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Analyse et Contrôle de Quelques Problèmes d'interaction Fluide-Structures

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Première partie

Introduction

Dans cette thèse, on s'intéresse au caractère bien posé et à la contrôlabilité de quelques systèmes d'interaction fluide-structure. Plus précisément, on considère le système constitué de structures déformables ou indéformables et d'un fluide visqueux incompressible. On suppose que le fluide satisfait les équations non linéaires de Navier-Stokes en dimension 2 ou 3 et de Burgers visqueux en dimension 1. Les équations du mouvement des structures sont obtenues en minimisant une énergie du système (principe de D'Alembert) ou en appliquant le principe fondamental de la dynamique (lois de Newton).

On rencontre ces problèmes dans de nombreux domaines d'application. Par exemple, en biologie, ces modèles peuvent être utilisés pour étudier l'écoulement du sang ou d'autres liquides dans le corps. L'étude de ce type de problèmes remonte à l'époque d'Euler, de D'Alembert, de Stokes etc... Cependant, leurs études sont en général faites dans des cas particuliers : les géométries considérées sont simples et/ou comportent de nombreuses symétries (domaine du fluide sphérique, carré ou $\mathbb{R}^d$, structures de forme sphérique, elliptique). De plus, les équations du fluide sont les équations de Laplace ou de Stokes au lieu des équations d'Euler (dans le cas d'un fluide parfait) ou de Navier-Stokes (dans le cas d'un fluide visqueux incompressible). Avec ces hypothèses, ils obtiennent des formules explicites pour les équations correspondantes. Ces formules sont encore utilisées par des mathématiciens, des physiciens et des ingénieurs.

Quand le domaine considéré n'a pas de symétrie, il est difficile d'obtenir une formule explicite de la solution du système. De plus, à cause du mouvement du solide, le domaine sur lequel les équations sont écrites dépend de la position de la structure, cette dernière étant inconnue. Heureusement, de nombreuses méthodes ont été trouvées ces dernières années et une des méthodes les plus générales consiste à transformer le problème à frontière libre en un problème à frontière fixe par un changement de variable qui est déterminé par le mouvement de la structure. Cette méthode est valable pour la majorité des modèles fluide-structure. Le premier problème (dans le chapitre 1) qu'on étudie est de ce type. Comme ce problème est en dimension 1, le changement de variable ne rend pas le système trop compliqué.

Au cours des 30 dernières années, la communauté de la physique a introduit une autre méthode pour modéliser le système fluide-structure qui s'appelle 'champ de phase'. D'une façon générale, ils introduisent une fonction ϕ qui s'appelle champ de phase et qui égale $+1$ à l'intérieur de la structure et à -1 à l'intérieur du fluide et dans un voisinage de l'interface entre le fluide et la structure, sa valeur change rapidement de -1 à $+1$ avec un profil de la forme $y = \frac{2}{\pi}\arctan(\frac{x}{\epsilon})$ avec $\epsilon > 0$ petit.

L'avantage de ce type de modèle est que, à l'aide de la fonction de champ de phase ϕ, les équations du système (celles du fluide et celles du champ de phase) sont écrites sur le domaine entier contenant le fluide, la structure et leur interface et il n'y a plus de frontière libre. Mais ce type de modèle n'est pas facile parce qu'il n'existe aucune loi physique précise pour établir ce type de système. Ceci explique pourquoi il existe des modèles différents. Dans la deuxième partie de la thèse on considère ce type de problème.

Dans la suite de cette introduction, nous présentons plus précisément le contenu de chaque chapitre de cette thèse.

0.1 Contrôlabilité de problèmes à frontière libre

Bien que l'objectif de cette section est de discuter d'un problème de contrôlabilité d'un système fluide-structure unidimensionnel, nous allons d'abord donner une introduction générale à ce domaine. Notons par $\mathcal{O} \subset \mathbb{R}^3$ le domaine contenant le fluide et le solide. Pour commencer, nous dérivons le système fluide-solide rigide en dimension 2 ou 3. Notons aussi par $B(t)$ le domaine contenant le solide seul et par $\Omega(t)$ celui contenant le fluide seul. Nous supposons toujours que l'ensemble $B(t)$ est compact, connexe, et de frontière régulière. Supposons de plus que le solide est indéformable et complètement immergé i.e. que le domaine $B(t)$ du solide reste à une distance strictement positive du bord de $\mathcal{O}$ ($\mathrm{dist}(\partial\mathcal{O}, B(t) > 0)$). Le fluide est décrit par les équations de Navier-Stokes incompressibles :

$$
\begin{aligned}
v_t + (v \cdot \nabla)v = \nu\Delta v + \nabla p + f, \quad x \in \Omega(t), \quad t \in [0,T], && \text{(0.1.1a)}\\
\nabla \cdot v = 0, \quad x \in \Omega(t), \quad t \in [0,T], && \text{(0.1.1b)}\\
v = 0, \quad x \in \partial\mathcal{O}, \quad t \in [0,T], && \text{(0.1.1c)}\\
v(0,\cdot) = v_0, \quad x \in \Omega(0), && \text{(0.1.1d)}
\end{aligned}
$$

où $v(t,x)$ désigne le vitesse du fluide et $p(t,x)$ est la pression du fluide. Des résultats mathématiques pour le système (0.1.1) peuvent être trouvés dans [14]. Le mouvement du corps rigide $B(t)$ est décrit par la trajectoire de son centre de masse $h(t)$ et sa vitesse angulaire $\omega(t)$. Notons par $\rho > 0$ et $m > 0$ la densité et la masse du solide rigide. Si on suppose toujours que ces deux quantités sont des constantes, on a $\rho \times \mathrm{Volume}(B(t)) = m$. Nous désignerons par J la matrice d'inertie du solide. C'est une fonction du temps $t \mapsto J(t) \in M(3)$ définie par

$$
(J(t)a) \cdot b = \rho \int_{B(t)} (a \wedge (x - h(t))) \cdot (b \wedge (x - h(t)))\mathrm{d}x, \quad a, b \in \mathbb{R}^3,
$$

où $M(3)$ désigne l'ensemble des matrices carrées réelles d'ordre 3 et $\wedge$ désigne le produit extérieur. Avec les notations précédentes, la loi fondamentale de la dynamique appliquée au solide s'écrit

$$m\ddot{h}(t) = -\int_{\partial B(t)} \sigma n \mathrm{d}s + \int_{B(t)} f(t)\mathrm{d}x, \tag{0.1.2a}$$

$$J\dot{\omega}(t) = J\omega \wedge \omega - \int_{\partial B(t)} (x-h(t)) \wedge \sigma n \mathrm{d}s + \int_{B(t)} (x-h(t)) \wedge f(t,x)\mathrm{d}x, \tag{0.1.2b}$$

$$h(0) = 0, \quad \dot{h}(0) = h_1 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{0.1.2c}$$

Dans le système ci-dessus, σ est le tenseur défini par

$$\sigma(t,x) = p(t,x)\mathbb{I} + 2\nu D(v(t,x)),$$

où $\mathbb{I}$ désigne le tenseur unitaire, $\nu > 0$ est la viscosité que l'on suppose constante et $D(v) = \frac{1}{2}(\nabla v +^t \nabla v)$. Pour coupler les équations du fluide et celles du corps rigide, nous devons ajouter la condition suivante d'adhérence du fluide à la paroi du solide :

$$v(t,x) = \dot{h}(t) + \omega(t) \wedge (x - h(t)), \quad x \in \partial B(t).$$

En rassemblant toutes les équations précédentes, on obtient le système couplé suivant

$$v_t + (v \cdot \nabla)v = \nu\Delta v + \nabla p + f, \quad x \in \Omega(t), \quad t \in [0,T], \tag{0.1.3a}$$

$$\nabla \cdot v = 0, \quad x \in \Omega(t), \quad t \in [0,T], \tag{0.1.3b}$$

$$v = 0, \quad x \in \partial\mathcal{O}, \quad t \in [0,T], \tag{0.1.3c}$$

$$v(t,x) = \dot{h}(t) + \omega(t) \wedge (x - h(t)), \quad x \in \partial B(t), \tag{0.1.3d}$$

$$m\ddot{h}(t) = -\int_{\partial B(t)} \sigma n \mathrm{d}s + \int_{B(t)} f(t)\mathrm{d}x, \tag{0.1.3e}$$

$$J\dot{\omega}(t) = J\omega \wedge \omega - \int_{\partial B(t)} (x-h(t)) \wedge \sigma n \mathrm{d}s + \int_{B(t)} (x-h(t)) \wedge f(t,x)\mathrm{d}x, \tag{0.1.3f}$$

$$v(0,\cdot) = v_0, \quad x \in \Omega(0), \tag{0.1.3g}$$

$$h(0) = 0, \quad \dot{h}(0) = h_1 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{0.1.3h}$$

Dans le cas bidimensionnel, ce système devient

$$v_t + (v\cdot\nabla)v = \nu\Delta v + \nabla p + f, \quad x \in \Omega(t), \quad t\in[0,T], \tag{0.1.4a}$$
$$\nabla\cdot v = 0, \quad x\in\Omega(t), \quad t\in[0,T], \tag{0.1.4b}$$
$$v = 0, \quad x\in\partial\mathcal{O}, \quad t\in[0,T], \tag{0.1.4c}$$
$$v(t,x) = \dot h(t) + \omega(t)(x-h(t))^{\perp}, \quad x\in\partial B(t), \tag{0.1.4d}$$
$$m\ddot h(t) = -\int\limits_{\partial B(t)} \sigma n \mathrm{d}s + \int\limits_{B(t)} f(t)\mathrm{d}x, \tag{0.1.4e}$$
$$J\dot\omega(t) = -\int\limits_{\partial B(t)} (x-h(t))^{\perp}\cdot\sigma n\mathrm{d}s + \int\limits_{B(t)} (x-h(t))^{\perp}\cdot f(t,x)\mathrm{d}x, \tag{0.1.4f}$$
$$v(0,\cdot) = v_0, \quad x\in\Omega(0), \tag{0.1.4g}$$
$$h(0) = 0, \quad \dot h(0) = h_1\in\mathbb{R}^2, \quad \omega(0) = \omega_0\in\mathbb{R}. \tag{0.1.4h}$$

Pour les travaux concernant le caractère bien posé du système (0.1.3) ou du système (0.1.4), nous mentionnons ici Desjardins et Esteban [15] et [16], Conca, San Martín et Tucsnak [51], Gunzburger, Lee et Seregin [31], Hoffmann et Starovoitov [32], San Martín, Starovoitov et Tucsnak [51], Takahashi [55] et [56] (dans le cas où le système fluide-rigide est confiné dans un domaine borné) et Serre [53], Judakov [39] (dans le cas où le système rigide-fluide remplit tout l'espace). Le problème stationnaire a été étudié dans [53] et Galdi [29]. Pour une description plus détaillée de la théorie mathématique du système fluide-structure, nous renvoyons à ces articles et à leurs références.

Maintenant, on considère le problème du contrôle pour les systèmes ci-dessus. Si on change f dans (0.1.4a) par $1_{\mathcal{C}}f$, où $1_{\mathcal{C}}$ est la fonction caractéristique d'un ouvert non vide $\mathcal{C}\subset\Omega(t)$, on peut poser le problème de la contrôlabilité à zéro : étant donné $(v_0, h_T, h_1, \omega_0)$, est-ce possible de trouver un contrôle f tel que la solution de (0.1.4) satisfait

$$v(T,\cdot) = 0, \quad \dot h(T) = 0, \quad h(T) = h_T, \quad \omega(T) = 0 \text{ ?}$$

Pour ce problème, une réponse positive a été donnée par Boulakia et Osses dans [11] et par Imanuvilov et Takahashi dans [34]. Nous allons esquisser brièvement la démonstration de ce dernier article. En utilisant le changement de variable introduit dans [55] et dans [56] au système (0.1.4), les auteurs de [34] obtiennent un système écrit dans un domaine fixe. Puis ils établissent une estimation de Carleman pour la linéarisation de ce système, ce qui implique l'observabilité et donc la contrôlabilité du système linéarisé. À la fin, une application du théorème de point fixe conduit à la contrôlabilité à zéro locale pour le système avec domaine fixe et donc pour le système original. Dans le Chapitre 1, on étudie la contrôlabilité d'un modèle simplifié de fluide-structure unidimensionnel, qui peut être considéré comme le cas à une dimension du problème de contrôle présenté ci-dessus. Plus précisément, on considère :

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + v(t,x)v_x(t,x) = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases} \tag{0.1.5}$$

Dans le système ci-dessus, v désigne la vitesse du fluide et h la trajectoire du point matériel qui bouge dans le fluide. Le système est contrôlé en imposant la vitesse du fluide sur le côté gauche et le contrôle correspondant est noté par $u(t)$. Le symbole $[f](x)$ représente le saut de f au point x, c'est-à-dire

$$[f](x) = f(x+) - f(x-),$$

où

$$f(x\pm) = \lim_{t>0, t\to 0} f(x \pm t).$$

Ce modèle a été introduit par Vázquez et Zuazua dans [61], où le caractère bien-posé est considéré dans le cas où l'équation de Burgers est écrite sur $\mathbb{R}\setminus\{h(t)\}$ au lieu de $(-1,1)\setminus\{h(t)\}$ (voir aussi [60]). On peux penser que le problème peut être résolu par la méthode introduite dans [34]. Mais en raison de la présence du corps rigide, le domaine sur lequel l'équation est écrite n'est plus connexe. À cause de ça, ce n'est pas facile d'établir l'estimation de Carleman globale. Dans [19], la contrôlabilité à zéro a été considérée par Doubova et Fernández-Cara. Dans leur article, ils l'approchent en utilisant l'estimation de Carleman globale pour l'équation de la chaleur, mais ils ont besoin du contrôle de chaque coté et la démonstration est très compliquée. Un problème ouvert proposé dans [19, Remarque 1.2] est : est-ce possible d'obtenir le même résultat en employant un seul contrôle ? On donne dans cette thèse une réponse positive à ce problème en introduisant une nouvelle méthode itérative pour la contrôlabilité à zéro d'équation parabolique, qui permet de se passer de l'estimation de Carleman.

0.1.1 Résultats principaux

Pour le système (0.1.5), nous montrons le théorème de la contrôlabilité à zéro locale suivant

Théorème 0.1.1 (Théorème Principal)**.** *Soit $\tau > 0$. Alors il existe $r > 0$ tel que, pour chaque (h_0, h_1, v_0) avec*

$$h_0 \in (-1,1), \ h_1 \in \mathbb{R}, \ v_0 \in H_0^1(-1,1), \ v_0(h_0) = h_1,$$

$$|h_0| + |h_1| + \|v_0\|_{H_0^1(-1,1)} \leqslant r,$$

il existe $u \in C[0,\tau]$ *tel que la solution de* (0.1.5) *vérifie*

$$v \in L^2([0,\tau], H^2((-1,1)\setminus\{h(t)\}) \cap C([0,\tau], H^1(-1,1)), \quad h \in C^1[0,\tau],$$

avec

$$\dot h(\tau) = 0, \quad v(\tau,\cdot) = 0, \quad h(\tau) = 0. \tag{0.1.6}$$

On considère d'abord la contrôlabilité interne, qui implique la contrôlabilité frontière ci-dessus par une méthodologie standard de la contrôlabilité des équations paraboliques (voir [59, Page 367]). Plus précisément, on étudie

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + v(t,x)v_x(t,x) = \hat u(t,x) \qquad (t \geqslant 0, \quad x \in [-2,1]\setminus\{h(t)\}),\\ v(t,h(t)) = \dot h(t) \qquad (t \geqslant 0),\\ m\ddot h(t) = [v_x](t,h(t)) \qquad (t \geqslant 0),\\ v(t,-2) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0),\\ h(0) = h_0, \quad \dot h(0) = h_1,\\ v(0,x) = v_0(x) \quad x \in [-2,1]\setminus\{h_0\}, \end{cases} \tag{0.1.7}$$

où $\hat u$ désigné le contrôle avec son support dans $(-2,-5/4)$. Pour le système (0.1.7), on a le résultat suivant

Théorème 0.1.2. *Le système* (0.1.7) *est localement contrôlable à zéro en tout temps* $\tau > 0$. *Plus précisément, pour tout* $\tau > 0$, *il existe* $r > 0$ *tel que, pour tout* $(h_0, h_1, v_0) \in \mathbb{R} \times \mathbb{R} \times H_0^1(-2,1)$ *avec*

$$|h_0| + |h_1| + \|v_0\|_{H_0^1(-2,1)} \leqslant r, \qquad v_0(h_0) = h_1,$$

il existe un contrôle $\hat u \in L^2([0,\tau], L^2(-2,-5/4))$ *tel que la solution de* (0.1.7) *satisfait* (0.1.6). *De plus,* $\widehat{u}$ *peut être choisi tel que* $h \in C[0,\tau]$ *et*

$$-1 < h(t) < 1 \qquad (t \in [0,\tau]).$$

Afin de transformer le système (0.1.7) en un système écrit sur un domaine fixe, on effectue le changement de variables suivant

$$y = \eta(t,x) = \begin{cases} \frac{x-h(t)}{1-h(t)}, x > h(t),\\ \frac{2(x-h(t))}{2+h(t)}, x < h(t). \end{cases} \tag{0.1.8}$$

$$\varphi(t,\eta(t,x)) = v(t,x) \qquad (t \in [0,\tau],\ x \in (-2,h(t)) \cup (h(t),1)). \tag{0.1.9}$$

Après quelques calculs, le système (0.1.7) donnent le système suivant d'inconnues $\varphi(t,y), g(t)$ et $h(t)$:

$$\begin{cases} \varphi_t = \varphi_{yy} + N_1(\varphi, h, g) + u \qquad (t \geqslant 0, \quad y \in [-2, -3/2] \setminus \{0\}), \\ \dot{g}(t) = m^{-1}[\varphi_y](0,t) + N_2(\varphi, h, g) \qquad (t \geqslant 0), \\ \varphi(t,0) = g(t) \qquad (t \geqslant 0), \\ \dot{h}(t) = g(t), \\ \varphi(t,-2) = 0, \quad \varphi(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad g(0) = h_1, \\ \varphi(0,y) = \varphi_0(y) \qquad (y \in [-2,1] \setminus \{0\}), \end{cases} \tag{0.1.10}$$

où N_1 et N_2 sont les termes non linéaires. Une étape très importante pour démontrer Théorème 0.1.2 est le résultat suivant pour le système (0.1.10) :

Proposition 0.1.3. *Il existe $r > 0$ tel que, pour tout $(h_0, h_1, \varphi_0) \in \mathbb{R} \times \mathbb{R} \times H_0^1(-2,1)$ avec*

$$|h_0| + |h_1| + \|\varphi_0\|_{H_0^1(-2,1)} \leqslant r, \qquad v_0(h_0) = h_1,$$

alors il existe un contrôle $u \in L^2([0,\tau], L^2(-2,-3/2))$ tel que la solution de (0.1.10) vérifie

$$\varphi(\tau, \cdot) = 0, \quad h(\tau) = 0, \quad \dot{h}(\tau) = 0. \tag{0.1.11}$$

De plus, $\widehat{u}$ peut être choisi tel que $h \in C[0,\tau]$ et

$$-1 < h(t) < 1 \qquad (t \in [0,\tau]).$$

Dans la suite de ce chapitre, on donne la démonstration du Proposition 0.1.3, qui implique Théorème 0.1.1.

0.1.2 Un algorithme itératif pour le contrôle de systèmes paraboliques

Dans cette section, on introduit d'abord un algorithme itératif qui on permet d'obtenir la contrôlabilité des équations paraboliques non homogènes à partir de la contrôlabilité des équation homogènes et qui évite d'utiliser l'estimation de Carleman. Puisque cette méthode est indépendante du système (0.1.5) et est applicable à d'autres équations paraboliques, on l'établit d'une façon abstraite. On suppose que le lecteur est familier avec la théorie de semi-groupe telle qu'elle est décrite dans le livre [50]. On adopte les notations de la monographie [59].

Soient X, U espaces de Hilbert et $A : \mathcal{D}(A) \to X$ un opérateur négatif qui engendre un semi-groupe fortement continu (ou C_0 semi-groupe) noté $\mathbb{T}$. On dit que l'opérateur A est **négatif** si $A : \mathcal{D}(A) \to X$ est autoadjoint et si $\langle Az, z\rangle \leqslant 0$ pour tout $z \in \mathcal{D}(A)$. Soit $B \in \mathcal{L}(U, X)$ un opérateur de contrôle. On considère le système suivant :

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0 \in X. \tag{0.1.12}$$

Définition 0.1.4 (Contrôlabilité à zéro)**.** *Le couple* (A, B) *est dit contrôlable à zéro en temps* $\tau > 0$ *si pour tout donnée initiale* $z_0 \in X$*, il existe un contrôle* $u \in L^2(0, T; U)$ *tel que la solution de* (0.1.12) *soit nulle au temps* $\tau > 0$ *:* $z(\tau) = 0$.

Supposons que (A, B) soit contrôlable à zéro en tout temps $\tau > 0$ et qu'il existe une fonction γ avec les propriétés suivantes

- $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction continue et non décroissante avec $\lim_{\tau \to 0^+} \gamma(\tau) = +\infty$.
- Le contrôle $u \in L^2([0, \tau], U)$ entraîne la solution de (0.1.12) à zéro au temps $\tau > 0$ satisfait

$$\|u\|_{L^2([0,\tau],U)} \leqslant \gamma(\tau)\|z_0\|_X \quad (z_0 \in X).$$

Remarque 0.1.5. *En fait, la fonction* γ *est un majoration du* ***coût de contrôle*** *de* (A, B)*. Plus précisément, si* (A, B) *est contrôlable à zéro en tout temps* $\tau > 0$*, l'ensemble*

$$\mathcal{C}_{\tau,z_0} := \{u \in L^2([0, \tau], U) \ \ | \ \ z(\tau) = 0\},$$

n'est pas vide et la fonction définie par

$$K(\tau) := \sup_{\|z_0\|=1} \inf_{u \in \mathcal{C}_{\tau,z_0}} \|u\|_{L^2([0,\tau],U)}.$$

est continue et non décroissante avec $\lim_{\tau \to 0^+} K(\tau) = +\infty$ *(voir, par exemple, [57]). Normalement, ce n'est pas facile de trouver une expression précise pour* $K(\tau)$*, mais on peut trouver une fonction* γ *telle que* $K(t) < \gamma(t), \quad t > 0$*.*

On considère maintenant le système non homogène suivant :

$$\left\{ \begin{array}{l} \dot{z} = Az + Bu + f, \\ z(0) = z_0. \end{array} \right. \tag{0.1.13}$$

où $f : [0, \infty) \to X$. On montre que, de manière générale, pour tout f qui tend vers zéro à un taux prefixé au temps τ, il existe un contrôle u tel que la solution de (0.1.13) tende vers zéro à un taux similaire au temps τ. Afin de rendre cette affirmation précise, nous avons besoin de quelques notations.

Soit $q > 1$ une constante dont la valeur sera déterminée plus tard. On considère une fonction $\rho_{\mathcal{F}} : [0, \infty] \to [0, \infty)$ avec

$$\rho_{\mathcal{F}} \ \text{ non croissante et } \ \rho_{\mathcal{F}}(\tau) = 0, \tag{0.1.14}$$

telle que il existe une fonction non croissante $\rho_0 : [0, \tau] \to [0, \infty)$ telle que

$$\rho_0(t) := \rho_{\mathcal{F}}(q^2(t - \tau) + \tau)\gamma\left((q - 1)(\tau - t)\right) \quad \left(t \in \left[\tau\left(1 - \frac{1}{q^2}\right), \tau\right]\right), \tag{0.1.15}$$

et

$$\rho_0(\tau) = 0. \tag{0.1.16}$$

L'existence d'une telle fonction peut être facilement vérifiée. On associe $\rho_{\mathcal{F}}$ et ρ_0 les espaces de Hilbert $\mathcal{F}$,$\mathcal{U}$ et $\mathcal{Z}$ définis par

$$\mathcal{F} = \left\{ f \in L^2([0,\tau], X) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2([0,\tau], X) \right\}, \tag{0.1.17a}$$

$$\mathcal{U} = \left\{ u \in L^2([0,\tau], U) \;\middle|\; \frac{u}{\rho_0} \in L^2([0,\tau], U) \right\}, \tag{0.1.17b}$$

$$\mathcal{Z} = \left\{ z \in L^2([0,\tau], X) \;\middle|\; \frac{z}{\rho_0} \in L^2([0,\tau], X) \right\}. \tag{0.1.17c}$$

Le produit scalaire sur $\mathcal{F}$ est donné par

$$\langle f_1, f_2 \rangle_{\mathcal{F}} = \int_0^\tau \rho_{\mathcal{F}}^{-2}(t) \langle f_1(t), f_2(t) \rangle \, \mathrm{d}t$$

et des définitions similaires sont utilisées pour $\mathcal{U}$ et $\mathcal{Z}$. Les normes induites sont désignées par $\|\cdot\|_{\mathcal{F}}$, $\|\cdot\|_{\mathcal{Z}}$ et $\|\cdot\|_{\mathcal{U}}$, respectivement. Le résultat le plus importance de ce travail est la proposition suivant :

Proposition 0.1.6. *Sous les hypothèses et notations ci-dessus, pour tout $z_0 \in X$ et $f \in \mathcal{F}$, il existe un contrôle $u \in \mathcal{U}$ tel que la solution de* (0.1.13) *soit $z \in \mathcal{Z}$. De plus, il existe une constante positive C, qui ne dépend pas de f ni de z_0, telle que*

$$\left\| \frac{z}{\rho_0} \right\|_{C([0,\tau],X)} + \|u\|_{\mathcal{U}} \leqslant C \left(\|f\|_{\mathcal{F}} + \|z_0\|_X \right). \tag{0.1.18}$$

Remarque 0.1.7. *Puisque ρ_0 est une fonction continue avec $\rho_0(\tau) = 0$, le fait que $\frac{z}{\rho_0} \in C([0,\tau], X)$ rend $z(\tau) = 0$.*

Considérons le système adjoint du système (0.1.13)

$$\begin{cases} -\dot{\zeta}(t) = A\zeta(t) + g \qquad (t \in (0,\tau)), \\ \zeta(\tau) = 0. \end{cases} \tag{0.1.19}$$

Par un argument de dualité (voir, par exemple, [34, Théorème 4.1]), nous obtenons la conséquence suivante de Proposition 0.1.6

Corollaire 0.1.8. *Sous les hypothèses et notations de la Proposition* 0.1.6*, il existe une constante $C > 0$ telle que, pour tout $g \in L^2([0,\tau], X)$, la solution de* (0.1.19) *satisfait :*

$$\|\zeta(0)\|^2 + \int_0^\tau \|\rho_{\mathcal{F}}\zeta\|^2 \mathrm{d}t \leqslant C \left(\int_0^\tau \|\rho_0 g\|^2 \, \mathrm{d}t + \int_0^\tau \|\rho_0 B^* \zeta\|_U^2 \mathrm{d}t \right). \tag{0.1.20}$$

La proposition suivante donne plus d'informations sur la régularité de la trajectoire contrôlée obtenue dans la Proposition 0.1.6.

Proposition 0.1.9. *Sous les hypothèses et notations de la Proposition* 0.1.6*, supposons qu'il existe une fonction continue et décroissante* $\rho : [0, \tau] \to \mathbb{R}^+$ *avec* $\rho(\tau) = 0$ *qui satisfait les inégalités suivantes*

$$\rho_0 \leqslant C\rho, \quad \rho_{\mathcal{F}} \leqslant C\rho, \quad |\dot{\rho}|\, \rho_0 \leqslant C\rho^2 \qquad (t \in [0, \tau]), \tag{0.1.21}$$

où $C > 0$ *est une constante qui ne dépend pas de* t*. Alors pour tout* $z_0 \in \mathcal{D}((-A)^{\frac{1}{2}})$ *il existe un contrôle* $u \in \mathcal{U}$ *tel que la solution* $z \in \mathcal{Z}$ *de* (0.1.13) *satisfait*

$$\frac{z}{\rho} \in L^2([0, \tau], \mathcal{D}(A)) \cap H^1((0, \tau), X) \cap C([0, \tau], \mathcal{D}((-A)^{\frac{1}{2}})).$$

De plus, il existe une constante $C > 0$ *qui ne dépend pas de* z_0 *ni de* f *telle que*

$$\left\| \frac{z}{\rho} \right\|_{C([0,\tau],\mathcal{D}((-A)^{1/2}))\cap H^1((0,\tau),X)\cap L^2([0,\tau],\mathcal{D}(A))} \leqslant C \left(\|z_0\|_{\mathcal{D}((-A)^{1/2})} + \|f\|_{\mathcal{F}} \right). \tag{0.1.22}$$

Dans la partie restante de cette section, nous considérons un système obtenu à partir de (0.1.13) en ajoutant un intégrateur simple. Plus précisément, nous considérons les équations

$$\begin{cases} \dot{z} = Az + Bu + f, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0, \end{cases} \tag{0.1.23}$$

où Y est un espace de dimension finie de Hilbert et $C \in \mathcal{L}(X, Y)$. Considérons le système adjoint de (0.1.23)

$$\begin{cases} -\dot{\xi}(t) = A\xi(t) + g(t) + C^* r, \\ \xi(\tau) = 0, \end{cases} \tag{0.1.24}$$

où $g : [0, \infty) \to X$ et $r \in Y$. Le dernier résultat dans cette section est une caractérisation par la dualité du fait que les solutions de (0.1.23) peuvent être amenées à zéro avec un taux de décroissance préfixé. En utilisant la notation introduite dans (0.1.15) - (0.1.17), ce résultat se lit comme suit :

Proposition 0.1.10. *Les deux instructions suivantes sont équivalentes*

(1) *Il existe une constante* $C > 0$ *telle que, pour tout* $(g, r) \in L^2([0, \tau], X) \times Y$*, la solution de* (0.1.24) *vérifie*

$$\|r\|_Y^2 + \|\xi(0)\|^2 + \int_0^\tau \|\rho_{\mathcal{F}}\xi\|^2 \mathrm{d}t \leqslant C \left(\int_0^\tau \|\rho_0 g\|^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^* \xi\|_U^2 \mathrm{d}t \right). \tag{0.1.25}$$

(2) *Il existe un opérateur linéaire borné $E_\tau : X \times Y \times \mathcal{F} \to \mathcal{U}$ tel que, pour tout $(z_0, h_0, f) \in X \times Y \times \mathcal{F}$, le contrôle $u = E_\tau(z_0, h_0, f)$ soit tel que la solution de* (0.1.23) *vérifie $z \in \mathcal{Z}$ et $h(\tau) = 0$.*

Pour la preuve de cette proposition, nous nous référons à [34, Théorème4.1] .

0.1.3 Contrôlabilité à zéro avec contrôle distribué pour un problème linéaire couplé

Afin de résoudre (0.1.10), nous utilisons la théorie des semi-groupes. Définissons

$$X = L^2(-2,1) \times \mathbb{R}, \quad U = L^2(-2,-3/2)$$

et $A : \mathcal{D}(A) \to X$ est un opérateur linéaire défini par

$$\mathcal{D}(A) = \left\{ z = \begin{bmatrix} \varphi \\ g \end{bmatrix} \in H_0^1(-2,1) \times \mathbb{R} \,|\, \varphi(0) = g,\ \varphi_{|_{(-2,0)}} \in H^2(-2,0),\ \varphi_{|_{(0,1)}} \in H^2(0,1) \right\},$$

$$A \begin{bmatrix} \varphi \\ g \end{bmatrix} = \begin{bmatrix} \varphi_{xx} \\ m^{-1}[\varphi_x](0) \end{bmatrix}. \qquad (0.1.26)$$

Pour tout $z_i = \begin{bmatrix} \varphi_i \\ g_i \end{bmatrix} \in X,\ i = 1, 2$, on définit un produit scalaire sur X par

$$\langle z_1, z_2 \rangle = \int_{-2}^{1} \varphi_1(x)\varphi_2(x)\,\mathrm{d}x + m g_1 g_2 \qquad (0.1.27)$$

et par $\|\cdot\|$ sa norme correspondante. De plus, on considère l'opérateur de contrôle $B \in \mathcal{L}(U; X)$ défini par

$$Bu = \begin{bmatrix} \mathbf{1}_{(-2,-3/2)} u \\ 0 \end{bmatrix} \qquad (u \in U). \qquad (0.1.28)$$

Lemme 0.1.11. *Avec les notations précédentes, l'opérateur A est négatif, en admettant une base orthonormée des vecteurs propres $(\Phi_k)_{k\geqslant 1}$ et avec la suite correspondante décroissante des valeurs propres $(-\lambda_k)_{k\geqslant 1}$ satisfaisant*

$$\inf_{k\geqslant 0} (\lambda_{k+1} - \lambda_k) = s > 0, \qquad (0.1.29)$$

$$\lambda_k = \frac{\pi^2 k^2}{9} + O(k) \qquad (k \to \infty), \qquad (0.1.30)$$

pour un certain $r > 0$. De plus, il existe une constante $m > 0$ telle que

$$\|B^*\Phi_k\|_U \geqslant m \qquad (k \geqslant 1). \qquad (0.1.31)$$

Pour la démonstration de ce lemme, nous nous référons aux Propositions 3.1 et 3.2 du Chapitre 1. En utilisant ce lemme, on peut démontrer le proposition suivante :

Proposition 0.1.12 (Coût de contrôle)**.** *Le couple (A, B) est contrôlable à zéro en tout temps $\tau > 0$ et son coût de contrôle vérifie $K(\tau) < M_0 e^{\frac{M_1}{\tau}}$ où M_0, M_1 sont des constantes positives.*

Le résultat ci-dessus est donnée Par Tenenbaum et Tucsnak dans [59, Theoreme 3.4 et Corollaire 3.6] et sa preuve peut être trouvée dans la Proposition 2.2 du Chapitre 1.

Maintenant, on considère le système non homogène suivant

$$\begin{cases} \dot{z} = Az + Bu + f, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0, \end{cases} \tag{0.1.32}$$

avec $C : X \to \mathbb{R}$ défini par $C \begin{bmatrix} \varphi \\ g \end{bmatrix} = g$. L'étape la plus importante pour démontrer la Proposition 0.1.3 est le résultat suivant pour le système (0.1.32) :

Proposition 0.1.13. *Soient M_0, M_1 les constantes dans la Proposition 0.1.12. Supposons que*

$$\rho_{\mathcal{F}},\ \rho_0 : [0, \tau) \to [0, \infty),$$

$$\begin{cases} \rho_{\mathcal{F}}(t) = \exp(-\frac{\alpha}{(\tau-t)^2}), \\ \rho_0(t) = M_0 \exp\left(\frac{M_1}{(q-1)(\tau-t)} - \frac{\alpha}{q^4(\tau-t)^2}\right), \\ \rho(t) = e^{-\frac{\beta}{(\tau-t)^2}}, \end{cases} \tag{0.1.33}$$

avec les paramètres q, α, β vérifiant :

$$1 < q < 2^{\frac{1}{4}}, \quad \alpha > \frac{M_1 q^4}{2(q-1)}\tau, \quad \beta < \frac{\alpha}{q^4}. \tag{0.1.34}$$

Soient $(\mathcal{F}, \mathcal{U}, \mathcal{Z})$ les espaces définis par (0.1.17)*. Alors il existe un opérateur linéaire borné, donné par*

$$E_\tau : X \times \mathbb{R} \times \mathcal{F} \to \mathcal{U} \tag{0.1.35}$$

tel que pour tout $(z_0, h_0, f) \in X \times \mathbb{R} \times \mathcal{F}$, $u = E_\tau(z_0, h_0, f)$ est le contrôle qui mène la solution de (0.1.32) *à zéro. De plus, si $z_0 \in \mathcal{D}((-A)^{\frac{1}{2}})$, alors $z \in C([0,\tau], \mathcal{D}((-A)^{\frac{1}{2}}))$ et il existe une constante $C > 0$ telle que*

$$\left\| \frac{z}{\rho} \right\|_{C([0,\tau];((-A)^{1/2}))\cap H^1((0,\tau),X)\cap L^2([0,\tau],\mathcal{D}(A))} \leqslant C \left(\|z_0\|_{((-A)^{1/2})} + |h_0| + \|f\|_{\mathcal{F}} \right), \tag{0.1.36}$$

pour toutes $z_0 \in ((-A)^{1/2})$, $f \in \mathcal{F}$ et $h_0 \in \mathbb{R}$.

On écrit ici justement l'idée de la démonstration :

Nous considérons le système adjoint du système (0.1.32) suivant :

$$\begin{cases} -\dot{\xi}(t) = A\xi(t) + g(t) + C^* r, \\ \xi(\tau) = 0, \end{cases} \tag{0.1.37}$$

où $g \in L^2([0,\tau], X)$ et $r \in \mathbb{R}$. On peut vérifier que les fonctions définies par (0.1.33) et par (0.1.34) vérifient les hypothèses de la Proposition 0.1.6 et du Corollaire 0.1.8. En utilisant (0.1.20), il existe une constante $C > 0$ qui ne dépend pas de g ni de r telle que

$$\|\xi(0)\|_X^2 + \int_0^\tau \|\rho_{\mathcal{F}}\xi\|_X^2 \mathrm{d}t \leqslant C \left(\int_0^\tau \|\rho_0(g + C^* r)\|_X^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^* \xi\|_U^2 \mathrm{d}t \right). \tag{0.1.38}$$

Par un argument de compacité (voir la Proposition 3.4 du Chapitre 1), on a

$$|r|^2 \leqslant C \left(\int_0^\tau \|\rho_0 g\|_X^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^* \xi\|_U^2 \mathrm{d}t \right). \tag{0.1.39}$$

Les deux inégalités au-dessus impliquent l'inégalité (0.1.25), qui nous donne l'existence de l'opérateur E_τ par la Proposition 0.1.10. La régularité de z est obtenue par la Proposition 0.1.9.

0.1.4 Démonstration de la Proposition 0.1.3

Dans cette section, on démontre la Proposition 0.1.3. Avec les notations de la section précédente, le système (0.1.10) peut être écrit comme

$$\begin{cases} \dot{z} = Az + N \begin{bmatrix} z \\ h \end{bmatrix} + Bu, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0, \end{cases} \tag{0.1.40}$$

où $N : \mathcal{D}(A) \times (-2,1) \to X$ est l'opérateur non linéaire défini par

$$N \begin{bmatrix} z \\ h \end{bmatrix} := \begin{bmatrix} N_1(\varphi, h, g) \\ N_2(\varphi, h, g) \end{bmatrix}.$$

Supposons $\mathcal{F}_r = \{f \in \mathcal{F} |\ \|f\|_{\mathcal{F}} \leqslant r\}$ et $|h_0| + |h_1| + \|\varphi_0\|_{H_0^1(-2,1)} \leqslant r$. On définit un opérateur $\mathcal{N}$ sur $\mathcal{F}_r$ par

$$\mathcal{N}(f)(t) := N \begin{bmatrix} z(t) \\ h(t) \end{bmatrix} \qquad (t \in [0,\tau]),$$

où $\begin{bmatrix} z \\ h \end{bmatrix}$ est la trajectoire de (0.1.32) correspondant à $u = E_\tau(z_0, h_0, f)$. Nous montrons que cet opérateur est bien défini de $\mathcal{F}_r$ dans lui-même et est une contraction si on choisit r assez petit. Par ceci, le système (0.1.40) et donc le système (0.1.10) possède un point fixe f et $u = E_\tau(h_0, h_1, f)$ est le contrôle qui amène le système (0.1.10) à zéro en tout temps τ.

0.1.5 Problème ouverts et perspectives

Une question naturelle, au vu du théorème (0.1.1), est de savoir si la position du point matériel est exactement contrôlable. Plus précisément, on peut se demander si l'on peut démontrer l'affirmation suivant :
Pour tout $\tau > 0$, il existe $r > 0$ tel que, pour chaque (h_0, h_τ, h_1, v_0) avec

$$h_0, h_\tau \in (-1,1),\ h_1 \in \mathbb{R},\ v_0 \in H_0^1(-1,1),\ v_0(h_0) = h_1,$$

$$|h_0 - h_\tau| + |h_1| + \|v_0\|_{H_0^1(-1,1)} \leqslant r,$$

il existe $u \in C[0,\tau]$ tel que la solution de (0.1.5) vérifie

$$v \in L^2([0,\tau], H^2((-1,1)\setminus\{h(t)\}) \cap C([0,\tau], H^1(-1,1)), \quad h \in C^1[0,\tau],$$

avec

$$\dot{h}(\tau) = 0, \quad v(\tau,\cdot) = 0, \quad h(\tau) = h_\tau. \tag{0.1.41}$$

Dans le cas bidimensionnel ou tridimensionnel, une réponse positive a été donnée par Imanuvilov et Takahashi dans [34, Corollary 1.2]. Dans leur article, le changement de variable pour transformer le problème à frontière libre en un problème à frontière fixe est valable pour tout point $x \in \Omega \setminus \mathcal{C}$. Mais dans notre travail, afin de simplifier la présentation, on applique un changement de variables qui marche seulement au point $x = 0$. Pourtant, puisque les résultats de la section (0.1.2) ne dependent que du coût du système, pour établir un résultat similaire avec celui de [34], il faut juste démontrer le Lemme 0.1.11 dans un cas plus général comme suit :

$$X = L^2(c,d) \times \mathbb{R}, \quad U = L^2(c, c+\epsilon)$$

avec $c < c+\epsilon < 0 < d$, $\frac{c}{d} \in \mathbb{Q}$ et $\epsilon \ll 1$ et $A : \mathcal{D}(A) \to X$ est un opérateur linéaire défini par

$$\mathcal{D}(A) = \left\{ z = \begin{bmatrix} \varphi \\ g \end{bmatrix} \in H_0^1(c,d) \times \mathbb{R} \,|\, \varphi(0) = g,\ \varphi_{|_{(c,0)\cup(0,d)}} \in H^2((c,0)\cup(0,d)) \right\},$$

$$A \begin{bmatrix} \varphi \\ g \end{bmatrix} = \begin{bmatrix} \varphi_{xx} \\ m^{-1}[\varphi_x](0) \end{bmatrix}. \tag{0.1.42}$$

De plus, l'opérateur de contrôle $B \in \mathcal{L}(U; X)$ dans ce cas est défini par

$$Bu = \begin{bmatrix} \mathbf{1}_{(c,c+\epsilon)} u \\ 0 \end{bmatrix} \qquad (u \in U). \tag{0.1.43}$$

En effet, la démonstration du Lemme 0.1.11 dans ce cas est plus compliquée que celui de la Proposition 1.3.2 du Chapitre 1, mais l'idée est similaire. On espère aussi trouver une méthode plus simple.

