial representation, we have the equivalence with the original system such that we directly obtain a friend of the subspace $\mathcal{V}^\star_{GA_{n_F}} = [1 \;\; 0]^\top \mathbb{R}$ *without a rational dependence. Therefore, for system* (3.31)*, we have an advantage in computing a generalized adaptively controlled invariant and output invisible subspace by considering the homogeneous polynomial representation instead of the affine one. It is interesting to highlight that if we consider the homogeneous polynomial representation and a degree* $n_F = 0$*, the largest subspace is given by* $\mathcal{V}^\star_G = \{\mathbf{0}\}$*. It means the subspace* $[1\ 0]^\top \mathbb{R}$ *is not controlled invariant and output invisible under parameter-independent inputs.*

To conclude this subsection, we emphasize that the construction yields the largest subspace $\mathcal{V}^\star_{GA_{n_F}}$ (resp. $\mathcal{V}^\star_G$) for the homogeneous polynomial representation, associated with a homogeneous polynomial friend $\hat{F}$ of degree n_F (resp. of degree $n_F = 0$).

Importantly, the subspaces $\mathcal{V}^\star_{GA_{n_F}}$ and $\mathcal{V}^\star_G$, viewed as subsets of $\mathbb{R}^n$, remain generalized adaptive (resp. generalized) controlled invariant and output invisible subspaces for the original representation (3.5) of Σ_θ. This equivalence follows from the relation between (3.5) and (3.16) through the composition $\alpha \circ \theta$. Hence, one can return to the original coordinates with the same subspaces $\mathcal{V}^\star_{GA_{n_F}}$ (resp. $\mathcal{V}^\star_G$), by choosing the friend $F = \hat{F} \circ \alpha$.

3.5 Generalized (adaptive) weakly unobservable input subspace

Let us now return to the original representation (3.5) of Σ_θ, and assume that we have already computed the subspace $\mathcal{V}^\star_{GA_{n_F}}$ (or $\mathcal{V}^\star_G$) together with an associated friend $F = \hat{F} \circ \alpha$. Since, in particular, $\mathcal{V}^\star_G = \mathcal{V}^\star_{GA_{n_F=0}}$, we shall refer in the sequel only to $\mathcal{V}^\star_{GA_{n_F}}$, with the understanding that the robust case $\mathcal{V}^\star_G$ is recovered by setting $n_F = 0$.

The main objective of this section is to develop a procedure which, given the n_F-generalized adaptively weakly unobservable subspace $\mathcal{V}^\star_{GA_{n_F}}$, computes the parameter-dependent input directions that yield zero output when the state is restricted to $\mathcal{V}^\star_{GA_{n_F}}$. This motivates the following definition.

Definition 3.10. *The n_F-generalized adaptively weakly unobservable input application $\mathcal{N}_{GA_{n_F}}$ of system* (3.5) *is the mapping which, for each $\xi \in \Theta$, associates the set of input directions producing zero output when the state is confined to the n_F-generalized adaptively weakly unobservable subspace $\mathcal{V}^\star_{GA_{n_F}}$. Formally,*

$$\mathcal{N}_{GA_{n_F}} : \xi \longmapsto \mathcal{N}_{GA_{n_F}}(\xi) = B^{-1}(\xi)\, \mathcal{V}^\star_{GA_{n_F}} \cap \ker\{D(\xi)\}. \tag{3.33}$$

For each $\xi \in \Theta$, the set $\mathcal{N}_{GA_{n_F}}(\xi)$ is a subspace of the input space $\mathbb{R}^m$, called the n_F-generalized adaptively weakly unobservable input subspace at ξ.

Finally, the generalized weakly unobservable input subspace $\mathcal{N}_G$ of system (3.5) *is obtained from* (3.33) *as*

$$\mathcal{N}_G \coloneqq \bigcap_{\xi\in\Theta} \mathcal{N}_{GA_{n_F=0}}(\xi) = \bigcap_{\xi\in\Theta} B^{-1}(\xi)\mathcal{V}^\star_G \cap \ker\{D(\xi)\}, \tag{3.34}$$

where $\mathcal{V}_G^\star$ *denotes the generalized weakly unobservable subspace of system* (3.5)*.*

Similarly to the LTI case, and as established in Section 3.6, obtaining a parameter-dependent application $L : \xi \mapsto L(\xi)$ such that $\operatorname{Im}\{L(\xi)\} = \mathcal{N}_{GA_{n_F}}(\xi)$ for all $\xi \in \Theta$ ensure that, for any initial condition $x_0 \in \mathcal{V}^\star_{GA_{n_F}}$, the input

$$u(t) = F(\theta(t))\, x_{x_0,u,\theta}(t) + L(\theta(t))\, w(t)$$

yields $y_{x_0,u,\theta} = 0$, for any trajectory w of appropriate dimension.

We would like to stress that, although we consider the same class of polynomial parameter-dependent systems described in (3.5), the scope here is restricted to systems with a scalar parameter, as described in Assumption 3.5. This simplification arises from technical challenges (described below) that limit the consideration of higher-dimensional parameter vectors.

Assumption 3.5. *The parameter dimension is reduced to* $l = 1$*:*

$$\Theta \subset \mathbb{R}.$$

3.5.1 A kernel based representation of the n_F-generalized adaptively weakly unobservable input subspaces

To derive conditions for constructing a basis of $\mathcal{N}_{GA_{n_F}}(\xi)$, we first present an equivalent reformulation of this subspace. The following lemma provides such a reformulation, valid for any subset $\mathcal{V} \subset \mathbb{R}^n$, and can therefore be applied in particular to $\mathcal{V} = \mathcal{V}^\star_{GA_{n_F}}$.

Lemma 3.3. *Given a parameter-dependent system* (3.5) *and* $\mathcal{V} \subset \mathbb{R}^n$*, the following relation holds:*

$$B^{-1}(\xi)\mathcal{V} \cap \ker\{D(\xi)\} = \ker\{W_{\mathcal{V}}(\xi)\}, \tag{3.35}$$

where

$$W_{\mathcal{V}}(\xi) = \begin{bmatrix} V_\perp^\top B(\xi) \\ D(\xi) \end{bmatrix}, \tag{3.36}$$

with $V_\perp$ *as an orthonormal basis of the orthogonal complement*[10] $\mathcal{V}_\perp$ *of* $\mathcal{V}$*.*

Proof. Let $\mathcal{V}_\perp$ be the orthogonal complement of $\mathcal{V}$. Let V and $V_\perp$ be a basis of $\mathcal{V}$ and $\mathcal{V}_\perp$, respectively, so that $\operatorname{Im}\{V\} = \mathcal{V}$ and $\operatorname{Im}\{V_\perp\} = \mathcal{V}_\perp$. First, we know that $B^{-1}(\xi)\mathcal{V} = \{u \in \mathbb{R}^m, B(\xi)u \in \mathcal{V}\}$. Then, since V is a basis of $\mathcal{V}$, $B^{-1}(\xi)\mathcal{V} = \{u \in \mathbb{R}^m, \exists z \in \mathbb{R}^q, B(\xi)u = Vz\}$. Multiplying at the left the expression $B(\theta)u = Vz$ by $V_\perp^\top$, we obtain $V_\perp^\top B(\xi)u = V_\perp^\top V z = \mathbf{0}$. Then it follows that, as shown in [7, Property 3.1.3],

$$B^{-1}(\xi)\mathcal{V} = \{u \in \mathbb{R}^m, V_\perp^\top B(\xi)u = \mathbf{0}\} = \ker\left\{V_\perp^\top B(\xi)\right\}.$$

Therefore,

$$B^{-1}(\xi)\mathcal{V} \cap \ker\{D(\xi)\} = \ker\left\{V_\perp^\top B(\xi)\right\} \cap \ker\{D(\xi)\},$$

which can be rewritten as $\ker\{W_{\mathcal{V}}(\xi)\}$. Then we can conclude that relation (3.35) holds. □

[10]The orthogonal complement of a subspace $\mathcal{V} \subset \mathbb{R}^n$ is defined as $\mathcal{V}_\perp := \{\, v \in \mathbb{R}^n : \langle v, w\rangle = 0 \text{ for all } w \in \mathcal{V}\,\}$.

3.5.2 Computation of the kernel of a parameter-dependent matrix through its Smith normal form

Computing the kernel of a parameter-dependent matrix is generally challenging, since both its direction and its dimension may vary with the parameter. A natural strategy would be to evaluate the kernel for many parameter values; however, as the parameter can take infinitely many values, this approach is computationally inefficient. Moreover, the kernel dimension typically changes only at isolated parameter values, i.e., on a set of measure zero. As a result, sampling the parameter space on a grid may fail to detect these critical points where the kernel structure changes.

As an original contribution of this work, we propose a systematic approach to this problem based on the Smith normal form of the parameter-dependent matrix. The Smith normal form is a classical tool in algebra that diagonalizes polynomial matrices through invertible polynomial transformations, thereby simplifying their structure while preserving key properties such as rank and kernel. In our setting, it can be applied to matrices with polynomial entries depending on a scalar parameter, as is the case for the matrices of system (3.5) under Assumption 3.5 [16]. A detailed presentation of the Smith normal form and its properties is provided below.

Proposition 3.2. *Smith normal form, [45, Theorem 6.3-16]. Let $\xi \in \mathbb{R}$ and consider a polynomial application $\xi \mapsto W(\xi) \in \mathbb{R}^{n\times m}$ of normal rank r, i.e.*[11]*, nrank$(W(\xi)) = r$. There exist non-singular (invertible) polynomial functions $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $Q : \mathbb{R} \to \mathbb{R}^{m\times m}$ such that*

$$P(\xi)W(\xi)Q(\xi) = S(\xi), \tag{3.37}$$

where

$$S(\xi) = \begin{bmatrix} \beta_1(\xi) & 0 & \cdots & 0 & 0 \\ 0 & \beta_2(\xi) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_r(\xi) & 0 \\ 0 & 0 & \cdots & 0 & 0_{(n-r)\times(m-r)} \end{bmatrix}, \tag{3.38}$$

with $\beta_i(\xi)$, for $i \in \{1, \dots, r\}$, nonzero polynomial functions of ξ satisfying the divisibility property $\beta_i \mid \beta_{i+1}$ for all $1 \le i \le r-1$ (that is, β_i divides β_{i+1}). The application $S : \xi \mapsto S(\xi)$ is called the Smith normal form *of W.*

See Appendix C for a detailed proof of Proposition 3.2, a constructive algorithm, and discussions about the Smith normal form.

This tool reduces the problem of computing the kernel of a general parameter-dependent matrix to that of analyzing the kernel of a diagonal matrix (corresponding to the non-zero diagonal block of S), which is considerably simpler. The result is formalized in the following lemma.

Lemma 3.4. *Consider the polynomial application $W : \xi \mapsto W(\xi) \in \mathbb{R}^{n\times m}$. Define its associated Smith normal form by $S : \xi \mapsto S(\xi) \in \mathbb{R}^{n\times m}$ as in* (3.38)*, $Q : \xi \mapsto Q(\xi) \in \mathbb{R}^{m\times m}$, and $P : \xi \mapsto P(\xi) \in \mathbb{R}^{n\times n}$ such that* (3.37) *holds. Then,*

$$\ker\{W(\xi)\} = Q(\xi)\ker\{S(\xi)\}, \tag{3.39}$$

[11] $\mathrm{nrank}(W(\xi)) := \max_{\xi\in\mathbb{R}} \mathrm{rank}(W(\xi))$.

holds, such that

$$\ker\{S(\xi)\} = \operatorname{Im}\left\{\begin{bmatrix}\operatorname{diag}_{i=1}^{r}\Delta_i & \mathbf{0}_{r\times m-r}\\ \mathbf{0}_{n-r\times r} & I_{n-r\times m-r}\end{bmatrix}\right\}, \tag{3.40}$$

$$\Delta_i = \sum_{j_i=1}^{\bar{d}_i}\Delta_{i,j_i}, \tag{3.41}$$

$$\Delta_{i,j_i} = \begin{cases}1, \text{ if } \xi = \delta_{i,j_i}\\ 0, \text{otherwise,}\end{cases} \tag{3.42}$$

where δ_{i,j_i} is such that $\beta_i(\delta_{i,j_i}) = 0$ for $i = \{1,\ldots,r\}$, the roots of $\beta_i(\xi)$, and $j_i = \{1,\ldots,\bar{d}_i\}$, with $\bar{d}_i$ equals the degree of $\beta_i(\xi)$.

Proof. From the Smith normal form given in Proposition 3.2, $W(\xi) = P^{-1}(\xi)S(\xi)Q^{-1}(\xi)$, then

$$\ker\{W(\xi)\} = \{u\in\mathbb{R}^m, P^{-1}(\xi)S(\xi)Q^{-1}(\xi)u = \mathbf{0}\}. \tag{3.43}$$

Since $Q(\xi)$ is non-singular, by applying a change of variables $z = Q^{-1}(\xi)u$, we obtain the set $\{z \in \mathbb{R}^m, P^{-1}(\xi)S(\xi)z = \mathbf{0}\}$ which is equivalent to $\ker\{S(\xi)\} = \{z\in\mathbb{R}^m, S(\xi)z = \mathbf{0}\}$, then $Q^{-1}(\xi)u \in \ker\{S(\xi)\}$. Therefore, back in the original variable u, the kernel of the parameter-dependent matrix can be represented as

$$\ker\{W(\xi)\} = Q(\xi)\ker\{S(\xi)\}.$$

Furthermore, since matrix $S(\xi)$ has some diagonal terms and zeros elsewhere as shown in (3.38), its kernel can be obtained from the values of ξ such that the diagonal term $\beta_i(\xi) = 0$ for $i = \{1,\ldots,r\}$, which are the roots of each polynomial $\beta_i(\xi)$ in the diagonal. Therefore, the direction related to each non-zero diagonal is activated when ξ meets one of the roots of the polynomial β_i, and the directions related to the zeros are always active. Then, the kernel of $S(\xi)$ can be defined as in (3.40), where Δ_i represents a function dependent on ξ that is equal to one only when ξ meets one of the roots of the correspondent diagonal term, activating the direction of the kernel, otherwise is equal to zero. □

Remark 3.7. *1. Since the parameter ξ is real-valued, only real roots δ_{i,j_i} contribute to the kernel. Complex roots can be ignored as their indicator functions Δ_{i,j_i} remain zero for all real ξ. Additionally, repeated roots correspond to repeated indicators, which can be ignored without loss of generality.*

2. The polynomials $\beta_i : \xi \mapsto \beta_i(\xi)$ satisfy a divisibility chain, i.e., β_{i+1} is divisible by β_i, $\forall i \in \{1,\ldots,r\}$. This implies that roots of β_i are also roots of all β_k with $k \geq i$. Hence, when $\xi = \delta_{i,j_i}$, the indicators Δ_k, for all $k \geq i$, simultaneously activate (equal 1).

3. Even though Δ_i is defined as a sum of Δ_{i,j_i}, it acts as a binary indicator (0 or 1) because, for a given ξ, at most one Δ_{i,j_i} is active after removing repeated roots.

Example 3.3. *Consider a polynomial matrix:*

$$W(\xi) = \begin{bmatrix}(\xi-1) & 0 & 0\\ 0 & (\xi-1)^2(\xi^2+1) & 0\end{bmatrix},$$

which is already in the Smith normal form such that $S(\xi) = W(\xi)$ and $P = Q = I_2$. The diagonal elements of $W(\xi)$ yield the polynomials:

- *$\beta_1(\xi) = \xi - 1$, which has a single root $\xi = 1$, so $\bar{d}_1 = 1$.*
- *$\beta_2(\xi) = (\xi - 1)^2(\xi^2 + 1)$, which has the repeated root $\xi = 1$ and a pair of complex conjugate roots $\xi = \pm j$, resulting in $\bar{d}_2 = 4$.*

Note that $\beta_1(\xi) \mid \beta_2(\xi)$ satisfies the divisible chain. Using Lemma 3.4, the kernel of $W(\xi)$ is expressed as:

$$\ker\{W(\xi)\} = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where the indicators functions are $\Delta_1 = \Delta_{1,1}$ and $\Delta_2 = \Delta_{2,1} + \Delta_{2,2} + \Delta_{2,3} + \Delta_{2,4}$ with

$$\Delta_{1,1} = \Delta_{2,1} = \Delta_{2,2} = \begin{cases} 1, & \text{if } \xi = 1, \\ 0, & \text{otherwise,} \end{cases} \quad \Delta_{2,3} = \begin{cases} 1, & \text{if } \xi = j, \\ 0, & \text{otherwise,} \end{cases} \quad \Delta_{2,4} = \begin{cases} 1, & \text{if } \xi = -j, \\ 0, & \text{otherwise.} \end{cases}$$

Let us analyse the cases stressed in Remark 3.7:

- *Complex roots: The roots $\xi = \pm j$ of $\beta_2(\xi)$ are complex. Since the parameter ξ is assumed to be real-valued, this root never activates the corresponding indicator, i.e., $\Delta_{2,3} = \Delta_{2,4} = 0$ for all $\xi \in \Theta$. Thus, this term can be safely ignored.*
- *The root $\xi = 1$ appears twice in $\beta_2(\xi)$, resulting in two indicators $\Delta_{2,1}$ and $\Delta_{2,2}$ that are both 1 when $\xi = 1$. However, since the individual indicators and sum of them are linearly dependent (they span the same subspace), one of them can be discarded without altering the resulting kernel subspace.*

Taking these simplifications into account, we can redefine: $\Delta_2 = \Delta_{2,1}$, excluding both the complex roots and the repeated root without loss of generality.

3.5.3 The basis of the n_F-generalized adaptively weakly unobservable input subspaces

Using Lemma 3.3 with $\mathcal{V} = \mathcal{V}^\star_{GA_{n_F}}$, we know that

$$\mathcal{N}_{GA_{n_F}}(\xi) = B^{-1}(\xi)\, \mathcal{V}^\star_{GA_{n_F}} \cap \ker\{D(\xi)\} = \ker\left\{W_{\mathcal{V}^\star_{GA_{n_F}}}(\xi)\right\},$$

where $W_{\mathcal{V}^\star_{GA_{n_F}}}(\xi)$ is given in (3.36). Applying the Smith normal form to $W_{\mathcal{V}^\star_{GA_{n_F}}}(\xi)$ (via Lemma 3.4), we obtain an application $L : \xi \mapsto L(\xi)$ such that

$$\operatorname{Im}\{L(\xi)\} = \ker\left\{W_{\mathcal{V}^\star_{GA_{n_F}}}(\xi)\right\} = \mathcal{N}_{GA_{n_F}}(\xi).$$

Moreover, the structure of such a matrix can be expressed as

$$L(\xi) \;=\; \begin{bmatrix} \Delta_1 L_1(\xi) & \cdots & \Delta_r L_r(\xi) & \tilde{L}(\xi) \end{bmatrix}, \tag{3.44}$$

where $L_i(\xi)$ for $i = 1, \ldots, r$, as well as $\tilde{L}(\xi)$, depend polynomially on ξ. Any of these blocks may be empty. This representation highlights that both the dimension and the directions of $\mathcal{N}_{GA_{n_F}}(\xi)$ may vary with the parameter ξ.

Singularities and parameter-dependence of $L(\xi)$: We can derive three different cases regarding the kind of dependence of the directions of the subspace $\mathcal{N}_{GA_{n_F}}(\xi)$ with respect to ξ, which are summarized in the following three points:

- *Independence:* Some directions are possibly independent of ξ, i.e., constant directions with a constant dimension. In this case, $\tilde{L}(\xi)$ contains constant columns.

- *Isolated dimension drop:* When directions depend on the value of ξ, such directions can be time-varying if ξ varies in time. The respective basis adapts according to the parameter value. The directions depend continuously on the parameter ξ. For some isolated values of ξ, some directions may vanish or be a linear combination of the others, leading to drops in the subspace dimension. This is represented by the columns of matrix $\tilde{L}(\xi)$ that depend on ξ.

- *Isolated dimension increase:* Differently from the second case, some directions are only active for a finite number of parameter values. It means that when ξ takes some specific values, a direction appears in the subspace. In Lemma 3.4, this is represented by the indicator Δ_i in (3.41). Therefore, it influences the subspace dimension which can change when such directions appear or not. This is represented by the columns $\Delta_i L_i(\xi)$, for $i = 1$ to $i = r$, of $L(\xi)$ in (3.44).

3.5.4 The basis of the generalized weakly unobservable input subspace

From the above discussion on the various parameter-dependent case of $L(\xi)$, we can also deduce a basis of the generalized weakly unobservable input subspace $\mathcal{N}_G$. This result is formalized in the next Lemma.

Lemma 3.5. *Let $n_F = 0$ and consider the n_F-generalized adaptively weakly unobservable input application $\mathcal{N}_{GA_{n_F=0}}$ of System* (3.5) *together with its generalized weakly unobservable input subspace $\mathcal{N}_G$. Let $L : \xi \mapsto L(\xi)$ be such that for all $\xi \in \Theta$, $\operatorname{Im}\{L(\xi)\} = \mathcal{N}_{GA_{n_F=0}}(\xi)$. Define $\bar{L}$ such that*

$$\operatorname{Im}\left\{\bar{L}\right\} = \bigcap_{\xi\in\Theta} \operatorname{Im}\left\{L(\xi)\right\}. \tag{3.45}$$

Then

$$\operatorname{Im}\left\{\bar{L}\right\} = \mathcal{N}_G.$$

Proof. Since $\mathcal{V}_G^\star = \mathcal{V}^\star_{GA_{n_F=0}}$, take $L : \xi \mapsto L(\xi)$ with $\operatorname{Im}\{L(\xi)\} = \mathcal{N}_{GA_{n_F=0}}(\xi)$ for all $\xi \in \Theta$. By Equation (3.34),

$$\mathcal{N}_G = \bigcap_{\xi\in\Theta} \mathcal{N}_{GA_{n_F=0}}(\xi) = \bigcap_{\xi\in\Theta} \operatorname{Im}\left\{L(\xi)\right\} = \operatorname{Im}\left\{\bar{L}\right\}.$$

□

This lemma yields a tractable way to compute $\mathcal{N}_G$: it suffices to extract the parameter-independent directions of $L(\xi)$, i.e. a basis of $\mathcal{N}_{GA_{n_F=0}}(\xi)$, which can be obtained using Proposition 3.2.

3.6 Characterization of robust and adaptive input redundancy

Now that we just developed strategies to compute all the essential objects to study input redundancy of parameter-dependent systems, we can develop the main result of this chapter. We first provide a technical lemma, which is an extension of Lemma 2.1 (coming from [70, Theorem 7.11]) and is a first step for the result of adaptive input redundancy. Then, we formalize in Theorem 3.2 the condition to check if a polynomial parameter-dependent linear system (3.5) under Assumption 3.5 is adaptively IR. The condition for robust input redundancy is obtained by particular choices on the results of adaptive input redundancy and is formalized in Corollary 3.2.

