THÈSE

présentée à l'Université de Metz
pour l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE METZ

EN MATHÉMATIQUES APPLIQUÉES

par

Abdelhak FERFERA

Titre de la thèse :

SUR QUELQUES PROBLÈMES RELATIFS AUX SYSTÈMES NON LINÉAIRES : LINÉARISATION STATIQUE ET SINGULARITÉS STABILISATION GLOBALE DE CERTAINS SYSTÈMES

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Résumé

Ce travail comporte trois parties. Dans la première, on s'intéresse au problème du découplage avec stabilité interne par commande statique pour les systèmes bilinéaires. Pour ces systèmes non linéaires, la matrice de découplage est singulière sur une surface algébrique contenant l'origine, ce qui pose un problème d'explosion de solutions : dans ce cas généralement les trajectoires du système bouclé ne sont pas complètes et/ou les commandes ne sont pas bornées. On considère ici des systèmes bilinéaires à deux entrées-deux sorties sans zéros dynamiques, pour lesquels on donne des conditions suffisantes de découplage avec stabilité par des commandes linéarisantes.

La deuxième partie est consacrée à des questions de stabilisation globale par retour d'état pour certains systèmes non linéaires. On s'intéresse d'une part aux systèmes partiellement linéaires pour lesquels divers auteurs ont donné des conditions suffisantes de stabilisation globale à partir d'une fonction de Lyapunov stricte. Il n'existe malheureusement pas de méthode systématique pour construire une telle fonction. On montre ici que la connaissance d'une fonction de Lyapunov large vérifiant le principe d'invariance de LaSalle suffit pour obtenir une commande stabilisante globale. L'intérêt de notre démarche est que pour de très larges classes de systèmes, dont les systèmes mécaniques, il est plus facile de construire une fonction de Lyapunov large plutôt qu'une stricte. On donne d'autre part une condition suffisante de stabilisation globale pour des systèmes non affines en contrôle généralisant celle de Jurdjevic-Quinn connue pour les systèmes affines en contrôle.

La dernière partie étend des résultats de stabilisation déterministes à des systèmes non linéaires stochastiques. On y donne une condition suffisante de stabilisation globale pour des systèmes partiellement linéaires stochastiques et une version stochastique de la condition de Jurdjevic-Quinn pour des systèmes stochastiques non nécessairement affines en contrôle.

En m'acceptant dans son équipe, G. Sallet m'a permis de reprendre une activité de recherche longuement interrompue. Je suis sensible à la confiance qu'il m'a témoignée et à l'aide qu'il m'a apportée. Je lui exprime ici toute ma reconnaissance.

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Introduction

Introduction

Dans cette thèse on traite de deux problèmes de la théorie des systèmes non linéaires. D'une part, le problème des singularités pour une classe de systèmes affines en contrôle linéarisables par feedback statique. D'autre part, le problème de la stabilisation globale par retour d'état pour certains systèmes non linéaires.

0.1 Problème des singularités

La question des singularités constitue l'objet des chapitres 1 et 2. Initialement, le problème posé est celui du découplage (ou commande non interactive) par feedback statique avec stabilité asymptotique interne pour des systèmes non linéaires affines en contrôle, qui est directement lié à celui de la linéarisation exacte par feedback statique de tels systèmes. Commençons par préciser la terminologie. Les notations sont celles que l'on peut trouver dans des ouvrages classiques de la théorie des systèmes non linéaires telles que [56] ou [88].

Soit un système de la forme

$$\begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \end{cases} \tag{0.1}$$

où l'état $x \in \mathbb{R}^n$, l'entrée $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$ et la sortie $y = (y_1, \ldots, y_m)^{\mathrm{T}} \in \mathbb{R}^m$. Les champs de vecteurs f, $g_1, \ldots, g_m$ et la fonction $h = (h_1, \ldots, h_m)^{\mathrm{T}}$ sont supposés de classe C^∞ sur $\mathbb{R}^n$.

Un feedback statique régulier défini sur un ouvert E de $\mathbb{R}^n$ est une loi de commande de la forme

$$u = \alpha(x) + \beta(x)v \tag{0.2}$$

où $v(t) \in \mathbb{R}^m$ est une nouvelle entrée et $\alpha : E \to \mathbb{R}^m$, $\beta : E \to \mathcal{M}_{m,m}(\mathbb{R})$ sont des fonctions de classe C^∞ sur E tel que la matrice $\beta(x)$ est inversible pour tout $x \in E$.

On dira que le problème du découplage statique pour le système (0.1) admet une solution sur E s'il existe un feedback de la forme (0.2) tel que pour toute condition initiale prise dans E, la composante y_i de la sortie du système bouclé

$$\begin{cases} \dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \\ y = h(x) \end{cases} \tag{0.3}$$

dépend uniquement de la composante v_i de l'entrée et non de v_j pour $j \neq i$.

Un tel feedback sera dit solution du problème du découplage statique avec stabilité asymptotique interne si, pour $v \equiv 0$, l'origine est un point d'équilibre asymptotiquement stable pour le système bouclé

$$\dot{x} = f(x) + g(x)\alpha(x)$$

On dira enfin que le système (0.1) est linéarisable sur E par un feedback de la forme (0.2) s'il existe sur E un changement de coordonnées locales $\xi = \Phi(x)$ transformant le système bouclé (0.3) en un système linéaire contrôlable et observable

$$\begin{cases} \dot{\xi} = F\xi + Gv \\ y = H\xi \end{cases}$$

Une bibliographie considérable a été consacrée aux problèmes du découplage et de la linéarisation exacte devenus classiques en automatique [29, 30, 31, 45, 51, 52, 58, 60, 81, 87, 90, 91, 95, 96, 100]. Des conditions nécessaires et suffisantes pour l'existence de feedbacks statiques réguliers solutions de ces problèmes sur un ouvert E de $\mathbb{R}^n$ ont été données grâce aux propriétés de la matrice de découplage associée au système.

Soit ρ_i le plus grand entier tel que pour tous $k < \rho_i$, $1 \leq j \leq m$ et $x \in \mathbb{R}^n$, on ait $L_{g_j} L_f^k h_i(x) = 0$, où La notation $L_X\lambda$ désigne la dérivée de Lie de la fonction λ suivant le champ de vecteurs X.

Si pour tout $1 \leq i \leq m$, $\rho_i < +\infty$, on appelle matrice de découplage la matrice $\mathbf{\Omega}(x) \in \mathcal{M}_{m,m}(\mathbb{R})$ définie pour tout $x \in R^n$ par

$$\mathbf{\Omega}(x) = \left(L_{g_j} L_f^{\rho_i} h_i(x)\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$$

Les résultats suivants ont été obtenus par différents auteurs dans les années 70 et 80 (*cf. e.g.* [56] et sa bibliographie).

Le problème du découplage statique admet des solutions locales sur un ouvert E de $\mathbb{R}^n$ si et seulement si $\rho_i < +\infty$ pour tout i et la matrice $\mathbf{\Omega}(x)$ est inversible sur E. Des solutions particulières sont données par

$$\alpha(x) = \mathbf{\Omega}^{-1}(x) \left[\begin{pmatrix} a_1(x) \\ \vdots \\ a_m(x) \end{pmatrix} - b(x) \right], \quad \beta(x) = \mathbf{\Omega}^{-1}(x) \tag{0.4}$$

où

$$
\begin{aligned}
a_i(x) &= c_{i1}h_i(x) + \cdots + c_{i\rho_i+1}L_f^{\rho_i}h_i(x), \quad c_{ij} \in \mathbb{R},\ 1 \leq i \leq m,\ 1 \leq j \leq \rho_i + 1 \\
b(x) &= \left(L_f^{\rho_1+1}h_1(x), \ldots, L_f^{\rho_m+1}h_m(x)\right)^{\mathrm{T}}
\end{aligned}
$$

Le problème de la linéarisation exacte par un feedback de la forme (0.2) admet des solutions locales sur un ouvert E de $\mathbb{R}^n$ si et seulement si $\sum_{i=1}^{m}(\rho_i + 1) = n$ et la matrice $\mathbf{\Omega}(x)$ est inversible sur E, les solutions étant données par

$$
\alpha(x) = \mathbf{\Omega}^{-1}(x)\left[C\Phi(x) - b(x)\right], \quad \beta(x) = \mathbf{\Omega}^{-1}(x) \tag{0.5}
$$

où $C \in \mathcal{M}_{m,n}(\mathbb{R})$ et

$$
\Phi(x) = \left(h_1(x), \ldots, L_f^{\rho_1}h_1(x), \ldots, h_m(x), \ldots, L_f^{\rho_m}h_m(x)\right)^{\mathrm{T}} \tag{0.6}
$$

est la fonction définissant le changement de coordonnées locales sur E.

Remarquons enfin que si $\sum_{i=1}^{m}(\rho_i + 1) = n$, tout feedback de la forme (0.4) assure en même temps la linéarisation exacte et le découplage de (0.1) sur E.

Ce qui précède montre que l'étude du comportement du système bouclé au voisinage des singularités de $\mathbf{\Omega}(x)$ (*i.e.* des points $x \in \mathbb{R}^n$ pour lesquels $\det \mathbf{\Omega}(x) = 0$) apparaît comme un problème important et difficile. En effet, l'existence de tels points fait qu'en général les trajectoires ne sont pas complètes et/ou la commande u n'est pas bornée sur ces trajectoires.

Le problème du découplage avec stabilité pour des systèmes non linéaires n'a quant à lui commencé à être étudié que récemment [4, 5, 55, 57, 107] et uniquement pour des systèmes avec matrice de découplage inversible à l'origine, et donc en l'absence de singularités au voisinage du point d'équilibre. Des conditions autant nécessaires que suffisantes sont données pour l'existence de solutions locales au voisinage de l'origine pour lesquelles ne se posent ni le problème des trajectoires non complètes ni celui des commandes non bornées. A notre connaissance, il n'existe pas de résultats de découplage avec stabilité pour des systèmes avec matrice de découplage singulière à l'origine.

Dans ce travail on s'intéresse donc au problème du découplage avec singularités pour des systèmes non linéaires particuliers.

Il est bien connu, maintenant, que les systèmes bilinéaires peuvent être (du point de vue du comportement non linéaire) aussi compliqués que l'on veut. D'ailleurs, tout système entrée-sortie peut être approché, sur tout intervalle compact de temps, par des systèmes bilinéaires [37, 101].

C'est pourquoi, dans ce travail, on considère des systèmes bilinéaires à deux entrées et deux sorties de la forme

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x \\ y = Hx \\ x \in \mathbb{R}^n, \ u = (u_1, u_2)^{\mathrm{T}} \in \mathbb{R}^2, \ y \in \mathbb{R}^2 \end{cases} \tag{0.7}$$

que l'on suppose linéarisables par un feedback statique de la forme (0.5). Pour cette classe de systèmes l'ensemble S des points singuliers (points annulant une forme quadratique sur $\mathbb{R}^n$) contient l'origine. Le problème du découplage avec stabilité par un feedback linéarisant de la forme (0.4) se pose donc en présence de singularités et conduit inévitablement à celui des trajectoires non complètes et/ou des commandes non bornées.

Par ailleurs, il apparaît que le changement de coordonnées $\xi = \Phi(x)$ défini par (0.6) est linéaire et global, et donc que le système bouclé est défini sur l'ouvert dense de $\mathbb{R}^n$ donné par $E = \mathbb{R}^n \backslash S$. Techniquement, il y a une difficulté pour la complétude des trajectoires du système bouclé : bien qu'il soit linéaire, il n'est défini que sur un ouvert dense $E \neq \mathbb{R}^n$. La question est alors de savoir s'il existe un feedback de la forme (0.5) (resp. (0.4)) tel que pour $v \equiv 0$, E est positivement invariant pour le système bouclé et le contrôle $u(t)$ est borné sur chaque trajectoire de ce système contenue dans E. Un tel feedback sera dit linéarisant sur E. Si de plus le système bouclé est asymptotiquement stable, il sera dit linéarisant et stabilisant sur E.

Dans le chapitre 1, on montre que, pour les systèmes bilinéaires plans de la forme (0.7), de tels feebacks existent toujours. On donne en outre, grâce à des techniques d'approximation bilinéaire, des conditions suffisantes de linéarisation et de stabilisation locales sur E pour des systèmes plans affines en contrôle

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \\ f(0) = g(0) = 0, \ h(0) = 0 \end{cases}$$

pour lesquels l'origine est encore un point singulier.

Pour les systèmes bilinéaires (0.7) de dimension supérieure, on donne dans le chapitre 2 une condition suffisante d'existence de feedback linéarisant sur E. Plus précisément, on montre que la structure particulière de l'ensemble des points singuliers (points annulant une forme quadratique sur $\mathbb{R}^n$) conduit à une condition simple sur la dérive linéaire du système bouclé caractérisant l'invariance de E. Dans ce cas, on donne alors une condition suffisante pour que la commande $u(t)$ soit bornée sur toute trajectoire du système bouclé sur E. Ceci conduit naturellement à une condition suffisante de découplage avec stabilité asymptotique interne.

Les résultats de cette première partie sont à notre connaissance les premiers sur les problèmes de linéarisation exacte et de découplage avec stabilité en présence de singularités. Ils ont fait l'objet des publications [17, 18, 19].

0.2 Stabilisation globale

La suite de cette thèse est consacrée à des questions de stabilisation globale par retour d'état pour certains systèmes non linéaires.

Systèmes déterministes. Le système non linéaire

$$\begin{cases} \dot{x} = f(x,u) \\ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ f(0,0) = 0 \end{cases}$$

est globalement stabilisable s'il existe un feedback $u = u(x)$ tel que l'origine soit un point d'équilibre globalement asymptotiquement stable pour le système bouclé

$$\dot{x} = f(x, u(x))$$

Là aussi commençons par préciser quelques définitions (*cf.* *e.g.* [53]). Soit un système différentiel autonome

$$\begin{cases} \dot{x} = F(x) \\ x \in \mathbb{R}^n, \ F(x_0) = 0 \end{cases} \tag{0.8}$$

où F est un champ de vecteur globalement lipschitzien sur $\mathbb{R}^n$. On désigne par $F_t(x)$ la solution de (0.8) issue de x à l'instant $t = 0$, *i.e.* $dF_t(x)/dt|_{t=0} = F(x)$ et $F_0(x) = x$. x_0 est un point d'équilibre stable pour (0.8) si :

$$\forall \varepsilon > 0, \ \exists \alpha > 0, \ \text{tel que } \|x - x_0\| < \alpha \Rightarrow \forall t \geq 0, \ \|F_t(x) - x_0\| < \varepsilon$$

x_0 est attractif pour (0.8) s'il existe un voisinage U de x_0 tel que, $\forall x \in U$, $F_t(x)$ existe pour tout $t \geq 0$ et $\lim_{t \to +\infty} F_t(x) = x_0$. Si $U = \mathbb{R}^n$, x_0 est dit globalement attractif. x_0 est un point d'équilibre asymptotiquement stable (resp. globalement asymptotiquement stable) pour (0.8) s'il est stable et attractif (resp. stable et globalement attractif).

Les propriétés de stabilité d'un point d'équilibre sont souvent étudiées sans avoir à résoudre explicitement le système, ce qui est en général impossible, et ce grâce aux résultats de Lyapunov connus en théorie qualitative sous le nom de méthode directe (*cf.* [53]).

Une fonction $V : U \to \mathbb{R}$, continue sur un voisinage U de x_0 et différentiable sur $U \backslash \{x_0\}$ est une fonction de Lyapunov large pour (0.8) en x_0 si :

$$\forall x \in U \backslash \{x_0\},\ V(x) > 0,\ V(x_0) = 0 \text{ et } \forall x \in U,\ \dot{V}(x) = F \cdot V(x) \leq 0$$

où $F \cdot V(x) = dV(F_t(x))/dt|_{t=0} = \langle \nabla V(x), F(x) \rangle$. Si de plus $\dot{V}(x) < 0,\ \forall x \in U \backslash \{x_0\}$, V est dite fonction de Lyapunov stricte pour (0.8) en x_0. Les théorèmes de Lyapunov s'énoncent alors ainsi.

Si (0.8) admet une fonction de Lyapunov large (resp. stricte) en x_0, alors x_0 est un point d'équilibre stable (resp. asymptotiquement stable) pour (0.8).

Si (0.8) admet en x_0 une fonction de Lyapunov stricte V définie sur $\mathbb{R}^n$ ($\dot{V}(x) < 0,\ \forall x \in \mathbb{R}^n \backslash \{x_0\}$) et propre (*i.e.* l'image réciproque d'un compact de $\mathbb{R}^+$ est un compact de $\mathbb{R}^n$), alors x_0 est un point d'équilibre globalement asymptotiquement stable pour (0.8).

On a aussi le résultat suivant connu sous le nom de théorème inverse (Kurzweil [69], Massera [82]). Si x_0 est un point d'équilibre asymptotiquement stable (resp. globalement asymptotiquement stable) alors (0.8) admet en x_0 une fonction de Lyapunov stricte de classe C^∞ sur un voisinage de x_0 (resp. sur $\mathbb{R}^n$).

Rappelons enfin le principe d'invariance de LaSalle [71]. Si (0.8) admet en x_0 une fonction de Lyapunov large V de classe C^1, alors toutes les trajectoires bornées pour $t \geq 0$ tendent vers Ω, le plus grand ensemble invariant par F et contenu dans $E = \{x \in \mathbb{R}^n \mid F \cdot V(x) = 0\}$. Pour montrer que x_0 est un point d'équilibre asymptotiquement stable, il suffit alors de montrer que $\Omega = \{x_0\}$. Si de plus V est propre alors toutes les trajectoires sont bornées et le résultat devient global.

Revenons au système controlé

$$\begin{cases} \dot{x} = f(x,u) \\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ f(0,0) = 0 \end{cases} \tag{0.9}$$

A la différence des systèmes linéaires ($f(x,u) = Ax + Bu$), le probème de la stabilisation est loin d'être complètement résolu pour les systèmes non linéaires et continue à faire l'objet d'une intense activité de recherche en automatique entamée il y a une vingtaine d'années.

Pour la stabilisation locale des conditions nécessaires ont été données par Brockett [10], Coron [32], Krasnosel'ski et Zabreiko [68], et diverses techniques (linéarisation, techniques de la variété centrale, zéro dynamique, techniques d'approximations,...) ont été utilisées pour obtenir des feedbacks stabilisants [1, 3, 10, 13, 25, 34, 54, 64, 65].

Ce sont les techniques de Lyapunov qui sont généralement à la base des résultats de stabilisation globale : méthode de Jurdjevic-Quinn [46, 62, 63, 74, 89, 103], utilisation des fonctions de Lyapunov contrôlées [2, 97, 103], et autres résultats [25, 98] (voir aussi la bibliographie de [98]).

Dans le chapitre 3 de ce travail, qui a fait l'objet de la publication [36], on s'intéresse au problème de la stabilization globale des systèmes partiellement linéaires de la forme

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = Ay + Bu \\ x \in \mathbb{R}^n, \ y \in \mathbb{R}^p, \ u \in \mathbb{R}^k \end{cases} \tag{0.10}$$

où la fonction f est supposée de classe C^∞ vérifiant $f(0,0) = 0$. Cette classe de systèmes a été beaucoup considérée ces dernières années. Le rôle important qu'elle joue dans la littérature est dû aux résultats des années 80 sur la linéarisation partielle par feedback [11, 12, 13, 59, 80], permettant sous certaines conditions de transformer un système non linéaire en un système en cascade de la forme (0.10).

Pour cette classe de systèmes, il est bien connu que si l'équation différentielle

$$\dot{x} = f(x,0) \tag{0.11}$$

est localement asymptotiquement stable en l'origine de $\mathbb{R}^n$ et si le système linéaire

$$\dot{y} = Ay + Bu \tag{0.12}$$

est localement stabilisable par un feedback linéaire $u(y) = Ky$, alors le système composite (0.10) est localement stabilisable par ce même feedback (Vidyasagar [106]).

Notons que ce résultat local n'a pas d'analogue global : la stabilisation globale de (0.10) ne se déduit pas nécessairement de celle de (0.12) et de la stabilité asymptotique globale de (0.11). A titre de contre exemple, considérons le système plan suivant (*cf.* [94]) :

$$\begin{cases} \dot{x} = x(x^2y^2 - 1) \\ \dot{y} = u \end{cases}$$

qui est localement (mais non globalement) stabilisable par le feedback $u(y) = -y$ alors que les deux sous-systèmes associés sont globalement asymptotiquement stables. On peut, en effet, vérifier que l'ensemble $\{(x,y) \in \mathbb{R}^2 \,|\, x^2y^2 = 2\}$ est invariant pour le système composite bouclé, ce qui exclut la stabilité asymptotique globale de ce dernier.

En fait, la stabilisation globale de (0.10) ne se déduit de celle de (0.12) et de la stabilité asymptotique globale de (0.11) qu'au prix de conditions supplémentaires [14, 67, 92, 102, 93]. Plutôt que d'imposer une restriction linéaire trop forte sur la croissance de $f(x,y)$, Saberi, Kokotovic et Sussmann supposent dans [92] que sa dépendance en y est de la forme

$$f(x,y) = f(x,0) + G(x,y)Cy, \quad \text{avec } B^{\mathrm{T}}P = C \tag{0.13}$$

où la matrice définie positive P est telle que $y^{\mathrm{T}}Py$ soit une fonction de Lyapunov pour le système linéaire bouclé asymptotiquement stable

$$\dot{y} = (A + BK)y$$

Ils montrent alors que le système composite (0.10) est globalement stabilisable par le feedback

$$u(x,y) = Ky - \frac{1}{2}G(x,y)^{\mathrm{T}}\nabla V(x)$$

où V est une fonction de Lyapunov stricte pour le système globalement asymptotiquement stable (0.11). Bien que l'existence de V soit assurée par le théorème inverse de Lyapunov, il n'existe malheureusement pas de méthode systématique pour construire de telles fonctions de Lyapunov.

On montre dans le chapitre 3 que la connaissance d'une fonction de Lyapunov large pour le système (0.11) vérifiant le principe d'invariance de LaSalle suffit pour obtenir une commande stabilisante pour (0.10). L'intérêt de notre démarche est que pour de très larges classes de systèmes, dont les systèmes mécaniques, il est plus facile de construire une fonction de Lyapunov large plutôt qu'une stricte. A titre d'exemple, considérons l'équation de Liénard dans $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - h(x_1)x_2 \end{cases}$$

où l'on suppose que pour tout $x \neq 0$, $xg(x) > 0$, $h(x) > 0$ et

$$G(x) = \int_0^x g(s)\,ds \ \to \infty \ \text{ lorsque } \ |x| \to \infty$$

Pour un tel système, il n'est pas évident de construire une fonction de Lyapunov stricte, alors que le principe d'invariance de LaSalle s'applique de façon immédiate avec la fonction de Lyapunov large $V(x_1,x_2) = (1/2)x_2^2 + G(x_1)$.

Dans le chapitre 4, on s'intéresse au problème de la stabilisation globale des systèmes non affines en contrôle de la forme générale

$$\begin{cases} \dot{x} = f(x,u) \\ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ f(0,0) = 0 \end{cases} \tag{0.14}$$

où f est un champ de vecteurs de classe C^∞. Les références bibliographiques citées plus haut sont consacrées en grande partie aux systèmes affines en contrôle

$$\dot{x} = X(x) + \sum_{i=1}^{m} u_i Y_i(x) \tag{0.15}$$

et on connaît peu de résultats, en particulier constructifs, sur les systèmes non affines. Dans ce travail, on donne une condition suffisante de stabilisation globale par une commande régulière pour (0.14) généralisant celle de Jurdjevic-Quinn [62] connue pour (0.14).

Historiquement, le résultat de [62] est l'un des premiers résultats significatifs sur la stabilisation. On peut le résumer ainsi (*cf.* [89]) : s'il existe une fonction de classe C^∞ $V : \mathbb{R}^n \to \mathbb{R}$ définie positive et propre telle que :

(i) la dérivée de Lie de V suivant le champ de vecteurs X satisfait

$$X \cdot V(x) \leq 0, \ \forall x \in \mathbb{R}^n$$

(ii) l'ensemble

$$W = \{x \in \mathbb{R}^n \mid X^{k+1} \cdot V(x) = X^k \cdot Y_i \cdot V(x) = 0, \ k \in \mathbb{N}, \ i = 1, \ldots, m\}$$

est réduit à $\{0\}$;

alors, la dérivée de V le long des trajectoires du système (0.15) étant donnée par

$$\dot{V}(x) = X \cdot V(x) + \sum_{i=1}^{m} u_i Y_i \cdot V(x)$$

le feedback

$$u(x) = -(Y_1 \cdot V(x), .., Y_m \cdot V(x))^{\mathrm{T}}$$

conduit à $\dot{V}(x) \leq 0$ et, par application du principe d'invariance de LaSalle, stabilise globalement (0.15).

Pour les systèmes non affines en contrôle de la forme générale (0.14), $\dot{V}(x)$ n'étant plus linéaire en u, la difficulté majeure est de prouver l'existence d'un feedback $u(x) \not\equiv 0$, $u(0) = 0$, conduisant à $\dot{V}(x) \leq 0$. On montre ici que si les champs de vecteurs :

$$X(x) = f(x, 0), \quad Y_i(x) = \frac{\partial f}{\partial u_i}(x, 0), \ i = 1, \ldots, m \tag{0.16}$$

satisfont (i) et (ii) alors, sans aucune condition supplémentaire, un tel feedback existe et stabilise globalement (0.14). On donne une preuve constructive de ce résultat permettant de calculer explicitement le feedback stabilisant. Mentionnons à ce titre qu'une version existentielle de ce résultat, ne permettant pas d'expliciter le feedback, a été donnée par Coron [33] avec une démonstration plus compliquée que celle proposée ici.