On peut étudier également le problème pour lequel le contrôle u dans (0.1.5) se situe au point materiel au lieu d'être sur le point $x = -1$. Plus précisément, on considère

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + v(t,x)v_x(t,x) = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) + u(t) \qquad (t \geqslant 0), \\ v(t,-1) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases} \tag{0.1.44}$$

Dans ce cas, il semble que l'on ne puisse pas passer de la contrôlabilité du système distribué à la contrôlabilité frontière. Par conséquent, il faut attaquer (0.1.44) directement. Mais, par rapport à l'opérateur définit par (0.1.28), l'opérateur de contrôle B pour le système (0.1.44) n'est plus borné, ce qui empêche d'appliquer les conditions d'employer les résultats de la Section 0.1.2.

Néanmoins, on peut aussi établir ces résultats quand B est une opérateur de contrôle admissible pour le semi-groupe $\mathbb{W}$ engendré par l'opérateur A (Voir [57, Section 2.2]).

0.2 Quelques modèles de champ de phase fluide-vésicule

Dans cette section et celle qui suit, on étudie quelques modèles de vésicules. En biologie cellulaire, une vésicule cytoplasmique est (d'après Wikipédia) un compartiment relativement petit, séparé du cytosol par au moins une bicouche lipidique. Circulant dans le cytoplasme, elle peut stocker, transporter ou encore digérer des produits et déchets cellulaires. L'étude des propriétés hydrodynamiques de la membrane de la vésicule est importante pour des nombreuses applications, notamment dans les disciplines biologiques et physiologiques. Des articles récents ont été consacrés à des études expérimentales (voir par exemple [40] ou [41]) à la modélisation et enfin à l'analyse mathématique des modèles obtenus (voir par exemple [20],[21], [22], [23] et [62]). Modéliser la déformation élastique de la vésicule dans un fluide n'est pas facile. Une possibilité pourrait consister à considérer les équations de l'élasticité de la membrane et les équations de Navier-Stokes pour le fluide qui la contient. Toutefois, l'analyse et la simulation de ce type de modèle pourrait être très difficile, en raison de la frontière libre correspondant aux interfaces entre le fluide et la vésicule. En outre, dans ce couplage, les équations de l'élasticité sont écrites en variables de Lagrange alors que les équations du fluide sont écrites en variable eulérienne (voir [8], [9] et les références des ces travaux). Notre approche de l'interaction fluide-vésicule consiste à utiliser la méthode de champ de phase : la membrane de la vésicule est décrite par une fonction φ qui s'appelle le champ de phase. Dans ce modèle, φ prend la valeur $+1$ à l'intérieur de la membrane vésiculaire et -1 à l'extérieur, avec une mince couche de transition de largeur caractérisée par un petit paramètre $\epsilon > 0$. Le fluide est modélisé par les équations de Navier-Stokes incompressibles écrites dans l'ensemble du domaine contenant le fluide et la vésicule. Comme la fonction φ satisfait une équation de diffusion avec le paramètre ϵ, il est important d'analyser le comportement asymptotique des solutions lorsque $\epsilon \to 0$. Ce type d'analyse asymptotique formelle peut être trouvé dans [7], [21] et [37]. Il convient de mentionner quelques références existantes qui utilisent la méthode de champ de phase pour régler d'autres problèmes d'interface. Dans [13], l'auteur considère un problème de frontière libre diffusive. Il obtient d'abord les résultats de régularité et puis il étudie comment la couche de la transition de phase devient progressivement plus mince quand $\epsilon \to 0$. Dans [1], la méthode de champ de phase est également utilisée pour étudier un modèle à interface diffusive pour le flux de deux phases et dans [2] les auteurs étudient son comportement asymptotique.

0.2.1 Une méthode générale pour modéliser les interactions fluide-vésicule

Selon Jamet et Misbah [37], la vésicule possède deux caractéristiques importantes :

- Elle est localement incompressible au sens où sa surface locale est constante au cours du temps.
- Elle possède une énergie élastique qui est proportionnelle à la variation de sa courbure moyenne .

Dans cette section, nous formulons le système fluide-vésicule d'une manière abstraite qui est introduite dans [21]. Dans un premier temps, nous considérons un fluide parfait décrit par les équations d'Euler. Dans ce cas, l'énergie totale du système est composée de l'énergie cinétique du fluide et de l'énergie élastique de la vésicule (nous négligeons l'énergie cinétique de la vésicule). Selon Arnold [3](voir aussi [24] et [4, Appendix 2]), la trajectoire du fluide parfait peut être considérée comme une famille des difféomorphismes à un paramètre conservant l'élément de volume. Plus précisément, soit Ω un ouvert borné de $\mathbb{R}^3$ avec une frontière lisse. Les points du Ω sont désignés par y et le temps est désigné par $t \in [0, T]$.

Un **mouvement** du domaine Ω est une **famille des difféomorphismes à un paramètre** $x = X(y,t) : \Omega \times [0,T] \to \Omega$ avec les propriétés suivantes : X est une application lisse de y et de t, $X(\cdot,t) : \Omega \to \Omega$ est un difféomorphisme avec $\det(\nabla_y X(\cdot,t)) > 0)$ pour tout $t \in [0,T]$ et $X(\cdot,0) = id_\Omega$. Pour $y \in \Omega$, la trajectoire d'une particule qui se trouve initialement en y est $\{X(y,t) : t \in [0,T]\}$. Nous associons à chaque mouvement X, le champ de vecteurs u défini par

$$u(X(y,t),t) := \partial_t X(y,t). \tag{0.2.1}$$

Avec les notations ci-dessus, on pose

$$\mathscr{X} = \{X \text{ est un mouvement de } \Omega,\ X(y,t)|_{\partial\Omega} = y,\ \det \nabla_y X(\cdot,t) = 1\}. \tag{0.2.2}$$

On voit facilement que, pour tout $X \in \mathscr{X}$, le champ de vecteurs u défini par (0.2.1) est nul sur $\partial\Omega$.

Soit X^s une famille de mouvements de $\mathscr{X}$, c'est-à-dire, $X^s \in \mathscr{X}$ pour chaque $s \in (-s_0, s_0)$ pour un certain s_0 positif. Alors on peut définir un champ de vecteurs $v : \Omega \times [0,T] \to \mathbb{R}^3$ par

$$v(y,t) = \frac{d}{ds} X^s(y,t)\Big|_{s=0}. \tag{0.2.3}$$

Le champ de vecteurs eulérien v^* associé à v est défini par

$$v^*(x,t) = v^*(X(y,t),t) := v(y,t). \tag{0.2.4}$$

De même pour u, les champs de vecteurs v et v^* s'annulent sur la frontière. On suppose que l'énergie élastique de la vésicule est donnée par

$$\mathscr{E}(\phi,\sigma) = \mathscr{E}(\phi(X(y,t),t), \sigma(X(y,t),t)), \tag{0.2.5}$$

qui dépend des fonctions de champ de phase ϕ, σ et du mouvement X. Pour la fonctionnelle (0.2.5), on peut définir la dérivée Euler-Lagrange de ϕ dans la direction ξ par

$$\int_\Omega \frac{\delta\mathscr{E}}{\delta\phi}\xi \mathrm{dx} = \frac{d}{dt}\Big|_{t=0} \mathscr{E}(\phi + t\xi, \sigma). \tag{0.2.6}$$

Une définition similaire est utilisée pour $\frac{\delta\mathscr{E}}{\delta\sigma}$ qui désigne la dérivée d'Euler-Lagrange par rapport à σ.

On considère la fonctionnelle d'action $\mathscr{A} : \mathscr{X} \to \mathbb{R}$ définie par :

$$\mathscr{A}(X) = \int_0^T \left(\int_\Omega \frac{1}{2}|\partial_t X|^2 \mathrm{dy} + \mathscr{E}(\phi,\sigma) \right) \mathrm{dt}. \tag{0.2.7}$$

Supposons qu'une mouvement $X \in \mathscr{X}$ est un point critique de la fonctionnelle $\mathscr{A}$. Il résulte que pour chaque courbe X^s qui passe X dans $\mathscr{X}$ avec

$$X^s(\cdot,0) = X(\cdot,0), \quad X^s(\cdot,T) = X(\cdot,T), \quad (-s_0 < s < s_0), \tag{0.2.8}$$

on a

$$\frac{d}{ds}\mathscr{A}(X^s)\Big|_{s=0} = 0. \tag{0.2.9}$$

D'après (0.2.8) et les définitions (0.2.3) et (0.2.4), on obtient que

$$v(\cdot,0) = v(\cdot,T) = 0. \tag{0.2.10}$$

D'après (0.2.9), on déduit que

$$\begin{aligned} 0 &= \frac{d}{ds}\mathscr{A}(X^s)\Big|_{s=0} \\ &= \int_0^T \left(\int_\Omega \partial_t X \cdot \partial_t \frac{d}{ds} X^s \Big|_{s=0} \mathrm{d}y + \frac{d}{ds}\mathscr{E}(\phi(X^s,t),\sigma(X^s,t))\Big|_{s=0} \right) \mathrm{dt}. \end{aligned} \tag{0.2.11}$$

Selon (0.2.3), (0.2.4), (0.2.5), (0.2.6) et la formule de la dérivée d'une application composée, il résulte que

$$\begin{aligned} \frac{d}{ds}\mathscr{E}(\phi(X^s,t),\sigma(X^s,t))\Big|_{s=0} &= \int_\Omega \left(\frac{\delta\mathscr{E}}{\delta\phi}\nabla\phi + \frac{\delta\mathscr{E}}{\delta\sigma}\nabla\sigma \right) \cdot \frac{d}{ds}X^s(y,t)\Big|_{s=0} \mathrm{dy} \\ &= \int_\Omega \left(\frac{\delta\mathscr{E}}{\delta\phi}\nabla\phi + \frac{\delta\mathscr{E}}{\delta\sigma}\nabla\sigma \right) \cdot v(y,t)\mathrm{dy}. \end{aligned} \tag{0.2.12}$$

En substituant (0.2.12) dans (0.2.11), on obtient que

$$0 = \int_0^T \left(\int_\Omega \partial_t X \cdot \partial_t \frac{d}{ds} X^s \Big|_{s=0} \mathrm{dy} + \int_\Omega \left(\frac{\delta\mathscr{E}}{\delta\phi}\nabla\phi + \frac{\delta\mathscr{E}}{\delta\sigma}\nabla\sigma \right) \cdot v(y,t)\mathrm{dy} \right) \mathrm{dt}.$$

D'après (0.2.3) et la condition au bord (0.2.10), il résulte que

$$\begin{aligned} &\int_0^T \left(\int_\Omega \partial_t X \cdot \partial_t \frac{d}{ds} X^s \Big|_{s=0} \mathrm{dy} + \int_\Omega \left(\frac{\delta\mathscr{E}}{\delta\phi}\nabla\phi + \frac{\delta\mathscr{E}}{\delta\sigma}\nabla\sigma \right) \cdot v(y,t)\mathrm{dy} \right) \mathrm{dt} \\ &\quad = \int_0^T \left(\int_\Omega \partial_t X \cdot \partial_t v \mathrm{dy} + \int_\Omega \left(\frac{\delta\mathscr{E}}{\delta\phi}\nabla\phi + \frac{\delta\mathscr{E}}{\delta\sigma}\nabla\sigma \right) \cdot v(y,t)\mathrm{dy} \right) \mathrm{dt} \\ &\quad = -\int_0^T \left(\int_\Omega \partial_{tt} X \cdot v \mathrm{dy} - \int_\Omega \left(\frac{\delta\mathscr{E}}{\delta\phi}\nabla\phi + \frac{\delta\mathscr{E}}{\delta\sigma}\nabla\sigma \right) \cdot v(y,t)\mathrm{dy} \right) \mathrm{dt}, \end{aligned}$$

qui implique

$$0 = \int_0^T \int_\Omega \left(\partial_{tt} X - \left(\frac{\delta \mathscr{E}}{\delta \phi} \nabla \phi + \frac{\delta \mathscr{E}}{\delta \sigma} \nabla \sigma \right) \right) \cdot v(y,t) \mathrm{dydt}$$

Comme X^s est arbitraire et $\det \nabla_y X^s(\cdot, t) = 1$ en tout $t \in [0, T]$, il résulte, par la définition (0.2.4) et le théorème de Liouville (voir [26, Page 476]), que v parcourt l'ensemble des champs de vecteurs vérifiant $\nabla \cdot v = 0 \quad (y \in \Omega, t \in [0, T])$, ce qui implique que,

$$\partial_{tt} X - \frac{\delta \mathscr{E}}{\delta \phi} \nabla \phi - \frac{\delta \mathscr{E}}{\delta \sigma} \nabla \sigma = \nabla p,$$

pour une certaine fonction ou une distribution p. Par (0.2.1), on obtient de la même façon que $\nabla \cdot u = 0$. De plus, comme

$$\partial_{tt} X = u_t + u \cdot \nabla u.$$

on déduit finalement, en ajoutant un terme du viscosité, que le système fluide-vésicule s'écrit comme

$$\begin{cases} u_t + u \cdot \nabla u - \frac{\delta \mathscr{E}}{\delta \phi} \nabla \phi - \frac{\delta \mathscr{E}}{\delta \sigma} \nabla \sigma = \nabla p + \Delta u, \\ \nabla \cdot u = 0. \end{cases}$$

Nous notons que la modélisation ci-dessus est très simplifiée. Par exemple, le mouvement vérifiant $X(y,t) = y \quad (y \in \partial \Omega)$ devrait être remplacé par un mouvement qui invariant sur le bord. Dans ce cas, le champ de vecteurs correspondant est parallèle au bord (voir [24]).

Nous examinons ensuite l'équation du champ de phase. Selon [6] ou [13], l'équation de ϕ s'écrit :

$$\frac{d\phi}{dt} = -\frac{\delta \mathscr{E}_1(\phi, \sigma)}{\delta \phi}, \tag{0.2.13}$$

où $\frac{d}{dt}$ désigne la dérivée particulaire de ϕ et $\mathscr{E}_1(\phi, \sigma)$ est l'énergie intrinsèque au sens où son rôle consiste uniquement à définir la frontière, et il n'affecte pas la physique. En conclusion, le système de champ de phase fluide-vésicule s'écrit

$$\begin{cases} u_t + u \cdot \nabla u - \frac{\delta \mathscr{E}}{\delta \phi} \nabla \phi - \frac{\delta \mathscr{E}}{\delta \sigma} \nabla \sigma = \nabla p + \Delta u, \\ \nabla \cdot u = 0, \\ \phi_t + \nabla \phi \cdot u = -\frac{\delta \mathscr{E}_1(\phi, \sigma)}{\delta \phi}, \\ u(0, \cdot) = u_0, \\ \phi(0, \cdot) = \phi_0. \end{cases} \tag{0.2.14}$$

Dans les sections suivantes, nous allons présenter les différents modèles en donnant des expressions concrètes des $\mathscr{E}, \mathscr{E}_1$ et du paramètre σ.

0.2.2 Un premier modèle élastique

Dans cette section, nous supposons que l'énergie élastique de la vésicule dépend d'une seule fonction de champ de phase notée ϕ. Plus précisément, nous supposons que

$$\mathscr{E}_1 = \mathscr{E} = \mathscr{E}(\phi) = E_\epsilon(\phi) + \frac{1}{2}M_1(V(\phi) - \alpha)^2 + \frac{1}{2}M_2(A(\phi) - \beta)^2,$$

où

$$\begin{aligned}
E_\epsilon(\phi) &= \frac{k}{2\epsilon}\int\limits_\Omega \left(\epsilon\Delta\phi + \left(\frac{1}{\epsilon}\phi + c_0\sqrt{2}\right)(1-\phi^2)\right)^2 \mathrm{dx},\\
A(\phi) &= \int\limits_\Omega \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2-1)^2 \mathrm{dx}, \quad V(\phi) = \int\limits_\Omega \phi \mathrm{dx},\\
\alpha &= V(\phi_0), \quad \beta = A(\phi_0),
\end{aligned}$$

et $\epsilon > 0$ est un paramètre caractérisant l'épaisseur de la couche limite autour de la vésicule. Ce modèle est introduit par Du et Liu dans [20] et [22]. Pour une vésicule $\mathcal{O}$ dans un domaine Ω, sa frontière $\partial\mathcal{O}$ peut être considérée comme une surface fermée (compacte sans bord) et ses formes d'équilibre sont souvent abstenues en minimisant l'énergie élastique de la membrane

$$E = \int\limits_{\partial\mathcal{O}} \frac{k}{2}(H - c_0)^2 ds \tag{0.2.15}$$

sous les contraintes que le volume $\int_\mathcal{O} dx$ et la surface de la vésicule $\int_{\partial\mathcal{O}} ds$ sont préservés. Dans (0.2.15), H est la courbure moyenne de $\partial\mathcal{O}$, c_0 la courbure spontanée et k le module de flexion. Dans [20] et [22], les auteurs approchent E , $\int_{\partial\mathcal{O}} ds$ et $\int_\mathcal{O} dx$ par E_ϵ, $A(\phi)$ et $V(\phi)$ respectivement. Pour imposer les deux contraintes, ils adoptent une formulation de pénalisation en choisissant M_1, M_2 assez grands tels qu'ils ne changeront pas de manière significative les valeurs de β et de α. En ajoutant les conditions aux bord, le système fluide-vésicule s'écrit

$$\begin{cases}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \nabla p + \Delta\mathbf{u} + \dfrac{\delta\mathscr{E}(\phi)}{\delta\phi}\nabla\phi & \text{in } [0,T]\times\Omega,\\
\nabla\cdot\mathbf{u} = 0 & \text{in } [0,T]\times\Omega,\\
\phi_t + \mathbf{u}\cdot\nabla\phi = -\frac{\delta\mathscr{E}(\phi)}{\delta\phi} & \text{in } [0,T]\times\Omega,\\
\mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega,\\
\mathbf{u}(t,x) = 0 & \text{on } [0,T]\times\partial\Omega,\\
\phi(0,x) = \phi_0(x) & \text{in } \Omega,\\
\phi(t,x) = -1 & \text{on } [0,T]\times\partial\Omega,\\
\Delta\phi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega.
\end{cases} \tag{0.2.16}$$

Il peut être vérifié que

$$\frac{\delta \mathscr{E}(\phi)}{\delta \phi} = kg(\phi) + M_1(V(\phi) - \alpha) + M_2(A(\phi) - \beta) f(\phi), \tag{0.2.17}$$

avec

$$f(\phi) = -\epsilon \Delta \phi + \frac{1}{\epsilon}(\phi^2 - 1)\phi, \quad g(\phi) = -\Delta f(\phi) + \frac{1}{\epsilon^2}(3\phi^2 - 1) f(\phi).$$

L'existence d'une solution faible globale de (0.2.16) est prouvée dans [20] en combinant des estimations a priori avec un argument de compacité. En effet, supposons que $(\mathbf{u}, \phi)$ soit une solution classique du système (0.2.16). On multiplie la première équation du système (0.2.16) par $\mathbf{u}$ et la troisième équation par $\frac{\delta \mathscr{E}(\phi)}{\delta \phi}$, puis, en intégrant sur Ω, il résulte que

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}|\mathbf{u}|^2 dx + \mathscr{E}(\phi) \right) + \int_\Omega |\nabla \mathbf{u}|^2 dx + \int_\Omega \left| \frac{\delta \mathscr{E}(\phi)}{\delta \phi} \right|^2 dx = 0,$$

d'où

$$\begin{aligned} \frac{1}{2}\|\mathbf{u}(T,\cdot)\|^2_{L^2(\Omega)} + \mathscr{E}(\phi) + \|\nabla \mathbf{u}\|^2_{L^2((0,T)\times\Omega)} + \left\| \frac{\delta \mathscr{E}(\phi)}{\delta \phi} \right\|^2_{L^2((0,T)\times\Omega)} \\ = \frac{1}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega)} + \mathscr{E}(\phi_0). \end{aligned} \tag{0.2.18}$$

Avant de rappeler les résultats principaux de [20], nous rappelons quelques notations standards sur le champ de vecteurs avec divergence nulle dans les espaces de Sobolev (Voir, par exemple [14, Page 1-10]). Notons

$$\mathscr{V} = \{\mathbf{v} \in (C_0^\infty(\Omega))^3 \mid \nabla \cdot \mathbf{v} = 0\},$$

et $H(\Omega)$, $V(\Omega)$ les adhérences de $\mathscr{V}$ dans $L^2(\Omega)$ et $H_0^1(\Omega)$ respectivement. On utilise aussi le terme trilinéaire :

$$B(u,v,w) = \int_\Omega u \cdot \nabla v \cdot w \mathrm{d}x.$$

Les théorèmes principaux de [20] sont les suivantes

Théorème 0.2.1 (Existence de solutions faibles)**.** *Soit Ω un ouvert borné de $\mathbb{R}^3$ ayant une frontière régulière. Il existe un couple de fonctions $(\mathbf{u}, \phi)$ avec*

- $\mathbf{u} \in L^2(0,T;V(\Omega)) \cap W^{1,\frac{4}{3}}(0,T;V^{-1}(\Omega))$,
- $\phi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$,

qui forme une solution faible du système en ce sens que

1. Pour chaque $\delta \in V(\Omega)$, $\xi \in L^2(\Omega)$ et presque tout $t \in (0,T)$, on a

$$\begin{cases} \langle \mathbf{u}_t, \delta \rangle + B(\mathbf{u}, \mathbf{u}, \delta) + \langle \nabla \mathbf{u}, \nabla \delta \rangle = \int_\Omega \frac{\delta \mathscr{E}(\phi)}{\delta \phi} \nabla \phi \cdot \delta dx, \\ \langle \phi_t, \xi \rangle + B(\mathbf{u}, \phi, \xi) = -\langle \frac{\delta \mathscr{E}(\phi)}{\delta \phi}, \xi \rangle. \end{cases} \tag{0.2.19}$$

2. $\mathbf{u}(0,\cdot) = \mathbf{u}_0$, $\phi(0,\cdot) = \phi_0$ où $\mathbf{u}_0 \in L^2(\Omega)$ et $\phi_0 + 1 \in H_0^2(\Omega)$.

Théorème 0.2.2 (Unicité de la solution faible)**.** *Pour les solutions faibles de l'équation* (0.2.16) *discutées dans le théorème d'existence précédent, si en plus, nous avons la solution satisfaisant $u \in L^8(0,T;L^4(\Omega))$, alors la solution faible est unique.*

Notre travail fournit l'existence locale en temps et l'unicité d'une solution forte. Cela est fait en combinant une estimation sur le système linéarisé de (0.2.16) avec une procédure de point fixe. Notre résultat est le suivant

Théorème 0.2.3. *Supposons que*

$$\begin{cases} \mathbf{u}_0 \in V(\Omega) \\ \phi_0 \in \tilde{H}^{2+\frac{3}{8}}(\Omega). \end{cases} \tag{0.2.20}$$

Alors il existe $T = T\left(\|\mathbf{u}_0\|_{H^1(\Omega)}, \|\phi_0\|_{H^{2+\frac{3}{8}}(\Omega)} \right) > 0$ tel que (0.2.16) *a une unique solution forte maximale :*

$$\begin{cases} \mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ p \in L^2(0,T;H^1(\Omega)), \\ \varphi \in L^2(0,T;H^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];H^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \end{cases} \tag{0.2.21}$$

De plus, l'une des affirmations suivantes a lieu :

1. $T = +\infty$,

2. $\lim_{t \to T-0} \mathcal{E}(t) = +\infty$.

où

$$\mathcal{E}(t) = \|\mathbf{u}(t)\|^2_{H^1(\Omega))} + \|\varphi(t)\|^2_{H^{2+\frac{3}{8}}(\Omega))}.$$

Remarque 0.2.4. *Dans le théorème ci-dessus, $\tilde{H}^s(\Omega)$ avec $s \in \mathbb{R}^+$ sont des espaces de Sobolev fractionnaires classiques $H^s(\Omega)$ qui seront définis dans le Chapitre 2.*

Nous remarquons que la partie principale de $\frac{\delta \mathscr{E}(\phi)}{\delta \phi}$ définie par (0.2.17) est $\Delta^2 \phi$. Puisque les valeurs des paramètres $\epsilon, \gamma, k, M_1, M_2$ n'affectent pas notre résultat, nous supposons que $\epsilon = \gamma = k = M_1 = M_2 = 1$ pour fixer les idées de la preuve. Ainsi on a

$$\frac{\delta \mathscr{E}(\phi)}{\delta \phi} = \Delta^2 \phi + \mathcal{L}(\phi),$$

où $\mathcal{L}(\phi)$ désigne les termes d'ordre inférieur. Dans ce cas, le système (0.2.16) s'écrit

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \nabla p + \Delta\mathbf{u} + \Delta^2\phi\nabla\phi + L(\phi)\nabla\phi & \text{in } [0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } [0,T]\times\Omega, \\ \phi_t + \mathbf{u}\cdot\nabla\phi = -\Delta^2\phi - L(\phi) & \text{in } [0,T]\times\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega, \\ \mathbf{u}(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \\ \phi(0,x) = \phi_0(x) & \text{in } \Omega, \\ \phi(t,x) = -1 & \text{on } [0,T]\times\partial\Omega, \\ \Delta\phi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \end{cases} \tag{0.2.22}$$

Le système ci-dessus peut être mis sous la forme abstraite :

$$\begin{cases} \dot{z} = Az + N(z), \\ z(0) = z_0, \end{cases}$$

où $z = \begin{bmatrix}\mathbf{u}\\ \phi\end{bmatrix}$, $A = \begin{bmatrix}-P\Delta \\ \Delta^2\end{bmatrix}$ avec $P : (L^2(\Omega))^3 \to H(\Omega)$ le projecteur orthogonal et $N(z)$ contenant les termes non linéaires. Afin de résoudre un tel système, nous avons d'abord étudié le linéarisé :

$$\begin{cases} \dot{z} = Az + f, \\ z(0) = z_0, \end{cases} \tag{0.2.23}$$

où $A : \mathcal{D}(A) \to E$ est un opérateur négatif avec $E = H(\Omega)\times H^{\frac{3}{8}}(\Omega)$. Il peut être démontré que, pour chaque $z_0 \in \mathcal{D}((-A)^{\frac{1}{2}})$ et $f \in L^2(0,T;E)$, le système (0.2.23) a une solution $z = \begin{bmatrix}\mathbf{u}\\ \phi\end{bmatrix}$ unique avec

$$\begin{cases} \mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap C([0,T];V(\Omega)) \cap H^1(0,T;H(\Omega)), \\ \phi \in L^2(0,T;H^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];H^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \end{cases}$$

De cette façon, nous pouvons définir, pour chaque $z_0 \in E$, un opérateur

$$\mathscr{F} : L^2(0,T;E) \to L^2(0,T;E)$$

par

$$\mathscr{F}(f) := N(z).$$

Il est prouvé dans le chapitre 2, avec des estimations non linéaires, que cet opérateur non linéaire possède un point fixe f sous la condition $T \ll 1$. Ainsi la solution z du système (0.2.23) est la solution du système (0.2.22). Dans notre demonstration, l'un des termes les plus difficiles à manipuler est $\Delta^2\phi\nabla\phi$ qui apparaît dans la premier équation du système (0.2.22). C'est l'une des raisons pour lesquelles nous utilisons les espaces de Sobolev d'ordre fractionnaire. Un autre résultat que l'on a obtenu est le suivant :

Théorème 0.2.5. *Sous les hypothèses du Théorème 0.2.3, pour tout $T > 0$, il existe $q = q(T) > 0$ tel que, si*

$$(\mu(\Omega) + \alpha)^2 + \|\mathbf{u}_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^2_{H^{2+\frac{3}{8}}(\Omega)} \leqslant q, \tag{0.2.24}$$

alors la solution locale obtenue dans le Théorème 0.2.3 est définie sur l'intervalle $[0, T]$.

Remarque 0.2.6. *On n'a pas obtenu l'existence globale à cause du terme $M_1(V(\phi)-\alpha)$ qui apparaît dans* (0.2.17)*. En fait, dans l'estimation a priori, le constante $M_1\alpha$ entraîne une augmentation linéaire qui ne peut pas être contrôlée par les termes visqueux.*

0.2.3 D'autres modèles concernant l'énergie de courbure

Dans cette section, nous introduisons un autre modèle pour l'énergie de courbure qui est introduit dans [25]. Comme dans le modèle de champ de phase, la frontière de la vésicule est représentée approximativement par les zéros de la fonction ϕ et le fluide se trouvant à l'intérieur (resp à l'extérieur) est représenté approximativement par $\{x \in \Omega | \ \phi(x) = -1\}$ (resp par $\{x \in \Omega | \ \phi(x) = 1\}$). On remarque que $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$ est le vecteur normal à la surface de la vésicule et que $|\nabla\phi|$ peut être utilisé pour passer d'une énergie localisée sur la surface à une énergie distribuée dans le volume (les lecteurs peuvent le considérer comme la fonction de Dirac sur la surface de la vésicule). Ainsi on peut définir la courbure moyenne par $c = c(\phi) = \nabla \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}\right)$ et le projecteur sur le plan tangent de la surface par

$$\mathbb{P} = \mathbb{I} - \mathbf{n} \otimes \mathbf{n} = \mathbb{I} - \frac{\nabla\phi \otimes \nabla\phi}{|\nabla\phi|^2}. \tag{0.2.25}$$

Selon [25], [38], [36] et [37], on donne l'expression concrète de l'énergie $\mathscr{E}(\phi, \sigma)$ par

$$\mathscr{E}(\phi,\sigma) = \frac{\kappa}{2}\int_\Omega (c - c_0)^2 \frac{|\nabla\phi|}{2}\mathrm{dx} + \int_\Omega \sigma|\nabla\phi|\mathrm{dx} + \int_\Omega W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2\mathrm{dx}. \tag{0.2.26}$$

Le premier terme dans l'expression ci-dessus peut être considéré comme l'approximation de l'énergie de courbure définie par (0.2.15). Le second terme est la contrainte d'incompressibilité locale, ce qui sera expliqué plus loin. Le troisième terme est lié au modèle à interface diffuse sans tension de surface où $W(\phi) = \frac{1}{4}(1 - \phi^2)^2$. Parmi les trois énergies dans (0.2.32), seulement les deux derniers sont intrinsèques (voir (0.2.13) pour la définition de l'equation du champ de phase et l'énergie intrinsèque). Ainsi on défini $\mathscr{E}_1(\phi, \sigma)$ par

$$\mathscr{E}_1(\phi,\sigma) = \int_\Omega \sigma|\nabla\phi|dx + \int_\Omega W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 dx. \tag{0.2.27}$$

Il reste à specifier le paramètre σ. Dans le modèle de champ de phase de la vésicule, l'équation pour le fluide est écrite sur l'ensemble du domaine. Donc la surface de la vésicule est considérée comme un fluide incompressible bidimensionnel. Dans ce cas, non seulement la superficie totale reste constante au cours du temps, mais la superficie doit aussi être conservée localement. Cela peut être réalisé en introduisant un multiplicateur de Lagrange σ. D'après [25], σ satisfait

$$\frac{1}{T}\frac{d\sigma}{dt} = (\mathbb{I} - \mathbf{n} \otimes \mathbf{n}) : \nabla \mathbf{u}. \tag{0.2.28}$$

Formellement, on a $(\mathbb{I} - \mathbf{n} \otimes \mathbf{n}) : \nabla \mathbf{u} \approx 0$ en choisissant T assez grand, ce qui est appelé **quasi-incompressibilité locale**. De plus, on a formellement, d'après (0.2.28)

$$\mathbb{P}\nabla \mathbf{u} := (\mathbb{I} - \mathbf{n} \otimes \mathbf{n}) : \nabla \mathbf{u} = 0, \quad T \to \infty. \tag{0.2.29}$$

La condition ci-dessus s'appelle **incompressibilité locale**. Cela est l'une des plus importantes différences entre le modèle de [20] et celui de [25]. L'énergie correspondant à σ est $\int_{\partial\mathcal{O}} \sigma ds$. Puisque $|\nabla\phi|$ est une fonction de type Dirac d'épaisseur ϵ, nous voyons que l'on peut l'approcher par

$$E_{\text{inc}} = \int_{\Omega} \sigma |\nabla\phi| dx. \tag{0.2.30}$$

En rassemblant toutes les équations précédentes, on obtient le système complet suivant

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \Delta \mathbf{u} + \nabla p - \nabla \cdot \left(\left(\lambda + \frac{\sigma}{|\nabla\phi|} \right) \nabla\phi \right) \nabla\phi + W'(\phi)\nabla\phi + \nabla\sigma|\nabla\phi| + F_c \nabla\phi, \\ \nabla \cdot \mathbf{u} = 0, \\ \phi_t - \nabla \cdot \left(\left(\lambda + \frac{\sigma}{|\nabla\phi|} \right) \nabla\phi \right) = -W'(\phi) - \mathbf{u} \cdot \nabla\phi, \\ \sigma_t + \mathbf{u} \cdot \nabla\sigma = T\,(\mathbb{I} - \mathbf{n} \otimes \mathbf{n}) : \nabla \mathbf{u}, \end{cases} \tag{0.2.31}$$

où $\mathbf{F}_c$ désigne la force extérieure due à l'énergie de flexion (énergie de Helfrich). Après quelques calculs, nous avons l'expression suivante (voir aussi [38])

$$F_c = -\frac{\kappa}{4} \left[(c - c_0)(-c(c + c_0) + 4H) - 2\nabla_s \cdot (\nabla_s c) \right].$$

d'où

$$\nabla_s = \mathbb{P} \cdot \nabla = (\mathbb{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla,$$

$$H = \frac{1}{2} [(\nabla_s \cdot \mathbf{n})^2 - \nabla_s \mathbf{n} : \nabla_s \mathbf{n}].$$

Le système (0.2.31) est assez similaire à celui présentée dans article [25]. Bien que la signification physique de ce système soit claire, il ne possède aucune loi de conservation, ce qui rend difficile de prouver l'existence de solutions.

Néanmoins, il possède une loi de conservation après quelques modifications. Une autre version du modèle ci-dessus qui permet d'obtention des estimation a priori a été proposée dans [38]. Dans ce travail, les auteurs supposent que

$$\mathscr{E} = \mathscr{E}_1 = \frac{\kappa}{2}\int_\Omega (c-c_0)^2\frac{|\nabla\phi|}{2}dx + \int_\Omega W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 dx. \qquad (0.2.32)$$

Avec (0.2.32), on obtient aisément que le système (0.2.14) s'écrit :

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \lambda\Delta\phi\nabla\phi + W'(\phi)\nabla\phi + F_c\nabla\phi, \\ \nabla\cdot\mathbf{u} = 0, \\ \phi_t - \lambda\Delta\phi = -W'(\phi) - \mathbf{u}\cdot\nabla\phi - F_c. \end{cases} \qquad (0.2.33)$$

La même étape que pour celle du système (0.2.16) ci-dessus donne l'estimation d'énergie comme (0.2.18). Mais en raison de la non linéarité causée par la courbure, il est encore difficile d'établir les résultats mathématiques pour sa solution.

0.2.4 L'étude mathématique du modèle simplifié

Les Systèmes (0.2.31) et (0.2.33) semblent très difficiles à discuter d'un point de vue mathématique. C'est pour cette raison que, dans ce paragraphe, nous introduisons un modèle simplifié. Tout d'abord, nous négligeons le premier terme dans (0.2.32) qui est lié à la courbure. Deuxièmement, on impose directement le condition (0.2.29) dans l'équation (0.2.30). Avec ces hypothèses, on obtient l'equation pour σ et les expressions des énergies potentielles comme suit :

$$\begin{cases} \frac{d\sigma}{dt} = \sigma_t + \mathbf{u}\cdot\nabla\sigma = 0, \\ \mathscr{E} = \mathscr{E}_1 = \int_\Omega \sigma|\nabla\phi|dx + \int_\Omega W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 dx. \end{cases}$$

Il peut être vérifié que

$$\begin{cases} \frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi} = -\nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\right) + W'(\phi), \\ \frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\sigma} = |\nabla\phi|. \end{cases}$$

Ainsi le système (0.2.14) peut être écrit comme

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p + \nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\right)\nabla\phi - W'(\phi)\nabla\phi - \nabla\sigma|\nabla\phi|, \\ \nabla\cdot\mathbf{u} = 0, \\ \phi_t - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi, \\ \sigma_t + \mathbf{u}\cdot\nabla\sigma = 0. \end{cases} \qquad (0.2.34)$$

Pour le système ci-dessus, nous imposons les conditions aux limites :

$$\begin{cases} \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega, \\ \phi = -1 \quad \text{on} \quad \partial\Omega. \end{cases}$$

En supposant que $(\mathbf{u}, \phi, \sigma)$ est une solution classique du système (0.2.34), on peut facilement verifier que :

$$\frac{d}{dt}\left(\frac{1}{2}\int\limits_{\Omega}|\mathbf{u}|^2dx + \mathscr{E}(\phi,\sigma)\right) + \int\limits_{\Omega}|\nabla\mathbf{u}|^2dx + \int\limits_{\Omega}\left|\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\right|^2 dx = 0. \quad (0.2.35)$$

En effet, cela peut être vérifié en prenant le produit scalaire $L^2(\Omega)$ de la première équation de (0.2.34) par $\mathbf{u}$. On obtient

$$\frac{d}{dt}\int\limits_{\Omega}|\mathbf{u}|^2dx + \int\limits_{\Omega}|\nabla\mathbf{u}|^2dx = \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\nabla\phi\cdot\mathbf{u}dx + \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\nabla\sigma\cdot\mathbf{u}dx. \quad (0.2.36)$$

On prend le produit scalaire $L^2(\Omega)$ de la troisième équation de (0.2.34) avec $\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}$ et il résulte que

$$\begin{aligned} -\int\limits_{\Omega}\left|\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\right|^2 dx &= \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\phi_t dx + \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\mathbf{u}\cdot\nabla\phi dx \\ &= \frac{d}{dt}\mathscr{E}(\phi,\sigma) - \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\sigma}\sigma_t dx + \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\mathbf{u}\cdot\nabla\phi dx. \quad (0.2.37)\end{aligned}$$

Dans les égalités ci-dessus, on utilise la formule suivante

$$\frac{d}{dt}\mathscr{E}(\phi,\sigma) = \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\sigma}\sigma_t dx + \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\phi_t dx.$$

En additionnant (0.2.36) et (0.2.37), on obtient que

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\int\limits_{\Omega}|\mathbf{u}|^2dx + \mathscr{E}(\phi,\sigma)\right) + \int\limits_{\Omega}|\nabla\mathbf{u}|^2dx + \int\limits_{\Omega}\left|\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\right|^2 dx \\ = \int\limits_{\Omega}\frac{\delta\mathscr{E}(\phi,\sigma)}{\delta\phi}\left(\sigma_t + \mathbf{u}\cdot\nabla\sigma\right)dx = 0.\end{aligned}$$

Pour obtenir la solution faible, il convient d'écrire les trois termes non linéaires à la droite de la première equation du système (0.2.34) comme

$$\begin{aligned} &\nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\right)\nabla\phi - W'(\phi)\nabla\phi - \nabla\sigma|\nabla\phi| \\ &= \nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\otimes\nabla\phi\right) - \left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi : \nabla^2\phi - W'(\phi)\nabla\phi - \nabla\sigma|\nabla\phi| \\ &= \nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\otimes\nabla\phi\right) - \nabla\left(\frac{\lambda}{2}|\nabla\phi|^2 + W(\phi) + \sigma|\nabla\phi|\right). \quad (0.2.38)\end{aligned}$$

Ainsi, on voit que les derniers termes ci-dessus peuvent être absorbés par ∇p. Comme la présence de $|\nabla\phi|$ rend le système (0.2.34) singulier, on l'approche par $\sqrt{|\phi|^2+\epsilon}$. Il résulte que, en combinant avec (0.2.38), le système approché de (0.2.34) s'écrit sur le domaine $(0,T)\times\Omega$:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\otimes\nabla\phi\right), \quad (0.2.39\text{a})$$

$$\nabla\cdot\mathbf{u} = 0, \quad (0.2.39\text{b})$$

$$\phi_t - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi, \quad (0.2.39\text{c})$$

$$\sigma_t + \mathbf{u}\cdot\nabla\sigma = 0, \quad (0.2.39\text{d})$$

avec les conditions aux bords :

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \nabla\cdot\mathbf{u}_0 = 0, \ \text{for } x\in\Omega, \quad (0.2.40\text{a})$$

$$\phi(0,x) = \phi_0(x), \quad \sigma(0,x) = \sigma_0(x), \ \text{for } x\in\Omega, \quad (0.2.40\text{b})$$

$$\mathbf{u}(t,x) = 0, \quad \phi(t,x) = 0, \quad \text{for } (t,x)\in(0,T)\times\partial\Omega. \quad (0.2.40\text{c})$$

Définition 0.2.7. *Soit $T>0$. On dit que $(\mathbf{u},\phi,\sigma)$ est une solution faible de* (0.2.39) *et* (0.2.40) *si et seulement si il satisfait les équations suivantes :*

$$\int_\Omega \mathbf{u}_0\cdot\mathbf{v}(0,x)\mathrm{d}x - \int_{Q^T}\mathbf{u}\cdot\mathbf{v}_t\mathrm{d}x\mathrm{d}t - \int_{Q^T}(\mathbf{u}\otimes\mathbf{u} - \nabla\mathbf{u}) : \nabla\mathbf{v}\mathrm{d}x\mathrm{d}t$$
$$= \int_{Q^T}\nabla\mathbf{v} : \left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\otimes\nabla\phi\right)\mathrm{d}x\mathrm{d}t,$$
$$\textit{pour toute} \quad \mathbf{v}\in C_0^\infty([0,T)\times\Omega), \nabla\cdot\mathbf{v} = 0,$$

$$\int_\Omega \phi_0\psi(0,x)\mathrm{d}x - \int_{Q^T}\phi\psi_t\mathrm{d}x\mathrm{d}t + \int_{Q^T}\nabla\psi\cdot\left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\right)\mathrm{d}x\mathrm{d}t$$
$$= \int_{Q^T}(\phi\mathbf{u}\cdot\nabla\psi - W'(\phi)\psi)\,\mathrm{d}x\mathrm{d}t, \quad \textit{pour tout} \quad \psi\in C_0^\infty([0,T)\times\Omega),$$

$$\int_\Omega \sigma_0\psi(0,x)\mathrm{d}x - \int_{Q^T}\sigma\psi_t\mathrm{d}x\mathrm{d}t = \int_{Q^T}\sigma\mathbf{u}\cdot\nabla\psi\mathrm{d}x\mathrm{d}t,$$
$$\textit{pour tout} \quad \psi\in C_0^\infty([0,T)\times\Omega),$$

où $Q^T = (0,T)\times\Omega$.