Lemma 3.6. *Let $\mathcal{V}^{\star}_{GA_{n_F}} \subset \mathbb{R}^n$ be the n_F-generalized adaptively controlled invariant and invisible subspace of* (3.5)*. Let $L : \Theta \to \mathbb{R}^{m\times q}$ such that*

$$\operatorname{Im}\{L(\xi)\} = Q(\xi)\ker\{S(\xi)\}, \tag{3.46}$$

where $Q(\xi)$ and $S(\xi)$ are the matrices of the Smith normal form of matrix $W_{\mathcal{V}^{\star}_{GA_{n_F}}}(\xi)$ defined in (3.36)*, and $\ker\{S(\xi)\}$ is defined in* (3.40)*. For all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$, let $x_0 \in \mathcal{V}^{\star}_{GA_{n_F}}$ and u be an input function of the form:*

$$u(t) = F(\theta(t))x(t) + L(\theta(t))w(t), \tag{3.47}$$

for some function w, then the output of Σ_θ resulting from u and x_0 verify

$$y_{x_0,u,\theta} = \mathbf{0}.$$

Proof. Applying the input in the form (3.47) to (3.5) leads to the following dynamics,

$$\begin{aligned}\dot{x}_{x_0,u,\theta}(t) &= (A(\theta(t)) + B(\theta(t))F(\theta(t)))x_{x_0,u,\theta}(t) + B(\theta(t))L(\theta(t))w(t),\\ y_{x_0,u,\theta}(t) &= (C(\theta(t)) + D(\theta(t))F(\theta(t)))x_{x_0,u,\theta}(t) + D(\theta(t))L(\theta(t))w(t).\end{aligned}$$

Since $\mathcal{N}_{GA_{n_F}}(\xi) = \ker\left\{W_{\mathcal{V}^{\star}_{GA_{n_F}}}(\xi)\right\}$ by Lemma 3.3 and $\ker\left\{W_{\mathcal{V}^{\star}_{GA_{n_F}}}(\xi)\right\} = Q(\xi)\ker\{S(\xi)\}$ by Lemma 3.4, then $\operatorname{Im}\{L(\xi)\} = B^{-1}(\xi)\mathcal{V}^{\star}_{GA_{n_F}} \cap \ker\{D(\xi)\}$. It follows $\forall t \geq 0$ that $L(\theta(t))w(t) \in \mathcal{N}_{GA_{n_F}}(\theta(t))$. Therefore,

$$B(\theta(t))L(\theta(t))w(t) \in \mathcal{V}^{\star}_{GA_{n_F}} \text{ and } D(\theta(t)))L(\theta(t))w(t) = \mathbf{0},$$

$\forall t \geq 0$. Thus, we can conclude that $y_{x_0,u,\theta} = C(\theta)x_{x_0,u,\theta} + D(\theta)u = \mathbf{0}$. $\square$

Theorem 3.2. *Let $\mathcal{V}^{\star}_{GA_{n_F}} \subset \mathbb{R}^n$ be the n_F-generalized adaptively weakly unobservable subspace of* (3.5) *and $\mathcal{N}_{GA_{n_F}}$ its n_F-generalized adaptively weakly unobservable input application. Under Assumption 3.5, System* (3.5) *is a-IR (resp. a-IR uniformly in θ) if $\exists\xi \in \Theta$ such that (resp. $\forall\xi \in \Theta$):*

$$\dim\left(\mathcal{N}_{GA_{n_F}}(\xi)\right) > 0, \tag{3.48}$$

or equivalently,

$$\dim\left(Q(\xi)\ker\{S(\xi)\}\right) > 0, \tag{3.49}$$

where $S(\xi)$ is the Smith normal form of matrix $W_{\mathcal{V}^{\star}_{GA_{n_F}}}(\xi)$ (defined in (3.36)*), and $Q(\xi)$ is a non-singular matrix obtained from the elementary operations to produce the Smith normal form. Both matrices respect the relation given in* (3.37)*, having that $W_{\mathcal{V}^{\star}_{GA_{n_F}}}(\xi) = W(\xi)$. Moreover, $\ker\{S(\xi)\}$ is defined in* (3.40)*.*

Proof. Let us consider an initial condition $x_0 \in \mathbb{R}^n$, two inputs $u_a, u_b : \mathbb{R}^+ \to \mathbb{R}^m$ and a parameter $\theta : \mathbb{R}^+ \to \Theta$. The errors between the two resulting state and output trajectories satisfy the following dynamics:

$$\begin{aligned}\dot{x}_{x_0,u_a,\theta}(t) - \dot{x}_{x_0,u_b,\theta}(t) &= A(\theta(t))(x_{x_0,u_a,\theta}(t) - x_{x_0,u_b,\theta}(t)) + B(\theta(t))(u_a(t) - u_b(t)),\\ y_{x_0,u_a,\theta}(t) - y_{x_0,u_b,\theta}(t) &= C(\theta(t))(x_{x_0,u_a,\theta}(t) - x_{x_0,u_b,\theta}(t)) + D(\theta(t))(u_a(t) - u_b(t)),\end{aligned}$$

for all $t \geq 0$, where the initial condition $x_{x_0,u_a,\theta}(0) - x_{x_0,u_b,\theta}(0) = \mathbf{0}$. To obtain the same output trajectories for u_a and u_b, we have to impose $y_{x_0,u_a,\theta} = y_{x_0,u_b,\theta}$, leading to $y_{x_0,u_a,\theta} - y_{x_0,u_b,\theta} = \mathbf{0}$, which is the output of the error dynamics. Then, since

$$x_{x_0,u_a,\theta}(0) - x_{x_0,u_b,\theta}(0) = \mathbf{0} \in \mathcal{V}^\star_{GA_{n_F}},$$

applying Lemma 3.6, we have that the output trajectory of the error, $y_{x_0,u_a,\theta} - y_{x_0,u_b,\theta}$, is zero if

$$u_a(t) - u_b(t) = F(\theta(t))(x_{x_0,u_a,\theta}(t) - x_{x_0,u_b,\theta}(t)) + L(\theta(t))w(t),$$

for any function w. Taking $\theta : t \mapsto \theta(t) = \xi$ such that $\dim(\mathrm{Im}\,\{L(\xi)\}) > 0$, thus we can take w such that $L(\theta(t))w(t) \neq \mathbf{0}$ for t in some non-zero measure interval, so that $u_a \neq u_b$ and System (3.5) is a-IR. For the uniformity in θ, $\forall \theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$, we have that $\dim(\mathrm{Im}\,\{L(\theta(t))\}) > 0$, $\forall t \geq 0$, leading to the choice of w such that $L(\theta(t))w(t) \neq \mathbf{0}$, $\forall t \geq 0$. By Lemma 3.3, and Lemma 3.6,

$$\mathrm{Im}\,\{L(\xi)\} = Q(\xi)\ker\{S(\xi)\} = \mathcal{N}_{GA_{n_F}}(\xi),$$

for all $\xi \in \Theta$, then these subspaces do not have a zero dimension either, and conditions (3.48) and (3.49) are equivalent. $\square$

From Lemma 3.6 and Theorem 3.2, the condition for robust input redundancy is obtained by considering the subspace $\mathcal{V}^\star_G$ instead of $\mathcal{V}^\star_{GA_{n_F}}$ and the constant columns of the basis of $B^{-1}(\xi)\mathcal{V}^\star_G \cap \ker\{D(\xi)\}$ to obtain $\mathcal{N}_G$, indicating that the inputs do not depend on the parameter. Such a result is established in Lemma 3.7, which is inspired by Theorem 9 from [48], and Corollary 3.2.

Lemma 3.7. *Let $\mathcal{V}^\star_G \subset \mathbb{R}^n$ be the generalized weakly unobservable subspace of System* (3.5) *with $F \in \mathbb{R}^{m\times n} \in \mathbf{F}(\mathcal{V}^\star_G)$, and $\bar{L} \in \mathbb{R}^{m\times\bar{q}}$ be such that*

$$\mathrm{Im}\,\{\bar{L}\} = \bigcap_{\xi\in\Theta} \mathrm{Im}\,\{L(\xi)\}, \text{ where } \mathrm{Im}\,\{L(\xi)\} = \mathcal{N}_{GA_{n_F}=0}(\xi), \tag{3.50}$$

with $\mathcal{N}_{GA_{n_F=0}}$ the $n_F = 0$-generalized adaptively weakly input application of System (3.5)*. Let $x_0 \in \mathcal{V}^\star_G$, $u : \mathbb{R}^+ \to \mathbb{R}^m$, then $\forall \theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$, the two following statements are equivalent:*

(i) $y_{x_0,u,\theta} = \mathbf{0}$*;*

(ii) $u = Fx_{x_0,u,\theta} + \bar{L}w$ *for some function w.*

Proof. $(i) \Rightarrow (ii)$: Let us assume that the output is zero, i.e., $y_{x_0,u,\theta} = \mathbf{0}$ for all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$. We can deduce that the state trajectory $x_{x_0,u,\theta}(t)$ belongs to the set $\mathcal{V}^\star_G$ for all $t \geq 0$ and all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$. We now introduce an auxiliary input $v = u - Fx_{x_0,u,\theta}$, resulting in the system trajectories having the following dynamics for all $t \geq 0$ and all $\theta \in \mathcal{PC}(\mathbb{R}^+, \Theta)$,

$$\begin{aligned}\dot{x}_{x_0,u,\theta}(t) &= (A(\theta(t)) + B(\theta(t))F(\theta(t)))x_{x_0,u,\theta}(t) + B(\theta(t))v(t),\\ y_{x_0,u,\theta}(t) &= (C(\theta(t)) + D(\theta(t))F(\theta(t)))x_{x_0,u,\theta}(t) + D(\theta(t))v(t).\end{aligned}$$

It yields that $B(\theta(t))v(t) \in \mathcal{V}_G^\star$ and $D(\theta(t))v(t) = \mathbf{0}$ for all $(t, \theta(t)) \in \mathbb{R}^+ \times \Theta$. Then, $v(t) \in B^{-1}(\theta(t))\mathcal{V}_G^\star \cap \ker\{D(\theta(t))\} = \mathcal{N}_{GA_{n_F=0}}(\theta(t)), \forall \xi \in \Theta$. Since the input functions are parameter-independent, v cannot depend on the parameter. Consequently, v rewrites as $v = \bar{L}w$ for some function w, where $\bar{L}$ is such as in (3.50).
$(ii) \Rightarrow (i)$: This implication follows from the computation of the state trajectory. □

Corollary 3.2. *Let $\mathcal{V}_G^\star \subset \mathbb{R}^n$ be the generalized weakly unobservable subspace of System* (3.5) *and let $\mathcal{N}_{GA_{n_F=0}}$ denote its generalized adaptively weakly unobservable input application. For each $\xi \in \Theta$, let $L(\xi)$ satisfy* $\operatorname{Im}\{L(\xi)\} = \mathcal{N}_{GA_{n_F=0}}(\xi)$*, and define $\bar{L}$ by*

$$\operatorname{Im}\left\{\bar{L}\right\} := \bigcap_{\xi \in \Theta} \operatorname{Im}\left\{L(\xi)\right\}.$$

Under Assumption 3.5, System (3.5) *is r-IR whenever*

$$\dim\!\left(\operatorname{Im}\left\{\bar{L}\right\}\right) > 0.$$

If, in addition, $\Theta \subset \mathbb{R}$ is a compact interval, the following statements are equivalent:

(i) System (3.5) *is r-IR;*

(ii) $\dim\!\left(\operatorname{Im}\left\{\bar{L}\right\}\right) > 0$.

Proof. The argument follows that of Theorem 3.2, relying on Lemma 3.7 and using the generalized subspace $\mathcal{V}_G^\star$, a constant $F \in \mathbb{R}^{m\times n}$, and a constant $L \in \mathbb{R}^{m\times q}$ formed by the parameter-independent directions of the subspace $Q(\xi)\ker\{S(\xi)\}$. Hence, if $\dim\!\left(\operatorname{Im}\left\{\bar{L}\right\}\right) > 0$ there exist nontrivial parameter-independent input directions producing zero output, which implies r-IR.

When Θ is a compact interval, the representations (3.5) and (3.16) are equivalent; therefore the existence of parameter-independent zero-output directions is also necessary, proving the equivalence of (i) and (ii). □

We now continue the HEV example to verify robust and adaptive input redundancy for the two parameter-dependent models (3.1) and (3.3) using the tools developed in this chapter.

Guideline Example - HEV. *First, we want to verify the **robust input redundancy** of the inertia dependent model* (3.1).

Previously in this chapter, we computed the largest generalized subspace of system (3.1)*:*

$$\mathcal{V}_G^\star = \operatorname{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & \frac{K}{L_a} \\ 0 & -\frac{1}{\tau} \\ 0 & 0 \end{bmatrix}\right\}.$$

Using now the tools based on the Smith normal form of a parameter-dependent matrix, we obtain

$$\mathcal{N}_G = \operatorname{Im}\left\{\begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} \end{bmatrix}\right\},$$

such that we can define

$$\bar{L} = \begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} \end{bmatrix}.$$

According to Corollary 3.2, since $\dim(\mathrm{Im}\left\{\bar{L}\right\}) > 0$*, the HEV described by model* (3.1) *is indeed r-IR, as envisioned in the beginning of the chapter.*

Previously in this chapter, we showed that the two distinct inputs, $u_a = \mathbf{0}$ *and* u_b *given in* (3.2)*, lead to the same output trajectory for distinct values of* θ*. Actually, by analyzing the structure of input* u_b*, we can note that it admits an alternative representation:*

$$u_b(t) = F\, x_{\mathbf{0},u_b,\theta}(t) + \bar{L}\, w(t), \tag{3.51}$$

where $w(t) = 1$ *for all* $t \geq 0$*, and*

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{4L_a}{K\tau t_d} & \frac{L_a - R_a\tau}{K\tau^2} & 0 & 0 \end{bmatrix} \in \mathbf{F}(\mathcal{V}_G^\star).$$

Consequently, the structure of $u_a - u_b$ *is consistent with the structure of an invisible input. Moreover,* w *can be chosen arbitrarily, as it represents the intrinsic degrees of freedom of the system.*

Considering the case where the parameter corresponds to the fueling delay t_d*, let us verify if the parameter-dependent model of the HEV* (3.3) *presents **robust or adaptive input redundancy**.*

Robust input redundancy test. *We first check whether system* (3.3) *preserves input redundancy independently of* $\theta(t)$*. We obtained that* $\mathcal{V}_G^\star = \{\mathbf{0}\}$*. This means there are no parameter-independent invariant subspaces. With the Smith normal form technique, we obtain also*

$$\mathcal{N}_G = \{\mathbf{0}\}.$$

We conclude that the HEV system (3.3) *is* not *r-IR.*

Adaptive input redundancy test. *When allowing the inputs to depend on the parameter, we obtained:*

$$\mathcal{V}_{GA_1}^\star = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & \frac{K}{L_a} \\ 0 & -\frac{1}{\tau} \\ 0 & 0 \end{bmatrix}\right\},$$

and its associated space

$$\mathcal{N}_{GA_1} : \xi \mapsto \mathrm{Im}\left\{\begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} \end{bmatrix}\right\}.$$

Since $\dim(\mathcal{N}_{GA_1}(\xi)) > 0$ *for all admissible* ξ*, Theorem 3.2 ensures that the HEV described by* (3.3) *is a-IR, uniformly in* θ*.*

Note that even if $\mathcal{N}_{GA_1}(\xi)$ *is independent of* ξ*, System* (3.3) *is not r-IR because every friend of* $\mathcal{V}_{GA,1}^\star$ *is parameter dependent. This is illustrated by the alternative representation of input*

(2.7) and (2.8) with $\frac{1}{t_d} = \theta(t)$*, leading to the same output trajectory:*

$$u_a = F(\theta)x_{\mathbf{0},u_a,\theta} + \left(L + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \right) \frac{m+b}{r} y_r, \tag{3.52}$$

and

$$u_b = F(\theta)x_{\mathbf{0},u_b,\theta} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \frac{m+b}{r} y_r, \tag{3.53}$$

where

$$F(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{4L_a}{K\tau}\theta & \frac{L_a - R_a\tau}{K\tau^2} & 0 & 0 \end{bmatrix} \in \mathbf{F}(\mathcal{V}^\star_{GA_1}), \qquad L = \begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} \end{bmatrix}.$$

Consequently, the structure of $u_a - u_b$ *is consistent with the structure of an invisible input.*

Now, we turn to an academic example that highlights the general phenomenon of adaptive input redundancy in the case of polynomial parameter dependence. This example is interesting since it captures features that do not appear in the HEV example.

Example 3.4. *To illustrate the methodology, we consider a simple parameter-dependent system. The aim is to verify whether input redundancy holds in an* adaptive *sense. This highlights how our framework can systematically uncover parameter-dependent redundancy.*

Consider the polynomial parameter-dependent system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ \theta^3(t) & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & \theta(t) \\ \theta(t) & 2\theta(t) & -\theta^2(t) \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \end{aligned} \tag{3.54}$$

with parameter $\theta(t) \in [0,1]$ *assumed known at each time.*

Step 1: Generalized adaptively invariant subspace. *Since the parameter dependence is polynomial and* $\theta(t)$ *lies in a compact set, Algorithm 2.2 can be applied. Choosing* $\mathcal{V}_0 = \mathbb{R}^n$*, a degree* $n_F = 1$ *for the friend* $F(\xi)$*, we obtain*

$$\mathcal{V}^\star_{GA_{n_F}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{R}, \qquad F(\xi) = \begin{bmatrix} 0 & 0 & \xi \\ 0 & 0 & 0 \end{bmatrix}^\top.$$

Thus, the largest generalized adaptively invariant subspace coincides with $\ker\{C\}$*.*

Step 2: Smith form reduction. *To check adaptive IR, we compute* $Q(\xi)\ker\{S(\xi)\}$*, where* $S(\xi)$ *and* $Q(\xi)$ *come from the Smith form of* $W_{\mathcal{V}^\star_{GA_{n_F}}}(\xi)$*. Since* $\mathcal{V}^\star_{GA_{n_F}} = [1 \;\; 0]^\top\mathbb{R}$*, we have*

$$W_{\mathcal{V}^\star_{GA_{n_F}}}(\xi) = \begin{bmatrix} \xi & 2\xi & -\xi^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Its Smith form is (See Appendix C):

$$S(\xi) = \begin{bmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \textit{with } Q(\xi) = \begin{bmatrix} 1 & -2 & \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \;\; P(\xi) = I_2.$$

From (3.40)*, the kernel of* $S(\xi)$ *is*

$$\ker\{S(\xi)\} = \mathrm{Im}\left\{ \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \qquad \Delta_1 = \begin{cases} 1, & \xi = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Hence,

$$Q(\xi)\ker\{S(\xi)\} = \mathrm{Im}\left\{ \begin{bmatrix} \Delta_1 & -2 & \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}. \tag{3.55}$$

Interpretation. *The subspace* (3.55) *reveals three invisible input directions: (i) a constant one, (ii) a* θ*-dependent one, (iii) a* Δ_1*-dependent one, which only exists at* $\theta = 0$*. Since* $\dim(Q(\xi)\ker\{S(\xi)\}) > 0$ *for all* $\xi \in [0,1]$*, the system is* adaptively IR*.*

Step 3: Simulation scenarios. *We illustrate two situations:*

(a) Δ_1***-dependent direction.*** *Consider first the case where only the* Δ_1*-dependent direction is active. Define*

$$\theta(t) = \begin{cases} 0, & 0 \le t < 2, \\ 1, & t \ge 2, \end{cases} \;\Rightarrow\; \Delta_1(t) = \begin{cases} 1, & 0 \le t < 2, \\ 0, & t \ge 2. \end{cases} \tag{3.56}$$

and the three inputs

$$\begin{aligned} u_a(t) &= F(\theta(t))x_{x_0,u_a,\theta}(t), \\ u_b(t) &= F(\theta(t))x_{x_0,u_b,\theta}(t) + \begin{bmatrix}\Delta_1(t) & 0 & 0\end{bmatrix}^\top \sin(t), \\ u_c(t) &= F(\theta(t))x_{x_0,u_c,\theta}(t) + \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top \sin(t). \end{aligned} \tag{3.57}$$

For $0 \le t < 2$*, we have* $\Delta_1 = 1$*, so* $u_b(t) = u_c(t) \neq u_a(t)$*. Since the term* $[\Delta_1\ 0\ 0]^\top \sin(t)$ *belongs to the* Δ_1*-dependent direction of* (3.55)*, Theorem 3.2 ensures that all three inputs generate the same output, as confirmed in Fig. 3.6.*

For $t \ge 2$*,* $\Delta_1 = 0$*, hence* $u_b(t) = u_a(t) \neq u_c(t)$*. Consequently,* $y_{x_0,u_a,\theta}(t) = y_{x_0,u_b,\theta}(t)$*, while* $y_{x_0,u_c,\theta}(t)$ *follows a different trajectory.*

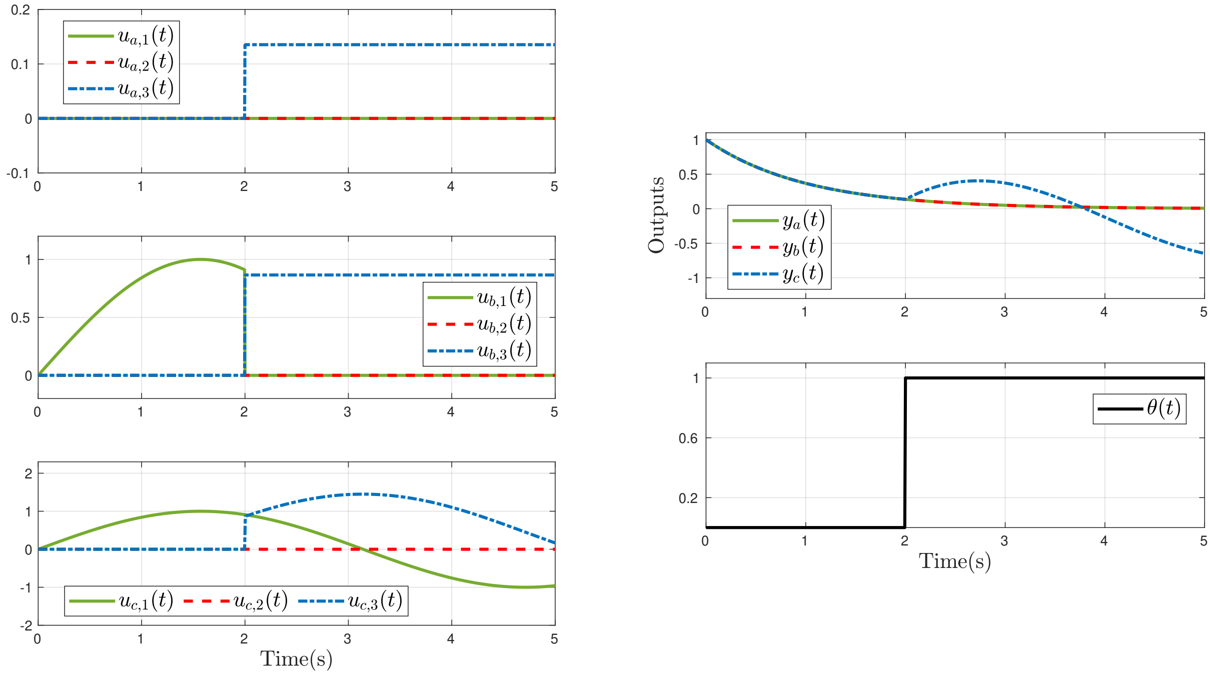

Figure 3.6: Input trajectories (3.57) and output trajectories in response to inputs in (3.57) and $\theta(t)$ in (3.56)

*(b) θ-**dependent direction.** Now consider the θ-dependent direction, with*

$$\theta(t) = \begin{cases} 0.2, & 0 \leq t < 2, \\ 0.2(t-2)+0.2, & 2 \leq t < 6, \\ 1, & t \geq 6, \end{cases} \tag{3.58}$$

and the inputs

$$\begin{aligned} u_a(t) &= F(\theta(t))x_{x0,u_a,\theta}(t), \\ u_b(t) &= F(\theta(t))x_{x0,u_b,\theta}(t) + \begin{bmatrix}\theta(t) & 0 & 1\end{bmatrix}^\top \sin(t), \\ u_c(t) &= F(\theta(t))x_{x0,u_c,\theta}(t) + \begin{bmatrix}0.2 & 0 & 1\end{bmatrix}^\top \sin(t). \end{aligned} \tag{3.59}$$

For $0 \leq t < 2$, $\theta = 0.2$, hence $u_b(t) = u_c(t) \neq u_a(t)$. Since $[\theta(t)\ 0\ 1]^\top \sin(t)$ is in the θ-dependent direction of (3.55), all three outputs coincide (Fig. 3.7).

For $t \geq 2$, $\theta(t)$ varies, so $u_b(t) \neq u_c(t)$. Nevertheless, $u_b(t)$ still lies in the θ-dependent direction, ensuring $y_{x0,u_a,\theta}(t) = y_{x0,u_b,\theta}(t)$, while $y_{x0,u_c,\theta}(t)$ differs since $[0.2\ 0\ 1]^\top \sin(t)$ is no longer in the subspace.

Figures 3.6 and 3.7 confirm the theoretical predictions: invisible inputs depend explicitly on the parameter value, which illustrates the adaptive IR phenomenon.

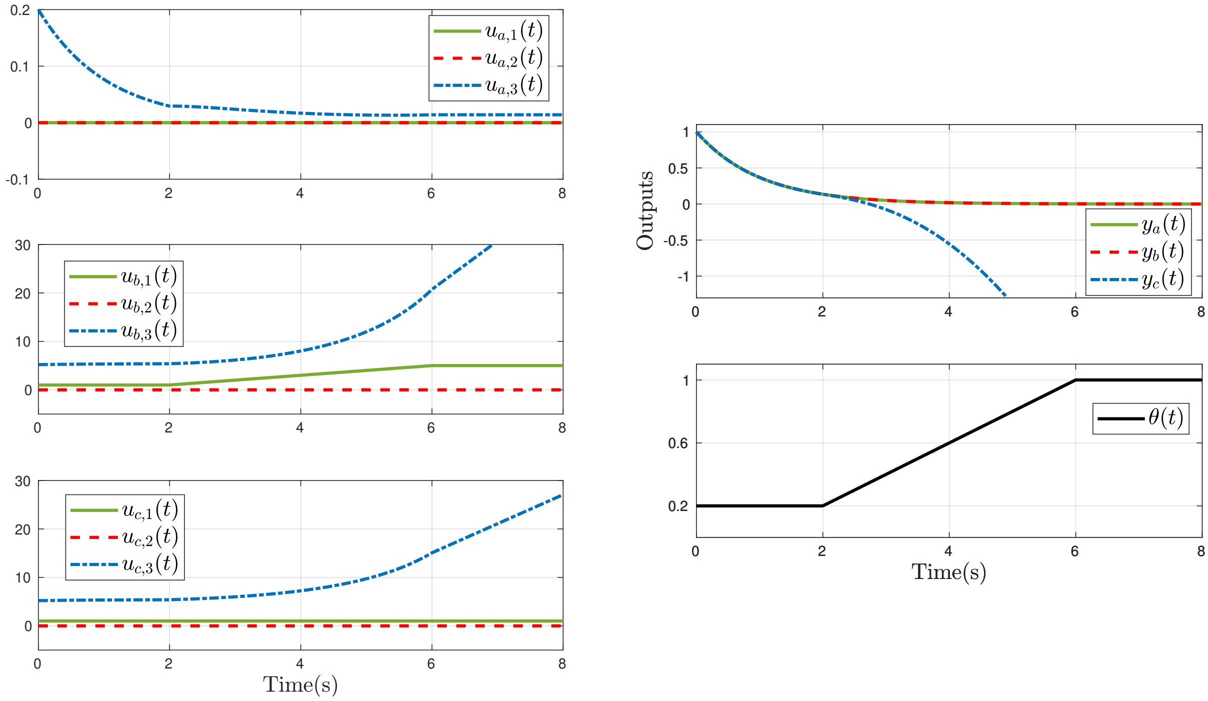

Figure 3.7: Input trajectories (3.59) and output trajectories in response to inputs in (3.59) and $\theta(t)$ given in (3.58).

