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Présentée par : **Sahbi Boussandel**

Sujet :

Méthodes de résolution d'équations algébriques et d'évolution en dimension finie et infinie

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Sahbi Boussandel

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Sahbi Boussandel

Table des matières

Introduction Générale

Dans ce travail, on s'intéresse à la résolution de problèmes algébriques et d'évolution en dimension finie et infinie.
On trouve dans la littérature un nombre considérable de méthodes permettant d'atteindre cet objectif. On cite à titre d'exemples :
- les méthodes de points fixes (Théorème de Banach, de Schauder,...) avec de nombreuses applications en théorie des équations différentielles et des équations aux dérivées partielles ; les théorèmes d'inversion locale et de fonction implicite peuvent-être considérés comme des conséquences et ont eux aussi beaucoup d'applications,
- les méthodes de degré topologiques (Brouwer, Leray–Schauder,...) qui sont très proches des méthodes de points fixes,
- les méthodes variationnelles comme application de l'analyse fonctionnelle abstraite (théorèmes de Riesz et de Lax-Milgram, minimisation des fonctions convexes sur les espaces réflexifs,...) et qui en revanche ont de nombreuses applications en théorie des équations aux dérivées partielles,
- la méthode d'approximation de Galerkin qui permette la résolution des équations différentielles et les équations aus dérivées partielles, cette méthode est courammnent utilisée en calcul numérique,
- les méthodes itératives comme par exemple la méthode de Newton et la méthode de la plus grande descente.

Dans le premier chapitre du présent manuscript, on étudie l'existence des solutions globales d'un système gradient abstrait.
Afin de formuler le problème, soit V un espace de Banach réel réfléxif séparable et soit H un espace de Hilbert tel que $V \hookrightarrow H$ avec injection continue. Soit $E : V \to \mathbb{R}$ une fonction continûement différentiable. Soit $g : [0,T] \times V \to \mathcal{L}_2(H,\mathbb{R}), (t,u) \to \langle\cdot,\cdot\rangle_{g(t,u)}$ une application telle que pour tout $t \in [0,T]$, $g(t,\cdot)$ est une métrique sur H.
Pour tout $t \in [0,T]$, on définit le gradient de E dans H par rapport à la

métrique $g(t,\cdot)$ par

$$D(\nabla_{g(t)}E) = \{u \in V : \exists\, w \in H, \forall v \in V, E'(u)\, v = \langle w, v\rangle_{g(t,u)}\},$$
$$\nabla_{g(t)}E(u) = w.$$

On considère le système gradient suivant :

$$\begin{cases} u'(t) + \nabla_{g(t)}E(u(t)) = f(t), & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{1}$$

où $u_0 \in V$ et $f \in L^2(0,T;H)$ sont données.
On montre un résultat de régularité L^2-maximale pour le système (1), en le sens que, pour tout $u_0 \in V$, pour tout $f \in L^2(0,T;H)$, il existe $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ tel que pour presque tout $t \in (0,T)$, $u(t) \in D(\nabla_{g(t)}E)$, qui est solution du système (1).
Plus précisément, on montre le théorème suivant :

Théorème 0.0.1. *Supposons que l'injection $V \hookrightarrow H$ est compacte. Supposons que E est H-elliptique et que $E' : V \to V'$ transforme les bornés de V en des bornés de V'. Supposons de plus que $g(t,\cdot)$ est une métrique pour tout $t \in [0,T]$ et que pour tout $u \in V$, pour tout v, $w \in H$, la fonction $t \to \langle v, w\rangle_{g(t,u)}$ est mesurable sur $[0,T]$. On suppose aussi qu'il existe c_1, $c_2 > 0$ tels que pour tout $u \in V$, $v \in H$, $t \in [0,T]$ on a*

$$c_1\|v\|_H \le \|v\|_{g(t,u)} \le c_2\|v\|_H.$$

Supposons finalement que la métrique $g(t,\cdot)$ est continue :

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ dans } W^{1,2}(0,T;H), \\ u_n \overset{\text{weak}*}{\to} u \text{ dans } L^\infty(0,T;V), \\ v_n \rightharpoonup v \text{ dans } L^2(0,T;H), \\ w \in L^2(0,T;H) \end{array}\right\} \Rightarrow \int_0^T \langle v_n, w\rangle_{g(t,u_n)}\, dt \to \int_0^T \langle v, w\rangle_{g(t,u)}\, dt. \tag{2}$$

Alors pour tout $u_0 \in V$, pour tout $f \in L^2(0,T;H)$, il existe $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ tel que pour presque tout $t \in (0,T)$, $u(t) \in D(\nabla_{g(t)}E)$, qui est solution du système (1).

Afin de démonter le Théorème 0.0.1, on utilise la méthode d'approximation de Galerkin, une méthode qui est très utilisée en analyse numérique et qui a l'avantage de donner une démonstration constructive d'existence de solutions.

La difficulté majeure pour la démonstration du Théorème 0.0.1 se présente au niveau du passage à la limite dans le terme non linéaire de l'équation en dimension finie associée à l'équation (1), surtout à cause de la présence de la métrique $g(t,\cdot)$. Afin de surmonter ce problème, on utilise la théorie de monotonie et une idée dûe à J.L. Lions [39] qui permet de prouver que la limite éventuelle du terme non linéaire de l'équation en dimension finie coincide avec le terme non linéaire de l'équation (1).

Une première application de ce théorème est la résolution d'un problème de diffusion non-linéaire qui fait intervenir l'opérateur p-Laplacien avec conditions aux limites de Dirichlet :

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\,\Delta_p u = f & \text{dans} \quad (0,T)\times\Omega, \\ u = 0 & \text{sur} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{dans} \quad \Omega, \end{cases} \tag{3}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $u_0 \in W_0^{1,p}(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ et

$$m : [0,T]\times\Omega\times\mathbb{R}^N \to [\varepsilon, \frac{1}{\varepsilon}] \;\; (\varepsilon \in (0,1) \text{ fixé})$$

est une fonction mesurable qui est continue par rapport à la troisième variable.
La résolution d'un problème de diffusion non-linéaire faisant intervenir l'opérateur p-Laplacien avec conditions aux limites de Neumann fait l'objet d'une deuxième application du Thèorème 0.0.1 :

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\,\Delta_p u = f & \text{dans} \quad (0,T)\times\Omega, \\ |\nabla u|^{p-2}\nabla u\cdot\overrightarrow{\nu} = 0 & \text{sur} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{dans} \quad \Omega, \end{cases} \tag{4}$$

où Ω est un ouvert borné régulier de $\mathbb{R}^N$, $u_0 \in W^{1,p}(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$, $\overrightarrow{\nu}$ est la normale sortante à Ω et m est comme dans l'application précédente.
Une troisième application de ce théorème est la résolution du problème d'évolution suivant :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(a(\cdot,u)\nabla u\right) = f & \text{dans} \quad (0,T)\times\Omega, \\ u = 0 & \text{sur} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{dans} \quad \Omega, \end{cases} \tag{5}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;H^{-1}(\Omega))$ et

$$a : \Omega \times L^2(\Omega) \to [\varepsilon, \frac{1}{\varepsilon}] \ \ (\varepsilon \in (0,1) \text{ fixé})$$

est une fonction mesurable par rapport à la première variable et qui satisfait une certaine continuité faible par rapport à la seconde variable.

Les systèmes gradients ont été étudié par plusieurs auteurs. Parmi ces auteurs, on cite J-.L. Lions [39] qui a étudié l'existence et l'unicité des solutions du problème suivant :

$$\begin{cases} u'(t) + A(u(t)) = f(t), \ \ t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{6}$$

où $u_0 \in H$ et $f \in L^{p'}(0,T;V')$ sont donnés.
Sous certaines hypothèses sur l'opérateur A, J.-L. Lions prouve un résultat de régularité $L^{p'}$-maximale dans V' pour le système (6). Plus précisément, il prouve le théorème suivant :

Théorème 0.0.2. *Supposons que l'injection $V \hookrightarrow H$ soit continue et que $V \hookrightarrow H = H' \hookrightarrow V'$. Soit $A : V \to V'$ un opérateur monotone, hémicontinu et vérifiant :*

$$\exists\, c_1 > 0,\ \forall\, u \in V,\ \|A(u)\| \le c_1 \|u\|^{p-1}$$

et

$$\exists\, c_2 > 0,\ \forall\, u \in V,\ \langle A(u), u\rangle_{V',V} \ge c_2 \|u\|^p \ \ (p > 1).$$

Alors pour tout $u_0 \in H$, pour tout $f \in L^{p'}(0,T;V')$, il existe un unique $u \in W^{1,p'}(0,T;V') \cap L^p(0,T;V)$ qui est solution de (6).

Remarquons que par rapport au Théorème 0.0.2, on perd au Théorème 0.0.1 l'unicité de la solution et on suppose en plus que l'injection $V \hookrightarrow H$ soit compacte. Par contre, on permet la présence d'une métrique non-constante.
Dans le cas particulier $p = 2$, W. Arendt et R. Chill [12] ont étudié l'existence des solutions pour le problème (3) pour des coefficients m qui dependent en plus du gradient de la solution u et pour des non-homogèniétés f qui dépendent en plus de la solution u et de son gradient. Ils prouvent l'existence des solutions pour ce problème dans l'espace $H^1_{loc}([0,\infty);L^2(\Omega)) \cap L^2_{loc}([0,\infty);D(\Delta_D)) \cap C([0,\infty);H^1_0(\Omega))$, où $D(\Delta_D)$ est le domaine de l'opérateur Laplacien dans $L^2(\Omega)$ avec conditions aux limites de Dirichlet.
Dans ce même cas, c'est-à dire $p = 2$, W. Arendt et R. Chill prouvent également l'existence des solutions pour le problème (4) dans l'espace

$H^1_{loc}([0,\infty);L^2(\Omega))\cap L^2_{loc}([0,\infty);D(\Delta_N))\cap C([0,\infty);H^1(\Omega))$, où $D(\Delta_N)$ est le domaine de l'opérateur Laplacien dans $L^2(\Omega)$ avec conditions aux limites de Neumann.
L'ingrédient essentiel de la preuve de ces résultats est l'utilisation du théorème du point fixe de Schaefer.
Remarquons que pour les problèmes quasi-linéaires étudiés par W. Arendt et R. Chill, l'ouvert Ω est supposé quelconque, pour les problèmes non-linéaires (problèmes (3) et (4)) on perd cette propriété et on suppose que Ω est borné.
Dans [7, 40], les auteurs ont étudié l'existence et l'unicité des solutions du problème quasi-linéaire

$$\begin{cases} u'(t)+A(u(t))u(t)+F(u(t))=f(t), \;\; t\in(0,T), \\ u(0)=u_0, \end{cases} \tag{7}$$

où $u_0\in Tr_p(X,D)$, $A:Tr_p(X,D)\to\mathcal{L}(X,D)$ et $F:Tr_p(X,D)\to X$; X et D sont deux espaces de Banach tel que $D\hookrightarrow X$ avec injection dense et continue.
Ils prouvent que si l'opérateur linéaire $A(u_0)$ admet une régularité L^p-maximale et si F est une fonction continue et localement lipschitizienne, alors le théorème du point fixe de Banach implique l'existence locale des solutions du problème (7).
Dans [38], les auteurs utilisent la méthode du point fixe de Schauder afin d'obtenir l'existence de solutions pour le problème (7).
Lorsque la métrique g est constante, c'est-à-dire indépendente de t et u, l'équation (1) peut-être généralisée en une équation faisant intervenir des opérateurs multivoques. Dans [18], H. Brézis a étudié l'existence et l'unicité des solutions de l'équation

$$\begin{cases} u'(t)+\partial\phi(u(t))\ni f(t), \;\; t\in(0,T), \\ u(0)=u_0, \end{cases} \tag{8}$$

où $\phi:H\to]-\infty,+\infty]$ est une fonction convexe, semi-continue inférieurement, définie sur un espace de Hilbert H, $f\in L^2(0,T;H)$, $u_0\in H$ et $\partial\phi(u)$ est la sous-différentielle de ϕ en u.
Sous l'hypothèse que $\partial\phi$ est un opérateur maximal monotone, H. Brézis prouve que pour tout $f\in L^2(0,T;H)$, pour tout $u_0\in\overline{D(\partial\phi)}$, l'équation (8) admet une unique solution forte, c'est-à-dire, il existe une fonction $u\in C([0,T];H)$ qui est absolument continue sur tout compact de $(0,T)$, vérifiant $u(t)\in D(\partial\phi)$ et $u'(t)+\partial\phi(u(t))\ni f(t)$ pour presque tout $t\in(0,T)$.

Dans le chapitre 2, on étudie l'existence locale, l'unicité et la régularité des

solutions de l'équation du raccourcissement des courbes .
La théorie de régularité maximale/optimale pour les problèmes d'évolution linéaires dans les espaces de Banach joue un rôle important dans la preuve de l'existence locale, l'unicité et la régularité des solutions des problèmes non liéaires du type

$$\begin{cases} u' + F(u) = f(t), \quad t \in (0, T), \\ u(0) = u_0. \end{cases} \tag{9}$$

Parmi les premiers travaux qui ont traité ce sujet, on cite par exemple ceux de Da Prato & Grisvard [25], Amann [5, 6], Angenent [8, 9], Clément & Li [23], Lunardi [40], et plus recemment les travaux de Escher, Prüss & Simonett [32], Prüss [46] et Amann [3, 4]. Dans ces travaux, mais aussi dans [19, 34, 35, 52, 53] (la liste n'est pas exhaustive), les auteurs utilisent essentiellement le théorème de point fixe de Banach afin de prouver l'existence locale et l'unicité des solutions.
Dans [8, 9], Angenent a remarqué que la régularité maximale pour les problèmes linéaires donne, en plus de l'existence locale et l'unicité des solutions, une dépendence régulière des solutions en temps. En fait, les solutions ont la même régularité que la fonction F. Ainsi l'équation (9) est similaire à une équation différentielle ordinaire pour laquelle ce genre de résultat est classique. Afin d'atteindre cet objectif, Angenent applique en plus du théorème de point fixe de Banach, le théorème des fonctions implicites (pour plus de detail, voir [32]).
Dans ce travail, on utilise le théorème d'inversion locale afin de résoudre l'équation du raccourcissement des courbes

$$\begin{cases} u_t - \dfrac{1}{\mid u_x \mid}(\dfrac{u_x}{\mid u_x \mid})_x = 0 & \text{in} \quad [0, T] \times \mathbb{R}, \\ u(t, x) = u(t, x + 2\pi) & \text{for} \quad (t, x) \in [0, T] \times \mathbb{R}, \\ u(0, x) = u_0(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{10}$$

L'application du théorème d'inversion locale représente un argument plus simple que celui du théorème de point fixe de Banach et fournit à la fois une dépendence régulière des solutions des données (donnée initiale, second membre). La régularité en temps des solutions est obtenue sans l'utilisation du théorème des fonctions implicites, mais c'est plutôt une conséquence de la dépendence régulière des solutions des données. Lors de l'application de ce théorème, on est amené naturellement à résoudre un problème de régularité maximale qui se traduit par le fait qu'un certain opérateur linéaire est un isomorphisme entre deux espaces de Banach bien appropriés.
L'équation du raccourcissement des courbes est parmi les équations les plus

simples dans les équations des flots géométriques. Les propriétés analytiques de cette équation et ses applications en physique et en théorie d'images étaient largement étudiés dans la littérature.
On trouve dans la littérature plusieurs approches permettant la résolution de l'équation (10). Parmi ces approches, on cite à titre d'exemples la théorie géométrique des mesures (Brakke [20]), la théorie des équations quasilinéaires paraboliques (se réferer à Ladyzhenskaya [38] et DeTurck [28], et aussi à Huisken & Polden [36]).