Nous sommes maintenant en mesure d'énoncer le principal résultat :

Théorème 0.2.8. *Soit* $(\mathbf{u}_0, \phi_0, \sigma_0) \in H(\Omega) \times H_0^1(\Omega) \times L^\infty(\Omega)$ *avec* $\sigma_0 \geqslant 0$ *presque partout. Alors le système* (0.2.39) *avec* (0.2.40) *possède une solution faible.*

Pour démontrer le théorème ci-dessus, on approche d'abord le système (0.2.39) en ajoutant un terme $\nu\Delta^6\mathbf{u}$ dans les équations de Navier-Stokes :

$$\begin{cases} \mathbf{u}_t + \nu\Delta^6\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p \\ \qquad -\nabla\cdot\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right) \quad \text{dans} \quad (0,T)\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{dans} \quad (0,T)\times\Omega, \\ \phi_t - \nabla\cdot\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi \quad \text{dans} \quad (0,T)\times\Omega, \\ \sigma_t + \mathbf{u}\cdot\nabla\sigma = 0 \quad \text{dans} \quad (0,T)\times\Omega, \\ \mathbf{u}|_{\partial\Omega} = 0,\ \Delta\mathbf{u}|_{\partial\Omega} = 0,\ \Delta^2\mathbf{u}|_{\partial\Omega} = 0, \\ \frac{\partial}{\partial\mathbf{n}}\mathbf{u}|_{\partial\Omega} = 0,\ \frac{\partial}{\partial\mathbf{n}}\Delta\mathbf{u}|_{\partial\Omega} = 0,\ \frac{\partial}{\partial\mathbf{n}}\Delta^2\mathbf{u}|_{\partial\Omega} = 0, \\ \phi|_{\partial\Omega} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{dans} \quad \Omega, \\ \phi(0,\cdot) = \phi_0 \quad \text{dans} \quad \Omega, \\ \sigma(0,\cdot) = \sigma_0 \quad \text{dans} \quad \Omega. \end{cases} \tag{0.2.41}$$

Pour simplifier la presentation, on note :

$$E(\phi,\sigma,\mathbf{u}) := \int_\Omega \left(W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 + |\mathbf{u}|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)dx,$$

et

$$A_\sigma(\phi) := -\nabla\cdot\left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\right).$$

Proposition 0.2.9. *Supposons que* $(\mathbf{u}_0, \phi_0, \sigma_0)$ *satisfisse*

$$\begin{cases} \mathbf{u}_0 \in H(\Omega), \\ \phi_0 \in H_0^1(\Omega), \\ \sigma_0 \in C^2(\bar\Omega), \quad \sigma_0 \geqslant 0. \end{cases}$$

Alors, pour tout $T > 0$*, le système* (0.2.41) *possède une unique solution*

$$\begin{cases} \mathbf{u} \in L^2(0,T;V(\Omega)\cap H_0^6(\Omega)) \cap C([0,T];H(\Omega)), \\ \phi \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap C([0,T];H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ \sigma \in C([0,T];C^2(\bar\Omega)) \cap H^1(0,T;C^1(\bar\Omega)), \end{cases} \tag{0.2.42}$$

et l'estimation suivante a lieu :

$$E(t) + \int_{Q^t} (|A_\sigma(\phi) + W'(\phi)|^2 + |\nabla\mathbf{u}|^2 + \nu|\Delta^3\mathbf{u}|^2)dxdt \leqslant E(0), \quad \forall t \in (0,T). \tag{0.2.43}$$

De plus, pour tous a, b avec $0 \leqslant a \leqslant b \leqslant \infty$,

$$\text{mes}\{x \in \Omega | a \leqslant \sigma(t,x) \leqslant b\} \quad \textit{est indépendant de } t. \tag{0.2.44}$$

La démonstration du théorème 0.2.8 en utilisant la proposition 0.2.9 est donnée dans le chapitre 3, section 6. Au cour de la démonstration, pour passer à la limite $\nu \mapsto 0$, on applique fréquemment des résultats de convergence faible et des propriétés des opérateurs monotones, qui sont introduits dans [46]. La démonstration de la proposition (0.2.9) est donnée dans le Chapitre 3, section 5 : on démontre tout d'abord de l'existence de solutions locales et ensuite, en combinant avec l'estimation a priori (0.2.43), on prouve que cette solution locale n'explose pas en temps finie, ce qui implique l'existence globale de solutions du système (0.2.41).

0.2.5 Problème ouverts et perspectives

On voit dans le Définition 0.2.7 que les solutions sont trop faibles. Par conséquent, on peut se demander si ces solutions peuvent être plus fortes sous d'autres hypothèses. Dans un cas très special, la réponse est positive : Remarquer que, si $\sigma_0(x) = c \in \mathbb{R}^+$, $\sigma(t,x) \equiv c$ est il-même une solution de la dernière equation du système (0.2.39) et donc on peut simplifier le système (0.2.39) en

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \nabla\cdot\left(\left(\lambda + \frac{c}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\otimes\nabla\phi\right), \tag{0.2.45a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{0.2.45b}$$

$$\phi_t - \nabla\cdot\left(\left(\lambda + \frac{c}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi. \tag{0.2.45c}$$

Pour ce système, on peut démontrer que la solution faible $\phi \in L^\infty(0,T;H_0^1(\Omega))$ est en fait dans $L^2(0,T;H^2(\Omega))$. De plus, si on prend $c = 0$, le système (0.2.45) devient un système très similaire avec celui de [43], [44] et [45]. On peut se demander s'il est possible d'établir la régularité partielle du solution faible du système (0.2.45) comme dans [45].

On peut aussi considérer la solution faible du système (0.2.33) en employant une méthode similaire à celle des système (0.2.39). Mais dans ce cas, la difficulté est de résoudre une EDP elliptique non linéaire d'ordre 4, au lieu de résoudre une EDP elliptique non linéaire d'ordre 2 comme dans la démonstration du Théorème 0.2.8.

Deuxième partie

Chapitre 1

Contrôlabilité d'un modèle fluide-structure

In this chapter, we study a controllability problem for a simplified one dimensional model for the motion of a rigid body in a viscous fluid. The control variable is the velocity of the fluid at one end. One of the novelties brought in with respect to the existing literature consist in the fact that we use a single scalar control. Moreover, we introduce a new methodology, which can be used for other nonlinear parabolic systems, independently of the techniques previously used for the linearized problem. This methodology is based on an abstract argument for the null controllability of parabolic equations in the presence of source terms and it avoids tackling linearized problems with time dependent coefficients.

1.1 Introduction

In this work we study the boundary null controllability of a simplified one dimensional model for the motion of a solid in a viscous fluid. More precisely, we consider the initial-boundary value problem

$$\begin{cases} v_t(t,x) - v_{xx}(t,x) + v(t,x)v_x(t,x) = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ M\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases} \tag{1.1.1}$$

In the above equations, v stands for the velocity field of the fluid, M is the mass of the solid, whereas h denotes the trajectory of the mass point moving in the fluid. The system is controlled by imposing the velocity of the fluid at the left-end of the tube and the corresponding control function is denoted by $u(t)$. The symbol $[f](x)$ represents the jump of f at x, i.e.

$$[f](x) = f(x+) - f(x-)$$

where $f(x-)$ and $f(x+)$ denote the left and right limits of f at x, respectively. We write v_t, v_x, v_{xx} for the derivatives of v with respect to t, x and for the second derivative of v with respect to x. If a function only depends on t, we just write $\dot{h}$ and $\ddot{h}$ for the first and second derivatives of h with respect to t.

This simplified model has been introduced by Vázquez and Zuazua in [61], where the well-posedness (with $(-1,1)$ replaced by $\mathbb{R}$) has been considered (see also [60]). The null controllability with controls at both ends has been considered by Doubova and Fernández-Cara in [19]. In this work the authors use a linearized problem with variable time-dependent coefficients, which is tackled by using the global Carleman estimates for the heat equation (see Imanuvilov and Fursikov [28]). As stated in [19, Remark 1.2] this methodology does not allow to extend the obtained results to the case of a control acting at only one end, due to the lack of connectivity of the fluid domain. In this work we tackle the case of controls acting at only one end by using a quite different strategy, which does not appeal at any Carleman estimate. Firstly, our linearized problem is with constant coefficients, so it can be tackled by spectral calculations combined with a classical method due to Fattorini and Russell. Secondly, we introduce a new iterative algorithm for the null-controllability in presence of source terms. The proposed iterative method for null-controllability in presence of source terms is of independent interest and therefore it is presented in an abstract setting. Note that this iterative method uses an idea going back to Lebeau and Robbiano [42], which consists in controlling a part of the state trajectory and using the dissipative character of parabolic equations to steer the remaining part to zero (see also Miller [49], Tenenbaum and Tucsnak [58]). Finally, we use a fixed point method introduced in Imanuvilov [35] and also used in Imanuvilov and Takahashi [34].

We also mention that similar controllability questions for the "full model" (i.e., involving two or three dimensional Navier-Stokes equations coupled with Newton's laws for rigid bodies) have been investigated in Boulakia and Osses [11] and in [34].

The main result in this paper asserts that the system (1.1.1) is locally null controllable in a neighborhood of zero and it can be stated as follows.

Theorem 1.1.1. *Let $\tau > 0$. There exists $r > 0$ such that for every $h_0 \in (-1,1)$, $h_1 \in \mathbb{R}$, $v_0 \in H_0^1(-1,1)$, $v_0(h_0) = h_1$, satisfying $|h_0| + |h_1| + \|v_0\|_{H_0^1(-1,1)} \leqslant r$, there exists $u \in C[0,\tau]$ such that the solution of* (1.1.1) *satisfies*

$$v \in L^2([0,\tau], H^2((-1,1) \setminus \{h(t)\}) \cap C([0,\tau], H^1(-1,1)), \quad h \in C^1[0,\tau],$$

together with

$$\dot{h}(\tau) = 0, \quad v(\tau,\cdot) = 0 \quad \textit{and} \quad h(\tau) = 0. \tag{1.1.2}$$

Remark 1.1.2. *The above result can be extended to show that, given $\tau > 0$, any initial state $\begin{bmatrix} v_0 \\ h_0 \\ h_1 \end{bmatrix}$ with small enough norm in $L^2(-1,1) \times \mathbb{R} \times \mathbb{R}$ can be steered to rest in time τ. To prove this fact, we take the control u equal to zero for $t \in [0,\tau/2]$ and we note that, by classical results on parabolic equations, we have*

$$v(\tau/2,\cdot) \in H_0^1(-1,1), \qquad v(\tau/2, h(\tau/2)) = \dot{h}(\tau/2).$$

We can thus conclude by controlling the system for $t \geqslant \tau/2$ (this fact is possible due to the above regularity property and to Theorem 1.1.1).

The outline of the remaining part of this work is the following. In Section 1.2 we first recall some known results on null-controllability of parabolic equations and then we introduce a new iterative method for null controllability in the presence of source terms. This method, whose precise statement is given in Proposition 1.2.3, gives a general iterative framework to tackle source terms and, when applied to a given PDE system, does not require the use of Carleman estimates. In Section 1.3 we prove the null internal controllability for an auxiliary linear system, associated to (1.1.1), by combining some spectral estimates with the results from Section 1.2. Finally, Section 1.4 is devoted to the proof of the main result.

1.2 Some background on null-controllability

We begin this section by briefly recalling some basic facts on the null-controllability of systems with negative generator. We next give the main result in this section, which concerns the controllability of these systems in the presence of a source term.

Throughout this section, X and U are Hilbert spaces which are identified with their duals, $\mathbb{T}$ is a strongly continuous semigroup on X, with generator $A : \mathcal{D}(A) \to X$ and $B \in \mathcal{L}(U, X)$. The inner product and the norm in X are simply denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We first consider classical linear control systems of the form

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0 \in X. \end{cases} \tag{1.2.1}$$

Definition 1.2.1. *For $\tau > 0$, the pair (A, B) is said null-controllable in time τ if for every $z_0 \in X$ there exists $u \in L^2([0, \tau], U)$ such that the solution of* (1.2.1) *satisfies $z(\tau) = 0$.*

This means that for every $z_0 \in X$, the set

$$\mathcal{C}_{\tau, z_0} := \{u \in L^2([0, \tau], U) \quad | \quad z(\tau) = 0\},$$

is non empty. The quantity

$$K(\tau) := \sup_{\|z_0\|=1} \inf_{u \in \mathcal{C}_{\tau, z_0}} \|u\|_{L^2([0,\tau],U)}$$

is then called the *control cost in time* τ. If the pair (A, B) is null-controllable in any time $\tau > 0$, then it is known (see, for instance, [30, Theorem 3.1]) that

$$K : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{is continuous, non-increasing and} \lim_{\tau \to 0^+} K(\tau) = +\infty. \tag{1.2.2}$$

The above definition for the control cost implies that for every function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, with $K(t) < \gamma(t)$ for every $t > 0$, and for every $\tau > 0$, there exists a control driving the solution of (1.2.1) to rest in time τ such that

$$\|u\|_{L^2([0,\tau],U)} \leqslant \gamma(\tau) \|z_0\| \qquad (z_0 \in X). \tag{1.2.3}$$

We first recall a result which essentially goes back to Fattorini and Russell [27]. However, in order to have the precise statement we need in this work and in order to make the control cost precise, we give a short proof below.

Proposition 1.2.2. *Assume that A is a negative operator, admitting an orthonormal basis of eigenvectors $(\varphi_k)_{k\geqslant 1}$ and with the corresponding decreasing sequence of eigenvalues $(-\lambda_k)_{k\geqslant 1}$ satisfying*

$$\inf_{k\geqslant 0}(\lambda_{k+1}-\lambda_k) = s > 0, \tag{1.2.4}$$

$$\lambda_k = rk^2 + O(k) \qquad (k\to\infty), \tag{1.2.5}$$

for some $r > 0$. Moreover, assume that U is a separable Hilbert space and that there exists a constant $m > 0$ such that the adjoint operator B^* satisfies

$$\|B^*\varphi_k\|_U \geqslant m \qquad (k\geqslant 1). \tag{1.2.6}$$

Then the pair (A, B) is null-controllable in any time $\tau > 0$ and there exist positive constants M, C, depending only on $(\lambda_k)_{k\geqslant 1}$ and on m, such that the control cost K satisfies

$$K(\tau) < Ce^{\frac{M}{\tau}} \quad (\tau > 0).$$

Proof. According to a classical result, which goes back to Dolecki and Russell [18] (see also Tucsnak and Weiss [59, Section 11.2]), the pair (A, B) is null-controllable in time τ at cost $K(\tau)$ iff (A^*, B^*) is final state observable in time τ at the same cost, i.e., if for every $K > K(\tau)$ we have

$$K^2\int_0^\tau \|B^*\mathbb{T}_t^*z_0\|_U^2\mathrm{d}t \geqslant \|\mathbb{T}_\tau^*z_0\|^2 \qquad (z_0\in X), \tag{1.2.7}$$

where $\mathbb{T}_t^*$ is the adjoint of $\mathbb{T}_t$, whose generator is A^*. Simple calculations show that

$$B^*\mathbb{T}_t^*z_0 = B^*\mathbb{T}_tz_0 = \sum_{k\geqslant 1} e^{-\lambda_k t}\langle z_0,\varphi_k\rangle B^*\varphi_k.$$

The above formula, combined with the fact that U admits an orthonormal basis $(\psi_l)_{l\geqslant 1}$, implies that

$$\int_0^\tau \|B^*\mathbb{T}_t^*z_0\|_U^2\mathrm{d}t = \sum_{l\geqslant 1}\int_0^\tau\left|\sum_{k\geqslant 1} e^{-\lambda_k t}\langle z_0,\varphi_k\rangle_X\langle B^*\varphi_k,\psi_l\rangle_U\right|^2\mathrm{d}t \qquad (\tau>0,\ z_0\in X). \tag{1.2.8}$$

On the other hand, it is known (see, for instance, Tenenbaum and Tucsnak [57, Corollary 3.6] and references therein) that, under the assumptions (1.2.4) and (1.2.5), there exist constants $M_1,\ M_2 > 0$ (depending only on s and on r) such that

$$M_1e^{\frac{M_2}{\tau}}\int_0^\tau\left|\sum_{k\geqslant 1} a_k e^{-\lambda_k t}\right|^2\mathrm{d}t \geqslant \sum_{k\geqslant 1}|a_k|^2e^{-2\lambda_k\tau} \qquad (\tau>0,\ \ (a_k)\in l^2).$$

The above estimate combined with (1.2.8) implies that

$$M_1 e^{\frac{M_2}{\tau}} \int\limits_0^\tau \|B^*\mathbb{T}_t^* z_0\|_U^2 \mathrm{d}t \geqslant \sum_{l\geqslant 1}\sum_{k\geqslant 1} e^{-2\lambda_k \tau} |\langle z_0, \varphi_k\rangle_X|^2 \; |\langle B^*\varphi_k, \psi_l\rangle_U|^2$$
$$(\tau > 0,\ z_0 \in X).$$

The last inequality, together with (1.2.6), gives

$$M_1 e^{\frac{M_2}{\tau}} \int\limits_0^\tau \|B^*\mathbb{T}_t^* z_0\|_U^2 \mathrm{d}t \geqslant m^2 \|\mathbb{T}_\tau^* z_0\|^2 \qquad (z_0 \in X),$$

so that (1.2.7) holds with $K = \sqrt{M_1} e^{\frac{M_2}{2\tau}}$. This fact clearly implies the conclusion of the proposition. $\square$

Assume that (A, B) is null controllable in any time $\tau > 0$. In the remaining part of this section we study a null-controllability problem associated to (A, B) in the presence of source terms, i.e., we consider the system

$$\begin{cases} \dot z = Az + Bu + f, \\ z(0) = z_0. \end{cases} \tag{1.2.9}$$

where $f : [0, \infty) \to X$.

We show below, roughly speaking, that if f vanishes, with a prescribed decay rate, for some $\tau > 0$, then there exists a control u such that the solution of (1.2.9) tends to zero at a similar rate when $t \to \tau$. To make this assertion precise, we need some notation.

Let $q > 1$ be a constant, let $\gamma : (0, \infty) \to [0, \infty)$ be a continuous non increasing function with $\lim_{t\to 0} \gamma(t) = +\infty$ and let $\tau > 0$. Consider a continuous function $\rho_{\mathcal{F}} : [0, \tau] \to [0, \infty)$ with

$$\rho_{\mathcal{F}} \ \text{ non increasing and } \ \rho_{\mathcal{F}}(\tau) = 0, \tag{1.2.10}$$

such that the function $\rho_0 : \left[\tau\left(1 - \frac{1}{q^2}\right), \tau\right] \to [0, \infty)$ defined by

$$\rho_0(t) := \rho_{\mathcal{F}}(q^2(t - \tau) + \tau)\gamma\left((q - 1)(\tau - t)\right) \quad \left(t \in \left[\tau\left(1 - \frac{1}{q^2}\right), \tau\right]\right), \tag{1.2.11}$$

is non increasing and

$$\rho_0(\tau) = 0. \tag{1.2.12}$$

The fact that, for each γ as above, such a function $\rho_{\mathcal{F}}$ exists is easy to check. Indeed, we can take, for instance,

$$\rho_{\mathcal{F}}(q^2(t - \tau) + \tau) = \left(\gamma\left((q - 1)(\tau - t)\right)\right)^{-(1+p)} \quad \left(t \in \left[\tau\left(1 - \frac{1}{q^2}\right), \tau\right]\right),$$

with $p > 0$. We associate to the functions $\rho_{\mathcal{F}}$ and ρ_0 the Hilbert spaces $\mathcal{F}$, $\mathcal{U}$ and $\mathcal{Z}$ defined by

$$\mathcal{F} = \left\{ f \in L^2([0,\tau], X) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2([0,\tau], X) \right\}, \tag{1.2.13a}$$

$$\mathcal{U} = \left\{ u \in L^2([0,\tau], U) \;\middle|\; \frac{u}{\rho_0} \in L^2([0,\tau], U) \right\}, \tag{1.2.13b}$$

$$\mathcal{Z} = \left\{ z \in L^2([0,\tau], X) \;\middle|\; \frac{z}{\rho_0} \in L^2([0,\tau], X) \right\}, \tag{1.2.13c}$$

where ρ_0 is any extension of the function in (1.2.11) (also denoted by ρ_0), initially defined on $\left[\tau\left(1-\frac{1}{q^2}\right), \tau\right]$, to a function which is continuous and non increasing on $[0,\tau]$. We can take, for instance,

$$\rho_0(t) = \rho_0\left(\tau\left(1-\frac{1}{q^2}\right)\right) \qquad \left(t \in \left[0, \tau\left(1-\frac{1}{q^2}\right)\right]\right),$$

but the choice of this extension plays no role in what follows.

The inner product in $\mathcal{F}$ is defined by

$$\langle f_1, f_2\rangle_{\mathcal{F}} = \int_0^\tau \rho_{\mathcal{F}}^{-2}(t)\langle f_1(t), f_2(t)\rangle \,\mathrm{d}t$$

and similar definitions are used for $\mathcal{U}$ and for $\mathcal{Z}$. The induced norms are denoted by $\|\cdot\|_{\mathcal{F}}$, $\|\cdot\|_{\mathcal{Z}}$ and $\|\cdot\|_{\mathcal{U}}$, respectively.

We are now in a position to prove the main result in this section.

Proposition 1.2.3. *Assume that the pair (A,B) is null-controllable in any time $t > 0$, with control cost $K(t)$. Let $\gamma : (0,\infty) \to [0,\infty)$ be a continuous non increasing function such that*

$$K(t) < \gamma(t) \qquad (t > 0). \tag{1.2.14}$$

Let $\tau > 0$ and let $\rho_{\mathcal{F}}$ and ρ_0 be two functions satisfying (1.2.10)-(1.2.12) *and let $(\mathcal{F}, \mathcal{Z}, \mathcal{U})$ be the corresponding Hilbert spaces, defined according to* (1.2.13)*. Then, for every $z_0 \in X$ and $f \in \mathcal{F}$, there exists $u \in \mathcal{U}$ such that the solution of* (1.2.9) *satisfies $z \in \mathcal{Z}$. Furthermore, there exists a positive constant C, not depending on f and on z_0, such that*

$$\left\|\frac{z}{\rho_0}\right\|_{C([0,\tau],X)} + \|u\|_{\mathcal{U}} \leqslant C\left(\|f\|_{\mathcal{F}} + \|z_0\|_X\right). \tag{1.2.15}$$

In particular, since ρ_0 is a continuous function satisfying $\rho_0(\tau) = 0$, the above relation yields $z(\tau) = 0$.

Remark 1.2.4. *Note that the statement of the above proposition depends on an arbitrary constant $q > 1$. The choice of this constant is important in applications to PDE systems. For instance to obtain Theorem 1.1.1, we will assume that $q^4 < 2$.*

Before proving the above proposition, we recall the following classical result (see, for instance, Lemma 3.3 and Theorem 3.1 of [5, p.80]).

Lemma 1.2.5. *Assume that A is negative and $f : [0, \infty) \to X$. Let $\tau_1,\ \tau_2 > 0$ and let z be the solution of*

$$\begin{cases} \dot{z} = Az + f & t \in (\tau_1, \tau_2), \\ z(\tau_1) = a \in X. \end{cases}$$

There exists a positive constant C, depending only on A, such that

$$\begin{aligned} &\|z\|^2_{C([\tau_1,\tau_2],X)} + \|(-A)^{1/2} z\|^2_{L^2([\tau_1,\tau_2],X)} \\ &\quad \leqslant \|a\|^2_X + C\, \|f\|^2_{L^2([\tau_1,\tau_2],X)} \qquad (a \in X, \quad f \in L^2([\tau_1, \tau_2], X)). \end{aligned} \tag{1.2.16}$$

Furthermore, if $a \in D((-A)^{\frac{1}{2}})$, then we have

$$\begin{aligned} &\|z\|_{H^1((\tau_1,\tau_2),X)} + \|(-A)^{\frac{1}{2}} z\|^2_{C([\tau_1,\tau_2],X)} + \|Az\|^2_{L^2([\tau_1,\tau_2],X)} \\ &\leqslant \|a\|^2_{D((-A)^{\frac{1}{2}})} + C\, \|f\|^2_{L^2([\tau_1,\tau_2],X)} \qquad (a \in D((-A)^{\frac{1}{2}}),\ \ f \in L^2([\tau_1, \tau_2], X)). \end{aligned}$$

Proof of Proposition 1.2.3. Consider the sequence

$$T_k = \tau - \frac{\tau}{q^k} \qquad (k \geqslant 0).$$

First of all, let us remark that in that case, (1.2.11) yields

$$\rho_0(T_{k+2}) = \rho_{\mathcal{F}}(T_k) \gamma \left(T_{k+2} - T_{k+1}\right). \tag{1.2.17}$$

We define the sequence $\{a_k\}_{k \geqslant 0}$ by $a_0 = z_0$ and

$$a_{k+1} = z_1(T_{k+1}-) \qquad (k \geqslant 0). \tag{1.2.18}$$

where z_1 satisfies

$$\begin{cases} \dot{z}_1 = Az_1 + f & t \in (T_k, T_{k+1}), \\ z_1(T_k+) = 0, \end{cases} \qquad (k \geqslant 0). \tag{1.2.19}$$

On the other hand, for each $k \geqslant 0$ we consider the control system

$$\begin{cases} \dot{z}_2 = Az_2 + Bu_k & (t \in (T_k, T_{k+1})), \\ z_2(T_k+) = a_k \in X, \end{cases} \qquad (k \geqslant 0),$$

where $u_k \in L^2([T_k, T_{k+1}], U)$ is such that

$$z_2(T_{k+1}-) = 0 \qquad (k \geqslant 0),$$

$$\|u_k\|^2_{L^2([T_k,T_{k+1}],U)} \leqslant \gamma^2 \left(T_{k+1} - T_k\right) \|a_k\|^2_X \qquad (k \geqslant 0). \tag{1.2.20}$$

The fact that such a control u_k exists for every $k \in \mathbb{N}$ follows from the null-controllability of (A, B) with a cost satisfying (1.2.14).

To estimate the solution z_1 of (1.2.19), we note that from Lemma 1.2.5 it follows that

$$\|z_1\|^2_{C([T_k,T_{k+1}];X)} + \|(-A)^{1/2} z_1\|^2_{L^2([T_k,T_{k+1}],X)} \leqslant C \, \|f\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0).$$

In particular, recalling (1.2.18), we have

$$\|a_{k+1}\|^2_X \leqslant C \, \|f\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0). \tag{1.2.21}$$

Combining (1.2.20), (1.2.21) and the fact that $\rho_{\mathcal{F}}$ is not increasing, we obtain that, for every $k \geqslant 0$,

$$\begin{aligned} \|u_{k+1}\|^2_{L^2([T_{k+1},T_{k+2}],U)} &\leqslant \gamma^2 \left(T_{k+2} - T_{k+1}\right) \|a_{k+1}\|^2_X \\ &\leqslant C\gamma^2 \left((q-1)\frac{\tau}{q^{k+2}}\right) \rho^2_{\mathcal{F}}(T_k) \left\| \frac{f}{\rho_{\mathcal{F}}} \right\|^2_{L^2([T_k,T_{k+1}],X)}. \end{aligned}$$

Recalling the definition of $\rho_0(t)$ in (1.2.11) and using (1.2.17), we obtain

$$\|u_{k+1}\|^2_{L^2([T_{k+1},T_{k+2}],U)} \leqslant C\rho_0^2(T_{k+2}) \left\| \frac{f}{\rho_{\mathcal{F}}} \right\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0).$$

Since ρ_0 is non-increasing, it follows that there exists a positive constant, still denoted by C, such that

$$\left\| \frac{u_{k+1}}{\rho_0} \right\|^2_{L^2([T_{k+1},T_{k+2}],U)} \leqslant C \left\| \frac{f}{\rho_{\mathcal{F}}} \right\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0). \tag{1.2.22}$$

We define the control u by

$$u := \sum_{k=0}^{\infty} u_k \mathbf{1}_{[T_k,T_{k+1})},$$

where $\mathbf{1}_I$ is the characteristic function of the set I. Combining (1.2.22) with (1.2.20) (for $k = 0$) implies that there exists a positive constant C such that

$$\left\| \frac{u}{\rho_0} \right\|_{L^2([0,\tau],U)} \leqslant C \left(\left\| \frac{f}{\rho_{\mathcal{F}}} \right\|_{L^2([0,\tau],X)} + \|z_0\| \right) \qquad (z_0 \in X, \quad f \in \mathcal{F}). \tag{1.2.23}$$

If we set $z := z_1 + z_2$ then, for every $k \geqslant 0$, we have

$$\left\{ \begin{array}{ll} \dot{z} = Az + Bu_k + f & \quad (t \in [T_k, T_{k+1}]), \\ z(T_k) = a_k. \end{array} \right. \tag{1.2.24}$$

Moreover,

$$z(T_k-) = z_1(T_k-) + z_2(T_k-) = a_k = z_1(T_k+) + z_2(T_k+) = z(T_k+) \qquad (k \geqslant 1),$$

so that z is continuous at T_k for every $k \geqslant 0$. Consequently, z satisfies (1.2.9) for $t \in [0, \tau]$.

Furthermore, applying Lemma 1.2.5 to (1.2.24), we obtain the existence of a positive constant C (depending only on A and B) such that

$$\|z\|^2_{C([T_k,T_{k+1}],X)} \leqslant \|a_k\|^2_X + C\,\|u\|^2_{L^2([T_k,T_{k+1}],X)} + C\,\|f\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0). \tag{1.2.25}$$

The above estimate, combined with (1.2.20) imply that

$$\|z\|^2_{C([T_k,T_{k+1}],X)} \leqslant \|a_k\|^2_X + C\gamma^2\,(T_{k+1} - T_k)\,\|a_k\|^2_X + C\,\|f\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0).$$

The above inequality and (1.2.21) imply that there exists a positive constant C, depending only on A and B, such that

$$\|z\|^2_{C([T_k,T_{k+1}],X)} \leqslant C\gamma^2(T_{k+1} - T_k)\|f\|^2_{L^2([T_{k-1},T_k],X)} \qquad (k \geqslant 1),$$

and thus, since $\rho_{\mathcal{F}}$ is not increasing (recall also (1.2.17)),

$$\begin{aligned}\|z\|^2_{C([T_k,T_{k+1}],X)} &\leqslant C\gamma^2(T_{k+1} - T_k)\rho^2_{\mathcal{F}}(T_{k-1}) \left\|\frac{f}{\rho_{\mathcal{F}}}\right\|^2_{L^2([T_{k-1},T_{k+1}],X)} \\ &= C\rho_0^2(T_{k+1}) \left\|\frac{f}{\rho_{\mathcal{F}}}\right\|^2_{L^2([T_{k-1},T_{k+1}],X)} \qquad (k \geqslant 1)\end{aligned} \tag{1.2.26}$$

Since ρ_0 is not increasing, we deduce from (1.2.26) that

$$\left\|\frac{z}{\rho_0}(t)\right\| \leqslant C \left\|\frac{f}{\rho_{\mathcal{F}}}\right\|_{L^2([T_{k-1},T_{k+1}],X)} \qquad (k \geqslant 1, \quad t \in [T_k, T_{k+1}]). \tag{1.2.27}$$

The above estimate, combined with (1.2.20) and (1.2.25) (both for $k = 0$), implies that

$$\left\|\frac{z}{\rho_0}\right\|_{C([0,\tau],X)} \leqslant C \left(\left\|\frac{f}{\rho_{\mathcal{F}}}\right\|_{L^2([0,\tau],X)} + \|z_0\|_X \right).$$

The above estimate and (1.2.23) imply the conclusion (1.2.15). ☐

Consider the backwards system

$$\begin{cases} -\dot\zeta(t) = A\zeta(t) + g \qquad (t \in (0, \tau)), \\ \zeta(\tau) = 0. \end{cases} \tag{1.2.28}$$

By a duality argument (see, for instance, Theorem 4.1 in [34]), we obtain the following consequence of Proposition 1.2.3.

Corollary 1.2.6. *Under the assumptions and notation of Proposition* 1.2.3*, for every* $g \in L^2([0, \tau], X)$ *the solution of* (1.2.28) *satisfies :*

$$\|\zeta(0)\|^2 + \int_0^\tau \|\rho_{\mathcal{F}}\zeta\|^2 \mathrm{d}t \leqslant C \left(\int_0^\tau \|\rho_0 g\|^2 \,\mathrm{d}t + \int_0^\tau \|\rho_0 B^*\zeta\|^2_U \mathrm{d}t \right). \tag{1.2.29}$$

Remark 1.2.7. *Looking to estimate* (1.2.29)*, one can easily think to Carleman estimates, such as those obtained in a PDE context in [35], [34]. However, the way in which we obtained* (1.2.29) *is very far from Carleman estimates based strategies. Indeed, our only assumption is the pair* (A, B) *is null controllable, without needing any knowledge on the methodolgy used to prove this controllability. In the application to our PDE system this gives the full choice of the method used to prove the null controllability of the pair* (A, B)*. In our application it turns out that a spectral method gives results which seem better than those obtained via Carleman estimates (see the next section).*

The next proposition gives more information on the regularity of the controlled trajectory obtained in Proposition 1.2.3.

Proposition 1.2.8. *Under the notation and assumptions in Proposition* 1.2.3*, assume that there exists a continuous not increasing function* $\rho : [0, \tau] \to \mathbb{R}^+$ *satisfying* $\rho(\tau) = 0$ *and the inequalities*

$$\rho_0 \leqslant C\rho, \quad \rho_{\mathcal{F}} \leqslant C\rho, \quad |\dot{\rho}|\, \rho_0 \leqslant C\rho^2 \qquad (t \in [0, \tau]) \tag{1.2.30}$$

for some positive constant which does not depend on t*. Then for every* $z_0 \in \mathcal{D}((-A)^{\frac{1}{2}})$ *there exists* $u \in \mathcal{U}$ *such that the solution* $z \in \mathcal{Z}$ *of* (1.2.9) *satisfies*

$$\frac{z}{\rho} \in L^2([0, \tau], \mathcal{D}(A)) \cap H^1((0, \tau), X) \cap C([0, \tau], \mathcal{D}((-A)^{\frac{1}{2}})).$$

Moreover, there exists a positive constant C*, depending only on* A *and on* B*, such that*

$$\left\| \frac{z}{\rho} \right\|_{C([0,\tau],\mathcal{D}((-A)^{1/2}))\cap H^1((0,\tau),X)\cap L^2([0,\tau],\mathcal{D}(A))} \leqslant C \left(\|z_0\|_{\mathcal{D}((-A)^{1/2})} + \|f\|_{\mathcal{F}} \right). \tag{1.2.31}$$

Proof. Let u be the control constructed in the proof of Proposition 1.2.3 and let z be the corresponding trajectory. Then $w := \frac{z}{\rho}$ satisfies

$$\dot{w} = Aw + \frac{Bu}{\rho} + \frac{f}{\rho} - \frac{\dot{\rho}\rho_0}{\rho^2}\frac{z}{\rho_0}. \tag{1.2.32}$$

Conditions (1.2.30) imply

$$\frac{Bu}{\rho} + \frac{f}{\rho} - \frac{\dot{\rho}\rho_0}{\rho^2}\frac{z}{\rho_0} \in L^2([0, \tau], X).$$

so that the conclusion easily follows by applying Lemma 1.2.5. □

In the remaining part of this section we consider a system obtained from (1.2.9) by adding a simple integrator. More precisely, we consider the equations

$$\begin{cases} \dot{z} = Az + Bu + f, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0, \end{cases} \tag{1.2.33}$$

where Y is a Hilbert space and $C \in \mathcal{L}(X, Y)$. Consider the adjoint system of (1.2.33)

$$\begin{cases} -\dot{\xi}(t) = A\xi(t) + g(t) + C^* r, \\ \xi(\tau) = 0, \end{cases} \tag{1.2.34}$$

where $g : [0, \infty) \to X$ and $r \in Y$. The last result in this section is a characterization by duality of the fact that the solutions of (1.2.33) can be steered to rest at a prescribed decay rate. Using the notation introduced in (1.2.11)-(1.2.13), this result reads as follows.

Proposition 1.2.9. *The following two statements are equivalent*

(1) *For every $(g, r) \in L^2([0, \tau], X) \times Y$, the solution of* (1.2.34) *satisfies*

$$\|r\|_Y^2 + \|\xi(0)\|^2 + \int_0^\tau \|\rho_{\mathcal{F}}\xi\|^2 \mathrm{d}t \leqslant C \left(\int_0^\tau \|\rho_0 g\|^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^* \xi\|_U^2 \mathrm{d}t \right). \tag{1.2.35}$$

(2) *There exists a bounded linear operator $E_\tau : X \times Y \times \mathcal{F} \to \mathcal{U}$ such that for any $(z_0, h_0, f) \in X \times Y \times \mathcal{F}$, the control $u = E_\tau(z_0, h_0, f)$ is such that the solution of* (1.2.33) *satisfy $z \in \mathcal{Z}$ and $h(\tau) = 0$.*

For the proof of this proposition, we refer to [34] Theorem 4.1.

1.3 Internal null controllability for a linear coupled problem

In this section, we apply the result from Section 1.2 to obtain the internal null controllability of a linear system coupling the heat equation with some simple ODE's. This result will be used as a first step in the proof of Theorem 1.1.1. More precisely, we begin by studying a "fictitious" internal controllability problem in order to be able to apply the techniques in Section 1.2. Since, in the proof of Theorem 1.1.1, we will just need the spatial domain in which the heat equation holds to be of the form $(c, 0) \cup (0, 1)$ with $c < -1$, we choose c here to our best convenience, i.e., we take $c = -2$. In this context, Propositions 1.3.1 and 1.3.2 show that this linear system satisfies the assumptions of Proposition 1.2.2 which allows to obtain its null-controllability (Proposition 1.3.3). Then in Proposition 1.3.4, we show that we can control this linear system when coupled with a simple integrator (which corresponds to the position of the particle here) as in system (1.2.33). The last result, given in Corollary 1.3.5, gives some regularity properties of the controlled solution obtained in Proposition 1.3.4.

Denote

$$X = L^2(-2,1) \times \mathbb{R},$$

and consider the operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) = \left\{ z = \begin{bmatrix} \varphi \\ g \end{bmatrix} \in H_0^1(-2,1) \times \mathbb{R} \,|\, \varphi(0) = g,\ \varphi_{|_{(-2,0)}} \in H^2(-2,0),\ \varphi_{|_{(0,1)}} \in H^2(0,1) \right\},$$

$$A \begin{bmatrix} \varphi \\ g \end{bmatrix} = \begin{bmatrix} \varphi_{xx} \\ M^{-1}[\varphi_x](0) \end{bmatrix}. \tag{1.3.1}$$

In the above formula φ_{xx} is not understood in the distribution sense but it is just the function in $L^2(-2,1)$ whose restrictions to $(-2,0)$ and to $(0,1)$ coincide with the second derivative of φ on each of those intervals. The space X is endowed with the inner product

$$\langle z_1, z_2 \rangle = \int_{-2}^{1} \varphi_1(x)\varphi_2(x)\,\mathrm{d}x + Mg_1g_2, \quad \text{where} \quad z_i = \begin{bmatrix} \varphi_i \\ g_i \end{bmatrix},\ i = 1,2. \tag{1.3.2}$$

We simply denote by $\|\cdot\|$ the induced norm.

The proof of the following result is quite standard so that we omit it.

Proposition 1.3.1. *The operator A is negative in X (this means that A is self-adjoint and that there exists $m_1 > 0$ such that $\langle Az, z\rangle \leqslant -m_1\|z\|^2$ for every $z \in \mathcal{D}(A)$).*

The last proposition implies, according to the Lumer-Phillips theorem, that A generates a contraction semigroup on X. Furthermore, since A obviously has compact resolvents, it follows that there exists an orthonormal basis (Φ_k) in X formed by eigenvectors of A such that the corresponding eigenvalues form a non-increasing sequence $(-\lambda_k)$ with $\lambda_k \to \infty$.

The result below gives more information about the eigenvectors and the eigenvalues of A. To give the precise statement we introduce the Hilbert space $U = L^2\left(-2, -\frac{3}{2}\right)$ and the operator $B \in \mathcal{L}(U, X)$ defined by

$$Bu = \begin{bmatrix} \mathbb{1}_{\left(-2,-\frac{3}{2}\right)} u \\ 0 \end{bmatrix} \qquad (u \in U), \tag{1.3.3}$$

where $\mathbb{1}_{\left(-2,-\frac{3}{2}\right)}$ stands for characteristic function of the interval $\left(-2, -\frac{3}{2}\right)$. Note here, for later use, that the adjoint B^* of B is given by

$$B^* \begin{bmatrix} \varphi \\ r \end{bmatrix} = \varphi|_{\left(-2,-\frac{3}{2}\right)} \qquad \begin{bmatrix} \varphi \\ r \end{bmatrix} \in X. \tag{1.3.4}$$

Proposition 1.3.2. *The sequence* $(-\lambda_k)_{k\geqslant 1}$ *is regular (i.e.,* $\inf_{k\geqslant 1}(\lambda_{k+1} - \lambda_k) > 0$*) and we have*

$$\lambda_k = \frac{k^2\pi^2}{9} + O(k) \qquad (k \to \infty). \tag{1.3.5}$$

Furthermore, there exists a positive constant c_1 *such that*

$$c_1 \leqslant \|B^*\Phi_k\|_U \qquad (k \geqslant 1). \tag{1.3.6}$$

Proof. Since A is negative, all its eigenvalues are negative numbers. Let $-\mu^2$, with $\mu > 0$ be an eigenvalue of A. This means that there exists a vector $\begin{bmatrix} \varphi \\ r \end{bmatrix} \in \mathcal{D}(A) \setminus \{0\}$ such that

$$\begin{cases} -\varphi_{xx} = \mu^2\varphi(x) & x \in (-2,0) \cup (0,1), \\ \varphi(-2) = \varphi(1) = 0, \\ \varphi(0) = -\dfrac{1}{M\mu^2}[\varphi_x](0). \end{cases} \tag{1.3.7}$$

From the first two equations in (1.3.7) we deduce that

$$\varphi(x) = \begin{cases} C\sin\left(\mu(2+x)\right), & x \in (-2,0), \\ D\sin\left(\mu(1-x)\right), & x \in (0,1), \end{cases} \tag{1.3.8}$$

with $C, D \in \mathbb{R}$ and $C^2 + D^2 \neq 0$. In order to use the continuity of φ at $x = 0$ and the last equation in (1.3.7), we distinguish two cases.