3.7 Conclusion

In this chapter, we extended the concept of input redundancy to parameter-dependent systems by introducing the notions of robust and adaptive input redundancy. We used the HEV example to illustrate these concepts and to motivate the need for their formal characterization. We then focused on linear parameter-dependent systems with polynomial parameter dependence, identifying the key ingredients required for characterization of, most notably, the controlled invariant and output invisible subspaces of parameter-dependent systems. Based on these elements, we proposed a tractable method for computing the relevant subspaces under structural assumptions on the system. This methodological step constitutes a central contribution of the chapter and forms the basis for the subsequent results on input redundancy. Building on this foundation, we employed the Smith normal form to compute a parameter-dependent subspace essential for the characterization of robust and adaptive input redundancy. We then formalized the corresponding tractable conditions, which represent the main theoretical contribution of the chapter. Finally, we demonstrated the applicability of our results through illustrative examples, highlighting both their theoretical significance and their practical relevance.

Chapter 4

Input redundancy for switched systems

4.1 Introduction

Among the natural extensions of the input redundancy framework, one cannot avoid the case of systems whose evolution combines continuous-time dynamics with discrete events, namely switched systems. Such systems are usually described as a collection of modes between which the system can switch, each mode being governed by its own continuous-time dynamics [71]. They typically involve two kinds of inputs: a continuous input acting on the continuous dynamics, and a switching signal governing the transitions between modes.

Extending the concept of input redundancy to switched systems is particularly relevant. Many power electronic systems, which inherently combine continuous and discrete dynamics, are known to exhibit input redundancy [15, 47, 65, 81]. Another illustrative example is the hybrid electric vehicle, where the gearbox induces discrete events that can naturally be modeled as mode switches. Despite the significant progress made in the theory of switched systems over the past three decades [24, 40, 52, 53], input redundancy has not yet been investigated in this context.

The objective of this chapter is to address this gap. In Section 4.2, we introduce two notions tailored to switched systems: continuous input redundancy, where the focus is placed solely on the continuous inputs, and switched input redundancy, where different switching signals may lead to identical outputs. Section 4.3 is devoted to the study of the first notion. The main observation is that a switched system can be represented as a parameter-dependent system polynomial in the parameter. This reformulation, achieved through Lagrange interpolation, allows us to reuse the tools developed in the previous chapter for parameter-dependent systems and to derive tractable conditions under which a system can be declared continuous input redundant, and also to reveal the system's degrees of freedom. Section 4.4 turns to the second notion, switched input redundancy. In this case, the problem is more intricate and cannot be addressed by a direct extension of previous results. To make the analysis tractable, we focus on the simple yet revealing case of switched systems with two modes and no continuous input. In this setting, we identify the main difficulties that arise in the derivation of general conditions, and we propose an approach based on the duality between input redundancy and left-invertibility. The results rely on existing work on left-invertibility of switched systems [44, 69], which we adapt to the present framework.

4.2 The concept of input redundancy in the switched context

4.2.1 System description

In this chapter, we consider systems represented as in Figure 4.1 and described according to the following definition.

Definition 4.1 (Switched system Σ_σ). *A switched system, denoted by Σ_σ is characterized by:*

- *a continuous input $u : \mathbb{R}^+ \to \mathcal{U}$, where $\mathcal{U}$ is the set of admissible input values,*
- *a switched input $\sigma : \mathbb{R}^+ \to [\![1, N]\!]$, where $N \in \mathbb{N}$ is the number of values that σ can take and $[\![1, N]\!] = [1, N] \cap \mathbb{Z}$,*
- *a state $x : \mathbb{R}^+ \to \mathcal{X}$ with initial condition $x(0) = x_0 \in \mathcal{X}$ where $\mathcal{X}$ is the set of admissible state values,*
- *and a performance output $y : \mathbb{R}^+ \to \mathcal{Y}$, where $\mathcal{Y}$ is the set of admissible output values. The output y reflects the control objective.*

For a given continuous input u, switched input σ, and initial condition x_0 that lead to the existence and uniqueness of state and output trajectories, we denote by $x_{x_0,u,\sigma}$ (resp. $y_{x_0,u,\sigma}$) the state (resp. output) trajectory generated by the system.

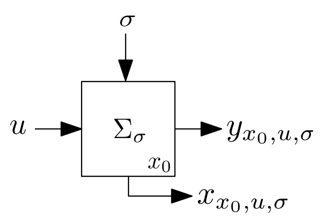

Figure 4.1: System Σ_σ with input u, state x, output y, and switching signal σ.

We adopt the following standing assumption to ensure the existence and uniqueness of solutions for Σ_σ throughout this chapter.

Assumption 4.1 (Existence and Uniqueness). *The dynamics of Σ_σ are assumed to be sufficiently regular to guarantee well-posedness. Moreover,*

- *The continuous input u is assumed to belong to $\mathcal{PC}(\mathbb{R}^+,\mathcal{U})$, the set of piecewise continuous functions from the time domain $\mathbb{R}^+$ to the set of input values $\mathcal{U}$;*
- *The switched input σ belongs to $\mathcal{S}$, the set of piecewise constant right-continuous functions from the time domain $\mathbb{R}^+$ to the set of modes $[\![1, N]\!]$ with a finite number of discontinuities in a finite length interval*[12]*.*

[12] A switched input $\sigma \in \mathcal{S}$ is assumed to prevent the occurrence of the Zeno phenomenon [42]. Combined with the regularity of the system dynamics and the piecewise–continuous nature of u, this assumption guarantees the existence and uniqueness of solutions [21].

4.2.2 Input redundancy definitions

In the context of switched systems, the general notion of input redundancy (Definition 2.2) must be adapted to this specific framework. Since such systems can be influenced both by the continuous input u and by the switching signal σ, it is natural to distinguish three cases: redundancy with respect to the continuous input, redundancy with respect to the switched signal, and redundancy with respect to both. To the best of our knowledge, a complete characterization of input redundancy for switched systems remains an open problem. In this chapter, we focus on the first two cases and defer the joint case to future research. We begin below by formally defining these notions.

Notion of continuous input redundancy

We introduce a definition of input redundancy for switched systems, particularly focusing on the continuous input.

Definition 4.2 (Continuous input redundancy). *The switched system Σ_σ is Continuous Input Redundant (c-IR) if there exists an output that can be produced by (at least) two distinct inputs, for some switching signal and some initial condition, i.e,*

$$\exists x_0 \in \mathcal{X},\ \sigma \in \mathcal{S},\ \textit{and}\ u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U}):\ u_a \neq u_b,\ y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma}.$$

It is clear from the definition that, for a given switching signal σ, the system Σ_σ may admit different input trajectories producing the same output, while this property might not hold for other signals σ. When the goal is to ensure this property *regardless of the switching signal*, a uniformity with respect to σ must be imposed, as formalized in the following definition.

Definition 4.3 (Continuous input redundancy uniformly in σ). *The switched system Σ_σ is c-IR, uniformly in σ if, for all switching signals, there exists an output that can be produced by (at least) two distinct inputs, for some initial condition, i.e,*

$$\forall \sigma \in \mathcal{S},\ \exists x_0 \in \mathcal{X},\ \textit{and}\ u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U}):\ u_a \neq u_b,\ y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma}.$$

Analogously to the LTI case, we now distinguish three types of input redundancy for switched systems, defined according to whether different inputs generating the same output can (or cannot) generate identical state trajectories.

Definition 4.4. *A switched system Σ_σ is c-IR of:*

- *The first kind if it is c-IR and for all initial conditions $x_0 \in \mathcal{X}$, all switching signal $\sigma \in \mathcal{S}$ and all input functions $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U})$ satisfying:*

  $$u_a \neq u_b \ \textit{and}\ y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma},$$

  the state trajectories are identical, i.e., $x_{x_0,u_a,\sigma} = x_{x_0,u_b,\sigma}$.

- *The second kind if it is c-IR and for all initial conditions x_0, all switching signal $\sigma \in \mathcal{S}$ and all input functions $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U})$ satisfying:*

  $$u_a \neq u_b \ \textit{and}\ y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma},$$

  the state trajectories are not identical, i.e., $x_{x_0,u_a,\sigma} \neq x_{x_0,u_b,\sigma}$.

- *The third kind if it is c-IR but not of kind 1 or kind 2.*

Notion of switched input redundancy

We now extend the notion of input redundancy of switched systems, regarding the switching signal as an input.

Definition 4.5 (Switched input redundancy)**.** *The switched system Σ_σ is Switched Input Redundant (σ-IR) if there exists an output trajectory that can be produced by, at least, two distinct switching signals for some initial condition and continuous input trajectory i.e,*

$$\exists x_0 \in \mathcal{X}, u \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U}),\ \sigma_a, \sigma_b \in \mathcal{S}:\ \sigma_a \neq \sigma_b,\ y_{x_0,u,\sigma_a} = y_{x_0,u,\sigma_b}.$$

Once again, we can define a uniform property that holds for all continuous input in Definition 4.6 and three different kinds of switched input redundancy in Definition 4.7.

Definition 4.6 (Switched input redundancy uniformly in u)**.** *The switched system Σ_σ is* input redundant w.r.t. the switching signal *(σ-IR) if for all continuous input trajectories, there exists an output trajectory that can be produced by, at least, two distinct switching signals for some initial condition i.e,*

$$\forall u \in \mathcal{PC}(\mathbb{R}^+,\mathcal{U}),\ \exists x_0 \in \mathcal{X},\ \sigma_a, \sigma_b \in \mathcal{S}:\ \sigma_a \neq \sigma_b,\ y_{x_0,u,\sigma_a} = y_{x_0,u,\sigma_b}.$$

Definition 4.7. *The switched system Σ_σ is σ-IR of:*

- *The first kind if it is σ-IR and for all initial conditions $x_0 \in \mathcal{X}$, all continuous input $u \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ and all switching law $\sigma_a, \sigma_b \in \mathcal{S}$ satisfying:*

  $$\sigma_a \neq \sigma_b \text{ and } y_{x_0,u,\sigma_a} = y_{x_0,u,\sigma_b},$$

  the state trajectories are identical, i.e., $x_{x_0,u,\sigma_a} = x_{x_0,u,\sigma_b}$.

- *The second kind if it is σ-IR and for all initial conditions $x_0 \in \mathcal{X}$, all continuous input $u \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ and all switching law $\sigma_a, \sigma_b \in \mathcal{S}$ satisfying:*

  $$\sigma_a \neq \sigma_b \text{ and } y_{x_0,u,\sigma_a} = y_{x_0,u,\sigma_b},$$

  the state trajectories are not identical, i.e., $x_{x_0,u,\sigma_a} \neq x_{x_0,u,\sigma_b}$.

- *The third kind if it is σ-IR but not of kind 1 or kind 2.*

To make these notions more intuitive, let us continue our discussion on the guiding example of the HEV.

Guideline Example - HEV. *Recall the model* (2.6) *of the HEV driveline Σ_{HEV}.*

The HEV example can also be represented as a switched system, where the switching signal corresponds to the gear position. In this case, the HEV switched system model is given by:

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} \frac{-2}{t_d} & 0 & 0 & 0 \\ \frac{4}{t_d} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & \frac{-K}{J} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & \frac{-(m+b)}{J} \end{bmatrix} x(t) + \begin{bmatrix} i & 0 \\ -i & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & 0 & 0 & \frac{r}{J} \end{bmatrix} x(t),
\end{aligned} \tag{4.1}$$

for $i \in [\![1, 6]\!]$, where i represents the six gear ratios that influence the torque input u_1. The switching signal is defined by $\sigma(t) \in [\![1, 6]\!]$ and may be interpreted in two ways: as an external signal (in an automatic transmission) or as a control input (in a manual transmission).

The central question is whether input redundancy remains valid under switched dynamics. In what follows, we restrict attention to the case where the switching signal is not a control input. This restriction is made because the forthcoming results on switched input redundancy do not apply to the system structure in this case, as will be clarified in Section 4.4.

Continuous input redundancy of the HEV. *To analyze continuous input redundancy, we need to find at least one switching law such that there exist two distinct inputs leading to the same output trajectory for some initial condition. We can note that by selecting $\sigma(t) = 1, \forall t \geq 0$, the LTI model* (2.6) *is obtained. As verified in Chapter 2, the LTI model of the HEV is IR. Consequently, system* (4.1) *is* c-IR.

However, we cannot conclude about c-IR*, uniformly in σ. To analyze this case, let us consider $u_a = \mathbf{0}$. For $x_0 = \mathbf{0}$ and $\sigma : t \mapsto \min(6, k+1)$, $\forall t \in [2k, 2(k+1))$, $k \in \mathbb{N}$, yielding $x_{\mathbf{0},u_a,\sigma} = \mathbf{0}$ and $y_{\mathbf{0},u_a,\sigma} = \mathbf{0}$.*

Next, we consider the input

$$u_b(t) = \begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau} i - \frac{-4L_a}{K\tau t_d} x_{b,1}(t) + \frac{L_a - R_a\tau}{K\tau^2} x_{b,2}(t) \end{bmatrix}, \forall i \in [\![1, 6]\!]. \tag{4.2}$$

The resulting trajectories, $x_b = x_{\mathbf{0},u_b,\sigma}$ and $y_b = y_{\mathbf{0},u_b,\sigma} = \mathbf{0}$, are shown in Fig. 4.2. We observe that the output trajectory remains identical to the output generated by u_a.

These results confirm that the chosen input structure achieves the same output trajectory despite the switching between all modes of the switching signal. This led us to think that the system may be c-IR*, uniformly in σ. However, to confirm that, we must check for all switching laws $\sigma \in \mathcal{S}$, leading to an infinite number of simulations.*

This practical example shows that when a switched dynamics appears, an IR system may effectively behave as a c-IR, uniformly or not in σ. Yet, to date there are no established methods in the literature for rigorously verifying input redundancy in such switched systems. The remainder of this chapter addresses this gap by developing an appropriate switched framework. Although the σ-IR case was not treated in this example, we will also address this case.

4.2.3 Problem statement

The objective of this chapter is to address the following two problems.

Problem 4.1. *Determine tractable conditions under which a switched system Σ_σ can be certified as c-IR and, whenever this is the case, rigorously characterize its inherent degrees of freedom.*

Problem 4.2. *Determine tractable conditions under which a switched system Σ_σ can be certified as σ-IR and, whenever this is the case, rigorously characterize its inherent degrees of freedom.*

Section 4.3.3 is devoted to Problem 4.1, whereas Section 4.4 focuses on Problem 4.2.

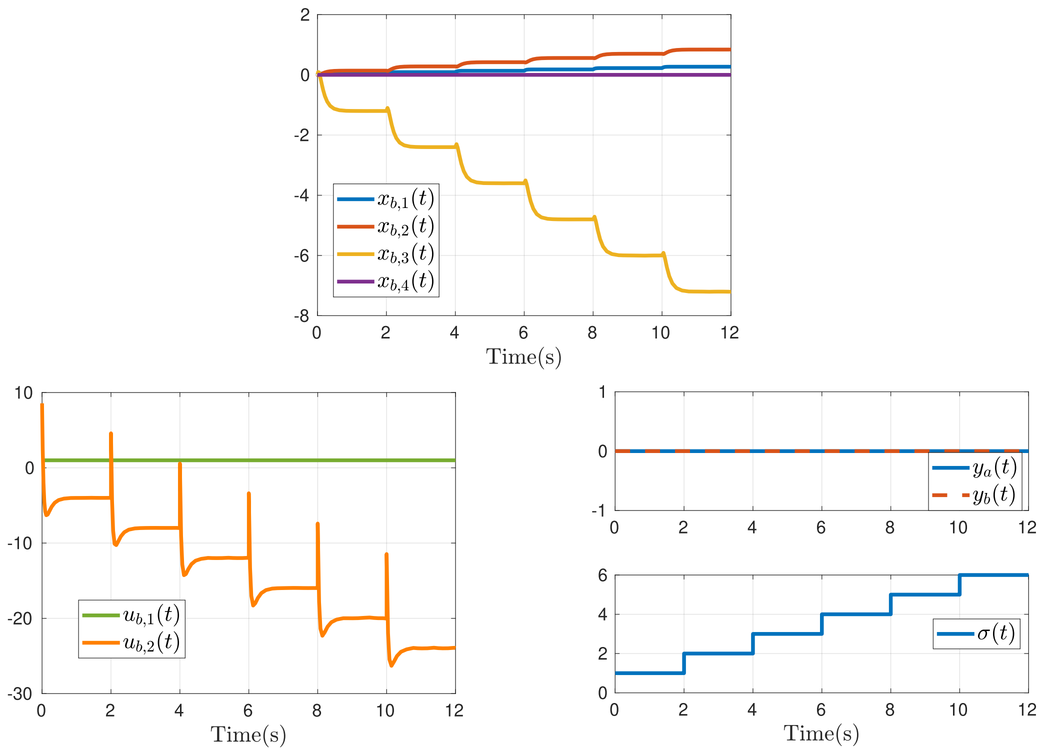

Figure 4.2: Input functions $u_a = \mathbf{0}$ and u_b given in (4.2) and respective state and output trajectories of system (4.1) for σ given in the guiding HEV example.

4.3 Characterization of continuous input redundancy

In this section, our objective is to analyze input redundancy with respect to the continuous input u, treating the switching signal σ as an external input over which no control can be exerted. Addressing Problem 4.1 within the general framework of Definition 4.1, which only relies on Assumption 4.1, is particularly challenging. To make the analysis tractable, we therefore introduce additional structural assumptions, as detailed below.

4.3.1 Considered framework

We begin by introducing the following assumptions.

Assumption 4.2 (Admissible sets)**.** *The continuous input u, state x, and output y are assumed to be unconstrained, i.e., $\mathcal{U} = \mathbb{R}^m$, $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^p$.*

Assumption 4.3 (Dependency on u and x)**.** *The system Σ_σ is assumed to be* linear *in the state x and the input u.*

Under these assumptions, System Σ_σ can be represented for all time $t \geq 0$, as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4.3a}$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \tag{4.3b}$$

where $\forall i \in [\![1, N]\!]$, matrices A_i, B_i, C_i, and D_i are of appropriate dimension. The quadruple (A_i, B_i, C_i, D_i), $i \in [\![1, N]\!]$, corresponds to an LTI subsystem, usually referred to as *the i-th mode* *of* Σ_σ. A mode $i \in [\![1, N]\!]$ for which the LTI system

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad x(0) = x_0 \in \mathbb{R}^n,$$
$$y(t) = C_i x(t) + D_i u(t),$$

is IR is referred to as an *IR mode.*

4.3.2 Preliminary results on continuous input redundancy

We already provide a characterization of the concept of continuous input redundancy, as well as its uniform corresponding notion (see Definitions 4.2-4.3). These results build on the finite-time properties of IR LTI systems presented in Section 2.4.2 of Chapter 2.

Theorem 4.1. *The following statements are equivalent:*

(i) The switched system Σ_σ *described by* (4.3) *is c-IR;*

(ii) There exists an index $i \in [\![1, N]\!]$ *such that mode* i *is IR.*

Proof. (i) $\Rightarrow$ (ii): Let $\{t_k\}_{k\geq 0}$ denote the switched instants of a switching signal σ, with $\sigma(t) = i$ for $t \in [t_k, t_{k+1})$ and $i \in [\![1, N]\!]$. Assume that the switched system (4.3) is c-IR. Then there exist $\sigma \in \mathcal{S}$, $x_0 \in \mathbb{R}^n$, and inputs $u_a \neq u_b$ such that $y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma}$. Consequently, there is at least one $k \in \mathbb{N}$ for which $\sigma(t) = i$ over the entire interval $[t_k, t_{k+1}]$ and $u_a|_{[t_k,t_{k+1}]} \neq u_b|_{[t_k,t_{k+1}]}$. Since $y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma}$, this situation satisfies property (S1) of Lemma 2.4 by a suitable time shift. It then follows from Lemma 2.4 that mode i is IR.
(ii) $\Rightarrow$ (i): Let us assume that the mode i of Σ_σ is IR. Let us select the particular switching signal $\sigma(t) = i$, $\forall t \geq 0$. Consequently, there exist $x_0 \in \mathbb{R}^n$, $u_a \neq u_b$ such that $y_{x_0,u_a,\sigma} = y_{x_0,u_a,i} = y_{x_0,u_b,i} = y_{x_0,u_b,\sigma}$. That implies that the switched linear system (4.3) is IR w.r.t. the continuous input. □

As indicated in the definition of c-IR, this property can be fragile, in the sense that it strongly depends on the switching signal σ, over which no control is exerted. Once the system leaves an IR mode i, the degrees of freedom associated with input redundancy can no longer be exploited. To address this limitation, we also provide a characterization of continuous input redundancy that holds uniformly over all switching signals σ.

Proposition 4.1. *The following statements are equivalent:*

(i) System Σ_σ *represented by* (4.3) *is c-IR uniformly in* σ*;*

(ii) The N *modes of System* (4.3) *are IR.*

Proof. $(i) \Rightarrow (ii)$ If the switched linear system (4.3) is c-IR, uniformly in σ, then for all $\sigma \in \mathcal{S}$, there exist two different input trajectories leading to the same output trajectories. This is in particular true for all constant switching signal $\sigma(t) = i \in [\![1, N]\!]$, that is, all the N modes of the switched system are IR.
$(ii) \Rightarrow (i)$ Let $\{t_k\}_{k\geq 0}$ denote the switching times of a switching signal σ, with $t_0 = 0$ and $\sigma(t) = i$ for $t \in [t_0, t_1)$, where $i \in [\![1, N]\!]$. Since the i-th mode is IR, there exist $x_0 \in \mathbb{R}^n$ and $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ such that $u_a|_{[t_0,t_1]} \neq u_b|_{[t_0,t_1]}$ and $y_{x_0,u_a,\sigma}|_{[t_0,t_1]} = y_{x_0,u_b,\sigma}|_{[t_0,t_1]}$. By virtue of Lemma 2.5, u_a and u_b can even be chosen so that $x_{x_0,u_a,\sigma}(t_1) = x_{x_0,u_b,\sigma}(t_1)$. Repeating the same argument interval by interval allows us to conclude that, for all $\sigma \in \mathcal{S}$, there exist $x_0 \in \mathbb{R}^n$ and $u_a \neq u_b$ such that $y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma}$. □

Guideline Example - HEV. *Consider again the switched model of the HEV* (4.1)*, where the switching signal is the gearbox position. Following the results of input redundancy for LTI systems recalled in Chapter 2, each LTI mode (A_i, B_i, C_i, D_i) of system* (4.1) *is IR. Then, according to Proposition 4.1, the switched system* (4.1) *is c-IR uniformly in σ.*

Another challenge arises from the fact that, even when continuous input redundancy is uniform in σ, it can still depend on the switching signal. In particular, difficulties occur when the state must assume specific values at the switching instants to reproduce the same output trajectories. Such constraints may limit the ability to exploit the available degrees of freedom. To illustrate this point, we consider the following academic example.

Example 4.1. *Consider System* (4.3) *with $N = 2$ and*

$$\big(A_i, B_i, C_i, D_i\big) = \Big(\mathbf{0}_{2\times 2},\ I_2,\ \begin{bmatrix} i-1 & 2-i \end{bmatrix},\ \mathbf{0}_{1\times 2}\Big), \qquad i \in \{1, 2\}. \tag{4.4}$$

Each mode corresponds to two decoupled integrators whose output selects either the first state component (for $i = 1$) or the second one (for $i = 2$). Since each individual mode is IR, the overall switched system can be regarded as c-IR.

Switching signal and initial state. *Let $x_0 = \mathbf{0}_2$ and choose the switching signal*

$$\sigma(t) = \begin{cases} 1, & 0 \le t < 1, \\ 2, & t \ge 1, \end{cases}$$

so that the output equals the first state component on $[0, 1)$ and switches to the second state component at $t = 1$.

Conditions for two inputs to give the same output. *Two input trajectories $u_a = [u_{a,1}, u_{a,2}]^\top$ and $u_b = [u_{b,1}, u_{b,2}]^\top$ generate identical outputs under σ if:*

(i) their first components coincide on $[0, 1)$, $u_{a,1}|_{[0,1)} = u_{b,1}|_{[0,1)}$, since the output depends on the first state driven by the first input;

(ii) their second components coincide on $[1, \infty)$, $u_{a,2}|_{[1,\infty)} = u_{b,2}|_{[1,\infty)}$, because after the switch the output depends on the second state/input;

(iii) the second state values match at the switching instant, $x_{x_0,u_a,\sigma,2}(1) = x_{x_0,u_b,\sigma,2}(1)$, otherwise the outputs differ immediately after $t = 1$.

Condition (iii) imposes a nontrivial coupling between the input histories before $t = 1$ to ensure the second states agree at the switching time.

Illustration. *Take for instance $u_a : t \mapsto \mathbf{1}_2$ (both inputs equal to one). Then any admissible u_b must satisfy the three conditions above. One valid choice is*

$$u_{b,1} : t \mapsto 1, \qquad u_{b,2} : t \mapsto \begin{cases} 0, & 0 \le t < 0.5, \\ 2, & 0.5 \le t < 1, \\ 1, & t \ge 1. \end{cases}$$

This design of $u_{b,2}$ guarantees the second states coincide at $t = 1$. The trajectories are depicted in Figure 4.3.

If, however, we replace the value 2 by $2 + \varepsilon$ with $\varepsilon \neq 0$, the matching at $t = 1$ fails and the outputs differ after the switch. This is shown in Figure 4.4 with $\varepsilon = 0.1$.

This example highlights how restrictive the input choice can become to preserve identical outputs across switching instants.