Dans le chapitre 3, on s'intérèsse à la résolution de l'équation algébrique

$$\phi(u) = v, \tag{11}$$

où $\phi : X \to Y$ est une fonction continûement différentiable et X et Y sont deux espaces de Banach.
La résolution du problème (11) représente l'un des problèmes fondamentaux en mathématique. Plusieurs méthodes peuvent-être utilisées afin de résoudre ce problème. On cite à titre d'exemples, les méthodes de points fixes, les méthodes de degrés topologiques, les méthodes variationnelles,...
A fin d'atteindre cet objectif, on choisit d'utiliser la méthode de Newton sous sa variante continue

$$\begin{cases} v'(t) = -\,\phi'(v(t))^{-1}(\phi(v(t))), \quad t \geq 0, \\ v(0) = v_0. \end{cases} \tag{12}$$

Par rapport aux méthodes mentionées ci-dessus, une démonstration d'existence utilisant la méthode continue de Newton a l'avantage d'être plus constructive.
Sous certaines hypothèses sur ϕ, on montre que la convergence de la méthode de Newton (12), c'est-à-dire l'existence des solutions pour le problème (12), implique l'existence des solutions pour l'équation (11). Ceci nous conduit à constater qu'une démonstration qui utilise la méthode continue de Newton à l'avantage d'être constructive.
Le résultat suivant donne des conditions suffisantes pour qu'une fonction $\phi : X \to Y$ soit un C^1-difféomorphisme et par conséquent sous ces conditions, le problème (11) admet une solution :

Théorème 0.0.3. *Soit $\phi : X \to Y$ une fonction continûement différentiable vérifiant :*

(i) ϕ est coércive, en le sens que,

$$\lim_{\|u\|_X \to \infty} \|\phi(u)\|_Y = \infty,$$

(ii) $\phi'(u)^{-1}$ *existe pour tout* $u \in X$*,*

(iii) la fonction $u \to \phi'(u)^{-1}$ *est bornée sur les bornés.*

Alors ϕ *est un* C^1*-difféomorphisme de* X *dans* Y*.*

Comme application du Théorème 0.0.3, on a pu trouver des solutions du problème périodique non-linéaire suivant :

$$\begin{cases} u \in W^{1,p}(0, 2\pi; \mathbb{R}^N), \\ u'(t) + F(u(t)) = f(t), \;\; t \in (0, 2\pi), \\ u(0) = u(2\pi), \end{cases} \tag{13}$$

où $F : \mathbb{R}^N \to \mathbb{R}^N$ est une fonction continûement différentiable, $p \geq 2$ et $f \in L^2(0, 2\pi; \mathbb{R}^N)$ sont données.

On peut ré-écrire l'équation (13) d'une façon abstraite comme une équation algébrique entre les espaces $X = \{u \in W^{1,p}(0, 2\pi; \mathbb{R}^N) \,:\, u(0) = u(2\pi)\}$ et $Y = L^p(0, 2\pi; \mathbb{R}^N)$, puis on résoud cette équation abstraite en utilisant le Théorème 0.0.3.

Pendant ce travail, on est amené à étudier des équations linéarisées périodiques et non-homogènes du type :

$$\begin{cases} u'(t) + A(t)u(t) = f(t), \;\; t \in (0, 2\pi), \\ u(0) = u(2\pi), \end{cases} \tag{14}$$

où $A(t) : \mathbb{R}^N \to \mathbb{R}^N$ est un opérateur linéaire.

Lors de l'étude du problème (13), l'existence et l'unicité de solutions pour le problème (14) est équivalente au fait que l'hypothèse (i) dans le Théorème 0.0.3 est satisfaite.

Le problème (14) a été étudié par W. Arendt et P. Rabier dans [14] dans un cas plus général lorsque $A(t) : D \to X$ est un opérateur non borné. Dans ce travail, les auteurs donnent des conditions suffisantes pour que l'opérateur $\frac{d}{dt} + A(\cdot) : W^{1,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D) \to L^p(0, 2\pi; X)$ soit inversible.

Vu que le problème (13) est non linéaire, il est tout à fait naturel d'avoir des hypothèses supplémentaires sur F pour assurer l'existence de solutions pour le problème (13) en plus des hypothèses qui garantissent l'existence de solutions pour le problème (14). Ces hypothèses supplémentaires assurent le fait que la condition (ii) dans le Théorème 0.0.3 soit satisfaite.

Afin de prouver que la condition (iii) soit satisfaite dans le cas du problème (13), on utilise essentiellement le fait qu'un problème linéarisé associé au problème (13) admet une unique solution.

Dans [54], l'auteur étudie le problème

$$\begin{cases} u'(t) = f(t, u(t)), \;\; t \in (0, T), \\ u(0) = u(T), \end{cases} \tag{15}$$

où $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction continue qui est périodique par rapport à la première variable. Il utilise une méthode de points fixes pour montrer l'existence de solutions pour ce problème.

Chapitre 1

Global existence and maximal regularity of solutions of gradient systems

Abstract : In this article, we use a Galerkin method to prove a maximal regularity result for the following abstract gradient system

$$\begin{cases} u'(t) + \nabla_{g(t)} E(u(t)) = f(t) \quad \text{for a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases}$$

This abstract result is applied to nonlinear diffusion equations and to non-degenerate quasilinear parabolic equations with nonlocal coefficients.

1.1 Introduction

In this work, we prove a global existence and maximal regularity result of solutions of an abstract gradient system.
In order to formulate the problem, let V be a real Banach space and

let H be a real Hilbert space such that V is densely and compactly embedded in H. Let $E : V \to \mathbb{R}$ be a continuously differentiable function. Consider the gradient system

$$\begin{cases} u'(t) + \nabla_g E(u(t)) = f(t) \quad \text{for a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u_0 \in V$, $f \in L^2(0,T;H)$ are given and $\nabla_g E$ denotes the gradient of E with respect to some metric g.
Under suitable conditions on V, g and E, we prove a maximal regularity

result for system (1.1), in the sense that, for every $u_0 \in V$ and every $f \in L^2(0,T;H)$, there exists $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ such that $u(t) \in D(\nabla_g E)$ for almost every $t \in (0,T)$, which is a solution of system (1.1). In particular, the two terms on the left-hand side of the above system have the same regularity as the right-hand side term.
Note that every function in $W^{1,2}(0,T;H)$ is continuous with values in H, which implies that the initial condition in system (1.1) makes sense.

Several authors have studied abstract gradient systems in various frameworks. In the case where the metric g is constant, that is, independent of u, we mention first J.-L. Lions [39, chapitre 2] who proved a maximal regularity result in V', that is, for right-hand sides $f \in L^{p'}(0,T;V')$ and initial values in H. The theory of subgradients gives maximal regularity results in H and includes the results obtained here if g is constant and E is convex ; two references are [18] and [50]. However, in both approaches it is not clear how to include general metrics ; concrete examples in which it is necessary to consider general metrics arise, for example in geometric evolution problems, that is, in the evolution of curves and surfaces. Usually such problems fall within the theory of quasilinear evolution problems and can be solved by maximal regularity results for linear problems and fixed point theorems ; see [7], [38] and [40] ; however, these approaches seem to fail if degenerate operators (p-Laplace operators) are involved. It is possible that our result is covered by the theory of gradient systems on general metric spaces (see [15]), however if it is, our approach by space discretization (Faedo-Galerkin approximation) is certainly not included, but probably important from the point of view of numerical analysis. Moreover, our result covers also nonautonomous problems in which the metric g depends on time and is only measurable in time.
We apply our result to the partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\,\Delta_p u = f & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\,(a(\cdot,u)\nabla u) = f & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

in which the coefficient m is measurable with respect to first two variables and continuous with respect to the third variable and the coefficient a is

measurable with respect to the first variable and satisfies a weak continuity assumption with respect to the second variable (see Examples 1 and 3).
Equation (1.2) has recently been considered by W. Arendt and R. Chill [12] in the special case $p = 2$. Like in [12], we can in general not prove uniqueness, neither for the abstract system (1.1), nor for the problems (1.2) and (1.3). The problem of uniqueness is open.

1.2 The main result

Let V be a real, separable and reflexive Banach space with norm $\|\cdot\|_V$, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and induced norm $\|\cdot\|_H$, and assume that V is densely and continuously embedded into H. The duality bracket between the dual space V' and V is denoted by $\langle\cdot,\cdot\rangle_{V',V}$. Let Inner(H) be the set of all inner products on H which are equivalent to the fixed one on H.

Definition 1.2.1. *A function $g : V \to \text{Inner(H)}$ is a* metric *if it satisfies the weak continuity condition*

$$u_n \rightharpoonup u \text{ in } V \Rightarrow \langle w, v\rangle_{g(u_n)} \to \langle w, v\rangle_{g(u)} \text{ for every } v, w \in H.$$

We denote by $\langle\cdot,\cdot\rangle_{g(u)}$ the inner product $g(u)$ at a point $u \in V$ and by $\|\cdot\|_{g(u)}$ the norm associated with this inner product.
Let $E : V \to \mathbb{R}$ be a continuously differentiable function. We denote by E' the Fréchet-derivative of E. Recall that the Fréchet-derivative of E at a point u is an element of V'.

Definition 1.2.2. *We define the* gradient *of E in H with respect to the metric g by*

$$D(\nabla_g E) = \{u \in V : \exists\, w \in H\, \forall\, v \in V, E'(u)\, v = \langle w, v\rangle_{g(u)}\},$$
$$\nabla_g E(u) = w.$$

Note that since V is densely embedded into H, the element $\nabla_g E(u)$ is uniquely determined.
Throughout the rest of this article we actually consider time dependent metrics, that is, given $T > 0$, we consider a function $g : [0, T] \times V \to \text{Inner(H)}$ such that $g(t, \cdot)$ is a metric for every $t \in [0, T]$. We define the gradient of the function E with respect to the metric $g(t, \cdot)$ like in the last definition and we denote it accordingly by $\nabla_{g(t)} E$.

Let $T > 0$, $f \in L^2(0,T;H)$ and $u_0 \in V$. We consider the problem of finding u such that

$$\begin{cases} u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \ u(t) \in D(\nabla_{g(t)}E) \text{ for a.e. } t \in (0,T), \\ u'(t) + \nabla_{g(t)}E(u(t)) = f(t) \text{ for a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{1.4}$$

We call the evolution equation in (1.4) an *abstract gradient system.*
By using the density of V in H, the separability of V and the definition of the gradient $\nabla_{g(t)}E$, problem (1.4) is equivalent to the variational problem of finding u such that

$$\begin{cases} u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \\ \langle u'(t), v\rangle_{g(t,u(t))} + E'(u(t))\,v = \langle f(t), v\rangle_{g(t,u(t))} \\ \qquad\qquad \text{for every } v \in V, \text{ and for a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{1.5}$$

Definition 1.2.3. *A function $E : V \to \mathbb{R}$ is H-elliptic if there exists $\omega \in \mathbb{R}$ such that the function $E_\omega : V \to \mathbb{R}$, $u \to E(u) + \frac{\omega}{2}\|u\|_H^2$ is*

(a) coercive : for every $R > 0$ there exists $C_R > 0$ such that for every $u \in V$ if $E_\omega(u) \le R$ then $\|u\|_V \le C_R$,

(b) convex : for every u, $v \in V$, $t \in [0,1]$, $E_\omega((1-t)u + tv) \le (1-t)E_\omega(u) + tE_\omega(v)$.

The main result of this article is the following theorem :

Theorem 1.2.4. *Suppose that V is a reflexive, separable Banach space which is compactly and densely embedded into H. Suppose that E is an H-elliptic, continuously differentiable function such that the derivative $E' : V \to V'$ maps bounded sets into bounded sets.*
Let $T > 0$. Suppose further that $g(t,\cdot)$ is a metric for every $t \in [0,T]$, and for every v, $w \in H$, $u \in V$, the function $t \to \langle v, w\rangle_{g(t,u)}$ is measurable on $[0,T]$. Suppose in addition that there exist two constants c_1, $c_2 > 0$ such that for every $u \in V$, every $v \in H$ and for every $t \in [0,T]$

$$c_1\|v\|_H \le \|v\|_{g(t,u)} \le c_2\|v\|_H. \tag{1.6}$$

Suppose finally that the metric $g(t,\cdot)$ is continuous in the sense that

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } W^{1,2}(0,T;H), \\ u_n \overset{\text{weak}*}{\to} u \text{ in } L^\infty(0,T;V), \\ v_n \rightharpoonup v \text{ in } L^2(0,T;H), \\ w \in L^2(0,T;H) \end{array}\right\} \Rightarrow \int_0^T \langle v_n, w\rangle_{g(t,u_n)}\,dt \to \int_0^T \langle v, w\rangle_{g(t,u)}\,dt. \tag{1.7}$$

Then, for every $u_0 \in V$ and every $f \in L^2(0,T;H)$, problem (1.4) admits a solution.

Remark 1.2.5. *We remark that Theorem 1.2.4 is an L^2-maximal regularity result for the nonlinear problem (1.4) in the sense that for every $f \in L^2(0,T;H)$ and every $u_0 \in V$, problem (1.4) admits a solution u (however not necessarily unique) such that the two members u' and $\nabla_{g(\cdot)}E(u)$ of the left-hand side of the evolution equation in problem (1.4) belong also to $L^2(0,T;H)$. Compare with the definition of the L^2-maximal regularity of linear problems in [10, 29, 37].*

Remark 1.2.6. *If $g(t,\cdot)$ is a metric for every $t \in [0,T]$, and if we suppose that assumption (1.6) holds then*

$$\left.\begin{array}{lll} u_n \to u & \text{in } V, \\ v_n \to v & \text{in } H, \\ w_n \to w & \text{in } H \end{array}\right\} \Rightarrow \langle v_n, w_n\rangle_{g(t,u_n)} \to \langle v, w\rangle_{g(t,u)}, \text{ for every } t \in [0,T].$$

In fact, this assertion is a simple consequence of the following inequality

$$\begin{aligned} |\langle v_n, w_n\rangle_{g(t,u_n)} - \langle v, w\rangle_{g(t,u)}| \le |\langle v, w\rangle_{g(t,u_n)} - \langle v, w\rangle_{g(t,u)}| + \\ + |\langle v_n, w_n - w\rangle_{g(t,u_n)}| + |\langle v_n - v, w\rangle_{g(t,u_n)}|. \end{aligned}$$

Remark 1.2.7. *Let $u : [0,T] \to V$, $v : [0,T] \to H$ and $w : [0,T] \to H$ be three measurable functions and assume that $g(t,\cdot)$ is a metric for every $t \in [0,T]$ which satisfies the measurability condition in Theorem 1.2.4. Then $t \to \langle v(t), w(t)\rangle_{g(t,u(t))}$ is a measurable function on $[0,T]$. In fact, by using the measurability condition on g in Theorem 1.2.4, this assertion holds for step functions in V and H. The general case is a consequence of the definition of measurable functions as the pointwise limit of step functions, and Remark 1.2.6.*
This result shows that the terms which appear under integral sign in the continuity assumption (1.7) are measurable. This result can also be used later in the proof of Theorem 1.2.4.

1.3 Proof of Theorem 1.2.4

To prove Theorem 1.2.4, we need the following two lemmas.

Lemma 1.3.1. *Assume that the embedding $V \hookrightarrow H$ is compact. Then the embedding*

$$W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \hookrightarrow C([0,T];H)$$

is compact, too.

Proof : We refer the reader to [51, Corollary 4, page 85] for the proof of this lemma.

Lemma 1.3.2. *Let $E : V \to \mathbb{R}$ be a continuously differentiable and convex function. Then $E' : V \to V'$ is a monotone operator, that is, for every u, $v \in V$ one has*

$$\langle E'(u) - E'(v), u - v\rangle_{V',V} \geq 0.$$

Proof : For the proof of this lemma, see [39, Proposition 1.1, page 158].

Proof of Theorem 1.2.4 : To prove the theorem, we use a Galerkin approximation.

Part 1 (Formulation of the finite dimensional approximating problems) : Let (w_n) be any sequence in V such that span $\{w_n : n \geq 1\}$ is dense in V ; such a sequence exists since V is a separable space. For every $m \in \mathbb{N}$, we put

$$V_m = \operatorname{span}\{w_n,\ 1 \leq n \leq m\},$$

and we choose $u_0^m \in V_m$ such that

$$u_0 = \lim_{m\to\infty} u_0^m \quad \text{in } V.$$

There indeed exists such a sequence (u_0^m) since $\bigcup_m V_m$ is dense in V and the sequence (V_m) is increasing.