On montre dans ce même chapitre que les techniques conduisant à ce résultat permettent aussi de considérer des systèmes non linéaires discrets de la forme

$$x(k+1) = f(x(k), u(k)), \qquad f(0,0) = 0, \qquad k = 0,\ 1,\ 2,\ \ldots$$

pour lesquels on donne une condition suffisante de stabilisation globale analogue à celle de Jurdjevic-Quinn.

L'importance en automatique des systèmes non linéaires discrets (*cf.* [61] et sa bibliographie) est dûe aux méthodes de discrétisation (sampling) pour les systèmes non linéaires en temps continu, ainsi qu'au fait que pour de nombreuses questions pratiques les équations aux différences fournissent un modèle plus naturel que les équations différentielles. A titre d'exemple, la commande explicite des systèmes industriels est souvent une commande échantillonnée. En effet, les mesures comme les entrées, pour des raisons physiques, sont souvent constantes soit en valeur, soit en pente (bloqueur d'ordre 0, ou d'ordre 1, etc...) sur un intervalle de temps égal à la constante d'échantillonnage. Par ailleurs, la puissance des instruments de calcul a remplacé la commande analogique par la commande numérique par ordinateur. De ce fait, on constate un intérêt croissant pour les systèmes discrets dans la littérature [26, 27, 28, 47, 48, 49, 50, 72, 73, 83, 84, 85, 86]. Notons que toutes les notions du continu ont leur problématique pour les systèmes discrets (commandabilité, observabilité, stabilisation,...).

Pour donner une idée des difficultés liées à l'étude des systèmes discrets non linéaires, notons qu'un système discret provenant de la discrétisation d'un système continu contrôlable peut très bien ne pas être lui-même contrôlable.

La stabilisation de ces systèmes n'est étudiée de manière soutenue que depuis quelques années [6, 15, 16, 78, 79, 99, 104, 105] et les résultats obtenus sont, à notre connaissance, à caractère local [99, 104, 105], ou restreints à des systèmes affines en contrôle [6, 15, 16].

Signalons, enfin, que les résultats de ce chapitre, qui ont fait l'objet de la communication [7] en Juin 1995 (voir aussi [9]), ont été obtenus à la même période, par des techniques proches des nôtres, par Lin en Septembre 1995 [75] pour les systèmes en temps continu à entrée scalaire (voir aussi [77] pour les systèmes à entrée vectorielle), et en Décembre 1995 [76] pour les systèmes discrets.

Systèmes stochastiques. Les deux derniers chapitres de cette thèse, qui correspondent respectivement aux publications [8, 35], sont consacrés à l'extension de résultats de stabilisation globale de systèmes non linéaires déterministes à des systèmes non linéaires stochastiques de la forme

$$\begin{cases} x_t = x_0 + \int_0^t f_0(x_s, u)ds + \sum_{j=1}^{p} \int_0^t f_j(x_s, u)d\omega_s^j \\ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ f_j(0,0) = 0 \end{cases} \tag{0.17}$$

où $\omega = (\omega^1, \dots, \omega^p)$ est un processus de Wiener à valeurs dans $\mathbb{R}^p$ défini sur un espace de probabilité usuel $(\Omega, \mathcal{F}, P)$, avec $\omega^1, \dots, \omega^p$ indépendants, et où les intégrales stochastiques sont prises au sens de Itô. Le système (0.17) est globalement stabilisable s'il existe une commande $u : \mathbb{R}^n \to \mathbb{R}^m$, $u(0) = 0$, telle que la solution $x_t \equiv 0$ soit globalement asymptotiquement stable en probabilité pour le système bouclé

$$x_t = x_0 + \int_0^t f_0(x_s, u(x_s))ds + \sum_{j=1}^{p} \int_0^t f_j(x_s, u(x_s))d\omega_s^j$$

La stabilisation par feedback de ces systèmes n'a commencé à être étudiée que récemment. Les versions stochastiques des théorèmes de Lyapunov [66] et du principe d'invariance de LaSalle [70] sont à la base des résultats obtenus. Rappelons les ici.

Soit x_t le processus à valeur dans $\mathbb{R}^n$ solution au sens de Itô de l'équation différentielle stochastique :

$$x_t = x_0 + \int_0^t \sigma_0(x_s)ds + \sum_{k=1}^{p} \int_0^t \sigma_k(x_s)d\omega_s^k, \quad \sigma_k(0) = 0, \ 0 \le k \le p \tag{0.18}$$

où les champs de vecteurs σ_k sont lipschitziens tels que pour tout $x \in \mathbb{R}^n$ on ait $\sum_{k=0}^{p} \|\sigma_k(x)\| \le K(1 + \|x\|)$, $K > 0$. Pour $x_0 \in \mathbb{R}^n$, $x_t(x_0)$, $t \ge 0$, désigne la solution à l'instant t de (0.18) démarrant de l'état x_0. La solution $x_t \equiv 0$ est stable en probabilité si

$$\forall \varepsilon > 0, \quad \lim_{x_0 \to 0} P\Big(\sup_{t>0} \|x_t(x_0)\| > \varepsilon\Big) = 0$$

S'il existe de plus un voisinage D de l'origine tel que

$$\forall x_0 \in D, \quad P\Big(\lim_{t \to +\infty} \|x_t(x_0)\| = 0\Big) = 1,$$

cette solution est asymptotiquement stable en probabilité. Elle est globalement asymptotiquement stable en probabilité (GASP) si $D = \mathbb{R}^n$.

Soit L le générateur infinitésimal associé à (0.18), défini pour toute fonction Ψ de classe C^2 sur $\mathbb{R}^n$ par

$$L\Psi(x) = \langle\, \sigma_0(x), \nabla\Psi(x)\,\rangle + \frac{1}{2}\sum_{k=1}^{p} \mathrm{Tr}\left[\sigma_k(x)\big(\sigma_k(x)\big)^{\mathrm{T}}\frac{\partial^2\Psi}{\partial x^2}(x)\right]$$

La version stochastique du théorème de Liapounov (*cf.* [66]) s'énonce ainsi. S'il existe un voisinage D de l'origine et une fonction $V : D \to \mathbb{R}$ de classe C^2 définie positive tel que $LV(x) \leq 0$ (resp. $LV(x) < 0$), $\forall x \in D,\ x \neq 0$, alors la solution $x_t \equiv 0$ de (0.18) est stable (resp. asymptotiquement stable) en probabilité. Elle est GASP si $D = \mathbb{R}^n$, V est propre et $LV(x) < 0$, $\forall x \in \mathbb{R}^n,\ x \neq 0$.

Rappelons enfin, que la version stochastique du principe d'invariance de LaSalle (*cf.* [70]) permet d'établir que si V est propre et $LV(x) \leq 0$, $\forall x \in \mathbb{R}^n$, le processus x_t converge en probabilité vers le plus grand ensemble invariant dont le support est contenu dans $\{x_t \,|\, LV(x_t) = 0, \forall t \geq 0\}$.

Les systèmes stochastiques contrôlés de la forme (0.17) considérés dans la littérature [20, 21, 22, 38, 39, 40, 41, 42, 43, 44] ont la particularité d'avoir l'entrée u non affectée par le bruit, *i.e.*

$$\frac{\partial f_j}{\partial u} = 0, \quad \text{pour } 1 \leq j \leq p$$

Ainsi l'intensité du bruit ne dépend pas de la commande du système, ce qui, dans la pratique, n'est pas très réaliste. En fait, cette hypothèse simplifie considérablement les choses et permet de transposer les résultats déterministes aux systèmes stochastiques de ce type de façon automatique. On peut ainsi écrire pour ces systèmes un dictionnaire traduisant les résultats de stabilisation déterministe en stochastique. A titre d'exemple (*cf.* [43]), soit un système non linéaire stochastique affine en contrôle, avec ω à valeurs dans $\mathbb{R}$ pour simplifier,

$$x_t = x_0 + \int_0^t \Big(X(x_s) + \sum_{i=1}^{m} u_i Y_i(x_s)\Big) ds + \int_0^t \tilde{X}(x_s) d\omega_s \tag{0.19}$$

dans lequel le coefficient associé au bruit est indépendant du contrôle. Un tel système peut être considéré comme associé au système déterministe affine en contrôle

$$x_t = x_0 + \int_0^t \Big(X(x_s) + \sum_{i=1}^{m} u_i Y_i(x_s)\Big) ds \tag{0.20}$$

où la dérive $\int_0^t X(x_s)ds$ est perturbée de façon aléatoire par le terme $\int_0^t \tilde{X}(x_s)d\omega_s$. Pour ces systèmes le générateur infinitésimal associé, $\mathcal{L}$, vérifie pour toute fonction V de classe C^2 sur $\mathbb{R}^n$,

$$\mathcal{L}V(x) = LV(x) + \sum_{i=1}^{m} u_i Y_i V(x)$$

où L est l'opérateur différentiel du second ordre défini par

$$LV(x) = XV(x) + \frac{1}{2}\mathrm{Tr}\left[\tilde{X}(x)\tilde{X}^{\mathrm{T}}(x)\frac{\partial^2 V}{\partial x^2}(x)\right]$$

Il s'ensuit que si V est définie positive et propre telle que :

(i') $LV(x) \leq 0$, pour tout $x \in \mathbb{R}^n$;

(ii') l'ensemble

$$\mathcal{K} = \{x \in \mathbb{R}^n \,|\, L^{k+1}V(x) = L^k Y_i V(x) = 0,\ k \in \mathbb{N},\ i = 1, \ldots, m\}$$

est reduit à $\{0\}$;

alors, le feedback $u_i(x) = -Y_i V(x)$ conduit à $\mathcal{L}V(x) \leq 0$, ce qui permet d'établir en [43], par application de la version stochastique du principe d'invariance de LaSalle, que ce feedback stabilise globalement (0.19). Ceci illustre bien que la technique de Jurdjevic-Quinn pour la stabilisation des systèmes déterministes de la forme (0.15) se transpose de façon systématique pour les systèmes stochastiques de la forme (0.19).

Si on se place à présent dans la situation plus générale où la dérive autant que la partie contrôlée du système déterministe (0.20) sont perturbées de façon aléatoire respectivement par les termes $\int_0^t \tilde{X}(x_s)d\omega_s^0$ et $\sum_{i=1}^m g_i(u) \int_0^t \tilde{Y}_i(x_s)d\omega_s^i$, on est conduit à associer à (0.20) le système non linéaire stochastique

$$x_t = x_0 + \int_0^t \Big(X(x_s) + \sum_{i=1}^m u_i Y_i(x_s)\Big)ds + \int_0^t \tilde{X}(x_s)d\omega_s^0 + \sum_{i=1}^m \int_0^t g_i(u)\tilde{Y}_i(x_s)d\omega_s^i \quad (0.21)$$

Pour un tel système le générateur infinitésimal associé vérifie

$$\mathcal{L}V(x) = LV(x) + \sum_{i=1}^m u_i Y_i V(x) + \frac{1}{2}\sum_{i=1}^m g_i^2(u)\mathrm{Tr}\left[\tilde{Y}_i(x)\tilde{Y}_i^{\mathrm{T}}(x)\frac{\partial^2 V}{\partial x^2}(x)\right]$$

On voit maintenant que le feedback de Jurdjevic-Quinn $u_i(x) = -Y_i V(x)$ stabilisant le système déterministe (0.15) sous les conditions (i) et (ii) n'assure plus le même résultat pour le système stochastique (0.21) sous les conditions (i') et (ii').

Les premiers résultats sur des systèmes prenant en compte des perturbation aléatoires autant sur la dérive que sur les termes contrôlés sont ceux de [23], [24], sur les systèmes stochastiques affines en contrôle (de la forme (0.21) avec $g_i(u) = u_i$). Pour cette classe de systèmes [23] donne un feedback stabilisant sous les conditions (i') et (ii'), et [24] donne une condition suffisante de stabilisation par un feedback obtenu à partir d'une fonction de Lyapunov contrôlée, généralisant le résultat déterministe de [97].

Dans le chapitre 5 de ce travail, on considère des systèmes stochastiques de la forme générale (0.17). A partir des champs $f_j(x,u)$, $0 \leq j \leq p$, on associe au système $m+1$ opérateurs différentiels du second ordre $L_0, \dots, L_m$. Ces opérateurs, qui constituent pour (0.17) l'analogue des opérateurs linéaires induits par les champs X, $Y_1, \dots, Y_m$ associés à (0.14) par (0.16), permettent alors d'établir la version stochastique du théorème de Jurdjevic-Quinn pour (0.17). S'il existe $V : \mathbb{R}^n \to \mathbb{R}$ de classe C^∞ définie positive et propre telle que $L_0 V(x) \leq 0$, $\forall x \in \mathbb{R}^n$, et l'ensemble

$$\{x \in \mathbb{R}^n \,|\, L_0^{k+1} V(x) = L_0^k L_i V(x) = 0,\ k \in \mathbb{N},\ i = 1, \dots, m\},$$

est réduit à $\{0\}$, alors le système stochastique (0.17) est globalement stabilisable par un feedback que l'on peut expliciter.

Notons que la forme (0.17) peut être considérée comme la plus générale (elle généralise en particulier (0.21)) dans le sens où, d'une part, le bruit affecte aussi bien la variable d'état que celle de commande, et d'autre part, les champs intervenant dans le système ne sont pas restreints à un type particulier de non linéarité comme pour les systèmes affines en contrôle ou ceux de la forme (0.21). Le résultat obtenu dans ce travail apparaît donc comme une généralisation de celui de [23].

Dans le dernier chapitre, on montre comment le résultat déterministe de [92] peut être étendu à des systèmes partiellement linéaires stochastiques obtenus à partir de (0.10) par adjonction d'un bruit multiplicatif à la partie non linéaire. Remarquons tout de suite que si ce bruit n'affecte pas la variable y (que l'on peut considérer comme un contrôle pour le sous système non linéaire), on obtient un système de la forme

$$\begin{cases} x_t = x_0 + \int_0^t f(x_s, y_s) ds + \int_0^t g(x_s) d\omega_s \\ y_t = y_0 + \int_0^t (A y_s + B u) ds \end{cases}$$

Les conditions données dans [92] conduisent alors directement à la stabilisation globale de ce système par le même feedback que dans [92]. Ceci n'est plus le cas pour les systèmes de la forme

$$\begin{cases} x_t = x_0 + \int_0^t f(x_s, y_s) ds + \int_0^t g(x_s, y_s) d\omega_s \\ y_t = y_0 + \int_0^t (A y_s + B u) ds \end{cases} \tag{0.22}$$

où le bruit affecte la variable y et où de ce fait le feedback stabilisant le système déterministe associé ($\omega_t \equiv 0$) n'assure plus la stabilisation de (0.22). On montre dans ce travail que si la dépendance de $g(x, y)$ en y est de la même forme (0.13) que $f(x, y)$, alors, sous les hypothèses de [92], le système (0.22) est globalement stabilisable par un feedback explicitement donné.

Pour terminer, notons que l'approche stochastique pour l'étude des systèmes contrôlés permet de prendre en considération les perturbations aléatoires auxquelles est soumis un système déterministe (erreurs de modélisation, bruits sur les variables du système). De ce fait, les lois de commande stabilisantes calculées dans ce travail à partir de systèmes stochastiques sont robustes aux bruits, tout en stabilisant le système déterministe idéal associé obtenu en l'absence de bruits.

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1

Noninteracting Control and Disturbance Decoupling with Singularity for Two Dimensional Bilinear Systems

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Noninteracting Control and Disturbance Decoupling with Singularity for Two Dimensional Bilinear Systems

Abstract: This paper investigates the noninteracting control with stability (and related problems) for two dimensional nonlinear systems for which singularities in the decoupling matrix are unavoidable.

Key words: noninteracting control, singularity, stability, nonlinear systems.

1.1 Introduction

This paper is a contribution to the study of the noninteracting control problem with stability for nonlinear systems for which singularities in the decoupling matrix are unavoidable. One of the basic questions in control theory is that of singularities and regularity of the trajectories. This problem has been addressed for the output tracking (see [4] and references therein).

For analytic control affine systems of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

if the decoupling matrix is invertible at the origin, the zero dynamics concept introduced and developped by Byrnes and Isidori, provides a strong tool to the study of the noninteracting problem with stability (see [5], [7], [9], [11]).

If $g(0) = 0$ the decoupling matrix is singular on a non empty set of measure zero. To our knowledge there is no results about this kind of systems. In this paper, our attention is restricted to planar control affine systems.

We consider bilinear systems of the form

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x \\ y = Hx \end{cases}$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$ and $\det H \neq 0$.

The set of singularities is determined as the set of points for which an homogeneous quadratic form vanishes. To overcome the singularities, the feedbacks laws which are considered here must not drive the state of the closed-loop system on the lines of singularities. In fact, those fedbacks are homogeneous of degree zero ($u(kx) = u(x)$ for $k \neq 0$) and consequently bounded on their definition set.

Later on, a bilinear approximation of more general systems is introduced and sufficient conditions are obtained for the noninteracting control with stability and the exact local linearization.

At the end, we deal in the same way with the disturbance decoupling problem with stability when the disturbance is available for measurements.

1.2 Some Previous Results

Consider a nonlinear system of the form

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y = h(x) \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}^m$; f, g_i, $1 \leq i \leq m$ and h are supposed to be analytic functions on a neighborhood U of the origin in $\mathbb{R}^n$.

Consider controls u defined via static state-feedback of the form

$$u = \alpha(x) + \beta(x)v \tag{1.2}$$

where $v(t) \in \mathbb{R}^m$, α and β are analytic functions on some generic open subset E in $\mathbb{R}^n$ such that $\beta(x)$ is an invertible matrix for any x in E.

The noninteracting control problem via static state-feedback is to find a control of the form (1.2) such that for any initial condition in E the output y_i of the closed-loop system

$$\begin{cases} \dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \\ y = h(x) \end{cases}$$

is affected only by the component v_i of the input and not by v_j if $j \neq i$.

Furthermore, notice that the noninteracting control problem with stability is solvable via static state-feedback if the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x) + g(x)\alpha(x)$$

Characteristic numbers and decoupling matrix of the system allow to express necessary and sufficient conditions for the existence of a static state-feedback law which solves the noninteracting control problem on a generic open subset in $\mathbb{R}^n$.

Definition 1.1 *The characteristic numbers* $\rho_1, \ldots, \rho_m$ *of system (1.1) are defined by*

$$\rho_i = \inf\left\{\mu \in \mathbb{N} / \ \exists\, 1 \leq j \leq m, L_{g_j} L_f^{\mu} h_i \not\equiv 0\right\}$$

Definition 1.2 *Assume that* $\rho_i < +\infty$ *for all* i, $1 \leq i \leq m$. *Then, the decoupling matrix* $\Omega(x) \in \mathcal{M}_m(\mathbb{R})$ *is defined by*

$$\Omega(x) = \left(L_{g_j} L_f^{\rho_i} h_i(x)\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, \forall x$$

Denote by R_h the complete ring generated by

$$h_1, L_f h_1, \ldots, L_f^{\rho_1} h_1, \ldots, h_m, \ldots, L_f^{\rho_m} h_m$$

and by R_{h_i}, $1 \leq i \leq m$, the complete ring generated by

$$h_i, L_f h_i, \ldots, L_f^{\rho_i} h_i$$

Notice that for any matrix C in $\mathcal{M}_m(\mathbb{R})$ the components of $Ch(x)$ are in R_h. If in addition C is diagonal, then the *i-th* component of $Ch(x)$ is in R_{h_i}.

Theorem 1.1 (see [3]):

1. $\forall i$, $1 \leq i \leq m$, $\rho_i < +\infty \Rightarrow \rho_i < n$.

2. Assume that $\rho_i < +\infty$ *for all* i, $1 \leq i \leq m$. *Then the noninteracting control problem via static state-feedback is solvable on some open subset* E *in* $\mathbb{R}^n$ *if and only if the matrix* $\Omega(x)$ *is nonsingular on* E.

3. Assume that $\Omega(x)$ *is nonsingular on* E *and let* φ_i, $\psi_i : E \rightarrow \mathbb{R}^n$ *be in* R_{h_i}, $1 \leq i \leq m$, *with* $\psi_i(x) \neq 0$ *for all* x *in* E. *Then the static state-feedback law defined on* E *by*

$$\begin{aligned} \alpha(x) &= \Omega^{-1}(x)\left(\varphi_1(x) - L_f^{\rho_1+1} h_1(x), \ldots, \varphi_m(x) - L_f^{\rho_m+1} h_m(x)\right)^T \\ \beta(x) &= \Omega^{-1}(x) \begin{pmatrix} \psi_1(x) & & \\ 0 & \ddots & 0 \\ & & \psi_m(x) \end{pmatrix} \end{aligned}$$

yields a noninteractive closed-loop system.

1.3 The Case of Planar Bilinear Systems with Two Inputs—Two Outputs

Consider a planar bilinear system

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x \\ y = Hx \end{cases} \tag{1.3}$$

with $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}^2$ and $\det H \neq 0$. Moreover, to matrices B_1 and B_2 associate the following quantity

$$\begin{aligned} \Delta(B_1, B_2) &= (\mathrm{Tr}B_1)^2 \mathrm{Tr}B_2^2 + \mathrm{Tr}B_1^2 (\mathrm{Tr}B_2)^2 + (\mathrm{Tr}B_1B_2)^2 \\ &\quad - \mathrm{Tr}B_1^2 \mathrm{Tr}B_2^2 - 2\mathrm{Tr}B_1B_2 \mathrm{Tr}B_1 \mathrm{Tr}B_2 \end{aligned}$$

Furthermore, without any loss of generality, assume that $\rho_1 = \rho_2 = 0$ and that $\det \mathbf{\Omega}(x)$ is non identically equal to zero on $\mathbb{R}^2$ (which is a generic assumption).

Proposition 1.1 *The noninteracting control problem via static state- feedback for system (1.3) can be achieved on the generic open subset E in $\mathbb{R}^2$ defined by*

$$\begin{array}{ll} E = \mathbb{R}^2 - \{0\} & \text{if } \Delta(B_1, B_2) < 0 \\ E = \mathbb{R}^2 - D & \text{if } \Delta(B_1, B_2) = 0 \\ E = \mathbb{R}^2 - (D_1 \cup D_2) & \text{if } \Delta(B_1, B_2) > 0 \end{array}$$

where D, D_1 and D_2 are straight lines passing through the origin and depending on matrices B_1 and B_2.

Proof: Since $\rho_1 = \rho_2 = 0$, yields $\mathbf{\Omega}(x) = H(B_1x \;\; B_2x)$. Hence,

$$\det \mathbf{\Omega}(x) = 0 \Leftrightarrow \det(B_1x \;\; B_2x) = 0$$

Furthermore, $\det(B_1x \;\; B_2x)$ is an homogeneous quadratic form on $\mathbb{R}^2$ whose discriminant is $\Delta(B_1, B_2)$. Then, in accordance with the sign of $\Delta(B_1, B_2)$, one can deduce that :

1. If $\Delta(B_1, B_2) < 0$, then the origin is the only singularity in the decoupling matrix $\mathbf{\Omega}(x)$.

2. If $\Delta(B_1, B_2) = 0$, then the set of singularities is a straight line D passing through the origin whose equation is given by

$$D \;:\quad 2ax_1 + bx_2 = 0$$

3. If $\Delta(B_1, B_2) > 0$, then the decoupling matrix is singular on two straight lines D_1 and D_2 passing through the origin whose equations are respectively given by

$$D_1 \ : \ 2ax_1 + \left(b + \sqrt{\Delta(B_1, B_2)}\right) x_2 = 0$$

$$D_2 \ : \ 2ax_1 - \left(b + \sqrt{\Delta(B_1, B_2)}\right) x_2 = 0$$

where

$$a = b_{11}^1 b_{21}^2 - b_{11}^2 b_{21}^1$$

$$b = b_{11}^1 b_{22}^2 - b_{11}^2 b_{22}^1 + b_{12}^1 b_{21}^2 - b_{12}^2 b_{21}^1$$

and

$$\left(b_{ij}^k\right)_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}} = B_k, \ 1 \leq k \leq 2$$

Remark: As in proposition 3.1 it has been proved that the noninteracting control problem for system (1.3) can be solved on an open subset E in $\mathbb{R}^2$ which is either $\mathbb{R}^2 - \{0\}$ or $\mathbb{R}^2$ except one or two straight lines passing through the origin, one can add the origin point to this set in order to get a connected set with the origin as an equilibrium point. Furthermore, to solve the noninteracting control problem with stability, one can choose a static state- feedback law on $E \cup \{0\}$ in order to get a noninteractive closed- loop system with the origin (globally) asymptotically stable. In particular, if $E \cup \{0\} \neq \mathbb{R}^2$, the selected feedback law must not drive the state of the closed-loop system on the lines of singularities. In fact, the lines of singularities will be considered as trajectories leading to the origin.