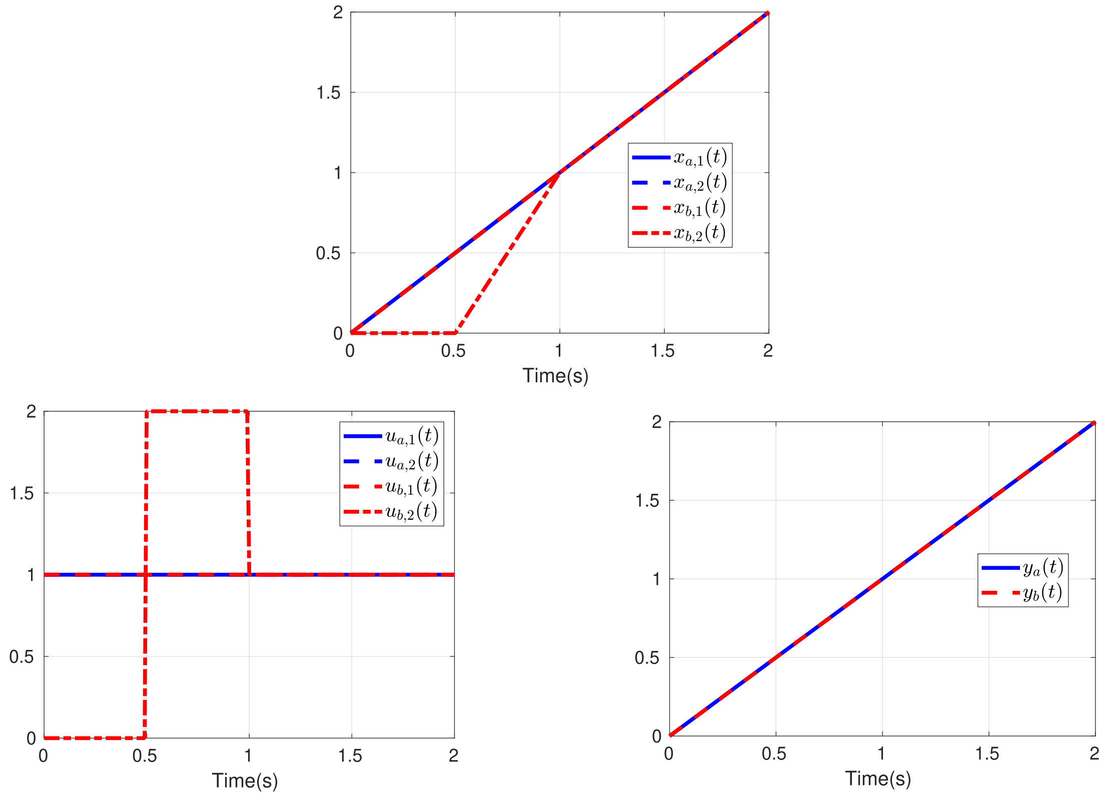

Figure 4.3: Trajectories of System (4.4) leading to the same output.

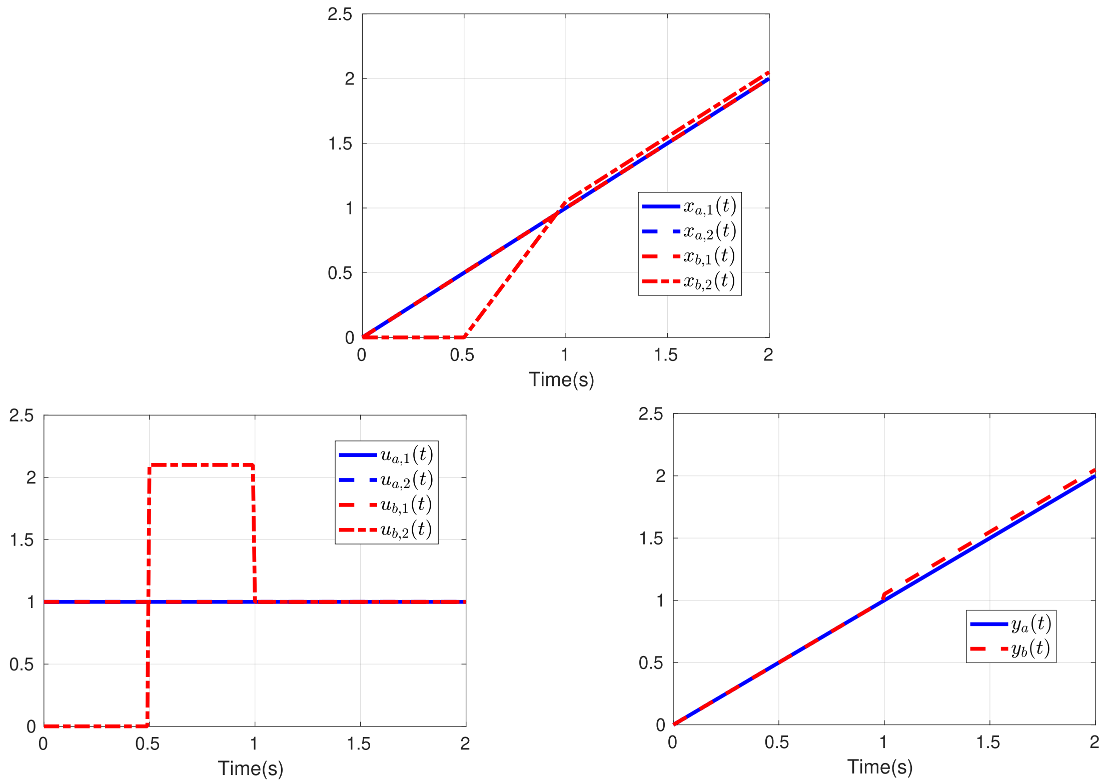

Figure 4.4: Trajectories of System (4.4) leading to different outputs.

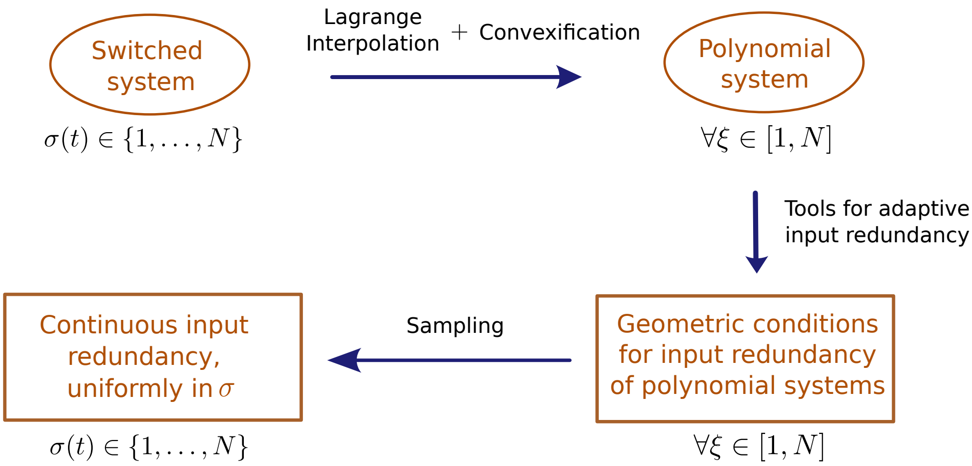

Figure 4.5: Diagram of the strategy for continuous input redundancy, uniformly in the switching signal.

This observation motivates us to characterize a form of continuous input redundancy that removes these constraints on the state values at specific instants. The remainder of this section is devoted to this objective, following the approach outlined below, which is represented in Figure 4.5. First, we derive a polynomial representation of Σ_σ with respect to σ by which a convexification of the index set $[\![1, N]\!]$ is performed. This allows us to exploit the results of Chapter 3 on adaptive input redundancy for parameter-dependent systems, treating σ as a time-varying parameter. Finally, the resulting conditions are presented in Section 4.3.4.

4.3.3 From switched systems to a polynomial representation

In this section we show that the switched system Σ_σ in (4.3) admits an equivalent polynomial representation of the form

$$\dot{x}(t) = A(\sigma(t))\, x(t) + B(\sigma(t))\, u(t), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4.5a}$$
$$y(t) = C(\sigma(t))\, x(t) + D(\sigma(t))\, u(t), \tag{4.5b}$$

where

$$A : [1, N] \to \mathbb{R}^{n\times n}, \quad B : [1, N] \to \mathbb{R}^{n\times m}, \quad C : [1, N] \to \mathbb{R}^{p\times n}, \quad D : [1, N] \to \mathbb{R}^{p\times m}$$

are matrix-valued polynomial maps of a single parameter, keeping in mind that $\sigma(t)$ takes values in $[\![1, N]\!] \subset [1, N]$, for all $t > 0$. This representation, obtained via Lagrange interpolation as in [56], is the basis for the developments that follow. We first define the elementary Lagrange interpolating polynomials.

Definition 4.8. *For each $i \in [\![1, N]\!]$, define the polynomial $P_i : \mathbb{R} \to \mathbb{R}$ by*

$$P_i(\xi) = \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{\xi - j}{i - j}. \tag{4.6}$$

The family $\{P_i\}_{i\in[\![1,N]\!]}$ forms the elementary Lagrange interpolating polynomials *associated with $[\![1, N]\!]$. In particular, for each $i \in [\![1, N]\!]$:*

- $\deg P_i = N - 1$;
- $P_i(i) = 1$, *and* $P_i(j) = 0$ *for all* $j \in [\![1, N]\!] \setminus \{i\}$.

The next lemma shows how to construct a matrix-valued polynomial that interpolates the matrices A_i of System (4.3).

Lemma 4.1. *Let* $\{P_i\}_{i \in [\![1,N]\!]}$ *be the elementary Lagrange interpolating polynomials of Definition 4.8. Then the map* $A : [1, N] \to \mathbb{R}^{n \times n}$ *defined by*

$$A(\xi) = \sum_{i=1}^{N} P_i(\xi) A_i \tag{4.7}$$

is the unique polynomial of degree at most $N - 1$ *satisfying*

$$A(i) = A_i \quad \text{for all } i \in [\![1, N]\!].$$

Proof. Each P_j is a polynomial of degree $N - 1$, hence $\deg A \leq N - 1$. Moreover, by the defining properties of P_j,

$$A(i) = \sum_{j=1}^{N} P_j(i) A_j = P_i(i) A_i = A_i \quad \forall i \in [\![1, N]\!].$$

Thus A is a polynomial of degree at most $N - 1$ interpolating the nodes (i, A_i), and by uniqueness of the Lagrange interpolant (see [25, Sec. 2.5]) no other polynomial of the same degree satisfies these conditions. ☐

Exactly the same construction yields the polynomial maps $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ appearing in (4.5).

Finally, note the one-to-one correspondence between the modes (A_i, B_i, C_i, D_i) of System (4.3) and the polynomial representation (4.5): for any switching signal $\sigma \in \mathcal{S}$,

$$(A(\sigma(t)), B(\sigma(t)), C(\sigma(t)), D(\sigma(t))) = (A_{\sigma(t)}, B_{\sigma(t)}, C_{\sigma(t)}, D_{\sigma(t)}) \quad \forall t \geq 0. \tag{4.8}$$

The following numerical example illustrates this transformation of a switched linear system to a polynomial parameter-dependent one.

Example 4.2. *Let us consider a switched linear system of three modes defined by the following matrices:*

$$\begin{aligned}
A_1 &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & 0 \\ 9 & -1 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \\
C_1 &= C_2 = C_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}, D_1 = D_2 = D_3 = \mathbf{0}_{1 \times 2}.
\end{aligned} \tag{4.9}$$

with $N = 3$. *To obtain the corresponding parameter-dependent matrices, we have to compute* $P_i(\xi)$, *given in* (4.6), *for* $i \in [\![1, 3]\!]$:

$$P_1(\xi) = \frac{1}{2}(\xi - 2)(\xi - 3), \;\; P_2(\xi) = -(\xi - 1)(\xi - 3), \;\; P_3(\xi) = \frac{1}{2}(\xi - 1)(\xi - 2).$$

Then, $A(\xi) = \sum_{i=1}^{3} P_i(\xi)A_i$*, and the same applies for matrices* $B(\xi)$ *and* $C(\xi)$*, such that*

$$A(\xi) = \begin{bmatrix} -1 & 0 \\ \xi^2 & -1 \end{bmatrix}, \ B(\xi) = \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}, \ C(\xi) = \begin{bmatrix} 0 & 1 \end{bmatrix}, D(\xi) = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

for $\xi \in [\![1, N]\!]$.

Let us also obtain the polynomial representation of the switched model of the HEV.

Guideline Example - HEV. *We have that matrices A and C are constant, only the input matrix depends on the switching signal. Then, to obtain the corresponding parameter-dependent matrices, we compute* $P_i(\xi)$*, given in* (4.6)*, for* $i \in [\![1, 6]\!]$ *to obtain* $B(\xi) = \sum_{i=1}^{6} P_i(\xi)B_i$*. The polynomial parameter-dependent representation of system* (4.1) *is given by:*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \frac{-2}{t_d} & 0 & 0 & 0 \\ \frac{4}{t_d} & \frac{-1}{\tau} & 0 & 0 \\ 0 & 0 & \frac{-R_a}{L_a} & \frac{-K}{J} \\ 0 & \frac{1}{\tau} & \frac{K}{L_a} & \frac{-(m+b)}{J} \end{bmatrix} x(t) + \begin{bmatrix} \xi & 0 \\ -\xi & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 0 & 0 & \frac{r}{J} \end{bmatrix} x(t), \end{aligned} \tag{4.10}$$

with a convex set $\xi \in [1, 6]$.

4.3.4 Degrees of freedom related to continuous input redundancy

The representation (4.5) expresses Σ_σ as a parameter-dependent system. Conditions for input redundancy, specifically, adaptive input redundancy, have already been established in Chapter 3. In what follows, we exploit these results to derive conditions on the continuous input redundancy of the switched system and to characterize the associated degrees of freedom.

Essential tools from the parameter-dependent context

The core tools required from the parameter-dependent framework are summarized below.

- n_F**-Generalized adaptively weakly unobservable subspace.** Using Algorithm 2.2, we compute $\mathcal{V}^\star_{GA_{n_F}} \subset \mathbb{R}^n$, the largest controlled invariant and output invisible subspace for the parameter-dependent system (3.5), under a polynomial friend $F(\xi)$ deriving from a homogeneous polynomial of a prescribed degree n_F (see Section 3.4 for more details). Since $[\![1, N]\!] \subset [1, N]$, the subspace $\mathcal{V}^\star_{GA_{n_F}}$ also satisfies the invariance and invisibility conditions for the original switched system (4.3).
- n_F**-Generalized adaptively weakly unobservable input subspaces.** The adaptive input redundancy condition for parameter-dependent systems hinges on the dimension of the parameter-dependent subspace

$$\mathcal{N}_{GA_{n_F}}(\xi) = B^{-1}(\xi)\mathcal{V}^\star_{GA_{n_F}} \cap \ker\{D(\xi)\}, \qquad \xi \in [1, N],$$

which contains the input directions that do not influence the outputs. This subspace can be computed efficiently using the procedure of Section 3.5. In the context of switched systems, we analyse its dimension for $\xi \in [\![1, N]\!]$, corresponding to the system modes.

Characterization of the degrees of freedom

Theorem 4.2. *Let $\mathcal{V}^{\star}_{GA_{n_F}} \subset \mathbb{R}^n$ be the n_F-generalized adaptively weakly unobservable subspace of System* (4.5) *and $\mathcal{N}_{GA_{n_F}}$ its n_F-generalized adaptively weakly unobservable input application. The switched linear system* (4.3) *is c-IR uniformly in σ if, for all $\xi \in [\![1, N]\!]$,*

$$\dim\left(\mathcal{N}_{GA_{n_F}}(\xi)\right) > 0. \tag{4.11}$$

Proof. Consider the parameter-dependent system (3.5) with $\theta(t) = \xi \in [1, N]$. Let $x_0 \in \mathbb{R}^n$, $\sigma \in \mathcal{S}$ $F : [1, N] \to \mathbb{R}^{m\times n}$, $L : [1, N] \to \mathbb{R}^{m\times m_w}$ where $m_w \in \mathbb{N}$ and $u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^m)$ such that $u_a - u_b = F(\sigma)(x_{x_0,u_a,\sigma} - x_{x_0,u_b,\sigma}) + L(\sigma)w$ for some $w \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R}^{m_w})$. Since $\sigma(t) \in [\![1, N]\!] \subset [1, N]$, $t \geq 0$, Lemma 3.6 implies that $y_{x_0,u_a,\sigma} = y_{x_0,u_b,\sigma}$ provided that the following hold: (i) $x_{x_0,u_a,\sigma}(0) - x_{x_0,u_b,\sigma}(0) \in \mathcal{V}^{\star}_{GA_{n_F}}$, (ii) F is a friend of $\mathcal{V}^{\star}_{GA_{n_F}}$ and (iii) L is such that $\operatorname{Im}\{L(\xi)\} = \mathcal{N}_{GA_{n_F}}(\xi)$, for all $\xi \in [1, N]$.

Condition (i) is automatic here because the initial conditions coincide and $\mathbf{0} \in \mathcal{V}^{\star}_{GA_{n_F}}$. Assume (ii) and (iii) hold. Then $u_a - u_b \neq \mathbf{0}$ can occur only if there exists a nontrivial time interval on which $L(\sigma(t))$ maps some $w(t)$ to a nonzero vector, i.e. $\operatorname{rank}\{L(\sigma(t))\} > 0$ and $w(t) \notin \ker\{L(\sigma(t))\}$ on a set of positive measure. Because this property must hold for every admissible switching signal $\sigma \in \mathcal{S}$, a necessary condition is $\dim\left(\mathcal{N}_{GA_{n_F}}(\xi)\right) > 0$ holds for all $\xi \in [\![1, N]\!]$. $\square$

Remark 4.1. *For any switched system, the process to obtain a polynomial representation can be applied. Nevertheless, due to the convexification, it may happen that a c-IR switched system does not satisfy condition* (4.11) *related to the convexified polynomial system. Theorem 4.2 is only conclusive when condition* (4.11) *holds. Yet, such a result is important since it enables us to characterize the degrees of freedom of an IR switched system using tools from geometric control theory.*

Remark 4.2. *The generalized subspace $\mathcal{V}^{\star}_{GA_{n_F}}$ used in the result for the characterization of the degrees of freedom represents a common invariant subspace, where switching between modes can be performed without leaving the subspace.*

Computation of matrices F_i: Given the parameter-dependent friend F of $\mathcal{V}^{\star}_{GA_{n_F}}$, we can represent it by the following matrix

$$F_i := F(\xi = i),\ \forall i \in [\![1, N]\!].$$

Computation of matrices L_i: Given the parameter-dependent matrix $L(\xi)$, it is possible that its evaluation at specific points $L(i)$, for $i \in [\![1, N]\!]$, may contain zero or linearly dependent columns. In these cases, we can determine reduced matrices $L_i \in \mathbb{R}^{m\times n_i}$, with possibly fewer columns than $L(i)$, such that

$$\operatorname{Im}\{L_i\} = \operatorname{Im}\{L(\xi = i)\},$$

but without the redundant or null columns. These matrices represent the bases for the images of $L(i)$.

We now show that our framework also recovers conditions for a continuous input redundancy that is *independent of the switching signal σ*. This is formalized below.

Corollary 4.1. *Let $\mathcal{V}_G^\star$ denote the largest generalized controlled invariant and output invisible subspace of System* (4.5)*. The switched linear system* (4.3) *is c-IR* independently of σ *if*

$$\dim\Big(\bigcap_{\xi\in[\![1,N]\!]} B^{-1}(\xi)\,\mathcal{V}_G^\star \cap \ker\{D(\xi)\}\Big) > 0. \tag{4.12}$$

The use of $\mathcal{V}_G^\star$ instead of $\mathcal{V}_{GA_{n_F}}^\star$ enforces the existence of a friend $F\in\mathbb{R}^{m\times n}$ *that does not depend on* σ. This can be ensured by setting $n_F=0$, as in Corollary 3.1, so that the bold matrices required by Algorithm 2.2 can be constructed. Condition (4.12) is then obtained by building a matrix $\bar{L}$ such that

$$\mathrm{Im}\left\{\bar{L}\right\} = \bigcap_{i=1}^{N}\mathrm{Im}\left\{L_i\right\}.$$

If $\dim\left(\mathrm{Im}\left\{\bar{L}\right\}\right)>0$, one can construct input directions that remain independent of the switching law while yielding the same output trajectory. This property is particularly useful when the switching signal is unknown.

Remark 4.3. *It is important to stress that the subspace*

$$\bigcap_{\xi\in[\![1,N]\!]} B^{-1}(\xi)\,\mathcal{V}_G^\star \cap \ker\{D(\xi)\}$$

appearing in Corollary 4.1 is not *the same as $\mathcal{N}_G$, the generalized weakly unobservable input subspace of* (4.5) *(see Definition 3.10). Even more interestingly, one can actually show the strict inclusion*

$$\mathcal{N}_G \subset \bigcap_{\xi\in[\![1,N]\!]} B^{-1}(\xi)\,\mathcal{V}_G^\star \cap \ker\{D(\xi)\}.$$

This distinction is significant: $\mathcal{N}_G$ is defined by intersecting over the entire continuous range $[1,N]$, whereas here the intersection is only over the discrete set $[\![1,N]\!]$. Consequently, the corollary provides a potentially larger and less conservative *subspace that still contains all generalized weakly unobservable input directions.*

4.3.5 Numerical example

In this section, we recall the switched model of the HEV example to show that our developed tools can be applied and to analyze whether we can compute an input redundancy independent of σ.

Guideline Example - HEV. *Consider again the switched model of the HEV* (4.1)*, where the switching signal is the gearbox position.*

Previously, we verified that this system is c-IR uniformly in σ. Now, the objective is to use the developed tools in this section to get information on the degrees of freedom.

First, consider the polynomial parameter-dependent representation of the system given in (4.10)*. Then, let us compute the subspaces $\mathcal{V}_{GA_{n_F}}^\star$ and $\mathcal{N}_{GA_{n_F}}(\xi)$ with the tools developed in Chapter 3. Applying Algorithm 2.2 with a constant friend (i.e., $n_F=0$), we compute:*

$$\mathcal{V}_{GA,0}^\star = \mathcal{V}_G^\star = \mathrm{Im}\left\{\begin{bmatrix}1 & 0\\ 0 & \frac{K}{L_a}\\ 0 & -\frac{1}{\tau}\\ 0 & 0\end{bmatrix}\right\}.$$

Then, by applying the tools developed in Section 3.5 to compute the subspace $B^{-1}(\xi)\mathcal{V}_G^\star \cap \ker\{D(\xi)\} = \operatorname{Im}\{L(\xi)\}$, *we obtain:*

$$L(\xi) = \begin{bmatrix} \Delta_\xi & \frac{K}{L_a} \\ 0 & \frac{1}{\tau}\xi \end{bmatrix}, \Delta_\xi = \begin{cases} 1, & \text{if } \xi = 0, \\ 0, & \text{otherwise}, \end{cases}$$

such that condition (4.11) *of Theorem 4.2 holds.*

Previously in this chapter, we showed that the two distinct inputs $u_a = \mathbf{0}$ *and* u_b *given in* (4.2) *lead to the same output trajectory for a switching law that passes through all modes. Actually, it can be shown that input* u_b *admits an alternative representation:*

$$u_b = F_\sigma x_{\mathbf{0},u_b,\sigma} + L_\sigma w_\sigma, \tag{4.13}$$

where $w_i(t) = 1$,

$$F_i = F,\ L_i = L(i) = \begin{bmatrix} \frac{K}{L_a} \\ \frac{1}{\tau}i \end{bmatrix}, \forall i \in [\![1, 6]\!].$$

This construction ensures that the difference between the two input signals satisfies:

$$u_a(t) - u_b(t) = F_{\sigma(t)}(x_{\mathbf{0},u_a,\sigma}(t) - x_{\mathbf{0},u_b,\sigma}(t)) - L_{\sigma(t)} w_{\sigma(t)}(t).$$

Consequently, the structure of $u_a - u_b$ *is consistent with the structure of an invisible input. Moreover,* w_i *can be chosen arbitrarily, as it represents the intrinsic degrees of freedom of the system. We can also verify that although the output trajectory remains the same, the state trajectories differ, highlighting c-IR of kinds 2 or 3.*

Now, let us analyze if we can obtain c-IR independent of the switching signal. We obtained $B^{-1}(\xi)\mathcal{V}_G^\star \cap \ker\{D(\xi)\}$ *only with parameter-dependent directions, consequently,*

$$\Big(\bigcap_{\xi \in [\![1,N]\!]} B^{-1}(\xi)\, \mathcal{V}_G^\star \cap \ker\{D(\xi)\} \Big) = \{\mathbf{0}\},$$

and System (4.1) *is not c-IR independently of* σ.

4.4 Characterization of switching input redundancy

In this section, we turn to the notion of *switching input redundancy* (see Definitions 4.5–4.7), with the goal of providing a formal characterization. The results presented below are still preliminary and developed within a deliberately restricted framework. Nevertheless, they already represent a significant step towards a systematic understanding of switching input redundancy. The considered framework is outlined next.

4.4.1 Considered framework

We begin by introducing the following assumptions for simplicity.

Assumption 4.4 (Dependence on u and x). *System* Σ_σ *is assumed to not be subject to continuous input* u, *and to be linear in the state* x.

Assumption 4.5 (Admissible sets). *The state* x *and output* y *are assumed to be unconstrained, i.e.,* $\mathcal{X} = \mathbb{R}^n$ *and* $\mathcal{Y} = \mathbb{R}^p$.

Assumption 4.6 (Number of modes)**.** *The switched system is assumed to have only two modes, i.e., $N = 2$.*

Under Assumptions 4.4-4.6, System Σ_σ can be represented for all $t \geq 0$, as follows:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t), \\ y(t) &= C_{\sigma(t)}x(t). \end{aligned} \tag{4.14}$$

where $\forall i \in \{1, 2\}$, matrices A_i and C_i are of appropriate dimension.

4.4.2 Fundamental limitation of switching input redundancy

A significant challenge arises from the linear nature of the modes, specifically concerning the choice of the initial condition. When the initial state is set to the origin ($x_0 = \mathbf{0}$), all admissible switching signals produce trivial state and output trajectories. Since Definition 4.5 relies on the existence of a particular initial condition, one could conclude that any linear autonomous switched system is input redundant with respect to the switching signal. Nevertheless, this highlights a lack of uniformity with respect to the initial condition, where the system may be σ-IR for some initial conditions but not for others.