For every $m \in \mathbb{N}$, we consider the variational problem of finding u_m such that

$$\begin{cases} u_m \in W^{1,2}_{loc}([0,T); V_m), \\ \langle u_m'(t), v\rangle_{g(t,u_m(t))} + E'(u_m(t))\, v = \langle f(t), v\rangle_{g(t,u_m(t))} \\ \qquad\qquad\qquad\qquad \text{for every } v \in V_m \text{ and a.e. } t \in (0,T), \\ u_m(0) = u_0^m. \end{cases} \tag{1.8}$$

Problem (1.8) is equivalent to the problem of finding a solution u_m of the following ordinary differential equation

$$\begin{cases} u_m \in W^{1,2}_{loc}([0,T); V_m), \\ u_m'(t) + \nabla_{g_m(t)} E_m(u_m(t)) = P_m(t, u_m(t)) f(t), \quad \text{a.e. } t \in (0,T), \\ u_m(0) = u_0^m, \end{cases} \tag{1.9}$$

where E_m is the restriction of E to V_m, $g_m : [0,T] \times V_m \to \mathrm{Inner}(\mathrm{V_m})$ is the function defined for every $u \in V_m$, for every $t \in [0,T]$ and for every v, $w \in V_m$ by

$$\langle v, w\rangle_{g_m(t,u)} = \langle v, w\rangle_{g(t,u)},$$

and $P_m(t, u_m) : H \to H$ is the orthogonal projection from H onto V_m with respect to the inner product $\langle \cdot, \cdot \rangle_{g(t,u_m)}$. By the Riesz-Fréchet theorem and since V_m is finite dimensional, for every $m \in \mathbb{N}$, every $u \in V_m$ and for every $t \in [0, T]$, the gradient $\nabla_{g_m(t)} E_m(u)$ exists and belongs to V_m.
In order to obtain existence of maximal solutions for problem (1.9), we check that the function $F : (0, T) \times V_m \to V_m$, $(t, u) \to \nabla_{g_m(t)} E_m(u) - P_m(t, u) f(t)$ satisfies the Carathéodory conditions :

(a) $F(\cdot, u)$ is measurable for every $u \in V_m$,

(b) $F(t, \cdot)$ is continuous for almost every $t \in (0, T)$, and

(c) for every $(t_0, u_0) \in (0, T) \times V_m$, there exist α, $r > 0$ and $m \in L^1(t_0, t_0 + \alpha)$ such that $\|F(t, u)\| \leq m(t)$ for almost every $t \in (t_0, t_0 + \alpha)$ and every $u \in V_m$ such that $\|u - u_0\|_{V_m} < r$.

For every $u \in V_m$ and for every $t \in [0, T]$, we consider the operator $Q_m(t, u) \in \mathcal{L}(V_m)$ defined for every v, $w \in V_m$ by

$$\langle Q_m(t, u)v, w \rangle_H = \langle v, w \rangle_{g_m(t,u)} = \langle v, w \rangle_{g(t,u)}. \tag{1.10}$$

For every u, $v \in V_m$ and for every $t \in [0, T]$ one has

$$\begin{aligned} \langle Q_m(t, u) P_m(t, u) f(t), v \rangle_H &= \langle P_m(t, u) f(t), v \rangle_{g(t,u)} \\ &= \langle f(t), P_m(t, u) v \rangle_{g(t,u)} \\ &= \langle f(t), v \rangle_{g(t,u)}. \end{aligned} \tag{1.11}$$

By Remark 1.2.7, for every u, $v \in V_m$, the function $t \to \langle f(t), v \rangle_{g(t,u)}$ is measurable on $(0, T)$. Using equality (1.11), we obtain that for every u, $v \in V_m$, the function $t \to \langle Q_m(t, u) P_m(t, u) f(t), v \rangle_H$ is measurable on $(0, T)$. This proves that for every $u \in V_m$, the function $t \to Q_m(t, u) P_m(t, u) f(t)$ is weakly measurable and then measurable on $(0, T)$ since V_m is a finite dimensional space. From the definition of the operator $Q_m(t, u)$ and since $\mathcal{L}(V_m)$ is a finite dimensional space, we obtain that for every $u \in V_m$, the function $t \to Q_m(t, u)$ is measurable on $[0, T]$. Using the fact that the operator $Q_m(t, u)$ is invertible and that taking the inverse is a homeomorphism in the set of all invertible operators, we deduce that for every $u \in V_m$, the function $t \to P_m(t, u) f(t)$ is measurable on $(0, T)$.
In addition, for every u, $v \in V_m$ and for every $t \in [0, T]$

$$\begin{aligned} \langle \nabla_{g_m(t)} E_m(u), v \rangle_H &= \langle \nabla_{g_m(t)} E_m(u), Q_m(t, u)\, Q_m(t, u)^{-1} v \rangle_H \\ &= \langle \nabla_{g_m(t)} E_m(u), Q_m(t, u)^{-1} v \rangle_{g_m(t,u)} \\ &= E'(u)(Q_m(t, u)^{-1} v). \end{aligned}$$

Since $t \to Q_m(t, u)^{-1} v$ is measurable on $[0, T]$ and $E'(u)$ is a continuous linear operator on V_m, we deduce that for every $u \in V$, the function $t \to$

$\nabla_{g_m(t)} E_m(u)$ is measurable on $[0,T]$. Hence, the Carathéodory condition (a) is satisfied.
By (1.10), the operator $Q_m(t,\cdot)$ is continuous on V_m for every $t \in [0,T]$. This yields that $Q_m(t,\cdot)^{-1}$ and hence $\nabla_{g_m(t)} E_m(\cdot)$ are continuous on V_m for every $t \in [0,T]$.
Similarly, by using (1.11), we see that $P_m(t,\cdot)f(t)$ is continuous on V_m for every $t \in [0,T]$. This proves that the Carathéodory condition (b) is satisfied.
Finally, it remains to check that the Carathéodory condition (c) is satisfied. Since V_m is finite dimensional and any two norms on V_m are equivalent, it suffices to estimate $\|\nabla_{g_m(t)} E_m(u)\|_H$ and $\|P_m(t,u)f(t)\|_H$.
Let $u_0 \in V_m$, $r > 0$ and $u \in V_m$ such that $\|u - u_0\|_{V_m} < r$. Note that by (1.6) and the definition of Q_m we obtain $\|Q_m(t,u)\| \leq c_2^2$ with respect to the H-norm. Since $\langle v, w\rangle_H = \langle Q_m(t,u)^{-1} v, w\rangle_{g(t,u)}$, one has for every $v \in V_m$

$$\begin{aligned} \|Q_m(t,u)^{-1} v\|_H &\leq \frac{1}{c_1} \|Q_m(t,u)^{-1} v\|_{g(t,u)} \\ &= \frac{1}{c_1} \sup_{\|w\|_{g(t,u)} \leq 1} |\langle v, w\rangle_H| \\ &\leq \frac{1}{c_1^2} \|v\|_H, \end{aligned}$$

hence $\|Q_m(t,u)^{-1}\| \leq \frac{1}{c_1^2}$ with respect to the H-norm. Now the bounds for condition (c) follow easily from the formulas used in part (a).
Hence, by [21, Theorem 4.1, Chapter 1], problem (1.9) admits a maximal solution $u_m \in W^{1,2}_{loc}([0,T_m); V_m)$ in the sense that either $T_m = T$, or $T_m < T$ and the solution u_m can not be extended to any larger interval. For every $m \in \mathbb{N}$, let u_m be a maximal solution of (1.9).
Part 2 (Bounds for the solutions u_m of the approximating problems) : We take $v = u_m'$ in equation (1.8). Then we integrate over the interval $(0,t)$, for $t \in (0,T_m)$, and we obtain

$$\int_0^t \|u_m'(s)\|^2_{g(s,u_m(s))}\, ds + E(u_m(t)) - E(u_0^m) = \int_0^t \langle f(s), u_m'(s)\rangle_{g(s,u_m(s))}\, ds.$$

Since $u_0^m \to u_0$ in V, and since E is continuous, we have $\lim_{m\to\infty} E(u_0^m) = E(u_0)$ and in particular the sequence $(E(u_0^m))$ is bounded. Hence, there exists a constant $c_3 > 0$ which is independent of m and t such that

$$\int_0^t \|u_m'(s)\|^2_{g(s,u_m(s))}\, ds + E(u_m(t)) \leq c_3 + \int_0^t \langle f(s), u_m'(s)\rangle_{g(s,u_m(s))}\, ds.$$

We employ assumption (1.6) in order to obtain

$$c_1^2 \int_0^t \|u_m'(s)\|_H^2\, ds + E(u_m(t)) \leq c_3 + c_2^2 \int_0^t \|f(s)\|_H \|u_m'(s)\|_H\, ds.$$

By using Young's inequality, we deduce that there exists a constant $c_4 > 0$ which is independent of m and t such that

$$\frac{c_1^2}{2}\int_0^t \|u_m'(s)\|_H^2\,ds + E(u_m(t)) \le c_4.$$

Let $\omega \ge 0$ such that E_ω is convex and coercive. Then the preceding inequality can be rewritten as

$$\frac{c_1^2}{2}\int_0^t \|u_m'(s)\|_H^2\,ds + \|u_m(t)\|_H^2 + E_\omega(u_m(t)) \le c_4 + \frac{\omega+2}{2}\|u_m(t)\|_H^2. \quad (1.12)$$

Moreover, there exist two constants c_5, $c_6 > 0$ which are independent of m and t such that the following estimate holds

$$\begin{aligned}
\|u_m(t)\|_H^2 &= \|u_m(0)\|_H^2 + \int_0^t \frac{d}{ds}\|u_m(s)\|_H^2\,ds \\
&= \|u_m(0)\|_H^2 + 2\int_0^t \langle u_m'(s), u_m(s)\rangle_H\,ds \\
&\le c_5 + \frac{c_1^2}{2(\omega+2)}\int_0^t \|u_m'\|_H^2\,ds + c_6\int_0^t \|u_m\|_H^2\,ds.
\end{aligned}$$

By combining this last inequality with inequality (1.12), we have the existence of two constants c_7, $c_8 > 0$ which are independent of m and t such that

$$\frac{c_1^2}{4}\int_0^t \|u_m'(s)\|_H^2\,ds + \|u_m(t)\|_H^2 + E_\omega(u_m(t)) \le c_7 + c_8\int_0^t \|u_m(s)\|_H^2\,ds.$$

Since E_ω is continuous, convex and coercive, E_ω is bounded from below (in fact, since V is reflexive, E_ω even attains a minimum). Hence, there exists a constant $c_9 > 0$ which is independent of m and t such that the last estimate implies

$$\frac{c_1^2}{4}\int_0^t \|u_m'(s)\|_H^2\,ds + \|u_m(t)\|_H^2 \le c_9 + c_8\int_0^t \|u_m(s)\|_H^2\,ds. \quad (1.13)$$

It follows that

$$\|u_m(t)\|_H^2 \le c_9 + c_8\int_0^t \|u_m(s)\|_H^2\,ds.$$

By Gronwall's lemma, there exists a positive constant c_{10} such that

$$\sup_{m\in\mathbb{N}}\sup_{t\in(0,T_m)} \|u_m(t)\|_H^2 \le c_{10}.$$

We return to inequality (1.12), we employ this last estimate, and we have the existence of a constant c_{11} which is independent of m and t such that

$$\frac{c_1^2}{2}\int_0^t \|u_m'(s)\|_H^2\,ds + \|u_m(t)\|_H^2 + E_\omega(u_m(t)) \le c_{11}. \tag{1.14}$$

This implies

$$\sup_{m\in\mathbb{N}}\ \sup_{t\in(0,T_m)} E_\omega(u_m(t)) \le c_{11}.$$

Using the fact that E_ω is coercive, this implies that there exists a constant $c_{12} > 0$ such that

$$\sup_{m\in\mathbb{N}}\ \sup_{t\in(0,T_m)} \|u_m(t)\|_V \le c_{12}.$$

Using again the fact that E_ω is bounded from below, we deduce from inequality (1.14) that

$$\sup_{m\in\mathbb{N}} \|u_m\|_{W^{1,2}(0,T_m;H)} < \infty.$$

Since $T_m \le T$ is finite, this implies that for each $m \in \mathbb{N}$ the function u_m' is integrable on $[0, T_m)$. Hence, u_m extends to a continuous function on the closed interval $[0, T_m]$, and [21, Theorem 1.1, Chapter 2] and the definition of the maximal solution imply that this is only possible if $T_m = T$, that is, the solutions u_m are global.
From the preceding two inequalities we obtain

$$(u_m) \ \text{ is bounded in } \ W^{1,2}(0,T;H) \cap L^\infty(0,T;V).$$

By assumption, E' maps bounded sets into bounded sets, so that the boundedness of (u_m) in $L^\infty(0,T;V)$ implies that

$$(E'(u_m)) \ \text{ is bounded in } \ L^\infty(0,T;V').$$

Part 3 (Extracting a convergent subsequence) : Since V is a reflexive space, the space $L^\infty(0,T;V)$ is isometrically isomorphic to the dual space of $L^1(0,T;V')$. Moreover $L^1(0,T;V')$ is a separable space since V' is a separable space. Then by the Banach-Alaoglu theorem and by Lemma 1.3.1, we can extract from (u_m) a sequence (which we denote again by (u_m)) such that

$$u_m \rightharpoonup u \quad \text{in} \quad W^{1,2}(0,T;H), \tag{1.15}$$

$$u_m \overset{w*}{\to} u \quad \text{in} \quad L^\infty(0,T;V), \tag{1.16}$$

$$u_m \to u \quad \text{in} \quad C([0,T];H), \text{ and} \tag{1.17}$$

$$E'(u_m) \overset{w*}{\to} \chi \quad \text{in} \quad L^\infty(0,T;V'). \tag{1.18}$$

Part 4 (Showing that the limit u is a solution of problem (1.4)) : Let $w \in V_m$ and $\varphi \in L^2(0,T)$. Then for every $n \geq m$ we have from equation (1.8)

$$\int_0^T \langle u_n', \varphi(t)\, w\rangle_{g(t,u_n)}\, dt + \int_0^T E'(u_n)\varphi(t)\, w\, dt = \int_0^T \langle f(t), \varphi(t)\, w\rangle_{g(t,u_n)}\, dt.$$

Letting $n \to \infty$ in this last equality, and using (1.15), (1.18) and the continuity assumption (1.7), we obtain

$$\int_0^T \langle u', \varphi(t)\, w\rangle_{g(t,u)}\, dt + \int_0^T \langle \chi, \varphi(t)\, w\rangle_{V',V}\, dt = \int_0^T \langle f(t), \varphi(t)\, w\rangle_{g(t,u)}\, dt. \tag{1.19}$$

Using the fact that $\{\varphi(\cdot)\, w,\ w \in \bigcup_m V_m,\ \varphi \in L^2(0,T)\}$ spans a dense subspace of $L^2(0,T;V)$, equality (1.19) implies for every $v \in L^2(0,T;V)$

$$\int_0^T \langle u', v\rangle_{g(t,u)}\, dt + \int_0^T \langle \chi, v\rangle_{V',V}\, dt = \int_0^T \langle f(t), v\rangle_{g(t,u)}\, dt. \tag{1.20}$$

We take $v = u \in L^2(0,T;V)$ in equality (1.20) and we obtain

$$\int_0^T \langle u', u\rangle_{g(t,u)}\, dt + \int_0^T \langle \chi, u\rangle_{V',V}\, dt = \int_0^T \langle f(t), u\rangle_{g(t,u)}\, dt. \tag{1.21}$$

We have also from equation (1.8)

$$\int_0^T E'(u_n)\, u_n\, dt = \int_0^T \langle f(t), u_n\rangle_{g(t,u_n)}\, dt - \int_0^T \langle u_n', u_n\rangle_{g(t,u_n)}\, dt. \tag{1.22}$$