Theorem 1.2 *The noninteracting control problem with stability via static state-feedback for system (1.3) can be achieved on $E \cup \{0\}$.*

Proof: Consider the static state-feedback law on $E \cup \{0\}$ defined by

$$u(x) = \begin{cases} \Omega^{-1}(x)\left[CHx - HAx + \begin{pmatrix} \psi_1(x) & 0 \\ 0 & \psi_2(x) \end{pmatrix} v\right] & \text{if } x \in E \\ \lambda & \text{if } x = 0 \end{cases}$$

where $\psi_i \in R_{H_i}$, $\psi_i(x) \neq 0$ for all x in E, $1 \leq i \leq 2$, $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ with $c_1 < 0,\ c_2 < 0$, and $\lambda = (\lambda_1, \lambda_2)^T$ is any constant vector in $\mathbb{R}^2$. Then,a noninteractive closed-loop system is obtained by application of the results of theorem 2.1 and proposition 3.1.

Furthermore, If $v(t) \equiv 0$, the closed-loop system is given by

$$\begin{cases} \dot{x} = H^{-1}CHx \\ y = Hx \end{cases} \qquad x \in E \cup \{0\}$$

and the feedback law $u = (u_1, u_2)^T$ is of the form

$$u_i(x) = \begin{cases} \frac{p_i(x)}{q(x)} & \text{if } x \in E \\ \lambda_i & \text{if } x = 0 \end{cases} \qquad 1 \leq i \leq 2$$

where p_1, p_2 and q are homogeneous quadradic forms on $\mathbb{R}^2$ such that $q(x) = 0$ if and only if $x \in \mathbb{R}^2 - E$

Notice that u is homogeneous of degree zero on $E \cup \{0\}$ and analytic on E. Hence,it follows that:

1. If $\Delta(B_1, B_2) < 0$, then for any $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$, $c_1 < 0$, $c_2 < 0$, the feedback u is bounded on $\mathbb{R}^2$ and the origin is a globally asymptotically stable equilibrium point of the closed-loop system.

2. If $\Delta(B_1, B_2) \leq 0$, then, to overcome the singularities, consider the asymptotically stable matrix C on the form $C = c\boldsymbol{I}$, $c < 0$, where $\boldsymbol{I}$ is the identity matrix in $\mathcal{M}_2(\mathbb{R})$. Then, for any x_0 in E, the solution $x(t, x_0)$ of the closed-loop system with the initial condition $x(0, x_0) = x_0$ is given for all $t \geq 0$ by

$$x(t, x_0) = e^{ct} x_0$$

Hence, for any $t \geq 0$, yields

$$u\left(x(t, x_0)\right) = u(x_0)$$

Then, one can deduce that the control law u is constant on the trajectories of the closed-loop system and consequently is bounded. The asymptotic stability of the closed-loop system at the origin follows easily.

Remark: When system (1.3) is a priori a noninteractive system, the problem amounts to the stabilization at the origin. Furthermore, the singularities in the decoupling matrix are in fact unreal singularities. Indeed, the feedbacks proposed in theorem 3.1 (with $v(t) \equiv 0$) are constant on E and, with a suitable choice of λ, it can be continued on $\mathbb{R}^2$.

The following exemple has been suggested by anonymous referee. Consider the planar noninteractive bilinear system

$$\begin{cases} \dot{x}_1 = u_1 x_1 \\ \dot{x}_2 = u_2 x_2 \end{cases}$$

where the obvious stabilizing feedback is $u_i = -x_i^2$. Let us applicate theorem 3.1. The decoupling matrix

$$\Omega(x) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

is singular for $x_1 = 0$ and $x_2 = 0$. Nevertheless, for any $\lambda_1 < 0$ and $\lambda_2 < 0$, the feedback law defined on $E \cup \{0\}$, where $E = \mathbb{R}^2 - \{(x_1, x_2) \mathbin{/} x_1 x_2 = 0\}$, by

$$u(x) = \begin{cases} \Omega^{-1}(x) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x & \text{if } x \in E \\ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} & \text{if } x = 0 \end{cases}$$

is a constant feedback

$$u(x) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \forall x \in E \cup \{0\}$$

Then it can be continued on $\mathbb{R}^2$ and we get a stabilizing constant feedback.

1.4 Bilinear Approximation and Noninteracting Control with Stability

Consider nonlinear systems defined on a neighborhood U of the origin in $\mathbb{R}^2$ of the form

$$\begin{cases} \dot{x} = f(x) + u_1\, g_1 x) + u_2\, g_2(x) \\ y = h(x) \end{cases} \tag{1.4}$$

and, suppose that $f, g_1, g_2, h : U \to \mathbb{R}^2$ are analytic functions such that

$$f(0) = g_1(0) = g_2(0) = h(0) = 0$$

With system (1.4) associate the bilinear approximation

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x \\ y = H(x) \end{cases} \tag{1.5}$$

where

$$A = \frac{\partial f}{\partial x}(0), \ B_j = \frac{\partial g_j}{\partial x}(0), \ 1 \le j \le 2, \ H = \frac{\partial h}{\partial x}(0)$$

On the other hand, recall that a function φ is said positively homogeneous of degree $m \ge 0$, if for any vector x and any nonnegative real k one has

$$\varphi(kx) = k^m \varphi(x)$$

Then one can state:

Theorem 1.3 *Assume that:*

1. The characteristic numbers of system (1.5) are equal to zero.

2. $\det H \neq 0$ *and* $\Delta(B_1, B_2) < 0$.

Then, the noninteracting control problem with stability via static state-feedback for system (1.4) can be achieved in a neighborhood of the origin.

Proof: It is easy to prove that for any i, j in $\{1,2\}$ and any x in U

$$L_{g_j} h_i(x) = H_i B_j x + o(\|x\|)$$

Then, assumption *1.* ensures that the characteristic numbers of system (1.4) are equal to zero and that its decoupling matrix

$$\Omega_1(x) = \begin{pmatrix} L_{g_1} h_1(x) & L_{g_2} h_1(x) \\ L_{g_1} h_2(x) & L_{g_2} h_2(x) \end{pmatrix}$$

satisfies

$$\det \Omega_1(x) = \det \Omega_2(x) + o(\|x\|^2)$$

where $\Omega_2(x) = H(B_1 x \ \ B_2 x)$ is the decoupling matrix of system (1.5). Furthermore, one can deduce from assumption *2.* that $\det \Omega_1(x) \neq 0$ for every $x \neq 0$ in a neighborhood of the origin $U' \subseteq U$.

On the other hand let $u(x)$ be a static state-feedback law defined on U' by

$$u(x) = \Omega^{-1}(x)\left[CHx - \begin{pmatrix} L_f h_1(x) \\ L_f h_2(x) \end{pmatrix} + \begin{pmatrix} \psi_1(x) & 0 \\ 0 & \psi_2(x) \end{pmatrix} v\right]$$

if $x \neq 0$, and $u(0) = \lambda$, where $\psi_i \in R_{h_i}$, $\psi_i(x) \neq 0$ for all x in $U' - \{0\}$, $1 \leq i \leq 2$, $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ with $c_1 < 0$, $c_2 < 0$, and λ is any constant vector in $\mathbb{R}^2$. Then, one can deduce from theorem 2.1, that system (1.4) yields a noninteractive closed-loop system.

Furthermore, if $v(t) \equiv 0$, the closed-loop system is

$$\dot{x} = F_1(x) \tag{1.6}$$

where

$$F_1(x) = f(x) + \frac{p_1(x) + \phi_1(x)}{\det \Omega_1(x)} g_1(x) + \frac{p_2(x) + \phi_2(x)}{\det \Omega_1(x)} g_2(x)$$

and $p_1(x)$, $p_2(x)$ are the homogeneous quadratic forms given by

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \det \Omega_2(x) \Omega_2^{-1}(x)\left[CHx - HAx\right]$$

Consider the following systems

$$\dot{x} = \det \Omega_1(x) F_1(x) \tag{1.7}$$
$$\dot{x} = F_2(x) \tag{1.8}$$
$$\dot{x} = \det \Omega_2(x) F_2(x) \tag{1.9}$$

where

$$F_2(x) = Ax + \frac{p_1(x)}{\det \Omega_2(x)} B_1(x) + \frac{p_2(x)}{\det \Omega_2(x)} B_2(x) = H^{-1}CHx$$

Then, the following properties hold:

(i) The origin is an asymptotically stable equilibrium point of system (1.9). Furthermore, without loss of generality, one can assume that

$$\det \Omega_2(x) > 0, \forall x \neq 0$$

which implies

$$\lim_{x \to 0} \det \Omega_2(x) F_2(x) = 0$$

Furthermore, according with [1], notice that systems (1.9) and (1.9) share the same phase portrait ; in particular system (1.9) is asymptotically stable in accordance with system (1.9). Moreover, arguing as above, one can also prove that systems (1.6) and (1.8) share the same phase portrait.

(ii) $\det \Omega_2(x)F_2(x)$ is a positively homogeneous analytic function of degree 3 such that

$$\det \Omega_1(x)F_1(x) = \det \Omega_2(x)F_2(x) + G(x)$$

where G satisfies $\|G(x)\| \leq M \|x\|^4$ for every x in a neighborhood of the origin $U'' \subset U'$.Hence, one can deduce from a theorem of Massera (see [10]), that the origin is an asymptotically stable equilibrium point for (1.8) and therefore for (1.6).

1.5 Exact Linearisation and Stabilisation of Planar Nonlinear Systems

Consider a nonlinear system defined on a neighborhood U of the origin in $\mathbb{R}^2$ by

$$\dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$$

where $f, g_1, g_2 : U \longrightarrow \mathbb{R}^2$ are analytic functions such that

$$f(0) = g_1(0) = g_2(0) = 0$$

On the other hand, let B_1 and B_2 be the matrices defined by

$$B_1 = \frac{\partial g_1}{\partial x}(0), B_2 = \frac{\partial g_2}{\partial x}(0)$$

Then one can prove:

Proposition 1.2 *Assume that $\Delta(B_1, B_2) < 0$. Then for any C in $\mathcal{M}_2(\mathbb{R})$, there exists a feedback law $u(x)$ defined on a neighborhood of the origin $U' \subseteq U$, such that*

$$f(x) + u_1 g_1(x) + u_2 g_2(x) = Cx, \ \forall x \in U'$$

Proof : Set $\Omega(x) = (g_1(x) \ g_2(x))$, then

$$\det \Omega(x) = \det(B_1 x \ B_2 x) + o(\|x\|^2), \ \forall x \in U$$

On the other hand, since $\Delta(B_1, B_2) < 0$, it follows that $\det \Omega(x) \neq 0$ for every $x \neq 0$ in a neighborhood of the origin $U' \subseteq U$.Then, for any C in $\mathcal{M}_2(\mathbb{R})$ and any constant vector λ in $\mathbb{R}^2$, the following feedback law

$$u(x) = \begin{cases} \Omega^{-1}(x)\,[Cx - f(x)] & \text{if } x \in U' - \{0\} \\ \lambda & \text{otherwise} \end{cases}$$

is bounded on U' and analytic on $U' - \{0\}$. Furthermore, one can prove easily that

$$f(x) + u_1 g_1(x) + u_2 g_2(x) = Cx, \ \forall x \in U'$$

Notice that if, in addition, C is asymptotically stable then the origin is a locally asymptotically stable equilibrium point for the closed-loop system.

Example: Consider the following bilinear system

$$\dot{x} = Ax + u_1 B_1 x + u_2 B_2 x \tag{1.10}$$

where

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}, \ B_2 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

Since $\Delta(B_1, B_2) < 0$, it follows from proposition 5.1 that, for any asymptotically stable matrix $C = (c_{ij})_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}}$ in $\mathcal{M}_2(\mathbb{R})$, and any constant vector $\lambda = (\lambda_1, \lambda_2)^T$ in $\mathbb{R}^2$, the homogeneous of degree zero feedback law $u = (u_1, u_2)^T$ defined on $\mathbb{R}^2$ by

$$u_i(x) = \begin{cases} \frac{p_i(x)}{q(x)} & \text{if } x \neq 0 \\ \lambda_i & \text{if } x = 0 \end{cases} \quad 1 \leq i \leq 2$$

where

$$\begin{aligned} p_1(x_1, x_2) &= (c_{21} - 2c_{11} + 1)x_1^2 + (c_{12} + 2c_{22} - 3)x_2^2 \\ &\quad + (c_{11} + c_{22} + 2c_{21} - 2c_{12} - 2)x_1 x_2 \\ p_2(x_1, x_2) &= (3c_{11} + c_{21} - 9)x_1^2 + (c_{12} - c_{22} - 3)x_2^2 \\ &\quad + (c_{11} + c_{22} + 3c_{12} - c_{21} - 8)x_1 x_2 \\ q(x_1, x_2) &= 5x_1^2 + 4x_1 x_2 + 3x_2^2 \end{aligned}$$

stabilizes system (1.10).

Remark: Stabilization of system (1.10) is not obvious :

(i) System (1.10) is not C^1-stabilizable. Indeed, for any C^1- feedback at the origin, the linear approximation of the closed-loop system is

$$\dot{x} = (A + u_1(0)B_1 + u_2(0)B_2)\,x$$

Since

$$\mathrm{Tr}\,(A + u_1(0)B_1 + u_2(0)B_2) = \mathrm{Tr}A + u_1(0)\mathrm{Tr}B_1 + u_2(0)\mathrm{Tr}B_2 = \mathrm{Tr}A > 0$$

then the linearisation has an uncontrollable unstable mode.

(ii) The pairs of linear vector fields (A, B_1) and (A, B_2) are not asympto tically controlable to the origin. Indeed, in a Jordan basis of B_1, matrices A and B_1 are respectively similar to

$$\begin{aligned} A' &= P^{-1}AP = \begin{pmatrix} -\frac{3}{2} & -\frac{15}{2} \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix} \\ B_1' &= P^{-1}B_1P = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

where the matrix P is given by

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$$

Set $\xi = P^{-1}x$. Then the system

$$\dot{x} = Ax + u_1B_1x \tag{1.11}$$

is equivalent to the system

$$\dot{\xi} = A'\xi + u_1B_1'\xi$$

Consider the following function

$$H(\xi_1, \xi_2) = -\xi_1\xi_2$$

The computation of $\dot{H}$, the derivative of H along the trajectories of the vector field $A' + u_1B_1'$ gives

$$\dot{H}(\xi_1, \xi_2) = \frac{1}{2}\xi_1^2 - 2\xi_1\xi_2 + \frac{15}{2}\xi_2^2$$

which is a positive definite homogeneous quadratic form on $\mathbb{R}^2$. Hence, the regions in the second orthant of the form

$$\{(\xi_1, \xi_2)/ - \xi_1\xi_2 > \alpha > 0\}$$

are stable under the action of the vector field $A' + u_1B_1'$ and one can deduce that system (1.11) is not controllable to the origin. Moreover, arguing as above, one can also prove that the system

$$\dot{x} = Ax + u_2B_2x$$

is not controllable to the origin.

1.6 Disturbance Decoupling with Stability for Some Planar Nonlinear Systems

Consider a nonlinear system defined on a neighborhood of the origin in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = f(x) + g(x)u + p(x)w \\ y = h(x) \end{cases}$$

in which w represents an undesired input, or disturbance. The disturbance decoupling problem is to find a static state-feedback $u = \alpha(x) + \beta(x)v$ yielding a closed-loop system in which the output y is completely independant of the disturbance w. Furthermore, one says that the disturbance decoupling problem with stability is solved if, setting $v(t) \equiv 0$ in the definition of the static state-feedback law u, the origin is an asymptotically stable equilibrium point of the closed-loop system. On the other hand, assuming that the disturbance w is available for measurements, one can seek for static state-feedback control of the form

$$u = \alpha(x) + \beta(x)v + \gamma(x)w$$

to solve the disturbance decoupling problem (with stability).

Consider a planar bilinear system

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x + wPx \\ y = Hx \end{cases} \tag{1.12}$$

where $x(t), y(t) \in \mathbb{R}^2$, $u_1(t), u_2(t), w(t) \in \mathbb{R}$ and $A, B_1, B_2, P, H \in \mathcal{M}_2(\mathbb{R})$ with $\det H \neq 0$.

Assume that the disturbance w is available for measurements. Furthermore, as in section 3, assume that $\rho_1 = \rho_2 = 0$ and that $\det \boldsymbol{\Omega}(x)$ is nonidentically equal to zero on $\mathbb{R}^2$, and define $\Delta(B_1, B_2)$, E, D, D_1 and D_2 as in section 3. Then one can prove :

Theorem 1.4 *The disturbance decoupling problem with stability for system (1.12) can be achieved on $E \cup \{0\}$ via static state-feedback of the form*

$$u(x) = \begin{cases} \boldsymbol{\Omega}_1(x)\,[CHx - HAx + \psi(x)v - wHPx] & \text{if } x \in E \\ \lambda & \text{if } x = 0 \end{cases}$$

where the matrix $\psi(x)$ has entries in R_H, λ is any constant vector in $\mathbb{R}^2$ and C is an asymptotically stable element in $\mathcal{M}_2(\mathbb{R})$ such that if $\Delta(B_1, B_2) \geq 0$, then, D (resp. D_1, D_2) is a characteristic direction of $H^{-1}CH$.

Remarks:

1. The feedback law described above, with

$$\psi(x) = \begin{pmatrix} \psi_1(x) & 0 \\ 0 & \psi_2(x) \end{pmatrix}, \ \psi_i \in R_{H_i}, \ \psi_i(x) \neq 0 \ \forall x \in E, \ 1 \leq i \leq 2$$

and

$$C = \begin{cases} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, & c_1 < 0, \ c_2 < 0, \quad \text{if } \Delta(B_1, B_2) < 0 \\ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, & c < 0, \qquad\qquad \text{if } \Delta(B_1, B_2) \geq 0 \end{cases}$$

solves the noninteracting control and disturbance decoupling problem with stability.

2. Recall (see [3]) that if $\rho_i < +\infty$, $1 \leq i \leq 2$, then the disturbance decoupling problem for system (1.12) via static state-feedback $u(x) = \alpha(x) + \beta(x)v + \gamma(x)w$ is solvable on some open subset E in $\mathbb{R}^2$ if and only if the decoupling matrix is nonsingular on E.

Consider now a nonlinear system defined on a neighborhood U of the origin in $\mathbb{R}^2$ by

$$\begin{cases} \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x) + w p(x) \\ y = h(x) \end{cases} \tag{1.13}$$

and assume that $f, g_1, g_2, p, h : U \longrightarrow \mathbb{R}^2$ are analytic functions such that

$$f(0) = g_1(0) = g_2(0) = p(0) = h(0) = 0$$

and that the disturbance w is available for measurements and is bounded on $[0, +\infty[$. With system (1.13) associate the bilinear approximation

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x + w P x \\ y = Hx \end{cases} \tag{1.14}$$

where

$$A = \frac{\partial f}{\partial x}(0), \ B_j = \frac{\partial g_j}{\partial x}(0), \ P = \frac{\partial p}{\partial x}(0), \ H = \frac{\partial h}{\partial x}(0)$$

Then one has:

Theorem 1.5 *Suppose that:*

1. The characteristic numbers of system (1.14) are equal to zero.

2. $\det H \neq 0$ *and* $\Delta(B_1, B_2) < 0$.

Then, the disturbance decoupling with stability for system (1.13) can be achieved on a neighborhood of the origin via static state-feedback of the form

$$u(x) = \mathbf{\Omega}_1^{-1}(x)\left[Ch(x) - \begin{pmatrix} L_f h_1(x) \\ L_f h_2(x) \end{pmatrix} + \psi(x)v - \begin{pmatrix} L_p h_1(x) \\ L_p h_2(x) \end{pmatrix} w\right]$$

if $x \neq 0$, *and* $u(0) = \lambda$, *where* $\mathbf{\Omega}_1(x)$ *is the decoupling matrix associated with system (1.13),* C *is an asymptotically stable matrix in* $\mathcal{M}_2(\mathbb{R})$, $\psi(x)$ *has entries in* R_h, *and* λ *is any constant vector in* $\mathbb{R}^2$.

Proof: The disturbance decoupling problem has been solved via static state- feedback such as above in [3]. Moreover, if $v(t) \equiv 0$, it can be shown that system (1.13) yields the closed-loop system defined on a neighborhood of the origin $U' \subseteq U$ by

$$\dot{x} = \frac{\det \mathbf{\Omega}_2(x) H^{-1} C H x + G_1(x) + w(t) G_2(x)}{\det \mathbf{\Omega}_1(x)}$$

where $\mathbf{\Omega}_2(x)$ is the decoupling matrix associated with system (1.14) and G_1, G_2 are such that

$$\|G_1(x)\| \leq M \|x\|^4, \ \|G_2(x)\| \leq M \|x\|^4$$

for every x in a neighborhood of the origin $U'' \subseteq U'$. Furthermore, since $w(t)$ is bounded on $[0, +\infty[$, by means of the same argument than in the proof of theorem 4.1, one can deduce from a theorem of Massera (see [10]), that the origin is an asymptotically stable equilibrium point of the closed-loop system.

Notice that the feedback described above with

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

$$\psi(x) = \begin{pmatrix} \psi_1(x) & 0 \\ 0 & \psi_2(x) \end{pmatrix}$$

and $c_1 < 0$, $c_2 < 0$, $\psi_i \in R_{h_i}$, $\psi_i(x) \neq 0 \ \forall x \in U' - \{0\}$, $1 \leq i \leq 2$, solves locally the noninteracting control and disturbance decoupling problem with stability.

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Singularity for Static State-Feedback Linearizable Bilinear Systems

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Singularity for Static State-Feedback Linearizable Bilinear Systems

Abstract: This paper deals with the problem of singularity for static state-feedback linearizable bilinear systems. For this class of nonlinear systems, the decoupling matrix is singular on an algebraic surface (which contains the origin), and this relates the static state-feedback linearization to the difficult problem of the completeness of the trajectories of the closed-loop system and/or the boundedness of the feedback laws. In this work, we give a sufficient condition under which all the trajectories of the closed-loop system are complete and the used feedback law is bounded on each of these trajectories.

Key words: Bilinear systems, Static state-feedback, Linearization, Singularity.

2.1 Introduction

Consider a bilinear system of the form

$$\begin{cases} \dot{x} = Ax + u_1 B_1 x + u_2 B_2 x \\ y = Hx \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) = (u_1(t), u_2(t))^{\mathrm{T}} \in \mathbb{R}^2$, $y(t) \in \mathbb{R}^2$, A, B_1 and B_2 are constant matrices in $\mathcal{M}_{n,n}(\mathbb{R})$ and H is a constant matrix in $\mathcal{M}_{2,n}(\mathbb{R})$ of full rank.

If one assumes that it can be found a regular static state-feedback law

$$u = \alpha(x) + \beta(x)v, \quad v \in \mathbb{R}^2 \tag{2.2}$$

defined on the open dense subset E of $\mathbb{R}^n$ where the decoupling matrix of (2.1) is nonsingular, which achieves linearization, thus, a question naturally arises: does there exists a feedback (2.2) so that, when $v = 0$, the set E is positively invariant for the closed-loop system and the input $u(t)$ is bounded for each trajectory of the closed-loop system evolving on E.

For planar bilinear systems, it is shown in [1] that such a feedback exists always and, using a bilinear approach, sufficient conditions are derived for more general nonlinear systems.

In this paper, we consider n-dimensional static state-feedback linearizable bilinear systems for $n \geq 3$. We show that, because of the particular structure of the set of singular points (the zero set of a homogeneous quadratic form) of the decoupling matrix of (2.1), the invariance of E lends itself to a relatively easy test on the structure of the linear drift term of the closed-loop system. We also give, when the invariance of E is met, a sufficient condition under wich the input $u(t)$ is bounded for each trajectory of the closed-loop system evolving on E.

Notice that this question is related to the problem of achieving static noninteracting control with internal asymptotic stability (see [5]). For control affine nonlinear systems with invertible decoupling matrix at the origin, the zero dynamics concept provides a strong tool for the study of this problem (see [3], [4], [6], [7]). That is not the case for the bilinear system (2.1) for which our results give naturally sufficient conditions for the static noninteracting control with stability.

2.2 Notations and preliminaries

For $1 \leq i \leq 2$, let H_i be the i-th row of H and

$$\rho_i = \inf\{\mu \in \mathbb{N} \mid \exists\, 1 \leq j \leq 2,\ H_i A^{\mu} B_j \neq 0\}$$

Throughout the paper it is assumed that:

(h1) *the system (2.1) has no zero dynamics, that is* $\rho_1 + \rho_2 = n - 2$.

(h2) *The matrix* $T \in \mathcal{M}_{n,n}(\mathbb{R})$ *defined by*

$$T = \begin{pmatrix} H_1 \\ \vdots \\ H_1 A^{\rho_1} \\ H_2 \\ \vdots \\ H_2 A^{\rho_2} \end{pmatrix}$$

is nonsingular.