To systematically investigate input redundancy with respect to the switching signal, we examine sets of initial conditions under which the switched system (4.14) exhibits input redundancy, distinguishing between cases where state trajectories coincide and where they differ.

We now introduce three sets of initial conditions that will play a central role in the following analysis. They are defined solely in terms of state and output trajectories under different switching signals, they will later be used to discuss the various kinds of σ–IR introduced in Definition 4.7.

- **Set $\mathcal{X}_{IR}$ — identical outputs under different switching signals:**

  $$\mathcal{X}_{IR} \coloneqq \Big\{x_0 \in \mathbb{R}^n \;:\; \exists\, \sigma_a, \sigma_b \in \mathcal{S},\ \sigma_a \neq \sigma_b,\ y_{x_0,\sigma_a} = y_{x_0,\sigma_b}\Big\}. \tag{4.15}$$

  This is the set of initial states for which at least two distinct switching signals produce the same output trajectory (every $x_0 \in \mathcal{X}_{IR}$ corresponds to some form of switching input redundancy).

- **Set $\mathcal{X}_{IR=}$ — identical states *and* outputs under different switching signals:**

  $$\mathcal{X}_{IR=} \coloneqq \Big\{x_0 \in \mathbb{R}^n \;:\; \exists\, \sigma_a, \sigma_b \in \mathcal{S},\ \sigma_a \neq \sigma_b,\ x_{x_0,\sigma_a} = x_{x_0,\sigma_b},\ y_{x_0,\sigma_a} = y_{x_0,\sigma_b}\Big\}. \tag{4.16}$$

  This subset collects the initial states for which distinct switching signals generate exactly the same state and output trajectories. It is the part of $\mathcal{X}_{IR}$ where the "first–kind" behavior of Definition 4.7 can occur.

- **Set $\mathcal{X}_{IR\neq}$ — different states but identical outputs under different switching signals:**

  $$\mathcal{X}_{IR\neq} \coloneqq \Big\{x_0 \in \mathbb{R}^n \;:\; \exists\, \sigma_a, \sigma_b \in \mathcal{S},\ \sigma_a \neq \sigma_b,\ x_{x_0,\sigma_a} \neq x_{x_0,\sigma_b},\ y_{x_0,\sigma_a} = y_{x_0,\sigma_b}\Big\}. \tag{4.17}$$

  This is the set of initial states for which non-equivalent switching signals yield different state trajectories but the same output trajectory. It corresponds to the region where the "second–kind" behavior of Definition 4.7 can occur.

Remark 4.4. *These three sets by themselves do* not *characterize the different kinds of σ–IR. A complete characterization of the various kinds of σ–IR will be given later in this chapter.*

The relation between sets defined in (4.15)-(4.17) is given by the following proposition.

Proposition 4.2. *Let us consider the sets $\mathcal{X}_{IR}$, $\mathcal{X}_{IR=}$ and $\mathcal{X}_{IR\neq}$, defined by* (4.15)-(4.17). *The following relation holds:*

$$\mathcal{X}_{IR=} \cup \mathcal{X}_{IR\neq} \quad = \quad \mathcal{X}_{IR}, \tag{4.18}$$

Proof. By the definitions of $\mathcal{X}_{IR}$, $\mathcal{X}_{IR=}$, and $\mathcal{X}_{IR\neq}$, it is clear that $\mathcal{X}_{IR=} \subset \mathcal{X}_{IR}$ and $\mathcal{X}_{IR\neq} \subset \mathcal{X}_{IR}$. We also have by definition that by picking $x_0 \in \mathcal{X}_{IR}$, x_0 is such that $x_0 \in \mathcal{X}_{IR=}$ or $x_0 \in \mathcal{X}_{IR\neq}$. Therefore, (4.18) holds. □

By analyzing such sets, we identify the following relations.

Proposition 4.3. *System* (4.14) *is always σ-IR and never σ-IR of kind 2.*

Proof. Let $x_0 = \mathbf{0}$. Due to the structure of system (4.14), for all $\sigma_a, \sigma_b \in \mathcal{S}$, we obtain $x_{\mathbf{0},\sigma_a} = x_{\mathbf{0},\sigma_b} = \mathbf{0}$ and $y_{\mathbf{0},\sigma_a} = y_{\mathbf{0},\sigma_b} = \mathbf{0}$. The latter relation still holds for the particular case of $\sigma_a \neq \sigma_b$. Consequently, $x_0 = \mathbf{0} \in \mathcal{X}_{IR}$, such that system (4.14) always exhibit $x_0 \in \mathbb{R}^n$ and $\sigma_a, \sigma_b \in \mathcal{S}$, with $\sigma_a \neq \sigma_b$, leading to $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$. Moreover, since there is always this zero initial condition leading to identical state and output trajectories for any switching signal, it cannot be σ-IR of kind 2. □

From Proposition 4.3, the switched system (4.14) is σ-IR of kind 1 or 3. The following theorem allows to determine its kind.

Proposition 4.4. *The following holds:*

- $\mathcal{X}_{IR\neq} = \emptyset \Leftrightarrow$ *System* (4.14) *is σ-IR of kind 1.*
- $\mathcal{X}_{IR\neq} \neq \emptyset \Leftrightarrow$ *System* (4.14) *is σ-IR of kind 3.*
- $\mathcal{X}_{IR}/\mathcal{X}_{IR=} \neq \emptyset \Rightarrow$ *System* (4.14) *is σ-IR of kind 3.*

Proof. By virtue of Proposition 4.2, we know that System (4.14) is either σ-IR of kind 1 or 3. Let us proceed point by point.

- A system that is σ-IR of kind 1 is characterized by (i) the existence of an initial condition for which every switching signal that leads to the same output trajectory, also produces the same state trajectories, and (ii) the absence of an initial condition for which one can select two different switching laws producing the same output trajectory but leading to different states. Therefore $\mathcal{X}_{IR\neq} = \emptyset$.
- By similar reasoning, one can easily conclude that $\mathcal{X}_{IR\neq} \neq \emptyset$ is equivalent to having a system that is σ-IR of kind 3.
- For the last point, when $\mathcal{X}_{IR} \setminus \mathcal{X}_{IR=} \neq \emptyset$, there exists at least an initial condition $x_0 \in \mathcal{X}_{IR}$ for which the different switching laws leading to the same output, leads to different state trajectories since it does not belong to $\mathcal{X}_{IR=}$. Then $x_0 \in \mathcal{X}_{IR\neq}$, which leads to switching input redundancy of kind 3 by using the second point.

□

Unfortunately, the conditions stated in Proposition 4.4 are not directly tractable, as they rely on the set $\mathcal{X}_{IR\neq}$, for which no explicit construction is currently available. In addition, the knowledge of all initial conditions of kinds 1 and 3 would provide insights of the degrees of freedom of system (4.14). Indeed, such initial conditions allows to know when we can compute distinct switching signals leading to the same output. The remainder of this section is therefore devoted to developing *tractable characterizations* of these sets, enabling a systematic procedure to decide whether System (4.14) is σ-IR of kind 1 or kind 3.

4.4.3 Computation of the sets $\mathcal{X}_{IR}$ and $\mathcal{X}_{IR=}$

Our method for characterizing $\mathcal{X}_{IR}$ and $\mathcal{X}_{IR=}$ relies on constructing extended LTI systems and analyzing, for these systems, the sets of initial conditions that produce identical output trajectories. This approach is inspired by the characterization of left-invertibility proposed in [44], a property closely related to input redundancy (see [49] for further details).

Computation of set $\mathcal{X}_{IR=}$

The main difficulty in characterizing the set $\mathcal{X}_{IR=}$ lies in the time-varying nature of the switching signals σ_a and σ_b. To overcome this, we exploit the defining property of $\mathcal{X}_{IR=}$: the state trajectories generated by σ_a and σ_b coincide at every instant. This observation enables us to reformulate the problem within the framework of LTI systems and to analyze whether their resulting output trajectories remain identically zero.

In particular, we propose to investigate four extended LTI systems, each gathering the dynamics of the two trajectories corresponding to σ_a and σ_b but initialized with the same initial state. For all $i, j \in \{1, 2\}$ and for all $t \geq 0$, these systems are defined as:

$$\Sigma_{ij=}: \quad \begin{cases} \dot{z}(t) = \mathbf{A}_{ij} z(t), \\ y_{ij=}(t) = \mathbf{C}_{ij=} z(t), \end{cases} \tag{4.19}$$

where $z(t) \in \mathbb{R}^{2n}$ is the augmented state, $y_{ij=}(t) \in \mathbb{R}^{p+n}$ is the augmented output, and

$$\mathbf{A}_{ij} = \begin{bmatrix} A_i & \mathbf{0} \\ \mathbf{0} & A_j \end{bmatrix}, \qquad \mathbf{C}_{ij=} = \begin{bmatrix} C_i & -C_j \\ \mathrm{I}_n & -\mathrm{I}_n \end{bmatrix}. \tag{4.20}$$

For any initial condition $z_0 \in \mathbb{R}^{2n}$ chosen as

$$z_0 = \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} x_0, \qquad x_0 \in \mathbb{R}^n,$$

the augmented state vector z captures the dynamics of modes i and j of the switched system (4.14). Moreover, the corresponding output can be written as

$$y_{ij=,z_0} = \begin{bmatrix} y_{i,x_0} - y_{j,x_0} \\ x_{i,x_0} - x_{j,x_0} \end{bmatrix},$$

where $x_{i,x_0}(t)$ (resp. $y_{i,x_0}(t)$) denotes the state trajectory (resp. the output trajectory) of the i-th mode of (4.14), starting from the initial condition x_0.

Therefore, our goal is to determine the conditions on z_0 such that $y_{ij=,z_0}$ remains identically zero, so that we conclude that $y_{i,x_0} = y_{j,x_0}$ and $x_{i,x_0} = x_{j,x_0}$, or saying differently, modes i and j have identical state and output trajectories. It is well known that the set of initial conditions z_0 for which the output $y_{ij=,z_0}$ remains null is the $(\mathbf{A}_{ij}, \mathbf{C}_{ij=})$-*unobservable subspace* that we recall the definition here.

Definition 4.9. *Given two matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, the (A, C)-unobservable subspace $\mathcal{N} \subset \mathbb{R}^n$ is the largest A-invariant subspace contained in* $\ker\{C\}$, *i.e.,*

$$\mathcal{N} := \bigcap_{i=0}^{n-1} \ker\left\{CA^i\right\}.$$

This leads to the following characterization of $\mathcal{X}_{IR=}$.

Lemma 4.2. *Consider the switched system Σ_σ given by (4.14), and let $\mathcal{N}_{12=}$ denote the $(\mathbf{A}_{12}, \mathbf{C}_{12=})$-unobservable subspace. Then:*

$$\mathcal{X}_{IR=} = \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \mathcal{N}_{12=}. \tag{4.21}$$

Proof. **(1)** $\left(\mathcal{X}_{IR=} \subset \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \mathcal{N}_{12=}\right)$**.** Take any $x_0 \in \mathcal{X}_{IR=}$. By definition, there exist switching signals $\sigma_a \neq \sigma_b$ such that $x_{x_0,\sigma_a} = x_{x_0,\sigma_b}$, $y_{x_0,\sigma_a}(t) = y_{x_0,\sigma_b}(t), \quad \forall t \geq 0$.

Let $\{t_k\}_{k\in\mathbb{N}}$ denote the ordered switching instants of σ_a and σ_b. Since $\sigma_a \neq \sigma_b$, there exists at least one interval $[t_k, t_{k+1})$ where $\sigma_a(t) = i \neq j = \sigma_b(t)$ with $i, j \in \{1, 2\}$. Over this interval, the augmented output satisfies

$$y_{12=}(t) = \mathbf{C}_{12=} e^{\mathbf{A}_{12}(t-t_k)} \begin{bmatrix} x(t_k) \\ x(t_k) \end{bmatrix} = \mathbf{0}, \quad \forall t \in [t_k, t_{k+1}).$$

By successive differentiation with respect to time, we obtain

$$\mathbf{C}_{12=}\mathbf{A}_{12}^q \begin{bmatrix} x(t_k) \\ x(t_k) \end{bmatrix} = \mathbf{0}, \quad \forall q \in \mathbb{N},$$

which implies

$$\begin{bmatrix} x(t_k) \\ x(t_k) \end{bmatrix} \in \mathcal{N}_{12=}.$$

Now we prove recursively that the initial state belongs to $\mathcal{N}_{12=}$. Suppose $t_k > 0$, and consider the previous switching interval $[t_{k-1}, t_k)$ with $\sigma_a(t) = i$, $\sigma_b(t) = j$. Then

$$\begin{bmatrix} x(t_{k-1}) \\ x(t_{k-1}) \end{bmatrix} = e^{-\mathbf{A}_{ij}(t_k - t_{k-1})} \begin{bmatrix} x(t_k) \\ x(t_k) \end{bmatrix}.$$

- If $i \neq j$, by the $\mathbf{A}_{12}$-invariance of $\mathcal{N}_{12-1}$, it follows that

$$\begin{bmatrix} x(t_{k-1}) \\ x(t_{k-1}) \end{bmatrix} \in \mathcal{N}_{12=}.$$

- If $i = j$, then $x(t_{k-1}) = e^{-A_i(t_k - t_{k-1})}x(t_k)$, and using the structure of $\mathbf{A}_{ii}$ and the invariance of $\mathcal{N}_{12=}$, we again obtain
$$\begin{bmatrix} x(t_{k-1}) \\ x(t_{k-1}) \end{bmatrix} \in \mathcal{N}_{12=}.$$

Repeating this argument recursively down to $t_0 = 0$, we get
$$\begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \in \mathcal{N}_{12=},$$
so that $x_0 \in \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \mathcal{N}_{12=}$.

(2) ($\begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \mathcal{N}_{12=} \subset \mathcal{X}_{IR=}$). Take any $x_0 \in \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \mathcal{N}_{12=}$. Define the augmented initial state
$$z_0 = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \in \mathcal{N}_{12=}.$$
By the definition of the unobservable subspace, the output $y_{12=,z_0}$ of the augmented system is identically zero. Hence, for $\sigma_a = 1$ and $\sigma_b = 2$, we have $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$ and $x_{x_0,\sigma_a}(t) = x_{x_0,\sigma_b}(t)$, so that $x_0 \in \mathcal{X}_{IR=}$.

Combining (**1**) and (**2**) gives the desired equality. □

Lemma 4.2 allows to obtain a tractable characterization of $\mathcal{X}_{IR=}$ since it is well-known that the unobservable subspace of an LTI system can be computed by an algorithm.

Computation of set $\mathcal{X}_{IR}$

When analyzing the set $\mathcal{X}_{IR}$ defined in (4.15), a crucial difference with respect to $\mathcal{X}_{IR=}$ is that the state trajectories are *not* required to coincide. They may evolve differently, including at the switching instants. Consequently, the approach developed in the previous section for characterizing $\mathcal{X}_{IR=}$ cannot be applied directly. To overcome this, we introduce the following *subset* of $\mathcal{X}_{IR}$, which will enable us to derive tractable conditions for switching input redundancy:
$$\hat{\mathcal{X}}_{IR} := \{x_0 \in \mathbb{R}^n \ : \ y_{x_0,1} = y_{x_0,2}\}. \tag{4.22}$$

To characterize $\hat{\mathcal{X}}_{IR}$, we rely on the following four LTI systems, for $i, j \in \{1, 2\}$, and for all $t \geq 0$,
$$\Sigma_{ij} \begin{cases} \dot{z}(t) = \mathbf{A}_{ij} z(t), \\ y_{ij}(t) = \mathbf{C}_{ij} z(t), \end{cases} \tag{4.23}$$
where $z(t) \in \mathbb{R}^{2n}$, $y_{ij}(t) \in \mathbb{R}^p$, $\mathbf{A}_{ij}$ is defined in (4.20) and
$$\mathbf{C}_{ij} = \begin{bmatrix} C_i & -C_j \end{bmatrix}. \tag{4.24}$$
We denote $\mathcal{N}_{ij}$ as the $(\mathbf{A}_{ij}, \mathbf{C}_{ij})$-unobservable subspace. Note that $\mathbf{C}_{ij}$ no longer contains $\begin{bmatrix} \mathrm{I}_n & -\mathrm{I}_n \end{bmatrix}$. Consequently,
$$\mathcal{N}_{ij} \not\subset \operatorname{Im} \left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\}.$$

Lemma 4.3. *Consider the switched system Σ_σ given by* (4.14)*, and let $\mathcal{N}_{12}$ denote the $(\mathbf{A}_{12}, \mathbf{C}_{12})$-unobservable subspace. Then*

$$\hat{\mathcal{X}}_{IR} = \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \left(\mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} \right). \tag{4.25}$$

Proof. **(1)** **(**$\hat{\mathcal{X}}_{IR} \subset \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \left(\mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\} \right)$**).** Let $x_0 \in \hat{\mathcal{X}}_{IR}$ and define

$$z_0 = \begin{bmatrix} x_0^\top & x_0^\top \end{bmatrix}^\top \in \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\}.$$

By definition of $\hat{\mathcal{X}}_{IR}$ we have $y_{x_0,1} = y_{x_0,2}$, which implies $y_{12,z_0} = \mathbf{0}$. From Definition 4.9 it follows that $z_0 \in \mathcal{N}_{12}$. Hence, $z_0 \in \mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\}$, and thus

$$x_0 \in \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \left(\mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\} \right).$$

(2) **(**$\begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \left(\mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\} \right) \subset \hat{\mathcal{X}}_{IR}$**).** Let

$$x_0 \in \begin{bmatrix} \mathrm{I}_n & \mathbf{0}_{n\times n} \end{bmatrix} \left(\mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\} \right),$$

and $z_0 = \begin{bmatrix} x_0^\top & x_0^\top \end{bmatrix}^\top$. Obviously, $z_0 \in \mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n & \mathrm{I}_n \end{bmatrix}^\top \right\}$, so by Definition 4.9 the output of the augmented system satisfies $y_{12,z_0} = \mathbf{0}$. Thus $y_{x_0,1} = y_{x_0,2}$ and therefore $x_0 \in \hat{\mathcal{X}}_{IR}$.

Combining (**1**) and (**2**) gives the desired equality. □

Similarly as for $\mathcal{X}_{IR=}$, the fact that $\hat{\mathcal{X}}_{IR}$ is characterized by the mean of unobservable subspaces of LTI systems, $\hat{\mathcal{X}}_{IR}$ can be systematically computed.

Due to the structure of $\Sigma_{12=}$ and Σ_{12}, we establish the following result.

Lemma 4.4. *For the switching system* (4.14)*, consider the sets $\mathcal{X}_{IR=}$ and $\hat{\mathcal{X}}_{IR}$ given by* (4.16) *and* (4.22)*, respectively. Then:*

$$\mathcal{X}_{IR=} \subset \hat{\mathcal{X}}_{IR}. \tag{4.26}$$

Proof. Since both systems Σ_{12} and $\Sigma_{12=}$ share the same state matrix $\mathbf{A}_{12}$ and since matrix $\mathbf{C}_{12}$ is one extraction of the rows of $\mathbf{C}_{12=}$, namely $\mathbf{C}_{12} = \begin{bmatrix} \mathrm{I}_p & \mathbf{0}_{p\times n} \end{bmatrix} \mathbf{C}_{12=}$, we have that

$$\mathcal{N}_{12=} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} \subset \mathcal{N}_{12} \cap \mathrm{Im}\left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\}.$$

Projecting the n-first components ends the proof. □

Finally, we derive the last technical result related to the unobservable subspaces.

Lemma 4.5. *For the switched system* (4.14)*, consider the set $\mathcal{X}_{IR\neq}$ and let $\mathcal{N}_{12=}$ and $\mathcal{N}_{12}$ denote the $(\mathbf{A}_{12}, \mathbf{C}_{12=})$- and $(\mathbf{A}_{12}, \mathbf{C}_{12})$-unobservable subspaces, respectively. The following statements are equivalent:*

(i) $\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=} \neq \emptyset$;

(ii) $\dim \left(\mathcal{N}_{12} \cap \operatorname{Im} \left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} /\mathcal{N}_{12=} \right) > 0$;

(iii) $\mathcal{X}_{IR\neq} \neq \emptyset$.

Proof. Equivalence between the first two points derive from the equalities obtained in Lemmas 4.2 and 4.3. For the second and third point, let us proceed by implications.
((ii)⇒(iii)) Assume $\dim \left(\mathcal{N}_{12} \cap \operatorname{Im} \left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} /\mathcal{N}_{12=} \right) > 0$. Then, we can take $x_0 \in \mathcal{N}_{12} \cap \operatorname{Im} \left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\}$ such that $x_0 \notin \mathcal{N}_{12=}$. Therefore, by applying $\sigma_a = 1$ and $\sigma_b = 2$. It is clear from the augmented systems (4.19), (4.23) that $y_{x_0,1} = y_{x_0,2}$ and $x_{x_0,1} \neq x_{x_0,2}$. This implies (iii).
((iii)⇒(ii)) Assume $\mathcal{X}_{IR\neq} \neq \emptyset$. Then there exist $x_0 \in \mathbb{R}^n$ and two switching signals $\sigma_a \neq \sigma_b$ such that $x_{x_0,\sigma_a} \neq x_{x_0,\sigma_b}$ and $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$. Let $\{t_k\}_{k\in\mathbb{N}}$ be the time-ordered sequence of switching instants of σ_a and σ_b. We can prove that there exists $k_0 \in \mathbb{N}$ and $t' \in [t_{k_0}, t_{k_0+1})$ such that

$$x_{x_0,\sigma_a}(t_{k_0}) = x_{x_0,\sigma_b}(t_{k_0}) = x_{t_{k_0}}, \quad x_{x_0,\sigma_a}(t') \neq x_{x_0,\sigma_b}(t').$$

This can be shown by contradiction: if such a k_0 does not exist, the state trajectories coincide on $\mathbb{R}^+$, which contradicts $x_0 \in \mathcal{X}_{IR\neq}$. On this time interval $[t_{k_0}, t_{k_0+1})$, $\sigma_a(t) = i \neq j = \sigma_b(t)$, $x_{x_0,\sigma_a}(t) = x_{x_0,\sigma_b}(t)$ and still $x_{x_0,\sigma_a}(t) = x_{x_0,\sigma_b}(t)$. Thus,

$$\begin{bmatrix} x_{t_{k_0}} \\ x_{t_{k_0}} \end{bmatrix} \in \mathcal{N}_{12} \setminus \mathcal{N}_{12=}.$$

□

4.4.4 Characterization of switching input redundancy

Building on the previous characterizations of the sets $\mathcal{X}_{IR=}$ and $\hat{\mathcal{X}}_{IR}$, we now move a step closer to a tractable description of switching input redundancy. Before stating our main result, we present the following lemma, which plays a central role. Its significance is worth emphasizing: even though we restrict our analysis to the subset $\hat{\mathcal{X}}_{IR} \subset \mathcal{X}_{IR}$, the structure uncovered by this lemma reveals that we can still recover an alternative characterization of switching input redundancy equivalent to that of Proposition 4.4. In other words, despite relying on a reduced set of initial conditions, we will still be able to recover equivalent relations for our main result. This elegant and somewhat unexpected property is formalized in the following lemma.

Lemma 4.6. *For the switching system* (4.14)*, consider the sets $\mathcal{X}_{IR}$, $\mathcal{X}_{IR=}$, $\mathcal{X}_{IR\neq}$, and $\hat{\mathcal{X}}_{IR}$ defined in* (4.15), (4.16), (4.17), *and* (4.22)*, respectively. The following statements are equivalent:*

(i) $\mathcal{X}_{IR}/\mathcal{X}_{IR=} \neq \emptyset$;

(ii) $\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=} \neq \emptyset$.

Proof. ((i)⇒(ii)) Assume that $\mathcal{X}_{IR}/\mathcal{X}_{IR=} \neq \emptyset$. Then there exist $x_0 \in \mathbb{R}^n$ and two switching signals $\sigma_a \neq \sigma_b$ such that $x_{x_0,\sigma_a} \neq x_{x_0,\sigma_b}$ and $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$. Let $\{t_k\}_{k\in\mathbb{N}}$ be the time-ordered sequence of switching instants of σ_a and σ_b. We can prove that there exists $k_0 \in \mathbb{N}$ and $t^{'} \in [t_{k_0}, t_{k_0+1})$ such that

$$x_{x_0,\sigma_a}(t_{k_0}) = x_{x_0,\sigma_b}(t_{k_0}) = x_{t_{k_0}}, \quad x_{x_0,\sigma_a}(t') \neq x_{x_0,\sigma_b}(t').$$

This can be shown by contradiction: if such a k_0 does not exist, the state trajectories coincide on $\mathbb{R}^+$, which contradicts $x_0 \in \mathcal{X}_{IR\neq}$. On this time interval $[t_{k_0}, t_{k_0+1})$, $\sigma_a(t) = i \neq j = \sigma_b(t)$, $x_{x_0,\sigma_a}(t) = x_{x_0,\sigma_b}(t)$ and still $x_{x_0,\sigma_a}(t) = x_{x_0,\sigma_b}(t)$. Thus,

$$\begin{bmatrix} x_{t_{k_0}} \\ x_{t_{k_0}} \end{bmatrix} \in \mathcal{N}_{12} \setminus \mathcal{N}_{12=}.$$

Consequently, setting an initial state $\tilde{x}_0 = x_{t_{k_0}}$, we have $x_{\tilde{x}_0,1} \neq x_{\tilde{x}_0,2}$ and $y_{\tilde{x}_0,1} = y_{\tilde{x}_0,2}$, which implies that $\tilde{x}_0 \in \mathcal{X}_{IR\neq}$ and in turns, $\mathcal{X}_{IR\neq} \neq \emptyset$. By virtue of Lemma 4.5 this is equivalent to $\hat{\mathcal{X}}_{IR} \setminus \mathcal{X}_{IR=} \neq \emptyset$, which implies (ii).