The continuity assumption (1.7) and (1.15) imply

$$\int_0^T \langle f(t), u_n\rangle_{g(t,u_n)}\, dt \to \int_0^T \langle f(t), u\rangle_{g(t,u)}\, dt. \tag{1.23}$$

One has the following equality

$$\int_0^T \langle u_n', u_n\rangle_{g(t,u_n)}\, dt = \int_0^T \langle u_n', u\rangle_{g(t,u_n)}\, dt + \int_0^T \langle u_n', u_n - u\rangle_{g(t,u_n)}\, dt. \tag{1.24}$$

Using again the continuity assumption (1.7) and (1.15), we obtain

$$\int_0^T \langle u_n', u\rangle_{g(t,u_n)}\, dt \to \int_0^T \langle u', u\rangle_{g(t,u)}\, dt. \tag{1.25}$$

By using the Cauchy-Schwarz inequality, assumption (1.6) and the fact that (u_n') is bounded in $L^2(0,T;H)$, there exists a constant $c_{13} > 0$ which is independent of n such that

$$\begin{aligned} |\int_0^T \langle u_n', u_n - u\rangle_{g(t,u_n)}\, dt| &\leq (\int_0^T \|u_n'\|^2_{g(t,u_n)} dt)^{\frac{1}{2}} (\int_0^T \|u_n - u\|^2_{g(t,u_n)} dt)^{\frac{1}{2}} \\ &\leq c_2^2 (\int_0^T \|u_n'\|^2_H dt)^{\frac{1}{2}} (\int_0^T \|u_n - u\|^2_H dt)^{\frac{1}{2}} \\ &\leq c_{13} (\int_0^T \|u_n - u\|^2_H dt)^{\frac{1}{2}}. \end{aligned}$$

Using (1.17), the preceding inequality implies

$$\int_0^T \langle u_n', u_n - u\rangle_{g(t,u_n)}\, dt \to 0.$$

By combining this convergence, (1.25) and (1.24), we deduce that

$$\int_0^T \langle u_n', u_n\rangle_{g(t,u_n)}\, dt \to \int_0^T \langle u', u\rangle_{g(t,u)}\, dt.$$

This convergence, (1.23), (1.22) and equality (1.21) yield

$$\int_0^T E'(u_n)\, u_n\, dt \to \int_0^T \langle \chi, u\rangle_{V',V}\, dt. \tag{1.26}$$

We have the following equality

$$\int_0^T E_\omega'(u_n)\, u_n\, dt = \int_0^T E'(u_n)\, u_n\, dt + \omega \int_0^T \|u_n\|^2_H\, dt.$$

This implies after using (1.17) and (1.26)

$$\begin{aligned} \int_0^T E_\omega'(u_n)\, u_n\, dt &\to \int_0^T \langle \chi, u\rangle_{V',V}\, dt + \omega \int_0^T \langle u, u\rangle_H\, dt \\ &= \int_0^T \langle \chi + \omega u, u\rangle_{V',V}\, dt. \end{aligned} \tag{1.27}$$

Let $v \in L^\infty(0,T;V)$ and $\lambda \in \mathbb{R}$. By applying Lemma 1.3.2 to the function E_ω and by integrating over $(0,T)$ we have

$$\int_0^T \langle E_\omega'(u_n), u_n - u - \lambda v\rangle_{V',V}\, dt \geq \int_0^T \langle E_\omega'(u + \lambda v), u_n - u - \lambda v\rangle_{V',V}\, dt.$$

Letting $n \to \infty$ in this last inequality, we obtain after using (1.16), (1.18) and (1.27) that

$$\int_0^T \langle \chi + \omega u, \lambda v \rangle_{V',V}\, dt \leq \int_0^T \langle E'_\omega(u + \lambda v), \lambda v \rangle_{V',V}\, dt.$$

We divide by $\lambda > 0$, let $\lambda \to 0^+$, and we use the continuity of E' in order to obtain

$$\int_0^T \langle \chi, v \rangle_{V',V}\, dt \leq \int_0^T \langle E'(u), v \rangle_{V',V}\, dt.$$

Since $v \in L^\infty(0, T; V)$ is arbitrary, this implies that

$$E'(u) = \chi.$$

Then equality (1.20) becomes for every $v \in L^2(0, T; V)$

$$\int_0^T \langle u', v \rangle_{g(t,u)}\, dt + \int_0^T E'(u)\, v\, dt = \int_0^T \langle f(t), v \rangle_{g(t,u)}\, dt.$$

This implies that the function u satisfies the evolution equation of system (1.4).

Finally, we check that the function u satisfies the initial condition of system (1.4). Since the point evaluation in 0 from $W^{1,2}(0, T; H)$ into H is bounded and linear, it maps weakly convergent sequences into weakly convergent sequences, one has $u_n(0) \rightharpoonup u(0)$ in H. Since $u_n(0) = u_0^n \to u_0$ in V by the choice of (u_0^n), we obtain that $u(0) = u_0$.

1.4 Applications

Example 1

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $1 < p < \infty$ such that $p > \dfrac{2N}{N+2}$. Let $\varepsilon \in (0, 1)$ and let

$$m : [0, T] \times \Omega \times \mathbb{R} \to [\varepsilon, \frac{1}{\varepsilon}]$$

be a measurable function such that $m(t, x, \cdot)$ is continuous for every $(t, x) \in [0, T] \times \Omega$.

We consider the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t, \cdot, u)\, \Delta_p u = f & \text{in} \quad (0, T) \times \Omega, \\ u = 0 & \text{on} \quad (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.28}$$

where Δ_p is the p-Laplace operator. This equation can be rewritten as a gradient system.
We put
$$V = W_0^{1,p}(\Omega),$$
which is a reflexive and separable Banach space for the norm
$$\|u\|_V = \|\nabla u\|_{L^p(\Omega)^N}.$$
Let $E : V \to \mathbb{R}$ be the function defined for every $u \in V$ by
$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx.$$
We let
$$H = L^2(\Omega),$$
equipped with the usual inner product and norm.
By the Fubini-Lebesgue theorem, for every $u \in W_0^{1,p}(\Omega)$, there exists a set $N_u \subset (0,T)$ of measure zero such that for every $t \in (0,T) \setminus N_u$, the function $\frac{1}{m(t,\cdot,u(\cdot))}$ is measurable on Ω. Since $W_0^{1,p}(\Omega)$ is separable and since $\frac{1}{m}$ is continuous with respect to the third variable, we may construct a set $N \subset (0,T)$ of measure zero such that for every $t \in (0,T) \setminus N$ and every $u \in W_0^{1,p}(\Omega)$, the function $\frac{1}{m(t,\cdot,u(\cdot))}$ is measurable on Ω. We may therefore consider the function $g : (0,T) \times V \to \text{Inner(H)}$ defined for every $t \in (0,T) \setminus N$, every $u \in W_0^{1,p}(\Omega)$ and every v, $w \in H$ by
$$\langle v, w \rangle_{g(t,u)} = \int_\Omega v \, w \, \frac{dx}{m(t,x,u(x))}.$$
We note that g is only defined for almost every $t \in (0,T)$; for $t \in N$, one might set $\langle \cdot, \cdot \rangle_{g(t,u)} = \langle \cdot, \cdot \rangle_{L^2(\Omega)}$.
Define the p-Laplace operator with Dirichlet boundary conditions on $L^2(\Omega)$ by
$$\begin{aligned} D(\Delta_p) &= \{u \in W_0^{1,p}(\Omega) : \exists\, w \in L^2(\Omega)\, \forall\, v \in W_0^{1,p}(\Omega), \\ &\qquad \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = -\int_\Omega w \, v \, dx\}, \\ \Delta_p u &= w. \end{aligned}$$
With this definition we have for every $u \in D(\Delta_p)$, $v \in V$ and for almost every $t \in (0,T)$
$$\begin{aligned} E'(u)\, v &= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \\ &= -\int_\Omega (\Delta_p u)\, v \, dx \\ &= \langle -m(t,\cdot,u)\, \Delta_p u, v \rangle_{g(t,u)}, \end{aligned}$$

that is, $u \in D(\nabla_{g(t)}E)$ and

$$\nabla_{g(t)}E(u) = -m(t,\cdot,u)\,\Delta_p u.$$

Similarly, one proves that $D(\nabla_{g(t)}E) \subseteq D(\Delta_p)$, and hence $D(\nabla_{g(t)}E) = D(\Delta_p)$.

Corollary 1.4.1. *For every $f \in L^2(0,T;L^2(\Omega))$ and every $u_0 \in W_0^{1,p}(\Omega)$, problem (1.28) admits a solution $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega))$ such that $u(t) \in D(\Delta_p)$ for almost every $t \in (0,T)$.*

Proof of Corollary 1.4.1 : It suffices to check that the assumptions of Theorem 1.2.4 are satisfied. Since $p > \dfrac{2N}{N+2}$, we obtain by the Rellich-Kondrachov theorem [17, Théorème IX.16, page 169] that $W_0^{1,p}(\Omega)$ is compactly embedded into $L^2(\Omega)$. The function E_ω is coercive and convex for all $\omega \geq 0$, so that E is an H-elliptic function. We have also by Hölder's inequality, for every $R \geq 0$, and for every u, $v \in V$ such that $\|u\|_V \leq R$

$$\begin{aligned} |E'(u)\,v| &= \Big|\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx\Big| \\ &\leq \int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx \\ &\leq \|\nabla u\|_{L^p(\Omega)^N}^{p-1}\|\nabla v\|_{L^p(\Omega)^N} \\ &= \|u\|_V^{p-1}\|v\|_V \\ &\leq R^{p-1}\|v\|_V. \end{aligned}$$

This proves that the derivative $E' : V \to V'$ maps bounded sets into bounded sets.
Let $(u_n) \subset V$ be such that $u_n \rightharpoonup u$ in V. Then (u_n) is bounded in V and by using the compact embedding $V \hookrightarrow H$ we can extract from (u_n) a subsequence (which we denote again (u_n)) such that $u_n \to u$ in H. This implies, after extracting a sequence again, that $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. Using the Lebesgue dominated theorem, we deduce that $g(t,\cdot)$ is a metric for every $t \in [0,T]$.
For every $u \in W_0^{1,p}(\Omega)$ and every v, $w \in L^2(\Omega)$, the function $\frac{1}{m(\cdot,\cdot,u(\cdot))}vw$ is integrable on $(0,T)\times\Omega$, then by the Fubini-Lebesgue theorem the function $\langle v,w\rangle_{g(\cdot,u)}$ is measurable on $(0,T)$.
Since m takes values in $[\varepsilon, \dfrac{1}{\varepsilon}]$, we obtain for every $t \in [0,T]$ and for every $v \in H$

$$\sqrt{\varepsilon}\,\|v\|_H \leq \|v\|_{g(t,u)} \leq \frac{1}{\sqrt{\varepsilon}}\,\|v\|_H.$$

Hence, assumption (1.6) is satisfied.
We let

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } W^{1,2}(0,T;L^2(\Omega)),\\
&u_n \overset{w*}{\rightharpoonup} u \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)),\\
&v_n \rightharpoonup v \text{ in } L^2(0,T;L^2(\Omega)), \text{and}\\
&w \in L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Since the embedding $C([0,T];H) \hookrightarrow L^2(0,T;H)$ is continuous, it follows from Lemma 1.3.1 that the embedding

$$W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$$

is compact. Hence we can extract from (u_n) a subsequence (which we denote again by (u_n)) such that

$$u_n \to u \text{ in } L^2(0,T;L^2(\Omega)).$$

Then we get (after passing to a subsequence again)

$$u_n(t,x) \to u(t,x) \text{ for a.e. } (t,x) \in (0,T)\times\Omega.$$

Since m is continuous with respect to the third variable and bounded away from 0, this implies

$$\frac{1}{m(t,x,u_n(t,x))} \to \frac{1}{m(t,x,u(t,x))} \text{ for a.e. } (t,x) \in (0,T)\times\Omega.$$

By the dominated convergence theorem, we obtain

$$\frac{1}{m(\cdot,\cdot,u_n)}\, w \to \frac{1}{m(\cdot,\cdot,u)}\, w \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence

$$\int_0^T \langle v_n, w\rangle_{g(t,u_n)}\, dt \to \int_0^T \langle v, w\rangle_{g(t,u)}\, dt.$$

This proves the continuity assumption (1.7) and the claim follows from Theorem 1.2.4.

Example 2

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1 and let ε, m and p be like in Example 1.

We consider the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot,u)\,\Delta_p u = f & \text{in} \quad (0,T)\times\Omega, \\ |\nabla u|^{p-2}\nabla u\cdot\overrightarrow{\nu} = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.29}$$

where $\overrightarrow{\nu}$ is the outer unit normal to the boundary $\partial\Omega$.
We put

$$V = W^{1,p}(\Omega),$$

which is a reflexive and separable Banach space for the norm

$$\|u\|_V = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N}.$$

We put further

$$H = L^2(\Omega).$$

Let $E : V \to \mathbb{R}$ be the function defined for every $u \in V$ by

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx,$$

and let $g : (0,T)\times V \to \text{Inner(H)}$ be the function defined for every $u \in V$, every v, $w \in H$ and for almost every $t \in (0,T)$ by

$$\langle v, w\rangle_{g(t,u)} = \int_\Omega v\,w\,\frac{dx}{m(t,x,u)}.$$

We define further the p-Laplace operator with Neumann boundary conditions on $L^2(\Omega)$ by

$$\begin{aligned} D(\Delta_p) &= \{u \in W^{1,p}(\Omega) : \exists\, w \in L^2(\Omega)\,\forall\, v \in W^{1,p}(\Omega), \\ &\qquad\qquad \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx = -\int_\Omega w\,v\,dx\}, \\ \Delta_p u &= w. \end{aligned}$$

Note that the Neumann type boundary condition $|\nabla u|^{p-2}\nabla u\cdot\overrightarrow{\nu} = 0$ is satisfied in a weak sense for every $u \in D(\Delta_p)$. In fact, if $u \in C^1(\overline{\Omega}) \cap D(\Delta_p)$ is such that $|\nabla u|^{p-2}\nabla u \in C^1(\overline{\Omega})$, then an integration by parts shows that $|\nabla u|^{p-2}\nabla u\cdot\overrightarrow{\nu} = 0$ on the boundary.
With the above definition of the p-Laplace operator we obtain like in Example 1 that $D(\nabla_{g(t)}E) = D(\Delta_p)$ and for every $u \in D(\Delta_p)$, for almost every $t \in (0,T)$

$$\nabla_{g(t)}E(u) = -m(t,\cdot,u)\,\Delta_p u.$$

Corollary 1.4.2. *For every $f \in L^2(0,T;L^2(\Omega))$ and every $u_0 \in W^{1,p}(\Omega)$, problem (1.29) admits a solution $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,p}(\Omega))$ such that $u(t) \in D(\Delta_p)$ for almost every $t \in (0,T)$.*

Proof of Corollary 1.4.2 : We check only that E is an H-elliptic function, the other assumptions of Theorem 1.2.4 are verified like in Corollary 1.4.1.
First, there exists a constant $C > 0$ such that for every $u \in V$ one has

$$\|u\|_{L^p(\Omega)} \leq C(\|\nabla u\|_{L^p(\Omega)^N} + \|u\|_{L^2(\Omega)}). \tag{1.30}$$

In fact, if $p \leq 2$, inequality (1.30) is clearly satisfied since the embedding $L^2(\Omega) \hookrightarrow L^p(\Omega)$ is continuous. If $p > 2$, we have the following embedding

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega),$$

where the first embedding is compact and the second is continuous. By [50, Lemma 1.1, p 106], for every $\delta > 0$, there exists $C_\delta > 0$ such that for every $u \in V$ one has

$$\|u\|_{L^p(\Omega)} \leq \delta(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)^N}) + C_\delta \|u\|_{L^2(\Omega)}.$$

Hence, inequality (1.30) follows by choosing $\delta < 1$ in this last inequality. Inequality (1.30) implies that for every $\omega > 0$ the function E_ω is coercive.