Let $\Omega(x) = (H_i A^{\rho_i} B_j x)_{1 \leq i,j \leq 2}$ be the decoupling matrix of (2.1) and assume that $\det \Omega(x) \not\equiv 0$. Set

$$S = \{x \in \mathbb{R}^n \mid \det \Omega(x) = 0\}$$

and consider on $E = \mathbb{R}^n \backslash S$ a feedback law of the form

$$\begin{aligned} u(x) &= \Omega^{-1}(x)\left[CTx + v - \begin{pmatrix} H_1 A^{\rho_1+1} \\ H_2 A^{\rho_2+1} \end{pmatrix} x\right] \\ C &= \begin{pmatrix} c_{11} & \dots & c_{1n} \\ c_{21} & \dots & c_{2n} \end{pmatrix} \end{aligned} \tag{2.3}$$

It is well known (see, e.g., [5]) that, under the assumptions (**h1**)-(**h2**), (2.3) is a linearizing feedback and, up to the linear change of coordinates $\xi = Tx$, the closed-loop system (2.1-2.3) is equivalent when $v = 0$ to the linear system

$$\dot{\xi} = F_c \xi$$

with

$$F_c = \begin{pmatrix} 0 & 1 & & & & & & & & \\ & \ddots & \ddots & 0 & & & & & & \\ & 0 & \ddots & \ddots & & & & \mathbf{0} & & \\ & & & 0 & 1 & & & & & \\ c_{11} & \cdots & \cdots & \cdots & c_{1\rho_1+1} & c_{1\rho_1+2} & \cdots & \cdots & \cdots & c_{1n} \\ & & & & & 0 & 1 & & & \\ & & & & & & \ddots & \ddots & 0 & \\ & & \mathbf{0} & & & & 0 & \ddots & \ddots & \\ & & & & & & & & 0 & 1 \\ c_{21} & \cdots & \cdots & \cdots & c_{2\rho_1+1} & c_{2\rho_1+2} & \cdots & \cdots & \cdots & c_{2n} \end{pmatrix}$$

Besides, Let $P = P^{\mathrm{T}} \in \mathcal{M}_{n,n}(\mathbb{R})$ be such that $\det \Omega(x) = x^{\mathrm{T}}Px$ and set $\tilde{P} = (T^{-1})^{\mathrm{T}}PT^{-1}$. Then one may easily check that there exists a regular matrix R in $\mathcal{M}_{n,n}(\mathbb{R})$ such that $R^{\mathrm{T}}\tilde{P}R = Q$ with the symetric matrix Q given by

$$Q = \begin{pmatrix} Q^1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.4}$$

where Q^1 is either of the form

$$Q^1 = \begin{pmatrix} \boldsymbol{I}_{n_1} & 0 \\ 0 & -\boldsymbol{I}_{n_2} \end{pmatrix}, \quad 1 \le n_1 \le 2,\ 1 \le n_2 \le 2 \tag{2.5}$$

or of the form

$$Q^1 = \boldsymbol{I}_m, \quad 1 \le m \le 2 \tag{2.6}$$

$\boldsymbol{I}_k$ being the identity matrix in $\mathcal{M}_{k,k}(\mathbb{R})$. Notice that

$$TS = \left\{\xi \in \mathbb{R}^n \,|\, \xi^{\mathrm{T}}\tilde{P}\xi = 0\right\} = R\mathcal{C}$$

where

$$\mathcal{C} = \left\{z \in \mathbb{R}^n \,|\, z^{\mathrm{T}}Qz = 0\right\}$$

We end this section by the following notations: G being a square matrix, let $\Lambda(G)$ be the spectrum of G and set $\lambda(G) = \sup\limits_{\sigma\in\Lambda(G)} \mathcal{R}e(\sigma)$ and $\mu(G) = \inf\limits_{\sigma\in\Lambda(G)} \mathcal{R}e(\sigma)$.

2.3 Invariance condition

Theorem 2.1 *Let R be any regular matrix such that $R^{\mathrm{T}}\tilde{P}R = Q$. If Q is of the form (2.4-2.5) then for $v = 0$, E is positively invariant for the closed-loop system (2.1-2.3) if and only if the matrix $F = R^{-1}F_{\mathcal{C}}R$ is of the form*

$$F = \begin{pmatrix} F^1 & 0 \\ F^2 & F^3 \end{pmatrix} \tag{2.7}$$

where F_1 is of the form

$$F^1 = s\boldsymbol{I}_m + \begin{pmatrix} M^1 & K \\ K^{\mathrm{T}} & M^2 \end{pmatrix} \tag{2.8}$$

with $s \in \mathbb{R}$, $m = n_1 + n_2$ and M^i a skew-symetric matrix in $\mathcal{M}_{n_i,n_i}(\mathbb{R})$.

If Q is of the form (2.4-2.6) then for $v = 0$, E is positively invariant for the closed-loop system (2.1-2.3) if and only if the matrix $F = R^{-1}F_{\mathcal{C}}R$ is of the form (2.7) with $F_1 \in \mathcal{M}_{m,m}(\mathbb{R})$.

Proof: According to the above preliminaries, E is positively invariant for the closed-loop system (2.1-2.3) if and only if $\mathcal{C}$ is invariant for the linear system

$$\dot{z} = Fz \tag{2.9}$$

Set $q(z) = z^{\mathrm{T}}Qz$ and, for a fixed $z \in \mathbb{R}^n$, consider the analytic function defined for $t \in \mathbb{R}$ by

$$h_z(t) = q(\mathrm{e}^{tF}z) = z^{\mathrm{T}}\mathrm{e}^{tF^{\mathrm{T}}}Q\,\mathrm{e}^{tF}z \tag{2.10}$$

By analyticity, one has, for all $t \in \mathbb{R}$,

$$h_z(t) = \sum_{k\geq 0} \frac{t^k}{k!} h_z^{(k)}(0)$$

and one can verify that $\forall\, t \in \mathbb{R}$ and $\forall\, k \in \mathbb{N}$,

$$h_z^{(k)}(t) = z^{\mathrm{T}} \mathrm{e}^{tF^{\mathrm{T}}} Q_k \mathrm{e}^{tF} z$$

where Q_k is defined recursively by

$$Q_0 = Q, \;\; Q_{k+1} = F^{\mathrm{T}} Q_k + Q_k F$$

so that, for any $t \in \mathbb{R}$,

$$h_z(t) = \sum_{k\geq 0} \frac{t^k}{k!} z^{\mathrm{T}} Q_k z$$

Assume now that F is of the form (2.7). Hence:

• If Q is of the form (2.4-2.5) and F^1 is of the form (2.8), then one can prove by induction that for any $k \in \mathbb{N}$, $Q_k = (2s)^k Q$ and for any $t \in \mathbb{R}$,

$$q(\mathrm{e}^{tF} z) = \sum_{k\geq 0} \frac{(2st)^k}{k!} z^{\mathrm{T}} Q z = \mathrm{e}^{2st} q(z) \tag{2.11}$$

So, one can deduce that the algebraic surface $\mathcal{C}$ is invariant for the linear system (2.9).

• Otherwise, if Q is of the form (2.4-2.6) then one may easily check that for any $z \in \mathcal{C}$ and any $k \in \mathbb{N}$ one has $z^{\mathrm{T}} Q_k z = 0$ and so, for any $t \in \mathbb{R}$, $q(\mathrm{e}^{tF} z) = 0$, which allows to state that $\mathcal{C}$ is invariant for (2.9).

Conversely, assume that $\mathcal{C}$ is invariant for (2.9). Then for any fixed $z \in \mathcal{C}$ the function h_z defined by (2.10) and all its derivatives are identically equal to zero. In particular, for $t = 0$, $z \in \mathcal{C}$ and $k \in \mathbb{N}$,

$$h_z^{(k)}(0) = z^{\mathrm{T}} Q_k z = 0 \tag{2.12}$$

Setting

$$F = \begin{pmatrix} F^1 & F^4 \\ F^2 & F^3 \end{pmatrix}, \quad z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

with $F^1 \in \mathcal{M}_{m,m}(\mathbb{R})$, $z^1 \in \mathbb{R}^m$ and $z^2 \in \mathbb{R}^{n-m}$, one has $z \in \mathcal{C}$ if and only if ${z^1}^{\mathrm{T}} Q^1 z^1 = 0$, and:

• If Q is of the form (2.4-2.5) then

$$Q_1 = F^{\mathrm{T}}Q + QF = \begin{pmatrix} F^{1\mathrm{T}}Q^1 + Q^1F^1 & Q^1F^4 \\ F^{4\mathrm{T}}Q^1 & 0 \end{pmatrix}$$

Hence, for $z^2 = 0$, one has from (2.12), $\forall z^1 \in \mathbb{R}^m$,

$$z^{1\mathrm{T}}Q^1z^1 = 0 \Rightarrow z^{1\mathrm{T}}(F^{1\mathrm{T}}Q^1 + Q^1F^1)z^1 = 0$$

that is equivalent to

$$F^{1\mathrm{T}}Q^1 + Q^1F^1 = \lambda Q^1, \ \lambda \in \mathbb{R} \tag{2.13}$$

and, by a simple computation, one gets (2.8). Furthermore, one has from (2.12) and (2.13), $\forall (z^1, z^2)$ in $\mathbb{R}^m \times \mathbb{R}^{n-m}$,

$$z^{1\mathrm{T}}Q^1z^1 = 0 \Rightarrow z^{1\mathrm{T}}Q^1F^4z^2 = (Q^1z^1)^{\mathrm{T}}F^4z^2 = 0$$

that is equivalent to

$$F^4 \cdot \mathbb{R}^{n-m} \subset \left(\mathrm{span}(Q^1 \cdot \mathcal{C}_1)\right)^{\perp}$$

with $\mathcal{C}_1 = \{z^1 \in \mathbb{R}^m \,|\, z^{1\mathrm{T}}Q^1z^1 = 0\}$. Besides, one can easily verify that $\mathrm{span}(Q^1 \cdot \mathcal{C}_1) = \mathbb{R}^m$, which implies that $F^4 \cdot \mathbb{R}^{n-m} = \{0\}$ and so $F^4 = 0$.

• If Q is of the form (2.4-2.6) then, on the one hand

$$Q_1 = F^{\mathrm{T}}Q + QF = \begin{pmatrix} F^{1\mathrm{T}} + F^1 & F^4 \\ F^{4\mathrm{T}} & 0 \end{pmatrix}$$

and $Q_2 = F^{\mathrm{T}}Q_1 + Q_1F$ is of the form

$$Q_2 = \begin{pmatrix} \star & \star \\ \star & 2F^{4\mathrm{T}}F^4 \end{pmatrix}$$

On the other hand, $z \in \mathcal{C}$ if and only if $z^1 = 0$, and it follows from (2.12) that $\forall z^2 \in \mathbb{R}^{n-m}$, $2z^{2\mathrm{T}}F^{4\mathrm{T}}F^4z^2 = 2\|F^4z^2\|^2 = 0$ and so $F^4 = 0$. ∎

Remark 2.1 From above we deduce that the matrix $F = R^{-1}F_cR$ either satisfies or not the conditions of theorem 2.1 independently of the choice of R such that $R^{\mathrm{T}}\tilde{P}R = Q$.

Now, for such a matrix R, set

$$\tilde{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_{\rho_1} \\ R_{\rho_1+2} \\ \vdots \\ R_{n-1} \end{pmatrix}, \quad \bar{R} = \begin{pmatrix} R_{\rho_1+1} \\ R_n \end{pmatrix}$$

where R_i denotes the i-th row of R, and let

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \mathcal{M}_{n-2,n}(\mathbb{R})$$

be defined by

$$\Phi_1 = \begin{pmatrix} 0 & 1 & & & & \\ & \ddots & \ddots & 0 & & \\ & 0 & \ddots & \ddots & & 0 \\ & & & 0 & 1 & \end{pmatrix} \in \mathcal{M}_{\rho_1,n}(\mathbb{R})$$

$$\Phi_2 = \begin{pmatrix} & 0 & 1 & & & \\ & & \ddots & \ddots & 0 & \\ 0 & & 0 & \ddots & \ddots & \\ & & & & 0 & 1 \end{pmatrix} \in \mathcal{M}_{\rho_2,n}(\mathbb{R})$$

Then, for any $C \in \mathcal{M}_{2,n}(\mathbb{R})$ and any $F \in \mathcal{M}_{n,n}(\mathbb{R})$, one has $R^{-1}F_C R = F$ if and only if

$$\tilde{R}F = \Phi R \tag{2.14}$$

$$C = \bar{R}FR^{-1} \tag{2.15}$$

Hence, by taking F in appropriate form of theorem 2.1, it turns out that there exists C in $\mathcal{M}_{2,n}(\mathbb{R})$ such that, for $v = 0$, E is positively invariant for the closed-loop system (2.1-2.3) if and only if the algebraic linear system (2.14) with entries of F as unknowns has solutions. Any solution, when there exists, provides a linearizing feedback (2.3) for (2.1) on E.

2.4 Bounded feedback sufficient condition

Theorem 2.2 *Let R be any regular matrix such that $R^{\mathrm{T}}\tilde{P}R = Q$. If Q is of the form (2.4-2.5) and $F = R^{-1}F_C R$ is of the form (2.7-2.8) with F^1 $\mathbb{C}$-diagonalizable, $\lambda(F^1) = s$ and $\lambda(F^3) < s$, then the feedback (2.3) with $v = 0$ is bounded for each trajectory of the closed-loop system (2.1-2.3) evolving on E.*

If Q is of the form (2.4-2.6) and $F = R^{-1}F_C R$ is of the form (2.7) with $F^1 \in \mathcal{M}_{m,m}(\mathbb{R})$ and $\lambda(F^3) < \mu(F^1)$, then the feedback (2.3) with $v = 0$ is bounded for each trajectory of the closed-loop system (2.1-2.3) evolving on E.

Proof: The feedback law $u(x) = (u_1(x), u_2(x))^{\mathrm{T}}$ given by (2.3), with $v = 0$, is of the form

$$u_i(x) = \frac{p_i(x)}{\det \Omega(x)}, \ \forall x \in E, \ 1 \le i \le 2$$

where p_1 and p_2 are homogeneous quadratic forms on $\mathbb{R}^n$. (Notice that u is analytic and homogeneous of degree zero on E). Hence, using the linear change of coordinates $x = T^{-1}Rz$, one has, according to section 2, for any $x = T^{-1}Rz$ in E

$$u_i(x) = \frac{p_i(T^{-1}Rz)}{z^{\mathrm{T}}Qz}$$

and setting $\tilde{p}_i(z) = p_i(T^{-1}Rz)$ and $z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$ with $z^1 \in \mathbb{R}^m$, $z^1 \notin C_1$ and $z^2 \in \mathbb{R}^{n-m}$, one gets from (2.4)

$$u_i(x) = \frac{\tilde{p}_i(z)}{{z^1}^{\mathrm{T}} Q^1 z^1}$$

So, the closed-loop system (2.1-2.3) being equivalent to the system (2.9) on E, it follows from (2.7) that, for any $x_0 = T^{-1}Rz_0$ in E

$$u_i\left(x(t, x_0)\right) = \frac{\tilde{p}_i(\mathrm{e}^{tF} z_0)}{{\mathrm{e}^{tF^1} z_0^1}^{\mathrm{T}} Q^1 \mathrm{e}^{tF^1} z_0^1}$$

where $x(t, x_0)$ is the solution of the closed-loop system with initial condition $x(0, x_0) = x_0$. Notice that $z_0^1 \notin C_1$ and so, for all $t \ge 0$, $\mathrm{e}^{tF^1} z_0^1 \notin C_1$, that is ${\mathrm{e}^{tF^1} z_0^1}^{\mathrm{T}} Q^1 \mathrm{e}^{tF^1} z_0^1 \neq 0$. Notice also that $\Lambda(F) = \Lambda(F_1) \cup \Lambda(F_3)$ so that $\lambda(F) = \lambda(F_1)$ because of $\lambda(F^3) < \lambda(F^1)$.

First, if Q is of the form (2.4-2.5) and $F = R^{-1}F_C R$ is of the form (2.7-2.8) then one can deduce from (2.11) that

$$u_i\left(x(t, x_0)\right) = \frac{\tilde{p}_i(\mathrm{e}^{tF} z_0)}{\mathrm{e}^{2st} {z_0^1}^{\mathrm{T}} Q^1 z_0^1} \tag{2.16}$$

and from the assumption F^1 is $\mathbb{C}$-diagonalizable and $\lambda(F^3) < \lambda(F^1) = s$, it follows that $e^{tF}z_0 = e^{st}\varphi(t)$ where the $\mathbb{R}^n$-valued function $\varphi(t)$ is bounded for $t \geq 0$. So, by homogeneity

$$u_i\left(x(t,x_0)\right) = \frac{\tilde{p}_i(\varphi(t))}{z_0^{1\,\mathrm{T}} Q^1 z_0^1}$$

and $u\left(x(t,x_0)\right)$ is bounded for $t \geq 0$.

Now, if Q is of the form (2.4-2.6) and $F = R^{-1}F_c R$ is of the form (2.7) with $\lambda(F^3) < \mu(F^1)$, then

$$u_i\left(x(t,x_0)\right) = \frac{\tilde{p}_i(e^{tF}z_0)}{\|e^{tF^1}z_0^1\|^2}$$

and one has:

• If $\lambda(F^1) = \mu(F^1) = s$ then, since $\lambda(F^3) < s$, it follows that $e^{tF}z_0 = e^{st}\varphi(t)$ with $\varphi(t)$ bounded for $t \geq 0$. Besides, on the one hand, when F^1 is $\mathbb{C}$-diagonalizable one has if $\Lambda(F^1) = \{s\}$ then $e^{tF^1}z_0^1 = e^{st}z_0^1$ and, by homogeneity

$$u_i\left(x(t,x_0)\right) = \frac{\tilde{p}_i(\varphi(t))}{\|z_0^1\|^2}$$

So, $u\left(x(t,x_0)\right)$ is bounded for $t \geq 0$. Otherwise, $\Lambda(F^1) = \{s+i\sigma, s-i\sigma\}$, $\sigma \in \mathbb{R}^*$, and there exists a regular matrix $K^1 \in \mathcal{M}_{2,2}(\mathbb{R})$ such that

$$K^1F^1(K^1)^{-1} = \begin{pmatrix} s & \sigma \\ -\sigma & s \end{pmatrix}$$

Hence, one may easily check that $\|e^{tF^1}z_0^1\| \geq ke^{st}$ with $k = \|K^1\|^{-1}\|(K^1)^{-1}\|^{-1}\|z_0^1\|$ and so $|u_i\left(x(t,x_0)\right)| \leq |\tilde{p}_i(\varphi(t))|/k^2$ which implies that $u\left(x(t,x_0)\right)$ is still bounded for $t \geq 0$. On the other hand, when F^1 is non-$\mathbb{C}$-diagonalizable there exists a regular matrix $K^1 \in \mathcal{M}_{2,2}(\mathbb{R})$ such that

$$K^1F^1(K^1)^{-1} = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}$$

and, setting $K^1z_0^1 = (\xi_1,\xi_2)^{\mathrm{T}} \neq 0$, one gets by a simple computation $\|e^{tF^1}z_0^1\| \geq ke^{st}$ with $k = \|K^1\|^{-1}|\xi_1|$ if $\xi_2 = 0$ and $k = \|K^1\|^{-1}|\xi_2|$ if $\xi_2 \neq 0$. So $|u_i\left(x(t,x_0)\right)| \leq |\tilde{p}_i(\varphi(t))|/k^2$ and $u\left(x(t,x_0)\right)$ is bounded for $t \geq 0$.

• If $s = \lambda(F^1) > \mu(F^1) = s'$ then $\Lambda(F^1) = \{s,s'\}$. Besides, $\Lambda(F_1) \cap \Lambda(F_3) = \emptyset$ because of $\lambda(F^3) < \mu(F^1)$, and one can show, by using the jordan forms of F^1 and F^3, that there exists a regular matrix $K \in \mathcal{M}_{n,n}(\mathbb{R})$ of the form

$$K = \begin{pmatrix} K^1 & 0 \\ K^2 & K^3 \end{pmatrix}, \quad K^1 \in \mathcal{M}_{2,2}(\mathbb{R})$$

such that

$$e^{tF} = \begin{pmatrix} e^{tF^1} & 0 \\ -(K^3)^{-1}K_2 e^{tF^1} + e^{tF^3}(K^3)^{-1}K_2 & e^{tF^3} \end{pmatrix} \tag{2.17}$$

Hence, if z_0^1 is an eigenvector of F^1 associated with s' then one has $e^{tF^1} z_0^1 = e^{s't} z_0^1$ and, by (2.17), $e^{tF} z_0 = e^{s't}\psi(t)$ with $\psi(t)$ bounded for $t \geq 0$ because of $\lambda(F^3) < s'$. So

$$u_i\left(x(t, x_0)\right) = \frac{\tilde{p}_i(\psi(t))}{\|z_0^1\|^2}$$

is bounded for $t \geq 0$. Otherwise, $z_0^1 = v_s + v_{s'}$, $v_s \neq 0$, with v_s and $v_{s'}$ being eigenvectors of F^1 associated respectively with s and s' so that

$$\|e^{tF^1} z_0^1\| = e^{st}\|v_s + e^{(s'-s)t}v_{s'}\| \geq k e^{st}$$

for some constante $k > 0$. Besides, $e^{tF} z_0 = e^{st}\varphi(t)$ with $\varphi(t)$ bounded for $t \geq 0$ because of $\lambda(F^3) < s$ and so

$$|u_i\left(x(t, x_0)\right)| \leq \frac{|\tilde{p}_i(\varphi(t))|}{k^2}$$

which implies that $u\left(x(t, x_0)\right)$ is still bounded for $t \geq 0$ and ends the proof. ∎

Remark 2.2 Homogeneous feedback of degree zero have been introduced in [2] and used to stabilize bilinear systems which are not stabilizable by continuous feedback at the origin.

Remark 2.3 A simple computation shows that if F^1 is of the form (2.8) then $\lambda(F^1) \geq s$. In case where Q is of the form (2.4-2.5) and $F = R^{-1}F_c R$ is of the form (2.7-2.8) with $\lambda(F^3) < \lambda(F^1)$ and $\lambda(F^1) > s$, let $\sigma > s$ be an eigenvalue of F^1 (and so of F) and $x_0 = T^{-1}Rz_0$ in E such that $Fz_0 = \sigma z_0$. Then it follows from (2.16) that

$$u_i\left(x(t, x_0)\right) = \frac{e^{2(\sigma - s)t}\tilde{p}_i(z_0)}{{z_0^1}^{\mathrm{T}} Q^1 z_0^1}$$

and $u\left(x(t, x_0)\right)$ is unbounded if $\tilde{p}_i(z_0) \neq 0$. In the same way, one may verify that if $\lambda(F^1) = s$ but F^1 non-$\mathbb{C}$-diagonalizable then $u\left(x(t, x_0)\right)$ is generally unbounded. At last, instead of $\lambda(F^3) < \lambda(F^1) = s$, one can assume, more generally, that $\lambda(F) = s$ and the characteristic roots of F with real part s have simple elementary divisors.

2.5 Example

Consider the following bilinear system which evolves on $\mathbb{R}^5$:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + 2x_2 + x_3 - 3x_4 + x_5 \\ \qquad + \frac{1}{3}u_1(3x_1 + x_2 + 3x_3 + 10x_4 + 3x_5) \\ \qquad + u_2(x_2 + 12x_3 + 4x_4) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = x_5 \\ \dot{x}_5 = -4x_1 + 3x_2 - 2x_3 + 6x_4 + x_5 \\ \qquad + \frac{1}{6}u_1(-4x_1 - 3x_2 + 18x_3 - 2x_5) \\ \qquad + u_2(x_1 - 9x_3 - 6x_4 - x_5) \\ y_1 = x_1, \; y_2 = x_3 \end{cases} \tag{2.18}$$

A simple computation shows that $\rho_1 = 1$, $\rho_2 = 2$ and $T = \boldsymbol{I}_5$. So, assumptions **(h1)** - **(h2)** are satisfied. Besides, the decoupling matrix is given by

$$\boldsymbol{\Omega}(x) = \begin{pmatrix} x_1 + \frac{1}{3}x_2 + x_3 + \frac{10}{3}x_4 + x_5 & x_2 + 12x_3 + 4x_4 \\ -\frac{2}{3}x_1 - \frac{1}{2}x_2 + 3x_3 - \frac{1}{3}x_5 & x_1 - 9x_3 - 6x_4 - x_5 \end{pmatrix}$$

and one can verify that the linear change of coordinates $x = Rz$ with

$$R = \frac{1}{8}\begin{pmatrix} 8 & 8 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & -7 & 1 & 9 \end{pmatrix}$$

transforms $\det \boldsymbol{\Omega}(x)$ to

$$q(z) = z_1^2 + z_2^2 - z_3^2 - z_4^2$$

So, by theorem 2.1, for $C \in \mathcal{M}_{2,5}(\mathbb{R})$ and $v = 0$, $E = \mathbb{R}^5 - \left\{x \in \mathbb{R}^5 \mid \det \boldsymbol{\Omega}(x) = 0\right\}$ is positively invariant for the closed-loop system (2.18 - 2.3) if and only if $F = R^{-1}F_c R$ is of the form

$$F = \begin{pmatrix} s & \alpha & a & b & 0 \\ -\alpha & s & c & d & 0 \\ a & c & s & \beta & 0 \\ b & d & -\beta & s & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \end{pmatrix}$$

Now, system (2.14), with unknowns $\{s, \alpha, \beta, a, b, c, d, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$, is given by

$$\begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 & -3 \end{pmatrix} F = \begin{pmatrix} -16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & -7 & 1 & 9 \end{pmatrix}$$

and by a simple computation, one may check that it has solutions given by

$$F = \begin{pmatrix} -1 & 1 & 0 & \gamma & 0 \\ -1 & -1 & 0 & -\gamma & 0 \\ 0 & 0 & -1 & -2 & 0 \\ \gamma & -\gamma & 2 & -1 & 0 \\ -\gamma & \gamma & 0 & 0 & -3 \end{pmatrix}, \qquad \gamma \in \mathbb{R}$$

Besides, for $\gamma \in \mathbb{R}$ such that $\gamma^2 < 1/2$ the matrices

$$F^1 = \begin{pmatrix} -1 & 1 & 0 & \gamma \\ -1 & -1 & 0 & -\gamma \\ 0 & 0 & -1 & -2 \\ \gamma & -\gamma & 2 & -1 \end{pmatrix} \quad \text{and} \quad F^3 = -3$$

satisfy the conditions of theorem 2.2. Hence, following (2.15), for any $\gamma \in \mathbb{R}$ such that $\gamma^2 < 1/2$ the feedback (2.3) with $v = 0$ and

$$C = \frac{1}{8}\begin{pmatrix} -16 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 1 & 9 \end{pmatrix} F R^{-1} = \begin{pmatrix} -2 & -2 & 6\gamma & 8\gamma & 2\gamma \\ \gamma & \gamma & -15 & -11 & -5 \end{pmatrix}$$

leaves E positively invariant for the closed-loop system (2.18 - 2.3) and it remains bounded on each trajectory of (2.18 - 2.3) evolving on E.