((ii)⇒(i)) Assume that $\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=} \neq \emptyset$. Then we can select $x_0 \in \hat{\mathcal{X}}_{IR} \setminus \mathcal{X}_{IR=}$. Such an initial condition is such that $x_{x_0,1} \neq x_{x_0,2}$ while $y_{x_0,1} = y_{x_0,2}$. Since the two outputs coincide, it follows that $x_0 \in \mathcal{X}_{IR}$. Since $x_0 \notin \mathcal{X}_{IR=}$, this implies $\mathcal{X}_{IR}/\mathcal{X}_{IR\neq} \neq \emptyset$.

□

This sets the stage for the theorem that follows, which reveals the full strength of our characterization.

Theorem 4.3. *The following holds:*

- $\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=} = \emptyset \Leftrightarrow$ *System* (4.14) *is σ-IR of kind 1.*
- $\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=} \neq \emptyset \Leftrightarrow$ *System* (4.14) *is σ-IR of kind 3.*

Proof. From Lemma 4.6, we know that $\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=} = \emptyset$ (resp. $\neq \emptyset$) is equivalent to $\mathcal{X}_{IR}/\mathcal{X}_{IR=} = \emptyset$ (resp. $\neq \emptyset$). Combining this with Proposition 4.4 ends the proof. □

4.4.5 Constructive design of the switching laws

The result based on the unobservable subspace of the augmented LTI system does not only provides a characterization of switching input redundancy of kinds 1 and 3 in terms of the initial conditions, but also offers a constructive framework for designing distinct switching laws that produce the same output trajectory. This is the focus of the present section.

Proposition 4.5. *Consider the switching system* (4.14) *with the associated extended matrices given in* (4.20) *and* (4.24)*. For each $i, j \in \{1, 2\}$, let $\mathcal{N}_{ij}$ (resp. $\mathcal{N}_{ij=}$) denote the $(\mathbf{A}_{ij}, \mathbf{C}_{ij})$-unobservable (resp. $(\mathbf{A}_{ij}, \mathbf{C}_{ij=})$-unobservable) subspace. Take any $x_0 \in \mathcal{X}_{IR}$, where $\mathcal{X}_{IR}$ is defined in* (4.15)*, and let $\sigma_a \neq \sigma_b$ be two switching signals such that $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$. Let $\{t_k\}_{k\in\mathbb{N}}$ be the time-ordered sequence of switching instants of σ_a and σ_b. Then, the following property holds:*

$$\forall k \in \mathbb{N},\ \exists\, i, j \in \{1, 2\} : \begin{bmatrix} x_{x_0,\sigma_a}(t_k) \\ x_{x_0,\sigma_b}(t_k) \end{bmatrix} \in \mathcal{N}_{ij}.$$

Moreover, when (4.14) *is* σ*-IR of kind 3 and* $x_0 \in \mathcal{X}_{IR\neq}$*, where* $\mathcal{X}_{IR\neq}$ *is given by* (4.17)*, we have:*

$$\exists k \in \mathbb{N} : \begin{bmatrix} x_{x_0,\sigma_a}(t_k) \\ x_{x_0,\sigma_b}(t_k) \end{bmatrix} \in \mathcal{N}_{12} \cap \operatorname{Im}\left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} \setminus \mathcal{N}_{12=}$$

Proof. **First property:** Pick any $x_0 \in \mathcal{X}_{IR}$ and any $k \in \mathbb{N}$. Since $\sigma_a, \sigma_b \in \mathcal{S}$, for all $t \in [t_k, t_{k+1})$, there exist $i, j \in \{1, 2\}$ such that $\sigma_a(t) = i$ and $\sigma_b(t) = j$. Because $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$, we have $y_{x_0,\sigma_a}(t) - y_{x_0,\sigma_b}(t) = 0$ for all $t \in [t_k, t_{k+1})$. This implies that

$$\begin{bmatrix} x_{x_0,\sigma_a}(t_k) \\ x_{x_0,\sigma_b}(t_k) \end{bmatrix} \in \mathcal{N}_{ij}.$$

Second property: Assume that System (4.14) is σ-IR of kind 3 and $x_0 \in \mathcal{X}_{IR\neq}$. Then σ_a, σ_b are such that $x_{x_0,\sigma_a} \neq x_{x_0,\sigma_b}$ and $y_{x_0,\sigma_a} = y_{x_0,\sigma_b}$. Since $x_{x_0,\sigma_a}(0) = x_{x_0,\sigma_b}(0)$, there exists $k_0 \in \mathbb{N}$ and $t' \in [t_{k_0}, t_{k_0+1})$ such that

$$x_{x_0,\sigma_a}(t_{k_0}) = x_{x_0,\sigma_b}(t_{k_0}) \quad \text{and} \quad x_{x_0,\sigma_a}(t') \neq x_{x_0,\sigma_b}(t').$$

On this interval, $\sigma_a(t) = i \neq j = \sigma_b(t)$ for all $t \in [t_{k_0}, t_{k_0+1})$. The fact that the two state trajectories differ at t' implies

$$\begin{bmatrix} x_{x_0,\sigma_a}(t_{k_0}) \\ x_{x_0,\sigma_b}(t_{k_0}) \end{bmatrix} \in \mathcal{N}_{12} \cap \operatorname{Im}\left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} \setminus \mathcal{N}_{12=}.$$

This concludes the proof. □

The significance of this proposition lies in its *sequential* interpretation. At each time t, we can examine which unobservable subspace $\mathcal{N}_{ij}$ contains the augmented state $\begin{bmatrix} x_{x_0,\sigma_a}(t) \\ x_{x_0,\sigma_b}(t) \end{bmatrix}$. Initially, the two trajectories may start from the same mode. Then, as the system evolves, whenever the augmented state reaches a particular subspace $\mathcal{N}_{ij}$, we are able to select two distinct modes for the next interval that will still produce the same output. By iterating this procedure at each time, one can construct switching laws that yield identical output trajectories despite differing state evolutions. This idea will be illustrated in more detail in the next section through a concrete example.

4.4.6 Numerical example

Unfortunately, at this stage we are not yet able to provide a characterization of switching input redundancy for the HEV. We therefore turn to the following academic, yet highly illustrative example.

Example 4.3. *Consider a switched linear system as* (4.14) *described by the following matrices:*

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We want to check if this system is switching input redundant of kinds 1 or 3. Then, from Theorem 4.3, we have to compute the subspaces $\hat{\mathcal{X}}_{IR}$ and $\mathcal{X}_{IR=}$. We obtain:

$$\mathcal{N}_{12} = \mathrm{Im}\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\},\ \mathcal{N}_{12=} = \{\mathbf{0}\},\ \mathcal{N}_{12}\cap\mathrm{Im}\left\{\begin{bmatrix}\mathrm{I}_n\\ \mathrm{I}_n\end{bmatrix}\right\} = \mathrm{Im}\left\{\mathbf{1}_4\right\},$$

such that

$$\hat{\mathcal{X}}_{IR} = \mathrm{Im}\left\{\mathbf{1}_2\right\},\ \mathcal{X}_{IR=} = \{\mathbf{0}\}.$$

Since $\dim(\hat{\mathcal{X}}_{IR}/\mathcal{X}_{IR=}) = 1 > 0$, *the system is σ-IR of kind 3.*

Analysis on the set $\mathcal{X}_{IR}$: *For this specific example, we are interested in finding the set of x_0 for which there exists a switching law σ that drives the state trajectory to the set $\hat{\mathcal{X}}_{IR}\setminus\mathcal{X}_{IR=}$ in finite time T, i.e., $x_{x_0,\sigma}(T)\in\hat{\mathcal{X}}_{IR}\setminus\mathcal{X}_{IR=}$. Since all the sets are invariant under mode 2, our approach is to use the invariant sets associated with mode 1 to characterize the set of initial conditions that can be steered to $\hat{\mathcal{X}}_{IR}$. By showing that these sets are complementary, i.e., they cover all the state space $\mathbb{R}^2$.*

Let us define the Jordan chain:

$$(v_1, v_2) = \left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right),$$

associated to the eigenvalue $\lambda = -1$ of A_1. Since v_1 and v_2 are orthogonal, they form a basis for $\mathbb{R}^2$, which allow us to express any initial condition uniquely as $x_0 = \alpha_1 v_1 + \alpha_2 v_2$, with $\alpha_1 = \frac{x_0^\top v_1}{|v_1|^2} = x_0^\top v_1$ and $\alpha_2 = \frac{x_0^\top v_2}{|v_2|^2} = x_0^\top v_2$. Moreover, the state response under mode 1 is given by

$$x(t) = e^{A_1 t}x_0 = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}e^{-t}x_0.$$

From this, we compute $v_2^\top x(t) = e^{-t}\alpha_2$, which does not change its sign. This implies that the invariant sets under mode 1 are the sets $\{x \mid v_2^\top x \le 0\}$ and $\{x \mid v_2^\top x \ge 0\}$.

Now that we have found the invariant sets, let us consider a basis $v^\perp = \begin{bmatrix}1\\-1\end{bmatrix}$ which spans the orthogonal complement of $\hat{\mathcal{X}}_{IR}$. To determine whether a given initial condition can be steered to $\hat{\mathcal{X}}_{IR}$, we analyze the behavior of the function

$$s(t) = (v^\perp)^\top x(t) = (v^\perp)^\top e^{A_1 t}x_0 = e^{-t}(\alpha_1 - \alpha_2 + t\alpha_2),$$

in Table 4.1. Thus, in cases 3., 4., and 5., there exists a finite t such that $x(t)\in\hat{\mathcal{X}}_{IR}$. From these analyses, we conclude that the set of initial conditions $\mathbb{R}^n\setminus\mathcal{X}_{IR}$ that cannot be steered to $\hat{\mathcal{X}}_{IR}$ is:

$$\mathbb{R}^n\setminus\mathcal{X}_{IR} = \left\{x_0 \mid v_2^T x_0 \ge 0 \wedge (v^\perp)^T x_0 > 0\right\}\cup\left\{x_0 \mid v_2^T x_0 \le 0 \wedge (v^\perp)^T x_0 < 0\right\}$$

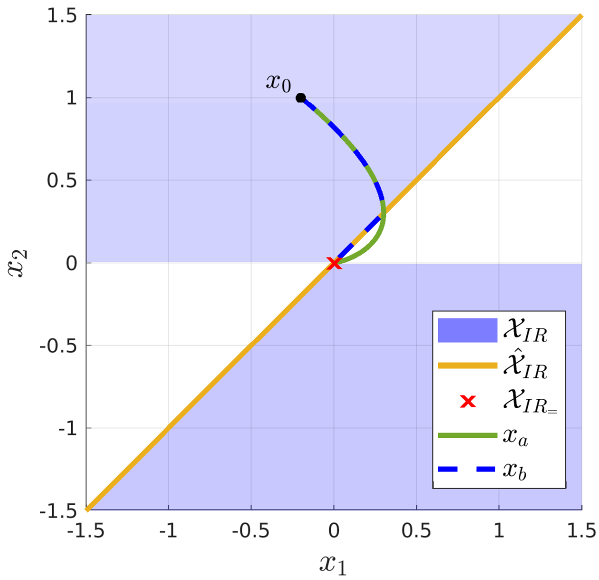

Figure 4.6: Sets $\mathcal{X}_{IR}$, $\hat{\mathcal{X}}_{IR}$, and $\mathcal{X}_{IR=}$, and phase portrait.

while the set $\mathcal{X}_{IR}$ of x_0 that can be steered to $\hat{\mathcal{X}}_{IR}$ is given by:

$$\mathcal{X}_{IR} = \left\{x_0 \mid v_2^T x_0 > 0 \wedge (v^{\perp})^T x_0 \leq 0\right\} \cup \left\{x_0 \mid v_2^T x_0 < 0 \wedge (v^{\perp})^T x_0 \geq 0\right\} \cup \{x_0 = \mathbf{0}\}.$$

As seen in Figure 4.6, they cover $\mathbb{R}^2$.

Cases	$s(t) = (v^{\perp})^\top x(t)$	$\hat{\mathcal{X}}_{IR}$
1.: $0 < \alpha_2 < \alpha_1$	$s(t) > 0, \forall t \geq 0$	$x(t) \notin \hat{\mathcal{X}}_{IR}, \forall t \geq 0$
2.: $\alpha_1 < \alpha_2 < 0$	$s(t) < 0, \forall t \geq 0$	$x(t) \notin \hat{\mathcal{X}}_{IR}, \forall t \geq 0$
3.: $(\alpha_1 < \alpha_2) \wedge \alpha_2 > 0$	$s(1 - \alpha_1/\alpha_2) = 0$	$x(1 - \alpha_1/\alpha_2) \in \hat{\mathcal{X}}_{IR}$
4.: $(\alpha_2 < \alpha_1) \wedge \alpha_2 < 0$	$s(1 - \lvert\alpha_1\rvert/\lvert\alpha_2\rvert) = 0$	$x(1 - \lvert\alpha_1\rvert/\lvert\alpha_2\rvert) \in \hat{\mathcal{X}}_{IR}$
5.: $\alpha_1 = \alpha_2 = 0$	$s(t) = 0, \forall t \geq 0$	$x(t) \in \hat{\mathcal{X}}_{IR}, \forall t \geq 0$

Table 4.1: Behavior of $s(t)$.

Let us select $x_0 = \begin{bmatrix} -0.2 & 1 \end{bmatrix}^\top \in \mathcal{X}_{IR\neq}$ to verify the output and state dynamics in response to distinct switching signals. We have that $x_0 \notin \hat{\mathcal{X}}_{IR} \setminus \mathcal{X}_{IR=}$, however, $x_0 = -0.2v_1 + v_2$ is included in case 3. in Table 4.1, such that $s(t_1) = 0$ for $t_1 = 1.2s$, which means that for a switching signal $\sigma(t) = 1$ for $t \in [0, t_1)$ we will obtain $x_{x_0,\sigma}(t_1) \in \hat{\mathcal{X}}_{IR} \setminus \mathcal{X}_{IR-1}$. Then, we selected two switching signals:

$$\sigma_a(t) = 1, \forall t \geq 0, \sigma_b(t) = \begin{cases} 1, t \in [0, t_1) \\ 2, t \in [t_1, +\infty), \end{cases} \tag{4.27}$$

which are depicted in Figure 4.7. Since $\sigma_a(t) = \sigma_b(t)$ for $t \in [0, t_1)$, of course, the state and output trajectories remain the same for the two switching signals. We can also verify that at time $t = t_1$, *the two components of the state became equal, confirming that* $x_{x_0,\sigma_a}(t_1) = x_{x_0,\sigma_b}(t_1) \in \hat{\mathcal{X}}_{IR} \backslash \mathcal{X}_{IR=}$. *Consequently, at this specific time,* σ_b *switches to mode 2 and produces different state trajectories but with the same output trajectory, as seen in Figure 4.7.*

Construction of two non-equivalent switching signals leading to the same output trajectory: *We now illustrate how to construct non-equivalent switching signals leading to the same output trajectory, using the unobservable subspaces* $\mathcal{N}_{ij}$ *and* $\mathcal{N}_{ij=}$ *associated with the augmented LTI systems* (4.23) *and* (4.19), *respectively.*

Consider the same initial condition $x_0 = \begin{bmatrix} -0.2 & 1 \end{bmatrix}^\top$ *and the same switching signals as in* (4.27) *for* $t \in [0, 1.2)$. *Figure 4.8 shows five binary indicator curves. The first four curves signals whether the concatenated state vector* $\begin{bmatrix} x_{x_0,\sigma_a}(t) \\ x_{x_0,\sigma_b}(t) \end{bmatrix}$ *lies inside the unobservable subspaces of the augmented systems* (4.23). *A value of 1 indicates inclusion in the subspace, and 0 otherwise. The final curve signals whether the vector belongs to the set*

$$\mathcal{N}_{12} \cap \operatorname{Im}\left\{ \begin{bmatrix} \mathrm{I}_n \\ \mathrm{I}_n \end{bmatrix} \right\} \setminus \mathcal{N}_{12=}, \tag{4.28}$$

which represents the subspace where we can switch to produce different state trajectories but remain with the same output.

From $t = 0s$ *to* $t = 1.2s$, *the vector lies only in* $\mathcal{N}_{11}$ *and* $\mathcal{N}_{22}$. *This implies that, although the output remains unchanged, no non-equivalent switching is possible since the system has not yet reached the set* $\hat{\mathcal{X}}_{IR} \setminus \mathcal{X}_{IR=}$.

At $t = 1.2s$, *the vector enters all the subspaces, including the one given in* (4.28). *This makes it a valid point for switching to non-equivalent signals. Thus, we define a new switching combination:* $\sigma_a(t) = 1, \sigma_b(t) = 2$, *for* $t \in [1.2, 3)$. *After this change, the indicator curves show a new situation: the first two indicators remain active, which implies we can either keep the current combination or switch* σ_b *back to mode 1 while still preserving the same output trajectory.*

Let us proceed to switch σ_b *to mode 1 at* $t = 3s$. *As shown in Figure 4.10, the output trajectory remains unchanged. Figure 4.9 confirms that, after the switch, the concatenated state vector lies solely in* $\mathcal{N}_{y_{11}}$. *This indicates that no further switching is allowed without affecting the output, as no other mode combination would satisfy the output trajectory invariance condition. Then, we obtain:*

$$\sigma_a(t) = 1, \forall t \geq 0, \;\; \sigma_b(t) = \begin{cases} 1, & t \in [0, 1.2) \cup [3, +\infty), \\ 2, & t \in [1.2, 3), \end{cases} \tag{4.29}$$

as a pair of non-equivalent switching signals leading to identical output trajectories. We can conclude that, by using these indicators, we can construct, over time, non-equivalent switching signals that lead to the same output trajectory. Of course, following this procedure will yield different non-equivalent pairs of switching signals, depending on the choices made.

4.5 Conclusion

In this chapter, we investigate input redundancy in switched linear systems, focusing on two complementary notions: continuous and switching input redundancy.

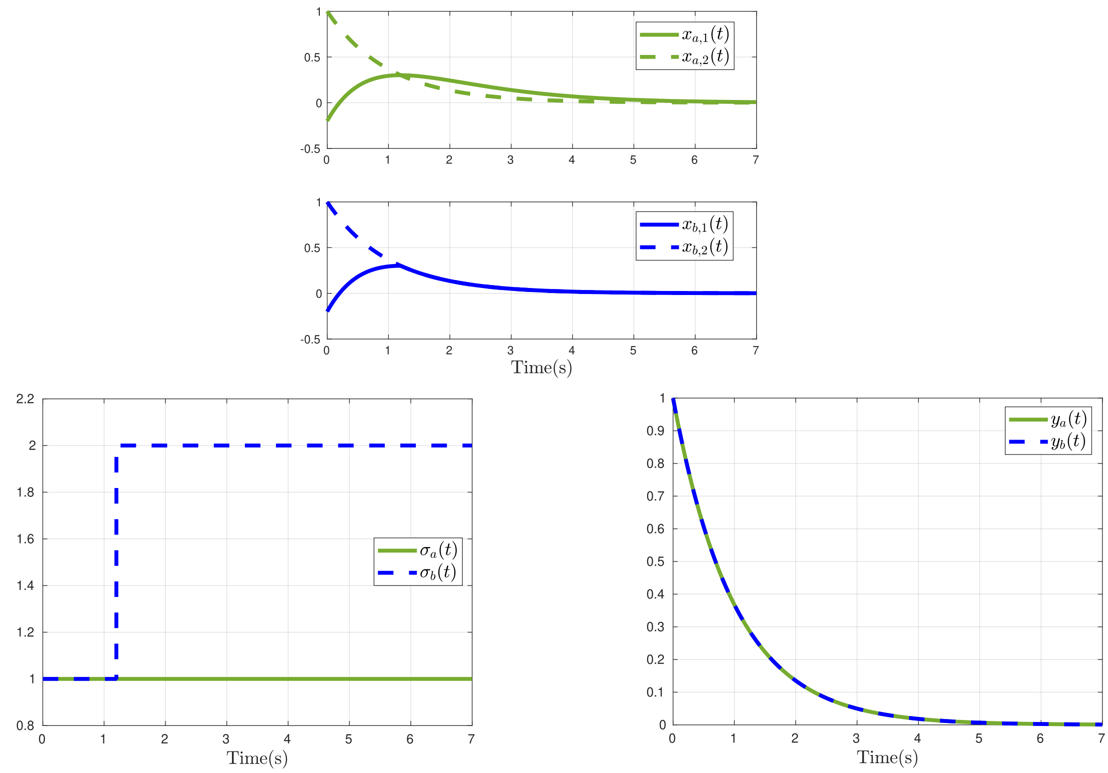

Figure 4.7: Switching signals (4.27), and resulting state and output trajectories.

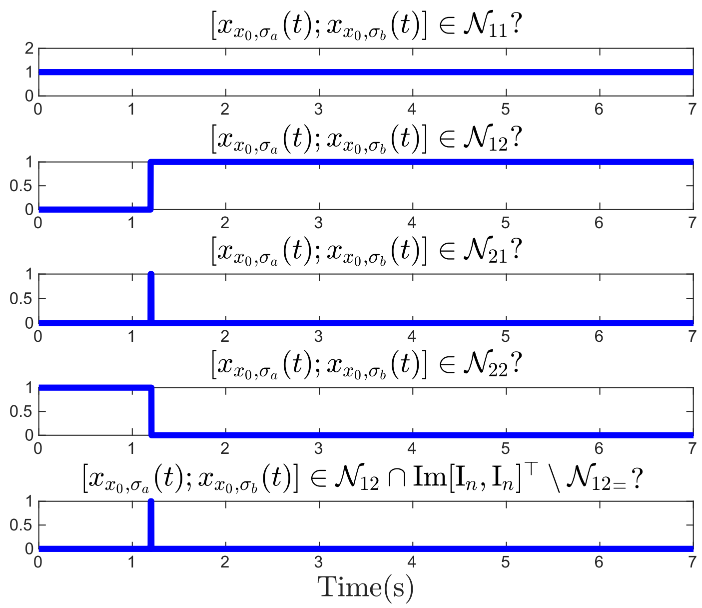

Figure 4.8: Indicators showing when the concatenated state vector belongs to various sets - switching signals as in (4.27) for $t \geq 0$.

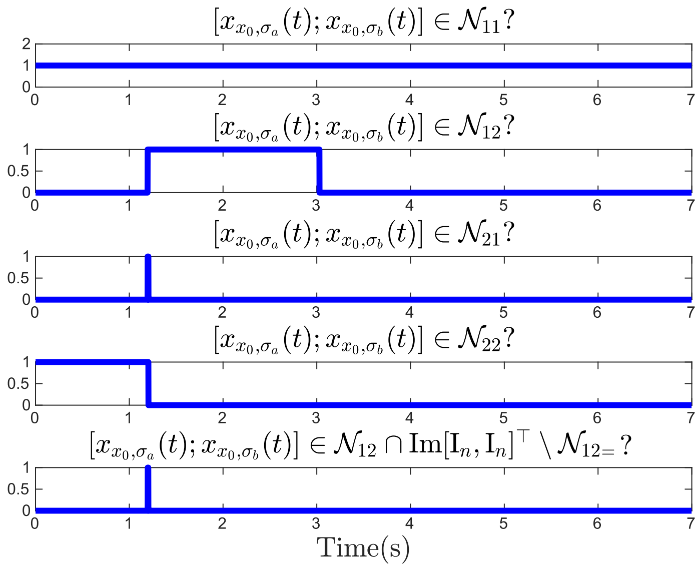

Figure 4.9: Indicators showing when the concatenated state vector belongs to various sets - switching signals as in (4.29) for $t \geq 0$.

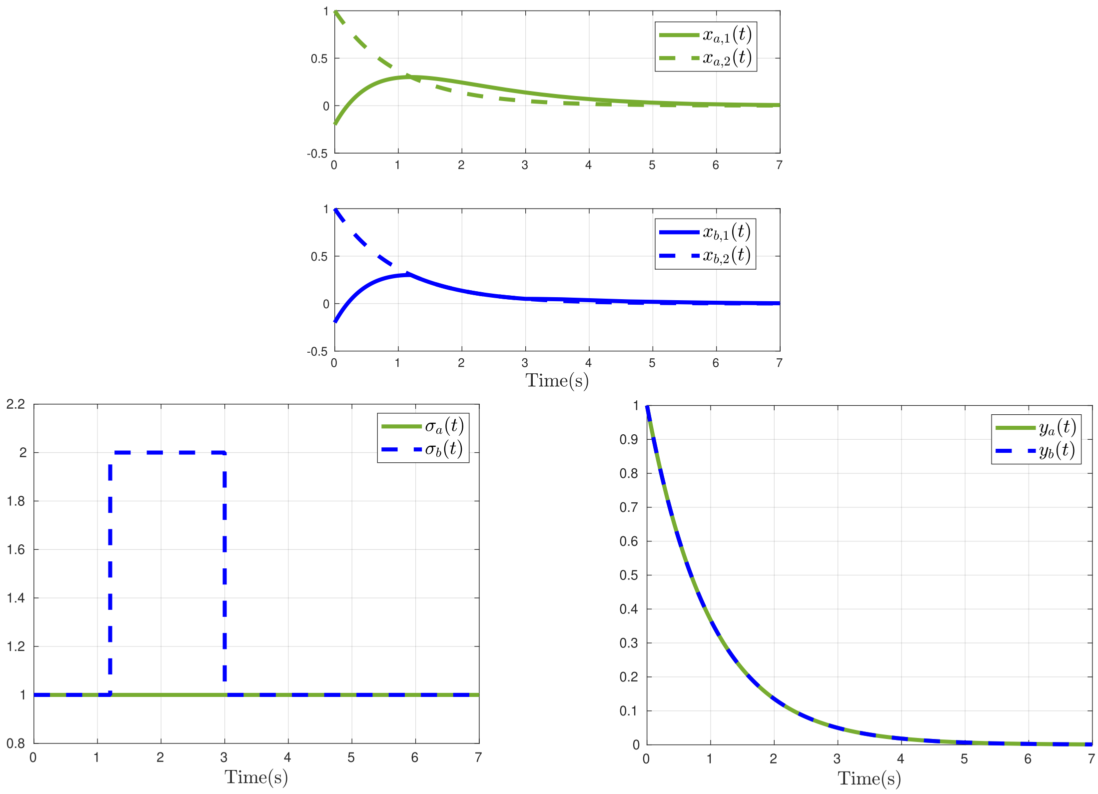

Figure 4.10: State and output trajectories resulting from the switching signals (4.29).