Example 3

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $\varepsilon \in (0,1)$ and let

$$a : \Omega \times L^2(\Omega) \to [\varepsilon, \frac{1}{\varepsilon}]$$

be a function such that
(a) $a(\cdot, u)$ is measurable for every $u \in L^2(\Omega)$,
(b) $a(x, \cdot)$ maps weakly convergent sequences in $L^2(\Omega)$ into convergent sequences in $\mathbb{R}$ for almost every $x \in \Omega$.
Consider the following evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(\cdot,u)\nabla u) = f & \text{in} \quad (0,T) \times \Omega, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{1.31}$$

Equation (1.31) can be rewritten as a gradient system. To see this, we put

$$V = L^2(\Omega)$$

and

$$H = H^{-1}(\Omega).$$

We consider the bounded and coercive inner products $l : L^2(\Omega) \to \mathrm{Inner}(\mathrm{H}_0^1(\Omega))$ defined for every $u \in L^2(\Omega)$ and every v, $w \in H_0^1(\Omega)$ by

$$\langle v, w\rangle_{l_u} = \int_\Omega a(x,u)\nabla v(x)\cdot\nabla w(x)\,dx.$$

By the Lax-Milgram theorem, the associated operators $L_u : H_0^1(\Omega) \to H^{-1}(\Omega)$, $L_u v := \langle v, \cdot\rangle_{l_u}$ are bounded and invertible. We denote by L_u^{-1} the inverses. By integrating by parts, we obtain for every $u \in L^2(\Omega)$ and every $v \in H_0^1(\Omega)$

$$L_u v = -\,\mathrm{div}\,(a(x,u)\nabla v) \;\text{ in }\; \mathcal{D}'(\Omega).$$

Let $g : V \to \mathrm{Inner}(\mathrm{H})$ be the function defined for every $u \in V$, and every v, $w \in H$ by

$$\langle v, w\rangle_{g(u)} = \langle v, L_u^{-1} w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Let $E : V \to \mathbb{R}$ be the function defined for every $u \in L^2(\Omega)$ by

$$E(u) = \frac{1}{2}\int_\Omega |u|^2\,dx.$$

Let $u \in D(\nabla_g E)$. Then there exists $w \in H^{-1}(\Omega)$ such that for every $v \in L^2(\Omega)$ one has

$$\begin{aligned}
\int_\Omega u\,v\,dx &= E'(u)\,v \\
&= \langle w, v\rangle_{g(t,u)} \\
&= \langle v, L_u^{-1} w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\
&= \int_\Omega (L_u^{-1} w)\,v\,dx.
\end{aligned}$$

Then we obtain $u = L_u^{-1} w \in H_0^1(\Omega)$ and $\nabla_g E(u) = w = L_u u$.
Similarly one proves that $H_0^1(\Omega) \subset D(\nabla_g E)$ and hence, $D(\nabla_g E) = H_0^1(\Omega)$.

Corollary 1.4.3. *For every $f \in L^2(0,T;H^{-1}(\Omega))$ and every $u_0 \in L^2(\Omega)$, problem (1.31) admits a solution $u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap C([0,T];L^2(\Omega))$ $\cap L^2(0,T;H_0^1(\Omega))$.*

To prove Corollary 1.4.3, we need the following lemma. We omit the proof which is straightforward when using Lemma 1.3.1 and the fact that every function in $W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ is weakly continuous with values in $L^2(\Omega)$.

Lemma 1.4.4. *Let* (u_n) *be a sequence such that*

$$u_n \rightharpoonup u \quad in \quad W^{1,2}(0,T;H^{-1}(\Omega)) \quad and$$
$$u_n \overset{w*}{\to} u \quad in \quad L^\infty(0,T;L^2(\Omega)).$$

Then we have

$$u_n(t) \rightharpoonup u(t) \quad in \quad L^2(\Omega) \quad for\ every\ t \in [0,T].$$

Proof of Corollary 1.4.3 : In order to prove that there exists a solution u in the space $W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, it suffices to check that the assumptions of Theorem 1.2.4 are satisfied. Since Ω is bounded, the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact. The function E is clearly continuously differentiable, H-elliptic and the derivative $E' : V \to V'$ maps bounded sets into bounded sets.
Let $(u_n) \subset L^2(\Omega)$ such that $u_n \rightharpoonup u$ in $L^2(\Omega)$. By the continuity assumption on a, we obtain

$$a(x,u_n) \to a(x,u) \text{ for almost every } x \in \Omega. \tag{1.32}$$

Using the Cauchy-Schwarz inequality, we have for every v, $w \in H_0^1(\Omega)$ such that $\|w\|_{H_0^1(\Omega)} \leq 1$

$$\begin{aligned} |\langle v,w\rangle_{l_{u_n}} - \langle v,w\rangle_{l_u}| &\leq \int_\Omega |a(x,u_n) - a(x,u)| \nabla v \cdot \nabla w \, dx \\ &\leq \Big(\int_\Omega |a(x,u_n) - a(x,u)|^2 |\nabla v|^2 \, dx \Big)^{\frac{1}{2}}. \end{aligned}$$

Using the convergence (1.32) and the dominated convergence theorem, this implies that for every $v \in H_0^1(\Omega)$

$$L_{u_n} v \to L_u v \ \text{ in } \ H^{-1}(\Omega). \tag{1.33}$$

By the Lax-Milgram theorem, one has for every $w \in H^{-1}(\Omega)$

$$\|L_u^{-1} w\|_{H_0^1(\Omega)} \leq \frac{1}{\varepsilon} \|w\|_{H^{-1}(\Omega)}. \tag{1.34}$$

This implies for every $w \in H^{-1}(\Omega)$

$$\begin{aligned} \|L_{u_n}^{-1} w - L_u^{-1} w\|_{H_0^1(\Omega)} &= \|L_{u_n}^{-1}(L_{u_n} - L_u) L_u^{-1} w\|_{H_0^1(\Omega)} \\ &\leq \|L_{u_n}^{-1}\| \, \|(L_{u_n} - L_u) L_u^{-1} w\|_{H^{-1}(\Omega)} \\ &\leq \frac{1}{\varepsilon} \|(L_{u_n} - L_u) L_u^{-1} w\|_{H^{-1}(\Omega)} \\ &\to 0 \quad (n \to \infty). \end{aligned} \tag{1.35}$$

This yields that g is a metric.
Let $v \in H^{-1}(\Omega)$. As a consequence of (1.34), one obtains

$$\begin{aligned}\|v\|^2_{g(u)} &= \langle v, L_u^{-1} v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \\ &\le \|v\|_{H^{-1}(\Omega)} \|L_u^{-1} v\|_{H^1_0(\Omega)} \\ &\le \frac{1}{\varepsilon} \|v\|^2_{H^{-1}(\Omega)}. \end{aligned} \tag{1.36}$$

It is easy to see that for every $w \in H^1_0(\Omega)$

$$\|L_u w\| \le \frac{1}{\varepsilon} \|w\|_{H^1_0(\Omega)}.$$

Using this last estimate, one obtains for every $u \in L^2(\Omega)$ and every $v \in H^{-1}(\Omega)$, $w \in H^1_0(\Omega)$

$$\begin{aligned}\langle v, w\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} &= \langle v, L_u^{-1} L_u w\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \langle v, L_u w\rangle_{g(u)} \\ &\le \|v\|_{g(u)} \|L_u w\|_{g(u)} \le \|v\|_{g(u)} \|L_u w\|^{\frac{1}{2}}_{H^{-1}(\Omega)} \|w\|^{\frac{1}{2}}_{H^1_0(\Omega)} \le \frac{1}{\sqrt{\varepsilon}} \|v\|_{g(u)},\end{aligned}$$

if $\|w\|_{H^1_0(\Omega)} \le 1$. We have thus proved that $\sqrt{\varepsilon}\|v\|_{H^{-1}(\Omega)} \le \|v\|_{g(u)}$.
Hence, by combining this last estimate with (1.36), we obtain

$$\sqrt{\varepsilon}\, \|v\|_{H^{-1}(\Omega)} \le \|v\|_{g(u)} \le \frac{1}{\sqrt{\varepsilon}}\, \|v\|_{H^{-1}(\Omega)},$$

and assumption (1.6) of Theorem 1.2.4 is satisfied.
We let

$$\begin{aligned}&u_n \rightharpoonup u \text{ in } W^{1,2}(0,T; H^{-1}(\Omega)),\\ &u_n \overset{w*}{\to} u \text{ in } L^\infty(0,T; L^2(\Omega)),\\ &v_n \rightharpoonup v \text{ in } L^2(0,T; H^{-1}(\Omega)), \text{and}\\ &w \in L^2(0,T; H^{-1}(\Omega)).\end{aligned}$$

We want to prove assumption (1.7) of Theorem 1.2.4. More precisely, we want to prove that

$$\int_0^T \langle v_n, L_{u_n}^{-1} w\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\, dt \to \int_0^T \langle v, L_u^{-1} w\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\, dt.$$

Since by assumption we have $v_n \rightharpoonup v$ in $L^2(0,T; H^{-1}(\Omega))$, it suffices to prove that

$$L_{u_n}^{-1} w \to L_u^{-1} w \text{ in } L^2(0,T; H^1_0(\Omega)).$$

By Lemma 1.4.4, we have

$$u_n(t) \rightharpoonup u(t) \text{ in } L^2(\Omega) \text{ for every } t \in [0,T].$$

By (1.35), one has

$$L_{u_n}^{-1} w(t) \to L_u^{-1} w(t) \text{ in } H_0^1(\Omega) \text{ for a.e. } t \in (0,T).$$

Using the fact that $L_{u_n}^{-1}$ is uniformly bounded, the dominated convergence theorem for Bochner-integrable functions yields that assumption (1.7) in Theorem 1.2.4 is satisfied and we deduce from Theorem 1.2.4 that there exists $u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ which is a solution of (1.31). From system (1.31) one has for almost every $t \in (0,T)$

$$L_u u = f(t) - \frac{\partial u}{\partial t}.$$

This implies that for almost every $t \in (0,T)$

$$u = L_u^{-1}(f(t) - \frac{\partial u}{\partial t}).$$

Then we obtain

$$\|u(t)\|_{H_0^1(\Omega)} \leq \frac{1}{\varepsilon}(\|f(t)\|_{H^{-1}(\Omega)} + \|\frac{\partial u}{\partial t}(t)\|_{H^{-1}(\Omega)})$$

and hence $u \in L^2(0,T;H_0^1(\Omega))$.
Using the fact that $W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ is a subspace of $C([0,T];L^2(\Omega))$ by [30, Theorem 3, page 287], we deduce that $u \in C([0,T];L^2(\Omega))$.

Remark 1.4.5. *Problem (1.31) was studied by A.A. Ovono and A. Rougirel [43] in an elliptic framework where the coefficient a is given by*

$$a(x,u) = \beta(\int_{\Omega \cap B(x,r)} u(y)\, dy).$$

The function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous bounded and verifies $\inf_{\mathbb{R}} \beta > 0$, and $B(x,r)$ denotes the open ball of center x and radius r.
We note that in this case, the coefficient a satisfies the weak continuity used in Example 3.

Remark 1.4.6. *The choice of the energy E and the metric g in Example 3 is inspired by F. Otto's work [42] on the porous medium equation.*

Chapitre 2

Maximal regularity, the local inverse theorem, and local well-posedness for the curve shortening flow

(joint work with Ralph Chill and Eva Fašangová)

2.1 Introduction

Optimal or maximal regularity results for linear evolution equations on Banach spaces are now widely used in order to prove local existence, uniqueness, regularity of solutions of abstract nonlinear parabolic evolution equations of the form

$$\dot{u} + F(u) = f \quad \text{on } (0, T), \quad u(0) = u_0; \tag{2.1}$$

among the first articles, we cite for example Da Prato & Grisvard [25], Amann [5, 6], Angenent [8, 9], Clément & Li [23], but we mention also the monograph by Lunardi [40] and more recent works by Escher, Prüss & Simonett [32], Prüss [46] and Amann [3, 4]. In these articles, but also in [19, 34, 35, 52, 53] (the list is not exhaustive), the authors applied the contraction mapping principle in order to prove local existence and uniqueness of solutions. The contraction mapping principle is of course a standard tool in nonlinear analysis, although its application sometimes requires the verification of technical details.

In [8, 9], Angenent remarked that optimal regularity of underlying linear evolution equations not only gives local existence and uniqueness of solutions of the nonlinear equation (2.1), but also time regularity of solutions and continuous / smooth dependence of solutions on data. The solutions are as regular and the dependence is as smooth as F is, that is, equation (2.1) behaves very much like an ordinary differential equation for which analoguous results are classical. In order to achieve his goal, Angenent applied – besides the contraction mapping principle – the Implicit Function Theorem in a most elegant way (see also [32] where the so-called parameter trick was further developped).

In this article, we show that the Local Inverse Theorem may be an efficient alternative to the contraction mapping principle : by applying the Local Inverse Theorem, the proof of local existence of solutions is even simpler *and* it gives continuous / smooth dependence on data at the same time, at least in the context of L^p-maximal regularity. Let us note that an application of the Local Inverse Theorem is perhaps natural since optimal / maximal regularity just translates the fact that a certain linear operator is an isomorphism between appropriate Banach spaces. In order to avoid much abstract notation, we illustrate the approach only in the particular case of the curve shortening flow equation

$$u_t - \kappa(u) = 0, \tag{2.2}$$

but the interested reader will certainly understand how to apply the Local Inverse Theorem in different, more abstract situations. For the curve shortening flow we show in addition that the approach via the Local Inverse Theorem yields – with very little additional effort – smooth solutions. Smoothness is here obtained without the use of the Implicit Function Theorem but follows from smooth dependence on data.

We have chosen the example of the curve shortening flow because it is one of the simplest examples of geometric flows. Analytic properties of the flow and its applications in physics or image analysis have been widely studied in the literature. Moreover, local existence and uniqueness of various types of solutions for appropriate initial data is well known ; at least this question is not an issue at all even in specialized monographs (see, for example, [22, 31, 57]). Among the possible approaches to obtain short time existence of solutions, we mention geometric measure theory (Brakke [20]), the theory of quasilinear parabolic equations (here one frequently refers to Ladyzhenskaya et al. [38] in combination with a reparametrization argument by DeTurck [28], but this approach comprises also the above mentioned use of optimal / maximal regularity and the contraction mapping principle / the Implicit Function Theorem, see Huisken & Polden [36]), the level set approach in combination with the theory of viscosity solutions (Giga [33] and references therein), or variational approaches (Almgren et al. [1], Deckelnick [27], Luckhaus & Sturzenhecker [41]).

2.2 The curve shortening flow equation – functional setting

The *curve shortening flow equation* for parametrizations of closed curves is the partial differential equation

$$\begin{cases} u_t - \kappa(u) = 0 & \text{in } [0,T] \times \mathbb{R}, \\ u(t,x) = u(t,x+2\pi) & \text{for } (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Here, each $u(t,\cdot) : \mathbb{R} \to \mathbb{R}^d$ is the parametrization of a closed curve Γ_t in $\mathbb{R}^d$. By definition, a *parametrization of a closed curve* $\Gamma \subseteq \mathbb{R}^d$ is a 2π-periodic, continuously differentiable function $u : \mathbb{R} \to \mathbb{R}^d$ such that $\Gamma = \{u(x) : x \in \mathbb{R}\}$ and $\inf_{x\in\mathbb{R}} |u_x(x)| > 0$ (the latter assumption on the derivative guarantees in particular that u is locally injective and therefore an immersion).