Remark 2.4 In the theorem 2.2, if one has $\lambda(F^1) < 0$ then the feedback law (2.3) is a stabilizing one on the open dense subset E. If in addition the matrix C is of the form

$$C = \begin{pmatrix} c_{11} & \dots & c_{1\rho_1+1} & 0 & \dots & 0 \\ 0 & \dots & 0 & c_{21} & \dots & c_{2\rho_2+1} \end{pmatrix}$$

then the feedback law (2.3) achieves the noninteracting control with stability (see [5]) for system (2.1). For instance, with $\gamma = 0$ one gets in the above example a linear noninteractive closed-loop system with internal stability by using the feedback

$$u_i(x) = \frac{p_i(x) + \varphi_i(x)v_1 + \psi_i(x)v_2}{\det \Omega(x)}$$

with

$$
\begin{aligned}
\det \Omega(x) &= 2\,x_1{}^2 + 2\,x_1\,x_2 - 90\,x_3{}^2 - 96\,x_3\,x_4 \\
&\quad - 12\,x_3\,x_5 - 40\,x_4{}^2 - 16\,x_4\,x_5 - 2\,x_5{}^2 + x_2{}^2 \\
p_1(x) &= -2\,x_1{}^2 - 16\,x_1\,x_2 - 80\,x_1\,x_3 - 14\,x_1\,x_4 + 6\,x_2{}^2 + 170\,x_2\,x_3 + 106\,x_2\,x_4 \\
&\quad + 20\,x_2\,x_5 + 330\,x_3{}^2 + 470\,x_3\,x_4 + 164\,x_3\,x_5 + 100\,x_4{}^2 + 54\,x_4\,x_5 + 2\,x_5{}^2 \\
p_2(x) &= +20\,x_1{}^2 - 29\,x_1\,x_2 - 40\,x_1\,x_3 - 10\,x_1\,x_4 - 18\,x_1\,x_5 \\
&\quad - 18\,x_2{}^2 + 25\,x_2\,x_3 - 85\,x_2\,x_4 - 41\,x_2\,x_5 - 60\,x_3{}^2 \\
&\quad - 416\,x_3\,x_4 - 98\,x_3\,x_5 - 340\,x_4{}^2 - 216\,x_4\,x_5 - 38\,x_5{}^2 \\
\varphi_1(x) &= 2\,x_1 - 18\,x_3 - 12\,x_4 - 2\,x_5 \\
\varphi_2(x) &= 4\,x_1 + 3\,x_2 - 18\,x_3 + 2\,x_5 \\
\psi_1(x) &= -2\,x_2 + 24\,x_3 + 8\,x_4 \\
\psi_2(x) &= 6\,x_1 + 2\,x_2 + 6\,x_3 + 20\,x_4 + 6\,x_5
\end{aligned}
$$

which is computed from (2.3).

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3

A Remark on the Stabilization of Partially Linear Composite Systems

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A Remark on the Stabilization of Partially Linear Composite Systems

Abstract: In this paper, we study the global stabilization, by means of smooth state feedback, of partially linear composite systems. We show how to compute the stabilizing feedback thanks to a weak Lyapunov function for a nonlinear subsystem instead of a stricte one.

Key words: Nonlinear systems, feedback, global stabilization, Lyapunov function.

3.1 Introduction

Many recent papers (see [1], [2], [6] and references therein) addressed the problem of The global stabilization, by means of state feedback, of nonlinear control systems of the form:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = Ay + Bu \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^p, u \in \mathbb{R}^k, A \in \mathcal{M}_{p,p}(\mathbb{R})$, $B \in \mathcal{M}_{p,k}(\mathbb{R})$ and f is a smooth vector field such that:

(h1) The pair (A, B) is stabilizable.

(h2) The equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is globally asymptotically stable (G.A.S).

In [6], the authors assumed that the dependence of $f(x, y)$ on y is of the form:

(h3) $f(x,y) = f(x,0) + G(x,y)\,Cy$, with $C \in \mathcal{M}_{k,p}(\mathbb{R})$ and both C and B of full rank.

They proved that there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ satisfying the following three conditions:

(H1) $P(A + BK) + (A + BK)^T P = -Q$, with Q symmetric positive (^{T}=transpose),

(H2) $(Q^{1/2}, A + BK)$ detectable,

(H3) $B^T P = C$,

if and only if the linear subsytem

$$\left\{ \begin{array}{ll} \dot{y} = Ay + Bu & \\ \tilde{y} = Cy\,, & \tilde{y} \in \mathbb{R}^k \end{array} \right. \tag{3.2}$$

is invertible, weakly minimum phase and with CB symetric positive definite.

Using these conditions, they showed that the system (3.1) is globally asymptotically stabilizable and they gave the stabilizing feedback:

$$u(x, y) = Ky - \frac{1}{2} G(x, y)^T \nabla V(x)$$

where V is a smooth Lyapunov function satisfying:

$$\langle\, \nabla V, f(x, 0) \,\rangle\ < 0\ \forall x \in \mathbb{R}^n\,,\ x \neq 0 \tag{3.3}$$

Notice that the existence of such a strict Lyapunov function V is assured by the condition **(h2)** and the inverse Lyapunov theorem (see [3], [5]). Unfortunately, there is no systematic method to compute a strict Lyapunov function for a given G.A.S system and it is often easier to constructe a weak Lyapunov function for which the hypotheses of LaSalle's invariance principle (see [4]) are satisfied. As an example one can consider the following system which evolves in $\mathbb{R}^2$ (Liénard's equation):

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - h(x_1) x_2 \end{array} \right. \tag{3.4}$$

where it is assumed that for all $x \neq 0$:

$$x g(x) > 0\,, \quad h(x) > 0$$

and

$$G(x) = \int_0^x g(s)\, ds\ \to \infty \text{ as } |x| \to \infty$$

For this system, it seems difficult to construct a strict Lyapunov function. However LaSalle's theorem can be applied in an obvious way by taking:

$$V(x_1, x_2) = \frac{1}{2} x_2^2 + G(x_1)$$

In this paper we show that to compute a stabilizing feedback for the system (3.1) , we do not need to have a strict Lyapunov function for:

$$\dot{x} = f(x, 0) \tag{3.5}$$

We also state that the stabilization procedure is still valid when, in the decomposition **(h3)** of f, the matrix C is of rank $m < k$, provided that CB is of full rank.

3.2 Notations and definitions

Before stating the main theorem let us introduce the following notations and definitions.

Definition 3.1 *A C^1 scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is a weak Lyapunov function for the system on $\mathbb{R}^n$:*

$$\dot{x} = X(x) \tag{3.6}$$

if V is positive definite proper and satisfies:

$$X.V(x) \leq 0, \ \forall x \in \mathbb{R}^n$$

where $X.V$ is the Lie-derivative of V along the trajectories of the vector field X ($X.V(x) = \langle \nabla V(x), X(x) \rangle$ where $\langle .,. \rangle$ is the inner product in $\mathbb{R}^n$).

By a *proper function* we mean a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $\{x \in \mathbb{R}^n | \, V(x) \leq \xi\}$ is compact for each $\xi > 0$. Notice that if the vector field X satisfies the definition 3.1 then all the trajectories of the system (3.6) are bounded because of V is proper and its derivative is non positive. For such a vector field, $X_t(.)$ will denote the flow of X defined on $\mathbb{R}^n$. A subset $E \in \mathbb{R}^n$ is said to be X-invariant if for any $x \in E$ on has $X_t(x) \in E$, $\forall t \geq 0$.

Definition 3.2 *We shall say that the system (3.6) is of LaSalle-type (L-T) if there exist a weak Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for (3.6) such that the largest X-invariant set contained in $E = \{x \in \mathbb{R}^n | \, X.V(x) = 0\}$ is reduced to the origin of $\mathbb{R}^n$.*

Remark 3.1 The system (3.6) is of (L-T) if and only if it is globally asymptotically stable about the origin of $\mathbb{R}^n$ (see [4]).

Remark 3.2 It is often easier to find a function V satisfying the definition 3.2 than one satisfying (3.3). This is typically the case for mechanical systems for example.

3.3 Stabilization by LaSalle's invariance principle

Theorem 3.1 *Assume that the pair (A,B) is stabilizable and* **(H1)**, **(H2)** *and* **(H3)** *hold. Then if the system (3.5) is of L-T, so is the closed-loop system (3.1) with the (stabilizing) feedback:*

$$u(x,y) = Ky - G(x,y)^T \nabla V(x) \tag{3.7}$$

where V is a weak Lyapunov function for (3.5) as in the definition 3.2.

Proof: First of all, if the linear subsystem (3.2) satisfies **(H1)**, **(H2)** and **(H3)**, then it is invertible, weakly minimum phase and with CB symetric positive definite, and it is possible to choose the matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and the symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ such that:

$$y^T Q\, y = 0 \;\Rightarrow\; C\, y = 0 \tag{3.8}$$

Indeed, as done in [6], one can assume, without loss of generality, that (3.2) is in the special ccordinate basis (see [7]):

$$\begin{cases} \dot{y}_{01} = A_{01} y_{01} + A_{11} y_1 \\ \dot{y}_{02} = A_{02} y_{02} + A_{12} y_1 \\ \dot{y}_1 = D_{01} y_{01} + D_{02} y_{02} + D_1 y_1 + CBu \\ \tilde{y} = y_1 \end{cases}$$

with A_{01} Hurwitz, $A_{02} + A_{02}^T = 0$, and take $K = (K_{01}, K_{02}, K_1)$ with:

$$\begin{aligned} K_{01} &= -(CB)^{-1} D_{01} + A_{11}^T P_{01} \\ K_{02} &= (CB)^{-1} D_{02} + A_{12}^T \\ K_1 &= (CB)^{-1} D_1 + \frac{1}{2} I \end{aligned}$$

and:

$$P = \begin{pmatrix} P_{01} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (CB)^{-1} \end{pmatrix}$$

with P_{01} symmetric positive definite such that $P_{01} A_{01} + A_{01}^T P_{01} = -I$. This particular choice of K and P leads to:

$$Q = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

Hence, $y^T Q\, y = \|y_{01}\|^2 + \|y_1\|^2$ and so:

$$y^T Q\, y = 0 \Rightarrow Cy = y_1 = 0$$

Assume now that the system (3.5) is of L-T, and set $X(x) = f(x, 0)$, $x \in \mathbb{R}^n$. Let V be a weak Lyapunov function for (3.5) as in the definition 3.2 and denote by Ω the largest invariant set by X contained in the locus $E = \{x \in \mathbb{R}^n |\, X.V(x) = 0\}$. By hypotheses $\Omega = \{0\}$. From **(H1)** – **(H3)**, for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, set:

$$Z(x,y) = \begin{pmatrix} f(x,y) \\ h(x,y) \end{pmatrix}$$

where:

$$\begin{aligned} h(x,y) &= Ay + Bu(x,y) \\ &= (A+BK)y - BG(x,y)^T \nabla V(x) \end{aligned}$$

and define (see [6]):

$$W(x,y) = V(x) + \frac{1}{2} y^T P y$$

W is of class C^1, definite positive and proper, and its derivative along the trajectories of the vector field Z is given by:

$$\begin{aligned} \dot{W}(x,y) &= Z.W(x,y) \\ &= \langle Z(x,y) \,,\, \nabla W(x,y) \rangle \\ &= X.V(x) + \langle \nabla V(x) \,,\, G(x,y)Cy \rangle - \frac{1}{2} y^T Q y + \Big\langle y \,,\, -PBG(x,y)^T \nabla V(x) \Big\rangle \\ &= X.V(x) - \frac{1}{2} y^T Q y + \Big\langle y \,,\, C^T G(x,y)^T \nabla V(x) \,-\, PBG(x,y)^T \nabla V(x) \Big\rangle \end{aligned}$$

So, by use of **(H3)** one has:

$$\dot{W}(x,y) = X.V(x) - \frac{1}{2} y^T Q y \leq 0$$

Notice that all the trajectories of the closed-loop system are bounded because of W is proper and its derivative is non positive. Set:

$$\begin{aligned} \tilde{E} &= \{(x,y) \in \mathbb{R}^{n+p} |\, Z.W(x,y) = 0\} \\ &= \{(x,y) \in \mathbb{R}^{n+p} |\, X.V(x) = 0, \text{ and } y^T Q y = 0\} \end{aligned}$$

According to LaSalle's theorem (see [4] pp. 66-67) all the solutions of the closed-loop system tend to $\tilde{\Omega}$ the largest invariant set by Z contained in $\tilde{E}$. in order to prove the theorem 1 let us show that $\tilde{\Omega}$ is the origin of $\mathbb{R}^{n+p}$. By (3.8), on $\tilde{E}$ the vector field Z is given by:

$$Z(x,y) = \begin{pmatrix} X(x) \\ Y(x,y) \end{pmatrix}$$

where $X(x) = f(x,0)$ and $Y(x,y) = (A+BK)y - BG(x,y)^T \nabla V(x)$, so that on $\tilde{E}$ the closed-loop system becomes:

$$\begin{cases} \dot{x} = f(x,0) = X(x) \\ \dot{y} = (A+BK)y - BG(x,y)^T \nabla V(x) \end{cases}$$

Let $(x(t), y(t))$ be a solution of the above system with $(x(0), y(0)) = (x,y) \in \tilde{\Omega}$. Since $\tilde{\Omega}$ is Z-invariant we have $(x(t), y(t)) \in \tilde{\Omega}$ for all $t \geq 0$. But one has:

$$\frac{d}{dt}(x(t)) = X(x(t))$$

so that $x(t) = X_t(x)$. Consider now the following set:

$$M = \{x \in \mathbb{R}^n | \, \exists \, y \in \mathbb{R}^p, \text{ such that } (x,y) \in \tilde{\Omega}\}$$

If $x \in M$ then $(x,y) \in \tilde{\Omega}$ for some $y \in \mathbb{R}^p$, and for all $t \geq 0$, $(x(t), y(t)) = (X_t(x), y(t)) \in \tilde{\Omega}$ since $\tilde{\Omega}$ is Z-invariant, that implies $X_t(x) \in M$. Then M is X-invariant wich implies that $M \subset \Omega = \{0\}$. So we have shown that:

$$(x,y) \in \tilde{\Omega} \Rightarrow x = 0$$

Since $x(t) = 0$ for all $t \geq 0$, $y(t)$ becomes a solution of $\dot{y} = (A+BK)y$ and $y(t)^T Q \, y(t) = 0$ for all $t \geq 0$. Hence, from **(H2)** one deduce that $y(t) = 0$ for all $t \geq 0$ and so $\tilde{\Omega} = \{(0,0)\}$ which completes the proof of theorem 1.

Example: Consider the following system evolving in $\mathbb{R}^4$:

$$\begin{cases} \dot{x}_1 = x_2 + (x_1 y_1)^{4/3} y_2 \\ \dot{x}_2 = -x_1^{5/3} - x_1^{4/3} x_2 + (x_1 y_2)^{4/3} y_2 \\ \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases} \tag{3.9}$$

The subsystem:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^{5/3} - x_1^{4/3} x_2 \end{cases}$$

is of the form (3.4) and so it is of L-T thanks to the weak Lyapunov function:

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{3}{8}x_1^{8/3}$$

Besides, the assumptions **H1** – **H3** hold for the linear subsystem:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \\ \tilde{y} = y_2 \end{cases}$$

with $K = (\ -1\ ,\ -\frac{1}{2}\ )$ and $P = I$. Hence the system (3.9) satisfies the conditions of the theorem 1 and so it is stabilizable thanks to the feedback:

$$u(x_1, x_2, y_1, y_2) = -y_1 - \frac{1}{2}y_2 - x_1^3 y_1^{4/3} - x_2(x_1 y_2)^{4/3}$$

Remark 3.3 Throughout all this work it is supposed that in **(h3)** one has $C \in \mathcal{M}_{k,p}(\mathbb{R})$, so that the linear subsytem (3.2) has the same number of inputs and outputs. This restriction can be relaxed by assuming that $C \in \mathcal{M}_{m,p}(\mathbb{R})$, $m \leq k$, and (3.2) is right invertible, weakly minimum phase and with CB of full rank. To make this, notice that, as mentionned in [6], if $m = k$ the assumption CB symetric positive definite can be replaced by CB nonsingular thanks to the use of a static precompensator $u = (CB)^{-1}\tilde{u}$. Then the remark is deduced from the following proposition.

Proposition 3.1 *Assume that $m < k \leq p$ and that both B and CB are of full rank. Then there exists a matrix function $G'(x,y) \in \mathcal{M}_{n,k}(\mathbb{R})$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and a constant matrix $C' \in \mathcal{M}_{k,p}(\mathbb{R})$ such that $C'B$ is nonsingular and:*

$$G(x,y)C = G'(x,y)C' \ , \quad \forall (x,y) \in \mathbb{R}^{n+p}$$

Furthermore, if (3.2) is weakly minimum phase, so is the linear system:

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y}' = C'y \, , \qquad \tilde{y}' \in \mathbb{R}^k \end{cases} \tag{3.10}$$

Proof: From the full rank property of B and CB, it is always possible to choose $\tilde{C} \in \mathcal{M}_{k-m,p}(\mathbb{R})$ in such a way that the block-matrix:

$$C' = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix} \tag{3.11}$$

satisfies $C'B$ nonsingular. For such a choice, and taking $G'(x,y) = (\, G(x,y)\ ,\ 0\,)$, one has:

$$G(x,y)C = G'(x,y)C'$$

Furthermore, from (3.11) one can deduce that the zero dynamics of (3.10) are included in those of (3.2) which completes the proof of the proposition.

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4

Global Stabilization of Continuous and Discrete-Time Nonaffine Control Systems

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Global Stabilization of Continuous and Discete-Time Nonaffine Control Systems

Abstract: We consider continuous-time and discete-time nonaffine control systems for which we give sufficient conditions for smooth stabilization. These conditions generalize the well known Jurdjevic-Quinn result for continuous-time affine control systems. Stabilizing feedback are explicitly computed.

Key words: Continuous-time and discete-time nonlinear systems, stabilization, feedback, Lyapunov functions.

4.1 Introduction

The stabilization of nonlinear control systems is one of the most important problems in control theory. Many techniques have been developed during the last two decays to study the stabilizability of control systems and to design stabilizing feedback. The local stabilizability of nonlinear systems has been extensively studied: necessary conditions have been derived by Brockett [2], Coron [3], Krasnosel'ski and Zabreiko [8], and many machineries (linearization, center manifold theory, zero dynamics, approximation techniques) have been developed for the construction of stabilizing feedback. For the global stabilization problem, one can cite the Jurdjevic-Quinn method ([5],[6],[7],[11],[12], [15]), the use of control Lyapunov functions introduced by Artstein [1] and used by Sontag [13] and Tsinias [15], and other results based on the reduction principle (see e.g. [14] and references therein). Most of the results cited above are restricted to smooth control systems which are affine in the control i.e. of the form

$$\dot{x} = X(x) + \sum_{i=1}^{m} u_i Y_i(x) \tag{4.1}$$

The goal of this paper is to study the global stabilization problem for smooth nonaffine control systems

$$\dot{x} = f(x, u) \tag{4.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function satisfying $f(0,0) = 0$. More precisely we generalize the well known Jurdjevic-Quinn result for affine control systems to nonaffine general systems of the form (4.2). In particular, we prove that if the vector fields

$$X(x) = f(x,0),\ Y_i(x) = \frac{\partial f}{\partial u_i}(x,0),\ i = 1,\dots,m \tag{4.3}$$

satisfy a Jurdjevic-Quinn type condition then system (4.2) is globally stabilizable by means of a bounded smooth state feedback control law $u = u(x)$ with $u(0) = 0$ and with an arbitrary choice of the bound. We also give an explicit design of stabilizing feedback control laws.

Historically, one of the first significant results is due to Jurdjevic and Quinn [6] who used the LaSalle's invariance principle to give a sufficient condition for the global stabilization of an affine nonlinear control system (4.1) with a linear (i.e. $X(x) = Ax$) and dissipative drift. Since then, various Jurdjevic-Quinn type sufficient conditions have been developed by several authors [5], [7], [11], [12], [15]. More specifically, in [12], Outbib and Sallet assumed that:

(c1) There exists a positive definite and proper smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that $X{\cdot}V(x) \leq 0,\ \forall x \in \mathbb{R}^n$. By $X{\cdot}V$ we denote the Lie-derivative of V along the trajectories of the vector field X:

$$X \cdot V(x) = \frac{d}{dt} V(X_t(x))|_{t=0} = \langle\ \nabla V(x), X(x)\ \rangle \tag{4.4}$$

where $\langle .,. \rangle$ is the inner product in $\mathbb{R}^n$ and, as usual in control theory, $X_t(.)$ denotes the flow of the vector field X defined on $\mathbb{R}^n$. By a proper function we mean a function V such that $\{x \in \mathbb{R}^n \mid V(x) \leq \xi\}$ is compact for each $\xi > 0$.

(c2) The set

$$W = \{x \in \mathbb{R}^n | X^{k+1}{\cdot}V(x) = X^k{\cdot}Y_i \cdot V(x) = 0 k \in \mathbb{N},\ i = 1,\dots,m\}$$

is reduced to $\{0\}$. In a similar manner to (4.4), higher order Lie-derivatives are defined inductively by $X^0{\cdot}V(x) = V(x),\ X^{k+1}{\cdot}V(x) = X{\cdot}X^k{\cdot}V(x),\ k \in \mathbb{N}$.

Then they proved that the affine system (4.1) is globally stabilizable by the smooth state feedback control law

$$u(x) = -(Y_1{\cdot}V(x), .., Y_m{\cdot}V(x))^{\mathrm{T}} \tag{4.5}$$

Besides, they remarked that the set W is also defined by

$$W = \{x \in \mathbb{R}^n \mid X^{k+1}{\cdot}V(x) = \mathrm{ad}_X^k Y_i \cdot V(x) = 0,\ k \in \mathbb{N},\ i = 1,\dots,m\}$$

corresponding to the familiar forms in the literature [5], [6], [7], [11], [15]. Clearly, the definition of W proposed in **(c2)** is computationally minimal.

Under assumption **(c1)**, the derivative of the function V along the solutions of the affine control system (4.1) is

$$\dot{V}(x) = X \cdot V(x) + \sum_{i=1}^{m} u_i Y_i \cdot V(x)$$

So Jurdjevic-Quinn feedback (4.5) yields

$$\dot{V}(x) = X \cdot V(x) - \sum_{i=1}^{m} (Y_i \cdot V(x))^2 \leq 0$$

that is the closed-loop system is Lyapunov stable.

For nonaffine control system (4.2), one of the most difficult problems is how to choose $u(x) \not\equiv 0$, $u(0) = 0$, in such a way that $Z \cdot V(x) \leq 0$ for $Z(x) = f(x, u(x))$, provided that the unforced dynamic system $\dot{x} = X(x)$, with $X(x) = f(x, 0)$, is Lyapunov stable and satisfies $X \cdot V(x) \leq 0$.

In this work we prove that if there exists a smooth positive definite and proper function $V(x)$ such that the vector field $X(x) = f(x, 0)$ satisfies $X \cdot V(x) \leq 0$, then, without any supplementary condition, there exists a feedback control law $u(x) \not\equiv 0$ such that $Z \cdot V(x) \leq 0$, and by the way, if the vector fields $X, Y_1, \cdots, Y_m$, determined by (4.3), satisfy **(c1)** and **(c2)** then the above feedback leeds to a globally asymptotically stable closed-loop system. This first statement is an existential result. Since for practical questions arising in automatic control, one generally needs to compute exactly the feedback $u(x)$, we give a second statement which provides an explicit design of such a stabilizing feedback control law. We have to notice that the existential result has been given by Coron [4] but our proof is simpler.

We also prove that the same ideas allow to consider discrete-time nonlinear systems of the general form

$$x(k+1) = f(x(k), u(k)), \qquad k = 0,\ 1,\ 2,\ \ldots$$

for which we give a discrete-time analogous Jurdjevic-Quinn sufficient condition for stabilization, the stabilizing feedback being explicitly computed.

The paper is organized as follows. In section 2, we state and prove the main theorems of this paper. In section 3, we give an interisting extension of previous section result in the single-input case. Section 4 presentes illustrating examples.

4.2 Stabilization of continuous-time systems

Throughout this paragraph it is assumed that the vector fields $X, Y_1, \cdots, Y_m$, defined by (4.3), satisfy **(c1)** and **(c2)**. Then one can state the following stabilizability sufficient condition for system (4.2).