We begin by addressing continuous input redundancy. For a fixed switching signal, we define continuous input redundancy as the existence of two distinct input trajectories producing the same output, and we classify the cases according to whether the associated state trajectories coincide. This framework allows us to systematically analyze how input redundancy manifests itself. We then extend the question and ask whether a switched system exhibits input redundancy with respect to the continuous input for *all* switching signals. To tackle this, we use a Lagrange–interpolation representation of the switched system, which enables us to leverage results of Chapter 3 on parameter-dependent systems and derive verifiable conditions for input redundancy under arbitrary switching signals.

The second part of the chapter turns to switching input redundancy. To keep the problem analytically tractable, we restrict attention to switched systems with $u = \mathbf{0}$ and two LTI modes. We formalize this notion, adopt a classification similar to that used for the continuous input, and show that every such switched system displays this type of redundancy. Because this result depends on the initial condition, we further characterize the sets of initial states that give rise to redundancy, distinguishing between Kind 1 and Kind 3 redundancy. We establish conditions for these sets to be nonempty, allowing for a fully characterization of switching input redundancy.

Chapter 5

Conclusion

This thesis focused on extending the concept of input redundancy beyond LTI systems, specifically for systems that depend on time-varying parameters and for switched systems with linear modes. In Chapter 1, we introduced the motivation for this thesis. Chapter 2 provided background results on input redundancy of LTI systems. The first results of this thesis consisted of analyzing input redundancy of parameter-dependent systems with polynomial dependence. Such results were presented in Chapter 3. In Chapter 4, we developed results on input redundancy for switched systems, first in the setting where the switching signal is arbitrary, and then in the setting where the switching signal serves as a control input. Below, we summarize the main contributions of the thesis and outline possible directions for future work.

5.1 Summary

- In Chapter 3, we introduced the concept of adaptive input redundancy, where redundancy depends on the values of known parameters. In order to investigate this concept for polynomial parameter-dependent systems using tools from geometric control theory, we first addressed the problem of computing controlled invariant and output invisible subspaces of polynomial parameter-dependent systems. Then, we proposed a computationally efficient procedure to compute controlled invariant and output invisible subspaces of such a class of systems. Given a friend of a fixed degree, the method returns the largest subspace satisfying the invariance and output invisibility conditions. Building upon these developed tools, we derived a sufficient condition to check if a polynomial parameter-dependent system is adaptive input redundant. Due to the application of the Smith normal form at this step, which is developed only for matrices depending on one parameter, our condition for input redundancy is only valid for systems depending on a single parameter. Additionally, this framework allows us to identify the directions in which inputs can be applied without influencing the output trajectories.

- In Chapter 4, we extended the notion of input redundancy to the class of switched systems. Two frameworks were considered: (i) arbitrary switching signal, (ii) controlled switching signals. Let us separately summarize the results for each case:

  - In Section 4.3, we considered switched systems with LTI modes and arbitrary switching signal. We defined continuous input redundancy, requiring that the system remains input redundant for all admissible switching sequences. To test this property, we transformed the switched system into a polynomial parameter-dependent system using Lagrange interpolation, where the interpolation nodes correspond to the system modes. This transformation allowed us to apply our earlier results of input redundancy for polynomial parameter-dependent systems from Chapter 3 and derive a sufficient condition for input redundancy w.r.t the continuous input.

– In Section 4.4, we analyzed switched systems with $u = \mathbf{0}$ where the switching signal acts as a control input. We introduced the concept of switched input redundancy. Unlike the linear setting, such a property is no longer uniform in the initial condition. Consequently, we studied whether there exist initial conditions for which two distinct switched signals generate identical outputs. Focusing on systems with only two modes, we developed a necessary and sufficient condition for the three kinds of switched input redundancy. The method involves analyzing the unobservable subspaces of augmented LTI systems derived from the modes of the original switched system.

5.2 Perspectives

Several interesting questions remain open and will be the subject of future work. Below, we list some of these questions:

- For the problem of computing controlled invariant and output invisible subspaces of parameter-dependent systems, we should explore whether there is a maximum degree n_F for the input dependence such that the dimension of the subspace remains constant even increasing n_F. Moreover, exploring more general dependence of the input on the parameters, beyond a polynomial one, appears promising.

- Another important direction is to consider subspaces that depend on the parameters themselves. This would enable the characterization of the largest subspace satisfying both invariance and output invisibility for parameter-dependent systems. Thereby, leading to a necessary and sufficient condition for adaptive input redundancy. Furthermore, extending the results of adaptive input redundancy to systems with multiple parameters is also desired.

- An additional perspective involves the application of control allocation techniques using the results of adaptive input redundancy developed in Chapter 3. This approach has the potential to open a wide range of interesting new results that do not yet exist in the literature.

- In the context of switched systems, extending the investigation of input redundancy concerning the continuous input beyond the confines of parameter-dependent systems will possibly provide an idea regarding the conservatism linked to the current strategy and allow us to derive less conservative results. Moreover, a direct extension of the results obtained for input redundancy concerning the switching signals is the extension of the results to switched systems with several modes. Investigating input redundancy w.r.t. both continuous input and switching signal seems also very interesting, as many switched systems involve both types of control inputs.

- In conclusion, the development of control allocation strategies for switched systems remains a largely unexplored area. Our goal is to leverage the input redundancy results obtained for this class of systems to design control allocation techniques adapted to them.

Appendix A

Quotient spaces

This Appendix presents a survey on quotient spaces.

A.1 Equivalence relation and quotient space

Definition A.1 (Equivalence relation)**.** *Let be $\mathcal{X}$ a set and $\sim$ a binary relation, that is two elements x and y of $\mathcal{X}$ can be in relation (we denote $x \sim y$) or not in relation (we denote $x \nsim y$). The relation $\sim$ is an equivalence relation if it verifies the three conditions:*

i) Reflexivity: $x \sim x,\ \forall x \in \mathcal{X}$;

ii) Symmetry: $x \sim y$ if and only if $y \sim x,\ \forall (x, y) \in \mathcal{X}^2$;

iii) Transitivity: if $x \sim y$ and $y \sim z$, then $x \sim z,\ \forall (x, y, z) \in \mathcal{X}^3$.

The *equivalence class* of $x \in \mathcal{X}$ is defined by the set $\{y \in \mathcal{X} : y \sim x\}$ and is denoted $\bar{x}$. It is clear by the definition of the equivalence class, that two equivalence classes are the same ($\bar{x} = \bar{y}$, if $x \sim y$) or disjointed ($\bar{x} \cap \bar{y} = \emptyset$, if $x \nsim y$).

Definition A.2. *The set of all the equivalence classes of $\mathcal{X}$ related to the equivalence relation $\sim$ is defined as the quotient set and denoted $\mathcal{X}/\sim$.*

Example: Let $\mathbb{Z}$ be the set of relative integers and $n \in \mathbb{N}, n \geq 2$. The relation *modulo n congruence* is defined as $x \sim y$ if $x - y$ is a multiple of n. The quotient set $\mathbb{Z}/\sim$ (also denoted $\mathbb{Z}/n\mathbb{Z}$) is the finite set of n elements: $\mathbb{Z}/\sim = \mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$. Its cardinality is n.

A.2 Quotient space of a vector space by a subspace

A.2.1 Definitions and properties

Let $\mathcal{X}$ be vector space over $\mathbb{R}$ and $\mathcal{V}$ be a subspace of $\mathcal{X}$. In particular, we have that $\mathcal{V} \subseteq \mathcal{X}$. One can associate to $\mathcal{V}$ a congruence relation defined as follows:

Definition A.3. *Let x and y two elements of $\mathcal{X}$. We define $x \sim_{\mathcal{V}} y$ if $x - y \in \mathcal{V}$.*

Proposition A.1. *The relation $\sim_{\mathcal{V}}$ is an equivalence relation.*

Proof.
- For any $x \in \mathcal{X}$, we have $x - x = 0_{\mathcal{X}} \in \mathcal{V}$ because $\mathcal{V}$ is a vector subspace. That implies that $x \sim_{\mathcal{V}} x$.

- For any $x, y \in \mathcal{X}$, if $x - y \in \mathcal{V}$, then $y - x = -(x - y) \in \mathcal{V}$ because $\mathcal{V}$ is a vector subspace. That is $x \sim_{\mathcal{V}} y$ implies $y \sim_{\mathcal{V}} x$.

- For any $x, y, z \in \mathcal{X}$, if $x - y \in \mathcal{V}$ and $y - z \in \mathcal{V}$ then $x - z = x - y + (y - z) \in \mathcal{V}$. That is $x \sim_{\mathcal{V}} y$ and $y \sim_{\mathcal{V}} z$ implies $x \sim_{\mathcal{V}} z$.

□

The equivalent relation $\sim_{\mathcal{V}}$ generates equivalence classes $\bar{x}, x \in \mathcal{X}$ and the set of these equivalence classes:

$$\mathcal{X}/\sim_{\mathcal{V}} = \{\bar{x},\, x \in \mathcal{X}\}.$$

By definition, we can reformulate:

$$\bar{x} = \{y \in \mathcal{X},\, x \sim_{\mathcal{V}} y\} = \{x + y,\, y \in \mathcal{V}\} =: x + \mathcal{V}.$$

By definition also,

$$\overline{0_{\mathcal{X}}} = \{0 + y,\, y \in \mathcal{V}\} = \mathcal{V}.$$

Because $\mathcal{X}$ is a vector space, we have the following properties:

- $\sim_{\mathcal{V}}$ is compatible with the summation: Let be x, x', y, y' be vectors of $\mathcal{X}$ such that $x \sim_{\mathcal{V}} x'$ and $y \sim_{\mathcal{V}} y'$, then $x - x' \in \mathcal{V}$ and $y - y' \in \mathcal{V}$, which implies $x - x' + y - y' \in \mathcal{V}$, that is $x + y \sim_{\mathcal{V}} x' + y'$. We can define the summation over $\mathcal{X}/\sim_{\mathcal{V}}$ by

$$\bar{x} + \bar{y} := \overline{x + y}, \quad \forall (\bar{x}, \bar{y}) \in \mathcal{X}/\sim_{\mathcal{V}}\,. \tag{A.1}$$

- $\sim_{\mathcal{V}}$ is compatible with the scalar multiplication: Let be x, x' be elements of $\mathcal{V}$ and $\alpha \in \mathbb{R}$. If $x \sim_{\mathcal{V}} x'$, then $x - x' \in \mathcal{V}$ and $\alpha(x - x') \in \mathcal{V}$. We can define the scalar multiplication over $\mathcal{X}/\sim_{\mathcal{V}}$ by

$$\overline{\alpha x} = \alpha x + \mathcal{V} = \alpha x + \alpha\mathcal{V} = \alpha(x + \mathcal{V}) =: \alpha\bar{x}, \quad \forall \bar{x} \in \mathcal{X}/\sim_{\mathcal{V}},\, \alpha \in \mathbb{R}. \tag{A.2}$$

- The quotient set $\mathcal{X}/\sim_{\mathcal{V}}$ is a vector space with the zero element $\overline{0_{\mathcal{X}}}$. We can check the 8 conditions of the definition of a vector space. It is thus denoted $\mathcal{X}/\sim_{\mathcal{V}} = \mathcal{X}/\mathcal{V}$.

Example: Let $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}$. $\mathcal{X}$ is the planar state-space and $\mathcal{V}$ is vertical line passing through the origin. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X}$, it generates the equivalent class

$$\overline{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R} = \overline{\begin{pmatrix} x_1 \\ 0 \end{pmatrix}} = x_1\overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \tag{A.3}$$

which is the vertical line (parallel to $\mathcal{V}$) passing through the point $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$.

A.2.2 Dimensions

Consider that the vector space $\mathcal{X}$ has a finite dimension. We can build a basis of the quotient space $\mathcal{X}/\mathcal{V}$ as follows.

Proposition A.2. *Let $(v_1, v_2, \ldots, v_k)$ be a basis of $\mathcal{V}$, with $\dim \mathcal{V} = k$. We can extend this basis to a basis of $\mathcal{X}$ as $(v_1, \ldots, v_k, w_1, \ldots, w_{n-k})$, with $\dim \mathcal{X} = n$. Then*

$$(\overline{w_1}, \ldots, \overline{w_{n-k}}) = (w_1 + \mathcal{V}, \ldots, w_{n-k} + \mathcal{V}) \tag{A.4}$$

is a basis of $\mathcal{X}/\mathcal{V}$. We conclude that

$$\dim \mathcal{X}/\mathcal{V} = \dim \mathcal{X} - \dim \mathcal{V}. \tag{A.5}$$

Proof. The proof is performed in two steps.

Let us prove first that $(\overline{w_1}, \ldots, \overline{w_{n-k}})$ are linearly independent equivalent classes. Suppose there exist $n-k$ scalars $(\alpha_1, \ldots, \alpha_{n-k}) \in \mathbb{R}^{n-k}$ such that $\alpha_1 \overline{w_1} + \cdots + \alpha_{n-k}\overline{w_{n-k}} = \overline{0_{\mathcal{X}}} = \mathcal{V}$. It leads to $\alpha_1 w_1 + \cdots + \alpha_{n-k} w_{n-k} \in \mathcal{V}$. Because $(v_1, \ldots, v_k)$ is a basis of $\mathcal{V}$, there exist scalars $(\beta_1, \ldots, \beta_k) \in \mathbb{R}^k$ such that $\alpha_1 w_1 + \cdots + \alpha_{n-k} w_{n-k} - \beta_1 v_1 - \cdots - \beta_k v_k = 0$. But because $(v_1, \ldots, v_k, w_1, \ldots, w_{n-k})$ is a basis of $\mathcal{X}$ we have that $\alpha_1 = \cdots = \alpha_{n-k} = \beta_1 = \cdots = \beta_k = 0$. We conclude that $(\overline{w_1}, \ldots, \overline{w_{n-k}})$ are linearly independent equivalent classes.

Secondly, let us prove that for any $x \in \mathcal{X}$, the equivalent class $\bar{x}$ can be expressed thanks to $(\overline{w_1}, \ldots, \overline{w_{n-k}})$. For that, let us express x in the basis of $\mathcal{X}$: there exist scalars $\alpha_1, \ldots, \alpha_{n-k}$ and $\beta_1, \ldots, \beta_k$ such that $x = \alpha_1 w_1 + \cdots + \alpha_{n-k} w_{n-k} + \beta_1 v_1 + \cdots + \beta_k v_k$. That allows $\bar{x} = x + \mathcal{V} = \alpha_1 w_1 + \cdots + \alpha_{n-k} w_{n-k} + \beta_1 v_1 + \cdots + \beta_k v_k + \mathcal{V} = \alpha_1 w_1 + \cdots + \alpha_{n-k} w_{n-k} + \mathcal{V} = \alpha_1 w_1 + \mathcal{V} + \cdots + \alpha_{n-k} w_{n-k} + \mathcal{V} = \alpha_1 \overline{w_1} + \cdots + \alpha_{n-k}\overline{w_{n-k}}$. Finally, $(\overline{w_1}, \ldots, \overline{w_{n-k}})$ is a basis of $\mathcal{X}/\mathcal{V}$.

In order to prove the relation between the dimension, the basis $(\overline{w_1}, \ldots, \overline{w_{n-k}})$ of $\mathcal{X}/\mathcal{V}$ has $n - k = \dim \mathcal{X} - \dim \mathcal{V}$ elements that concludes the proof. $\square$

Appendix B

Homogeneous polynomials

In this appendix we briefly recall the main notions and properties of *homogeneous polynomials* that are used throughout the chapter. We explain how matrix-valued functions can be rewritten in a homogeneous form and illustrate the construction on a simple example.

B.1 Construction of a homogeneous polynomial representation

For simplicity, consider an application $\Phi : \Theta \to \mathbb{R}^{n_1 \times n_2}$ that is piecewise continuous in its argument. This includes, in particular, the class of matrix-valued polynomials defined on the compact set Θ. Under the following assumption that the compact set $\Theta \subset \mathbb{R}^l$ can be rescaled into the unit simplex Λ_H of some dimension H, Lemma B.1 ensures that Φ can then be expressed in homogeneous polynomial form.

Assumption B.1. *Given a piecewise continuous application $\Phi : \Theta \to \mathbb{R}^{n_1 \times n_2}$. We assume that there exist an integer $H \in \mathbb{N}^*$, an vector-valued function $\alpha : \Theta \to \Lambda_H$ with*

$$\Lambda_H := \left\{ \alpha \in \mathbb{R}^H : \ \alpha_i \geq 0, \forall i \in [\![1, H]\!] \ \textit{and} \ \sum_{i=1}^{H} \alpha_i = 1 \right\},$$

and a polynomial matrix-valued function $P : \Lambda_H \to \mathbb{R}^{n_1 \times n_2}$ such that $\Phi = P \circ \alpha$.

Lemma B.1. *Consider a piecewise continuous application $\Phi : \Theta \to \mathbb{R}^{n_1 \times n_2}$. Under Assumption B.1, the associated polynomial matrix-valued function $P : \Lambda_H \to \mathbb{R}^{n_1 \times n_2}$ can be represented as a homogeneous polynomial $\hat{P} : \Lambda_H \to \mathbb{R}^{n_1 \times n_2}$ of the same degree that the one of P.*

Proof. Since Assumption B.1 holds, take $\alpha : \Theta \to \Lambda_H$ and $P : \Lambda_H \to \mathbb{R}^{n_1 \times n_2}$ of the assumption. Let n_P be the degree of P. P can be written as:

$$P(\alpha) = \sum_{k \in \mathbb{N}^H : \sum_{i=1}^{H} k_i \leq n_P} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_H^{k_H} P_{(k_1, k_2, \cdots, k_H)} \tag{B.1}$$

for some coefficients $P_{(k_1,k_2,\cdots,k_H)} \in \mathbb{R}^{n_1 \times n_2}$. Since $\alpha \in \Lambda_H$, it follows that $\sum_{i=1}^{H} \alpha_i = 1$, and in turns $(\sum_{i=1}^{H} \alpha_i)^d = 1$, for any $d \in \mathbb{N}$. In Equation (B.1), it is possible to multiply each term of the sum by $(\sum_{i=1}^{H} \alpha_i)^{n_P - \sum_{i=1}^{H} k_i}$. This yields that each developed term can be identified uniquely as a monomial of a common degree n_P and P can be reformulated as an homogeneous polynomial $\hat{P}$ as in Definition 3.9, where its coefficients $\hat{P}_k$ in (3.15) are suitable combinations of the matrices P_k in (B.1). $\square$

It is worth noting that for every application Φ that is piecewise continuous on the compact set Θ, Assumption B.1 holds. Moreover, for a broad class of such applications, no conservatism is introduced when obtaining an homogeneous polynomial matrix-valued function $\hat{P}$ of Φ. This point is illustrated in the following example, where we show that a homogeneous polynomial of degree $n_P = 2$ can be obtained without conservatism.

Example B.1. *Consider $\Phi : [0,1] \times [0,\pi/2] \to \mathbb{R}$ defined by*

$$\Phi : \xi \mapsto \xi_1 + \xi_1^2 + \sin(\xi_2).$$

We compare two approaches:

Case 1: Classical linear representation. *Define $\alpha_{a,1} : \xi \mapsto \xi_1/3$, $\alpha_{a,2} : \xi \mapsto \xi_1^2/3$, $\alpha_{a,3} : \xi \mapsto \sin(\xi_2)/3$, and $\alpha_{a,4} : \xi \mapsto 1-(\alpha_{a,1}(\xi)+\alpha_{a,2}(\xi)+\alpha_{a,3}(\xi))$, so that $\alpha_a = [\alpha_{a,1}, \alpha_{a,2}, \alpha_{a,3}, \alpha_{a,4}]^\top \in \Lambda_4$. Then*

$$\hat{P}_a : \alpha_a \mapsto 3(\alpha_{a,1} + \alpha_{a,2} + \alpha_{a,3})$$

is homogeneous of degree $n_P = 1$, and $\Phi = \hat{P}_a \circ \alpha_a$. However, in this representation the structural relation between ξ_1 and ξ_1^2 is lost. As a result, this homogeneous representation is conservative, since it includes many values that are not attainable by Φ.

Case 2: Homogeneous quadratic representation. *Define $\alpha_{b,1} : \xi \mapsto \xi_1/2$, $\alpha_{b,2} : \xi \mapsto \sin(\xi_2)/2$, and $\alpha_{b,3} : \xi \mapsto 1 - (\alpha_{b,1}(\xi) + \alpha_{b,2}(\xi))$, so that $\alpha_b = [\alpha_{b,1}, \alpha_{b,2}, \alpha_{b,3}]^\top \in \Lambda_3$. Then*

$$P_b : \alpha_b \mapsto 2\alpha_{b,1} + 4\alpha_{b,1}^2 + 2\alpha_{b,2}$$

is such that $\Phi = P_b \circ \alpha_b$. Although P_b is not homogeneous, we can exploit the identity $\alpha_{b,1} + \alpha_{b,2} + \alpha_{b,3} = 1$ to define

$$\hat{P}_b : \alpha_b \mapsto 4\alpha_{b,1}^2 + 4\alpha_{b,1}\alpha_{b,2} + 2\alpha_{b,1}\alpha_{b,3} + 2\alpha_{b,2}\alpha_{b,3} + 2\alpha_{b,2}^2,$$

which is homogeneous of degree $n_P = 2$ and such that $\hat{P}_b(\alpha_b) = P_b(\alpha_b)$ for all $\alpha_b \in \Lambda_3$. In this form, the structural relation between ξ_1 and ξ_1^2 is preserved, and no conservatism is introduced.

Interestingly, the dimension of α_b required to parametrize $\hat{P}_b$ is smaller than that of α_a, suggesting that α_b provides a more natural basis for expressing the homogeneous polynomial representation of Φ.

Remark B.1. *Note that if the compact set Θ is a polytope and the application Φ is piecewise continuous and bounded on Θ, this set can be represented as the convex hull of the vertices of the polytope*

$$\Theta = \left\{ \xi \in \mathbb{R}^l : \xi = \sum_{i=1}^{H} \alpha_i h_i = M\alpha, \alpha \in \Lambda_H \right\},$$

where h_i are the vertices of the polytope [17, Section 2.2.4]. Consequently, system (3.5) can always be rewritten in a homogeneous polynomial form since the re-parameterization $\alpha(\xi)$ in Assumption B.1 always exists. However, such $\alpha(\xi)$ is not unique, and there is no generic or systematic way to calculate it, such a choice will depend on each specific numerical system. [31] exhibits, in specific frameworks, a procedure to obtain $\alpha(\xi)$ thanks to the Wachspress's construction [77].

Appendix C

Smith Normal Form

This appendix presents the main tools on *polynomial Euclidean division* and on the *Smith normal form* of a polynomial matrix. The exposition draws largely on [45, Section 6.3.3] and [16, Section 14.2.3].

We begin by recalling some technical results on polynomial Euclidean division and on Bézout's (or linear Diophantine) equation, which will serve as key ingredients in the proof of Proposition 3.2 concerning the Smith normal form.

C.1 Polynomial Euclidean division and Bézout's equation

The following theorem and definition recall the classical result of polynomial Euclidean division.

Theorem C.1. *Let $a(\xi)$ and $b(\xi)$ be two univariate polynomials in the variable ξ. Then there exist unique polynomials $q(\xi)$ (the* quotient*) and $r(\xi)$ (the* remainder*) such that*

$$a(\xi) = q(\xi)b(\xi) + r(\xi), \tag{C.1}$$

with

$$\deg r < \deg b. \tag{C.2}$$

Definition C.4. *The process of computing the polynomials q and r satisfying* (C.1) *is called the* polynomial Euclidean division *of the* dividend *a by the* divisor *b.*

We next recall a theorem on Bézout's equation together with the associated algorithm for solving it.

Theorem C.2. *Let $a(\xi)$, $b(\xi)$ and $c(\xi)$ be three univariate polynomials. The equation, known as Bézout's equation, in the unknown polynomials $r(\xi)$ and $s(\xi)$,*

$$a(\xi)r(\xi) + b(\xi)s(\xi) = c(\xi), \tag{C.3}$$

admits a (polynomial) solution $(r(\xi), s(\xi))$ if and only if the Greatest Common Divisor (GCD) of $a(\xi)$ and $b(\xi)$ divides $c(\xi)$.

Proof. $\Rightarrow$ Suppose, for contradiction, that there exists a solution $(r(\xi), s(\xi))$ to Equation (C.3), but that the GCD of $a(\xi)$ and $b(\xi)$ does not divide $c(\xi)$. Let $d(\xi) = \gcd(a(\xi), b(\xi))$. Then there exist polynomials $\tilde{a}(\xi)$ and $\tilde{b}(\xi)$ such that $a(\xi) = \tilde{a}(\xi)d(\xi)$ and $b(\xi) = \tilde{b}(\xi)d(\xi)$. Substituting into (C.3) gives

$$\big(\tilde{a}(\xi)r(\xi) + \tilde{b}(\xi)s(\xi)\big)d(\xi) = c(\xi),$$

implying that $d(\xi)$ divides $c(\xi)$, leading to a contradiction.