The first case corresponds to the situation in which $\varphi(0) = 0$. In this case, by the continuity of φ at $x = 0$ we obtain

$$C\sin 2\mu = 0, \quad D\sin\mu = 0. \tag{1.3.9}$$

The above equation and the last equation in (1.3.7) yield

$$C\cos 2\mu = D\cos\mu.$$

From the above formula and (1.3.9), it easily follows that

$$C^2 = D^2. \tag{1.3.10}$$

This, combined with the fact that $C^2 + D^2 \neq 0$ and with (1.3.9), indicates that we have a first family $(-\lambda_m^{(1)})_{m\geqslant 1}$ of eigenvalues of A given by

$$\lambda_m^{(1)} = (\mu_m^{(1)})^2 = m^2\pi^2 \qquad (m \in \mathbb{N}^*). \tag{1.3.11}$$

The second case corresponds to the situation in which $\varphi(0) \neq 0$. In this case, we have

$$\sin(2\mu) \neq 0, \quad \sin\mu \neq 0,$$

and the constants C and D in the expression (1.3.8) of the eigenvectors can be chosen such that

$$D = C\frac{\sin 2\mu}{\sin \mu} = 2C\cos \mu. \tag{1.3.12}$$

With this choice, the last condition in (1.3.7) yields the characteristic equation

$$M\mu = \text{cotan}\mu + \text{cotan}2\mu. \tag{1.3.13}$$

All the eigenvalues of A not belonging to the family $(-\lambda_m^{(1)})$, obtained in our first case, are of the form $\lambda = -\mu^2$, where μ is a positive root of (1.3.13). To locate the roots of this transcendental equation, note first that the expression in the right hand side of (1.3.13), suggests to study the function $f(\mu) = \text{cotan}2\mu + \text{cotan}\mu$, which is defined on the union of all intervals of the form $\left((m-1)\pi, (m-1)\pi + \frac{\pi}{2}\right)$ and $\left((m-1)\pi + \frac{\pi}{2}, m\pi\right)$, with $m \in \mathbb{N}^*$. Moreover, f is decreasing on each of these intervals and

$$\lim_{\substack{\mu\to(m-1)\pi \\ \mu>(m-1)\pi}} f(\mu) = +\infty, \quad f\left((m-1)\pi + \frac{\pi}{3}\right) = 0, \quad \lim_{\substack{\mu\to(m-1)\pi+\pi/2 \\ \mu<(m-1)\pi+\pi/2}} f(\mu) = -\infty,$$

$$\lim_{\substack{\mu\to(m-1)\pi+\frac{\pi}{2} \\ \mu>(m-1)\pi+\frac{\pi}{2}}} f(\mu) = +\infty, \quad f\left((m-1)\pi + \frac{2\pi}{3}\right) = 0, \quad \lim_{\substack{\mu\to m\pi \\ \mu<m\pi}} f(\mu) = -\infty$$

Therefore, we have two other families of eigenvalues of A, denoted $(-\lambda_m^{(2)})_{m\geqslant 1}$ and $(-\lambda_m^{(3)})_{m\geqslant 1}$ where

$$\lambda_m^{(j)} = (\mu_m^{(j)})^2 \qquad (j \in \{2,3\},\ m \in \mathbb{N}^*)$$

and $\mu_m^{(2)},\ \mu_m^{(3)}$ can be written

$$\mu_m^{(2)} = \frac{\pi}{2} + (m-1)\pi + \omega_m^{(2)} \quad (m \in \mathbb{N}^*), \tag{1.3.14}$$

and

$$\mu_m^{(3)} = (m-1)\pi + \omega_m^{(3)} \quad (m \in \mathbb{N}^*), \tag{1.3.15}$$

where $\omega_m^{(2)} \in \left(0, \frac{\pi}{6}\right)$, $\omega_m^{(3)} \in \left(0, \frac{\pi}{3}\right)$ and

$$\lim_{m\to\infty} \omega_m^{(i)} = 0, \qquad i \in \{2,3\}. \tag{1.3.16}$$

It is easily seen from (1.3.11), (1.3.14) and (1.3.15) that, for each $m \in \mathbb{N}^*$,

$$\mu_m^{(3)} < \mu_m^{(2)} < \mu_m^{(1)} < \mu_{m+1}^{(3)} < \mu_{m+1}^{(2)} < \dots, \tag{1.3.17}$$

which implies that the sequences $(\mu_m^{(j)})_{m\geqslant 1}$, with $j\in\{1,2,3\}$ have no elements in common. Furthermore, using that $\omega_m^{(2)}\in(0,\frac{\pi}{6})$, $\omega_m^{(3)}\in(0,\frac{\pi}{3})$, we obtain that for each $m\in\mathbb{N}^*$ we have

$$\mu_m^{(2)}-\mu_m^{(3)}\geqslant\frac{\pi}{2}+\omega_m^{(2)}-\omega_m^{(3)}>\frac{\pi}{6}, \tag{1.3.18a}$$

$$\mu_m^{(1)}-\mu_m^{(2)}\geqslant\frac{\pi}{2}-\omega_m^{(2)}>\frac{\pi}{3}, \tag{1.3.18b}$$

$$\mu_{m+1}^{(3)}-\mu_m^{(1)}\geqslant\omega_{m+1}^{(3)}, \tag{1.3.18c}$$

In order to give a lower bound for $\omega_m^{(3)}$ we note that using (1.3.15), equation (1.3.13) writes

$$\begin{aligned}&M((m-1)\pi+\omega_m^{(3)})=\text{cotan}((m-1)\pi+\omega_m^{(3)})\\&+\text{cotan}[2((m-1)\pi+\omega_m^{(3)})]=\text{cotan}(\omega_m^{(3)})+\text{cotan}(2\omega_m^{(3)})\qquad(m\in\mathbb{N}^*).\end{aligned} \tag{1.3.19}$$

Since

$$\text{cotan}x=\frac{1}{x}-\frac{1}{3}x+o(x^2)\qquad(x\to 0), \tag{1.3.20}$$

equation (1.3.19) writes

$$M[(m-1)\pi+\omega_m^{(3)}]=\frac{3}{2\omega_m^{(3)}}-\omega_m^{(3)}+o((\omega_m^{(3)})^2). \tag{1.3.21}$$

Multiplying both sides of the above formula by $\omega_m^{(3)}/M$ we get

$$(m-1)\pi\omega_m^{(3)}=\frac{3}{2M}+o((\omega_m^{(3)})^2)\qquad(m\to\infty).$$

Dividing both sides of the above formula by $(m-1)\pi$ we obtain

$$\omega_m^{(3)}=\frac{3}{2\pi M(m-1)}+o\left(\frac{1}{m}\right)\qquad(m\to\infty), \tag{1.3.22}$$

which gives the desired lower bound for $\omega_m^{(3)}$.

Let now $(\mu_m)_{m\geqslant 1}$ denote the increasing sequence formed by rearranging all the values of the sequences $(\mu_m^{(i)})_{m\geqslant 1}$, with $i\in\{1,2,3\}$. The the spectrum of A is the set $-\Lambda$, where $\Lambda=(\lambda_m)_{m\geqslant 1}:=(\mu_m^2)_{m\geqslant 1}$. Combining (1.3.17), (1.3.18) and (1.3.22), we can see that Λ is regular, which yields the first conclusion of our proposition.

We still have to show that Λ satisfies (1.3.5). To achieve this goal denote

$$\nu(K)=\max\{m\ \mid\ \mu_m\leqslant K\}\qquad(K\geqslant 0).$$

It is clear that

$$\nu(K)=\sum_{i=1}^{3}\nu^{(i)}(K)\qquad(K\geqslant 0), \tag{1.3.23}$$

where

$$\nu^{(i)}(K) = \max\{m \mid \mu_m^{(i)} \leqslant K\} \qquad (i \in \{1,2,3\}, \quad K \geqslant 0).$$

From (1.3.11), (1.3.14) and (1.3.15), it follows that

$$\nu^{(i)}(K) = \frac{K}{\pi} + O(1) \qquad (i \in \{1,2,3\}, \quad K \to \infty). \tag{1.3.24}$$

Combining (1.3.23) with (1.3.24), it follows that

$$\nu(K) = \frac{3K}{\pi} + O(1) \qquad (K \to \infty). \tag{1.3.25}$$

Using the fact that

$$\mu_{\nu(K)} = K + O(1) \qquad (K \to \infty),$$

setting $\nu(K) = m$ in the last formula and using (1.3.25) we obtain that

$$\mu_m = \frac{m\pi}{3} + O(1) \qquad (m \to \infty), \tag{1.3.26}$$

which clearly implies (1.3.5).

We finally check estimate (1.3.6). For every $m \in \mathbb{N}^*$, the unitary eigenvector associated to the eigenvalue $(-\mu_m^2)$ is given by $\Phi_m = \begin{bmatrix} \varphi_m \\ \varphi_m(0) \end{bmatrix}$ with

$$\varphi_m(x) = \begin{cases} C_m \sin\left(\mu_m(2+x)\right), & x \in (-2,0), \\ D_m \sin\left(\mu_m(1-x)\right), & x \in (0,1), \end{cases} \tag{1.3.27}$$

where, in order to ensure the normalization condition, we have

$$C_m^2\left(1 - \frac{\sin 4\mu_m}{4\mu_m}\right) + D_m^2\left(\frac{1}{2} - \frac{\sin 2\mu_m}{4\mu_m}\right) + MC_m^2 \sin^2(2\mu_m) = 1. \tag{1.3.28}$$

We deduce from (1.3.4) that

$$B^*\Phi_m = \varphi_m|_{\left(-2,-\frac{3}{2}\right)}$$

so that

$$\|B^*\Phi_m\|_U^2 = \frac{C_m^2}{4}\left(1 - \frac{\sin \mu_m}{\mu_m}\right). \tag{1.3.29}$$

From (1.3.26), we deduce that there exists $d_1 > 0$ such that

$$d_1 < 1 - \frac{\sin \mu_m}{\mu_m} \qquad (m \in \mathbb{N}^*). \tag{1.3.30}$$

Therefore, we see from (1.3.29) and (1.3.30) that, in order to obtain (1.3.6), it suffices to estimate the sequence $(C_m)_{m\geqslant 1}$. This will be done by distinguishing two cases.

First case : Assume that μ_m is a term of the sequence $(\mu_k^{(1)})$, which has been defined in (1.3.11). We deduce from (1.3.10) and from (1.3.28) that $C_m^2 = 2/3$. This fact, combined with (1.3.29) implies that (1.3.6) holds in the considered case.

Second case : Assume that μ_m is not a term of the sequence $(\mu_k^{(1)})$. In this case, we know from (1.3.12) that $D_m = 2C_m \cos\mu_m$. Using this information in (1.3.28) it follows that

$$C_m^2 = \left[1 - \frac{\sin 4\mu_m}{4\mu_m} + 2\cos^2(\mu_m) - \cos^2(\mu_m)\frac{\sin 2\mu_m}{\mu_m} + M\sin^2(2\mu_m)\right]^{-1}. \tag{1.3.31}$$

On the other hand, by using (1.3.14), (1.3.15) and (1.3.16), we deduce

$$\sin^2(2\mu_m^{(i)}) \to 0, \quad \frac{\sin 4\mu_m^{(i)}}{4\mu_m^{(i)}} \to 0, \quad \cos^2(\mu_m^{(i)})\frac{\sin 2\mu_m^{(i)}}{\mu_m^{(i)}} \to 0 \quad (i \in \{2,3\}) \tag{1.3.32}$$

$$1 + 2\cos^2(2\mu_m^{(2)}) \to 1, \quad 1 + 2\cos^2(2\mu_m^{(3)}) \to 3. \tag{1.3.33}$$

Gathering (1.3.31) and (1.3.32) yields

$$|C_m| \geqslant c_1 \qquad (m \in \mathbb{N}^*)$$

for some constant $c_1 > 0$.

We conclude that (1.3.6) holds for the two cases and the proof is now complete. $\square$

Combining Proposition 1.2.2, Proposition 1.3.1 and Proposition 1.3.2, we conclude

Proposition 1.3.3. *The pair (A, B) is null-controllable in any time $\tau > 0$ with control cost $K(\tau) < M_0 e^{\frac{M_1}{\tau}}$ for some positive constants M_0 and M_1.*

Now we turn to the controllability of the system (1.2.33) in the particular case where A, B are defined by (1.3.1) and (1.3.3) and $C : X \to \mathbb{R}$ is defined by

$$C\begin{bmatrix}\varphi \\ g\end{bmatrix} = g \qquad \left(\begin{bmatrix}\varphi \\ g\end{bmatrix} \in X\right), \tag{1.3.34}$$

i.e. we consider the controllability of the nonhomogeneous system

$$\begin{cases} \dot{z} = Az + Bu + f, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0. \end{cases} \tag{1.3.35}$$

Proposition 1.3.4. *With the above notation for A, B and C, let $\tau > 0$ and let $M_0,\ M_1$ be the constants in Proposition 1.3.3. Let $\rho_{\mathcal{F}},\ \rho_0 : [0,\tau) \to [0,\infty)$*

$$\begin{cases} \rho_{\mathcal{F}}(t) = \exp(-\frac{\alpha}{(\tau-t)^2}), \\ \rho_0(t) = M_0 \exp\left(\frac{M_1}{(q-1)(\tau-t)} - \frac{\alpha}{q^4(\tau-t)^2}\right), \end{cases} \tag{1.3.36}$$

where the parameters q, α satisfy :

$$q > 1, \quad \alpha > \frac{M_1 q^4}{2(q-1)}\tau. \tag{1.3.37}$$

Let $(\mathcal{F},\mathcal{U},\mathcal{Z})$ be the spaces obtained from $\rho_{\mathcal{F}}$ and ρ_0 according to (1.2.13). *Then there exists a bounded linear operator*

$$E_\tau : X \times \mathbb{R} \times \mathcal{F} \to \mathcal{U} \tag{1.3.38}$$

such that for any $(z_0, h_0, f) \in X \times \mathbb{R} \times \mathcal{F}$, the control $u = E_\tau(z_0, h_0, f)$ is such that the solution of (1.3.35) *satisfies $z \in \mathcal{Z}$ and $h(\tau) = 0$.*

Proof. We consider the adjoint system (see (1.2.34)) of (1.3.35)

$$\begin{cases} -\dot{\xi}(t) = A\xi(t) + g(t) + C^* r, \\ \xi(\tau) = 0, \end{cases} \tag{1.3.39}$$

where $g : [0,\infty) \to X$ and $r \in \mathbb{R}$. It is not difficult to check that our assumptions (1.3.37) imply that the functions defined in (1.3.36) satisfy (1.2.10)-(1.2.12). This fact, combined with Proposition 1.2.3 (with $\gamma(t) = M_0 e^{\frac{M_1}{t}}$) and Corollary 1.2.6 yields that there exists a $C > 0$ with

$$\|\xi(0)\|_X^2 + \int_0^\tau \|\rho_{\mathcal{F}}\xi\|_X^2 \mathrm{d}t \leqslant C\left(\int_0^\tau \|\rho_0(g + C^* r)\|_X^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^*\xi\|_U^2 \mathrm{d}t\right), \tag{1.3.40}$$

for every solution of (1.3.39). In order to eliminate r from the right-hand side of the above inequality, we show that

$$|r|^2 \leqslant C\left(\int_0^\tau \|\rho_0 g\|_X^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^*\xi\|_U^2 \mathrm{d}t\right), \tag{1.3.41}$$

for every solution of (1.3.39). This is done by employing a compactness-uniqueness argument. Assume that (1.3.41) is false, then there exists a sequence $(g_n, r_n) \in L^2([0,\tau], X) \times \mathbb{R}$ such that $|r_n|^2 = 1$ for every $n \geqslant 1$ and

$$\int_0^\tau \|\rho_0 g_n\|_X^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^*\xi_n\|_U^2 \mathrm{d}t \to 0, \tag{1.3.42}$$

where ξ_n, g_n and r_n satisfy

$$\begin{cases} -\dot{\xi}_n(t) = A\xi_n(t) + g_n(t) + C^* r_n, \\ \xi_n(\tau) = 0. \end{cases} \tag{1.3.43}$$

By compactness we deduce that, up to the extraction of a subsequence,

$$r_n \to r, \quad |r| = 1. \tag{1.3.44}$$

Moreover, (1.3.42) implies that

$$\rho_0 g_n \to 0 \quad \text{strongly in} \qquad L^2([0,\tau], X). \tag{1.3.45}$$

Let

$$\rho_1(t) = M_0 \exp\left(\frac{M_1}{(q-1)(\tau-t)} - \frac{2\alpha}{(\tau-t)^2}\right) \qquad (t \in (0,\tau)).$$

It can be easily verified that

$$\rho_1(t) \leqslant \rho_0(t) \qquad t \in [0,\tau],$$

so that, using (1.3.45), we obtain that

$$\rho_1 g_n \to 0 \quad \text{strongly in} \qquad L^2([0,\tau], X). \tag{1.3.46}$$

On the other hand, it is not difficult to check that there exists a $C_1 > 0$ with

$$|\dot{\rho}_1(t)| \leqslant C_1 \rho_{\mathcal{F}}(t) \qquad t \in [0,\tau].$$

This last estimate, combined with (1.3.40) and (1.3.42), implies that

$$\|\dot{\rho}_1 \xi_n\|_{L^2([0,\tau],X)} \leqslant C_1 \|\rho_{\mathcal{F}} \xi_n\|_{L^2([0,\tau],X)} \leqslant C \qquad (n \geqslant 1), \tag{1.3.47}$$

for some positive constant C. Multiplying both sides of (1.3.43) by $\rho_1(t)$, we deduce that

$$\begin{cases} -(\rho_1 \xi_n)_t = A(\rho_1 \xi_n(t)) + \rho_1 g_n(t) + \rho_1 C^* r_n - \dot{\rho}_1 \xi_n(t), \\ \xi_n(\tau) = 0. \end{cases} \tag{1.3.48}$$

Let $f_n = \rho_1 g_n + \rho_1 C^* r_n - \dot{\rho}_1 \xi_n$. Combining (1.3.44), (1.3.46) and (1.3.47) implies that (f_n) is bounded in $L^2([0,\tau], X)$. Hence, using Lemma 1.2.5 for (1.3.48), we deduce that $(\rho_1 \xi_n)$ is bounded in $L^2([0,\tau], \mathcal{D}(A)) \cap H^1((0,\tau), X)$. Since ρ_1 is bounded away from zero on $[0, \tau - \epsilon]$ for every $\epsilon > 0$, we have

$$\xi_n \to \xi \quad \text{weakly in} \quad L^2([0,\tau-\epsilon], \mathcal{D}(A)) \cap H^1((0,\tau-\epsilon), X),$$

up to the extraction of a subsequence. The function ξ above obviously satisfies the first equation (1.3.39) for $t \in [0, \tau - \epsilon]$. Therefore denoting $\xi = \begin{bmatrix} \phi \\ k \end{bmatrix}$, after passing to the limit in (1.3.43), we have

$$\xi \in L^2([0, \tau - \epsilon], \mathcal{D}(A)) \cap H^1((0, \tau - \epsilon), X), \tag{1.3.49}$$

$$\begin{cases} -\phi_t(t,x) = \phi_{xx}(t,x), \quad (0, \tau - \epsilon) \times \{(-2,0) \cup (0,1)\} \\ \dot{k}(t) = [\phi_x](t,0) + r, \\ k(t) = \phi(t,0), \\ \phi(t,-2) = 0, \\ \phi(t,1) = 0. \end{cases} \tag{1.3.50}$$

Recalling the relation (1.3.42), the definition (1.3.36) of ρ_0 and the formula (1.3.4) for B^*, we see that for every $\epsilon > 0$, $\phi = 0$ in $(0, \tau - \epsilon) \times (-2, -3/2)$. Using a unique continuation property for the heat equation (see, for instance, [52, Theorem 1.1]), we deduce that $\phi = 0$ on $(0, \tau - \epsilon) \times (-2, 0)$. Recalling the definition of A and $\mathcal{D}(A)$ from (1.3.1) together with (1.3.49), we see that

$$\phi(t, 0+) = \phi(t, 0-) = 0, \tag{1.3.51}$$

for almost every $t \in (0, \tau - \epsilon)$, so that ϕ satisfies the heat equation with homogeneous Dirichlet boundary conditions for $t \in (0, \tau - \epsilon)$ and $x \in (0,1)$. It follows that ϕ can be written

$$\phi(t,x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{-\pi^2 n^2 (\tau - t - \epsilon)} \qquad (t \in [0, \tau - \epsilon], \quad x \in (0,1)), \tag{1.3.52}$$

for some real sequence (a_n) with $(n a_n) \in l^2$.

On the other hand, using the third equation in (1.3.50) and (1.3.51), it follows that $k(t) = 0$ for every $t \in [0, \tau - \epsilon)$. By using the second equation of (1.3.50) and the fact that ϕ vanishes for $x < 0$, yields

$$\phi_x(t, 0+) = -r \qquad (t \in [0, \tau - \epsilon]),$$

The above equation and (1.3.52) give

$$r + \pi \sum_{n=1}^{\infty} n a_n e^{-\pi^2 n^2 (\tau - t - \epsilon)} = 0 \qquad (t \in [0, \tau - \epsilon]).$$

which clearly implies that $a_n = 0$ for every $n \geqslant 1$ and that $r = 0$. The last assertion contradicts (1.3.44). We can thus conclude that (1.3.41) holds.

Using next (1.3.41) in (1.3.40) it follows that there exists a positive constant C such that

$$|r|^2 + \|\xi(0)\|^2 + \int_0^\tau \|\rho_{\mathcal{F}} \xi\|^2 \mathrm{d}t \leqslant C \left(\int_0^\tau \|\rho_0 g\|^2 \mathrm{d}t + \int_0^\tau \|\rho_0 B^* \xi\|_U^2 \mathrm{d}t \right),$$

for every solution of (1.3.39). By applying Proposition 1.2.9 (with $Y = \mathbb{R}$) we obtain the desired result. $\square$

Corollary 1.3.5. *With the assumptions and notation in Proposition 1.3.4, denote*

$$\rho(t) = e^{-\frac{\beta}{(\tau-t)^2}} \qquad (t \in [0,\tau)), \tag{1.3.53}$$

where the positive constant β is such that $\beta < \frac{\alpha}{q^4}$. Then, for every $z_0 \in \mathcal{D}((-A)^{1/2})$, $f \in \mathcal{F}$ and $h_0 \in \mathbb{R}$, the trajectory z obtained by solving (1.3.35) *with $u = E_\tau(z_0, h_0, f)$ is in $C([0,\tau], \mathcal{D}((-A)^{1/2}))$. Moreover, there exists a positive constant C such that*

$$\left\|\frac{z}{\rho}\right\|_{C([0,\tau];\mathcal{D}((-A)^{1/2}))\cap H^1((0,\tau),X)\cap L^2([0,\tau],\mathcal{D}(A))} \leqslant C\left(\|z_0\|_{\mathcal{D}((-A)^{1/2})} + |h_0| + \left\|\frac{f}{\rho_{\mathcal{F}}}\right\|_{L^2([0,\tau],X)}\right), \tag{1.3.54}$$

for every $z_0 \in \mathcal{D}((-A)^{1/2})$, $f \in \mathcal{F}$ and $h_0 \in \mathbb{R}$. Finally, if the parameter q from the definition (1.3.36) *of ρ_0 satisfies $q^4 < 2$ then*

$$\frac{\rho^2}{\rho_{\mathcal{F}}} \in C[0,\tau]. \tag{1.3.55}$$

Proof. Since $\beta < \frac{\alpha}{q^4}$, we see that the functions ρ_0, $\rho_{\mathcal{F}}$ and ρ satisfy (1.2.30). Therefore (1.3.54) follows by applying Proposition 1.2.8.

Furthermore, if we choose q such that $1 < q^4 < 2$, then $\frac{\rho^2}{\rho_{\mathcal{F}}} \in C[0,\tau]$ since

$$\frac{\rho^2}{\rho_{\mathcal{F}}}(t) = \exp\left(\frac{\alpha - 2\beta}{(\tau-t)^2}\right),$$

is bounded on $[0,\tau]$ if $\alpha - 2\beta < 0$. Using again the fact that $\beta < \frac{\alpha}{q^4}$, we deduce that we can achieve this by choosing such a β iff $q^4 < 2$. $\square$

1.4 Proof of the main result

This section is devoted to the proof of Theorem 1.1.1. Using a standard methodology for parabolic control problems we first study, instead of the boundary control problem (1.1.1), a distributed control problem (see (1.4.1) below). To deal with this system with moving boundary, we first use a change of variable to transform it (see (1.4.4) below). This system can be written as system (1.3.35) but with a right-hand side f depending on z and h. Lemma 1.4.1 gives properties of this nonlinear term and thus using a fixed-point argument and the results of the previous section, we deduce the null-controllability of (1.4.4) for small initial data. Using our change of variables, it implies the null-controllability of (1.4.4) for small initial data of the distributed control problem (1.4.1) (Theorem 1.4.3). We end this section by proving that the controllability of (1.4.1) implies the controllability of (1.1.1), which proves Theorem 1.1.1.

First, let us consider a distributed control problem associated to (1.1.1) :

$$\left\{\begin{array}{l} v_t(t,x) - v_{xx}(t,x) + v(t,x)v_x(t,x) = \hat{u}(t,x) \qquad (t \geqslant 0, \quad x \in [-2,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ M\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-2) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-2,1] \setminus \{h_0\}. \end{array}\right. \tag{1.4.1}$$

where $\hat{u}(t,x)$ is the input function, supported the subinterval of $(-2,-5/4)$, and the state of the system is the triple $\begin{bmatrix} v \\ h \\ \dot{h} \end{bmatrix}$. The study of the above internal controllability problem should be seen only as a tool in the proof of our main boundary controllability result. Therefore, we choose here the very particular interval $[-2,1]$ for technical purposes (as already explained in the previous section), but this will be enough to obtain the proof of our main result.

To study the control problem (1.4.1) we reduce it to a problem in a fixed domain by using an appropriate change of variables. More precisely, let $\tau > 0$ and let $h : [0,\tau] \to (-1,1)$ be a function in $H^1(0,\tau)$. For every $t \in [0,\tau]$ we define a diffeomorphism $\eta(t,\cdot)$ from $(-2,h(t)) \cup (h(t),1)$ onto $(-2,0) \cup (0,1)$ by

$$y = \eta(t,x) = \left\{\begin{array}{l} \frac{x-h(t)}{1-h(t)}, x > h(t), \\ \frac{2(x-h(t))}{2+h(t)}, x < h(t). \end{array}\right. \tag{1.4.2}$$

We define a new function φ by setting

$$\varphi(t,\eta(t,x)) = v(t,x) \qquad (t \in [0,\tau],\ x \in (-2,h(t)) \cup (h(t),1)). \tag{1.4.3}$$

It is not difficult to check that, using the change of variables defined by (1.4.2) and (1.4.3), the system (1.4.1) becomes

$$\left\{\begin{array}{l} \varphi_t - \left(\eta_x \circ \eta^{-1}\right)^2 \varphi_{yy} + (\eta_x \circ \eta^{-1})\varphi_y\varphi \\ \qquad\qquad\qquad\qquad + (\eta_t \circ \eta^{-1})\varphi_y = u, \qquad (t \geqslant 0, \quad y \in [-2,1] \setminus \{0\}), \\ M\dot{g}(t) = [(\eta_x \circ \eta^{-1})\varphi_y](0,t), \qquad (t \geqslant 0), \\ \varphi(t,0) = g(t), \qquad (t \geqslant 0), \\ \dot{h}(t) = g(t), \qquad (t \geqslant 0), \\ \varphi(t,-2) = 0, \quad \varphi(t,1) = 0, \qquad (t \geqslant 0), \\ h(0) = h_0, \quad g(0) = h_1, \\ \varphi(0,y) = \varphi_0(y) \quad y \in [-2,1] \setminus \{0\}, \end{array}\right. \tag{1.4.4}$$

where u is a new input function and

$$[(\eta_x \circ \eta^{-1})\varphi_y](t,0) = [(\eta_x \circ \eta^{-1})](t,0)\varphi_y(t,0+) + (\eta_x \circ \eta^{-1})(t,0-)[\varphi_y](t,0).$$

We note that, since the function η is given by (1.4.2), we have

$$(\eta_x \circ \eta^{-1})(t,y) = \frac{1}{1-\kappa(y)h(t)} = \begin{cases} \frac{1}{1-h(t)}, & y > 0, \\ \frac{2}{2+h(t)}, & y < 0. \end{cases} \tag{1.4.5}$$

$$(\eta_t \circ \eta^{-1})(t,y) = -g(t)\frac{1-\kappa(y)y}{1-\kappa(y)h(t)} = \begin{cases} -\frac{1-y}{1-h(t)}g(t) & \text{for} \quad y > 0, \\ -\frac{2+y}{2+h(t)}g(t) & \text{for} \quad y < 0. \end{cases} \tag{1.4.6}$$

where

$$\kappa(y) = \begin{cases} 1 & \text{for} \quad y > 0, \\ -\frac{1}{2} & \text{for} \quad y < 0. \end{cases} \tag{1.4.7}$$

Moreover, we have

$$[\eta_x \varphi_y](t,0) = \left(\frac{1}{1-h(t)} - \frac{2}{2+h(t)}\right)\varphi_y(t,0+) + \frac{2}{2+h(t)}[\varphi_y](t,0)$$

Using the above formula, combined with (1.4.5) and (1.4.6), we obtain that (1.4.4) writes in the more explicit form

$$\begin{cases} \varphi_t = \varphi_{yy} + \left[\left(\frac{1}{1-\kappa h}\right)^2 - 1\right]\varphi_{yy} \\ \qquad\qquad - \left(\frac{1}{1-\kappa h}\right)\varphi_y\varphi - \frac{1-\kappa y}{1-\kappa h}g\varphi_y + u \qquad (t \geqslant 0, \quad y \in [-2,1]\setminus\{0\}), \\ M\dot{g} = [\varphi_y](t,0) - \frac{h(t)}{2+h(t)}[\varphi_y](t,0) + \left(\frac{1}{1-h(t)} - \frac{2}{2+h(t)}\right)\varphi_y(t,0+) \qquad (t \geqslant 0), \\ \varphi(t,0) = g(t) \qquad (t \geqslant 0), \\ \dot{h}(t) = g(t), \\ \varphi(t,-2) = 0, \quad \varphi(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad g(0) = h_1, \\ \varphi(0,y) = \varphi_0(y) \qquad (y \in [-2,1]\setminus\{0\}). \end{cases} \tag{1.4.8}$$

The above system suggests the introduction of the nonlinear operator

$$N : \mathcal{D}(A) \times (-2,1) \to X$$

defined by

$$N\begin{bmatrix} z \\ h \end{bmatrix} = \begin{bmatrix} \left[\left(\frac{1}{1-\kappa h}\right)^2 - 1\right]\varphi_{yy} - \left(\frac{1}{1-\kappa h}\right)\varphi_y\varphi - \frac{1-\kappa y}{1-\kappa h}g\varphi_y \\ -\frac{h}{2+h}[\varphi_y](0) + \left(\frac{1}{1-h} - \frac{2}{2+h}\right)\varphi_y(0+) \end{bmatrix},$$

where $z = \begin{bmatrix} \varphi \\ g \end{bmatrix}$ is a generic element of $\mathcal{D}(A)$. Lemma below yields some important properties of N.

Lemma 1.4.1. *For every $a \in (0,1)$, there exists a positive constant $C = C(a)$ such that*

$$\left\| N \begin{bmatrix} z \\ h \end{bmatrix} \right\|_X \leqslant C \left(|g|^2 + \|\varphi\|_{H^1}^2 + |h| \ \|\varphi_{yy}\|_{L^2} \right) \qquad \left(\begin{bmatrix} z \\ h \end{bmatrix} \in \mathcal{D}(A) \times (-a,a) \right), \tag{1.4.9}$$

$$\left\| N \begin{bmatrix} z_1 \\ h_1 \end{bmatrix} - N \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} \right\|_X \leqslant C \Big((|h_1 - h_2| + |g_1 - g_2|) \left(\|\varphi_{1,yy}\|_{L^2} + \|\varphi_1\|_{H^1}^2 + \|\varphi_2\|_{H^1} \right)$$

$$+ |h_2| \|\varphi_1 - \varphi_2\|_{H^2} + \|\varphi_1^2 - \varphi_2^2\|_{H^1} \Big) \qquad \left(\begin{bmatrix} z_i \\ h_i \end{bmatrix} \in \mathcal{D}(A) \times (-a,a), i = 1,2 \right). \tag{1.4.10}$$

Proof. The proof of (1.4.9) follows easily from standard embedding and trace results for Sobolev spaces.

To prove (1.4.10), we note that

$$N = \begin{bmatrix} F_1 + F_2 + F_3 \\ F_4 \end{bmatrix}, \tag{1.4.11}$$

where

$$F_1 \begin{bmatrix} z \\ h \end{bmatrix} = \left[\left(\frac{1}{1 - \kappa h} \right)^2 - 1 \right] \varphi_{yy}, \quad F_2 \begin{bmatrix} z \\ h \end{bmatrix} = - \left(\frac{1}{1 - \kappa h} \right) \varphi_y \varphi, \tag{1.4.12}$$

$$F_3 \begin{bmatrix} z \\ h \end{bmatrix} = - \frac{1 - \kappa y}{1 - \kappa h} g \varphi_y, \tag{1.4.13}$$

$$F_4 \begin{bmatrix} z \\ h \end{bmatrix} = - \frac{h}{2 + h} [\varphi_y](0) + \left(\frac{1}{1-h} - \frac{2}{2+h} \right) \varphi_y(0+). \tag{1.4.14}$$

Simple calculations, combined with standard embedding and trace results for Sobolev spaces, imply that there exists a constant $C > 0$, depending only on a, such that for every $\begin{bmatrix} z_1 \\ h_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} \in \mathcal{D}(A) \times (-a,a)$ we have

$$\left\| F_1 \begin{bmatrix} z_1 \\ h_1 \end{bmatrix} - F_1 \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} \right\|_{L^2} \leqslant C \left(|h_1 - h_2| \ \|\varphi_{1,yy}\|_{L^2} + |h_2| \ \|\varphi_{1,yy} - \varphi_{2,yy}\|_{L^2} \right), \tag{1.4.15}$$

$$\left\| F_2 \begin{bmatrix} z_1 \\ h_1 \end{bmatrix} - F_2 \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} \right\|_{L^2} \leqslant C \left(|h_1 - h_2| \ \|\varphi_1\|_{H^1}^2 + \ \|\varphi_1^2 - \varphi_2^2\|_{H^1} \right), \tag{1.4.16}$$

$$\left\| F_3 \begin{bmatrix} z_1 \\ h_1 \end{bmatrix} - F_3 \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} \right\|_{L^2}$$
$$\leqslant C \left(|h_1 - h_2| \ \ |g_1| \ \|\varphi_1\|_{H^1} + |g_1| \ \|\varphi_1 - \varphi_2\|_{H^1} + |g_1 - g_2| \ \|\varphi_2\|_{H^1} \right), \tag{1.4.17}$$

$$\left| F_4 \begin{bmatrix} z_1 \\ h_1 \end{bmatrix} - F_4 \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} \right| \leqslant C \left(|h_1 - h_2| \, \|\varphi_{1,yy}\|_{L^2} + |h_2| \, \|\varphi_{1,yy} - \varphi_{2,yy}\|_{L^2} \right). \tag{1.4.18}$$

The above estimates clearly imply the conclusion. □

The main step in the proof of Theorem 1.1.1 is the following result.

Proposition 1.4.2. *Let $\tau > 0$. Then there exists $r > 0$ such that for every $h_0, \; h_1, \; \varphi_0$ satisfying*

$$|h_0| + |h_1| + \|\varphi_0\|_{H_0^1(-2,1)} \leqslant r, \tag{1.4.19}$$

$$\varphi_0(0) = h_1.$$

the nonlinear system (1.4.8) *is null controllable in time τ with a control u supported in* $(-2, -3/2)$ *: there exists $u \in L^2([0,\tau] \times [-2, -3/2])$ such that the corresponding solution satisfies*

$$\varphi(\tau, \cdot) = 0, \quad h(\tau) = 0, \quad \dot{h}(\tau) = 0. \tag{1.4.20}$$

Moreover, the control u can be chosen so that

$$-1 < h(t) < 1 \qquad (t \in [0, \tau]). \tag{1.4.21}$$

Proof. Since we look for controls such that $u(t, \cdot)$ is supported in $(-2, -3/2)$, the problem can be rewritten

$$\begin{cases} \dot{z} = Az + N \begin{bmatrix} z \\ h \end{bmatrix} + Bu, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0, \end{cases} \tag{1.4.22}$$

with the same choice of the operators A, B and C as in Section 1.3 (see (1.3.1), (1.3.3) and (1.3.34)).

In order to prove the null controllability of (1.4.22) we consider the linearized problem

$$\begin{cases} \dot{z} = Az + Bu + f, \\ \dot{h} = Cz, \\ z(0) = z_0, \\ h(0) = h_0, \end{cases} \tag{1.4.23}$$

with $f : [0, \tau] \to X$. Let $\rho_{\mathcal{F}}$, ρ_0, ρ be the functions introduced in Proposition 1.3.4 and in Corollary 1.3.5 and let $(\mathcal{F}, \mathcal{U}, \mathcal{Z})$ be the corresponding function spaces, defined according to (1.2.13). According to Proposition 1.3.4 and Corollary 1.3.5 there exists a control $u = E_\tau(z_0, h_0, f)$ such that the corresponding trajectory (z, h) satisfies $h(\tau) = 0$ together with (1.3.54). It follows that, denoting

$$\mathcal{F}_r = \{ f \in \mathcal{F} \;\; | \;\; \|f\|_{\mathcal{F}} \leqslant r \}, \tag{1.4.24}$$

with $r > 0$ small enough, we can define an operator $\mathcal{N}$ acting on $\mathcal{F}_r$ by

$$\mathcal{N}(f)(t) := N \begin{bmatrix} z(t) \\ h(t) \end{bmatrix} \qquad (t \in [0, \tau]),$$

where $\begin{bmatrix} z \\ h \end{bmatrix}$ is the state trajectory of (1.4.23) corresponding to the input $u = E_\tau(z_0, h_0, f)$. Indeed, the fact that, provided that r is small enough, we have $h(t) \in (-1, 1)$ for every $t \in [0, \tau]$ (so that $\mathcal{N}$ is well-defined on $\mathcal{F}_r$) follows from Corollary 1.3.5. More precisely, let ρ the function defined in (1.3.53). Since $\rho(\tau) = h(\tau) = 0$ and $\dot\rho(t) = -\frac{\beta}{(\tau-t)^3}\rho(t)$, by the Cauchy mean value theorem, for every $t \in [0, \tau]$ we have

$$\left|\frac{h(t)}{\rho(t)}\right| = \left|\frac{h(t) - h(\tau)}{\rho(t) - \rho(\tau)}\right| \leqslant \left\|\frac{g}{\dot\rho}\right\|_{C[0,\tau]} \leqslant C \left\|\frac{g}{\rho}\right\|_{C[0,\tau]}, \tag{1.4.25}$$

where C is a constant depending only on β and on τ. By (1.3.54), if we choose r small enough, we have (1.4.21), so that $\mathcal{N}$ is indeed well defined.

In order to obtain the conclusion of the proposition, it suffices to check that, for $r > 0$ small enough, $\mathcal{N}$ is contractive mapping from $\mathcal{F}_r$ into itself. We first check that $\mathcal{F}_r$ is invariant for $\mathcal{N}$, provided that r is small enough. This follows from Lemma 1.4.1, noting that, for almost every $t \in [0, \tau]$, we have

$$\left\|\frac{\mathcal{N}(f)}{\rho_{\mathcal{F}}}(t)\right\|_X \leqslant C \left|\frac{\rho^2(t)}{\rho_{\mathcal{F}}(t)}\right| \left(\left|\frac{g(t)}{\rho(t)}\right|^2 + \left|\frac{h(t)}{\rho(t)}\right| \left\|\frac{\varphi(t,\cdot)}{\rho(t)}\right\|_{H^2} + \left\|\frac{\varphi(t,\cdot)}{\rho(t)}\right\|_{H^1}^2\right),$$

Since ρ and $\frac{\rho^2}{\rho_{\mathcal{F}}}$ are continuous functions on $[0, \tau]$ (see (1.3.36) and (1.3.55)), it follows that

$$\|\mathcal{N}(f)\|_{\mathcal{F}} \leqslant C \left(\left\|\frac{g}{\rho}\right\|_{C[0,\tau]}^2 + \left\|\frac{h}{\rho}\right\|_{C[0,\tau]} \left\|\frac{\varphi}{\rho}\right\|_{L^2([0,\tau],H^2)} + \left\|\frac{\varphi}{\rho}\right\|_{C([0,\tau],H^1)}^2\right), \tag{1.4.26}$$

Using (1.3.54), (1.4.19), (1.4.25) and (1.4.26), it follows that

$$\|\mathcal{N}(f)\|_{\mathcal{F}} \leqslant Cr^2 \qquad (f \in \mathcal{F}_r),$$

for some positive constant C. The last estimate implies indeed that $\mathcal{N}$ invariates $\mathcal{F}_r$ for $r > 0$ small enough.

We still have to show that $\mathcal{N}$ is contracting for r small enough. Using (1.4.10) it follows that there exists $C > 0$, independent of $r > 0$, such that the inequality

$$
\begin{aligned}
&\frac{1}{\rho_{\mathcal{F}}}\left\|\mathcal{N}(f_1)-\mathcal{N}(f_2)\right\|_X \\
&\quad\leqslant \frac{C\rho^2}{\rho_{\mathcal{F}}}\left(\frac{|h_1-h_2|}{\rho}+\frac{|g_1-g_2|}{\rho}\right)\left(\left\|\frac{\varphi_{1,yy}}{\rho}\right\|_{L^2}+\rho\left\|\frac{\varphi_1}{\rho}\right\|_{H^1}^2+\left\|\frac{\varphi_2}{\rho}\right\|_{H^1}\right) \\
&+\frac{C\rho^2}{\rho_{\mathcal{F}}}\frac{|h_2|+|g_1|}{\rho}\left(\left\|\frac{\varphi_{1,yy}-\varphi_{2,yy}}{\rho}\right\|_{L^2}+\rho\left\|\frac{\varphi_1^2-\varphi_2^2}{\rho^2}\right\|_{H^1}\right) \qquad (f_1,\ f_2\in\mathcal{F}_r),
\end{aligned}
\tag{1.4.27}
$$

holds for $t\in[0,\tau]$. Integrating the last estimate on $[0,\tau]$ and using (1.4.25) it follows that there exists $r^*>0$ and a constant $C>0$, not depending on $r\in(0,r^*)$, such that

$$
\begin{aligned}
&\left\|\mathcal{N}(f_1)-\mathcal{N}(f_2)\right\|_{\mathcal{F}} \\
&\leqslant Cr\left(\left\|\frac{g_1-g_2}{\rho}\right\|_{C[0,\tau]}+\left\|\frac{\varphi_{1,yy}-\varphi_{2,yy}}{\rho}\right\|_{L^2([0,\tau],L^2)}+\left\|\frac{\varphi_1-\varphi_2}{\rho}\right\|_{C([0,\tau],H^1)}\right).
\end{aligned}
\tag{1.4.28}
$$

Since $z=z_1-z_2,\ h=h_1-h_2$ satisfy (1.3.35), with $z_0=0$, $h_0=0$ and $f=f_1-f_2$, we can use (1.3.54) to obtain that $\mathcal{N}$ is indeed contracting on $\mathcal{F}_r$, provided that r is small enough. □

As a consequence of the last proposition we obtain the following result :

Theorem 1.4.3. *The system* (1.4.1) *is locally null controllable in any time* $\tau>0$. *More precisely, for every* $\tau>0$ *there exists* $r>0$ *such that every initial data* h_0,h_1,v_0 *satisfying*

$$
|h_0|+|h_1|+\|v_0\|_{H_0^1(-2,1)}\leqslant r, \qquad v_0(h_0)=h_1,
$$

can be steered to rest by means of a control, $\hat{u}\in L^2([0,\tau],L^2(-2,-5/4))$. *Moreover,* $\widehat{u}$ *can be chosen such that* $h\in C[0,\tau]$ *and*

$$
-1<h(t)<1 \qquad (t\in[0,\tau]). \tag{1.4.29}
$$

Proof. According to Proposition 1.4.2, we can find

$$
u\in L^2(0,\tau;L^2(-2,-3/2))
$$

such that the solution of (1.4.8) satisfies $h\in C[0,\tau]$, together with (1.4.29) and

$$
\varphi(\tau,\cdot)=0, \qquad g(\tau)=h(\tau)=0.
$$

Using the change of variables (1.4.2), we obtain a controlled trajectory of the system (1.4.1), which is vanishing for $t = \tau$ and with a control $\widehat{u}$ such that

$$\operatorname{supp} \widehat{u}(t, \cdot) \subset (-2, \hat{a}) \qquad (t \in [0, \tau]).$$

It can be seen by (1.4.2) that

$$\frac{2(\hat{a} - h(t))}{2 + h(t)} = \eta(t, \hat{a}) \leqslant -\frac{3}{2}.$$

This, combined with (1.4.29) implies that $\hat{a} \leqslant -\frac{5}{4}$, which finishes the proof. $\square$

We are now in a position to prove our main result.