Theorem 4.1 *If the assumptions* **(c1)** *and* **(c2)** *hold, then, for any positive constant* η*, system (4.2) is globally asymptotically stabilizable by means of a smooth feedback law* $u(x)$ *satisfying* $u(0) = 0$ *and* $\|u(x)\| \leq \eta$*,* $\forall\, x \in \mathbb{R}^n$.

Proof: By smoothness, $f(x,u) = f(x,0) + g(x,u)u$ with

$$g(x,u) = \int_0^1 \frac{\partial f}{\partial u}(x,tu)dt \tag{4.6}$$

If one computes the derivative of the Lyapunov function V along the trajectories of system (4.2), one gets :

$$\begin{aligned} \dot{V}(x) &= \langle \nabla V(x), f(x,0) + g(x,u)u \rangle \\ &= X \cdot V(x) - u^{\mathrm{T}} \varphi(x,u) \end{aligned} \tag{4.7}$$

where, for any $x \in \mathbb{R}^n$ and any $u \in \mathbb{R}^m$

$$\varphi(x,u) = -(g(x,u))^{\mathrm{T}}\, \nabla V(x) \tag{4.8}$$

For a fixed $\eta > 0$, let $\alpha : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$\alpha(x,u) = \frac{\eta \varphi(x,u)}{K_1(x) + 2\eta K_2(x)}$$

where K_1 and K_2 are any smooth nonnegative real valued functions satisfying, $\forall x \in \mathbb{R}^n$, $K_1(x) + K_2(x) \neq 0$ and

$$\begin{aligned} K_1(x) &\geq \sup_{\|u\| \leq \eta} \|\varphi(x,u)\| \\ K_2(x) &\geq \sup_{\|u\| \leq \eta} \left\| \frac{\partial \varphi}{\partial u}(x,u) \right\| \end{aligned}$$

Then, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ such that $\|u\| \leq \eta$, one has

$$\begin{aligned} \|\alpha(x,u)\| &\leq \eta \\ \left\| \frac{\partial \alpha}{\partial u}(x,u) \right\| &\leq 1/2 \end{aligned}$$

So, on the one hand, applying the fixed point theorem one can deduce that there exists a unique continuous function $k : \mathbb{R}^n \to \mathbb{R}^m$, with $k(0) = 0$, satisfying for all $x \in \mathbb{R}^n$, $\|k(x)\| \leq \eta$ and $\alpha(x, k(x)) = k(x)$. One the other hand, the implicit function theorem apply to the function $\psi(x, u) = \alpha(x, u) - u$ in each $x_0 \in \mathbb{R}^n$ since $\psi(x_0, k(x_0)) = 0$ and the jacobian matrix

$$\frac{\partial \psi}{\partial u}(x_0, k(x_0)) = \frac{\partial \alpha}{\partial u}(x_0, k(x_0)) - I_m$$

is invertible. So, there exist a neighbourhood $\mathcal{V} \times \mathcal{U}$ of $(x_0, k(x_0))$ and $v : \mathcal{V} \to \mathcal{U}$ such that $v(x_0) = k(x_0)$ and $\psi(x, v(x)) = 0$, $\forall x \in \mathcal{V}$. Now $v \in \mathcal{C}^\infty(\mathcal{V}, \mathcal{U})$ because of ψ is $\mathcal{C}^\infty$, but the equation $\psi(x, u) = 0$ has a unique solution $k(x)$ defined on $\mathbb{R}^n$, and so, $k_{|\mathcal{V}} = v$ and then k is $\mathcal{C}^\infty$. Besides, by (4.7) and

$$\varphi(x, k(x)) = \frac{1}{\eta}(K_1(x) + 2\eta K_2(x))k(x) \tag{4.9}$$

one gets

$$\dot{V}(x) = X \cdot V(x) - \frac{1}{\eta}\left(K_1(x) + 2\eta K_2(x)\right) k^{\mathrm{T}}(x) k(x) \leq 0$$

which implies that the state feedback control law $u = k(x)$ leeds to a Lyapunov-stable closed-loop system.

Notice that all the trajectories of the closed-loop system are bounded because of V is proper and its derivative is non positive. Now, set

$$\begin{aligned} E &= \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\} \\ &= \{x \in \mathbb{R}^n \mid X \cdot V(x) = 0, \text{ and } k(x) = 0\} \end{aligned} \tag{4.10}$$

According to LaSalle's invariance principle (*cf.* [10]) all the solutions of the closed-loop system tend to Ω the largest invariant set contained in E. In order to prove the global asymptotic stability let us show that Ω is the origin of $\mathbb{R}^n$. By (4.3), (4.6) and (4.8), $\varphi(x, 0) = -(Y_1 \cdot V(x), \ldots, Y_m \cdot V(x))^{\mathrm{T}}$, and by (4.9) it turns out that

$$\begin{aligned} k(x) = 0 &\Rightarrow \varphi(x, 0) = 0 \\ &\Rightarrow Y_i \cdot V(x) = 0, \; i = 1, \ldots, m \end{aligned} \tag{4.11}$$

Let $x(t)$ be a solution of the closed-loop system with $x(0) = x \in \Omega$. $k(x)$ vanishing on E and Ω being invariant for the closed-loop system, we have $x(t) = X_t(x) \in \Omega$, $\forall t \geq 0$, and by (4.10) and (4.11), $X \cdot V(X_t(x)) = Y_i \cdot V(X_t(x)) = 0$, $\forall t \geq 0$, $i = 1, \ldots, m$. It follows that for all $k \in \mathbb{N}$

$$\frac{d^k}{dt^k} X \cdot V(X_t(x))\Big|_{t=0} = \frac{d^k}{dt^k} Y_i \cdot V(X_t(x))\Big|_{t=0} = 0$$

Hence, using the Lie derivative definition, one can deduce by induction that

$$\begin{aligned} X^{k+1}\cdot V(x) &= \frac{d^k}{dt^k} X\cdot V(X_t(x))\Big|_{t=0} = 0 \\ X^k\cdot Y_i\cdot V(x) &= \frac{d^k}{dt^k} Y_i\cdot V(X_t(x))\Big|_{t=0} = 0 \end{aligned}$$

and so $x \in W$ which implies, from assumption **(c2)**, that $x = 0$ and ends the proof. ∎

Notice that, as established above, theorem 4.1 gives an existential stabilizability result in the sense that, even if in some particular cases the fixed point can be exactly computed, it does not yield, in general, explicitly the stabilizing feedback control law. By providing an explicit design of such a feedback, the second main theorem, stated bellow, is more close to practical preoccupations in automatic control.

Let now $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the smooth function defined by

$$\varphi(x,v,w) = \left(\int_0^1 (1-t) \frac{\partial^2 f}{\partial u^2}(x,tv)w^2 dt \right)^{\mathrm{T}} \nabla V(x) \tag{4.12}$$

where $(\partial^2 f/\partial u^2)(x,tv) \in \mathcal{L}_2(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ (the space of bilinear applications from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^n$) being the second order derivative of f with respect to u at (x,tv), and $w^2 = (w,w)$:

$$\frac{\partial^2 f}{\partial u^2}(x,tv)w^2 = \left(w^{\mathrm{T}} \frac{\partial^2 f_1}{\partial u^2}(x,tv)w, \ldots, w^{\mathrm{T}} \frac{\partial^2 f_n}{\partial u^2}(x,tv)w \right)^{\mathrm{T}}$$

For a fixed $\eta > 0$, let $K_1(x)$ and $K_2(x)$ be any nonnegative smooth real valued functions satisfying, for any $x \in \mathbb{R}^n$, $K_1(x) + K_2(x) \neq 0$ and

$$K_1(x) \geq \sup_{\|v\| \leq \eta, \|w\|=1} |\varphi(x,v,w)| \tag{4.13}$$

$$K_2(x) \geq \|(Y_1\cdot V(x), \ldots, Y_m\cdot V(x))\| \tag{4.14}$$

and set

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)} \tag{4.15}$$

Notice that the real valued function φ is homegeneous of degree 2 with respect to w. The following statement gives explicitly a bounded stabilizing feedback for system (4.2).

Theorem 4.2 *If the assumptions* **(c1)** *and* **(c2)** *hold, then, for any positive constant* η, *system (4.2) is globally asymptotically stabilizable by means of the feedback law*

$$u(x) = -K(x)(Y_1 \cdot V(x), \ldots, Y_m \cdot V(x))^{\mathrm{T}} \tag{4.16}$$

which satisfies $\|u(x)\| \leq \eta$, $\forall x \in \mathbb{R}^n$

Proof: The inequality $\|u(x)\| \leq \eta$ is an immediate consequence of (4.14), (4.15) and (4.16). Moreover, from (4.3) and (4.12), and the Taylor expansion formula, it follows that the derivative of the Lyapunov function V along the trajectories of the closed-loop system (4.2-4.16) satisfies

$$\begin{aligned} \dot{V}(x) &= \left\langle \nabla V(x), f(x,0) + \frac{\partial f}{\partial u}(x,0)u(x) + \int_0^1 (1-t)\frac{\partial^2 f}{\partial u^2}(x, tu(x))u^2(x)dt \right\rangle \\ &= X \cdot V(x) + \sum_{i=1}^{m} u_i(x) Y_i \cdot V(x) + \varphi(x, u(x), u(x)) \end{aligned}$$

One can deduce that, for $x \in \mathbb{R}^n$ such that $u(x) = 0$ one has $\dot{V}(x) = X \cdot V(x)$, and otherwise, from the homogeneity property of $\varphi(x, v, w)$ with respect to w one gets

$$\begin{aligned} \dot{V}(x) &= X \cdot V(x) - \frac{1}{K(x)}\|u(x)\|^2 + \|u(x)\|^2 \varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right) \\ &= X \cdot V(x) - \frac{1}{K(x)}\|u(x)\|^2 \left[1 - K(x)\varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right)\right] \end{aligned}$$

By (4.13) and (4.15),

$$1 - K(x)\varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right) \geq 0$$

and so one gets, from the assumption **(c1)**, $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$, which implies that the feedback law (4.16) leeds to a Lyapunov-stable closed-loop system. Furtheremore, by (4.13), (4.14) and (4.16), if $u(x) \neq 0$ then $K_2(x) \neq 0$ and

$$1 - K(x)\varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right) \neq 0$$

So, from (4.16), it turns out that $\dot{V}(x) = 0$ if and only if $X \cdot V(x) = Y_i \cdot V(x) = 0$, $i = 1, \ldots, m$. So, as in theorem 4.1, one can deduce from the LaSalle's invariance principle that feedback (4.16) stabilizes globally system (4.2). ∎

4.3 Stabilization of discrete-time systems

Consider now a discrete-time nonlinear system of the form

$$x(k+1) = f(x(k), u(k)) \tag{4.17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^2$ function satisfying $f(0,0) = 0$.

Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ be the $\mathcal{C}^2$ function defined on $\mathbb{R}^n$ by

$$\tilde{f}(x) = f(x, 0) \tag{4.18}$$

and assume that:

(d1) The unforced dynamic system

$$x(k+1) = \tilde{f}(x(k))$$

is Lyapunov-stable and that a $\mathcal{C}^2$ proper Lyapunov function

$$V(x) > 0, \quad x \neq 0, \qquad V(0) = 0$$

is known such that

$$V(\tilde{f}(x)) \leq V(x), \quad \forall x \neq 0$$

(d2) The sets

$$\begin{aligned} W_1 &= \left\{ x \in \mathbb{R}^n \,\middle|\, V(\tilde{f}^{k+1}(x)) - V(\tilde{f}^k(x)) = 0, \forall k \in \mathbb{N} \right\} \\ W_2 &= \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{\partial V}{\partial x}(\tilde{f}^{k+1}(x)) \frac{\partial f}{\partial u}(\tilde{f}^k(x), 0) = 0, \forall k \in \mathbb{N} \right\} \end{aligned}$$

where $\tilde{f}^k : \mathbb{R}^n \to \mathbb{R}^n$, $k \geq 0$, are given recursively by

$$\begin{aligned} \tilde{f}^0(x) &= x \\ \tilde{f}^k(x) &= \tilde{f}(\tilde{f}^{k-1}(x)), \quad \text{for } k \geq 1 \end{aligned}$$

satisfy $W_1 \cap W_2 = \{0\}$.

Let $\tilde{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be defined respectively by

$$\tilde{V}(x, u) = V(f(x, u)) \tag{4.19}$$

$$\varphi(x, u, v) = \int_0^1 (1-t) v^{\mathrm{T}} \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu)\, v \, dt \tag{4.20}$$

For a fixed $\eta > 0$, let $K_1(x)$ and $K_2(x)$ be any nonnegative continuous real valued functions satisfying $K_1(x) + K_2(x) \neq 0$, $\forall x \in \mathbb{R}^n$ and

$$K_1(x) \geq \sup_{\|u\|\leq\eta, \|v\|=1} |\varphi(x,u,v)|, \quad \forall x \in \mathbb{R}^n \tag{4.21}$$

$$K_2(x) \geq \left\| \frac{\partial V}{\partial x}(\tilde{f}(x)) \frac{\partial f}{\partial u}(x,0) \right\|, \quad \forall x \in \mathbb{R}^n \tag{4.22}$$

and set

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)} \tag{4.23}$$

The following result is a discrete-time analogous of theorem 4.2.

Theorem 4.3 *If the assumptions* **(d1)** *and* **(d2)** *hold, then, for any positive constant η, system (4.17) is globally asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = -K(x) \left(\frac{\partial V}{\partial x}(\tilde{f}(x)) \frac{\partial f}{\partial u}(x,0) \right)^{\mathrm{T}} \tag{4.24}$$

which satisfies $\|u(x)\| \leq \eta$, $\forall x \in \mathbb{R}^n$

Proof: If one computes the difference of the Lyapunov function V along the trajectories of the closed-loop system (4.17-4.24), one gets from (4.19) and the Taylor expansion formula:

$$\begin{aligned} \Delta V(x) &= V(f(x,u(x))) - V(x) \\ &= \tilde{V}(x,u(x)) - V(x) \\ &= \tilde{V}(x,0) - V(x) + \frac{\partial \tilde{V}}{\partial u}(x,0)\, u(x) + \int_0^1 (1-t) u^{\mathrm{T}}(x) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu(x))\, u(x)\, dt \end{aligned}$$

Notice that

$$\tilde{V}(x,0) = V(\tilde{f}(x)) \tag{4.25}$$

and

$$\frac{\partial \tilde{V}}{\partial u}(x,0) = \frac{\partial V}{\partial x}(\tilde{f}(x)) \frac{\partial f}{\partial u}(x,0)$$

so that, from (4.20) and (4.24),

$$\Delta V(x) = V(\tilde{f}(x)) - V(x) - \frac{1}{K(x)} u^{\mathrm{T}}(x) u(x) + \varphi(x, u(x), u(x))$$

It follows that, for $x \in \mathbb{R}^n$ such that $u(x) = 0$ one has

$$\Delta V(x) = V(\tilde{f}(x)) - V(x)$$

and otherwise, $\varphi(x, u, v)$ being homogeneous of degree 2 with respect to v one gets:

$$\Delta V(x) = V(\tilde{f}(x)) - V(x) - \frac{1}{K(x)}\|u(x)\|^2 + \|u(x)\|^2 \varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right)$$

As in the proof of theorem 4.2, for any $x \in \mathbb{R}^n$, one has $\|u(x)\| \leq \eta$ and one can deduce that $\Delta V(x) \leq 0$, which leeds to a Lyapunov-stable closed-loop system, and that

$$\begin{aligned} \Delta V(x) = 0 \;\; &\Leftrightarrow \;\; V(\tilde{f}(x)) - V(x) = 0 \text{ and } u(x) = 0 \\ &\Leftrightarrow \;\; V(\tilde{f}(x)) - V(x) = 0 \text{ and } \frac{\partial V}{\partial x}(\tilde{f}(x))\frac{\partial f}{\partial u}(x, 0) = 0 \qquad (4.26) \end{aligned}$$

Notice that all the trajectories of the closed-loop system are bounded because of V is proper and its difference along the trajectories of the closed-loop system is non positive. According to LaSalle's invariance principle for difference equations (see [9]), all the solutions of the closed-loop system tend to Ω the largest invariant set contained in

$$E = \{x \in \mathbb{R}^n \,|\, \Delta V(x) = 0\}$$

Let $x(k)$ be a solution of the closed-loop system with $x(0) = x \in \Omega$. Since Ω is invariant for the closed-loop system we have $x(k) \in \Omega$ for all $k \geq 0$. But, $u(x)$ vanishing on Ω, one has, from (4.18), $x(k) = \tilde{f}^k(x)$ and so, by (4.26),

$$V(\tilde{f}^{k+1}(x)) - V(\tilde{f}^k(x)) = 0$$

and

$$\frac{\partial V}{\partial x}(\tilde{f}^{k+1}(x))\frac{\partial f}{\partial u}(\tilde{f}^k(x), 0) = 0$$

It follows that $x \in W_1 \cap W_2$ which implies, by assumption **(d2)**, that $x = 0$ and ends the proof. ■

Remark 4.1 If in addition, f and V are both C^∞ functions, by choosing K_1 and K_2 C^∞, the stabilizing feedback law (4.24) becomes also C^∞.

In the next section, we turn our interest to the single-input case ($m = 1$) for which we give more general sufficient condition for stabilization.

4.4 Single-input case

Consider the continuous-time nonlinear system (4.2) with the control $u \in \mathbb{R}$, and set

$$\mu = \inf \left\{ k \in \mathbb{N}^* \,\middle|\, \left\langle \nabla V(x), \frac{\partial^k f}{\partial u^k}(x,0) \right\rangle \not\equiv 0 \right\}$$

Replace assumption **(c2)** by

(c'2) The positive integer μ is odd and the set W given by

$$W = \{x \in \mathbb{R}^n | X^{k+1} \cdot V(x) = X^k \cdot Y \cdot V(x) = 0,\ k \in \mathbb{N}\}$$

where Y is the vector field defined by

$$Y(x) = \frac{\partial^\mu f}{\partial u^\mu}(x,0)$$

is reduced to $\{0\}$.

Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be defined by

$$\varphi(x,u) = \frac{1}{\mu!} \int_0^1 (1-t)^\mu \frac{\partial^{\mu+1} f}{\partial u^{\mu+1}}(x,tu) \tag{4.27}$$

The following statement generalizes theorem 4.2 for single-input continuous-time systems.

Theorem 4.4 *If $m = 1$ and the assumptions* **(c1)** *and* **(c'2)** *hold, then, for any positive constant η, system (4.2) is globally asymptotically stabilizable by means of the feedback law $u(x)$ — which satisfies $|u(x)| \leq \eta$, $\forall x \in \mathbb{R}^n$ — given by*

$$u(x) = -\frac{1}{\mu!} \frac{\eta Y \cdot V(x)}{\eta K_1(x) + K_2(x)} \tag{4.28}$$

where $K_1(x)$ and $K_2(x)$ are any nonnegative smooth real valued functions satisfying, for any $x \in \mathbb{R}^n$, $K_1(x) + K_2(x) \neq 0$ and

$$K_1(x) \geq \sup_{|u| \leq \eta} |\langle \nabla V(x), \varphi(x,u) \rangle| \tag{4.29}$$

$$K_2(x) \geq \frac{1}{\mu!} |Y \cdot V(x)| \tag{4.30}$$

Proof: The derivative of the Lyapunov function V along the trajectories of the closed-loop system (4.2-4.28) is given by

$$\begin{aligned}
\dot{V}(x) &= \Big\langle \nabla V(x), f(x,0) + u(x)\frac{\partial f}{\partial u}(x,0) + \cdots + \frac{u^{\mu}(x)}{\mu!}\frac{\partial^{\mu} f}{\partial u^{\mu}}(x,0) \\
&\quad + \frac{u^{\mu+1}(x)}{\mu!}\int_0^1 (1-t)^{\mu}\frac{\partial^{\mu+1} f}{\partial u^{\mu+1}}(x, tu(x)dt)\Big\rangle \\
&= X\cdot V(x) + \frac{u^{\mu}(x)}{\mu!} Y\cdot V(x) + u^{\mu+1}(x)\,\langle \nabla V(x), \varphi(x,u(x))\rangle \\
&= X\cdot V(x) - \frac{u^{\mu+1}(x)}{K(x)}\,[1 - K(x)\,\langle \nabla V(x), \varphi(x,u(x))\rangle]
\end{aligned}$$

where

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)}$$

But, from (4.28), (4.29) and (4.30) it follows that $|u(x)| \leq \eta$, $\forall x \in \mathbb{R}^n$, and that $1 - K(x)\,\langle \nabla V(x), \varphi(x,u(x))\rangle \geq 0$ and so, by assumption **(c1)** and μ being odd, one has $\dot{V}(x) \leq 0$ which implies that the closed-loop system is Lyapunov-stable. As in theorem 4.2, one can deduce the global asymptotic stability from the LaSalle's invariance principle. ∎

Remark 4.2 Notice that if, in addition to **(c1)** and **(c'2)**, f is of the form

$$f(x,u) = f(x,0) + u\frac{\partial f}{\partial u}(x,0) + \cdots + \frac{u^{\mu}}{\mu!}\frac{\partial^{\mu} f}{\partial u^{\mu}}(x,0)$$

that is the function φ determined by (4.27) vanishes identically on $\mathbb{R}^n \times \mathbb{R}$, then the feedback $u(x) = -Y\cdot V(x)$ also globally stabilizes system (4.2).

As for theorem 4.2, a discrete-time analogous of theorem 4.4 may be stated. Consider a discrete-time nonlinear single-input system of the form 4.17 with $u \in \mathbb{R}$ and, for sake of simplicity, f a smooth function. In addition, $\tilde{V}$ being determined by (4.19) that is $\tilde{V} = V \circ f$, assume also that the assumption **(d2)** is replaced by

(d'2) The positive integer $\mu \in \mathbb{N}^*$ defined by

$$\mu = \inf\left\{ k \in \mathbb{N}^* \;\middle|\; \frac{\partial^k \tilde{V}}{\partial u^k}(x,0) \not\equiv 0 \right\} \tag{4.31}$$

is odd and the set

$$W = \left\{ x \in \mathbb{R}^n \,\middle|\, V(\tilde{f}^{k+1}(x)) - V(\tilde{f}^k(x)) = \frac{\partial^\mu \tilde{V}}{\partial u^\mu}(\tilde{f}^k(x), 0) = 0, \ \forall k \in \mathbb{N} \right\}$$

is reduced to $\{0\}$.

Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x,u) = \frac{1}{\mu!} \int_0^1 (1-t)^\mu \frac{\partial^{\mu+1} \tilde{V}}{\partial u^{\mu+1}}(x, tu)\, dt$$

Then one can state that:

Theorem 4.5 *If $m = 1$ and the assumptions* **(d1)** *and* **(d'2)** *hold, then, for any positive constant η, system (4.17) is globally asymptotically stabilizable by means of the feedback law $u(x)$ — which satisfies $|u(x)| \leq \eta$, $\forall\, x \in \mathbb{R}^n$ — given by*

$$u(x) = -\frac{1}{\mu!} \frac{\eta}{\eta K_1(x) + K_2(x)} \frac{\partial^\mu \tilde{V}}{\partial u^\mu}(x, 0) \tag{4.32}$$

where $K_1(x)$ and $K_2(x)$ are any nonnegative smooth real valued functions satisfying $K_1(x) + K_2(x) \neq 0$ and

$$K_1(x) \ \geq \ \sup_{|u| \leq \eta} |\varphi(x,u)|, \quad \forall x \in \mathbb{R}^n \tag{4.33}$$

$$K_2(x) \ \geq \ \frac{1}{\mu!} \left| \frac{\partial^\mu \tilde{V}}{\partial u^\mu}(x,0) \right|, \quad \forall x \in \mathbb{R}^n \tag{4.34}$$

Proof: Frome (4.19) and (4.31), the difference of the Lyapunov function V along the trajectories of the closed-loop system (4.17-4.32) is given by

$$\begin{aligned} \Delta V(x) &= V(f(x, u(x))) - V(x) \\ &= \tilde{V}(x, u(x)) - V(x) \\ &= \tilde{V}(x,0) - V(x) + \frac{u(x)^\mu}{\mu!} \frac{\partial^\mu \tilde{V}}{\partial u^\mu}(x,0) + u(x)^{\mu+1} \varphi(x, u(x)) \end{aligned}$$

and one gets from (4.25) and (4.32):

$$\Delta V(x) = V(\tilde{f}(x)) - V(x) - \frac{u^{\mu+1}(x)}{K(x)} \left[1 - K(x)\, \varphi(x, u(x)) \right]$$

where

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)}$$

So, like in the proof of theorem 4.4, one gets $\Delta V(x) \leq 0$. As for theorem 4.2, the global asymptotic stability is deduced from the LaSalle's invariance principle for difference equations. ■

4.5 Examples

Example 1. Consider the system evolving in $\mathbb{R}^3$

$$\begin{cases} \dot{x}_1 = x_2^3 + (x_2^2 - 2x_2x_3)u + x_1u^2 \\ \dot{x}_2 = -x_1^3 - (x_2^2 - x_3^2)u + u^3 \\ \dot{x}_3 = -x_3^3 + (x_1^2 + x_2^2 - 3x_3^2)u + (x_2 - x_1)u^2 + 2u^3 \end{cases} \tag{4.35}$$

The vector fields X and Y defined in (4.3) are given by

$$X(x) = \begin{pmatrix} x_2^3 \\ -x_1^3 \\ -x_3^3 \end{pmatrix} \quad \text{and} \quad Y(x) = \begin{pmatrix} x_2^2 - 2\,x_2\,x_3 \\ -x_2^2 + x_3^2 \\ x_1^2 + x_2^2 - 3\,x_3^2 \end{pmatrix}$$

By taking the Lyapunov function $V(x) = (x_1^4 + x_2^4 + x_3^4)/4$, one has $X{\cdot}V(x) = -x_3^6 \leq 0$, and

$$\begin{aligned} Y{\cdot}V(x) &= x_1^3\,x_2^2 - x_2^5 - 2\,x_1^3\,x_2\,x_3 + x_2^3\,x_3^2 + x_1^2\,x_3^3 + x_2^2\,x_3^3 - 3\,x_3^5 \\ X{\cdot}Y{\cdot}V(x) &= -2\,x_1^6\,x_2 + 5\,x_1^3\,x_2^4 + 3\,x_1^2\,x_2^5 + 2\,x_1^6\,x_3 - 6\,x_1^2\,x_2^4\,x_3 \\ &\quad - 3\,x_1^3\,x_2^2\,x_3^2 + 2\,x_1\,x_2^3\,x_3^3 - 2\,x_2^3\,x_3^4 - 3\,x_1^2\,x_3^5 - 3\,x_2^2\,x_3^5 + 15\,x_3^7 \\ X^2{\cdot}Y{\cdot}V(x) &= 2\,x_1^9 - 20\,x_1^6\,x_2^3 - 27\,x_1^5\,x_2^4 + 15\,x_1^2\,x_2^7 + 6\,x_1\,x_2^8 \\ &\quad + 36\,x_1^5\,x_2^3\,x_3 - 12\,x_1\,x_2^7\,x_3 + 6\,x_1^6\,x_2\,x_3^2 - 9\,x_1^2\,x_2^5\,x_3^2 - 2\,x_1^6\,x_3^3 \\ &\quad - 6\,x_1^4\,x_2^2\,x_3^3 + 6\,x_1^2\,x_2^4\,x_3^3 + 2\,x_2^6\,x_3^3 + 12\,x_1^3\,x_2^2\,x_3^4 + 6\,x_1^3\,x_2\,x_3^5 \\ &\quad - 12\,x_1\,x_2^3\,x_3^5 + 8\,x_2^3\,x_3^6 + 15\,x_1^2\,x_3^7 + 15\,x_2^2 x_3^7 - 105\,x_3^9 \end{aligned}$$

So one gets $X{\cdot}V(x_1, x_2, x_3) = 0$ if and only if $x_3 = 0$, and

$$\begin{aligned} Y{\cdot}V(x_1, x_2, 0) &= (x_1 - x_2)\;x_2^2\,\left(x_1^2 + x_1\,x_2 + x_2^2\right) \\ X{\cdot}Y{\cdot}V(x_1, x_2, 0) &= x_1^2\,x_2\,\left(-2\,x_1^4 + 5\,x_1\,x_2^3 + 3\,x_2^4\right) \\ X^2{\cdot}Y{\cdot}V(x_1, x_2, 0) &= x_1\,\left(2\,x_1^8 - 20\,x_1^5\,x_2^3 - 27\,x_1^4\,x_2^4 + 15\,x_1\,x_2^7 + 6\,x_2^8\right) \end{aligned}$$

Hence, $X\cdot V(x) = Y\cdot V(x) = X\cdot Y\cdot V(x) = X^2\cdot Y\cdot V(x) = 0$ imply $x_1 = x_2 = x_3 = 0$. So we have $W = \{0\}$, and system (4.35) satisfies the assumptions **(c1)** and **(c2)**. Then it can be globally stabilized by a bounded smooth state feedback. Besides, the function $\varphi : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined in (4.12) is given by

$$\varphi(x, v, w) = w^2 \left[x_1^4 + v\, x_2^3 + (x_2 - x_1 + 2\, v)x_3^3\right]$$

and one may easily check that, for a given positive real number η, the real valued functions K_1 and K_2 defined on $\mathbb{R}^3$ by

$$\begin{aligned} K_1(x) &= 1 + 3\eta + x_1^4 + \eta x_2^6 + 2\eta x_3^6 + (x_2 - x_1)^2 x_3^6 \\ K_2(x) &= 1 + (Y\cdot V(x))^2 \end{aligned}$$

satisfies (4.13) and (4.14). So, by theorem 4.2, system (4.35) can be globally stabilized by the feedback law

$$u(x) = \frac{-\eta\, Y\cdot V(x)}{\eta K_1(x) + K_2(x)}$$

Notice that the linearization of (4.35) at the origin gives no answer to the local stabilization question.

Example 2. Consider the planar system

$$\begin{cases} \dot{x}_1 = -x_2 + x_2^2 u + x_1 x_2 u^3 + x_1 u^3 \sin u \\ \dot{x}_2 = x_1 - x_1 x_2 u + x_2 u^3 + x_2 u^3 \sin u \end{cases} \tag{4.36}$$

With the notations introduced in **(c'2)** and for $V(x) = (x_1^2 + x_2^2)/2$ one has immediattely $\mu = 3$ and

$$\begin{aligned} X\cdot V(x) &= 0 \\ Y\cdot V(x) &= 6(x_1^2 + x_2)x_2 \\ X\cdot Y\cdot V(x) &= 6(-2x_1x_2^2 + x_1^3 + 2x_1x_2) \end{aligned}$$

which yields $W = \{0\}$. According with (4.27) one has $\varphi(x, u) = (\sin u)x$ and for a given $\eta > 0$, (4.29) and (4.30) hold with

$$\begin{aligned} K_1(x) &= x_1^2 + x_2^2 \\ K_2(x) &= x_1^2(1 + x_2^2) + x_2^2 + 1 \end{aligned}$$

So, following theorem 4.4, the feedback control law defined by

$$u(x) = \frac{-\eta x_2(x_1^2 + x_2^2)}{(\eta + 1)(x_1^2 + x_2^2) + x_1^2 x_2^2 + 1}$$

globally stabilizes system (4.36). Notice that the "approximation"

$$\begin{cases} \dot{x}_1 = -x_2 + x_2^2 u \\ \dot{x}_2 = x_1 - x_1 x_2 u \end{cases} \tag{4.37}$$

of (4.36) is not stabilizable. Indeed, one may easily check that $V(x) = (x_1^2 + x_2^2)/2$ is a first integral for (4.37) independently of u.

Example 3. Consider the smooth system affine in control evolving in $\mathbb{R}^3$

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) + x_2^2(k) \\ x_2(k+1) = u_1(k) + u_2(k) \\ x_3(k) = x_3(k) + u_2(k)x_1(k) \end{cases} \tag{4.38}$$

With the notations introduced in section 2, one has

$$\tilde{f}(x) = f(x,0) = \begin{pmatrix} x_1 + x_2 + x_2^2 \\ 0 \\ x_3 \end{pmatrix} \quad \text{and} \quad \frac{\partial f}{\partial u}(x,u) = g(x) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & x_1 \end{pmatrix}$$

By taking the Lyapunov function $V(x) = (x_1 + x_2 + x_2^2 + x_3^2)^2 + x_2^2 + x_3^2$ one may easily check that

$$V(\tilde{f}(x)) - V(x) = -x_2^2 \leq 0$$

and that

$$\frac{\partial V}{\partial x}(\tilde{f}(x))g(x) = \big(2(x_1 + x_2 + x_2^2 + x_3^2), 2(x_1 + x_2 + x_2^2 + x_3^2)(1 + 2x_1x_3) + 2x_1x_3\big)$$

So one gets

$$V(\tilde{f}(x)) - V(x) = 0 \Leftrightarrow x_2 = 0$$

and

$$\frac{\partial V}{\partial x}(\tilde{f}(x_1,0,x_3))g(x_1,0,x_3) = \big(2(x_1 + x_3^2), 2(x_1 + x_3^2)(1 + 2x_1x_3) + 2x_1x_3\big)$$

Hence, it is immediate that $V(\tilde{f}(x)) - V(x) = 0$ and $(\partial V/\partial x)(\tilde{f}(x))g(x) = 0$ imply $x_1 = x_2 = x_3 = 0$. So we have $W_1 \cap W_2 = \{0\}$, and system (4.38) satisfies the assumptions **(d1)** and **(d2)**. Then, by using theorem 4.3, it can be globally stabilized by a bounded smooth state feedback. Besides, the function $\varphi : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined in (4.20) is given by

$$\varphi(x,u,v) = a(x,u)v_1^2 + b(x,u)v_1v_2 + c(x,u)v_2^2$$

with

$$\begin{aligned}
a(x,u) &= 2(1+x_1+x_2+x_2^2+x_3^2)+2u_1+2(1+\frac{2}{3}x_1x_3)u_2 \\
&\quad +u_1^2+2u_1u_2+(1+\frac{1}{3}x_1^2)u_2^2 \\
b(x,u) &= 4(1+x_1+x_2+x_2^2+x_3^2+x_1x_3)+4(1+\frac{2}{3}x_1x_3)u_1+4(1+\frac{1}{3}x_1^2+\frac{4}{3}x_1x_3)u_2 \\
&\quad +2u_1^2+4(1+\frac{1}{3}x_1^2)u_1u_2+2(1+x_1^2)u_2^2 \\
c(x,u) &= 2+2x_1+2x_2+x_1^2+4x_1x_3+2x_2^2+2x_3^2+2x_1^3+2x_1^2x_2+2x_1^2x_2^2+6x_1^2x_3^2 \\
&\quad +2(1+\frac{1}{3}x_1^2+\frac{4}{3}x_1x_3)u_1+2(1+x_1^2+2x_1x_3+2x_1^3x_3)u_2 \\
&\quad +(1+\frac{1}{3}x_1^2)u_1^2+2(1+x_1^2)u_1u_2+(1+2x_1^2+x_1^4)u_2^2
\end{aligned}$$

and one may easily check that, for a given $\eta > 0$, the real valued functions K_1 and K_2 defined on $\mathbb{R}^3$ by

$$\begin{aligned}
K_1(x) &= \left(2+\eta^2\right)x_1{}^4+4\eta x_3{}^2x_1{}^4+4\,x_2\,x_1{}^3+\left(8\,\eta^2+11+8\,\eta\right)x_1{}^2+4\,x_1{}^2x_2{}^2 \\
&\quad +(14+16\,\eta)\,x_3{}^2x_1{}^2+16\,x_2\,x_1+24+32\,\eta+16\,x_2{}^2+8\,x_3{}^2+16\,\eta^2 \\
K_2(x) &= 1+4\left(x_1+x_2+x_2{}^2+x_3{}^2\right)^2 \\
&\quad +\left[2\left(x_1+x_2+x_2{}^2+x_3{}^2\right)(1+2\,x_1\,x_3)+2\,x_1\,x_3\right]^2
\end{aligned}$$

satisfy (4.21) and (4.22). So, by application of theorem 4.3, system (4.38) can be globally stabilized by the feedback law

$$u(x)=\frac{-\eta}{\eta\,K_1(x)+K_2(x)}\begin{pmatrix} 2(x_1+x_2+x_2^2+x_3^2) \\ 2(x_1+x_2+x_2^2+x_3^2)(1+2x_1x_3)+2x_1x_3 \end{pmatrix}$$

Notice that the linearization of (4.38) at the origin gives no answer to the local stabilization question.

Example 4. Consider the planar system

$$\left\{\begin{array}{l} x_1(k+1)=x_1(k)+x_1^3(k)u(k) \\ x_2(k+1)=x_2(k)+x_2(k)u(k)+x_2(k)x_1^2(k)u^2(k) \end{array}\right. \tag{4.39}$$

whose unforced dynamic system

$$\left\{\begin{array}{l} x_1(k+1)=x_1(k) \\ x_2(k+1)=x_2(k) \end{array}\right.$$

is Lyapunov-stable but not asymptotically stable. Furthermore, the linear approximation of (4.39) has two uncontrollable critical modes associated with the eigenvalue $\lambda = 1$, and thus, the principle of stability in the first approximation does not apply.

Using the 2nd section's notations one has

$$\tilde{f}(x) = f(x,0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \frac{\partial f}{\partial u}(x,0) = \begin{pmatrix} x_1^3 \\ x_2 \end{pmatrix}$$

For the Lyapunov function $V(x) = x_1^2 + x_2^2$ one has immediately $W_1 = \mathbb{R}^2$ and

$$\frac{\partial V}{\partial x}(\tilde{f}(x))\frac{\partial f}{\partial u}(x,0) = 2x_1^4 + 2x_2^2$$

which yields $W_2 = \{0\}$ and so **(d1)** and **(d2)** hold. Moreover, a simple computation gives

$$\varphi(x,u,v) = (x_1^6 + 2x_1^2x_2^2 + x_2^2 + 2x_1^2x_2^2u + x_1^4x_2^2u^2)v^2$$

and one may easily check that, for a given $\eta > 0$, the real valued functions K_1 and K_2 defined on $\mathbb{R}^2$ by

$$\begin{aligned} K_1(x) &= x_1^6 + 2x_1^2x_2^2 + x_2^2 + 2x_1^2x_2^2\eta + x_1^4x_2^2\eta^2 \\ K_2(x) &= 1 + 2(x_1^4 + x_2^2) \end{aligned}$$

satisfy (4.21) and (4.22). Hence, by application of theorem 4.3, system (4.39) can be globally stabilized by the feedback law

$$u(x) = \frac{-2\eta(x_1^4 + x_2^2)}{\eta\,(x_1^6 + 2x_1^2x_2^2 + x_2^2 + 2x_1^2x_2^2\eta + x_1^4x_2^2\eta^2) + 2(x_1^4 + x_2^2) + 1}$$

Example 5. Consider the planar single-input system

$$\begin{cases} x_1(k+1) = -x_2(k) + x_1^2(k)u(k) + \frac{1}{2}x_1^2(k)x_2(k)u^2(k) \\ x_2(k+1) = x_1(k) + x_1(k)x_2(k)u(k) - \frac{1}{2}x_1^3(k)u^2(k) + u^3(k) \end{cases} \tag{4.40}$$

As for the above example, the unforced dynamic system is Lyapunov-stable but not asymptotically stable and the principle of stability in the first approximation does not apply.

With the notations introduced in section 3 and for $V(x) = x_1^2 + x_2^2$ one has

$$\tilde{f}(x) = f(x,0) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

and by a simple computation

$$\tilde{V}(x,u) = x_1^2 + x_2^2 + 2x_1u^3 + u^4\varphi(x,u)$$

with

$$\varphi(x,u) = 2x_1x_2 + \frac{1}{4}x_1^4(x_1^2+x_2^2) - x_1^3u + u^2$$

It follows immediately that $\mu = 3$ and

$$\begin{aligned} V(\tilde{f}(x)) - V(x) &= 0 \\ \frac{\partial^3\tilde{V}}{\partial u^3}(x,0) &= 12x_1 \\ \frac{\partial^3\tilde{V}}{\partial u^3}(\tilde{f}(x),0) &= -12x_2 \end{aligned}$$

which yields $W = \{0\}$. Besides, one can verify that for a given $\eta > 0$, (4.33) and (4.34) hold with

$$\begin{aligned} K_1(x) &= 2(1+x_1^2x_2^2) + \frac{1}{4}x_1^4(x_1^2+x_2^2) + \eta x_1^2(1+x_1^2) + \eta^2 \\ K_2(x) &= 2(1+x_1^2) \end{aligned}$$

So, following theorem 4.5, the feedback control law

$$u(x) = \frac{-2\eta x_1}{2(1+x_1^2) + \eta(2(1+x_1^2x_2^2) + \frac{1}{4}x_1^4(x_1^2+x_2^2) + \eta x_1^2(1+x_1^2) + \eta^2)}$$

globally stabilizes system (4.40). Notice that the "approximations"

$$\begin{cases} x_1(k+1) = -x_2(k) + x_1^2(k)u(k) \\ x_2(k+1) = x_1(k) + x_1(k)x_2(k)u(k) \end{cases} \tag{4.41}$$

and

$$\begin{cases} x_1(k+1) = -x_2(k) + x_1^2(k)u(k) + \frac{1}{2}x_1^2(k)x_2(k)u^2(k) \\ x_2(k+1) = x_1(k) + x_1(k)x_2(k)u(k) - \frac{1}{2}x_1^3(k)u^2(k) \end{cases} \tag{4.42}$$

of (4.40) are both not stabilizable. Indeed, one may easily check that either for (4.41) or (4.42) $V(x) = x_1^2 + x_2^2$ satisfies $V(f(x,u)) - V(x) = 0$, $\forall\, x \in \mathbb{R}^2$, $\forall\, u \in \mathbb{R}$.

4.6 References

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5

Stabilisation Globale de Systèmes Non Linéaires Stochastiques

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Stabilisation Globale de Systèmes Non Linéaires Stochastiques

Résumé : Nous donnons une condition suffisante de stabilisation globale pour des systèmes stochastiques non linéaires, généralisant celle de Jurdjevic-Quinn, connue pour les systèmes déterministes.

5.1 Introduction

Soient $(\Omega, \mathcal{F}, P)$ un espace de probabilité usuel, $\{z_t,\ t \geq 0\}$ un processus de Wiener sur cet espace à valeur dans $\mathbb{R}^p$ et $(\mathcal{F}_t)_{t\geq 0}$ la filtration engendré par z. Le but de cette Note est de donner une condition suffisante de stabilisation globale par un retour d'état pour des systèmes non linéaires stochastiques (pris au sens de Itô) :

$$x_t = x_0 + \int_0^t f_0(x_s, u)ds + \sum_{j=1}^{p} \int_0^t f_j(x_s, u)dz_s^j, \tag{5.1}$$

avec $u = (u^1, \dots, u^m)^{\mathrm{T}}$ un contrôle à valeur dans R^m et $f_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ des fonctions lipschitziennes de classe C^∞ telles que $f_j(0,0) = 0,\ 0 \leq j \leq p$.

Cette condition généralise celle de Jurdjevic-Quinn, connue pour les systèmes non linéaires déterministes affines en contrôle (voir [3, 5]) :

$$\dot{x} = X(x) + \sum_{i=1}^{m} u^i Y^i(x). \tag{5.2}$$

En [5], il est prouvé que s'il existe $V : \mathbb{R}^n \to \mathbb{R}$ définie positive et propre telle que :

(i) la dérivée de Lie de V suivant le champ X est telle que $XV(x) \leq 0$, $\forall x \in \mathbb{R}^n$,

(ii) l'ensemble

$$\{x \in \mathbb{R}^n | X^{k+1}V(x) = X^k Y^i(V) = 0,\ k \in \mathbb{N},\ i = 1, \dots, m\}$$

est réduit à $\{0\}$,

alors (5.2) est globalement stabilisable par le feedback $u(x) = -Y^i V(x)$.

Ce résultat est étendu en [1] à des systèmes stochastiques affines en contrôle :

$$x_t = x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^m u^i Y_0^i(x_s)\Big) ds + \sum_{j=1}^p \int_0^t X_j(x_s) dz_s^j, \tag{5.3}$$

dans lesquels les coefficients associés au bruit sont indépendants du contrôle. Pour ces systèmes le générateur infinitésimal associé, $\mathcal{L}$, vérifie

$$\mathcal{L}V(x) = LV(x) + \sum_{i=1}^m u^i Y_0^i V(x),$$

où L est l'opérateur différentiel du second ordre :

$$L = X_0 + \frac{1}{2} \sum_{j=1}^p X_j^2,$$

et V une fonction de Liapounov. Ainsi, le feedback $u^i(x) = -Y_0^i V(x)$ conduit à $\mathcal{L}V(x) \leq 0$ si $LV(x) \leq 0$, ce qui permet d'établir en [1], sous des conditions analogues à (i) et (ii), que ce feedback stabilise globalement (5.3).

Pour les systèmes stochastiques de la forme générale (5.1), $\mathcal{L}V(x)$ n'etant plus linéaire en u, la difficulté majeure est de prouver l'existence d'un feedback $u(x) \not\equiv 0$ conduisant à $\mathcal{L}V(x) \leq 0$. Dans cette Note, on montre que si les champs de vecteurs

$$X_j(x) = f_j(x, 0), \; Y_j^i(x) = \frac{\partial f_j}{\partial u^i}(x, 0), \quad 0 \leq j \leq p, \; 1 \leq i \leq m,$$

satisfont l'analogue stochastique de (i) et (ii) le système (5.1) est globalement stabilisable par un feedback borné explicitement donné.

5.2 Stabilité stochastique

On rappelle dans ce paragraphe quelques résultats classiques de stabilité stochastique (voir [2]). Soit x_t le processus à valeur dans $\mathbb{R}^n$ solution au sens de Itô de l'équation différentielle stochastique :

$$x_t = x_0 + \int_o^t \sigma_0(x_s) ds + \sum_{k=1}^p \int_0^t \sigma_k(x_s) dz_s^k, \quad \sigma_k(0) = 0, \; 0 \leq k \leq p. \tag{5.4}$$

où les champs de vecteurs σ_k sont lipschitziens tels que pour tout $x \in \mathbb{R}^n$ on ait :

$$\sum_{k=0}^{p} \|\sigma_k(x)\| \leq K(1 + \|x\|), \ K > 0. \tag{5.5}$$

Pour $x_0 \in \mathbb{R}^n$, $x_t(x_0)$, $t \geq 0$, désigne la solution à l'instant t de (5.4) démarrant de l'état x_0. La solution $x_t \equiv 0$ est stable en probabilité si

$$\forall \varepsilon > 0, \quad \lim_{x_0 \to 0} P\left(\sup_{t>0} \|x_t(x_0)\| > \varepsilon\right) = 0.$$

S'il existe de plus un voisinage D de l'origine tel que

$$\forall x_0 \in D, \quad P\left(\lim_{t \to +\infty} \|x_t(x_0)\| = 0\right) = 1,$$

cette solution est asymptotiquement stable en probabilité. Elle est globalement asymptotiquement stable en probabilité (GASP) si $D = \mathbb{R}^n$.

Soit L le générateur infinitésimal associé à (5.4), défini pour toute fonction Ψ de classe C^2 sur $\mathbb{R}^n$ par

$$L\Psi(x) = \langle \sigma_0(x), \nabla\Psi(x) \rangle + \frac{1}{2} \sum_{k=1}^{p} \mathrm{Tr}(\sigma_k(x)\, (\sigma_k(x))^{\mathrm{T}} \frac{\partial^2 \Psi}{\partial x^2}(x)),$$

où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire dans $\mathbb{R}^n$. On peut alors rappeler la version stochastique du théorème de Liapounov (voir [2]).