$\Leftarrow$ If $d(\xi)$ divides $c(\xi)$, then there exists a polynomial $\tilde{c}(\xi)$ such that $c(\xi) = \tilde{c}(\xi)d(\xi)$. Equation (C.3) then reduces to

$$\tilde{a}(\xi)r(\xi) + \tilde{b}(\xi)s(\xi) = \tilde{c}(\xi), \tag{C.4}$$

where $\tilde{a}(\xi)$ and $\tilde{b}(\xi)$ are coprime. Using Algorithm C.1 below, we can construct a solution $(\hat{r}(\xi), \hat{s}(\xi))$ of

$$\tilde{a}(\xi)\hat{r}(\xi) + \tilde{b}(\xi)\hat{s}(\xi) = 1. \tag{C.5}$$

Then

$$(r(\xi), s(\xi)) = \big(\hat{r}(\xi)\tilde{c}(\xi),\ \hat{s}(\xi)\tilde{c}(\xi)\big)$$

is a solution of Equation (C.4).

Algorithm C.1: Solving Bézout's Equation (C.5)

input : $\tilde{a}(\xi), \tilde{b}(\xi)$ coprime
output : $\hat{r}(\xi), \hat{s}(\xi)$ satisfying Equation (C.5)

$u_0(\xi) := 1; v_0(\xi) := 0; r_0(\xi) := \tilde{a}(\xi)$;
$u_1(\xi) := 0; v_1(\xi) := 1; r_1(\xi) := \tilde{b}(\xi)$;
repeat
  Let $r_{k+2}(\xi)$ and $q_k(\xi)$ be respectively the remainder and the quotient of the Euclidean division of $r_k(\xi)$ by $r_{k+1}(\xi)$, satisfying

$$r_{k+2}(\xi) = r_k(\xi) - q_k(\xi)r_{k+1}(\xi); \tag{C.6}$$

  and update

$$\begin{aligned} u_{k+2}(\xi) &:= u_k(\xi) - q_k(\xi)u_{k+1}(\xi); && \text{(C.7)} \\ v_{k+2}(\xi) &:= v_k(\xi) - q_k(\xi)v_{k+1}(\xi); && \text{(C.8)} \end{aligned}$$

  $k = k + 1$;
until $\deg(r_{k+2}) = 0$;
return $\hat{r}(\xi) = u_{k+2}(\xi)/r_{k+2}(\xi), \quad \hat{s}(\xi) = v_{k+2}(\xi)/r_{k+2}(\xi)$.

In Algorithm C.1, three sequences of polynomials $\{u_k(\xi)\}$, $\{v_k(\xi)\}$ and $\{r_k(\xi)\}$ are built so that

$$\tilde{a}(\xi)u_k(\xi) + \tilde{b}(\xi)v_k(\xi) = r_k(\xi), \quad \forall k \in \mathbb{N}.$$

The initialization satisfies $\tilde{a}(\xi)u_0(\xi) + \tilde{b}(\xi)v_0(\xi) = \tilde{a}(\xi) = r_0(\xi)$ and $\tilde{a}(\xi)u_1(\xi) + \tilde{b}(\xi)v_1(\xi) = \tilde{b}(\xi) = r_1(\xi)$.

Because Equation (C.6) is an Euclidean division, we have $\deg(r_{k+2}(\xi)) < \deg(r_{k+1}(\xi))$, hence the degrees strictly decrease after finitely many steps. We eventually obtain $\deg(r_{k+2}(\xi)) = 0$, so $r_{k+2}(\xi)$ is a nonzero constant (it cannot be zero since $\tilde{a}(\xi)$ and $\tilde{b}(\xi)$ are coprime). Finally, normalizing $u_{k+2}(\xi)$ and $v_{k+2}(\xi)$ by $r_{k+2}(\xi)$ yields a solution to Equation (C.5). $\square$

Remark C.2. *If Equation* (C.3) *admits one solution* $(r(\xi), s(\xi))$, *then it actually admits infinitely many solutions of the form*

$$\big(r(\xi) - \chi(\xi)b(\xi),\ s(\xi) + \chi(\xi)a(\xi)\big),$$

for any polynomial $\chi(\xi)$.

C.2 Proof of Proposition 3.2

This section presents a detailed proof of Proposition 3.2.

Proof. The proof of Proposition 3.2 proceeds by constructing the Smith normal form through suitable row and column permutations together with elementary row and column operations. For clarity, the argument is organized into five steps.

Step 1: If $W(\xi) = 0$, i.e. $r = 0$, or if $W(\xi)$ is scalar, then $W(\xi)$ is already in Smith normal form. We therefore assume in the following that $W(\xi)$ is not the zero matrix.

Step 2: Select a nontrivial entry of $W(\xi)$ with the smallest degree. By suitable row and column permutations, bring this entry to the $(1,1)$-position.

Step 3: Denote by $q_{i,j}(\xi)$ the quotient and by $r_{i,j}(\xi)$ the remainder of the Euclidean division of $W_{i,j}(\xi)$ by $W_{1,1}(\xi)$ for all $(i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}$. Apply the following row operations:

$$W_{i,\bullet}(\xi) \leftarrow W_{i,\bullet}(\xi) - q_{i,1}(\xi) W_{1,\bullet}(\xi), \quad \forall i \geq 2, \tag{C.9}$$

which corresponds to the invertible matrix operation

$$W(\xi) \leftarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -q_{2,1}(\xi) & & & \\ \vdots & & I_{n-1} & \\ -q_{n,1}(\xi) & & & \end{bmatrix} W(\xi). \tag{C.10}$$

Next, apply the following column operations:

$$W_{\bullet,j}(\xi) \leftarrow W_{\bullet,j}(\xi) - q_{1,j}(\xi) W_{\bullet,1}(\xi), \quad \forall j \geq 2, \tag{C.11}$$

which corresponds to the invertible matrix operation

$$W(\xi) \leftarrow W(\xi) \begin{bmatrix} 1 & -q_{1,2}(\xi) & \cdots & -q_{1,m}(\xi) \\ 0 & & & \\ \vdots & & I_{m-1} & \\ 0 & & & \end{bmatrix}. \tag{C.12}$$

After these operations, $W(\xi)$ takes the form

$$W(\xi) = \begin{bmatrix} W_{1,1}(\xi) & r_{1,2}(\xi) & \cdots & r_{1,m}(\xi) \\ r_{2,1}(\xi) & & \bullet & \\ \vdots & & & \\ r_{m,1}(\xi) & & & \end{bmatrix}. \tag{C.13}$$

Step 4: If $W_{i,1}(\xi) = 0$ for all $i \geq 2$ and $W_{1,j}(\xi) = 0$ for all $j \geq 2$, then $W(\xi)$ has the block form

$$W(\xi) = \begin{bmatrix} W_{1,1}(\xi) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{W}(\xi) & \\ 0 & & & \end{bmatrix}. \tag{C.14}$$

If $\tilde{W}$ is scalar, proceed to Step 5; otherwise restart from Step 1 with W replaced by $\tilde{W}$. Otherwise, repeat Step 2 with the same W. At each iteration, $\deg W_{1,1}(\xi)$ strictly decreases (since $\deg r_{i,j}(\xi) < \deg W_{1,1}(\xi)$), and only finitely many iterations are needed.

Step 5: At this point $W(\xi)$ is diagonal. We now enforce the divisibility property. Consider any 2×2 diagonal submatrix

$$\begin{bmatrix} W_{i,i}(\xi) & 0 \\ 0 & W_{j,j}(\xi) \end{bmatrix}.$$

Let $\gamma_{ij}(\xi) = \gcd(W_{i,i}(\xi), W_{j,j}(\xi))$. Then there exist $\hat{W}_{i,i}(\xi)$ and $\hat{W}_{j,j}(\xi)$ such that $W_{i,i}(\xi) = \gamma_{ij}(\xi)\hat{W}_{i,i}(\xi)$ and $W_{j,j}(\xi) = \gamma_{ij}(\xi)\hat{W}_{j,j}(\xi)$. By Theorem C.2, there exist $r(\xi)$ and $s(\xi)$ satisfying $W_{i,i}(\xi)r(\xi) + W_{j,j}(\xi)s(\xi) = \gamma_{ij}(\xi)$, and we can write

$$\begin{aligned} \begin{bmatrix} 1 & s(\xi) \\ -\hat{W}_{j,j}(\xi) & -\hat{W}_{j,j}(\xi) + 1 \end{bmatrix} \begin{bmatrix} W_{i,i}(\xi) & 0 \\ 0 & W_{j,j}(\xi) \end{bmatrix} \begin{bmatrix} r(\xi) & r(\xi)\hat{W}_{i,i}(\xi) - 1 \\ 1 & \hat{W}_{i,i}(\xi) \end{bmatrix} \\ = \begin{bmatrix} \gamma_{ij}(\xi) & 0 \\ 0 & \hat{W}_{i,i}(\xi)W_{j,j}(\xi) \end{bmatrix}, \end{aligned} \tag{C.15}$$

where $\gamma_{ij}(\xi) \mid W_{j,j}(\xi)$ and

$$\begin{bmatrix} 1 & s(\xi) \\ -\hat{W}_{j,j}(\xi) & -\hat{W}_{j,j}(\xi) + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\hat{W}_{j,j}(\xi) & 1 \end{bmatrix} \begin{bmatrix} 1 & s(\xi) \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} r(\xi) & r(\xi)\hat{W}_{i,i}(\xi) - 1 \\ 1 & \hat{W}_{i,i}(\xi) \end{bmatrix} = \begin{bmatrix} r(\xi) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \hat{W}_{i,i}(\xi) \\ 0 & -1 \end{bmatrix}$$

are invertible matrices whose inverses are also polynomial in ξ. Applying this procedure to all diagonal entries yields the Smith normal form (3.38).

To conclude the proof, note that the matrices $P(\xi)$ and $Q(\xi)$ are products of elementary matrices that are polynomial and invertible, with polynomial inverses as well. $\square$

Remark C.3. *In the Smith normal form of a matrix $W(\xi)$ in Proposition 3.2, the matrix $S(\xi)$ is unique. This is not the case of the matrices $P(\xi)$ and $Q(\xi)$, because the order of operations in the Algorithm to obtain the Smith normal form may be different.*

Appendix D

Résumé en français

Un système dynamique est un processus qui évolue au cours du temps selon des lois naturelles. Il est généralement décrit par un modèle mathématique spécifiant comment les variables du système, c'est-à-dire les états, évoluent en réponse aux fonctions d'entrée. La sortie du système dépend en général à la fois de ses états et de ses entrées. En théorie du contrôle, l'objectif principal consiste à concevoir des entrées capables de modifier le comportement du système pour qu'il présente le comportement desiré, tels que la stabilité ou le suivi de référence. Sous des conditions simples de régularité de la dynamique, une fonction d'entrée donnée génère une et une seule trajectoire de sortie. Cependant, l'application entrée–sortie n'est pas toujours injective. Certains systèmes présentent une *redondance d'entrée*, c'est-à-dire que des entrées distinctes produisent la même sortie. En pratique, cela procure des degrés de liberté supplémentaires dans le choix des commandes nécessaires pour atteindre une sortie donnée.

La redondance d'entrée n'est pas limitée à une classe restreinte de systèmes, elle est en réalité très répandue dans les applications d'ingénierie. Cela provient du fait que de nombreux systèmes sont délibérément conçus avec davantage d'actionneurs que nécessaire pour atteindre les objectifs de commande principaux. Par exemple, en aéronautique, en électronique de puissance ou dans les systèmes automobiles, des actionneurs redondants sont introduits pour améliorer les performances en réduisant les harmoniques, en accélérant la réponse ou en diminuant la consommation globale [51, 28, 60]. La redondance est également exploitée afin d'étendre les contraintes des actionneurs, permettant un fonctionnement dans des régimes qui ne seraient pas réalisables avec un seul actionneur [18, 29]. Enfin, elle constitue un élément fondamental de la tolérance aux fautes, garantissant la continuité du fonctionnement malgré la défaillance d'actionneurs grâce à la réallocation de l'effort de commande parmi ceux restants [1, 63]. Pour toutes ces raisons, la redondance d'entrée n'est pas une simple propriété abstraite, mais un choix de conception intentionnel dans de nombreux systèmes critiques en performance ou en sécurité.

Afin de fournir une définition précise et générale de la redondance d'entrée, considérons un système dynamique Σ, défini ci-dessous.

Definition D.5 (Système Σ)**.** *Un* système*, noté Σ, est caractérisé par :*

- *une entrée $u : \mathbb{R}^+ \to \mathcal{U}$, où $\mathcal{U}$ désigne l'ensemble des valeurs d'entrée admissibles,*
- *un état $x : \mathbb{R}^+ \to \mathcal{X}$ avec condition initiale $x(0) = x_0 \in \mathcal{X}$, où $\mathcal{X}$ est l'ensemble des valeurs d'état admissibles,*
- *et une sortie de performance $y : \mathbb{R}^+ \to \mathcal{Y}$, où $\mathcal{Y}$ désigne l'ensemble des valeurs de sortie admissibles. La sortie y reflète l'objectif de commande.*

Dès lors que, pour une condition initiale x_0 et une entrée u données, les trajectoires d'état et de sortie correspondantes sont bien définies et uniques, nous les notons respectivement $x_{x_0,u}$ et $y_{x_0,u}$.

Nous imposons l'hypothèse suivante afin de garantir l'existence et l'unicité des trajectoires d'état et de sortie de Σ.

Assumption D.2 (Existence et unicité)**.** *La dynamique du système Σ est supposée suffisamment régulière pour garantir le caractère bien-posé des trajectoires. De plus, les entrées admissibles de Σ appartiennent à l'ensemble des fonctions continues par morceaux $\mathcal{PC}(\mathbb{R}^+, \mathcal{U})$ de $\mathbb{R}^+$ vers $\mathcal{U}$.*

La redondance d'entrée est donc définie formellement comme suit:

Definition D.6 ([49])**.** *Le système Σ est dit redondant en entrée s'il existe une condition initiale et deux trajectoires d'entrée distinctes telles que les trajectoires de sortie résultantes soient identiques, c'est-à-dire :*

$$\exists x_0 \in \mathcal{X},\ u_a, u_b \in \mathcal{PC}(\mathbb{R}^+, \mathcal{U}) :\ u_a \neq u_b,\ y_{x_0,u_a} = y_{x_0,u_b}. \tag{D.1}$$

Cette définition, qui s'applique à une classe très générale de systèmes (aucune hypothèse n'est faite sur la structure de Σ, hormis celles garantissant l'existence et l'unicité des trajectoires d'état et de sortie), met en évidence l'existence de degrés de liberté dans les systèmes redondants en entrée. Soit une conditions initiale x_0 et une commande u_a qui génèrent une trajectoire de sortie désirée y_{x_0,u_a} qui peut être générée par une autre entrée u_b distincte. L'objectif principal du contrôle est associé à la trajectoire de sortie y_{x_0,u_a}. La possibilité de choisir entre au moins deux trajectoires d'entrée (u_a, et u_b voir d'autres) constitue un degré de liberté. Il est alors possible de concevoir un critère de choix qui est un objectif secondaire.

Pour les systèmes redondants en entrée, il est alors possible de proposer une architecture de commande en deux étapes successives: la première étape consiste à faire la synthèse d'un contrôleur primaire associé à un effort total de commande requis pour optimiser un objectif principal; la seconde étape détermine l'entrée qui optimise un objectif secondaire parmi toutes les entrées solutions du problème d'optimisation principal. Cette approche, largement connue sous le nom de *allocation de commande* (*control allocation* en anglais) [11, 29, 41], repose sur l'identification des entrées produisant la même sortie. Une caractérisation rigoureuse de la redondance d'entrée est donc essentielle : elle permet d'établir si un système est ou non redondant en entrée et de déterminer les degrés de liberté associés. Sans une telle caractérisation, une partie de la redondance peut rester inexploitable. Si la redondance d'entrée a été caractérisée pour les systèmes linéaires invariants dans le temps (LTI), le problème reste en grande partie ouvert dans des cadres plus généraux.

L'objectif de cette thèse est d'étendre le concept de redondance d'entrée au-delà du cadre LTI. Une caractérisation complète pour des systèmes non linéaires généraux serait idéale, mais demeure un défi considérable, voir impossible. Afin de proposer de potentielles extensions, il est donc nécessaire de considérer des classes particulières de systèmes qui présentent une structure suffisante pour obtenir des caractérisations et qui soient en même temps des classes suffisamment larges en représentant de nombreuses applications. En suivant cette démarche, nous nous sommes concentrés dans cette thèse sur les *systèmes linéaires dépendant polynomialement de paramètres* et les *systèmes linéaires commutés*.

D.0.1 Extension de la redondance d'entrée pour les systèmes linéaires dépendant de paramètres

Dans la pratique, de nombreux systèmes présentant une redondance d'entrée sont également soumis à des variations de paramètres, éventuellement inconnues, voir, par exemple, [27, 30, 47, 54].

Un cas typique est la masse d'un véhicule hybride, qui dépend du nombre de passagers et est donc incertaine. Des situations similaires apparaissent dans d'autres domaines, comme la variation des valeurs de composants avec la température en électronique de puissance, ou les changements du point de fonctionnement à partir duquel un modèle linéaire est dérivé en aéronautique.

Pour traiter de telles variations et incertitudes, il est courant d'employer un modèle dépendant de paramètres, pour lequel une large gamme d'outils existe déjà, tant pour l'analyse de la stabilité [19, 23, 26] que pour la synthèse de commande [3, 10, 50]. Cependant, à notre connaissance, le problème de la redondance d'entrée dans ce contexte dépendant de paramètres reste largement inexploré.

L'un des objectifs de la thèse, traité dans le Chapitre 3, est de combler cette lacune. En nous appuyant sur les définitions, fondées sur les signaux, de la redondance d'entrée, nous introduisons tout d'abord, deux nouvelles notions en adéquation avec le cadre des systèmes dépendant de paramètres : la *redondance d'entrée adaptative* et la *redondance d'entrée robuste*. Parce que nous estimons que ces définitions ne doivent reposer sur aucune hypothèse simplificatrice, nous les formulons sans imposer de restrictions sur la dépendance en paramètres, au-delà de l'existence et de l'unicité des solutions du système. Nous nous concentrons ensuite sur un cadre simplifié mais pertinent en pratique : les systèmes linéaires dépendant de paramètres, avec une dépendance polynomiale.

Dans ce cadre, nous établissons des conditions vérifiables permettant de déterminer si un système dépendant de paramètres est (de manière adaptative ou de manière robuste) redondant en entrée. Notre approche étend le raisonnement développé dans le contexte LTI. La redondance d'entrée dans les systèmes LTI est caractérisée au moyen d'une analyse incrémentale impliquant des *entrées invisibles*, qui comportent deux composantes : (i) un terme garantissant que l'état demeure dans un sous-espace d'état particulier, le sous-espace faiblement inobservable, et (ii) un terme qui peut être librement choisi dans un sous-espace d'entrée particulier, le sous-espace d'entrée faiblement inobservable. Nous étendons ensuite la notion de sous-espace faiblement inobservable au cadre dépendant de paramètres et discutons sa construction pratique. Ces extensions introduisent inévitablement un certain degré de conservatisme. Enfin, nous combinons les résultats précédents afin de fournir des conditions vérifiables pour la redondance d'entrée adaptative et robuste.

D.0.2 Extension de la redondance d'entrée pour les systèmes linéaires commutés

Parmi les extensions naturelles du cadre de la redondance d'entrée, nous proposons d'étudier la redondance d'entrée pour les systèmes dont la dynamique est à chaque instant parmi un nombre fini de dynamiques, à savoir les systèmes commutés [71]. Ils font typiquement intervenir deux types d'entrées : une entrée continue agissant sur la dynamique continue, et un signal de commutation régissant les transitions entre les modes.

Étendre le concept de redondance d'entrée aux systèmes commutés est particulièrement pertinent. De nombreux systèmes d'électronique de puissance, qui combinent intrinsèquement dynamiques continues et dynamiques discrètes, sont connus pour présenter une redondance d'entrée [15, 47, 65, 81]. Un autre exemple illustratif est le véhicule hybride, où la boîte de vitesses génère des événements discrets pouvant naturellement être modélisés comme des commutations de mode. Malgré les progrès significatifs réalisés dans la théorie des systèmes commutés au cours des trois dernières décennies [24, 40, 52, 53], la redondance d'entrée n'a pas encore été étudiée dans ce contexte.

Un deuxième objectif de la thèse, traité dans le Chapitre 4, est également de combler cette lacune. Pour cela, nous introduisons deux notions adaptées aux systèmes commutés : la redondance d'entrée continue, où l'attention est portée exclusivement sur les entrées continues, et la redondance d'entrée commutée, où différents signaux de commutation peuvent conduire à des sorties identiques. Nous étudions d'abord la première notion. L'observation principale est qu'un système commuté peut être représenté comme un système dépendant de paramètres, polynomial en ce paramètre. Cette reformulation, obtenue via l'interpolation de Lagrange, permet de réutiliser les outils développés au chapitre précédent pour les systèmes dépendant de paramètres, et d'établir des conditions vérifiables sous lesquelles un système peut être déclaré redondant en entrée continue, tout en révélant ses degrés de liberté. Nous nous concentrons ensuite sur la seconde notion, la redondance d'entrée commutée. Dans ce cas, le problème est plus délicat et ne peut être traité par une extension directe des résultats précédents. Pour rendre l'analyse faisable, nous nous concentrons sur le cas simple mais révélateur des systèmes commutés à deux modes et sans entrée continue. Dans ce cadre, nous identifions les principales difficultés qui émergent dans la dérivation de conditions générales, et nous proposons une approche fondée sur la dualité entre redondance d'entrée et inversibilité à gauche. Les résultats s'appuient sur des travaux existants relatifs à la inversibilité à gauche des systèmes commutés [44, 69], que nous adaptons au cadre présent.

D.1 Contributions

Les principales contributions concernent l'extension de la redondance d'entrée et le développement de caractérisations pour les systèmes linéaires dépendant de paramètres et les systèmes commutés. Des résultats techniques relatifs au cas LTI sont également développés. Elles peuvent être résumées comme suit :

1. Systèmes linéaires invariants dans le temps

1.1. Extension des algorithmes pour la détermination numérique de sous-espaces contrôlés invariants et invisibles en sortie, afin de gérer des matrices de dimension étendue.

1.2. Analyse des trajectoires en temps fini et de leur lien avec la redondance d'entrée.

2. Systèmes dépendant de paramètres

2.1. Extension de la redondance d'entrée aux systèmes dépendant de paramètres. Introduction des notions de redondance *adaptative* et *robuste*.

2.2. Outils de caractérisation pour les systèmes linéaires en l'état et l'entrée et polynomiaux en le paramètre :

   (a) Algorithme pour la détermination du *sous-espace faiblement inobservable*.

   (b) Algorithme pour la détermination du *sous-espace des entrées faiblement inobservables*.

2.3. Caractérisation de la redondance d'entrée adaptative et robuste.

3. Systèmes linéaires commutés

3.1. Extension de la redondance d'entrée aux systèmes commutés, introduction des notions de redondance continue et commutée.

3.2. Redondance d'entrée continue :

  (a) Caractérisation vis-à-vis des entrées continues.
  (b) Construction d'entrées distinctes produisant des sorties identiques.

3.3. Redondance d'entrée commutée :

  (a) Étude du cas à deux modes autonomes.
  (b) Caractérisation complète de la redondance commutée.

Contributions 1.1 et 2.2(a) issues de [74]. Contributions 2.2(b) et 2.3 issues de [75]. Contribution 3.2 issue de [73]. Contribution 3.3 issue de [76]. Les autres contributions (1.2, 2.1, 3.1) sont entièrement originales.

D.2 Publications

- **Articles de revue:**

  1. V. V. Viana, J. Kreiss, M. Jungers, On the computation of controlled invariant and output invisible subspaces for parameter-dependent systems, IEEE Transactions on Automatic Control, vol. 69, no. 7, pp. 4695-4701, 2024. [74]
     https://doi.org/10.1109/TAC.2024.3356170
  2. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched linear systems via polynomial parameter-dependent systems, IEEE Control Systems Letters, vol. 8, pp. 1066-1071, 2024. [73]
     https://dx.doi.org/10.1109/LCSYS.2024.3405380
  3. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched systems concerning the switching signal, IEEE Control Systems Letters, vol. 9, pp. 1520-1525, 2025. [76]
     https://doi.org/10.1109/LCSYS.2025.3582170
  4. V. V. Viana, J. Kreiss and M. Jungers, Adaptive input redundancy of polynomial parameter-dependent systems, in IEEE Transactions on Automatic Control, 2025. [75]
     https://dx.doi.org/10.1109/TAC.2025.3615034

- **Articles de conférence:**

  1. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched linear systems via polynomial parameter-dependent systems, IEEE Conference on Decision and Control (CDC), Milan, Italy, 2024. (Also in L-CSS).
  2. V. V. Viana, S. Galeani, M. Sassano, On the construction of a minimal order annihilator and its role in dynamic control allocation, IEEE Conference on Decision and Control (CDC), Rio de Janeiro, Brazil, 2025. [Cette contribution a été préparée lors d'une visite de deux mois de Valessa V. Viana à l'Université de Rome, Tor Vergata, cofinancée par le projet DREAM de l'Université de Lorraine et par le laboratoire CRAN.]
  3. V. V. Viana, J. Kreiss, M. Jungers, Input redundancy of switched systems concerning the switching signal, IEEE Conference on Decision and Control (CDC), Rio de Janeiro, Brazil, 2025. (Also in L-CSS).

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