In the following, we consider the Sobolev spaces

$$H^k_{per} := \{u \in H^k_{loc}(\mathbb{R};\mathbb{R}^d) : u(x) = u(x+2\pi)\},$$

and we denote analogously by C^k_{per} the space of all 2π-periodic, k times continuously differentiable functions. We assume that the initial value u_0 in (2.3) is a parametrization in the Sobolev space H^2_{per}. This space has the simple advantage of being a Hilbert space. Moreover, this Sobolev space is continuously embedded into C^1_{per} so that the set of all parametrizations in H^2_{per} is open. Finally, given a parametrization $u \in H^2_{per}$, we can define the associated *curvature vector field* $\kappa(u)$ by

$$\begin{aligned} \kappa(u) &:= \frac{1}{|u_x|}\Big(\frac{u_x}{|u_x|}\Big)_x \\ &= \frac{u_{xx}}{|u_x|^2} - \frac{u_x}{|u_x|}\,\langle\frac{u_{xx}}{|u_x|^2}, \frac{u_x}{|u_x|}\rangle \\ &= P^\perp \frac{u_{xx}}{|u_x|^2}. \end{aligned}$$

Here,

$$P^\perp v := v - \frac{u_x}{|u_x|}\,\langle v, \frac{u_x}{|u_x|}\rangle$$

is the orthogonal projection along the tangent space $\langle u_x\rangle$ onto the normal space along u.

2.3 Reduction of the curve shortening flow equation

Following an idea of DeTurck in [28], one may introduce reparametrizations of a solution u of the curve shortening flow equation in such a way that the reparametrizations satisfy a strictly parabolic equation (see, for example, Zhu [57]). This strictly parabolic equation can be obtained by "projecting" the time derivative of the function u into the space which is normal along u_0 (the equation thus obtained therefore depends on u_0). Here, we somehow proceed in the opposite way (see also Deckelnick [27]) : instead of projecting the curve shortening flow equation into normal direction we rather leave out the normal projection $P^\perp$ which appears in the definition of the curvature vector $\kappa(u)$. That is, instead of the curve shortening flow equation (2.3) we consider the following problem

$$\begin{cases} v_t - \dfrac{v_{xx}}{|v_x|^2} = 0 & \text{in } [0,T]\times\mathbb{R}, \\ v(t,x) = v(t,x+2\pi) & \text{for } (t,x)\in[0,T]\times\mathbb{R}, \\ v(0,x) = u_0(x) & \text{for } x\in\mathbb{R}; \end{cases} \tag{2.4}$$

(same initial value u_0 as in (2.3) !). In this problem, $(v(t,\cdot))$ is again a family of parametrizations of closed curves in $\mathbb{R}^d$. We show in this section that a smooth solution v of the problem (2.4) and a smooth solution u of the curve shortening flow equation (2.3) parametrize the same family of curves. As a consequence, if one is only interested in the evolution of the associated curves, it suffices to solve the reduced problem (2.4). For simplicity, we work only with C^∞ solutions here and we do not try to find the weakest possible regularity on v which ensures existence and uniqueness of sufficiently regular reparametrizations θ (see the following lemma).

Lemma 2.3.1. *Let $v \in C^\infty([0,T];C^\infty_{per})$ be a solution of the problem* (2.4). *Then there exists a unique function*

$$\theta \in C^\infty([0,T]\times\mathbb{R}), \quad \theta = \theta(t,x),$$

(same existence time as for v !) satisfying

$$\begin{cases} \theta_t + \dfrac{1}{|v_x(t,\theta)|}\langle \dfrac{v_{xx}(t,\theta)}{|v_x(t,\theta)|^2}, \dfrac{v_x(t,\theta)}{|v_x(t,\theta)|}\rangle = 0 & \text{in } [0,T]\times\mathbb{R}, \\ \theta(0,x) = x & \text{for } x\in\mathbb{R}. \end{cases} \tag{2.5}$$

Démonstration. For every fixed $x \in \mathbb{R}$ the equation (2.5) is an ordinary differential equation for the function $\theta(\cdot,x)$. For this ordinary differentiable equation, the classical results for local / global existence and uniqueness of solutions and smooth dependence on initial data apply and yield the claim. □

Let v and θ be as in Lemma 2.3.1 and define

$$u(t,x) := v(t,\theta(t,x)) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}.$$

Note carefully that u is 2π-periodic in the second variable. Moreover, by the chain rule and since v and θ are solutions of (2.4) and (2.5), respectively,

$$\begin{aligned} u_t(t,x) &= v_t(t,\theta(t,x)) + v_x(t,\theta(t,x))\,\theta_t(t,x) \\ &= \frac{v_{xx}(t,\theta(t,x))}{|v_x(t,\theta(t,x))|^2} - \frac{v_x(t,\theta(t,x))}{|v_x(t,\theta(t,x))|}\langle \frac{v_{xx}(t,\theta(t,x))}{|v_x(t,\theta(t,x))|^2}, \frac{v_x(t,\theta(t,x))}{|v_x(t,\theta(t,x))|}\rangle \\ &= \kappa(v(t,\theta(t,x))) \\ &= \kappa(u(t,x)). \end{aligned}$$

In the last equality we have used the equality $\kappa(v(t,\theta(t,x))) = \kappa(u(t,x))$, that is, the curvature vector at the point $v(t,\theta(t,x)) = u(t,x)$ does not depend on the particular parametrization. Since we have also that

$$u(0,x) = v(0,\theta(0,x)) = v(0,x) = u_0(x),$$

the function u defined above is indeed a solution of the curve shortening flow equation.

2.4 Existence and regularity for the reduced problem by the local inverse theorem

In this section we solve the reduced problem (2.4). More precisely, we prove existence and uniqueness of solutions which belong to the *maximal regularity space*

$$MR := H^1(0,T;H^1_{per}) \cap L^2(0,T;H^3_{per}).$$

This space is equipped with the natural norm, so that it becomes a Banach (or : Hilbert) space. One has two continuous embeddings

$$MR \subseteq C([0,T];H^2_{per}) \subseteq C([0,T];C^1_{per}).$$

The second embedding follows from the Sobolev embedding $H^2_{per} \subseteq C^1_{per}$, while the first embedding follows from interpolation theory and the fact that

H^2_{per} is the trace space (or : interpolation space) between H^1_{per} and H^3_{per} associated with the maximal regularity space MR. The subset

$$U := \{u \in MR : \inf_{(t,x)} |u_x(t,x)| > 0\}$$

is, by the above embeddings, an open subset of the maximal regularity space. For every parametrization $u_0 \in H^2_{per}$ there exists an element $u \in U$ such that $u(0) = u_0$. In fact, since H^2_{per} is the trace space associated with MR, there exists an element $\tilde{u} \in MR$ such that $\tilde{u}(0) = u_0$. Then, by a simple continuity and compactness argument, there exists $T' \in (0,T]$ such that $\inf_{(t,x)\in[0,T']\times\mathbb{R}} |\tilde{u}_x(t,x)| > 0$. Now, the function $u(t,x) = \tilde{u}(\frac{T'}{T}t, x)$ belongs to U and satisfies $u(0) = u_0$.

Theorem 2.4.1 (Local existence and smooth dependence of local solutions on data)**.** *For every parametrization $u_0 \in H^2_{per}$ and every $f \in L^2(0,T;H^1_{per})$ there exists a local existence time $T' \in (0,T]$ and a constant $r > 0$ such that for every $v_0 \in H^2_{per}$ and every $g \in L^2(0,T';H^1_{per})$ with $\|v_0 - u_0\|_{H^2_{per}} < r$ and $\|g - f\|_{L^2(0,T';H^1_{per})} < r$ the problem*

$$\begin{cases} v_t - \dfrac{v_{xx}}{|v_x|^2} = g & \text{in } [0,T'] \times \mathbb{R}, \\ v(t,x) = v(t,x+2\pi) & \text{for } (t,x) \in [0,T'] \times \mathbb{R}, \\ v(0,x) = v_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.6}$$

admits a unique solution

$$v \in H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per}).$$

Moreover, the mapping which maps every pair $(g, v_0) \in B(f,r) \times B(u_0,r)$ (the open balls in $L^2(0,T';H^1_{per})$ and H^2_{per}, respectively) to the unique solution $v \in H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per})$ is analytic (in the sense of [56, Definition 8.8, p.362]).

In particular, for every parametrization $u_0 \in H^2_{per}$ the problem (2.4) *admits a unique local solution $v \in H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per})$.*

Démonstration. Consider the function

$$G : U \to L^2(0,T;H^1_{per}) \times H^2_{per}$$
$$v \mapsto (v_t - \frac{v_{xx}}{|v_x|^2}, v(0)).$$

It is analytic in the sense of [56]. We show that G is a local diffeomorphism. Denote by G' the Fréchet derivative of G. For every $\bar{v} \in U$ and every $w \in MR$,

$$G'(\bar{v})w = (w_t - \frac{w_{xx}}{|\bar{v}_x|^2} + \frac{\bar{v}_{xx}}{|\bar{v}_x|^4}\langle \bar{v}_x, w_x\rangle, w(0)).$$

Saying that $G'(\bar{v})$ is a linear isomorphism from MR onto $L^2(0,T;H^1_{per}) \times H^2_{per}$ is then clearly equivalent to saying that for every right-hand side $h \in L^2(0,T;H^1_{per})$ and every initial value $w_0 \in H^2_{per}$ the problem

$$\begin{cases} w_t - \dfrac{w_{xx}}{|\bar{v}_x|^2} + \dfrac{\bar{v}_{xx}}{|\bar{v}_x|^4}\langle \bar{v}_x, w_x\rangle = h & \text{in } [0,T]\times\mathbb{R}, \\ w(t,x) = w(t,x+2\pi) & \text{for } (t,x)\in[0,T]\times\mathbb{R}, \\ w(0,x) = w_0 & \text{for } x\in\mathbb{R}, \end{cases} \tag{2.7}$$

admits a unique solution $w \in MR$. We take this fact for granted, or we refer to Section 2.5 below, where we briefly sketch why this linear, nonautonomous problem has L^2-maximal regularity in H^1_{per}.

Now the problem (2.6) can be solved in the following way. Given a parametrization $u_0 \in H^2_{per}$ and a function $f \in L^2(0,T;H^1_{per})$, there exists an element $\bar{v} \in U$ such that $\bar{v}(0) = u_0$. Since $G'(\bar{v})$ is linear and continuously invertible (by the above granted assumption), and by the Local Inverse Theorem [56, Theorem 4.F, p.172], there exists a neighbourhood $\bar{V} \subseteq U$ of $\bar{v}$ and a neighbourhood $\bar{W} \subseteq L^2(0,T;H^1_{per}) \times H^2_{per}$ of $G(\bar{v}) =: (\bar{f}, u_0)$ such that G is a diffeomorphism between $\bar{V}$ and $\bar{W}$. More precisely, G and its local inverse G^{-1} are analytic [56, Corollary 4.37, p.172].

Now, choose first $r > 0$ so small that $B(\bar{f}, 2r)\times B(u_0, r) \subseteq \bar{W}$, and choose then $T' \in (0,T]$ so small such that

$$\|f - \bar{f}\|_{L^2(0,T';H^1_{per})} < r;$$

here it is crucial that we work with L^2-maximal regularity, since this ensures that such a time T' exists. Let $v_0 \in H^2_{per}$ and $g \in L^2(0,T';H^1_{per})$ be such that $\|v_0 - u_0\|_{H^2_{per}} < r$ and $\|g - f\|_{L^2(0,T';H^1_{per})} < r$. We extend g by $\bar{f}$ on $(T',T]$ and we denote this extension by Eg. Then $Eg \in L^2(0,T;H^1_{per})$ and

$$\begin{aligned} \|Eg - \bar{f}\|_{L^2(0,T;H^1_{per})} &= \|g - \bar{f}\|_{L^2(0,T';H^1_{per})} \\ &\le \|g - f\|_{L^2(0,T';H^1_{per})} + \|f - \bar{f}\|_{L^2(0,T';H^1_{per})} \\ &< 2r. \end{aligned}$$

In particular, $(Eg, v_0) \in B(\bar{f}, 2r) \times B(u_0, r) \subseteq \bar{W}$. Since $G : \bar{V} \to \bar{W}$ is invertible, there exists $v \in \bar{V} \subseteq MR$ such that $G(v) = (Eg, v_0)$. By definition of G and Eg, this implies that the restriction of v to $[0,T'] \times \mathbb{R}$ is a local solution of (2.6).

The mapping which maps every

$$(g, v_0) \in B(f,r) \times B(u_0,r) \subseteq L^2(0,T';H^1_{per}) \times H^2_{per}$$

to the local solution $v \in H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per})$ is the composition of the affine extension operator E (the sum of a linear operator and a constant), the inverse G^{-1} and a linear restriction operator, and it is thus analytic.

Since the parametrization $u_0 \in H^2_{per}$ and the right-hand side $f \in L^2(0,T;H^1_{per})$ were arbitrary (so that we may take $f = 0$), the above arguments yield in particular the existence of a local solution v of (2.4). □

The smooth dependence on the data implies higher space regularity if the data are more regular, too.

Corollary 2.4.2. *Let $u_0 \in H^2_{per}$, $f \in L^2(0,T;H^1_{per})$ and $T' \in (0,T]$ be as in Theorem 2.4.1, and let $v \in H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per})$ be the local solution of* (2.6) *with $g = f$ and $v_0 = u_0$. If $u_0 \in H^{2+k}_{per}$ and $f \in L^2(0,T;H^{1+k}_{per})$ for some integer $k \geq 0$, then $v \in H^1(0,T';H^{1+k}_{per}) \cap L^2(0,T';H^{3+k}_{per}) \subseteq C([0,T'];H^{2+k}_{per})$.*

Démonstration. Note that $u_0 \in H^{2+k}_{per}$ if and only if the mapping $h \mapsto u_0(\cdot + h)$ is k times continuously differentiable from $\mathbb{R}$ into H^2_{per}. Similarly, $f \in L^2(0,T;H^{1+k}_{per})$ if and only if the mapping $h \mapsto f(\cdot, \cdot + h)$ is k times continuously differentiable from $\mathbb{R}$ into $L^2(0,T;H^1_{per})$. Since $v(\cdot, \cdot + h)$ is the (unique) solution of (2.6) with initial value u_0 replaced by $u_0(\cdot + h)$ and right-hand side f replaced by $f(\cdot, \cdot + h)$, the smooth dependence of solutions on initial data (Theorem 2.4.1) implies that the mapping $h \mapsto v(\cdot, \cdot + h)$ is k times continuously differentiable from $\mathbb{R}$ into $H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per}) \cap C([0,T'];H^2_{per})$. This gives the desired regularity. □

With little additional effort, we now show that the unique local solution v found in Theorem 2.4.1 is smooth for $t > 0$. Note that the following corollary may also be proved by applying the beautiful argument of Angenent (see [8], [32]) ; there, the Implicit Function Theorem first gives time regularity while the space regularity can for example be obtained by using the equation (2.4). Here, the smooth dependence on data implies first the space regularity, and the time regularity is obtained in the second place.