Proof of Theorem 1.1.1. Since $v_0 \in H_0^1(-1, 1)$, we can extend it by zero on $(-2, -1)$ such that $v_0 \in H_0^1(-2, 1)$. We consider system (1.4.1) with initial data (v_0, h_0, h_1). Then, there exists a constant $r > 0$ such that if

$$|h_0| + |h_1| + \|v_0\|_{H_0^1((-2,1)\setminus\{h_0\})} \leqslant r,$$

there exists a control

$$\hat{u} \in L^2([0, \tau], L^2(-2, -5/4))$$

such that the solution of (1.4.1) satisfies (1.4.29) and

$$v \in L^2([0, \tau], H^2(-2, 1)) \cap C([0, \tau], H^1(-2, 1)) \cap H^1((0, \tau), L^2(-2, 1)),$$

$$v(\tau, \cdot) = 0, \qquad h(\tau) = \dot{h}(\tau) = 0.$$

Define

$$u(t) = v(-1, t) \qquad t \in [0, \tau].$$

We clearly have $u \in C[0, \tau]$ and u drives the system (1.1.1) to rest in time τ. $\square$

Chapitre 2

Existence d'une Solution Forte pour un Modèle de la Vésicule

2.1 Introduction

The study of hydrodynamical properties of vesicle membranes is important for many applications, in particular in biological and physiological subjects. A vesicle is an elastic membrane containing a liquid and surrounded by another liquid. Such a vesicle can be found in nature or it can be created in laboratory. They can store and/or transport substances. Modelling vesicle is also a first step in order to study and understand the behavior of more complex cell such as red cells. Recent papers have been devoted to both experimental studies, see for instance [41] or [40], to the modelling and finally to the mathematical analysis of the obtained models, see for instance [20], [22], [23], [21] and [62].

To model the elastic deformation of the vesicle in the fluid is not an easy task. One possibility could be to consider equations of elasticity for the membrane and the Navier–Stokes equations outside the membrane. However this model could be very difficult to study and to simulate numerically due to the free-boundary corresponding the interfaces between the fluid and the vesicle. Moreover, in this coupling, the equations of elasticity are written in Lagrangian variables whereas the equations for the fluid are written in Eulerian variables. As a consequence, another approach to model this vesicle-fluid interaction is to use the phase-field theory : the membrane of the vesicle is described by a phase field function φ. In this model, φ takes value $+1$ inside the vesicle membrane and -1 outside, with a thin transition layer of width characterized by a small positive parameter ϵ. The fluid is modelled by the incompressible Navier–Stokes equations written in the whole domain containing the fluid and the vesicle.

We use the model introduced in [20] which is described by the following system of equations of unknowns $\mathbf{u}$ (the velocity of the fluid) and φ (the phase field function) :

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \nabla p + \nu\Delta\mathbf{u} + W(\varphi)\nabla\varphi & \text{in } [0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } [0,T]\times\Omega, \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi = -\gamma W(\varphi) & \text{in } [0,T]\times\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega, \\ \mathbf{u}(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \\ \varphi(0,x) = \varphi_0(x) & \text{in } \Omega, \\ \varphi(t,x) = -1 & \text{on } [0,T]\times\partial\Omega, \\ \Delta\varphi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega. \end{cases} \tag{2.1.1}$$

In the above equations we denote :

$$\begin{aligned} W(\varphi) &= kg(\varphi) + M_1[V(\varphi)-\alpha] + M_2[B(\varphi)-\beta]f(\varphi), \\ &= k\epsilon\Delta^2\varphi - \frac{k}{\epsilon}\Delta(\varphi^3) + \frac{k}{\epsilon}\Delta\varphi + \frac{k}{\epsilon^2}(3\varphi^2-1)f(\varphi) \\ &\quad + M_1[V(\varphi)-\alpha] + M_2[B(\varphi)-\beta]f(\varphi), \end{aligned} \tag{2.1.2}$$

with

$$V(\varphi) = \int_\Omega \varphi \, dx, \tag{2.1.3a}$$

$$B(\varphi) = \int_\Omega \left(\frac{\epsilon}{2}|\Delta\varphi|^2 + \frac{1}{4\epsilon}(\varphi^2-1)^2\right) dx, \tag{2.1.3b}$$

$$f(\varphi) = -\epsilon\Delta\varphi + \frac{1}{\epsilon}(\varphi^2-1)\varphi, \tag{2.1.3c}$$

$$g(\varphi) = -\Delta f(\varphi) + \frac{1}{\epsilon^2}(3\varphi^2-1)f(\varphi). \tag{2.1.3d}$$

where ϵ is a small positive parameter depending on the thickness of the membrane of the vesicle. The constant $k > 0$ is the bending modules of the vesicle. The constant α is defined by $\alpha = \int_\Omega \varphi_0 dx$. Note that $\alpha < 0$ if the vesicle is "smaller" compared to the remaining part of Ω. The constant β represents the total surface area of the vesicle. Moreover, M_1 and M_2 are two penalty constants used to enforce the volume and the surface of the vesicle to remain constant. More precisely, in this model, the energy of the vesicle is

$$\mathcal{E}(\varphi) := \frac{k}{2\varepsilon}\int_\Omega |f(\varphi)|^2 \, dx + \frac{1}{2}M_1[V(\varphi)-\alpha]^2 + \frac{1}{2}M_2[B(\varphi)-\beta]^2,$$

where

$$\frac{k}{2\varepsilon}\int_\Omega |f(\varphi)|^2 \, dx$$

is an approximation of the Helfrich bending elasticity energy of the membrane and where $B(\varphi)$ is an approximation of the surface area of the vesicle. Formally, the term $W(\varphi)$ is obtained by differentiating $\mathcal{E}$ with respect to φ. For more details on the derivation of the above equations, see [20] and [21].

The well-posedness problem of the system (2.1.1) has been first studied in [20]. In this work the authors obtain the existence of weak solutions by using Galerkin's method, in the flavor of the classical techniques introduced in [14]. In the same work the uniqueness of solutions has been proved in a class smaller than the one used for existence. The aim of this work is to prove the local in time existence and uniqueness of a stronger concept of solution. Note that our existence and uniqueness results are valid in the same spaces. We also show the existence of almost global in time strong solutions provided that the initial data and the constant $(\mu(\Omega)+\alpha)^2$ are small enough.

In order to give the precise statement of our main result, we need some notations. In the remaining part of this work, $\Omega \subset \mathbb{R}^3$ is an open bounded set with smooth boundary $\partial\Omega$. We set

$$E(t) = \|\mathbf{u}(t)\|^2_{H^1(\Omega))} + \|\varphi(t)\|^2_{H^{2+\frac{3}{8}}(\Omega))}. \tag{2.1.4}$$

Our first result is :

Theorem 2.1.1. *(Local existence and uniqueness of strong solutions) Suppose that*

$$\begin{cases} \mathbf{u}_0 & \in H^1_0(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0 \quad \text{in } \Omega, \\ \varphi_0 & \in \tilde{H}^{2+\frac{3}{8}}(\Omega). \end{cases} \tag{2.1.5}$$

Then there exists $T = T\left(\|\mathbf{u}_0\|_{H^1(\Omega)}, \|\varphi_0\|_{H^{2+\frac{3}{8}}(\Omega)}\right) > 0$ *such that* (2.1.1) *has a unique maximal strong solution :*

$$\begin{cases} \mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ p \in L^2(0,T;H^1(\Omega)), \\ \varphi \in L^2(0,T;\tilde{H}^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];\tilde{H}^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \end{cases} \tag{2.1.6}$$

Moreover, one of the following results holds :

1. $T = +\infty$,
2. $\lim\limits_{t\to T-0} E(t) = +\infty$.

The spaces $\tilde{H}^s(\Omega)$ used above are closed subspaces of the classical Sobolev spaces $H^s(\Omega)$. We refer to the next section for the precise definition of these spaces (see (2.2.4)).

Remark 2.1.2. *The power* $3/8$ *appearing in the spaces for* φ *comes from the estimates on the nonlinear terms. More precisely, one of the most difficult* *terms to handle is $\Delta^2\varphi\nabla\varphi$ which appear in the right-hand side of the Navier–Stokes equations. To obtain strong solutions we need that this term is L^2 in time and space and for this, we have to work with φ in*

$$L^2(0,T;\tilde{H}^{4+s}(\Omega)) \cap C([0,T];\tilde{H}^{2+s}(\Omega)) \cap H^1(0,T;H^s(\Omega)),$$

with $s \geqslant \frac{1}{4}$ (see the proof of Lemma 2.4.3 and more precisely inequality (2.4.8)*). In fact, to obtain use a Banach fixed point theorem, we need $s > \frac{1}{4}$.*

On the other hand, we can not consider s too large. In fact, to prove the existence of strong solution, we use a fixed point strategy where all the nonlinear terms are put in the right-hand sides. Thus a first step is to solve a linear problem with given right-hand sides : these are Corollaries 2.2.7 and 2.2.8. However these right-hand sides coming from the nonlinear terms do not satisfy the compatibility conditions (at the boundary) needed to obtain very regular solutions. In particular, the right-hand side for φ could not be in $L^2(\tilde{H}^s)$ for $s \geqslant \frac{1}{2}$ which implies the restriction $s < \frac{1}{2}$.

For all these considerations, it appears that we need to impose $\frac{1}{4} < s < \frac{1}{2}$. We take here $s = \frac{3}{8}$ but all the theory should work for s in the interval $(\frac{1}{4}, \frac{1}{2})$.

Remark 2.1.3. *In [20], the authors prove the existence of weak solution of* (2.1.1) *with the following regularity :*

$$\mathbf{u} \in L^2(0,T;V(\Omega)) \cap W^{1,\frac{4}{3}}(0,T;V(\Omega)')$$

and

$$\varphi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)).$$

Our second main result says : if the initial data and the volume of the vesicle are small in certain sense, then we have almost global in time solution :

Theorem 2.1.4. *(Almost global existence of strong solutions) Given $T > 0$, there exists $q = q(T) > 0$ such that if*

$$(\mu(\Omega) + \alpha)^2 + \|\mathbf{u}_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^2_{H^{2+\frac{3}{8}}(\Omega)} \leqslant q, \tag{2.1.7}$$

then the solutions in Theorem 2.1.1 exist in the interval $[0,T]$.

Remark 2.1.5. *We do not obtain a result of global in time existence. The main result for this comes from the definition of $W(\varphi)$ and more precisely from the term $M_1\,[V(\varphi) - \alpha]$. Indeed, when dealing with the a priori estimates (Section 2.3), this term gives some constants in the right-hand side which can not be absorbed by the viscous term as for the classical Navier–Stokes equations (see* (2.3.10) *for details).*

2.2 Notation and preliminaries

We first recall for later use some interpolation inequalities involving Sobolev spaces, see [47, pp.22-23].

Proposition 2.2.1. *Assume $s_1 > 0$, $s_2 > 0$ and set $s = (1-\vartheta)s_1 + \vartheta s_2$ for $\vartheta \in (0,1)$. Then there exists C=$C(\vartheta, s_1, s_2, \Omega)$ such that, $\forall\ u \in H^{s_1}(\Omega) \cap H^{s_2}(\Omega)$, we have*

$$\|u\|_{H^s(\Omega)} \leqslant C\|u\|^{1-\vartheta}_{H^{s_1}(\Omega)}\|u\|^{\vartheta}_{H^{s_2}(\Omega)}.$$

Remark 2.2.2. *In particular for $s_1 = 4 + \frac{3}{8}$, $s_2 = 2 + \frac{1}{4}$ and $\vartheta = \frac{16}{17}$, we have :*

$$\|\phi\|_{H^4(\Omega)} \leqslant \|\phi\|_{H^{4+\frac{1}{4}}(\Omega)} \leqslant C\|\phi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}(\Omega)}\|\phi\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}(\Omega)}. \tag{2.2.1}$$

For $s_1 = 2$, $s_2 = 1$ and $\vartheta = \frac{3}{4}$, we have :

$$\|u\|_{H^{1+\frac{3}{4}}(\Omega)} \leqslant C\|u\|^{\frac{3}{4}}_{H^2(\Omega)}\|u\|^{\frac{1}{4}}_{H^1(\Omega)} \leqslant C\|u\|^{\frac{16}{17}}_{H^2(\Omega)}\|u\|^{\frac{1}{17}}_{H^1(\Omega)}. \tag{2.2.2}$$

The following proposition will be used to estimate the nonlinear terms.

Proposition 2.2.3. *(Banach Algebra) If $s > \frac{3}{2}$, then $H^s(\Omega)$ is a Banach Algebra. More precisely, there exists a positive constant $C = C(\Omega)$ such that for all $u \in H^s(\Omega)$ and $v \in H^s(\Omega)$, we have $uv \in H^s(\Omega)$ and*

$$\|uv\|_{H^s(\Omega)} \leqslant C\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}.$$

For the proof of the above proposition, we refer to [47, pp.45-46]. Let $\mathcal{A}$ be the Dirichlet Laplacian in Ω, that is :

$$\mathcal{A} : D(\mathcal{A}) = H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega), \quad u \mapsto -\Delta u. \tag{2.2.3}$$

It is well-known that $\mathcal{A}$ is a positive self-adjoint operator and we set

$$\tilde{H}^s(\Omega) = D(\mathcal{A}^{\frac{s}{2}}).$$

Using classical results on $\mathcal{A}$ and on interpolation theory (see, for instance, [5, pp.111–115]), we have the following characterization of $\tilde{H}^s(\Omega)$, for $s-\frac{1}{2} \notin \mathbb{N}$:

$$\tilde{H}^s(\Omega) = \left\{ u \in H^s(\Omega)\ ;\ (-\Delta)^j u = 0 \text{ on } \partial\Omega, \quad \text{for } 0 \leqslant 2j < s - \frac{1}{2} \right\}. \tag{2.2.4}$$

If $s - \frac{1}{2} \in \mathbb{N}$, the characterization of $\tilde{H}^s(\Omega)$ use the space $L^2_{-\frac{1}{2}}(\Omega)$ of u such that $\operatorname{dist}(x, \partial\Omega)^{-\frac{1}{2}}u \in L^2(\Omega)$. In particular, we use in this paper the spaces

$$\tilde{H}^{\frac{3}{8}}(\Omega) = H^{\frac{3}{8}}(\Omega),$$
$$\tilde{H}^{2+\frac{3}{8}}(\Omega) = H^{2+\frac{3}{8}}(\Omega) \cap H^1_0(\Omega),$$
$$\tilde{H}^{4+\frac{3}{8}}(\Omega) = \left\{ u \in H^{4+\frac{3}{8}}(\Omega)\ ;\ u = \Delta u = 0 \text{ on } \partial\Omega \right\}.$$

Proposition 2.2.4. *The space $\tilde{H}^s(\Omega)$ is a closed subspace of $H^s(\Omega)$. Moreover, if we set*

$$\|\phi\|_{\tilde{H}^s(\Omega)} := \left\|A^{\frac{s}{2}}\phi\right\|_{L^2(\Omega)},$$

then we have the following inequality

$$\|\phi\|_{H^s(\Omega)} \leqslant C\|\phi\|_{\tilde{H}^s(\Omega)} \quad (\phi \in \tilde{H}^s(\Omega)). \tag{2.2.5}$$

For the proof of the above proposition, we refer for instance [47, page 31-33]. The inequality (2.2.5) comes from elliptic regularity of $\mathcal{A}$ and from classical interpolation theorems.

Remark 2.2.5. *If there is no confusion, we simply denote $H^s(\Omega)$ by H^s and $\tilde{H}^s(\Omega)$ by $\tilde{H}^s$. We also denote $L^2(0,T;H^s(\Omega))$ and $C([0,T];H^s(\Omega))$ by $L^2(H^s)$ and $C(H^s)$, respectively.*

We also recall some classical notations from [14]. Define

$$\mathscr{V}(\Omega) = \left\{\mathbf{u} \in (C_0^\infty(\Omega))^3 \ \mid \nabla\cdot\mathbf{u} = 0\right\}.$$

Let $H(\Omega)$ be the closure of $\mathscr{V}(\Omega)$ in $[L^2(\Omega)]^3$. We denote by $V(\Omega)$ the closure of $\mathscr{V}(\Omega)$ in $[H_0^1(\Omega)]^3$. We also denote by P the orthogonal projector from $[L^2(\Omega)]^3$ onto $H(\Omega)$ (the Leray projector). Consider the Stokes operator $A : D(A) \to H(\Omega)$, where $D(A) = V(\Omega)\cap H^2(\Omega)$, $A\mathbf{u} = -P\Delta\mathbf{u}$, $\forall\mathbf{u} \in D(A)$. If we apply the projector P to the first equation in (2.1.1) and if we set $\phi = \varphi + 1$, then we can rewrite (2.1.1) in the following form :

$$\begin{cases} \mathbf{u}_t + \nu A\mathbf{u} = -P(\mathbf{u}\cdot\nabla\mathbf{u}) + P(W(\phi-1)\nabla\phi) & \text{in } [0,T]\times\Omega, \\ \phi_t + k\gamma\epsilon\mathcal{A}^2\phi = -\mathbf{u}\cdot\nabla\phi - \gamma L(\phi-1) & \text{in } [0,T]\times\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega, \\ \phi(0,x) = \phi_0(x) & \text{in } \Omega, \\ \mathbf{u}(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \\ \phi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega, \\ \Delta\phi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega. \end{cases} \tag{2.2.6}$$

We write for later use $W(\phi) = k\epsilon\Delta^2\phi + L(\phi)$, with $L(\phi)$ given by :

$$\begin{aligned} L(\phi) = \quad & -\frac{k}{\epsilon}\Delta(\phi^3) + \frac{k}{\epsilon}\Delta\phi + \frac{k}{\epsilon^2}(3\phi^2-1)f(\phi) \\ & + M_1[V(\phi)-\alpha] + M_2[B(\phi)-\beta]f(\phi) \\ = \quad & J_1(\phi) + J_2(\phi) + J_3(\phi) + J_4(\phi) + J_5(\phi). \end{aligned} \tag{2.2.7}$$

where $J_1(\phi) = -\frac{k}{\epsilon}\Delta(\phi^3)$, $J_2(\phi) = \frac{k}{\epsilon}\Delta\phi$, $J_3(\phi) = \frac{k}{\epsilon^2}(3\phi^2-1)f(\phi)$, $J_4(\phi) = M_1[V(\phi)-\alpha]$ and $J_5(\phi) = M_2[B(\phi)-\beta]f(\phi)$.

We next recall some tools of functional analysis and semigroup theory. For the proof, see Lemma 3.3 and Theorem 3.1 of [5]. Denote by E a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $A_0 : D(A_0) \to E$ be a strictly positive operator (this means that A_0 is self-adjoint and $(A_0\phi,\phi) \geqslant c\|\phi\|^2$ for some $c>0$, $\forall \phi \in D(A_0)$).

Proposition 2.2.6. *With the above notation, for every* $f \in L^2(0,T;E)$, $u_0 \in D(A_0^{\frac{1}{2}})$ *, the abstract Cauchy problem :*

$$\begin{cases} u_t + A_0 u = f, \\ u(0) = u_0. \end{cases} \tag{2.2.8}$$

has a unique solution $u \in L^2(0,T;D(A_0)) \cap C([0,T];D(A_0^{\frac{1}{2}}) \cap H^1(0,T;E)$*. Moreover, there exists a constant* $K>0$*, such that :*

$$\begin{aligned} &\|u\|^2_{L^2(0,T;D(A_0))} + \|u\|^2_{C([0,T];D(A_0^{\frac{1}{2}}))} + \|u\|^2_{H^1(0,T;E)} \\ \leqslant\ & K\left(\|u_0\|^2_{D(A_0^{\frac{1}{2}})} + \|f\|^2_{L^2(0,T;E)}\right). \end{aligned}$$

As a consequence, we have :

Corollary 2.2.7. *Suppose* $\mathbf{u}_0 \in V(\Omega)$ *and* $f \in L^2(0,T;H(\Omega))$ *and* A *is the Stokes operator defined in Section 1. Then the Cauchy problem*

$$\begin{cases} \mathbf{u}_t + \nu A\mathbf{u} = \mathbf{f}, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.2.9}$$

has a unique solution $\mathbf{u} \in L^2(0,T;D(A)) \cap C([0,T];V(\Omega)) \cap H^1(0,T;H(\Omega))$*. Moreover, there exists* $K>0$ *such that :*

$$\begin{aligned} &\|\mathbf{u}\|^2_{L^2(0,T;H^2)} + \|\mathbf{u}\|^2_{C([0,T];H^1)} + \|\mathbf{u}\|^2_{H^1(0,T;H(\Omega))} \\ \leqslant\ & K(\|\mathbf{u}_0\|^2_{H^1} + \|\mathbf{f}\|^2_{L^2(0,T;L^2)}). \end{aligned} \tag{2.2.10}$$

Proof. It suffice to notice that the operator $A : D(A) \to H$ is self-adjoint and strictly positive (see [14, pp.31-34] for the proof) and to apply the above proposition. □

Corollary 2.2.8. *Suppose* $\phi_0 \in \tilde{H}^{2+\frac{3}{8}}(\Omega)$ *and* $g \in L^2(0,T;H^{\frac{3}{8}}(\Omega))$*, then the Cauchy problem*

$$\begin{cases} \phi_t + k\gamma\epsilon\mathcal{A}^2\phi = g, \\ \phi(0,x) = \phi_0(x). \end{cases} \tag{2.2.11}$$

has a unique solution

$$\phi \in L^2(0,T;\tilde{H}^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];\tilde{H}^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;\tilde{H}^{\frac{3}{8}}(\Omega))$$

with the following estimate :

$$\begin{aligned} &\|\phi\|^2_{L^2(0,T;H^{4+\frac{3}{8}})} + \|\phi\|^2_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|^2_{H^1(0,T;H^{\frac{3}{8}})} \\ \leqslant\ & K\left(\|\phi_0\|^2_{\tilde{H}^{2+\frac{3}{8}}} + \|g\|^2_{L^2(0,T;H^{\frac{3}{8}})}\right). \end{aligned} \tag{2.2.12}$$

Proof. First let us recall that $H^{\frac{3}{8}}(\Omega) = \tilde{H}^{\frac{3}{8}}(\Omega)$ (Proposition 2.2.4). We also recall that $\mathcal{A}$ is the Dirichlet Laplacian defined by (2.2.3). By the definition of $\tilde{H}^s(\Omega)$, we have :

$$\mathcal{A}^2 : \quad D(\mathcal{A}^{2+\frac{3}{16}}) = \tilde{H}^{4+\frac{3}{8}} \to D(\mathcal{A}^{\frac{3}{16}}) = \tilde{H}^{\frac{3}{8}}.$$

Applying Proposition 2.2.6 with $A_0 = \mathcal{A}^2$ and $E = H^{\frac{3}{8}}(\Omega)$, we have :

$$\begin{aligned} &\|\phi\|^2_{L^2(0,T;D(\mathcal{A}^{2+\frac{3}{16}}))} + \|\phi\|^2_{C([0,T];D(\mathcal{A}^{1+\frac{3}{16}}))} + \|\phi\|^2_{H^1(0,T;D(\mathcal{A}^{\frac{3}{16}}))} \\ \leqslant\ & K\left(\|\phi_0\|^2_{D(\mathcal{A}^{1+\frac{3}{16}})} + \|g\|^2_{L^2(0,T;D(\mathcal{A}^{\frac{3}{16}}))}\right). \end{aligned} \tag{2.2.13}$$

By Proposition 2.2.4, the above inequality implies

$$\begin{aligned} &\|\phi\|^2_{L^2(0,T;H^{4+\frac{3}{8}})} + \|\phi\|^2_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|^2_{H^1(0,T;H^{\frac{3}{8}})} \\ \leqslant\ & K\left(\|\phi_0\|^2_{\tilde{H}^{2+\frac{3}{8}}} + \|g\|^2_{L^2(0,T;H^{\frac{3}{8}})}\right). \end{aligned}$$

□

2.3 Proof of the main theorems

We first give some estimates of the nonlinear terms in (2.1.1). These estimates are essential in our fixed point procedure and will be proved in the next sections. Suppose :

$$\mathbf{u} \in L^2(0,T;[H^2(\Omega)]^3) \cap C([0,T];[H^1(\Omega)]^3) \cap H^1(0,T;[L^2(\Omega)]^3), \tag{2.3.1}$$
$$\phi \in L^2(0,T;H^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];H^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \tag{2.3.2}$$

Recall the definition of $W(\phi)$ and $L(\phi)$ from Section 1. We have the following two theorems.

Theorem 2.3.1. *Suppose $\psi \in H^{4+\frac{3}{8}}(\Omega)$ and $\mathbf{v} \in H^2(\Omega)$, then there exists a constant $C = C(\Omega, \epsilon, k, M_1, M_2, \beta, \gamma)$ such that :*

$$\begin{aligned} \|W(\psi-1)\nabla\psi\|_{L^2} \leqslant C\Big[&\|\psi\|_{H^{2+\frac{3}{8}}}(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^4\|\psi\|_{H^{4+\frac{3}{8}}} \\ &+(\|\psi\|_{H^{2+\frac{3}{8}}} + |\alpha+\mu(\Omega)|)^7\|\psi\|_{H^{2+\frac{3}{8}}}\Big], \end{aligned} \tag{2.3.3a}$$

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2} \leqslant C\|\mathbf{v}\|_{H^1}\|\mathbf{v}\|_{H^2}, \tag{2.3.3b}$$

$$\|\mathbf{v}\cdot\nabla\psi\|_{H^{\frac{3}{8}}} \leqslant C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \tag{2.3.3c}$$

$$\|L(\psi-1)\|_{H^{\frac{3}{8}}} \leqslant C\left[(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^6\|\psi\|_{H^{2+\frac{3}{8}}}+\|\psi\|^2_{H^2}\|\psi\|_{H^4}+|\alpha+\mu(\Omega)|\right]. \tag{2.3.3d}$$

Theorem 2.3.2. *Under the assumptions* (2.3.1), (2.3.2) *and* $T\leqslant 1$*, there exists a positive constant* $C=C(\Omega,\epsilon,k,M_1,M_2,\beta,\gamma)$ *such that :*

$$\|W(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant C\left(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}+\|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}+1\right)^8, \tag{2.3.4a}$$

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant C\left(\|\mathbf{u}\|_{C([0,T];H^1)}+\|\mathbf{u}\|_{L^2(0,T;H^2)}\right)^2, \tag{2.3.4b}$$

$$\|\mathbf{u}\cdot\nabla\phi\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \leqslant C\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}\left(\|\mathbf{u}\|_{C([0,T];H^1)}+\|\mathbf{u}\|_{L^2(0,T;H^2)}\right), \tag{2.3.4c}$$

$$\|L(\phi-1)\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \leqslant C\left(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}+\|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}+1\right)^7. \tag{2.3.4d}$$

We also need the estimates of differences of the nonlinear terms. Let $T\leqslant 1$ and let $R>1$ be a constant, which will be made precise later. We assume that

$$\begin{cases} \mathbf{u}_i \in L^2(0,T;H^2)\cap C([0,T];H^1)\cap H^1(0,T;L^2), \\ \phi_i \in L^2(0,T;H^{4+\frac{3}{8}})\cap C([0,T];H^{2+\frac{3}{8}})\cap H^1(0,T;H^{\frac{3}{8}}). \end{cases} \tag{2.3.5}$$

$$\begin{cases} \max\left\{\|\mathbf{u}_i\|_{L^2(0,T;H^2)},\|\mathbf{u}_i\|_{C([0,T];H^1)},\|\mathbf{u}_i\|_{H^1(0,T;L^2)}\right\} \leqslant CR, \\ \max\left\{||\phi_i||_{L^2(0,T;H^{4+\frac{3}{8}})},||\phi_i||_{C([0,T];H^{2+\frac{3}{8}})},||\phi_i||_{H^1(0,T;H^{\frac{3}{8}})}\right\} \leqslant CR, \end{cases} \tag{2.3.6}$$

where C is a positive constant.

Set $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$ and $\phi=\phi_1-\phi_2$. We have the following estimates :

Theorem 2.3.3. *There exists a constant* $C=C(\epsilon,k,\Omega,M_1,M_2,\beta,\gamma)$ *such* *that :*

$$\|L(\phi_1)-L(\phi_2)\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \leqslant CR^6\left(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}+\|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}\right), \tag{2.3.7a}$$

$$\|L(\phi_1)\nabla\phi_1-L(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant CR^7\left(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}+\|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}\right), \tag{2.3.7b}$$

$$\|\Delta^2\phi_1\nabla\phi_1-\Delta^2\phi_2\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant CR\left(\|\phi\|_{C([0,T];H^{2+\frac{1}{4}})}+\|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}\right), \tag{2.3.7c}$$

$$\|\mathbf{u}_1\cdot\nabla\mathbf{u}_1-\mathbf{u}_2\cdot\nabla\mathbf{u}_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant CR\left(\|\mathbf{u}\|_{L^2(0,T;H^2)}+\|\mathbf{u}\|_{C([0,T];H^1)}\right), \tag{2.3.7d}$$

$$\|\mathbf{u}_1\cdot\nabla\phi_1-\mathbf{u}_2\cdot\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \leqslant CR\left(\|\mathbf{u}\|_{C([0,T];H^1)}+\|\mathbf{u}\|_{L^2(0,T;H^2)}+\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}\right). \tag{2.3.7e}$$

Note that we have the same estimate for $L(\phi_1-1)-L(\phi_2-1)$ as for $L(\phi_1)-L(\phi_2)$ by applying (2.3.7a) to ϕ_1-1 and ϕ_2-1. Since the proof of the above three theorems are technical, they are postponed at Section 4 and Section 5. We use them to prove here our main results.

Proof of Theorem 2.1.1 : Without loss of generality, we can assume that $T\leqslant 1$. By the classical results on the Leray projector (see for instance Chapter 5 of [14]), we only need to prove that (2.2.6) has a unique strong solution :

$$\begin{cases} \mathbf{u}\in L^2(0,T;H^2(\Omega))\cap C([0,T];V(\Omega))\cap H^1(0,T;H(\Omega)), \\ \phi\in L^2(0,T;\tilde{H}^{4+\frac{3}{8}}(\Omega))\cap C([0,T];\tilde{H}^{2+\frac{3}{8}}(\Omega))\cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \end{cases}$$

Consider the Hilbert space

$$E=\left[L^2(\Omega)\right]^3\times H^{\frac{3}{8}}(\Omega),$$

and

$$B(0,R)=\left\{v=(\mathbf{u},\phi)\in L^2(0,T;E)\middle|\ \|v\|_{L^2(0,T;E)}\leqslant R\right\}.$$

We define the following nonlinear map from B(0,R) to $L^2(0,T;E)$:

$$F:\begin{pmatrix}\mathbf{f}\\ g\end{pmatrix}\longmapsto\begin{pmatrix}-P(\mathbf{u}\cdot\nabla\mathbf{u})+P(W(\phi-1)\nabla\phi)\\ -\mathbf{u}\cdot\nabla\phi-\gamma L(\phi-1)\end{pmatrix},$$

where $\mathbf{u}$ and ϕ satisfy (2.2.9) and (2.2.11) respectively. The next proposition states that F is well-defined and that B(0,R) is invariant through F for T small enough.

Proposition 2.3.4. *There exists a positive constant $C = C(\Omega, k, \epsilon, \beta, \gamma, M_1, M_2)$ such that, if we choose $R \geqslant 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$ and $T \leqslant CR^{-238}$, then F maps B(0,R) into itself.*

Proof. By using the estimates (2.3.4), (2.2.10) and (2.2.12) (since $\mathbf{u}$ and ϕ satisfy (2.2.9) and (2.2.11)) we have :

$$\begin{aligned}
& \|W(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^{\frac{17}{8}}(L^2)} \\
& +\|\mathbf{u}\cdot\nabla\phi\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} + \|L(\phi-1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \\
\leqslant\ & C(\|\mathbf{u}\|_{L^2(H^2)} + \|\mathbf{u}\|_{C(H^1)} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})} + 1)^8 \\
\leqslant\ & C(\|\mathbf{f}\|_{L^2(L^2)} + \|g\|_{L^2(H^{\frac{3}{8}})} + \|\phi_0\|_{H^{2+\frac{3}{8}}} + \|\mathbf{u}_0\|_{H^1} + 1)^8.
\end{aligned}$$

If we choose $R \geqslant 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$, then we have :

$$\|F(\mathbf{f}, g)\|_{L^{\frac{17}{8}}(0,T;E)} \leqslant CR^8.$$

Applying Hölder inequality, we obtain :

$$\|F(\mathbf{f}, g)\|_{L^2(0,T;E)} \leqslant T^{\frac{1}{34}}\|F(\mathbf{f}, g)\|_{L^{\frac{17}{8}}(0,T;E)} \leqslant T^{\frac{1}{34}}CR^8.$$

Thus, if we choose $T \leqslant CR^{-238}$, then F actually maps B(0,R) into itself. □

Proposition 2.3.5. *Assume that $R \geqslant 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$. There exists a positive constant $C = C(\Omega, k, \epsilon, \beta, \gamma, M_1, M_2)$ such that, if we choose $T \leqslant CR^{-238}$, then $F_{|B(0,R)}$ is a contraction.*

Proof. Assume ($\mathbf{f}_1$,g_1), ($\mathbf{f}_2$, g_2)$\in L^2(0, T; E)$, and consider the solution ($\mathbf{u}_1$,ϕ_1) and ($\mathbf{u}_2$,ϕ_2) of (2.2.9) and (2.2.11) associated with ($\mathbf{f}_1$,g_1) and ($\mathbf{f}_2$, g_2). By using (2.2.10), (2.2.12) and that $R \geqslant 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$, we deduce that (2.3.6) holds true.

We also have that the difference $\phi = \phi_1 - \phi_2$ and $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ satisfy :

$$\begin{cases}
\mathbf{u}_t + \nu A\mathbf{u} = \mathbf{f} \\
\phi_t + k\gamma\epsilon\mathcal{A}^2\phi = g \\
\mathbf{u}(0, x) = 0 \\
\phi(0, x) = 0
\end{cases}$$

where $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$ and $g = g_1 - g_2$. Using the estimates (2.2.10) and (2.2.12), we obtain,

$$\begin{gathered}
\|\mathbf{u}\|_{L^2(H^2)} + \|\mathbf{u}\|_{C(H^1)} \leqslant C\|\mathbf{f}\|_{L^2(L^2)}, \\
\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{3}{8}})} \leqslant C\|g\|_{L^2(H^{\frac{3}{8}})}.
\end{gathered}$$

Applying (2.3.7) to $(\phi_1 - 1, \phi_2 - 1)$, we obtain :

$$\begin{aligned}
& \|F(\mathbf{f}_1, g_1) - F(\mathbf{f}_2, g_2)\|_{L^{\frac{17}{8}}(E)} \\
\leqslant\ & \|\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2\|_{L^{\frac{17}{8}}(L^2)} + \|W(\phi_1 - 1)\nabla\phi_1 - W(\phi_2 - 1)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \\
& + \|\mathbf{u}_1 \cdot \nabla\phi_1 - \mathbf{u}_2 \cdot \nabla\phi_2\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} + \|L(\phi_1 - 1) - L(\phi_2 - 1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \\
\leqslant\ & CR^7 \left(\|\mathbf{u}\|_{L^2(H^2)} + \|\mathbf{u}\|_{C(H^1)} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})} \right) \\
\leqslant\ & CR^7 \left(\|\mathbf{f}\|_{L^2(L^2)} + \|g\|_{L^2(H^{\frac{3}{8}})} \right).
\end{aligned}$$

We deduce from the above estimates that :

$$\begin{aligned}
& \|F(\mathbf{f}_1, g_1) - F(\mathbf{f}_2, g_2)\|_{L^2(0,T;E)} \\
\leqslant\ & T^{\frac{1}{34}} \|F(\mathbf{f}_1, g_1) - F(\mathbf{f}_2, g_2)\|_{L^{\frac{17}{8}}(0,T;E)} \\
\leqslant\ & T^{\frac{1}{34}} CR^7 (\|\mathbf{f}\|_{L^2(L^2)} + \|g\|_{L^2(H^{\frac{3}{8}})}) \\
=\ & T^{\frac{1}{34}} CR^7 \|(\mathbf{f}_1, g_1) - (\mathbf{f}_2, g_2)\|_{L^2(0,T;E)}.
\end{aligned}$$

So if we choose $T \leqslant CR^{-238}$, then F is a contraction.

Using the above proposition, we can apply the Banach fixed point theorem and obtain the local in time existence of strong solution. □

Next we prove Theorem 2.1.4. We first prove the following proposition :

Proposition 2.3.6. *Let* $\mathbf{u}$ *and* ϕ *be the strong solution constructed in Theorem 2.1.1, then there exists a positive constant* $C > 0$ *which does not depend on* T *such that :*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{H^1}^2 + \frac{3}{4}\nu\|A\mathbf{u}(t)\|_{L^2}^2 \leqslant C\|\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\|_{L^2}^2 + C\|W(\phi(t)-1)\nabla\phi(t)\|_{L^2}^2, \tag{2.3.8}$$

$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|_{H^{2+\frac{3}{8}}}^2 + \frac{3}{4}k\gamma\epsilon\|\mathcal{A}^{2+\frac{3}{16}}\phi(t)\|_{L^2}^2 \leqslant C\|\mathbf{u}(t)\cdot\nabla\phi(t)\|_{H^{\frac{3}{8}}}^2 + C\|L(\phi(t)-1)\|_{H^{\frac{3}{8}}}. \tag{2.3.9}$$

Proof. Note that $\mathbf{u}_t \in L^2(0,T;H)$, $A\mathbf{u} \in L^2(0,T;H)$. Taking the inner product in $L^2(\Omega)$ of the first equation in (2.2.6) by $A\mathbf{u}$, we get :

$$< \mathbf{u}_t, A\mathbf{u} > + \nu < A\mathbf{u}, A\mathbf{u} > = - < \mathbf{u}\cdot\nabla\mathbf{u}, A\mathbf{u} > + < W(\phi - 1)\nabla\phi, A\mathbf{u} > .$$

Here we have denoted by $< \cdot, \cdot >$ the standard $L^2(\Omega)$ inner product.

It follows from the above equation that :

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}\mathbf{u}\|_{L^2}^2 + \nu\|A\mathbf{u}\|_{L^2}^2 \\
\leqslant\ & \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\|A\mathbf{u}\|_{L^2} + \|W(\phi-1)\nabla\phi\|_{L^2}\|A\mathbf{u}\|_{L^2} \\
\leqslant\ & \frac{2}{\nu}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \frac{\nu}{8}\|A\mathbf{u}\|_{L^2}^2 + \frac{2}{\nu}\|W(\phi-1)\nabla\phi\|_{L^2}^2 + \frac{\nu}{8}\|A\mathbf{u}\|_{L^2}^2,
\end{aligned}
$$

which implies :

$$
\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}\mathbf{u}\|_{L^2}^2 + \frac{3}{4}\nu\|A\mathbf{u}\|_{L^2}^2 \leqslant \frac{2}{\nu}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \frac{2}{\nu}\|W(\phi-1)\nabla\phi\|_{L^2}^2.
$$

We have proved (2.3.8).

To prove the other inequity, we first recall that $-\mathcal{A}$ is the Dirichlet Laplacian and that there exists a constant C such that

$$
\|\phi\|_{H^{4+\frac{3}{8}}}^2 \leqslant C\|\mathcal{A}^{2+\frac{3}{16}}\phi\|_{L^2}^2.
$$

According to (2.3.2), $\phi_t \in L^2(0,T;H^{\frac{3}{8}})$ and $\mathcal{A}^{2+\frac{3}{8}}\phi \in L^2(0,T;H^{-\frac{3}{8}})$. Taking the duality product between the second equation in (2.2.6) and $\mathcal{A}^{2+\frac{3}{8}}\phi$ yields

$$
\begin{aligned}
&(\phi_t, \mathcal{A}^{2+\frac{3}{8}}\phi) + k\gamma\epsilon(\mathcal{A}^2\phi, \mathcal{A}^{2+\frac{3}{8}}\phi) \\
&\qquad = -(u\cdot\nabla\phi, \mathcal{A}^{2+\frac{3}{8}}\phi) - \gamma(L(\phi-1), \mathcal{A}^{2+\frac{3}{8}}\phi) \\
&\qquad = - < \mathcal{A}^{\frac{3}{16}}(u\cdot\nabla\phi), \mathcal{A}^{2+\frac{3}{16}}\phi > -\gamma < \mathcal{A}^{\frac{3}{16}}(L(\phi-1)), \mathcal{A}^{2+\frac{3}{16}}\phi > \\
&\qquad \leqslant \|\mathbf{u}\cdot\nabla\phi\|_{H^{\frac{3}{8}}}\|\phi\|_{H^{4+\frac{3}{8}}} + \gamma\|L(\phi-1)\|_{H^{\frac{3}{8}}}\|\phi\|_{H^{4+\frac{3}{8}}} \\
&\leqslant \frac{k\gamma\epsilon}{8C}\|\phi\|_{H^{4+\frac{3}{8}}}^2 + \frac{2C}{k\gamma\epsilon}\|\mathbf{u}\cdot\nabla\phi\|_{H^{\frac{3}{8}}}^2 + \frac{k\gamma\epsilon}{8C}\|\phi\|_{H^{4+\frac{3}{8}}} + \frac{2\gamma C}{k\epsilon}\|L(\phi-1)\|_{H^{\frac{3}{8}}}.
\end{aligned}
\tag{2.3.10}
$$

Here we have denoted by $(\cdot,\cdot)$ the pairing of $H^{3/8}(\Omega)$ and $H^{-3/8}(\Omega)$.