Théorème 5.1 *S'il existe un voisinage D de l'origine et une fonction $V : D \to \mathbb{R}$ de classe C^2 définie positive tel que $LV(x) \leq 0$ (resp. $LV(x) < 0$), $\forall x \in D$, $x \neq 0$, alors la solution $x_t \equiv 0$ de (5.4) est stable (resp. asymptotiquement stable) en probabilité. Elle est GASP si $D = \mathbb{R}^n$, V est propre et $LV(x) < 0$, $\forall x \in \mathbb{R}^n$, $x \neq 0$.*

Rappelons enfin, que la version stochastique du principe d'invariance de LaSalle (voir [4]) permet d'établir que si $LV(x) \leq 0$, $\forall x \in D$, $x \neq 0$, le processus x_t converge en probabilité vers le plus grand ensemble invariant dont le support est contenu dans

$$\{x_t \,|\, LV(x_t) = 0, \forall t \geq 0\}.$$

5.3 Condition suffisante de stabilisation

Le système (5.1) est globalement stabilisable s'il existe une commande $u : \mathbb{R}^n \to \mathbb{R}^m$, $u(0) = 0$, telle que la solution $x_t \equiv 0$ soit GASP pour le système bouclé :

$$x_t = x_0 + \int_0^t f_0(x_s, u(x_s))ds + \sum_{j=1}^{p} \int_0^t f_j(x_s, u(x_s))dz_s^j.$$

Pour $0 \leq j \leq p,\ 1 \leq i \leq m$, on associe au système (5.1) les champs de vecteurs

$$X_j(x) = f_j(x,0),\ Y_j^i(x) = \frac{\partial f_j}{\partial u^i}(x,0),$$

et les opérateurs différentiels du second ordre L_0, L_i, définis pour toute fonction Ψ de $C^2(\mathbb{R}^n)$ par :

$$L_0\Psi(x) = \langle X_0(x), \nabla\Psi(x) \rangle + \frac{1}{2}\sum_{j=1}^{p} \mathrm{Tr}\Big(X_j(x)\,(X_j(x))^{\mathrm{T}}\ \frac{\partial^2\Psi}{\partial x^2}(x)\Big), \tag{5.6}$$

$$\begin{aligned} L_i\Psi(x) &= \langle Y_0^i(x), \nabla\Psi(x) \rangle \\ &\quad + \frac{1}{2}\sum_{j=1}^{p} \mathrm{Tr}\Big(\Big[X_j(x)\left(Y_j^i(x)\right)^{\mathrm{T}} + Y_j^i(x)\,(X_j(x))^{\mathrm{T}}\Big]\ \frac{\partial^2\Psi}{\partial x^2}(x)\Big). \end{aligned} \tag{5.7}$$

Pour $x \in \mathbb{R}^n$ et $v, w \in \mathbb{R}^m$ on pose

$$\tilde{f}_j(x,v,w) = \int_0^1 (1-t)\frac{\partial^2 f_j}{\partial u^2}(x,tv)(w,w)dt,$$

où

$$\frac{\partial^2 f_j}{\partial u^2}(x,tv)(w,w) = \left(w^{\mathrm{T}}\frac{\partial^2 f_j^1}{\partial u^2}(x,tv)w, \ldots, w^{\mathrm{T}}\frac{\partial^2 f_j^n}{\partial u^2}(x,tv)w\right)^{\mathrm{T}}$$

et on considère les matrices d'ordre n données par :

$$\begin{aligned} A_j(x,v,w) &= X_j(x)\left(\tilde{f}_j(x,v,w)\right)^{\mathrm{T}} + \tilde{f}_j(x,v,w)\,(X_j(x))^{\mathrm{T}} + \sum_{i=1}^{m} v^i\Big[Y_j^i(x)\left(\tilde{f}_j(x,v,w)\right)^{\mathrm{T}} \\ &\quad + \tilde{f}_j(x,v,w)\left(Y_j^i(x)\right)^{\mathrm{T}}\Big] + \sum_{i_1,i_2=1}^{m} w^{i_1}w^{i_2}Y_j^{i_1}(x)\left(Y_j^{i_2}(x)\right)^{\mathrm{T}} + \tilde{f}_j(x,v,v)\left(\tilde{f}_j(x,v,w)\right)^{\mathrm{T}}. \end{aligned}$$

On suppose enfin qu'il existe $V : \mathbb{R}^n \to \mathbb{R}$ C^∞ définie positive et propre telle que :

(h1) $L_0 V(x) \leq 0, \ \forall x \in \mathbb{R}^n$;

(h2) l'ensemble

$$\{x \in \mathbb{R}^n \mid L_0^{k+1} V(x) = L_0^k L_i V(x) = 0, \ k \in \mathbb{N}, \ i = 1, \ldots, m\},$$

où $L_0^{k+1}\Psi(x) = L_0 L_0^k \Psi(x), \ L_0^0 \Psi(x) = \Psi(x)$, est réduit à $\{0\}$;

et on pose :

$$\varphi_V(x, v, w) = \langle \tilde{f}_0(x, v, w), \nabla V(x) \rangle + \frac{1}{2} \sum_{j=1}^{p} \mathrm{Tr}\Big(A_j(x, v, w) \, \frac{\partial^2 V}{\partial x^2}(x) \Big), \tag{5.8}$$

Notons que φ_V est homogène de degré 2 par rapport à w. On peut alors énoncer :

Théorème 5.2 *Si les conditions* **(h1)** *et* **(h2)** *sont satisfaites, alors, pour tout* $\eta > 0$ *et toutes fonctions* $K_1(x)$ *et* $K_2(x)$ *de classe* $\mathcal{C}^\infty$ *telles que* $\forall x \in \mathbb{R}^n$, $K_1(x) + K_2(x) \neq 0$, *et*

$$K_1(x) \geq \sup_{\|v\| \leq \eta, \|w\| = 1} |\varphi_V(x, v, w)|, \quad K_2(x) \geq \|(L_1 V(x), .., L_m V(x))\|, \tag{5.9}$$

le système stochastique (5.1) est globalement stabilisable par la commande :

$$u(x) = \frac{-\eta}{\eta K_1(x) + K_2(x)} (L_1 V(x), .., L_m V(x))^{\mathrm{T}} \tag{5.10}$$

qui satisfait $\|u(x)\| \leq \eta, \ \forall x \in \mathbb{R}^n$.

Preuve : L'inegalité $\|u(x)\| \leq \eta$ est immédiate et entraine aisément (5.5) pour $\sigma_j = f_j(x, u(x))$. Par ailleurs, le système bouclé s'écrit :

$$\begin{aligned} x_t &= x_0 + \int_0^t \Big(X_0(x_s) + \sum_{i=1}^{m} u^i(x_s) Y_0^i(x_s) + \tilde{f}_0(x_s, u(x_s), u(x_s)) \Big) ds \\ &\quad + \sum_{j=1}^{p} \int_0^t \Big(X_j(x_s) + \sum_{i=1}^{m} u^i(x_s) Y_j^i(x_s) + \tilde{f}_j(x_s, u(x_s), u(x_s)) \Big) dz_s^j. \end{aligned} \tag{5.11}$$

Si $\mathcal{L}$ est le générateur infinitésimal associé à (5.11), on a :

$$\begin{aligned} \mathcal{L}V(x) &= \Big\langle X_0(x_s) + \sum_{i=1}^{p} u^i(x_s) Y_0^i(x_s) + \tilde{f}(x_s, u(x_s), u(x_s)), \nabla V(x) \Big\rangle \\ &\quad + \frac{1}{2} \sum_{j=1}^{p} \mathrm{Tr}\Big(\Big[X_j(x_s) + \sum_{i=1}^{m} u^i(x_s) Y_j^i(x_s) + \tilde{f}_j(x_s, u(x_s), u(x_s)) \Big] \\ &\quad \times \Big[X_j(x_s) + \sum_{i=1}^{m} u^i(x_s) Y_j^i(x_s) + \tilde{f}_j(x_s, u(x_s), u(x_s)) \Big]^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2}(x) \Big) \end{aligned}$$

et, grâce à (5.6), (5.7) et (5.8) :

$$\mathcal{L}V(x) = L_0V(x) + \sum_{i=1}^{m} u^i(x)L_iV(x) + \varphi_V(x, u(x), u(x)),$$

soit, $\mathcal{L}V(x) = L_0V(x)$ si $u(x) = 0$ et, sinon, en utilisant (5.10), :

$$\mathcal{L}V(x) = L_0V(x) - \frac{1}{K(x)}\|u(x)\|^2 \left[1 - K(x)\,\varphi_V\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right)\right],$$

où

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)}\,.$$

En outre, à partir de $\|u(x)\| \leq \eta$ et de (5.9) on a :

$$1 - K(x)\,\varphi_V\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right) \geq 0.$$

La condition **(h1)** donne alors $\mathcal{L}V(x) \leq 0,\ \forall\, x \in \mathbb{R}^n$, d'où la stabilité en probabilité de la solution $x_t \equiv 0$. Par application de la version stochastique du principe d'invariance de LaSalle (see [4]), il suffit, pour montrer qu'elle est GASP, d'établir que pour toute solution complète x_t de (5.11) vérifiant $\mathcal{L}V(x_t) = 0,\ \forall t \geq 0$, on a nécessairement $x_t \equiv 0$. On vérifie aisément que :

$$u(x) \neq 0 \Rightarrow K_2(x) \neq 0 \Rightarrow 1 - K(x)\,\varphi_V\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right) \neq 0,$$

et donc que $\mathcal{L}V(x) = 0$ si et seulement si $L_iV(x) = 0,\ i = 0, \ldots, m$. Ainsi, pour toute solution complète x_t de (5.11) vérifiant $\mathcal{L}V(x_t) = 0,\ \forall t \geq 0$, la formule de Itô permet d'avoir, par dérivations sucssésives,

$$L_0^{k+1}V(x_t) = L_0^k L_i V(x_t) = 0,\ \forall t \geq 0,\ \forall k \in \mathbb{N},\ i = 1, \ldots, m.$$

La condition **(h2)** entraine alors que $x_t \equiv 0$.

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6

Feedback Stabilization of Stochastic Nonlinear Composite Systems

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Feedback Stabilization of Stochastic Nonlinear Composite Systems

Abstract: In this paper, we study the global stabilization, by means of smooth state feedback, of partially linear composite stochastic systems.

Key words: Nonlinear stochastic systems, Feedback, Global stabilization, Lyapunov's function.

6.1 Introduction

Many recent papers (see [1, 2, 3] and references therein) addressed the problem of The global stabilization, by means of state feedback, of deterministic nonlinear control systems of the form :

$$\begin{cases} \dot{x} = f(x,y) & x \in \mathbb{R}^n \\ \dot{y} = Ay + Bu & y \in \mathbb{R}^p \end{cases} \tag{6.1}$$

where $u \in \mathbb{R}^k$ is the control, $A \in \mathcal{M}_{p,p}(\mathbb{R}), B \in \mathcal{M}_{p,k}(\mathbb{R})$ and f is a smooth vector field such that :

(h1) The pair (A, B) is stabilizable.

(h2) The equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is globally asymptotically stable (G.A.S).

In [3], the authors assumed that the dependence of $f(x, y)$ on y is of the form :

(h3) $f(x,y) = f(x,0) + G(x,y).Cy.$

with $C \in \mathcal{M}_{k,p}(\mathbb{R})$. They gave conditions on the linear subsytem

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y} = Cy\,, \qquad \tilde{y} \in \mathbb{R}^k \end{cases}$$

under which there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ satisfying the following three conditions :

(H1) $P(A+BK)+(A+BK)^T P = -Q$, with Q symmetric positive (^{T} = transpose).

(H2) $(Q^{1/2}, A+BK)$ detectable.

(H3) $B^T P = C$.

Using the above assumptions, they proved that the system (6.1) is globally asymptotically stabilizable and they gave the stabilizing feedback

$$u(x,y) = Ky - \frac{1}{2}\,(G(x,y))^T \nabla V(x)$$

where V is a smooth Lyapunov function satisfying

$$\langle\, \nabla V, f(x,0)\,\rangle < 0, \quad \forall x \in \mathbb{R}^n\,,\ x \neq 0 \tag{6.2}$$

The goal of our work is to show that the result of [3] can be extended when the nonlinear part of the system (6.1) is corrupted by a noise which satisfies the same hypothesis **(h3)** as f. We prove that the stochastic system

$$\begin{cases} dx_t = f(x_t, y_t)dt + g(x_t, y_t)dw_t \\ dy_t = (Ay_t + Bu)dt \end{cases}$$

where both f and g are of the form **(h3)**, is globally asymptotically stabilizable in probability, if **(H1)**, **(H2)**, **(H3)** and the condition

(h'2) the solution $x_t \equiv 0$ of $dx_t = f(x_t, 0)dt + g(x_t, 0)dw_t$ is globally asymptotically stable in probability,

hold.

Notice that the systems of the form

$$\begin{cases} dx_t = f(x_t, y_t)dt + g(x_t)dw_t \\ dy_t = (Ay_t + Bu)dt \end{cases} \tag{6.3}$$

have been studied in [4]. Under conditions on the dependence on y of the vector field f, the authors proved that (6.3) is exponentially stabilizable in mean square if

(h"2) the solution $x_t \equiv 0$ of $dx_t = f(x_t, 0)dt + g(x_t)dw_t$ is exponentially stable in mean square.

Remark that **(h"2)** is stronger than **(h'2)**.

6.2 Stochastic stability

The aim of this section is to recall the main definitions and results proved by Has'minskii (see [5], chapter V) for the zero state of a stochastic differential equation to be stable in probability.

Let $(\Omega, \mathcal{F}, P)$ be an usual probability space and denote by w a standard $\mathbb{R}^m$–valued Wiener process defined on this space. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the complete right–continuous filtration generated by the standard Wiener process w. Let $x_t \in \mathbb{R}^n$ be the stochastic process solution of the stochastic differential equation written in the sense of Itô,

$$x_t = x_0 + \int_o^t b(x_s)\, ds + \sum_{k=1}^m \int_0^t \sigma_k(x_s) \circ dw_s^k \tag{6.4}$$

where b and σ_k, $1 \leq k \leq m$, are Lipschitz functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ such that

1. $b(0) = 0$, $\sigma_k(0) = 0$, $1 \leq k \leq m$.
2. There exists a non–negative constant K such that

$$|b(x)| + \sum_{k=1}^m |\sigma_k(x)| \leq K(1 + |x|)$$

   for every x in $\mathbb{R}^n$.

Furthermore, for any $t \geq 0$ and $x_0 \in \mathbb{R}^n$, denote by $x_t(x_0)$, the solution at time t of the equation (6.4) starting from the state x_0 .

Then, the main notions of stochastic stability we are dealing with in this paper may be defined by

Definition 6.1 *The solution $x_t \equiv 0$ of the stochastic differential equation (6.4) is said to be stable in probability if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|x_0| < \delta \Rightarrow P\left(\sup_{t>0} |x_t(x_0)| > \varepsilon\right) < \varepsilon.$$

If, in addition, there exists a neighbourhood D of the origin such that

$$P\left(\lim_{t\to+\infty} |x_t(x_0)| = 0\right) = 1, \ \forall x_0 \in D$$

the solution $x_t \equiv 0$ of the stochastic differential equation (6.4) is said to be asymptotically stable in probability. It is globally asymptotically stable in probability (G.A.S.P) if

$$P\left(\lim_{t\to+\infty} |x_t(x_0)| = 0\right) = 1, \ \forall x_0 \in \mathbb{R}^n$$

Therefore, denoting by L the infinitesimal generator associated with the stochastic differential equation (6.4) defined for any function Ψ in $C^2(\mathbb{R}^n)$ by

$$L\Psi(x) = \sum_{i=1}^{n} b^i(x)\frac{\partial \Psi}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} a^{i,j}(x)\frac{\partial^2 \Psi}{\partial x_i \partial x_j}(x) \tag{6.5}$$

where $a^{i,j}(x) = \sum_{k=1}^{m} \sigma_k^i(x)\sigma_k^j(x)$, $1 \le i,j \le n$, one can prove the following stochastic Lyapunov Theorem (see [5], [6]).

Theorem 6.1 *Let D be a neighbourhood of the point $x = 0$ which is contained in $\mathbb{R}^n$ together with its boundary, and assume that there exists a Lyapunov function V defined in D (i.e. a proper function V positive definite mapping D into $\mathbb{R}$) such that*

$$LV(x) \le 0 \ \ (\textit{respectively} \ \ LV(x) < 0), \ \ \forall x \in D, \ x \ne 0$$

Then, the solution $x_t \equiv 0$ of the stochastic differential equation (6.4) is stable (respectively asymptotically stable) in probability. It is G.A.S.P if

$$LV(x) < 0, \ \forall x \in \mathbb{R}^n, \ x \ne 0$$

In this paper, we shall make use of the latter Theorem and a stochastical version of Lassalle's invariance principle (see [7]), in order to prove that the class of nonlinear stochastic control systems introduced in the following section is globally asymptotically stabilizable in probability.

6.3 Main result

The systems considered here are of the form

$$\begin{cases} x_t = x_0 + \int_0^t f(x_s, y_s)\, ds + \int_0^t g(x_s, y_s)\, dw_s \\ y_t = y_0 + \int_0^t (Ay_s + Bu)\, ds \end{cases} \tag{6.6}$$

where the dependance of g on y is analogous to the one of f given by **(h3)**, that is

$$g(x,y) = g(x,0) + H(x,y)\,Cy \tag{6.7}$$

We assume that the solution $x_t \equiv 0$ is G.A.S.P for

$$x_t = x_0 + \int_0^t f(x_s,0)\,ds + \int_0^t g(x_s,0)\,dw_s$$

and a positive definite and proper function satisfying

$$L.V(x) = \langle\, f(x,0), \nabla V(x)\,\rangle + \frac{1}{2}\mathrm{Tr}\left(g(x,0)(g(x,0))^T\frac{\partial^2 V}{\partial x^2}(x)\right) < 0$$

is known. Then we can state :

Theorem 6.2 *If there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ such that* **(H1)**, **(H2)** *and* **(H3)** *hold then the system (6.6) is globally asymptotically stabilzable in probability thanks to the following feedback*

$$u = Ky - (G(x,y))^T\,\nabla V(x) - \frac{1}{2}(H(x,y))^T\,\frac{\partial^2 V}{\partial x^2}(x)[g(x,0) + g(x,y)] \tag{6.8}$$

Proof Let

$$W(x,y) = V(x) + \frac{1}{2}y^T P\,y.$$

Denoting by $\mathcal{L}$ the infinitesimal generator associated with the stochastic differential equation (6.6-6.8), and setting

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad Z(z) = \begin{pmatrix} f(x,y) \\ Ay + Bu(x,y) \end{pmatrix}, \quad \tilde{g}(z) = \begin{pmatrix} g(z) \\ 0 \end{pmatrix},$$

one has

$$\begin{aligned} \mathcal{L}.W(z) &= Z.W(z) + \frac{1}{2}\mathrm{Tr}\left(\tilde{g}(z)(\tilde{g}(z))^T\frac{\partial^2 W}{\partial z^2}(z)\right) \\ &= Z.W(z) + \frac{1}{2}\mathrm{Tr}\left(g(z)(g(z))^T\frac{\partial^2 V}{\partial x^2}(x)\right) \end{aligned}$$

According to the decomposition of f given by **(h3)** and the one of g given by (6.7), one gets

$$
\begin{aligned}
Z.W(x,y) &= \langle f(x,0)\,,\,\nabla V(x)\rangle - \frac{1}{2}y^T Q\,y + \langle \nabla V(x)\,,\,G(x,y)Cy\rangle \\
&\quad + \left\langle y\,,\,-PB\left((G(x,y))^T\nabla V(x) + \frac{1}{2}(H(x,y))^T\frac{\partial^2 V}{\partial x^2}(x)[g(x,0)+g(x,y)]\right)\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left(g(z)(g(z))^T\frac{\partial^2 V}{\partial x^2}(x)\right) &= \frac{1}{2}\mathrm{Tr}\left(g(x,0)(g(x,0))^T\frac{\partial^2 V}{\partial x^2}(x)\right. \\
&\quad + g(x,0)y^T C^T(H(z))^T\frac{\partial^2 V}{\partial x^2}(x) \\
&\quad + H(z)Cy(g(x,0))^T\frac{\partial^2 V}{\partial x^2}(x) \\
&\quad \left. + H(z)Cy\,y^T C^T(H(z))^T\frac{\partial^2 V}{\partial x^2}(x)\right).
\end{aligned}
$$

So, from $PB = C^T$ one has

$$
\begin{aligned}
\mathcal{L}.W(z) &= \langle f(x,0)\,,\,\nabla V(x)\rangle - \frac{1}{2}y^T Q\,y + \frac{1}{2}\mathrm{Tr}\left(g(x,0)(g(x,0))^T\frac{\partial^2 V}{\partial x^2}(x)\right) \\
&\quad - \frac{1}{2}\left\langle y\,,\,C^T(H(x,y))^T\frac{\partial^2 V}{\partial x^2}(x)[g(x,0)+g(x,y)]\right\rangle \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(g(x,0)y^T C^T(H(z))^T\frac{\partial^2 V}{\partial x^2}(x)\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(H(z)Cy(g(x,0))^T\frac{\partial^2 V}{\partial x^2}(x)\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(H(z)Cy\,y^T C^T(H(z))^T\frac{\partial^2 V}{\partial x^2}(x)\right)
\end{aligned}
$$

But using the fact that for all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\mathrm{Tr}(\xi_1 \xi_2^T) = \langle \xi_1, \xi_2 \rangle$, one gets

$$
\begin{aligned}
\mathrm{Tr}\left(g(x,0) y^T C^T (H(z))^T \frac{\partial^2 V}{\partial x^2}(x)\right) &= \left\langle g(x,0)\,,\, \frac{\partial^2 V}{\partial x^2}(x) H(z) C y \right\rangle \\
&= \left\langle y\,,\, C^T (H(z))^T \frac{\partial^2 V}{\partial x^2}(x) g(x,0) \right\rangle \\
\mathrm{Tr}\left(H(z) C y (g(x,0))^T \frac{\partial^2 V}{\partial x^2}(x)\right) &= \left\langle H(z) C y\,,\, \frac{\partial^2 V}{\partial x^2}(x) g(x,0) \right\rangle \\
&= \left\langle y\,,\, C^T (H(z))^T \frac{\partial^2 V}{\partial x^2}(x) g(x,0) \right\rangle \\
\mathrm{Tr}\left(H(z) C y\, y^T C^T (H(z))^T \frac{\partial^2 V}{\partial x^2}(x)\right) &= \left\langle H(z) C y\,,\, \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(z) C y \right\rangle \\
&= \left\langle y\,,\, C^T (H(z))^T \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^T H(z) C y \right\rangle
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
\mathcal{L}.W(z) &= L.V(x) - \frac{1}{2} y^T Q\, y - \frac{1}{2}\left\langle y\,,\, C^T (H(z))^T \frac{\partial^2 V}{\partial x^2}(x)[g(x,0) + g(x,y)] \right\rangle \\
&= +\frac{1}{2}\left\langle y\,,\, C^T (H(z))^T \frac{\partial^2 V}{\partial x^2}(x)[2g(x,0) + H(z) C y] \right\rangle \\
&= L.V(x) - \frac{1}{2} y^T Q\, y \leq 0
\end{aligned}
$$

According to the stochastical version of Lassale's invariance principle (see [7]), the processus z_t converges in probability to Ω the largeste invariant set whose support is contained in the locus $\mathcal{L}.W(z_t) = 0$. Let (x_t, y_t) be a complete solution of the closed-loop system (6.6) along which $\mathcal{L}.W(x_t, y_t) = 0$, we must show that $(x_t, y_t) = (0,0)$ for all $t \geq 0$. Since

$$\mathcal{L}.W(x,y) = 0 \Leftrightarrow x = 0 \text{ and } y^T Q y = 0,$$

x_t must be zero for all $t \geq 0$ and y_t will be a solution of

$$\dot{y}_t = (A + BK) y_t$$

and must satisfy $y_t^T Q y_t = 0$ for all $t \geq 0$. By the detectability assumption (H2) this implies $y_t = 0$ for all $t \geq 0$ and, hence, $(x_t, y_t) = (0, 0)$. This completes the proof.

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Résumé. Ce travail comporte trois parties. Dans la première, on s'intéresse au problème du découplage avec stabilité interne par commande statique pour les systèmes bilinéaires. Pour ces systèmes non linéaires, la matrice de découplage est singulière sur une surface algébrique contenant l'origine, ce qui pose un problème d'explosion de solutions : dans ce cas généralement les trajectoires du système bouclé ne sont pas complètes et/ou les commandes ne sont pas bornées. On considère ici des systèmes bilinéaires à deux entrées-deux sorties sans zéros dynamiques, pour lesquels on donne des conditions suffisantes de découplage avec stabilité par des commandes linéarisantes.

La deuxième partie est consacrée à des questions de stabilisation globale par retour d'état pour certains systèmes non linéaires. On s'intéresse d'une part aux systèmes partiellement linéaires pour lesquels divers auteurs ont donné des conditions suffisantes de stabilisation globale à partir d'une fonction de Lyapunov stricte. Il n'existe malheureusement pas de méthode systématique pour construire une telle fonction. On montre ici que la connaissance d'une fonction de Lyapunov large vérifiant le principe d'invariance de LaSalle suffit pour obtenir une commande stabilisante globale. L'intérêt de notre démarche est que pour de très larges classes de systèmes, dont les systèmes mécaniques, il est plus facile de construire une fonction de Lyapunov large plutôt qu'une stricte. On donne d'autre part une condition suffisante de stabilisation globale pour des systèmes non affines en contrôle généralisant celle de Jurdjevic-Quinn connue pour les systèmes affines en contrôle.

La dernière partie étend des résultats de stabilisation déterministes à des systèmes non linéaires stochastiques. On y donne une condition suffisante de stabilisation globale pour des systèmes partiellement linéaires stochastiques et une version stochastique de la condition de Jurdjevic-Quinn pour des systèmes stochastiques non nécessairement affines en contrôle.

Mots clés. Systèmes non linéaires, linéarisation, découplage, singularités, stabilisation, fonction de Lyapunov, systèmes stochastiques.

Résumé. Ce travail comporte trois parties. Dans la première, on s'intéresse au problème du découplage avec stabilité interne par commande statique pour les systèmes bilinéaires. Pour ces systèmes non linéaires, la matrice de découplage est singulière sur une surface algébrique contenant l'origine, ce qui pose un problème d'explosion de solutions : dans ce cas généralement les trajectoires du système bouclé ne sont pas complètes et/ou les commandes ne sont pas bornées. On considère ici des systèmes bilinéaires à deux entrées-deux sorties sans zéros dynamiques, pour lesquels on donne des conditions suffisantes de découplage avec stabilité par des commandes linéarisantes.

La deuxième partie est consacrée à des questions de stabilisation globale par retour d'état pour certains systèmes non linéaires. On s'intéresse d'une part aux systèmes partiellement linéaires pour lesquels divers auteurs ont donné des conditions suffisantes de stabilisation globale à partir d'une fonction de Lyapunov stricte. Il n'existe malheureusement pas de méthode systématique pour construire une telle fonction. On montre ici que la connaissance d'une fonction de Lyapunov large vérifiant le principe d'invariance de LaSalle suffit pour obtenir une commande stabilisante globale. L'intérêt de notre démarche est que pour de très larges classes de systèmes, dont les systèmes mécaniques, il est plus facile de construire une fonction de Lyapunov large plutôt qu'une stricte. On donne d'autre part une condition suffisante de stabilisation globale pour des systèmes non affines en contrôle généralisant celle de Jurdjevic-Quinn connue pour les systèmes affines en contrôle.

La dernière partie étend des résultats de stabilisation déterministes à des systèmes non linéaires stochastiques. On y donne une condition suffisante de stabilisation globale pour des systèmes partiellement linéaires stochastiques et une version stochastique de la condition de Jurdjevic-Quinn pour des systèmes stochastiques non nécessairement affines en contrôle.

Mots clés. Systèmes non linéaires, linéarisation, découplage, singularités, stabilisation, fonction de Lyapunov, systèmes stochastiques.