Corollary 2.4.3. *Let $v \in H^1(0,T';H^1_{per}) \cap L^2(0,T';H^3_{per})$ be a solution of the homogeneous problem* (2.4) *(the existence of a local solution is guaranteed by Theorem 2.4.1). Then*

$$v \in C^\infty((0,T'];C^\infty_{per}).$$

Démonstration. We start by showing space regularity. Note that if

$$v \in H^1_{loc}((0,T'];H^{1+k}_{per}) \cap L^2_{loc}((0,T'];H^{3+k}_{per}) \quad \text{for some } k \geq 0,$$

then, for almost every $t \in (0,T')$, $v(t) \in H^{3+k}_{per}$. By Corollary 2.4.2, this implies that $v \in H^1([t,T']; H^{2+k}_{per}) \cap L^2([t,T']; H^{4+k}_{per})$ for almost every $t \in (0,T')$, and therefore

$$v \in H^1_{loc}((0,T']; H^{2+k}_{per}) \cap L^2_{loc}((0,T']; H^{4+k}_{per}) \subseteq C((0,T']; H^{3+k}_{per}).$$

Hence, an induction on $k \geq 0$ shows that $v \in C((0,T']; C^\infty_{per})$. This regularity and the equality

$$v_t = \frac{v_{xx}}{|v_x|^2}$$

imply first that $v \in C^1((0,T']; C^\infty_{per})$, and then, by iterating this argument, that $v \in C^\infty((0,T']; C^\infty_{per})$. □

2.5 Maximal regularity for the linear, nonautonomous problem

In this section, we briefly sketch an idea why the linear problem (2.7) has L^2-maximal regularity in H^1_{per}. Let us first note that the problem (2.7) is a special case of the linear, nonautonomous problem

$$\begin{cases} w_t - m(t,x)w_{xx} + b(t,x,w_x) = h & \text{in } [0,T] \times \mathbb{R}, \\ w(t,x) = w(t,x+2\pi) & \text{for } (t,x) \in [0,T] \times \mathbb{R}, \\ w(0,x) = w_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.8}$$

where $m : [0,T] \times \mathbb{R} \to [\varepsilon, 1/\varepsilon]$ ($\varepsilon > 0$ fixed) and $b : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ are two measurable functions such that

(i) $m \in C([0,T]; H^1_{per}(\mathbb{R}))$,

(ii) b is linear in the third variable and 2π-periodic in the second variable,

(iii) $|b(t,x,p)| \leq \beta_1(t,x)\,|p|$ for some $\beta_1 \in L^2(0,T; L^\infty_{per}(\mathbb{R}))$ and every (t,x,p),

(iv) $|b_x(t,x,p)| \leq \beta_2(t,x)\,|p|$ for some $\beta_2 \in L^2(0,T; L^2_{per}(\mathbb{R}))$ and every (t,x,p).

In order to prove L^2-maximal regularity of the problem (2.8) in H^1_{per}, it is convenient to proceed in several steps and to consider the following three cases :

1. the case when $m(t,\cdot) = m(\cdot) \in H^1_{per}$ does not depend on time and b vanishes identically (autonomous case) ;
2. the case when m is arbitrary (but satisfies the conditions above) and b vanishes identically ;

3. the general case (m and b satisfy the conditions above).

The first case is of course the simplest one. In order to prove L^2-maximal regularity in H^1_{per}, it suffices to know that the operator $-mw_{xx}$ with domain H^3_{per} generates an analytic C_0-semigroup on the Hilbert space H^1_{per} and to refer to [26]. Alternatively, one can show by variational methods that the operator $-mw_{xx}$ with domain H^1_{per} has L^2-maximal regularity on H^{-1}_{per} and then to use a similarity argument.

Once the first case is settled, the cases 2 and 3 follow by perturbation arguments (using the Neumann series, for example). For the second case, due to assumption (i), one may apply either [45, Theorem 2.5], [2, Theorem 7.1], or [11, Theorem 2.7] in combination with the first case. The third, general case follows similarly [2, Theorem 7.1].

Chapitre 3

Continuous Newton's method and existence and uniqueness of solutions for periodic boundary value problems

3.1 Introduction and the main result

In this paper, we are interested in solving the algebraic equation

$$\phi(u) = v, \tag{3.1}$$

where $\phi : X \to Y$ is a continuously differentiable function and X and Y are two Banach spaces.
The problem of finding a solution for the algebraic equation (3.1) is one of the fundamental problems in mathematics. Many methods can be used to solve this problem, e.g., fixed point methods, degree theory, variational methods.
In order to solve problem (3.1), we choose to use the continuous Newton's method

$$\begin{cases} v\,'(t) = -\,\phi\,'(v(t))^{-1}(\phi(v(t))), \quad t \geq 0, \\ v(0) = v_0. \end{cases} \tag{3.2}$$

Compared with the methods mentioned above, a proof using the continuous Newton's method (3.2) has the advantage to be more constructive.
We prove that if the continuous Newton's method (3.2) converges, that is, if the problem (3.2) admits a global solution which converges to some limit as $t \to \infty$, then the algebraic equation (3.1) admits a solution (namely the limit of Newton's method).

The main result of this paper is the following theorem which gives sufficient conditions for ϕ to be a global C^1 diffeomorphism and hence for every $v \in Y$, there exists a unique $u \in X$ which is a solution of equation (3.1). This result is well known in the literature (see for example [44], [49] and [54]), and in [54], the author used also the continuous Newton's method in order to prove a global existence result between Banach spaces.

Theorem 3.1.1. *Let X and Y be two Banach spaces and let $\phi : X \to Y$ be a continuously differentiable function. We suppose that the function ϕ satisfies the following assumptions :*
(i) for every $u \in X$, $\phi'(u)$ is invertible,
(ii) the function ϕ is coercive, in the sense that,

$$\lim_{\|u\|_X \to \infty} \|\phi(u)\|_Y = \infty,$$

(iii) the function $u \to \phi'(u)^{-1}$ maps bounded sets of X into bounded sets of $\mathcal{L}(Y, X)$.
Then ϕ is a C^1 diffeomorphism from X into Y.

We note that the coercivity assumption (ii) is equivalent to the following statement : for every $R > 0$ there exists $c_R > 0$ such that for every $u \in X$ the implication

$$\|\phi(u)\|_Y \leq R \quad \Rightarrow \quad \|u\|_X \leq c_R$$

holds true. In other words, ϕ is coercive if and only if preimages of bounded sets are bounded sets.
Global homeomorphism theorems were the subject of many papers. We refer the reader to [44], [49], [54] and to the references therein for more details.

We apply Theorem 3.1.1 in order to solve the nonlinear periodic differential equation

$$\begin{cases} u \in W^{1,p}(0, 2\pi; \mathbb{R}^N), \\ u'(t) + F(u(t)) = f(t) \text{ for a.e. } t \in (0, 2\pi), \\ u(0) = u(2\pi), \end{cases} \tag{3.3}$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is a continuously differentiable function and $p \geq 2$.
Equation (3.3) can be abstractly rewritten as an algebraic equation between Banach spaces, and we use Theorem 3.1.1 in order to solve this equation.
In order to solve problem (3.3), we need to solve a linearized periodic problem of type

$$\begin{cases} u'(t) + A(t)u(t) = f(t), \quad t \in (0, 2\pi), \\ u(0) = u(2\pi), \end{cases} \tag{3.4}$$

where $A(t) : \mathbb{R}^N \to \mathbb{R}^N$ is a linear operator. Existence and uniqueness of solutions for problem (3.4) is equivalent to the fact that assumption (i) in Theorem 3.1.1 is satisfied. Problem (3.4) was studied by Arendt and Rabier [14] in a more general case where $A(t) : D \to X$ is an unbounded operator on a Banach space X. Sufficient conditions for $\frac{d}{dt} + A(\cdot) : W^{1,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D) \to L^p(0, 2\pi; X)$ to be invertible are given in [14]. In our framework (finite-dimensional case), we use the Floquet theory to show that the operator $A(t)$ is similar to an operator with constant coefficient and then we use a result due to Arendt and Bu [13] to prove that the operator $\frac{d}{dt} + A(\cdot)$ is an isomorphism between some Banach spaces.
Since problem (3.3) is nonlinear, in addition to assumptions ensuring the well-posedness of the linear problem (3.4) in [14], it is natural that we need further assumptions on the function F to guarantee the existence of solutions for problem (3.3). These additional assumptions ensure the coercivity condition (ii) in Theorem 3.1.1.
To prove that assumption (iii) is satisfied in the case of problem (3.3), we use essentially the fact that a linearized problem associated to problem (3.3) with bounded nonhomegenous terms admits a unique solution.
In [55], the author studied the problem

$$\begin{cases} u'(t) = f(t, u(t)), & t \in (0, T), \\ u(0) = u(T), \end{cases}$$

where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function which is periodic with respect to the first variable. The author presented an existence result based on the search of fixed points for some operator. We refer the reader to this work and to the references therein for more details.

3.2 Proof of Theorem 3.1.1

In this section, we give the proof of Theorem 3.1.1. For this aim, we prove in a first step that the continuous Newton's methods admits global solutions, that is solutions defined for all $t \geq 0$, and this fact implies that ϕ is injective (see the proofs of [54, Theorem 2.1 and Theorem 3.1] for more details). In a second step, we prove that the global solutions converge to a limit as $t \to \infty$, and this limit is a solution of the problem (3.1).
Proof of Theorem 3.1.1 : We consider the continuous Newton's method

$$\begin{cases} v'(t) = -\,\phi'(v(t))^{-1}(\phi(v(t))), & t \geq 0, \\ v(0) = v_0. \end{cases} \tag{3.5}$$

A function $v \in C^1([0,T],X)$ is a solution of system (3.5) if and only if

$$\phi(v(t)) = e^{-t}\phi(v_0) \text{ for every } t \in [0,T] \text{ and } v(0) = v_0. \tag{3.6}$$

In fact, if $v \in C^1([0,T],X)$ is a solution of system (3.5) then one has

$$\begin{aligned} \frac{d}{dt}(\phi(v(t))) &= \phi'(v(t))\,v'(t) \\ &= -\,\phi(v(t)). \end{aligned}$$

Hence, the function $\phi(v)$ is a solution of the differential equation

$$\begin{cases} \dfrac{d}{dt}(\phi(v(t))) = -\,\phi(v(t)),\ t \in [0,T], \\ \phi(v(0)) = \phi(v_0), \end{cases}$$

and the relation (3.6) follows.
Conversely, if a function $v \in C^1([0,T],X)$ satisfies the relation (3.6) then one has

$$\begin{aligned} \phi'(v(t))\,v'(t) &= \frac{d}{dt}(\phi(v(t))) \\ &= \frac{d}{dt}(e^{-t}\phi(v_0)) \\ &= -\,e^{-t}\phi(v_0) \\ &= -\,\phi(v(t)), \end{aligned}$$

and $v(0) = v_0$. Hence, the function v is a solution of (3.5).
We use the equivalence of the systems (3.5) and (3.6) in order to show that the continuous Newton's method (3.5) admits a solution.
Since ϕ is a local C^1 diffeomorphism by assumption (i) and the local inverse theorem, there exists an open neighborhood V_0 of v_0 and an open neighborhood W_0 of $\phi(v_0)$ such that $\phi : V_0 \to W_0$ is a C^1 diffeomorphism. Since $\lim_{t\to 0} e^{-t}\phi(v_0) = \phi(v_0)$, there exists $T > 0$ such that $e^{-t}\phi(v_0) \in W_0$ for every $t \in [0,T]$. This implies that $\phi^{-1}(e^{-t}\phi(v_0)) \in V_0$ for every $t \in [0,T]$. Now, define the function v on $[0,T]$ by

$$v(t) = \phi^{-1}(e^{-t}\phi(v_0)),\ t \in [0,T].$$

Then $v \in C^1([0,T];X)$ is a solution of (3.6) and hence of (3.5).
Let $v_{max} : [0,T_{max}) \to V$ be a maximal solution of system (3.5). Since $\phi(v(t)) = e^{-t}\phi(v_0)$ remains bounded, and by assumption (ii), the function

v_{max} is bounded on $[0, T_{max})$.
Assumption (iii) implies that there exists a positive constant M such that

$$\sup_{t\in[0,T_{max})} \|\phi'(v_{max}(t))^{-1}\| \leq M.$$

Then one obtains

$$\begin{aligned}\|v'_{max}(t)\|_X &= \|\phi'(v_{max}(t))^{-1}(\phi(v_{max}(t)))\|_X \\ &\leq M\, e^{-t}\|\phi(v_0)\|_Y.\end{aligned}$$

Hence, v'_{max} is absolutely integrable on $[0, T_{max})$ and by Cauchy's criterion, v_{max} admits a finite limit as t tends to T_{max}. This proves that $T_{max} = \infty$, that is v_{max} is a global solution, and this implies injectivity of ϕ as mentioned at the bigening of this paragraph.
Let $u = \lim_{t\to\infty} v_{max}(t)$. Then one has

$$\begin{aligned}\phi(u) &= \lim_{t\to\infty} \phi(v_{max}(t)) \\ &= \lim_{t\to\infty} e^{-t}\phi(v_0) \\ &= 0,\end{aligned}$$

and hence the limit u is the solution of the problem $\phi(u) = 0$.
The second step consists to prove that the equation $\phi(u) = v$ admits a solution. For this, it suffices to apply the first step with the function $\psi(u) = \phi(u) - v$. Hence, ϕ is surjective and we conclude that ϕ is a global C^1 diffeomorphism.

3.3 Application

We now give an example of application of Theorem 3.1.1.
Example : Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuously differentiable function and let $p \geq 2$. We consider the problem of finding u such that

$$\begin{cases} u \in W^{1,p}(0, 2\pi; \mathbb{R}^N), \\ u'(t) + F(u(t)) = f(t) \text{ for a.e. } t \in (0, 2\pi), \\ u(0) = u(2\pi), \end{cases} \tag{3.7}$$

where $f \in L^p(0, 2\pi; \mathbb{R}^N)$ is given. Recall, from the Sobolev embedding theorem, that $W^{1,p}(0, 2\pi; \mathbb{R}^N)$ is a subspace of $C([0, 2\pi]; \mathbb{R}^N)$ so that the condition $u(0) = u(2\pi)$ in problem (3.7) makes sense.
We define the Banach spaces

$$X = \{u \in W^{1,p}(0, 2\pi; \mathbb{R}^N) : \ u(0) = u(2\pi)\}$$

and

$$Y = L^p(0, 2\pi; \mathbb{R}^N).$$

We define further the Nemytskii operator

$$\begin{aligned} \mathbf{F} : X &\rightarrow Y \\ u &\rightarrow \mathbf{F}(u) \end{aligned}$$

where for every $u \in X$ and every $t \in [0, 2\pi]$

$$(\mathbf{F}(u))(t) = F(u(t)).$$

Lemma 3.3.1. *The Nemytskii operator $\boldsymbol{F}$ is well defined, continuously differentiable on X and for every $u \in X$, the derivative $\boldsymbol{F}'(u)$ is given by*

$$\boldsymbol{F}'(u)v = F'(u(\cdot))v(\cdot), \;\; v \in X.$$

Proof : For the proof of this lemma, see [48, Theorem 3.1] where the function F depends further on the time parameter in the first variable.

By the preceding lemma, for every $u \in X$, we may consider the linearized problem

$$\begin{cases} h'(t) + F'(u(t))h(t) = f(t) \text{ for a.e. } t \in (0, 2\pi), \\ h(0) = h(2\pi). \end{cases} \tag{3.8}$$

We suppose that the function F satisfies the following assumptions :

(a) For every $u \in X$, there exists a t-independent $N \times N$-matrix B_u such that $\sigma(-B_u) \cap i\mathbb{Z} = \emptyset$ and there exists an invertible matrix $\Phi(t)$ with absolutely continuous 2π-periodic elements such that

$$h(t) = \Phi(t)y(t), \;\; t \in [0, 2\pi]$$

reduces problem (3.8) to the autonomous problem

$$\begin{cases} y'(t) + B_u y(t) = f(t) \text{ for a.e. } t \in (0, 2\pi), \\ y(0) = y(2\pi). \end{cases} \tag{3.9}$$

(b) $\exists\, c_1 > 0, \forall\, x \in \mathbb{R}^N, c_1|\,x| \leq |\,F(x)|$,
(c) $\forall x \in \mathbb{R}^N, |F(x)|^2 + 2\langle F'(x)F(x), x\rangle \geq 0$,
(d) $\exists\, c_2 > 0, \forall\, x \in \mathbb{R}^N, c_2|\,F(x)| \leq e(x)$,
(e) $\exists\, c_3 > 0, \forall\, x, y \in \mathbb{R}^N, \langle F'(x)y, x\rangle \leq c_3\, e(x)|\,y|$,
where $e(x) = \sqrt{|F(x)|^2 + 2\langle F'(x)F(x), x\rangle}$,
(f) $\forall r > 0, \exists c(r) > 0, \forall x, y \in \mathbb{R}^N, |y| \leq r \Rightarrow c(r)|x|^2 \leq \langle F'(y)x, x\rangle$.