Since $\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{1+\frac{3}{16}}\phi\|_{L^2}^2 = (\phi_t, \mathcal{A}^{2+\frac{3}{8}}\phi)$, we deduce from the above inequality that

$$
\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{1+\frac{3}{16}}\phi\|_{L^2}^2 + \frac{3}{4}k\gamma\epsilon\|\mathcal{A}^{2+\frac{3}{16}}\phi\|_{L^2}^2 \leqslant \frac{2C}{k\gamma\epsilon}\|\mathbf{u}\cdot\nabla\phi\|_{H^{\frac{3}{8}}}^2 + \frac{2\gamma C}{k\epsilon}\|L(\phi-1)\|_{H^{\frac{3}{8}}}.
\tag{2.3.11}
$$

As a consequence, (2.3.9) is proved. □

Proof of Theorem 2.1.4. Using Theorem 2.1.1, we can consider a maximal strong solution $(\mathbf{u},\phi)$ of (2.1.1) on the time interval $[0,T_{\max})$. Assume $T>0$. We prove here that if we choose the initial data small enough, then we have $T_{\max} > T$ i.e. the solution exists on [0,T].

Assume by contradiction that $T_{\max} \leqslant T$, Adding (2.3.8) to (2.3.9) and using (2.3.3), we obtain :

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_{H^1}^2 + \|\phi\|_{H^{2+\frac{3}{8}}}^2) + \frac{3}{4}\nu\|\mathbf{u}\|_{H^2}^2 + \frac{3}{4}k\gamma\epsilon\|\phi\|_{H^{4+\frac{3}{8}}}^2 \\
\leqslant\ & C\|\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\|_{L^2}^2 + C\|W(\phi-1)\nabla\phi(t)\|_{L^2}^2 + C\|\mathbf{u}(t)\cdot\nabla\phi(t)\|_{H^{\frac{3}{8}}}^2 + C\|L(\phi-1)\|_{H^{\frac{3}{8}}} \\
\leqslant\ & C\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2 + C\|\phi\|_{H^{2+\frac{3}{8}}}^2(\|\phi\|_{H^{2+\frac{3}{8}}} + 1)^8\|\phi\|_{H^{4+\frac{3}{8}}}^2 \\
& + C(\|\phi\|_{H^{2+\frac{3}{8}}} + |\alpha + \mu(\Omega)|)^{14}\|\phi\|_{H^{2+\frac{3}{8}}}^2 + C\|\mathbf{u}\|_{H^2}^2\|\phi\|_{H^{2+\frac{3}{8}}}^2 \\
& + C(\|\phi\|_{H^{2+\frac{3}{8}}} + 1)^{12}\|\phi\|_{H^{2+\frac{3}{8}}}^2 + C\|\phi\|_{H^2}^4\|\phi\|_{H^4}^2 + C(\alpha+\mu(\Omega))^2 \\
\leqslant\ & C\left(\|\mathbf{u}\|_{H^1}^2 + \|\phi\|_{H^{2+\frac{3}{8}}}^2\right)\|\mathbf{u}\|_{H^2}^2 + C\|\phi\|_{H^{2+\frac{3}{8}}}^2(\|\phi\|_{H^{2+\frac{3}{8}}} + 1)^8\|\phi\|_{H^{4+\frac{3}{8}}}^2 \\
& + C(\|\phi\|_{H^{2+\frac{3}{8}}} + |\alpha + \mu(\Omega)|)^{14}\|\phi\|_{H^{2+\frac{3}{8}}}^2 + C(\alpha+\mu(\Omega))^2. \qquad (2.3.12)
\end{aligned}
$$

Here the constant C does not depend on α and which will be fixed for the end of the proof. We also consider $q > 0$ small enough so that

$$q \leqslant \frac{\nu}{4C}, \quad q(\sqrt{q}+1)^8 \leqslant \frac{k\gamma\epsilon}{4C} \qquad (2.3.13)$$

We recall that E is defined by (2.1.4) and we set $\mathbf{G} = \left\{t \in [0, T_{\max}) | \sup_{[0,t]} E \leqslant q\right\}$. We prove below that if assume

$$E(0) + 2TC\left((\sqrt{q} + |\alpha + \mu(\Omega)|)^{14}\, q + (\alpha + \mu(\Omega))^2\right) < q, \qquad (2.3.14)$$

then $\mathbf{G} = [0, T_{\max})$. To prove this, we use a continuity argument : we verify the following three properties :

1. $0 \in \mathbf{G}$,
2. $\mathbf{G}$ is closed in $[0, T_{\max})$,
3. $\mathbf{G}$ is open in $[0, T_{\max})$.

Conditions 1. and 2. are easy to check so we skip their proof and we only prove Condition 3. Assume $t_1 \in \mathbf{G}$ then from (2.3.13), we deduce that for $t \in [0, t_1]$

$$\begin{cases} C(\|\mathbf{u}(t)\|_{H^1}^2 + \|\phi(t)\|_{H^{2+\frac{3}{8}}}^2) \leqslant \dfrac{\nu}{4}, \\ C\|\phi(t)\|_{H^{2+\frac{3}{8}}}^2(\|\phi(t)\|_{H^{2+\frac{3}{8}}} + 1)^8 \leqslant \frac{k\gamma\epsilon}{4}. \end{cases}$$

With these inequalities, (2.3.12) can be written as :

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^1}^2 + \|\phi(t)\|_{H^{2+\frac{3}{8}}}^2) + \frac{1}{2}\nu\|\mathbf{u}(t)\|_{H^2}^2 + \frac{1}{2}k\gamma\epsilon\|\phi(t)\|_{H^{4+\frac{3}{8}}}^2 \\
\leqslant\ & C(\sqrt{q} + |\alpha + \mu(\Omega)|)^{14} q + C(\alpha + \mu(\Omega))^2. \qquad (2.3.15)
\end{aligned}
$$

Relation (2.3.15) implies that for $t \in [0, t_1]$:

$$\frac{d}{dt}E(t) \leqslant 2C\left((\sqrt{q} + |\alpha + \mu(\Omega)|)^{14}\, q + (\alpha + \mu(\Omega))^2\right)$$

Using (2.3.14), the above relation implies

$$E(t_1) \leqslant E(0) + 2C\left((\sqrt{q} + |\alpha + \mu(\Omega)|)^{14} q + (\alpha + \mu(\Omega))^2\right) T < q.$$

It follows that there exists $\delta > 0$ such that $E(t') \leqslant q$, $\forall t' \in (-\delta + t_1, \delta + t_1)$. Consequently $\mathbf{G}$ is open in $[0, T_{\max})$. Since $[0, T_{\max})$ is a connected set, we have $\mathbf{G} = [0, T_{\max})$. On the other hand, by the definition of $\mathbf{G}$, we have $E(t)$ is bounded on $[0, T_{\max})$, which contradicts Theorem 2.1.1. We thus have $T_{\max} > T$, which proves Theorem 2.1.4. $\square$

Remark 2.3.7. *In the inequality* (2.3.10)*, one can see the obstacle to get the global existence for small data as for the Navier–Stokes equations. In fact, contrary to the Navier–Stokes equations, the estimates of the nonlinear terms*

$$C\left(\|\mathbf{u}\|_{H^1}^2 + \|\phi\|_{H^{2+\frac{3}{8}}}^2\right)\|\mathbf{u}\|_{H^2}^2 + C\|\phi\|_{H^{2+\frac{3}{8}}}^2 (\|\phi\|_{H^{2+\frac{3}{8}}} + 1)^8 \|\phi\|_{H^{4+\frac{3}{8}}}^2$$
$$+ C(\|\phi\|_{H^{2+\frac{3}{8}}} + |\alpha + \mu(\Omega)|)^{14}\|\phi\|_{H^{2+\frac{3}{8}}}^2 + C(\alpha + \mu(\Omega))^2.$$

can not be absorbed by the viscous term $\frac{3}{4}\nu\|\mathbf{u}\|_{H^2}^2 + \frac{3}{4}k\gamma\epsilon\|\phi\|_{H^{4+\frac{3}{8}}}^2$, even for $\|\mathbf{u}\|_{H^1}$ and $\|\phi\|_{H^{2+\frac{3}{8}}}$ small. We can control the two first terms of the right-hand side of (2.3.10) *by using this strategy but the two other terms and in particular $C(\alpha+\mu(\Omega))^2$ have no reason to be smaller than the viscous term.*

2.4 Proof of Theorems 2.3.1 and 2.3.2

In this part, we will detail the proof of the estimates of the nonlinear terms defined by (2.1.2). We use the notation C for any positive constant depending on Ω and physical parameters $\epsilon, \gamma, \beta, k, \Omega, M_1, M_2, \nu$ (but not depending on α).

We first estimate $f(\phi)$, $B(\phi)$ defined by (2.1.3) :

Lemma 2.4.1. *There exists a constant $C = C(\Omega, \epsilon)$ such that :*

$$\max\{B(\psi), B(\psi - 1)\} \leqslant C(\|\psi\|_{H^2}^4 + 1), \tag{2.4.1}$$
$$\max\left\{\|f(\psi)\|_{H^{\frac{3}{8}}}, \|f(\psi-1)\|_{H^{\frac{3}{8}}}\right\} \leqslant C\left(\|\psi\|_{H^{2+\frac{3}{8}}} + \|\psi\|_{H^{2+\frac{3}{8}}}^3\right), \tag{2.4.2}$$

for $\psi \in H^2(\Omega)$. Furthermore, we have

$$\max\{\|f(\psi)\|_{H^2}, \|f(\psi-1)\|_{H^2}\} \leqslant C(\|\psi\|_{H^4} + \|\psi\|_{H^2}^3), \quad \psi \in H^4(\Omega). \tag{2.4.3}$$

Proof. The proof is essentially based on the fact that $H^2(\Omega)$ is a Banach Algebra. Since the proof of the four estimates are similar, we only prove (2.4.2). We have :

$$\|f(\psi)\|_{H^{\frac{3}{8}}} \leqslant \epsilon\|\Delta\psi\|_{H^{\frac{3}{8}}} + \frac{1}{\epsilon}\|\psi^3\|_{H^{\frac{3}{8}}} + \frac{1}{\epsilon}\|\psi\|_{H^{\frac{3}{8}}} \leqslant C\left(\|\psi\|_{H^{2+\frac{3}{8}}} + \|\psi\|_{H^{2+\frac{3}{8}}}^3\right).$$

Since

$$f(\psi-1) = -\epsilon\Delta\psi + \frac{1}{\epsilon}(\psi^3 - 3\psi^2 + 2\psi),$$

we have the same estimate for $f(\psi-1)$. $\square$

We next estimate the term $W(\phi)\nabla\phi = k\epsilon\Delta^2\phi\nabla\phi + L(\phi)\nabla\phi$, where $L(\phi)$ is defined by (2.2.7).

Proposition 2.4.2. *We have :*

$$\begin{aligned}\|W(\psi-1)\nabla\psi\|_{L^2} \leqslant C\|\psi\|_{H^{2+\frac{3}{8}}}(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^4\|\psi\|_{H^{4+\frac{3}{8}}} \\ + C(\|\psi\|_{H^{2+\frac{3}{8}}} + |\alpha+\mu(\Omega)|)^7\|\psi\|_{H^{2+\frac{3}{8}}},\end{aligned} \quad (2.4.4)$$

for every $\psi \in H^{4+\frac{3}{8}}(\Omega)$. Moreover, for every ϕ satisfying (2.3.2) *and $T \leqslant 1$, we have :*

$$\|W(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \leqslant C\left(\|\phi\|_{C(H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + 1\right)^8. \quad (2.4.5)$$

The proof of the above proposition will be split into six lemmas.

Lemma 2.4.3. *We have :*

$$\|\Delta^2\psi\ \nabla\psi\|_{L^2(\Omega)} \leqslant C\|\psi\|_{H^{4+\frac{3}{8}}}\|\psi\|_{H^{2+\frac{3}{8}}},\ \psi \in H^{4+\frac{3}{8}}(\Omega). \quad (2.4.6)$$

Moreover, for $T \leqslant 1$ and ϕ satisfying (2.3.2)*, we have*

$$\|\Delta^2\phi\ \nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \leqslant C\|\phi\|^{\frac{16}{17}}_{L^2(H^{4+\frac{3}{8}})}\|\phi\|^{\frac{18}{17}}_{C(H^{2+\frac{1}{4}})}. \quad (2.4.7)$$

Proof. By Hölder's inequality and Sobolev inequality, we have :

$$\begin{aligned}\|\Delta^2\psi\ \nabla\psi\|_{L^2(\Omega)} \leqslant \|\Delta^2\psi\|_{L^{\frac{12}{5}}}\|\nabla\psi\|_{L^{12}} \leqslant C\|\Delta^2\psi\|_{H^{\frac{1}{4}}}\|\nabla\psi\|_{H^{\frac{5}{4}}} \\ \leqslant C\|\psi\|_{H^{4+\frac{1}{4}}}\|\psi\|_{H^{2+\frac{1}{4}}},\end{aligned} \quad (2.4.8)$$

which implies (2.4.6). Using next (2.2.1), we have :

$$\begin{aligned}\|\Delta^2\psi\ \nabla\psi\|_{L^2(\Omega)} \leqslant C\|\psi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}}\|\psi\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}}\|\psi\|_{H^{2+\frac{1}{4}}} \\ = C\|\psi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}}\|\psi\|^{\frac{18}{17}}_{H^{2+\frac{1}{4}}}.\end{aligned} \quad (2.4.9)$$

Applying next (2.4.9) with $\psi = \phi(t,\cdot)$, we have :

$$\|\Delta^2\phi(t)\ \nabla\phi(t)\|_{L^2(\Omega)} \leqslant C\|\phi(t)\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}}\|\phi(t)\|^{\frac{18}{17}}_{H^{2+\frac{1}{4}}}, t \in [0,T]\ a.e.$$

This implies (2.4.7). $\square$

Recall from (2.2.7) that $J_1(\phi) = -\dfrac{k}{\epsilon}\Delta(\phi^3)$.

Lemma 2.4.4. *We have :*

$$\|J_1(\psi-1)\ \nabla\psi\|_{L^2(\Omega)} \leqslant C\|\psi\|^4_{H^{2+\frac14}} + C\|\psi\|^3_{H^{2+\frac14}},\ \psi \in H^{2+\frac14}(\Omega). \tag{2.4.10}$$

Moreover, for $\phi \in C([0,T]; H^{2+\frac14}(\Omega))$ and $T \leqslant 1$ we have :

$$\|J_1(\phi-1)\ \nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \leqslant C\|\phi\|^4_{C(H^{2+\frac14})} + C\|\phi\|^3_{C(H^{2+\frac14})}. \tag{2.4.11}$$

Proof. Using Hölder's inequality, Sobolev inequality and the fact that $H^{2+\frac14}(\Omega)$ is a Banach Algebra, we have :

$$\begin{aligned}
\|\Delta(\psi-1)^3\ \nabla\psi\|_{L^2} &\leqslant \|\Delta(\psi^3+3\psi^2+3\psi)\|_{L^{\frac{12}{5}}}\|\nabla\psi\|_{L^{12}} \\
&\leqslant C\left(\|\psi\|^3_{H^{2+\frac14}}+1\right)\|\psi\|_{H^{2+\frac14}},
\end{aligned}$$

which proves (2.4.10). Applying it with $\psi = \phi(t,\cdot)$, we have :

$$\|J_1(\phi-1)\ \nabla\phi\|_{L^2} \leqslant C\left(\|\phi\|^3_{H^{2+\frac14}}+1\right)\|\phi\|_{H^{2+\frac14}},\ t\in[0,T]\ a.e.$$

which clearly implies (2.4.11). □

Recall from (2.2.7) that $J_2(\phi) = \dfrac{k}{\epsilon}\Delta\phi$, where $f(\phi)$ is defined by (2.1.3c).

Lemma 2.4.5. *We have*

$$\|J_2(\psi-1)\ \nabla\psi\|_{L^2} \ \leqslant\ C\|\psi\|^2_{H^{2+\frac14}},\ \psi\in H^{2+\frac14}(\Omega). \tag{2.4.12}$$

Moreover, for every $\phi \in C([0,T]; H^{2+\frac14}(\Omega))$, we have

$$\|J_2(\phi-1)\nabla\phi\|_{C(L^2)} \ \leqslant\ C\|\phi\|^2_{C(H^{2+\frac14})}. \tag{2.4.13}$$

Proof. With the same arguments as in Lemma 2.4.4, we have :

$$\|\Delta\psi\ \nabla\psi\|_{L^2} \leqslant \|\Delta\psi\|_{L^{\frac{12}{5}}}\|\nabla\psi\|_{L^{12}} \leqslant C\|\Delta\psi\|_{H^{\frac14}}\|\nabla\psi\|_{H^{\frac54}} \leqslant C\|\psi\|^2_{H^{2+\frac14}},$$

which clearly implies (2.4.12) and (2.4.13). □

Recall from (2.2.7) that $J_3(\phi) = \dfrac{k}{\epsilon^2}(3\phi^2-1)f(\phi)$.

Lemma 2.4.6. *We have :*

$$\|J_3(\psi-1)\nabla\psi\|_{L^2} \leqslant C\left(\|\psi\|^3_{H^{2+\frac38}} + \|\psi\|_{H^{2+\frac38}}\right)^2,\ \psi\in H^{4+\frac38}(\Omega). \tag{2.4.14}$$

Moreover, for ϕ satisfying (2.3.2) *and $T \leqslant 1$, we have*

$$\begin{aligned}
&\|J_3(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \\
\leqslant\ & C\left(\|\phi\|^3_{C([0,T];H^{2+\frac38})} + \|\phi\|_{C([0,T];H^{2+\frac38})}\right)^2.
\end{aligned} \tag{2.4.15}$$

Proof. We only prove (2.4.14) since it easily leads to (2.4.15). By Sobolev inequality, we have :

$$\begin{aligned}\|f(\psi-1)(3(\psi-1)^2-1)\nabla\psi\|_{L^2} &\leqslant \|f(\psi-1)\|_{L^{\frac{8}{3}}}\|\nabla((\psi-1)^3-\psi+1)\|_{L^8}\\ &\leqslant C\|f(\psi-1)\|_{H^{\frac{3}{8}}}\|\psi^3-3\psi^2+2\psi\|_{H^{2+\frac{3}{8}}}.\end{aligned}$$

then (2.4.14) is proved by applying (2.4.2) and the fact that $H^{2+\frac{3}{8}}(\Omega)$ is a Banach algebra.

□

Recall from (2.2.7) that $J_4(\phi)=M_1[V(\phi)-\alpha]$, the following lemma holds :

Lemma 2.4.7. *We have*

$$\|J_4(\psi-1)\nabla\psi\|_{L^2(\Omega)}\leqslant C(\|\psi\|_{L^2}+|\mu(\Omega)+\alpha|)\|\psi\|_{H^1},\ (\psi\in H^1(\Omega)).$$

Moreover, for every ϕ satisfying (2.3.2) *and $T\leqslant 1$, we have*

$$\|J_4(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)}\leqslant C\|\phi\|_{C(H^1)}(\|\phi\|_{C(L^2)}+|\mu(\Omega)+\alpha|).$$

The proof of this lemma is similar as the lemmas above, so we skip them.
Recall from (2.2.7) that $J_5(\psi)=M_2\,(B(\psi)-\beta)\,f(\psi)$.

Lemma 2.4.8. *We have*

$$\|J_5(\psi-1)\nabla\psi\|_{L^2}\leqslant C\left(\|\psi\|^8_{H^{2+\frac{3}{8}}}+\|\psi\|^2_{H^{2+\frac{3}{8}}}\right),\ \psi\in H^{2+\frac{3}{8}}(\Omega).\quad(2.4.16)$$

Moreover, for ϕ satisfying (2.3.2) *and $T\leqslant 1$, we have*

$$\|J_5(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)}\leqslant C\left(\|\psi\|^8_{C([0,T];H^{2+\frac{3}{8}})}+\|\psi\|^2_{C([0,T];H^{2+\frac{3}{8}})}\right)\quad(2.4.17)$$

Proof. Since (2.4.17) can be easily derived from (2.4.16), we only prove the later. By Hölder's inequality and Sobolev inequality :

$$\begin{aligned}\|(B(\psi-1)-\beta)f(\psi-1)\cdot\nabla\psi\|_{L^2} &\leqslant |B(\psi-1)-\beta|\|f(\psi-1)\|_{L^{\frac{8}{3}}}\|\nabla\psi\|_{L^8}\\ &\leqslant |B(\psi-1)-\beta|\|f(\psi-1)\|_{H^{\frac{3}{8}}}\|\nabla\psi\|_{H^{\frac{9}{8}}}.\end{aligned}$$

By (2.4.1) and (2.4.2), we have :

$$\|J_5(\psi-1)\nabla\psi\|_{L^2}\leqslant C\left(\|\psi\|^4_{H^2}+1\right)\left(\|\psi\|_{H^{2+\frac{3}{8}}}+\|\psi\|^3_{H^{2+\frac{3}{8}}}\right)\|\psi\|_{H^{2+\frac{3}{8}}},$$

which implies (2.4.16) and (2.4.17). □

We next recall some classical estimates.

Proposition 2.4.9. *We have*

$$\|\mathbf{v}_1 \cdot \nabla \mathbf{v}_2\|_{L^2} \leqslant C\|\mathbf{v}_1\|_{H^1}\|\mathbf{v}_2\|_{H^2}, \ (\mathbf{v}_1, \mathbf{v}_2 \in [H^2(\Omega)]^3). \quad (2.4.18)$$

Moreover, for every $\mathbf{u}_1, \mathbf{u}_2$ *satisfying* (2.3.1)*, we have*

$$\|\mathbf{u}_1 \cdot \nabla \mathbf{u}_2\|_{L^{\frac{17}{8}}(L^2)} \leqslant C\|\mathbf{u}_1\|_{C(H^1)}\left(\|\mathbf{u}_1\|_{C(H^1)} + \|\mathbf{u}_2\|_{L^2(H^2)}\right) (2.4.19)$$

Proof. By Hölder's inequality, Sobolev inequality and (2.2.2), we have

$$\begin{aligned}\|\mathbf{v}_1 \cdot \nabla \mathbf{v}_2\|_{L^2} \leqslant \|\mathbf{v}_1\|_{L^4}\|\nabla \mathbf{v}_2\|_{L^4} &\leqslant C\|\mathbf{v}_1\|_{H^1}\|\mathbf{v}_2\|_{H^{1+\frac{3}{4}}} \\ &\leqslant C\|\mathbf{v}_1\|_{H^1}\|\mathbf{v}_2\|_{H^1}^{\frac{1}{17}}\|\mathbf{v}_2\|_{H^2}^{\frac{16}{17}},\end{aligned}$$

which implies (2.4.18). Estimate (2.4.19) is proved by applying the above estimate with $\mathbf{v}_1 = \mathbf{u}_1(t,\cdot)$ and $\mathbf{v}_2 = \mathbf{u}_2(t,\cdot)$. □

We next estimate the term $\mathbf{u} \cdot \nabla \phi$.

Proposition 2.4.10. *We have*

$$\|\mathbf{v} \cdot \nabla \psi\|_{H^{\frac{3}{8}}} \leqslant C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \ \mathbf{v} \in [H^2(\Omega)]^3, \ \psi \in H^{2+\frac{3}{8}}(\Omega). (2.4.20)$$

Moreover, for $\mathbf{u}$*,* ϕ *satisfying* (2.3.1) *and* (2.3.2) *respectively, we have*

$$\|\mathbf{u} \cdot \nabla \phi\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \leqslant C\|\phi\|_{C(H^{2+\frac{3}{8}})}\left(\|\mathbf{u}\|_{C(H^1)} + \|\mathbf{u}\|_{L^2(H^2)}\right). \quad (2.4.21)$$

Proof. Denote $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$. By Hölder's inequality, Sobolev inequality and (2.2.2), we have for $i, j, k \in \{1, 2, 3\}$,

$$\|v_k \partial_i \partial_j \psi\|_{L^2} \leqslant \|v_k\|_{L^8}\|\partial_i \partial_j \psi\|_{L^{\frac{8}{3}}} \leqslant C\|\mathbf{v}\|_{H^{\frac{9}{8}}}\|\psi\|_{H^{2+\frac{3}{8}}}. \quad (2.4.22)$$

$$\|\partial_i v_j \partial_k \psi\|_{L^2} \leqslant \|\partial_i v_j\|_{L^{\frac{12}{5}}}\|\partial_k \psi\|_{L^{12}} \leqslant C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{1}{4}}}. \quad (2.4.23)$$

We also have

$$\|v_i \partial_j \psi\|_{L^2} \leqslant \|v_i\|_{L^4}\|\partial_j \psi\|_{L^4} \leqslant C\|\mathbf{v}\|_{H^1}\|\psi\|_{H^2}.$$

By the Leibnitz formula, $\partial_k(v_i \partial_j \psi) = \partial_k v_i \partial_j \psi + v_i \partial_k \partial_j \psi$. With the above estimates, we have :

$$\|v_k \partial_j \psi\|_{H^1} \leqslant \|v_k \partial_j \psi\|_{L^2} + \sum_{i=1}^{3} \|\partial_i(v_k \partial_j \psi)\|_{L^2} \leqslant C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{3}{8}}},$$

which clearly implies (2.4.20). Applying (2.2.2) to (2.4.22) and (2.4.23), we obtain

$$\begin{aligned}\|v_k \partial_i \partial_j \psi\|_{L^2} &\leqslant C\|\mathbf{v}\|_{H^2}^{\frac{16}{17}}\|\mathbf{v}\|_{H^1}^{\frac{1}{17}}\|\psi\|_{H^{2+\frac{3}{8}}}, \\ \|\partial_i v_j \partial_k \psi\|_{L^2} &\leqslant C\|\mathbf{v}\|_{H^2}^{\frac{16}{17}}\|\mathbf{v}\|_{H^1}^{\frac{1}{17}}\|\psi\|_{H^{2+\frac{1}{4}}},\end{aligned}$$

Applying to $\mathbf{v} = \mathbf{u}(t, \cdot)$ and $\psi = \phi(t, \cdot)$ we have

$$\begin{aligned}
\|u_i \partial_j \partial_k \phi\|_{L^{\frac{17}{8}}(L^2)} &\leqslant C\|\mathbf{u}\|_{C(H^1)}^{\frac{1}{17}} \|\mathbf{u}\|_{L^2(H^2)}^{\frac{16}{17}} \|\phi\|_{C(H^{2+\frac{3}{8}})}, \\
\|\partial_i u_j \partial_k \phi\|_{L^{\frac{17}{8}}(L^2)} &\leqslant C\|\mathbf{u}\|_{C(H^1)}^{\frac{1}{17}} \|\mathbf{u}\|_{L^2(H^2)}^{\frac{16}{17}} \|\phi\|_{C(H^{2+\frac{1}{4}})},
\end{aligned}$$

which implies (2.4.21). □

We next estimate the term $L(\phi)$ defined by (2.2.7).

Proposition 2.4.11. *We have*

$$\begin{aligned}
&\|L(\psi - 1)\|_{H^{\frac{3}{8}}} \\
\leqslant\ & C(\|\psi\|_{H^{2+\frac{3}{8}}} + 1)^6 \|\psi\|_{H^{2+\frac{3}{8}}} + C\|\psi\|_{H^2}^2 \|\psi\|_{H^4} + C\,|\mu(\Omega) + \alpha|,
\end{aligned} \tag{2.4.24}$$

for $\psi \in H^4(\Omega)$. Moreover, for every ϕ satisfying (2.3.2) *and $T \leqslant 1$, we have*

$$\|L(\phi - 1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \leqslant C\left(\|\phi\|_{C(H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + 1\right)^7. \tag{2.4.25}$$

Proof. By the fact that $H^{2+\frac{3}{8}}(\Omega)$ is a Banach Algebra, we have for $J_1(\psi - 1)$:

$$\|J_1(\psi - 1)\|_{H^{\frac{3}{8}}} = C\|\Delta(\psi - 1)^3\|_{H^{\frac{3}{8}}} \leqslant C\|\psi\|_{H^{2+\frac{3}{8}}}^3 + C\|\psi\|_{H^{2+\frac{3}{8}}}. \tag{2.4.26}$$

For $J_2(\psi - 1)$, we have

$$\|J_2(\psi - 1)\|_{H^{\frac{3}{8}}} \leqslant C\|\psi\|_{H^{2+\frac{3}{8}}}. \tag{2.4.27}$$

By using (2.4.2) and the Banach algebra property of $H^2(\Omega)$, we obtain

$$\begin{aligned}
&\|(3(\psi - 1)^2 - 1) f(\psi - 1)\|_{H^{\frac{3}{8}}} \\
\leqslant\ & 3\|\psi^2 f(\psi - 1)\|_{H^2} + C\|f(\psi - 1)\|_{H^{\frac{3}{8}}} \\
\leqslant\ & 3\|\psi^2\|_{H^2} \|f(\psi - 1)\|_{H^2} + C\|f(\psi - 1)\|_{H^{\frac{3}{8}}} \\
\leqslant\ & C\|\psi\|_{H^2}^2 \left(\|\psi\|_{H^4} + \|\psi\|_{H^2}^3\right) + C\|f(\psi - 1)\|_{H^{\frac{3}{8}}},
\end{aligned}$$

which implies :

$$\|J_3(\psi - 1)\|_{H^{\frac{3}{8}}} \leqslant C\|\psi\|_{H^2}^2 \left(\|\psi\|_{H^4} + \|\psi\|_{H^2}^3\right) + C\|\psi\|_{H^{2+\frac{3}{8}}} + C\|\psi\|_{H^{2+\frac{3}{8}}}^3. \tag{2.4.28}$$

For $J_4(\psi - 1)$, we have

$$\|J_4(\psi - 1)\|_{H^{\frac{3}{8}}} \leqslant C(\|\psi\|_{H^1} + |\alpha + \mu(\Omega)|). \tag{2.4.29}$$

Similarly we have the following :

$$\begin{aligned}\|J_5(\psi-1)\|_{H^{\frac{3}{8}}} \leqslant \quad & C|B(\psi-1)-\beta|\cdot\|f(\psi-1)\|_{H^{\frac{3}{8}}} \\ \leqslant \quad & C\left(\|\psi\|_{H^2}^4+1\right)\left(\|\psi\|_{H^{2+\frac{3}{8}}}+\|\psi\|^3_{H^{2+\frac{3}{8}}}\right) \quad (2.4.30)\end{aligned}$$

Then (2.4.24) is proved by adding inequalities (2.4.26)-(2.4.30). Applying (2.4.26), (2.4.27) and (2.4.29) with $\psi=\phi(t,\cdot)$, we obtain :

$$\|J_1(\phi-1)\|_{C(H^{\frac{3}{8}})} \leqslant C\|\phi\|_{C(H^{2+\frac{3}{8}})}\left(\|\phi\|^2_{C(H^{2+\frac{3}{8}})}+1\right), \tag{2.4.31a}$$

$$\|J_2(\phi-1)\|_{C(H^{\frac{3}{8}})} \leqslant C\|\phi\|_{C(H^{2+\frac{3}{8}})}, \tag{2.4.31b}$$

$$\|J_4(\phi-1)\|_{C(H^{\frac{3}{8}})} \leqslant C(\|\phi\|_{C(H^1)}+|\alpha+\mu(\Omega)|). \tag{2.4.31c}$$

Applying (2.4.28) with $\psi=\phi(t,\cdot)$, we have the following

$$\begin{aligned}& \|J_3(\phi-1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \\ \leqslant \quad & C\|\phi\|^2_{C(H^2)}\left(\|\phi\|^{\frac{16}{17}}_{L^2(H^{4+\frac{3}{8}})}\|\phi\|^{\frac{1}{17}}_{C(H^{2+\frac{1}{4}})}+\|\phi\|^3_{C(H^2)}\right) \\ & +C\|\phi\|_{C(H^{2+\frac{3}{8}})}+C\|\phi\|^3_{C(H^{2+\frac{3}{8}})} \\ \leqslant \quad & C\left(\|\phi\|^2_{C(H^{2+\frac{3}{8}})}+1\right)\left(\|\phi\|^{\frac{16}{17}}_{L^2(H^{4+\frac{3}{8}})}\|\phi\|^{\frac{1}{17}}_{C(H^{2+\frac{1}{4}})}+\|\phi\|^3_{C(H^{2+\frac{3}{8}})}\right)\end{aligned} \tag{2.4.32}$$

By (2.4.30), we have

$$\|J_5(\phi-1)\|_{C(H^{\frac{3}{8}})} \leqslant C\left(\|\psi\|^4_{C(H^2)}+1\right)\left(\|\psi\|_{C(H^{2+\frac{3}{8}})}+\|\psi\|^3_{C(H^{2+\frac{3}{8}})}\right). \tag{2.4.33}$$

Then (2.4.25) is proved by adding (2.4.31), (2.4.32) and (2.4.33) together. □

2.5 Proof of Theorem 2.3.3

In this section, we detail the proof of Theorem 2.3.3. We begin with a lemma. Recall the definitions of $f(\phi)$ and $B(\phi)$ from (2.1.3), we have the following result.

Lemma 2.5.1. *For $\psi_1,\psi_2\in H^4(\Omega)$, we have*

$$\|\Delta(\psi_1^3)-\Delta(\psi_2^3)\|_{H^{\frac{3}{8}}} \leqslant C\|\psi\|^3_{H^{2+\frac{3}{8}}}+C\|\psi_1\|_{H^{2+\frac{3}{8}}}\|\psi_2\|_{H^{2+\frac{3}{8}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \tag{2.5.1}$$

$$\|f(\psi_1) \; - \; f(\psi_2)\|_{H^2} \;\; \leqslant \;\; C\left(\|\psi\|_{H^4} + \|\psi\|_{H^2}^2 + \|\psi_1\|_{H^2}\|\psi_2\|_{H^2}\|\psi\|_{H^2}\right), \tag{2.5.2}$$

$$|B(\psi_1) \;\; - \;\; B(\psi_2)| \;\;\; \leqslant \;\;\; C\left(\|\psi_1\|_{H^2}^3 + \|\psi_2\|_{H^2}^3 + 1\right)\|\psi\|_{H^2}. \tag{2.5.3}$$

where $\psi = \psi_1 - \psi_2$.

Proof. we have :

$$\begin{aligned} \|\Delta(\psi_1^3) - \Delta(\psi_2^3)\|_{H^{\frac{3}{8}}} &\leqslant \|\psi_1^3 - \psi_1^3\|_{H^{2+\frac{3}{8}}} \\ &\leqslant \|(\psi_1 - \psi_2)^3\|_{H^{2+\frac{3}{8}}} + 3\|\psi_1\psi_2(\psi_1 - \psi_2)\|_{H^{2+\frac{3}{8}}}. \end{aligned}$$

Then (2.5.1) is proved by the fact that $H^s(\Omega)$ is a Banach Algebra if $s > \frac{3}{2}$. For $f(\psi)$ we have

$$\|f(\psi_1) - f(\psi_2)\|_{H^2} \leqslant C\|\Delta\psi\|_{H^2} + C\|\psi_1^3 - \psi_2^3\|_{H^2},$$

which implies (2.5.2). Estimate (2.5.3) can be proved similarly, so we skip the details. $\square$

We next estimate the leading order term $\Delta^2\phi\nabla\phi$.

Proposition 2.5.2. *Under the condition* (2.3.6)*, there exists a constant $C = C(\Omega)$ such that :*

$$\|\Delta^2\phi_1\nabla\phi_1 - \Delta^2\phi_2\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \leqslant CR\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})}\right).$$

Proof. Since $\Delta^2\phi_1\nabla\phi_1 - \Delta^2\phi_2\nabla\phi_2 = \Delta^2\phi\nabla\phi_1 + \Delta^2\phi_2\nabla\phi$, the lemma is proved by applying (2.4.7). $\square$

Recall the definition of $L(\phi)$ from (2.2.7) We have

Proposition 2.5.3. *Under the assumption* (2.3.6)*, there exists a constant C depending on parameters $\epsilon, k, \Omega, M_1, M_2, \beta$ such that :*

$$\|L(\phi_1)\nabla\phi_1 - L(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant CR^7(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}).$$

Proof.

$$L(\phi_1)\nabla\phi_1 - L(\phi_2)\nabla\phi_2 = \sum_{i=1}^{5}\left[J_i(\phi_1)\nabla\phi_1 - J_i(\phi_2)\nabla\phi_2\right]$$

We will estimate term by term. Recall from (2.2.7) the definition of J_1,

$$\|\Delta(\phi_1^3)\nabla\phi_1 - \Delta(\phi_2^3)\nabla\phi_2\|_{L^2} \leqslant \|[\Delta(\phi_1^3) - \Delta(\phi_2^3)]\nabla\phi_2 + \Delta(\phi_1^3)(\nabla(\phi))\|_{L^2}.$$

By Hölder's inequality and Sobolev inequality,

$$\begin{aligned}
& \|\Delta(\phi_1^3)\nabla\phi_1 - \Delta(\phi_2^3)\nabla\phi_2\|_{L^2} \\
\leqslant \quad & \|(\Delta(\phi_1^3) - \Delta(\phi_2^3))\nabla\phi_2 + \Delta(\phi_1^3)(\nabla(\phi))\|_{L^2} \\
\leqslant \quad & \|\Delta(\phi_1^3) - \Delta(\phi_2^3)\|_{L^{\frac{8}{3}}}\|\nabla\phi_2\|_{L^8} + \|\Delta(\phi_1^3)\|_{L^{\frac{8}{3}}}\|\nabla\phi\|_{L^8} \\
\leqslant \quad & C\|\Delta(\phi_1^3) - \Delta(\phi_2^3)\|_{H^{\frac{3}{8}}}\|\phi_2\|_{H^2} + C\|\phi_1^3\|_{H^{\frac{3}{8}}}\|\phi\|_{H^2}.
\end{aligned}$$

By the Banach algebra property of $H^s(\Omega)$ and (2.5.1),

$$\begin{aligned}
& \|\Delta(\phi_1^3)\nabla\phi_1 - \Delta(\phi_2^3)\nabla\phi_2\|_{L^2} \\
\leqslant \ & C\left(\|\phi\|^3_{H^{2+\frac{3}{8}}} + \|\phi_1\|_{H^{2+\frac{3}{8}}}\|\phi_2\|_{H^{2+\frac{3}{8}}}\|\phi\|_{H^{2+\frac{3}{8}}}\right)\|\phi_2\|_{H^{2+\frac{3}{8}}} \\
& + C\|\phi_1\|^3_{H^{2+\frac{3}{8}}}\|\phi\|_{H^{2+\frac{3}{8}}},
\end{aligned} \tag{2.5.4}$$

which clearly implies

$$\|J_1(\phi_1)\nabla\phi_1 - J_1(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \leqslant CR^3\|\phi\|_{C(H^{2+\frac{3}{8}})}. \tag{2.5.5}$$

Recall from (2.2.7) for the definition of J_3. By Sobolev inequality, (2.4.3) and (2.5.2),

$$\begin{aligned}
& \|(3\phi_1^2 - 1)f(\phi_1)\cdot\nabla\phi_1 - (3\phi_2^2 - 1)f(\phi_2)\cdot\nabla\phi_2\|_{L^2} \\
\leqslant \ & \|f(\phi_1) - f(\phi_2)\|_{H^2}\|\nabla(\phi_1^3 - \phi_1)\|_{L^2} + \|\nabla(\phi_1^3 - \phi_1) - \nabla(\phi_2^3 - \phi_2)\|_{L^2}\|f(\phi_2)\|_{H^2} \\
\leqslant \ & C\left(\|\phi\|_{H^4} + \|\phi\|^2_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2}\right)\left(\|\phi_1\|^3_{H^2} + \|\phi_1\|_{H^2}\right) \\
& + C\left(\|\phi\|^3_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2} + \|\phi\|_{H^2}\right)\left(\|\phi_2\|^3_{H^2} + \|\phi_2\|_{H^4}\right).
\end{aligned}$$

Using (2.2.1) to deal with the term with $H^4(\Omega)$ norm,

$$\begin{aligned}
& \|J_3(\phi_1)\nabla\phi_1 - J_3(\phi_2)\nabla\phi_2\|_{L^2(\Omega)} \\
\leqslant \ & C\left(\|\phi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}(\Omega)}\|\phi\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}(\Omega)} + \|\phi\|^2_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2}\right) \\
& \left(\|\phi_1\|^3_{H^2} + \|\phi_1\|_{H^2}\right) + C\left(\|\phi\|^3_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2} + \|\phi\|_{H^2}\right) \\
& \left(\|\phi_2\|^3_{H^2} + \|\phi_2\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}(\Omega)}\|\phi_2\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}(\Omega)}\right).
\end{aligned} \tag{2.5.6}$$

By (2.3.6) we have :

$$\|J_3(\phi_1)\nabla\phi_1 - J_3(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \leqslant CR^5\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})}\right). \tag{2.5.7}$$

Recall from (2.2.7) for the definition of J_5. Note that

$$\begin{aligned}
& M_2^{-1}[J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2] \\
= \; & [B(\phi_1)-\beta]f(\phi_1)\nabla\phi_1 - [B(\phi_2)-\beta]f(\phi_2)\nabla\phi_2 \\
= \; & [B(\phi_1)-\beta]\{[f(\phi_1)-f(\phi_2)]\nabla\phi_1 + f(\phi_2)\nabla\phi\} + [B(\phi_1)-B(\phi_2)]\, f(\phi_2)\nabla\phi_2.
\end{aligned}$$

From the above formula and the Sobolev inequality, it follows that :

$$\begin{aligned}
& M_2^{-1}\|J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2\|_{L^2(\Omega)} \\
\leqslant \; & |B(\phi_1)-\beta|\,\{\|f(\phi_1)-f(\phi_2)\|_{L^\infty}\|\nabla\phi_1\|_{L^2} + \|f(\phi_2)\|_{L^\infty}\|\nabla\phi\|_{L^2}\} \\
& +|B(\phi_1)-B(\phi_2)|\|f(\phi_2)\|_{L^\infty}\|\nabla\phi_2\|_{L^2} \\
\leqslant \; & |B(\phi_1)-\beta|\,[\|f(\phi_1)-f(\phi_2)\|_{H^2}\|\phi_1\|_{H^1} + \|f(\phi_2)\|_{H^2}\|\phi\|_{H^1}] \\
& +|B(\phi_1)-B(\phi_2)|\|f(\phi_2)\|_{H^2}\|\phi_2\|_{H^1},
\end{aligned}$$

By (2.4.1), (2.4.3), (2.5.2) and (2.5.3) we have :

$$\begin{aligned}
& \|J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2\|_{L^2(\Omega)} \\
\leqslant \; & C(\|\phi_1\|_{H^2}^4+1)C\left(\|\phi\|_{H^4} + \|\phi\|_{H^2}^2 + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2}\right) \\
& +C\left(\|\phi_1\|_{H^2}^3 + \|\phi_2\|_{H^2}^3 + 1\right)\|\phi\|_{H^2}(\|\phi_2\|_{H^2} + \|\phi_2\|_{H^2}^3)\|\phi_2\|_{H^1},
\end{aligned}$$

Using the same steps as those used to derive (2.5.6) we obtain

$$\|J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \leqslant CR^7\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})}\right). \tag{2.5.8}$$

Since the terms J_2 and J_4 are of lower order and thus easier, we don't detail the proofs of the corresponding estimates,

$$\|J_2(\phi_1)\nabla\phi_1 - J_2(\phi_1)\nabla\phi_1\|_{C(H^{2+\frac{3}{8}})} \leqslant CR\|\phi\|_{C(H^{2+\frac{3}{8}})} \tag{2.5.9}$$

$$\|J_4(\phi_1)\nabla\phi_1 - J_4(\phi_1)\nabla\phi_1\|_{C(H^{2+\frac{3}{8}})} \leqslant CR\|\phi\|_{C(H^{2+\frac{3}{8}})} \tag{2.5.10}$$

Then the proposition is proved by combining (2.5.5), (2.5.7), (2.5.8), (2.5.9) and (2.5.10). □

The proof of the following proposition is similar to the previous proposition, so we skip the details.