Remark 3.3.2. *We note that the existence of the matrices B_u and $\Phi(t)$ as in assumption (a) can be guaranteed by the Floquet theory. We refer the reader to [16, Theorem 1.2] and [24, Theorem 4.1] for more details.*
We note also that the condition $\sigma(-B_u) \cap i\mathbb{Z} = \emptyset$ is not ensured by the Floquet theory. We need this supplementary condition to ensure the invertibility of some differential operator between the spaces X and Y.

Theorem 3.3.3. *Under the assumptions (a)–(f), problem (3.7) admits a unique solution.*

To prove Theorem 3.3.3, we need the following lemma which gives a necessarily and a sufficient condition for the operator $\frac{d}{dt} + B_u$ to be an isomorphism between X and Y.

Lemma 3.3.4. *The operator $\frac{d}{dt} + B_u$ is an isomorphism from X into Y if and only if $\sigma(-B_u) \cap i\mathbb{Z} = \emptyset$.*

Proof : For the proof of this lemma, see [13, Theorem 2.3].

Remark 3.3.5. *Assumption (a) and Lemma 3.3.4 imply that the operator $\frac{d}{dt} + F'(u(\cdot))$ is an isomorphism from X into Y*

Proof of Theorem 3.3.3 : For the proof, we apply Theorem 3.1.1. We study the cases $p = 2$ and $p > 2$ separately.
First case : $p = 2$.
We consider the function $\phi : X \to Y$ defined for every $u \in X$ by

$$\phi(u) = u' + \mathbf{F}(u).$$

By Lemma 3.3.1, the function ϕ is well defined, continuously differentiable on X and for every $u \in X$, the Fréchet-derivative $\phi'(u)$ is given by

$$\phi'(u)v = v' + \mathbf{F}'(u)v, \ v \in X.$$

Using Remark 3.3.5, we deduce that $\phi'(u)$ is invertible from X into Y for every $u \in X$. This proves that ϕ satisfies assumption (i) of Theorem 3.1.1.
Now, we are proving that ϕ is coercive. Let $R > 0$ and $u \in X$ be such that $\|\phi(u)\|_Y \leq R$.
We have the following equality

$$\|\phi(u)\|_Y^2 = \|u'\|_Y^2 + \|F(u)\|_Y^2 + 2\int_0^{2\pi} \langle F(u), u' \rangle \, dt.$$

This implies after integrating by parts, using the periodic boundary conditions

$$\begin{aligned}\|\phi(u)\|_Y^2 &= \|u'\|_Y^2 + \|F(u)\|_Y^2 - 2\int_0^{2\pi} \langle F'(u)u', u\rangle\, dt \\ &= \|u'\|_Y^2 + \|F(u)\|_Y^2 + 2\int_0^{2\pi} \langle F'(u)F(u), u\rangle\, dt \\ &\quad -2\int_0^{2\pi} \langle F'(u)\phi(u), u\rangle\, dt \\ &\leq R^2.\end{aligned}$$

By using assumption (e) and the Cauchy-Schwarz inequality we obtain

$$\|e(u)\|_{L^2(0,2\pi)}^2 + \|u'\|_Y^2 - 2c_3\|e(u)\|_{L^2(0,2\pi)}\|\phi(u)\|_Y \leq R^2.$$

This implies

$$\|e(u)\|_{L^2(0,2\pi)}^2 - 2c_3R\|e(u)\|_{L^2(0,2\pi)} + \|u'\|_Y^2 \leq R^2.$$

Hence, we obtain

$$\|e(u)\|_{L^2(0,2\pi)} \leq R(1+2c_3).$$

Using assumption (d), we deduce

$$\|F(u)\|_Y \leq Rc_2^{-1}(1+2c_3).$$

Assumption (b) implies

$$\|u\|_Y \leq Rc_1^{-1}c_2^{-1}(1+2c_3).$$

Using the identity $u' = \phi(u) - F(u)$ we obtain

$$\|u'\|_Y \leq R + Rc_2^{-1}(1+2c_3).$$

Hence

$$\|u\|_X \leq c_R,$$

where $c_R = R + Rc_1^{-1}(1+2c_3) + Rc_1^{-1}c_2^{-1}(1+2c_3)$.
This proves assumption (ii) of Theorem 3.1.1.
Let $R > 0$. Our aim is to prove that there exists a constant $c(R) > 0$ such that for every $u \in X$ with $\|u\|_X \leq R$ one has $\|\phi'(u)^{-1}\|_{\mathcal{L}(Y,X)} \leq c(R)$.
Let $u \in X$ with $\|u\|_X \leq R$. First observe that since X is continuously embedded into $C([0,2\pi];\mathbb{R}^N)$, there exists a constant $c > 0$ which is independent on u such that

$$\sup_{t\in[0,2\pi]} |u(t)| \leq cR. \tag{3.10}$$

Since F' is continuous from $\mathbb{R}^N$ into $\mathcal{L}(\mathbb{R}^N)$, this implies that there exists a constant $c_1(R) > 0$ which is independent on u such that

$$\sup_{t\in[0,2\pi]} \|F'(u(t))\|_{\mathcal{L}(\mathbb{R}^N)} \leq c_1(R).$$

Now let $k \in Y$. Then there exists a unique $h \in X$ such that $\phi'(u)h = k$, that is, there exists a unique solution h of the linear differential equation

$$\begin{cases} h'(t) + F'(u(t))h(t) = k(t) \text{ for a.e. } t \in (0, 2\pi), \\ h(0) = h(2\pi). \end{cases} \tag{3.11}$$

We multiply equation (3.11) by $h(t)$, then we integrate over $(0, 2\pi)$ in order to obtain

$$\int_0^{2\pi} \langle F'(u(t))h(t), h(t)\rangle \, dt = \int_0^{2\pi} \langle k(t), h(t)\rangle \, dt.$$

Now, we use assumption (f) and the estimate (3.10) and we have the existence of a positive constant $c_2(R)$ such the last inequality implies

$$c_2(R)\|h\|_Y^2 \leq \int_0^{2\pi} \langle k(t), h(t)\rangle \, dt.$$

By using the Young's inequality, we deduce that there exists a positive constant $c_3(R)$ such that

$$\|h\|_Y \leq c_3(R)\|k\|_Y.$$

This estimate and equation (3.11) yield

$$\begin{aligned} \|\frac{d}{dt}(\phi'(u)^{-1}k)\|_Y &= \|h'\|_Y \\ &\leq \|F'(u(\cdot))h\|_Y + \|k\|_Y \\ &\leq (\sup_{t\in[0,2\pi]} \|F'(u(t))\|_{\mathcal{L}(\mathbb{R}^N)} c_3(R) + 1)\, \|k\|_Y. \end{aligned}$$

This estimate and the preceding estimate together prove that assumption (iii) of Theorem 3.1.1 is satisfied.
By Theorem 3.1.1, ϕ is a global C^1 diffeomorphism and hence problem (3.7) admits a unique solution for $p = 2$.
Second case : $p > 2$.
We are proving that this case is a consequence of the first case. Let $f \in L^p(0, 2\pi; \mathbb{R}^N)$. Since $p > 2$, $f \in L^2(0, 2\pi; \mathbb{R}^N)$ and by the first case, there exists a unique $u \in H^1(0, 2\pi; \mathbb{R}^N)$ which is solution of

$$\begin{cases} u'(t) + F(u(t)) = f(t) \text{ for a.e. } t \in (0, 2\pi), \\ u(0) = u(2\pi). \end{cases} \tag{3.12}$$

Since $H^1(0, 2\pi; \mathbb{R}^N)$ is a subspace of $C([0, 2\pi]; \mathbb{R}^N)$, the function $t \to F(u(t))$ is continuous on $[0, 2\pi]$ and then $F(u) \in L^p(0, 2\pi; \mathbb{R}^N)$. It follows by equation (3.12) that $u' \in L^p(0, 2\pi; \mathbb{R}^N)$. Hence, $u \in W^{1,p}(0, 2\pi; \mathbb{R}^N)$ is the unique solution of problem (3.7).

Remark 3.3.6. *Assumptions (b)–(e) and the method of the proof of the coercivity of the function ϕ in Example 1 are inspired by Rabier and Stuart's work in [47].*

Bibliographie

[1] F. Almgren, J. E. Taylor, and Lihe Wang, *Curvature-driven flows : a variational approach*, SIAM J. Control Optim. **31** (1993), no. 2, 387–438.

[2] H. Amann, *Maximal regularity for nonautonomous evolution equations*, Advanced Nonlinear Studies **4** (2004), 417–430.

[3] H. Amann, *Maximal regularity and quasilinear parabolic boundary value problems*, Recent advances in elliptic and parabolic problems, World Sci. Publ., Hackensack, NJ, 2005, pp. 1–17.

[4] H. Amann, *Quasilinear parabolic problems via maximal regularity*, Adv. Differential Equations **10** (2005), 1081–1110.

[5] H. Amann, *Existence and regularity for semilinear parabolic evolution equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), no. 4, 593–676.

[6] H. Amann, *Quasilinear evolution equations and parabolic systems*, Trans. Amer. Math. Soc. **293** (1986), no. 1, 191–227.

[7] H. Amann, *Quasilinear evolution equations and parabolic systems*, Trans. Amer. Soc. 293 (1986), no. 1, 191-227.

[8] S. B. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh **115A** (1990), 91–107.

[9] S. B. Angenent, *Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature*, Ann. of Math. (2) **132** (1990), no. 3, 451–483.

[10] W. Arendt, *Semigroups and evolution equations : functional calculus, regularity and kernel estimates*, Evolutionary equations. Vol.I, 1–85, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[11] W. Arendt, R. Chill, S. Fornaro, and C. Poupaud, *L^p-maximal regularity for non-autonomous evolution equations*, J. Differential Equations **237** (2007), 1–26.

[12] W. Arendt and R. Chill, *Global existence for quasilinear diffusion equations in isotropic nondivergence form*, Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

[13] W. Arendt and S. Bu, *The operator-valued Marcinkiewiecz multiplier theorem and maximal regularity*, Math. Z. 240 (2002) 311-344.

[14] W. Arendt and P. J. Rabier, *Linear evolution operators on spaces of periodic functions*, Commun. Pure Appl. Anal. 8 (2009), no. 1.

[15] L. Ambrosio, N. Gigli, G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Birkhäuser Verlag, Basel, Boston, Berlin, 2003.

[16] R. Agarwal, M. Bohner, A. Domoshnitsky and Y. Goltser, *Floquet theory and stability of nonlinear integro-differential equations*, Acta Math. Hungar. 109 (4) (2005), 305–330.

[17] H. Brézis, *Analyse fonctionnelle. Théorie et applications*, Dunod, Paris, 1999.

[18] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. Vol. 5 of North Holland Mathematics Studies. North-Holland, Amsterdam, London, 1973.

[19] D. Bothe and J. Prüss, *L_P-theory for a class of non-Newtonian fluids*, SIAM J. Math. Anal. **39** (2007), no. 2, 379–421 (electronic).

[20] K. A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, vol. 20, Princeton University Press, Princeton, N.J., 1978.

[21] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, New York, Toronto, London, 1955.

[22] Kai-Seng Chou and Xi-Ping Zhu, *The Curve Shortening Problem*, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[23] Ph. Clément and S. Li, *Abstract parabolic quasilinear problems and application to a groundwater flow problem*, Adv. Math. Sci. Appl. **3** (1994), 17–32.

[24] J. Chi and C. Zu, *Generalization of the Floquet theory*, Nonlinear analysis 71 (2009) 1100–1107.

[25] G. Da Prato and P. Grisvard, *Equations d'évolution abstraites non linéaires de type parabolique*, Ann. Mat. Pura Appl. (4) **120** (1979), 329–396.

[26] L. De Simon, *Un applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine*, Rend. Sem. Mat. Univ. Padova **34** (1964), 547–558.

[27] K. Deckelnick, *Weak solutions of the curve shortening flow*, Calc. Var. Partial Differential Equations **5** (1997), no. 6, 489–510.

[28] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18** (1983), no. 1, 157–162.

[29] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. 166 (2003), no. 788, viii+114pp.

[30] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, American Mathematical Society, Providence, Rhode Island.

[31] K. Ecker, *Regularity Theory for Mean Curvature Flow*, Progress in Nonlinear Differential Equations and Their Applications, vol. 57, Birkäuser, Boston, Basel, Berlin, 2004.

[32] J. Escher, J. Prüss, and G. Simonett, *A new approach to the regularity of solutions of parabolic equations*, Evolution Equations : Proceedings in Honor of J. A. Goldstein's 60th Birthday, Marcel Dekker, New York, 2003, pp. 167–190.

[33] Y. Giga, *Surface Evolution Equations*, Monographs in Mathematics, vol. 99, Birkhäuser, Basel, 2006.

[34] D. Guidetti, *A maximal regularity result with applications to parabolic problems with nonhomogeneous boundary conditions*, Rend. Sem. Mat. Univ. Padova **84** (1990), 1–37 (1991).

[35] M. Hieber and J. Rehberg, *Quasilinear parabolic systems with mixed boundary conditions on nonsmooth domains*, SIAM J. Math. Anal. **40** (2008), no. 1, 292–305.

[36] G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Math., vol. 1713, Springer, Berlin, 1999, pp. 45–84.

[37] P.C. Kunstmann and L. Weis, *Maximal L^p-regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-Functional Calculus*, Functional analytic methods for evolution equations, 65-311, Lectures Notes in Math, 1855, Springer, Berlin, 2004.

[38] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[39] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 2002.

[40] A. Lunardi : *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Vol. 16 of Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser, Basel, 1995.

[41] S. Luckhaus and Th. Sturzenhecker, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations **3** (1995), no. 2, 253–271.

[42] F. Otto, *The geometry of dissipative evolution equations : the porous medium equation*, Comm. Partial Differential Equations 26 (2001), no.1-2, 101-174.

[43] A. A. Ovono and A. Rougirel, *Elliptic equations with diffusion parametrized by the range of nonlocal interactions*, Annali di Matematica Pura ed Applicata, (2009).

[44] R. Plastock, *Homeomorphisms between Banach spaces*, Trans. Amer. Math. Soc. 200 (1974), 169–183.

[45] J. Prüss and R. Schnaubelt, *Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time*, J. Math. Anal. Appl. **256** (2001), 405–430.

[46] J. Prüss, *Maximal regularity for evolution equations in L_p-spaces*, Conf. Semin. Mat. Univ. Bari (2002), no. 285, 1–39 (2003).

[47] P. J. Rabier and C.A. Stuart, *A Sobolev space approach to boundary value problems on the half-line*, Commun. Contemp. Math. 7 (2005), no. 1, 1–36.

[48] P. J. Rabier and C.A. Stuart, *Boundary value problems for first order systems on the half-line*, Topl. Methods Nonlinear Anal. 25 (2005), no. 1, 101–133.

[49] M. Radulescu and S. Radulescu, *Global inversion theorems and applications to differential equations*, Nonlinear Analysis, Theory, Methods and Applications, Vol 4, No. 4, pp 951-965, 1980.

[50] R.E. Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.

[51] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) 146 (1987), 65-96.

[52] J. Saal, *Strong solutions for the Navier-Stokes equations on bounded and unbounded domains with a moving boundary*, Proceedings of the

Sixth Mississippi State–UBA Conference on Differential Equations and Computational Simulations (San Marcos, TX), Electron. J. Differ. Equ. Conf., vol. 15, Southwest Texas State Univ., 2007, pp. 365–375.

[53] G. Simonett, *The Willmore flow near spheres*, Differential Integral Equations **14** (2001), 1005–1014.

[54] G. Zampieri, *Diffeomorphisms with Banach space domains*, Nonlinear Analysis, Theory, Methods and Applications, Vol 19, No. 10, pp 923-932, 1992.

[55] F. Zanolin, *Continuation theorems for the periodic problem via the translation operator*, Rend. Sem. Math. Univ. Pol. Torino, Vol. 54, (1996).

[56] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. I*, Springer Verlag, New York, Berlin, Heidelberg, 1990.

[57] Xi-Ping Zhu, *Lectures on mean curvature flows*, AMS/IP Studies in Advanced Mathematics, vol. 32, American Mathematical Society, Providence, RI, 2002.