Proposition 2.5.4. *Under the condition* (2.3.6), *there exists a constant* $C = C(\epsilon, \beta, k, \Omega, M_1, M_2)$ *such that we have the following estimates :*

$$\|L(\phi_1) - L(\phi_2)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \leqslant CR^6\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{3}{8}})}\right).$$

The following proposition can be easily derived from (2.4.19) and (2.4.21).

Proposition 2.5.5. *Under the assumption* (2.3.6), *there exists a constant* $C = C(\Omega)$ *such that*

$$\|\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \leqslant CR(\|\mathbf{u}\|_{L^2(0,T;H^2)} + \|\mathbf{u}\|_{C([0,T];H^1)}). \tag{2.5.11}$$

$$\begin{aligned}&\|\mathbf{u}_1 \cdot \nabla \phi_1 - \mathbf{u}_2 \cdot \nabla \phi_2\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \\ &\qquad \leqslant CR\left(\|\mathbf{u}\|_{C([0,T];H^1)} + \|\mathbf{u}\|_{L^2(0,T;H^2)} + \|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}\right).\end{aligned} \tag{2.5.12}$$

Chapitre 3

Existence de solutions faibles pour un modèle fluide-vésicule

3.1 Introduction

In this work, we study the well-posedness of a phase-field fluid-vesicle model. More precisely, we consider the following nonlinear system written in the domain $(0,T)\times\Omega$:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\otimes\nabla\phi\right), \tag{3.1.1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3.1.1b}$$

$$\phi_t - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi, \tag{3.1.1c}$$

$$\sigma_t + \mathbf{u}\cdot\nabla\sigma = 0, \tag{3.1.1d}$$

with the initial and boundary conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \nabla\cdot\mathbf{u}_0 = 0, \text{ for } x\in\Omega, \tag{3.1.2a}$$

$$\phi(0,x) = \phi_0(x), \quad \sigma(0,x) = \sigma_0(x), \text{ for } x\in\Omega, \tag{3.1.2b}$$

$$\mathbf{u}(t,x) = 0, \quad \phi(t,x) = 0, \quad \text{for } (t,x)\in(0,T)\times\partial\Omega, \tag{3.1.2c}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^3$ and λ, ϵ are positive constants. For $\mathbf{u}$, $\mathbf{v}$ vector fields and $M=(m_{ij})$ a second order tensor field on $\mathbb{R}^3$, we use the standard notation

$$\mathbf{u}\otimes\mathbf{v} = \{u_iv_j\}_{1\leqslant i,j\leqslant 3}, \quad (\nabla\cdot M)_i := \sum_j \frac{\partial m_{ij}}{\partial x_j}, \quad (\nabla v)_{ij} := \frac{\partial v_i}{\partial x_j}. \tag{3.1.3}$$

The functional W is defined by $W(\phi) = ((\phi-1)^2-1)^2$ and W' denotes its derivative which is given by

$$W'(\phi) = 4\phi((\phi-1)^2-1). \tag{3.1.4}$$

The aim of this chapter is to prove the following result

Theorem 3.1.1 (Main result)**.** *Assume that*

$$\mathbf{u}_0\in V^0(\Omega),\ \phi_0\in H_0^1(\Omega),\ \sigma_0\in L^\infty(\Omega). \tag{3.1.5}$$

and $\sigma_0\geqslant 0$ *almost everywhere in* Ω*. Then system* (3.1.1)*,* (3.1.2) *admits a weak solution.*

We recall that the space $V^0(\Omega)$ used in the above theorem is defined by the closure of $\mathscr{V}(\Omega)$ in $[L^2(\Omega)]^3$ where

$$\mathscr{V} = \{\mathbf{v}\in(C_0^\infty(\Omega))^3 \mid \nabla\cdot\mathbf{v} = 0\}.$$

The precise definition of a weak solution for system (3.1.1), (3.1.2) is given in the next section (Section 3.2).

System (3.1.1) is a version of a model which has been introduced by Jamet and Misbah in [36]. Our model can be seen as a regularization of the model proposed in the above mentioned work. The original model corresponds to the one obtained by taking $\epsilon = 0$ in (3.1.1). Our motivation to introduce this regularization term comes from the difficulty of tackling the "singular" terms containing $\frac{\nabla\phi}{|\nabla\phi|}$. More precisely, the system considered in [36] writes :

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\otimes\nabla\phi\right), \\ \nabla\cdot\mathbf{u} = 0, \\ \phi_t - \nabla\cdot\left(\left(\lambda + \frac{\sigma}{|\nabla\phi|}\right)\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi, \\ \sigma_t + \mathbf{u}\cdot\nabla\sigma = 0. \end{cases} \tag{3.1.6}$$

The above System (3.1.6) is itself a simplification, obtained by neglecting the nonlinear terms containing the curvature, of a more general phase-field model of vesicle introduced in [37]. In the above mentioned works, the boundary condition imposed on ϕ is $\phi(t,x) = -1, \quad x\in\partial\Omega$ and the expression for W is $W(\phi) = (\phi^2-1)^2$. However, after a change of variable $\phi\to\phi-1$, we have the homogeneous boundary condition for ϕ. It is worth noticing the fact that, $\sigma\equiv 0$ is a solution of the last equation of (3.1.6) provided that $\sigma_0\equiv 0$. In this case, system (3.1.6) can be written as follows :

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \lambda\nabla\cdot(\nabla\phi\otimes\nabla\phi), \\ \nabla\cdot\mathbf{u} = 0, \\ \phi_t + \mathbf{u}\cdot\nabla\phi = \Delta\phi - W'(\phi). \end{cases} \tag{3.1.7}$$

It is worth mentioning that a quite similar system was proposed by Lin (see [43]) many years ago to model the flow of nematic liquid crystals with varying director lengths, or with variable degree of orientation :

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p - \lambda\nabla\cdot(\nabla\mathbf{d}\bigodot\nabla\mathbf{d}), \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} - W'(\mathbf{d}), \end{cases} \tag{3.1.8}$$

where $\mathbf{d} : (0,T)\times\Omega\to\mathbb{R}^3$ is the optical molecule direction and

$$\nabla\mathbf{d}\bigodot\nabla\mathbf{d} = \left\{\sum_{j=1}^{3}\partial_i\mathbf{d}_j\partial_j\mathbf{d}_k\right\}_{1\leqslant i,j\leqslant 3}. \tag{3.1.9}$$

In [44], the authors studied the well-posedness of (3.1.8) with Dirichlet boundary condition. In this work, they obtain several energy inequalities which enable them to obtain the existence of global weak solutions by employing an improved Galerkin method. They also discuss the uniqueness, regularity and some stability properties of solutions. Moreover, they proved in [45] that if the domain and the initial-boundary condition in problem (3.1.8) are smooth enough, then there exists a suitable weak solution whose singular set has one-dimension Hausdorff measure zero in space-times. Since system (3.1.8) contains the Navier-Stokes equations as a subsystem, this result can be considered as a natural generalization of an earlier work of Caffarelli-Kohn-Nirenberg (see [12]).

For general initial σ_0, the method introduced in [44] seems less effective for system (3.1.1)-(3.1.2) since there are some difficulties in dealing with the transport equation. More precisely, the classical theory of Diperna and Lions for transport equation seems less effective for the approximate solutions constructed by Galerkin method. For this reason, we will instead choose another approximate method which can be found in [46, pp. 97].

Many constants arise in the course of our work. The symbol C denotes a generic constant whose value may change from line to line. We reserve subscripts ($\epsilon, \lambda, etc.$) for those constants to which repeated reference must be made. Since the parameters λ and ϵ play no role in our analysis and since we will repeatedly use Sobolev imbedding theorem, **the dependence of C on λ, ϵ, Ω will not be specified throughout this work**. Unless otherwise indicated, $(\cdot,\cdot)$ will denote the inner product of $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ will denote the dual product of $H_0^1(\Omega)$ with $H^{-1}(\Omega)$.

The plan of this paper is as follows. In Section 2, we give a formal prove of an energy estimate for the system (3.1.1) and introduce the definition of its weak solution. In Section 3, we recall some classical inequality and compact result which will be used later on. Section 4 consists in proving some results on phase-field equation (a type of nonlinear parabolic equation) and viscous Navier-Stokes equations. In Section 5, we construct a approximate solution of system (3.1.1)-(3.1.2) and in the last section (Section 6), we show that the approximate solutions obtained in Section 5 converge weakly to a weak solution of system (3.1.1)-(3.1.2), which gives the proof of Theorem 3.1.1.

3.2 A priori estimates and definition of weak solution

In this section, we first derive some a priori estimates and then we introduce the definition of the solution for the system (3.1.1), (3.1.2).

3.2.1 A priori estimates

First of all, we compute formally some a priori estimates for the problem (3.1.1), (3.1.2).

We introduce the operators A_σ and J_σ :

$$A_\sigma(\phi) := -\nabla \cdot \left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2 + \epsilon}})\nabla\phi \right), \tag{3.2.10}$$

and

$$J_\sigma(\phi) := \left(\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}} \right) \nabla\phi. \tag{3.2.11}$$

The operator A_σ already appears in (3.1.1c). By using the following simple calculation, we can also write the right-hand side of (3.1.1a) by using A_σ. More precisely, using the Leibniz rule and recalling the notation (3.1.3), we have the following relation :

$$\nabla\cdot(u\otimes v) = (\nabla u)\, v + (\nabla\cdot v)\, u. \tag{3.2.12}$$

By using (3.2.12), we obtain

$$\begin{aligned}\nabla\cdot &\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi \right) \\ &= \nabla^2\phi \left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right) + \nabla\cdot \left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right)\nabla\phi,\end{aligned} \tag{3.2.13}$$

where $\nabla^2\phi = \left(\frac{\partial\phi}{\partial x_i \partial x_j}\right)_{1\leqslant i,j\leqslant 3}$.

On the other hand, using again Leibniz rule, we have

$$\begin{aligned}\nabla\left(\frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right) = \nabla^2\phi &\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right) \\ &+ \left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma.\end{aligned}$$

Combining the above equation with (3.2.13) and using the definition (3.2.10) of A_σ yield

$$\begin{aligned}\nabla\cdot &\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi \right) \\ &= -A_\sigma(\phi)\nabla\phi + \nabla\left(\frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right) - \left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma.\end{aligned} \tag{3.2.14}$$

Consequently, if we denote by $q_{\phi,\sigma}$ the quantity

$$q_{\phi,\sigma} = \frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon},$$

then we can rewrite (3.1.1) as

$$\begin{aligned}\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} &= \Delta\mathbf{u} + \nabla(p+q) \\ &+ A_\sigma(\phi)\nabla\phi + \left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma \quad \text{in} \quad (0,T)\times\Omega,\end{aligned} \tag{3.2.15}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{3.2.16}$$
$$\phi_t + \mathbf{u}\cdot(\nabla\phi) = -W'(\phi) - A_\sigma(\phi) \quad \text{in} \quad (0,T)\times\Omega, \tag{3.2.17}$$
$$\sigma_t + \mathbf{u}\cdot\nabla\sigma = 0 \quad \text{in} \quad (0,T)\times\Omega. \tag{3.2.18}$$

- We multiply (3.2.15) by u and integrate by parts :

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 \mathrm{d}x + \int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}x \\ = \int_\Omega A_\sigma(\phi)\nabla\phi\cdot\mathbf{u}\mathrm{d}x + \int_\Omega \nabla\sigma\cdot\mathbf{u}\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x.\end{aligned} \tag{3.2.19}$$

- We multiply (3.2.17) by $\phi_t + \mathbf{u}\cdot\nabla\phi$ and integrate by parts :

$$\begin{aligned}\int_\Omega |\phi_t + \mathbf{u}\cdot\nabla\phi|^2 \mathrm{d}x \\ = -\frac{d}{dt}\int_\Omega \left(W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)\mathrm{d}x \\ - \int_\Omega A_\sigma(\phi)\nabla\phi\cdot\mathbf{u}\mathrm{d}x + \int_\Omega \sigma_t\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x.\end{aligned} \tag{3.2.20}$$

Summing (3.2.19) and (3.2.20), we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega |\mathbf{u}|^2 + W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)\mathrm{d}x \\ + \int_\Omega |\nabla\mathbf{u}|^2 + \int_\Omega |\phi_t + \mathbf{u}\cdot\nabla\phi|^2 \mathrm{d}x \\ = \int_\Omega (\sigma_t + \nabla\sigma\cdot\mathbf{u})\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x.\end{aligned}$$

The above equation and (3.2.18) imply the following energy estimates :

$$E(t) + \int_0^t \int_\Omega |\nabla\mathbf{u}|^2 + \left|W'(\phi) + A_\sigma(\phi)\right|^2 \mathrm{d}x\mathrm{d}s \leqslant E(0), \tag{3.2.21}$$

where the energy associated to our system is defined by

$$E(t) := \int_\Omega \left(|\mathbf{u}(t)|^2 + \frac{\lambda}{2}|\nabla\phi(t)|^2 + W(\phi(t)) + \sigma\sqrt{|\nabla\phi(t)|^2+\epsilon}\right)\mathrm{d}x. \tag{3.2.22}$$

3.2.2 Function spaces and Definition of Weak Solutions

In this work, we adopt standard notations of divergence-free vector fields of Sobolev spaces (see, for instance, [14, pp. 1-10]). Let us denote by $\mathscr{V}$ the set

$$\mathscr{V} = \{\mathbf{v} \in (C_0^\infty(\Omega))^3 \mid \nabla \cdot \mathbf{v} = 0\}.$$

and by $V^k(\Omega)$ the closure of $\mathscr{V}$ in $H^k(\Omega)$ for $k \in \mathbb{N}^*$ and by $V^0(\Omega)$ the closure of $\mathscr{V}$ in $L^2(\Omega)$. It can be verified that for $k \geqslant 1$,

$$V^k(\Omega) = H_0^k(\Omega) \cap V(\Omega). \tag{3.2.23}$$

In particular, with this notation, the classical spaces $H(\Omega)$ and $V(\Omega)$ are denoted respectively by $V^0(\Omega)$ and $V^1(\Omega)$. We also denote by $V^{-k}(\Omega)$ the dual space of $V^k(\Omega)$ with respect to $V^0(\Omega)$. Finally, we set $Q^t = (0,t) \times \Omega$ for $t > 0$.

With the above notation, we can state the definition of weak solution of (3.1.1).

Definition 3.2.1 (Definition of weak solution)**.** *The triple* $(\mathbf{u}, \phi, \sigma)$ *is a weak solution of* (3.1.1) *with initial data* (3.1.2) *provided that*

$$\begin{cases} \mathbf{u} \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;V^0(\Omega)), \\ \phi \in L^\infty(0,T;H_0^1(\Omega)), \\ \sigma \in C([0,T];L^p(\Omega)), \ (1 \leqslant p < \infty), \end{cases} \tag{3.2.24}$$

$(\mathbf{u}, \phi, \sigma)$ *satisfies the energy estimate* (3.2.21) *and if the following relations hold :*

$$\begin{aligned} &\int_\Omega \mathbf{u}_0 \cdot \mathbf{v}(0,x)\mathrm{d}x - \int_{Q^T} \mathbf{u} \cdot \mathbf{v}_t \mathrm{d}x\mathrm{d}t - \int_{Q^T} (\mathbf{u} \otimes \mathbf{u} - \nabla \mathbf{u}) : \nabla \mathbf{v} \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_{Q^T} \nabla \mathbf{v} : \left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2 + \epsilon}} \right) \nabla\phi \otimes \nabla\phi \right) \mathrm{d}x\mathrm{d}t, \\ &\qquad\qquad \textit{for all} \quad \mathbf{v} \in C^1([0,T];V^6(\Omega)), \ \mathbf{v}(T,\cdot) = 0, \end{aligned} \tag{3.2.25}$$

$$\begin{aligned} &\int_\Omega \phi_0 \psi(0,x)\mathrm{d}x - \int_{Q^T} \phi\psi_t \mathrm{d}x\mathrm{d}t + \int_{Q^T} \nabla\psi \cdot \left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2 + \epsilon}} \right) \nabla\phi \right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_{Q^T} \left(\phi \mathbf{u} \cdot \nabla\psi - W'(\phi)\psi \right) \mathrm{d}x\mathrm{d}t, \\ &\qquad\qquad \textit{for all} \quad \psi \in C^1([0,T];V^3(\Omega)), \ \psi(T,\cdot) = 0, \end{aligned} \tag{3.2.26}$$

$$\int_{\Omega} \sigma_0 \eta(0,x)\mathrm{d}x - \int_{Q^T} \sigma \eta_t \mathrm{d}x\mathrm{d}t = \int_{Q^T} \sigma \mathbf{u} \cdot \nabla \eta \mathrm{d}x\mathrm{d}t,$$
$$\textit{for all} \quad \eta \in C^1([0,T]; V^3(\Omega)), \ \eta(T,\cdot) = 0. \quad (3.2.27)$$

Remark 3.2.2. *The above definition is obtained by multiplying formally* (3.1.1a)*,* (3.1.1c) *and* (3.1.1d) *by smooth function and integrating by parts. In particular, a smooth solution of* (3.1.1)*,* (3.1.2) *is a weak solution in the above sense. Note that in* (3.2.25)*,* (3.2.26) *and* (3.2.27)*, the terms appearing in the integral on Q^T and involving $\mathbf{u}, \phi, \sigma$ are in $L^1(Q^T)$ if $(\mathbf{u}, \phi, \sigma)$ satisfies* (3.2.24)*. This comes from Sobolev embedding theorems and Hölder inequalities, as it will be seen in Subsection 3.3.1.*

3.3 Preliminaries

In this section we recall some classical result and derive some basic inequalities which are useful in our problem.

3.3.1 Some inequalities

Let us assume $(\mathbf{u}, \phi, \sigma)$ satisfies the regularity assumptions of Definition 3.2.1, i.e.

$$\begin{cases} \mathbf{u} \in L^2(0,T; V^1(\Omega)) \cap L^\infty(0,T; V^0(\Omega)), \\ \phi \in L^\infty(0,T; H_0^1(\Omega)), \\ \sigma \in C([0,T]; L^p(\Omega)), \ (1 \leqslant p < \infty). \end{cases} \quad (3.3.1)$$

Using Hölder's inequalities, we obtain

$$\left((\lambda + \frac{\sigma}{\sqrt{|\nabla \phi|^2 + \epsilon}}) \nabla \phi \right) \in L^\infty(0,T; L^2(\Omega)),$$

$$\left((\lambda + \frac{\sigma}{\sqrt{|\nabla \phi|^2 + \epsilon}}) \nabla \phi \otimes \nabla \phi \right) \in L^\infty(0,T; L^1(\Omega))$$

with

$$\left\| \left((\lambda + \frac{\sigma}{\sqrt{|\nabla \phi|^2 + \epsilon}}) \nabla \phi \right) \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C \left(\|\phi\|_{L^\infty(0,T;H^1(\Omega))} + \|\sigma\|_{L^\infty(0,T;L^2(\Omega))} \right), \quad (3.3.2)$$

and

$$\left\| \left((\lambda + \frac{\sigma}{\sqrt{|\nabla \phi|^2 + \epsilon}}) \nabla \phi \otimes \nabla \phi \right) \right\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C \left(\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\sigma\|^2_{L^\infty(0,T;L^2(\Omega))} \right). \quad (3.3.3)$$

We recall that by using Sobolev embeddings, we have $H^1(\Omega) \subset L^6(\Omega)$. As a consequence, since

$$W'(\phi) = 4\phi^3 - 12\phi^2 + 8\phi,$$

we deduce that if $\phi \in L^\infty(0,T;H_0^1(\Omega))$, then

$$W'(\phi) \in L^\infty(0,T;L^2(\Omega))$$

and

$$\|W'(\phi)\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C\left(\|\phi\|^3_{L^\infty(0,T;H^1(\Omega))} + \|\phi\|_{L^\infty(0,T;H^1(\Omega))}\right). \quad (3.3.4)$$

Using (3.3.3) and Sobolev embeddings results, we can deduce the following proposition which gives some estimates of the nonlinear terms appearing in the right hand side of (3.1.1a) :

Proposition 3.3.1. *Let $\sigma \in C([0,T];L^2(\Omega))$, $\phi \in L^4(0,T;H_0^1(\Omega))$ and*

$$\mathbf{F}(\sigma,\phi) = -\nabla\cdot\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right), \quad (3.3.5)$$

Then $\mathbf{F}(\phi,\sigma) \in L^2(0,T;V^{-3}(\Omega))$ and there exists a constant C depending only on Ω such that

$$\begin{aligned}&\|\mathbf{F}(\phi,\sigma)\|_{L^2(0,T;V^{-3}(\Omega))}\\ &\qquad\leqslant C\|\phi\|^2_{L^4(0,T;H_0^1(\Omega))} + C\|\sigma\|_{C([0,T];L^2(\Omega))}\|\phi\|_{L^2(0,T;H_0^1(\Omega))}.\end{aligned}$$

3.3.2 Compactness sets

We need in what follows of several arguments of compactness. We recall here some well-known results adapted in our framework.

Lemma 3.3.2. *Let $\{f_n\}$ be a bounded sequence in $L^\infty(0,T;H_0^1(\Omega))$ such that $\{\frac{\partial f_n}{\partial t}\}$ is bounded in $L^2(0,T;L^1(\Omega))$. Then $\{f_n\}$ is relatively compact in $C([0,T];L^4(\Omega))$.*

The proof of the above lemma is given in [54, Corollary 8].

We also need the following lemma :

Lemma 3.3.3. *Let $\{f_n\}$ be a bounded sequence in $L^2(0,T;V^1(\Omega))$ such that $\{\frac{\partial f_n}{\partial t}\}$ is bounded in $L^2(0,T;V^{-k}(\Omega))$ for some $k \geqslant 1$, $k \in \mathbb{N}^*$. Then $\{f_n\}$ is relatively compact in $L^2([0,T];V^0(\Omega))$.*

The proof of the above lemma is given in [46, pp.57-60].

3.4 Existence results for the Navier–Stokes equations and the phase-field equation

Now we consider the following viscous Navier-Stokes equations introduced in [46, pp. 97] :

$$\begin{cases} \mathbf{u}_t + \nu\Delta^m \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p + \mathbf{f}, \quad (t,x) \in (0,T)\times\Omega, \\ \nabla\cdot\mathbf{u} = 0, \quad (t,x)\in(0,T)\times\Omega, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \\ \mathbf{u}|_{\partial\Omega}, \cdots, \Delta^{\hat{m}-1}\mathbf{u}|_{\partial\Omega} = 0, \\ \frac{\partial}{\partial\mathbf{n}}\mathbf{u}|_{\partial\Omega}, \cdots, \frac{\partial}{\partial\mathbf{n}}\Delta^{\hat{m}-1}\mathbf{u}|_{\partial\Omega} = 0, \end{cases} \tag{3.4.1}$$

where $\nu \in \mathbb{R}^+$ and $m = 2\hat{m}$ for $\hat{m} \in \mathbb{N}^* = \{1, 2, \cdots\}$. For the above equations, we have the following result about the existence of weak solutions which is similar to that for Navier-Stokes equations.

Proposition 3.4.1. *Let $V^m(\Omega)$ be the space defined by (3.2.23) and $V^{-m}(\Omega)$ be its dual space with respect to $H(\Omega)$ for $m \geqslant 2$. Assume that $\mathbf{f} \in L^2(0,T;V^{-m}(\Omega))$ and $\mathbf{u}_0 \in H(\Omega)$, then (3.4.1) admits a unique weak solution*

$$\mathbf{u} \in L^2(0,T;V^m(\Omega)) \cap H^1(0,T;V^{-m}(\Omega))$$

in the sense that

$$\nu\int_0^T (\mathbf{u},\mathbf{v})_{V^m(\Omega)}\mathrm{d}t - \int_\Omega \mathbf{u}_0(\cdot)\mathbf{v}(0,\cdot)\mathrm{d}x - \int_{Q^T}\mathbf{u}\cdot\mathbf{v}_t\mathrm{d}x\mathrm{d}t - \int_{Q^T}(\mathbf{u}\otimes\mathbf{u} - \nabla\mathbf{u}) : \nabla\mathbf{v}\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \langle\mathbf{f},\mathbf{v}\rangle_{V^{-m},V^m}\mathrm{d}t,$$
$$\textit{for all} \quad \mathbf{v}\in C^1([0,T];V^m(\Omega)),\ \mathbf{v}(T,\cdot) = 0. \tag{3.4.2}$$

Furthermore, we have the following estimate for $\mathbf{u}$:

$$\|\mathbf{u}_t\|_{L^2(0,T;V^{-m}(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;V^m(\Omega))} + \|\mathbf{u}\|_{C([0,T];H(\Omega))}$$
$$\leqslant C(\nu)\|\mathbf{f}\|_{L^2(0,T;V^{-m}(\Omega))} + \|\mathbf{u}_0\|_{H(\Omega)}. \tag{3.4.3}$$

Remark 3.4.2. *Unless otherwise indicated, the weak solution of viscous Navier-Stokes equations or Navier-Stokes equations ($\nu = 0$) in the remaining part of this work will be understood by (3.4.2).*

Since the proof of the above proposition for $\hat{m} \in \mathbb{N}^*$ is quite similar to the case without $\nu\Delta^m\mathbf{u}$ in (3.4.1) (see for instance [46, pp. 75-77]), we only give a formal proof here. Before giving the proof, we first state a lemma which can be found in [46, pp. 74, Corollary 6.1]

Lemma 3.4.3. *There exists a bounded self-adjoint linear operator* $A \in \mathscr{L}(V^m(\Omega), V^{-m}(\Omega))$ *such that*

$$\nu(u,v)_{V^m(\Omega)} + (u,v)_{V^1(\Omega)} = \langle Au, v\rangle_{V^{-m}(\Omega),V^m(\Omega)}. \tag{3.4.4}$$

Moreover, there exists an positive non-decreasing series $\{\lambda_k\}_{k=1}^{\infty}$ *with* $\lim_{k\to\infty}\lambda_k = \infty$ *and an orthonormal basis* $\{\omega_k\}_{k=1}^{\infty} \subset V^m(\Omega)$ *of* $H(\Omega)$ *such that*

$$\langle Au, v\rangle_{V^{-m}(\Omega),V^m(\Omega)} = \lambda_k(\omega_k, v)_{H(\Omega)}, \quad \forall v \in V^m(\Omega). \tag{3.4.5}$$

Proof of Proposition 3.4.1. For every $k \in \mathbb{N}^*$, by the above lemma, we can construct the approximate solution of (3.4.3) by

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t)\omega_j(x),$$

$$\begin{aligned}(u_k'(t), \omega_j)_{H(\Omega)} &+ \nu(u_k(t),\omega_j)_{V^m(\Omega)} + (u_k(t),\omega_j)_{V^1(\Omega)}\\ &+ b(u_k(t),u_k(t),\omega_j) = \langle f(t),\omega_j\rangle_{V^{-m}(\Omega),V^m(\Omega)}, \quad 1 \leqslant j \leqslant k,\end{aligned} \tag{3.4.6}$$

$$u_k(0) = u_{0k}, \quad u_{0j} \in \operatorname{span}\{\omega_1,\ldots,\omega_k\}, \quad u_{0k} \mapsto u_0 \quad \text{in} \quad H(\Omega), \tag{3.4.7}$$

where

$$b(u,v,w) = \int\limits_{\Omega} u\cdot\nabla v\omega \mathrm{d}x.$$

By the fundamental theorem of ODE, (3.4.6) has a unique solution on $[0,t_k]$ for each k. However, by a similar argument as those for Navier-Stokes equations, we have the following a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_{H(\Omega)}^2 + \nu\|u_k(t)\|_{V^m(\Omega)}^2 \leqslant \langle f(t), u_k(t)\rangle_{V^{-m}(\Omega),V^m(\Omega)}, \tag{3.4.8}$$

which implies, combining Cauchy inequality and Gronwall inequality :

$$\|u_k(t)\|_{H(\Omega)} + \nu\int\limits_0^t \|u_k(\tau)\|_{V^m(\Omega)}^2 \mathrm{d}\tau \leqslant \|u_{0k}\|_{H(\Omega)}^2 + C(\nu)\int\limits_0^t \|f(\tau)\|_{V^{-m}(\Omega)}^2\mathrm{d}\tau. \tag{3.4.9}$$

The above inequality combined with (3.4.6) implies that $t_k = T$ and

$$\|u_k\|_{L^2(0,T;V^m(\Omega))} + \|u_k\|_{L^\infty(0,T;H(\Omega))} \leqslant C(\nu)\|f\|_{L^2(0,T;V^{-m}(\Omega))} + \|u_0\|_{H(\Omega)}. \tag{3.4.10}$$

Now we turn to estimate of u_k' : Let $P_k : H(\Omega) \mapsto \operatorname{span}\{\omega_1,\cdots,\omega_k\}$ be the projector $P_k h = \sum_{i=1}^k (h,\omega_i)_{H(\Omega)}\omega_i$. Using Lemma 3.4.3, we deduce that $P_k \in \mathscr{L}(V^m(\Omega), V^m(\Omega))$ and

$$\|P_k\|_{\mathscr{L}(V^m(\Omega),V^m(\Omega))} \leqslant 1. \tag{3.4.11}$$

By a dual argument, we have $P_k \in \mathscr{L}(V^{-m}(\Omega), V^{-m}(\Omega))$ and

$$\|P_k\|_{\mathscr{L}(V^{-m}(\Omega),V^{-m}(\Omega))} \leqslant 1. \tag{3.4.12}$$

Applying P_k to (3.4.6), we deduce that

$$u_k' = -P_k(u_k \cdot \nabla u_k) - \nu P_k A u_k + P_k f. \tag{3.4.13}$$

It can be shown by combining (3.4.10) with Sobolev imbedding theorem and Hölder's inequality that

$$\|P_k(u_k \cdot \nabla u_k)\|_{L^2(0,T;V^{-m}(\Omega))} \leqslant C\|u_k\|_{L^\infty(0,T;H(\Omega))}\|u_k\|_{L^2(0,T;V^m(\Omega))}, \tag{3.4.14}$$

which implies

$$\|u_k'\|_{L^2(0,T;V^{-m}(\Omega))} \leqslant C(\nu)\|f\|_{L^2(0,T;V^{-m}(\Omega))} + \|u_0\|_{H(\Omega)}. \tag{3.4.15}$$

Choosing a function $\tilde{v} \in C^1([0,T]; V^m(\Omega))$ having the form

$$\tilde{v}(t) = \sum_{k=1}^{N} d^k(t)\omega_k, \tag{3.4.16}$$

and $\tilde{v}(T) = 0$. Substituting the above equation into (3.4.6), it follows

$$\int_0^T \langle u_k' + u_k \cdot \nabla u_k - A u_k - f, \tilde{v}\rangle_{V^{-m}(\Omega),V^m(\Omega)} \mathrm{d}t = 0. \tag{3.4.17}$$

Integrating by part, we obtain

$$\begin{aligned}\nu \int_0^T (u_k, \tilde{v})_{V^m(\Omega)} \mathrm{d}t - \int_\Omega u_{0k}(\cdot)\tilde{v}(0,\cdot)\mathrm{d}x - \int_{Q^T} u_k \cdot \tilde{v}_t \mathrm{d}x\mathrm{d}t \\ - \int_{Q^T} (u_k \otimes u_k - \nabla u_k) : \nabla\tilde{v}\mathrm{d}x\mathrm{d}t = \int_0^T \langle f, \tilde{v}\rangle_{V^{-m},V^m} \mathrm{d}t.\end{aligned} \tag{3.4.18}$$

Combining (3.4.10) with (3.4.15), up to the extraction of a subsequence,

$$u_k \to u \quad \text{weakly in} \quad L^2(0,T;V^m(\Omega)) \tag{3.4.19}$$

and

$$u_k' \to u' \quad \text{weakly in} \quad L^2(0,T;V^{-m}(\Omega)). \tag{3.4.20}$$

Combining the above two statements with the Aubin-Lions type interpolation theorem, we can pass to the limit $k \mapsto \infty$ in (3.4.18). Since the functions of the form (3.4.16) are dense in $C^1([0,T]; V^m(\Omega))$, we obtain (3.4.2).

By weak sequential lower semicontinuity (See, for instance, [63, pp. 235],), we obtain by combining (3.4.19), (3.4.20), (3.4.10) and (3.4.15) that

$$
\begin{aligned}
\|u\|_{L^2(0,T;V^m(\Omega))} &+ \|u'\|_{L^2(0,T;V^{-m}(\Omega))} \\
&\leqslant \liminf_{k\to\infty} \|u_k\|_{L^2(0,T;V^m(\Omega))} + \liminf_{k\to\infty} \|u_k'\|_{L^2(0,T;V^{-m}(\Omega))} \\
&\leqslant C(\nu)\|f\|_{L^2(0,T;V^{-m}(\Omega))} + \|u_0\|_{H(\Omega)},
\end{aligned}
$$

which is (3.4.3). With such regularity of u, the classical method implies the uniqueness, which is different from the corresponding result for Navier-Stokes equations. □

Now we turn to the phase-field equation 3.1.1c. More precisely, we consider the initial-boundary value problem of the following nonlinear parabolic equation :

$$
\begin{cases}
\phi_t - \nabla\cdot\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi, \quad (t,x)\in(0,T)\times\Omega, \\
\phi(0,\cdot) = \phi_0, \quad \phi(t,x)|_{\partial\Omega} = 0.
\end{cases}
\tag{3.4.21}
$$

We will establish the H^2-regularity of the solution of the above equation. Pay attention to the fact that the main part of (3.4.21) (which is denoted by $A_\sigma(\phi)$) has the form of nonlinear monotone operator. The following proposition shows that

$$
\phi \in H_0^1(\Omega) \mapsto A_\sigma(\phi) \in H^{-1}(\Omega)
$$

is a monotone operator provided that σ is non-negative.

Proposition 3.4.4. *There exists a positive constant C depending only on Ω such that*

$$
\|A_\sigma(\phi)\|_{H^{-1}(\Omega)} \leqslant \lambda\|\phi\|_{H_0^1(\Omega)} + C\|\sigma\|_{L^\infty(\Omega)} \quad (\phi\in H_0^1(\Omega), \sigma\in L^\infty(\Omega)).
\tag{3.4.22}
$$

Moreover, if $\sigma\in L^\infty(\Omega)$ and $\sigma\geqslant 0$ a.e. in Ω, then

$$
\langle A_\sigma(\phi_1) - A_\sigma(\phi_2), \phi_1-\phi_2\rangle \geqslant \lambda\|\phi_1-\phi_2\|^2_{H_0^1(\Omega)} \quad (\phi_1,\phi_2\in H_0^1(\Omega)).
\tag{3.4.23}
$$

Proof. We first prove (3.4.22). Notice that, under the assumption of ϕ and σ in the proposition, we have $\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi \in L^2(\Omega)$, which implies that $A_\sigma(\phi)\in H^{-1}(\Omega)$. Recalling (3.2.11), for every $g\in H_0^1(\Omega)$

$$
\begin{aligned}
\langle A_\sigma(\phi), g\rangle &= \int_\Omega (\lambda\nabla\phi\cdot\nabla g + J_\sigma(\phi)\cdot\nabla g)\mathrm{d}x \\
&\leqslant \lambda\|\nabla\phi\|_{L^2(\Omega)}\|\nabla g\|_{L^2(\Omega)} + C(\Omega)\|\sigma\|_{L^\infty(\Omega)}\|\nabla g\|_{L^2(\Omega)},
\end{aligned}
$$

which deduces (3.4.22). For inequality (3.4.23), we have

$$
\begin{aligned}
&\langle A_\sigma(\phi_1) - A_\sigma(\phi_2), \phi_1 - \phi_2 \rangle \\
&= \int_\Omega (J_\sigma(\phi_1) - J_\sigma(\phi_2)) \cdot (\nabla\phi_1 - \nabla\phi_2) \mathrm{d}x + \lambda \|\nabla\phi_1 - \nabla\phi_2\|^2_{L^2(\Omega)}. \qquad (3.4.24)
\end{aligned}
$$

Noticing that, for $\alpha > 0$, the function $z \mapsto \frac{z}{\sqrt{|z|^2+\alpha}}$ is increasing in $\mathbb{R}$, it follows

$$
\left(\frac{x}{\sqrt{|x|^2+\epsilon}} - \frac{\bar{x}}{\sqrt{|\bar{x}|^2+\epsilon}} \right) \cdot (x - \bar{x}) \geqslant 0 \quad (x, \bar{x} \in \mathbb{R}^3).
$$

which combined with (3.2.11) and $\sigma \geqslant 0$ yields

$$
(J_\sigma(\phi_1) - J_\sigma(\phi_2)) \cdot (\nabla\phi_1 - \nabla\phi_2) \geqslant 0 \quad \text{in } \Omega.
$$

The above inequality and (3.4.24) imply (3.4.23). □

In order to study (3.4.21), we first consider the corresponding nonlinear elliptic equation :

$$
-\sum_{i=1}^{n} (L_{p_i}(D\phi))_{x_i} = f \quad \text{in} \quad \Omega. \qquad (3.4.25)
$$

Proposition 3.4.5 (H^2-regularity of nonlinear elliptic equation)**.** *Let $\phi \in H^1(\Omega)$ be a weak solution of the nonlinear partial differential equations* (3.4.25)*, where L satisfies the following structure condition*

$$
|D_p L(p)| \leqslant C(|p| + 1) \quad (p \in \mathbb{R}^n),
$$

$$
|D^2 L(p)| \leqslant C \quad (p \in \mathbb{R}^n)
$$

$$
\sum_{i,j=1}^{n} L_{p_i p_j}(p)\xi_i\xi_j \geqslant \theta|\xi|^2 \quad (p, \xi \in \mathbb{R}^n),
$$

for some $\theta > 0$, then $\phi \in H^2_{loc}(\Omega)$. Furthermore, assume $\phi \in H^1_0(\Omega)$ and that $\partial\Omega$ is C^2, then $u \in H^2(\Omega)$ and

$$
\|\phi\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.
$$

For the proof of the above proposition, we refer to [26, pp. 459]. The following result is essentially contained in [46, pp. 159-161] which can be considered as the application of monotone operator theory to parabolic equation.

Proposition 3.4.6 (Monotone trick for parabolic equation)**.** *Let* $\sigma_n, \sigma \in L^\infty((0,T)\times\Omega)$, $\phi_n, \phi \in L^\infty(0,T;H_0^1(\Omega))$ *and* $\chi \in L^2(0,T;H^{-1}(\Omega))$ *and satisfy*

$$\begin{cases} \sigma_n \to \sigma \quad \text{strongly in} \quad C([0,T];L^p(\Omega)),\ 1<p<\infty, \\ \sigma_n, \sigma \geqslant 0 \quad a.e. \quad (t,x)\in[0,T]\times\Omega, \\ \phi_n \to \phi \quad \text{strongly in} \quad C([0,T];L^4(\Omega)), \\ \phi_n \to \phi \quad \text{weakly-star in} \quad L^\infty(0,T;H_0^1(\Omega)), \\ A_{\sigma_n}(\phi_n) \to \chi \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)), \end{cases} \tag{3.4.26}$$

$$\frac{1}{2}\int_\Omega |\phi_n(T)|^2 \mathrm{d}x + \int_0^T \langle A_{\sigma_n}(\phi_n), \phi_n\rangle \mathrm{d}t + \int_{Q^T} \phi_n W'(\phi_n)\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Omega |\phi_n(0)|^2\mathrm{d}x, \tag{3.4.27}$$

$$\frac{1}{2}\int_\Omega |\phi(T)|^2 \mathrm{d}x + \int_0^T \langle \chi, \phi\rangle \mathrm{d}t + \int_{Q^T} \phi W'(\phi)\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Omega |\phi(0)|^2\mathrm{d}x. \tag{3.4.28}$$

Then $A_\sigma(\phi) = \chi$ *and*

$$\nabla\phi_n \to \nabla\phi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.4.29}$$

Remark 3.4.7. *We note that* (3.4.29) *is the most important result for this work. For general monotone operator (like those introduced in [46]), we only have*

$$\nabla\phi_n \to \nabla\phi \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

Proof. Let $\hat{\phi} \in L^2(0,T;H_0^1(\Omega))$. We first claim that, under condition (3.4.26), we have

$$\int_0^T \langle A_{\sigma_n}(\hat{\phi}), \phi_n - \hat{\phi}\rangle \mathrm{d}t \to \int_0^T \langle A_\sigma(\hat{\phi}), \phi - \hat{\phi}\rangle \mathrm{d}t \tag{3.4.30}$$

Indeed, using integrate by part, we deduce

$$\lim_{n\to\infty} \int_0^T \langle A_{\sigma_n}(\hat{\phi}), \phi_n - \hat{\phi}\rangle \mathrm{d}t$$
$$= \lim_{n\to\infty} \int_{Q^T} (\lambda\nabla\hat{\phi} + J_{\sigma_n}(\hat{\phi}))\cdot(\nabla\phi_n - \nabla\hat{\phi})\mathrm{d}x\mathrm{d}t. \tag{3.4.31}$$

Combining the first statement of (3.4.26) with Lebesgue's Dominated convergence theorem, we have

$$J_{\sigma_n}(\hat{\phi}) \to J_\sigma(\hat{\phi}) \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)),$$

which combined with (3.4.31) and the fourth statement of (3.4.26) indicates that

$$\lim_{n\to\infty}\int_0^T \langle A_{\sigma_n}(\hat{\phi}), \phi_n - \hat{\phi}\rangle \mathrm{d}t = \int_{Q^T} (\lambda\nabla\hat{\phi} + J_\sigma(\hat{\phi}))\cdot(\nabla\phi - \nabla\hat{\phi})\mathrm{d}x\mathrm{d}t,$$

which implies (3.4.30) after integrating by part.

Let

$$X_n = \int_0^T \langle A_{\sigma_n}(\phi_n) - A_{\sigma_n}(\hat{\phi}), \phi_n - \hat{\phi}\rangle \mathrm{d}t. \tag{3.4.32}$$

By (3.4.23), we have for every $n \in \mathbb{N}^*$,

$$X_n \geqslant \lambda \|\phi_n - \hat{\phi}\|^2_{L^2(0,T;H_0^1(\Omega))}. \tag{3.4.33}$$

By (3.1.4), we have $|\phi_n W'(\phi_n)| \leqslant C(|\phi_n|^4 + 1)$, which combined with the third statement of (3.4.26) indicates

$$\int_{Q^T} \phi_n W'(\phi_n)\mathrm{d}x\mathrm{d}t \to \int_{Q^T} \phi W'(\phi)\mathrm{d}x\mathrm{d}t. \tag{3.4.34}$$

Substituting (3.4.27) into (3.4.32) yields

$$\begin{aligned} X_n = \frac{1}{2}\int_\Omega |\phi_n(0)|^2\mathrm{d}x - \frac{1}{2}\int_\Omega |\phi_n(T)|^2\mathrm{d}x - \int_{Q^T} \phi_n W'(\phi_n)\mathrm{d}x\mathrm{d}t \\ - \int_0^T \langle A_{\sigma_n}(\phi_n), \hat{\phi}\rangle\mathrm{d}t - \int_0^T \langle A_{\sigma_n}(\hat{\phi}), \phi_n - \hat{\phi}\rangle\mathrm{d}t, \end{aligned} \tag{3.4.35}$$

which combined with (3.4.26), (3.4.30) and (3.4.34) implies that

$$\begin{aligned} \lim_{n\to\infty} X_n = \frac{1}{2}\int_\Omega |\phi(0)|^2\mathrm{d}x - \frac{1}{2}\int_\Omega |\phi(T)|^2\mathrm{d}x - \int_{Q^T} \phi W'(\phi)\mathrm{d}x\mathrm{d}t \\ - \int_0^T \langle \chi, \hat{\phi}\rangle\mathrm{d}t - \int_0^T \langle A_\sigma(\hat{\phi}), \phi - \hat{\phi}\rangle\mathrm{d}t. \end{aligned}$$

Substituting (3.4.28) into the above inequality implies

$$\int_0^T \langle \chi - A_\sigma(\hat{\phi}), \phi - \hat{\phi}\rangle\mathrm{d}t = \lim_{n\to\infty} X_n \geqslant \lim_{n\to\infty} \lambda\|\phi_n - \hat{\phi}\|^2_{L^2(0,T;H_0^1(\Omega))}. \tag{3.4.36}$$

The last inequality of (3.4.36) is due to (3.4.33). Choosing $\hat{\phi} = \phi$ in (3.4.36) gives

$$\nabla\phi_n \to \nabla\phi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)).$$

We now use hemicontinuity to prove $\chi = A_\sigma(\phi)$. Choose $\hat{\phi} = \phi - \alpha w$, with $\alpha > 0$ and $w \in L^2(0,T; H_0^1(\Omega))$, it follows from (3.4.36) that

$$\alpha \int_0^T \langle \chi - A_\sigma(\phi - \alpha w), w \rangle \mathrm{d}t \geqslant 0$$

and thus

$$\int_0^T \langle \chi - A_\sigma(\phi - \alpha w), w \rangle \mathrm{d}t \geqslant 0$$

Taking $\alpha \to 0$, we deduce

$$\int_0^T \langle \chi - A_\sigma(\phi), w \rangle \mathrm{d}t \geqslant 0, \quad \forall w \in L^2(0,T; H_0^1(\Omega)),$$

which implies, by density theorem of Sobolev spaces, $\chi = A_\sigma(\phi)$. □

With the help of the above proposition, we can state and prove the following proposition which shows that (3.4.21) has a strong solution provided that σ, $\mathbf{u}$ and the initial data ϕ_0 are regular enough.

Proposition 3.4.8. *Let*

$$\begin{cases} \mathbf{u} \in L^2(0,T; H^2(\Omega) \cap V(\Omega)), \\ \sigma \in C([0,T]; C^2(\bar{\Omega})), \\ \sigma(t,x) \geqslant 0 \quad a.e. \quad (t,x) \in [0,T] \times \Omega), \\ \sigma_t \in L^2(0,T; L^2(\Omega)), \\ \phi_0 \in H_0^1(\Omega). \end{cases} \tag{3.4.37}$$

and assume that there exists a positive constant C_0 such that

$$\|\mathbf{u}\|_{L^2(0,T;H^2(\Omega)\cap V(\Omega))} + \|\sigma\|_{C([0,T];C^2(\bar{\Omega}))} + \|\sigma_t\|_{L^2(0,T;L^2(\Omega))} + \|\phi_0\|_{H_0^1(\Omega)} \leqslant C_0,$$

then equation (3.4.21) *has a unique solution*

$$\phi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega)).$$

Furthermore, there exists a positive constant $C = C(C_0, \Omega, T)$ such that

$$\|\phi\|_{L^2(0,T;H^2(\Omega)} + \|\phi\|_{C([0,T];H_0^1(\Omega))} + \|\phi\|_{H^1(0,T;L^2(\Omega))} \leqslant C. \tag{3.4.38}$$

Remark 3.4.9. *The proof of Proposition 3.4.8 will be separated into three parts. In the first one, we construct the approximating solutions by Galerkin method and give some a priori estimates. Then we pass to the limit by the compact argument to obtain the existence of weak solutions. At last, we prove that the weak solution is actually regular by solving an elliptic equation. The first two parts are technical. A similar but shorter proof can be found in [46, pp. 158].*

Proof. **Part 1 : Construction of approximating solutions**
Let $\{w_i\}_{i=1}^{\infty}$ be a orthonormal base of $H_0^1(\Omega)$. We introduce the approximating solutions $\phi_m(t)$ by

$$\begin{cases} \phi_m(t) \in \mathrm{span}\{w_1, \cdot, w_m\}, \\ (\phi_m'(t), w_j) + \langle A_\sigma(\phi_m(t)), w_j\rangle = -(W'(\phi_m(t)) + \mathbf{u}\cdot\nabla\phi_m(t), w_j) \quad (1 \leqslant j \leqslant m), \\ \phi_m(0) = \phi_{0m} \in \mathrm{Span}\{w_1, \cdot, w_m\}, \quad \phi_{0m} \to \phi_0 \in L^2(\Omega), \end{cases} \tag{3.4.39}$$

For each $m \in \mathbb{N}^*$, by the fundamental theorem of ODE, the system (3.4.39) has a unique smooth solution $\phi_m(t)$ on $[0, t_m)$ with $t_m > 0$. Using (3.4.39) (replace ω_j by ϕ_m' in the second equation of (3.4.39)), it follows that

$$\begin{aligned} &\int_\Omega |\phi_m'|^2 \mathrm{d}x + \partial_t \int_\Omega \left(\frac{\lambda}{2}|\nabla\phi_m|^2 + W(\phi_m) + \sigma\sqrt{|\nabla\phi_m|^2+\epsilon}\right)\mathrm{d}x \\ &= -\int_\Omega \mathbf{u}\cdot\nabla\phi_m\phi_m'\mathrm{d}x - \int_\Omega \sigma_t\sqrt{|\nabla\phi_m|^2+\epsilon}\mathrm{d}x \\ &\leqslant \frac{1}{2}\|\phi_m'\|^2_{L^2(\Omega)} + 2\|\mathbf{u}\cdot\nabla\phi_m\|^2_{L^2(\Omega)} + \|\sigma_t\|^2_{L^2(\Omega)} + \|\nabla\phi_m\|^2_{L^2(\Omega)} + C(\epsilon) \end{aligned} \tag{3.4.40}$$

Using Sobolev imbedding theorem and (3.4.37), we obtain

$$\|\mathbf{u}\cdot\nabla\phi_m\|^2_{L^2(\Omega)} \leqslant \|\mathbf{u}\|^2_{H^2(\Omega)}\|\nabla\phi_m\|^2_{L^2(\Omega)},$$

Combining this with (3.4.37), (3.4.40) and Gronwall inequality induces that

$$\begin{aligned} \int_{Q^T} |\phi_m'|^2\mathrm{d}x\mathrm{d}t + \max_{t\in[0,T]}\int_\Omega \left(\frac{\lambda}{2}|\nabla\phi_m(t)|^2 + W(\phi_m(t)) + \sigma\sqrt{|\nabla\phi_m(t)|^2+\epsilon}\right)\mathrm{d}x \\ \leqslant e^{\int_0^T \|\mathbf{u}(s)\|^2_{H^2(\Omega)}\mathrm{d}s}\int_\Omega\left(\frac{\lambda}{2}|\nabla\phi_0|^2 + W(\phi_0) + \sigma\sqrt{|\nabla\phi_0|^2+\epsilon}\right)\mathrm{d}x \\ + e^{\int_0^T \|\mathbf{u}(s)\|^2_{H^2(\Omega)}\mathrm{d}s}\left(C(\epsilon, T, \Omega) + \|\sigma_t\|^2_{L^2(0,T;L^2(\Omega))}\right), \end{aligned} \tag{3.4.41}$$

which implies that the solutions of the ODE (3.4.39) don't blow up in finite time and can be extend on $[0, T]$. Besides

$$\phi_m \quad \text{is bounded in} \quad H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)). \tag{3.4.42}$$

Therefore, using interpolation, we have

$$\phi_m \quad \text{is bounded in} \quad C([0, T]; H^{\frac{1}{2}}(\Omega)), \tag{3.4.43}$$

which indicates that $\phi_m(0)$ and $\phi_m(T)$ have senses. Using (3.4.22) and the second condition of (3.4.37), we know that

$$A_\sigma(\phi_m) \quad \text{is bounded in} \quad L^2(0, T; H^{-1}(\Omega)). \tag{3.4.44}$$

Combining (3.4.42), (3.4.43) and (3.4.44), up to the extraction of a subsequence, we have

$$\begin{cases} \phi_n \to \phi & \text{weak-star in} & L^\infty(0,T;H_0^1(\Omega)), \\ \phi_n(T) \to \xi & \text{weakly in} & L^2(\Omega), \\ A_\sigma(\phi_n) \to \chi & \text{weakly in} & L^2(0,T;H^{-1}(\Omega)), \\ \phi_m' \to \varphi & \text{weakly in} & L^2(0,T;L^2(\Omega)). \end{cases} \tag{3.4.45}$$

It can be easily verified that $\phi' = \varphi$. Moreover, combining (3.4.42) with Lemma 3.3.2, up to the extraction of a subsequence, we have

$$\phi_n \to \phi \quad \text{strongly in} \quad C([0,T];L^4(\Omega)). \tag{3.4.46}$$

Part 2 : Pass to the limit
We introduce now $\phi_n^*, \phi^*, A_\sigma(\phi_n)^*, \cdots$ be the extension of $\phi_n, \phi, A_\sigma(\phi_n), \cdots$ by 0 on $(T,+\infty)$. Then it follows from (3.4.39) that

$$\begin{aligned} \left(\frac{d}{dt}\phi_m^*(t), w_j\right) + \langle A_\sigma(\phi_m(t))^*, w_j\rangle + (W'(\phi_m(t))^* + \mathbf{u}^* \cdot \nabla\phi_m(t)^*, w_j) \\ = (\phi_{0m}, w_j)\delta(t-0) - (\phi_m(T), w_j)\delta(t-T), \quad \forall j \geqslant 1, \end{aligned} \tag{3.4.47}$$

where δ denotes the Dirac delta function. Using (3.4.45), we can now pass to the limit in (3.4.47) with $m = n$ by fixing j, by which we deduce

$$\begin{aligned} \left(\frac{d}{dt}\phi^*, w_j\right) + (\chi^*, w_j) + (W'(\phi(t))^* + \mathbf{u}^* \cdot \nabla\phi(t)^*, w_j) \\ = (\phi_0, w_j)\delta(t-0) - (\xi, w_j)\delta(t-T), \quad j \geqslant 1. \end{aligned}$$

In consequence,

$$\frac{d\phi^*}{dt} + \chi^* + W'(\phi^*) + \mathbf{u}^* \cdot \nabla\phi^* = \phi_0\delta(t-0) - \xi\delta(t-T). \tag{3.4.48}$$

Restricting the above equation on $(0,T)$, we deduce

$$\phi_t + \chi + W'(\phi) + \mathbf{u} \cdot \nabla\phi = 0. \tag{3.4.49}$$

It can be seen from the above equation and the last statement of (3.4.45) that $\phi_t \in L^2(0,T;L^2(\Omega))$, which combined with the first statement of (3.4.45) implies that $\phi(0)$ and $\phi(T)$ have senses. Furthermore, we have $\phi(0) = \phi_0$ and $\phi(T) = \xi$.

It can be seen that we can prove the existence of weak solution provided that we can show $\chi = A_\sigma(\phi)$. This is done by applying Proposition 3.4.6. Indeed, using (3.4.39) and (3.4.49), we have (3.4.27) and (3.4.28) respectively. Moreover, combining (3.4.37), (3.4.45) with (3.4.46) (in this case $\sigma_n = \sigma$), we have (3.4.26). Finally, we obtain

$$A_\sigma(\phi_n) \to \chi = A_\sigma(\phi) \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)).$$

In conclusion, we have the existence of weak solution of (3.4.21) :

$$\phi \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)). \tag{3.4.50}$$

Part 3 : Regularization
In this part, we prove the H^2-regularity of ϕ about variable x. Let

$$f = \phi_t + W'(\phi) + \mathbf{u} \cdot \nabla\phi.$$

By (3.4.37) and (3.4.50), it can be verified by applying Hölder's inequality and Sobolev imbedding theorem that $f \in L^2(0,T;L^2(\Omega))$. According to (3.4.49), for almost every $t \in (0,T)$

$$\begin{cases} \nabla \cdot \left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right) = f, \quad (t,x) \in (0,T) \times \Omega. \\ \phi(t,x)|_{\partial\Omega} = 0. \end{cases}$$

The Lagrangian corresponding to this elliptic equation is

$$L(p,z,x) = \frac{\lambda}{2}|p|^2 + \sigma\sqrt{|p|^2+\epsilon},$$

which satisfies the structure condition in Proposition 3.4.5 (The process of the verification is direct and tedious, so we omit them). In this case, we have for almost every $t \in (0,T)$

$$\|\phi(t)\|_{H^2(\Omega)} \leqslant C\|f(t)\|_{L^2(\Omega)},$$

where C is a positive constant depending only on $\|\sigma\|_{C([0,T],C^2(\bar{\Omega}))}$. Therefore

$$\|\phi\|_{L^2(0,T;H^2(\Omega))} \leqslant C\|f\|_{L^2(0,T;L^2(\Omega))}. \tag{3.4.51}$$

In conclusion, we have

$$\phi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)). \tag{3.4.52}$$

The energy estimate (3.4.38) can be proved by (3.4.41) and (3.4.51). □

We end this section by recalling some classical results on the transport equation :

$$\begin{cases} \sigma_t + \nabla \cdot (\mathbf{u}\,\sigma) = 0, \\ \sigma(0,x) = \sigma_0(x), \end{cases} \tag{3.4.53}$$

where $\mathbf{u}$ is a given divergence-free vector field, smooth or non-smooth. We first consider the weak solution of (3.4.53). The following proposition is the classical result of Diperna and Lions and is essentially contained in [17]. A simplified proof can be found in [48, pp. 41].

Proposition 3.4.10. *Let*

$$\begin{cases} \sigma_n \in C([0,T];L^2(\Omega)), \\ 0 \leqslant \sigma_n(t,x) \leqslant C, \quad a.e.\ on \quad (0,T)\times\Omega, \\ \mathbf{u}_n \in L^2(0,T;V(\Omega)), \quad \|\mathbf{u}_n\|_{L^2(0,T;V(\Omega))} \leqslant C, \\ \mathbf{u}_n \to \mathbf{u} \quad weakly\ in \quad L^2(0,T;V(\Omega)), \\ \sigma_{n,t} + \nabla\cdot(\sigma_n \mathbf{u}_n) = 0, \quad in \quad \mathscr{D}'((0,T)\times\Omega), \\ \sigma_0^n \to \sigma_0 \quad in \quad L^2(\Omega), \\ 0 \leqslant \sigma_0(x) \leqslant C, \quad a.e.\ on\ \Omega. \end{cases} \tag{3.4.54}$$

Then for every $1 \leqslant p < \infty$*,*

$$\sigma_n \to \sigma \quad strongly\ in \quad C([0,T];L^p(\Omega)),$$

where $\sigma \in L^\infty((0,T)\times\Omega)$ *is the unique weak solution of the following equation*

$$\begin{cases} \sigma_t + \nabla\cdot(\sigma\mathbf{u}) = 0 \quad in \quad \mathscr{D}'(\Omega\times(0,T)), \\ \sigma(0) = \sigma_0 \quad a.e.\ on \quad \Omega. \end{cases} \tag{3.4.55}$$

Now we turn to the classical solution of (3.4.53).

Proposition 3.4.11. *Assume* $m \in \mathbb{N}^*,\ m \geqslant 5$ *and*

$$\begin{cases} \mathbf{u} \in L^2(0,T;V^m(\Omega)), \\ \sigma_0 \in C^2(\bar{\Omega}), \quad \sigma_0 \geqslant 0. \end{cases}$$

Then (3.4.53) *has a unique solution* σ *satisfying*

$$\begin{cases} \sigma \in C([0,T];C^2(\bar{\Omega})), \\ \sigma_t \in L^2(0,T;C^1(\bar{\Omega})), \\ 0 \leqslant \sigma \leqslant \|\sigma_0\|_{C(\bar{\Omega})}. \end{cases}$$

The proof of the above Proposition can be found in [48, p.p. 67]. We rewrite it here for the readers' convenience.

Proof. The solution of (3.4.53) by classical (and elementary) considerations on (divergence-free) transport equations is given by a simple integration along 'particle paths', i.e. solutions of the following ordinary differential equation

$$\begin{cases} \frac{dX}{ds} = \mathbf{u}(s,X), \\ X(s;t,x) = x, \quad x \in \bar{\Omega}, \quad t \in [0,T]. \end{cases} \tag{3.4.56}$$

In view of the properties of $\mathbf{u}$, there exists a unique solution X of (3.4.56), continuous in $(s,t) \in [0,T]^2$, smooth in $x \in \bar{\Omega}$ such that

$$\partial_x^\alpha X \in C([0,T]\times\bar{\Omega}\times[0,T])$$

for $|\alpha| \leqslant 4$ and $X(s;x,t) \in \bar{\Omega}$ for all $(s,t) \in [0,T]^2, x \in \bar{\Omega}$. Then we have

$$\sigma(x,t) = \sigma_0(X(0;x,t)), \quad \forall x \in \bar{\Omega}, \forall t \in [0,T].$$

Obviously, we deduce from the above formula that $0 \leqslant \sigma \leqslant \|\sigma_0\|_{C(\bar{\Omega})}$ and $\sigma \in C([0,T]; C^2(\bar{\Omega}))$. Since $\sigma_t = -\nabla \cdot (\mathbf{u}\sigma) = -\nabla\sigma \cdot \mathbf{u}$, we have for every $t \in [0,T]$,

$$\|\sigma_t(t)\|_{C^1(\bar{\Omega})} \leqslant \|\sigma(t)\|_{C^2(\bar{\Omega})} \|\mathbf{u}(t)\|_{C^1(\bar{\Omega})},$$

which implies

$$\|\sigma_t\|_{L^2(0,T;C^1(\bar{\Omega}))} \leqslant \|\sigma\|_{C([0,T];C^2(\bar{\Omega}))} \|\mathbf{u}\|_{L^2(0,T;C^1(\bar{\Omega}))}.$$

□

3.5 Construction of Approximate Solutions

In this section, we consider the approximate equations of (3.1.1) by adding a viscous term $\Delta^6 \mathbf{u}$. More precisely, we will study the following initial-boundary value problem :

$$\begin{cases} \mathbf{u}_t + \nu\Delta^6\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p \\ \qquad -\nabla\cdot\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right) \quad \text{in} \quad (0,T)\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad (0,T)\times\Omega, \\ \phi_t - \nabla\cdot\left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi \quad \text{in} \quad (0,T)\times\Omega, \\ \sigma_t + \mathbf{u}\cdot\nabla\sigma = 0 \quad \text{in} \quad (0,T)\times\Omega, \\ \mathbf{u}|_{\partial\Omega} = 0, \ \Delta\mathbf{u}|_{\partial\Omega} = 0, \ \Delta^2\mathbf{u}|_{\partial\Omega} = 0, \\ \frac{\partial}{\partial\mathbf{n}}\mathbf{u}|_{\partial\Omega} = 0, \ \frac{\partial}{\partial\mathbf{n}}\Delta\mathbf{u}|_{\partial\Omega} = 0, \ \frac{\partial}{\partial\mathbf{n}}\Delta^2\mathbf{u}|_{\partial\Omega} = 0, \\ \phi|_{\partial\Omega} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \\ \phi(0,\cdot) = \phi_0 \quad \text{in} \quad \Omega, \\ \sigma(0,\cdot) = \sigma_0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.5.1}$$

Remark 3.5.1. *In the previous section, we use Galerkin approximation for the viscous Navier-Stokes equation and nonlinear parabolic equation separately. A natural ideal is to approximate the full system* (3.1.1) *by a uniform Galerkin scheme. However, to our best knowledge, this seems hopeless for two reasons : first, it is not easy to approximate the transport equation by a Galerkin scheme such that the classical result of Diperna and Lions can be applied directly. Second, since the unknown σ appears in the coefficient of the nonlinear elliptic operator in (3.1.1c), it is difficult to approximate the full system* (3.1.1) *by a finite dimensional nonlinear system. Due to the above reasons we introduce a regularizing term $\Delta^6\mathbf{u}$ whose role is to obtain a high* *order spatial derivatives estimate for* $\mathbf{u}$ *such that the transport equation can be solved by the Proposition 3.4.11. We note that such kind of regularizing method have been widely used in the references (see, for instance, [10] and [33]).*

We will show that the above system has an a priori estimate which will play a key role in guaranteeing the existence of weak solution for (3.5.1) and which is quite similar to (3.2.21). For

$$(\mathbf{u}, \phi, \sigma) \in L^\infty(0,T; H(\Omega)) \times L^\infty(0,T; H_0^1(\Omega)) \times C([0,T]; L^2(\Omega)),$$

we can define $E(t)$ for almost every $t \in (0,T)$ by (3.2.22). The following proposition shows that $E(t)$ is a non-increasing functional for the system (3.5.1).

Proposition 3.5.2 (A priori estimates)**.** *Let*

$$\begin{cases} \mathbf{u} \in L^2(0,T; V^6(\Omega)) \cap H^1(0,T; V^{-6}(\Omega)), \\ \phi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega)), \\ \sigma \in H^1(0,T; C^1(\bar{\Omega})) \cap C([0,T]; C^2(\bar{\Omega})). \end{cases} \tag{3.5.2}$$

be a solution of system (3.5.1) *(in the sense of distribution for the equations of* $\mathbf{u}$*), then we have the following a priori estimate*

$$E(t) + \int_{Q^t} (|A_\sigma(\phi) + W'(\phi)|^2 + |\nabla \mathbf{u}|^2 + \nu |\Delta^3 \mathbf{u}|^2) \mathrm{d}x\mathrm{d}t = E(0), \quad a.e.\ t \in (0,T). \tag{3.5.3}$$

Remark 3.5.3. *We say* $\mathbf{u} \in L^2(0,T; V^6(\Omega))$ *is a solution of the first equation of* (3.5.1) *in the sense of distribution provided that*

$$\nu \int_{Q^T} \Delta^3 \mathbf{u} \cdot \Delta^3 \mathbf{v} \mathrm{d}x\mathrm{d}t - \int_\Omega \mathbf{u}_0(\cdot)\mathbf{v}(0,\cdot)\mathrm{d}x - \int_{Q^T} \mathbf{u} \cdot \mathbf{v}_t \mathrm{d}x\mathrm{d}t - \int_{Q^T} (\mathbf{u} \otimes \mathbf{u} - \nabla \mathbf{u}) : \nabla \mathbf{v} \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \langle \mathbf{F}(\phi,\sigma) \cdot \mathbf{v} \rangle_{V^{-3},V^3} \mathrm{d}t, \quad \textit{for all} \quad \mathbf{v} \in C^1([0,T]; V^6(\Omega)),\ \mathbf{v}(T,\cdot) = 0. \tag{3.5.4}$$

Recall that $\mathbf{F}(\phi,\sigma)$ *is defined by* (3.3.5)*.*

Before giving the proof, we first state a lemma whose proof is a repeated application of Sobolev imbedding theorem and Hölder inequality.

Lemma 3.5.4. *Under the assumption* (3.5.2), *we have*

$$\begin{cases} \mathbf{u}\otimes\mathbf{u}\in L^2(0,T;H(\Omega)), \\ A_\sigma(\phi)\nabla\phi\in L^2(0,T;L^1(\Omega)\cap V^{-2}(\Omega)), \\ A_\sigma(\phi)\nabla\phi\cdot\mathbf{u}\in L^1(Q^T), \\ \nabla\sigma\sqrt{|\nabla\phi|^2+\epsilon}\in L^2(Q^T). \end{cases} \tag{3.5.5}$$

Proof of Proposition 3.5.2. First, let us recall some notation : if $M=(m_{ij})$, then we set

$$(\nabla\cdot M)_i := \sum_j \frac{\partial m_{ij}}{\partial x_j}.$$

If $v=(v_i)$, then we write

$$(\nabla v)_{ij} := \frac{\partial v_i}{\partial x_j}.$$

In particular, using the Leibniz rule, we have the following relation :

$$\nabla\cdot(u\otimes v) = (\nabla u)\,v + (\nabla\cdot v)\,u. \tag{3.5.6}$$

We first rewrite the right-hand side of the first equation of (3.5.1). It can be seen from Lemma 3.5.4 that we can calculate directly : Using (3.5.6), we obtain

$$\begin{aligned} &\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right) \\ &\quad = \nabla^2\phi\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)+\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)\nabla\phi, \end{aligned} \tag{3.5.7}$$

where $\nabla^2\phi=\left(\frac{\partial\phi}{\partial x_i\partial x_j}\right)_{1\leqslant i,j\leqslant 3}$.

On the other hand, using again Leibniz rule, we have

$$\begin{aligned} \nabla\left(\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}\right) = \nabla^2\phi&\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right) \\ &+\left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma. \end{aligned}$$

Combining the above equation with (3.5.7) and using the definition (3.2.10) of A_σ yield

$$\begin{aligned} &\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right) \\ &= -A_\sigma(\phi)\nabla\phi+\nabla\left(\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)-\left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma. \end{aligned} \tag{3.5.8}$$

Consequently, if we denote by $q_{\phi,\sigma}$ the quantity

$$q_{\phi,\sigma} = \frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon},$$

then we can rewrite (3.5.1) as

$$\begin{aligned}\mathbf{u}_t + \nu\Delta^6\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= \Delta\mathbf{u} + \nabla(p+q)\\ &+ A_\sigma(\phi)\nabla\phi + \left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma \quad \text{in} \quad (0,T)\times\Omega, \quad (3.5.9)\end{aligned}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{3.5.10}$$

$$\phi_t + \mathbf{u}\cdot(\nabla\phi) = -W'(\phi) - A_\sigma(\phi) \quad \text{in} \quad (0,T)\times\Omega, \tag{3.5.11}$$

$$\sigma_t + \mathbf{u}\cdot\nabla\sigma = 0 \quad \text{in} \quad (0,T)\times\Omega. \tag{3.5.12}$$

- We first claim that :

$$\begin{aligned}\int_\Omega |\mathbf{u}(t,\cdot)|^2 dx + \|\nabla\mathbf{u}\|^2_{L^2(0,t;H(\Omega))} + \nu\|\Delta^3\mathbf{u}\|^2_{L^2(0,t;H(\Omega))}\\ - \int_{Q^t} A_\sigma(\phi)\nabla\phi\cdot\mathbf{u}dxd\tau - \int_{Q^t}\nabla\sigma\cdot\mathbf{u}\sqrt{|\nabla\phi|^2+\epsilon}dxd\tau\\ = \int_\Omega |\mathbf{u}_0|^2 dx. \quad a.e.\ t\in(0,T). \quad (3.5.13)\end{aligned}$$

Formally, it can be done by multiplying (3.5.9) by $\mathbf{u}$ and integrate by parts. However, we can obtain it rigorously : for almost every $t\in(0,T)$, since $\mathbf{u}\in L^2(0,T;V^6(\Omega))$, it is clear that $\mathbf{u}\in L^2(0,t;V^6(\Omega))$. So there exists $\mathbf{v}_n\in C^1([0,t];V^6(\Omega))$, $\mathbf{v}(t,\cdot)=0$ such that

$$\mathbf{v}_n \to \mathbf{u} \in L^2(0,t;V^6(\Omega)). \tag{3.5.14}$$

Substituting equation (3.5.8) into (3.5.4) and replacing $\mathbf{v}$ by $\mathbf{v}_n$ and T by t, it follows that

$$\begin{aligned}\nu\int_{Q^t}\Delta^3\mathbf{u}\cdot\Delta^3\mathbf{v}_n\mathrm{d}x\mathrm{d}\tau - \int_\Omega \mathbf{u}_0(\cdot)\mathbf{v}_n(0,\cdot)\mathrm{d}x - \int_{Q^t}\mathbf{u}\cdot\mathbf{v}_{n,t}\mathrm{d}x\mathrm{d}\tau - \int_{Q^t}(\mathbf{u}\otimes\mathbf{u}-\nabla\mathbf{u}):\nabla\mathbf{v}_n\mathrm{d}x\mathrm{d}\tau\\ = \int_0^t \langle A_\sigma(\phi)\nabla\phi,\mathbf{v}_n\rangle_{V^{-2},V^2}\mathrm{d}\tau + \int_{Q^t}\nabla\sigma\cdot\mathbf{v}_n\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x\mathrm{d}\tau. \quad (3.5.15)\end{aligned}$$

Since $\mathbf{u}_t\in L^2(0,T;V^{-6}(\Omega))$, after integrating by part, we have

$$-\int_0^t\langle\mathbf{u}_t,\mathbf{v}_n\rangle_{V^{-6},V^6}\mathrm{d}\tau = \int_\Omega\mathbf{u}_0(\cdot)\mathbf{v}_n(0,\cdot)\mathrm{d}x + \int_{Q^t}\mathbf{u}\cdot\mathbf{v}_{n,t}\mathrm{d}x\mathrm{d}\tau, \tag{3.5.16}$$

Substituting the above equation into (3.5.15) yields

$$\nu \int_{Q^t} \Delta^3\mathbf{u}\cdot\Delta^3\mathbf{v}_n \mathrm{d}x\mathrm{d}\tau + \int_0^t \langle \mathbf{u}_t, \mathbf{v}_n\rangle_{V^{-6},V^6}\mathrm{d}\tau - \int_{Q^t} (\mathbf{u}\otimes\mathbf{u} - \nabla\mathbf{u}) : \nabla\mathbf{v}_n \mathrm{dxd}\tau$$

$$= \int_0^t \langle A_\sigma(\phi)\nabla\phi, \mathbf{v}_n\rangle_{V^{-2},V^2}\mathrm{d}\tau + \int_{Q^t} \nabla\sigma\cdot\mathbf{v}_n\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x\mathrm{d}\tau. \quad (3.5.17)$$

By Lemma 3.5.4, we can pass to the limit $n\to\infty$ in the above equation, which gives (3.5.13).

- We multiply (3.5.11) by $\phi_t + \mathbf{u}\cdot\nabla\phi$ and integrate by parts :

$$\int_\Omega |\phi_t + \mathbf{u}\cdot\nabla\phi|^2 \,\mathrm{d}x = -\frac{d}{dt}\int_\Omega \left(W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)\mathrm{d}x$$

$$- \int_\Omega A_\sigma(\phi)\nabla\phi\cdot\mathbf{u}\mathrm{d}x + \int_\Omega \sigma_t\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x, \quad (3.5.18)$$

which implies that

$$\int_{Q^t} |\phi_t + \mathbf{u}\cdot\nabla\phi|^2 \,\mathrm{d}x\mathrm{d}\tau + \int_0^t \frac{d}{dt}\int_\Omega \left(W(\phi) + \frac{\lambda}{2}|\nabla\phi|^2 + \sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)\mathrm{d}x\mathrm{d}\tau$$

$$= -\int_{Q^t} A_\sigma(\phi)\nabla\phi\cdot\mathbf{u}\mathrm{d}x\mathrm{d}\tau + \int_{Q^t} \sigma_t\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x\mathrm{d}\tau, \quad (3.5.19)$$

Summing (3.5.13) and (3.5.19), we obtain, for almost every $t\in(0,T)$

$$E(t) + \int_{Q^t} |\nabla\mathbf{u}|^2 + \nu|\Delta^3\mathbf{u}|^2\mathrm{d}x\mathrm{d}\tau + \int_{Q^t} |\phi_t + \mathbf{u}\cdot\nabla\phi|^2 \,\mathrm{d}x\mathrm{d}\tau$$

$$= E(0) + \int_{Q^t} (\sigma_t + \nabla\sigma\cdot\mathbf{u})\sqrt{|\nabla\phi|^2+\epsilon}\mathrm{d}x\mathrm{d}\tau.$$

The above equation and (3.5.12) imply the result. □

Remark 3.5.5. *The readers who are familiar with compressible Navier-Stokes equations or density-dependent Navier-Stokes equations (see, for example, [48, Chapter 1]) might have doubts why we don't use the following approximate method to enhance the regularity of σ instead of adding the viscous terms $\nu\Delta^6\mathbf{u}$:*

$$\sigma_t + \mathbf{u}_\epsilon\cdot\nabla\sigma = 0, \quad (3.5.20)$$

where $\mathbf{u}^\epsilon = \mathbf{u} * \omega_\epsilon$ with ω_ϵ a regularizing kernel. The reason can be found in the last step of the proof of the above proposition : if we adopt (3.5.20)*, we will not have the a priori estimate* (3.5.3)*.*

Basing on the a priori estimate (3.5.3), we will show that system (3.5.1) has a unique generalized solution provided that the initial data is regular.

Proposition 3.5.6. *Assume that the initial data* $(\mathbf{u}_0, \phi_0, \sigma_0)$ *satisfy*

$$\begin{cases} \mathbf{u}_0 \in H(\Omega), \\ \phi_0 \in H_0^1(\Omega), \\ \sigma_0 \in C^2(\bar{\Omega}), \quad \sigma_0 \geqslant 0. \end{cases} \tag{3.5.21}$$

Then, for every $T > 0$, *system* (3.5.1) *admet a unique global in time solution*

$$\begin{cases} \mathbf{u} \in L^2(0,T;V^6(\Omega)) \cap C([0,T];H(\Omega)), \\ \phi \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,T];H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ \sigma \in C([0,T];C^2(\bar{\Omega})) \cap H^1(0,T;C^1(\bar{\Omega})). \end{cases} \tag{3.5.22}$$

Furthermore, the solution satisfies (3.5.3) *and that, for every* $0 \leqslant a \leqslant b \leqslant \infty$,

$$\text{meas}\{x \in \Omega | a \leqslant \sigma(t,x) \leqslant b\} \quad \textit{is independent of } t. \tag{3.5.23}$$

Proof. **Part 1 : Local existence**
Let $0 \leqslant \hat{T} \leqslant T$ and

$$\mathscr{B} = \{(\phi, \sigma) \in L^4(0,\hat{T};H_0^1(\Omega)) \times C([0,\hat{T}];L^2(\Omega))\},$$

$$\mathscr{B}_R = \{(\phi,\sigma) \in \mathscr{B} \quad | \quad \|\phi\|_{L^4(0,\hat{T};H_0^1(\Omega))} + \|\sigma\|_{C([0,\hat{T}];L^2(\Omega))} \leqslant R\}. \tag{3.5.24}$$

Let $F(\bar{\phi}, \bar{\sigma})$ be a nonlinear operator defined by (3.3.5). We define a map by

$$\mathscr{F}(\bar{\phi}, \bar{\sigma}) = (\phi, \sigma), \quad \forall (\bar{\phi}, \bar{\sigma}) \in \mathscr{B}_R$$

where (ϕ, σ) is solutions of the following nonlinear system

$$\begin{cases} \mathbf{u}_t + \nu\Delta^6\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \Delta\mathbf{u} + \nabla p + F(\bar{\phi},\bar{\sigma}) \quad \text{in} \quad (0,\hat{T}) \times \Omega, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad (0,\hat{T}) \times \Omega, \\ \phi_t - \nabla \cdot \left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right) = -W'(\phi) - \mathbf{u}\cdot\nabla\phi \quad \text{in} \quad (0,\hat{T}) \times \Omega, \\ \sigma_t + \mathbf{u}\cdot\nabla\sigma = 0 \quad \text{in} \quad (0,\hat{T}) \times \Omega, \\ \mathbf{u}|_{\partial\Omega} = 0, \ \Delta\mathbf{u}|_{\partial\Omega} = 0, \ \Delta^2\mathbf{u}|_{\partial\Omega} = 0, \\ \frac{\partial}{\partial\mathbf{n}}\mathbf{u}|_{\partial\Omega} = 0, \ \frac{\partial}{\partial\mathbf{n}}\Delta\mathbf{u}|_{\partial\Omega} = 0, \ \frac{\partial}{\partial\mathbf{n}}\Delta^2\mathbf{u}|_{\partial\Omega} = 0, \\ \phi|_{\partial\Omega} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \\ \phi(0,\cdot) = \phi_0, \\ \sigma(0,\cdot) = \sigma_0. \end{cases} \tag{3.5.25}$$

Since the viscous constant ν plays no important role here, we set $\nu = 1$ in (3.5.25) and in (3.5.1). Under the above notation, we will show that $\mathscr{F}$ has a fixed point.

For every $(\bar{\phi}, \bar{\sigma}) \in \mathscr{B}_R$,

$$\|\bar{\phi}\|_{L^4(0,\hat{T};H_0^1(\Omega))} + \|\bar{\sigma}\|_{C([0,\hat{T}];L^2(\Omega))} \leqslant R. \tag{3.5.26}$$

Using Proposition 3.3.1, there exists a positive constant $C = C(\Omega)$ such that

$$\begin{aligned}&\|F(\bar{\phi}, \bar{\sigma})\|_{L^2(0,\hat{T};V^{-3}(\Omega))} \\ &\quad \leqslant C\left(\|\bar{\phi}\|^2_{L^4(0,\hat{T};H_0^1(\Omega))} + \|\bar{\sigma}\|_{C([0,\hat{T}];L^2(\Omega))}\|\bar{\phi}\|_{L^2(0,\hat{T};H_0^1(\Omega))}\right).\end{aligned} \tag{3.5.27}$$

Applying Proposition 3.4.1 to the viscous NSE in (3.5.25) yields

$$\begin{aligned}&\|\mathbf{u}\|_{L^2(0,\hat{T};V^6(\Omega))} + \|\mathbf{u}\|_{C([0,\hat{T}];H(\Omega))} \\ &\qquad \leqslant C\|F(\bar{\phi}, \bar{\sigma})\|_{L^2(0,\hat{T};V^{-3}(\Omega))} + \|\mathbf{u}_0\|_{H(\Omega)}.\end{aligned}$$

Combining the above estimate with (3.5.27) and (3.5.24), we deduce that

$$\|\mathbf{u}\|_{L^2(0,\hat{T};V^6(\Omega))} + \|\mathbf{u}\|_{C([0,\hat{T}];H(\Omega))} \leqslant C(R, \|\mathbf{u}_0\|_{H(\Omega)}). \tag{3.5.28}$$

Now we are in a position of solving the transport equation in (3.5.25) by using Proposition 3.4.11 :

$$\|\sigma\|_{C([0,\hat{T}];C^2(\bar{\Omega}))} + \|\sigma_t\|_{L^2(0,\hat{T};C^1(\bar{\Omega}))} \leqslant C(R, \|\sigma_0\|_{C^2(\bar{\Omega})}, \|\mathbf{u}_0\|_{H(\Omega)}, T). \tag{3.5.29}$$

With the above regularity of σ, we can solve the parabolic equation of ϕ in (3.5.25) by applying Proposition 3.4.8, which yields :

$$\begin{aligned}&\|\phi\|_{L^2(0,\hat{T};H^2(\Omega))} + \|\phi\|_{C([0,\hat{T}];H_0^1(\Omega))} + \|\phi\|_{H^1(0,\hat{T};L^2(\Omega))} \\ &\qquad \leqslant C(R, \|\sigma_0\|_{C^2(\bar{\Omega})}, \|\phi_0\|_{H_0^1(\Omega)}, \|\mathbf{u}_0\|_{H(\Omega)}, T).\end{aligned} \tag{3.5.30}$$

Combining (3.5.28), (3.5.29) and (3.5.30), we conclude that, for every $(\bar{\phi}, \bar{\sigma}) \in \mathscr{B}_R$, system (3.5.25) has a solution $(\mathbf{u}, \phi, \sigma)$ satisfying (3.5.22), which induces that

$$\mathscr{F} : \mathscr{B}_R \to \mathscr{B} \quad \text{is compact.}$$

Furthermore, using (3.5.30) and (3.5.29), it follows that :

$$\begin{aligned}\|\phi\|_{L^4(0,\hat{T};H_0^1(\Omega))} &\leqslant \hat{T}^{\frac{1}{4}}\|\phi\|_{C([0,\hat{T}];H_0^1(\Omega))} \\ &\leqslant \hat{T}^{\frac{1}{4}} C(R, \|\phi_0\|_{H_0^1(\Omega)}, \|\mathbf{u}_0\|_{H(\Omega)}, \|\sigma_0\|_{C^2(\bar{\Omega})}, T),\end{aligned}$$

and

$$\begin{aligned}\|\sigma\|_{C([0,\hat{T}];L^2(\Omega))} &\leqslant \|\sigma_0\|_{L^2(\Omega)} + \int_0^{\hat{T}} \|\sigma_t(\tau, \cdot)\|_{L^2(\Omega)} \mathrm{d}\tau \\ &\leqslant \|\sigma_0\|_{L^2(\Omega)} + \hat{T}^{\frac{1}{2}}\|\sigma_t\|_{L^2(0,\hat{T};L^2(\Omega))} \\ &\leqslant \|\sigma_0\|_{L^2(\Omega)} + \hat{T}^{\frac{1}{2}} C(R, \|\sigma_0\|_{C^2(\bar{\Omega})}, \|\mathbf{u}_0\|_{H(\Omega)}, T).\end{aligned}$$

It follows from the above two inequalities that, by choosing

$$R \geqslant \max\{\|\phi_0\|_{H_0^1(\Omega)}, \|\mathbf{u}_0\|_{H(\Omega)}, 2\|\sigma_0\|_{C^2(\bar{\Omega})}\},$$

we have

$$\|\phi\|_{L^4(0,\hat{T};H_0^1(\Omega))} \leqslant \hat{T}^{\frac{1}{4}} C(R), \quad \|\sigma\|_{C([0,\hat{T}];L^2(\Omega))} \leqslant \frac{R}{2} + \hat{T}^{\frac{1}{2}} C(R).$$

By choosing $\hat{T}$ small enough, we deduce from the above estimates that $\mathscr{F}$ maps $\mathscr{B}_R$ into itself, which combined with Schauder's fixed point theorem shows that $\mathscr{F}$ has a fixed point

$$\mathscr{F}(\phi, \sigma) = (\phi, \sigma).$$

By solving $\mathbf{u}$ from the first two equations of (3.5.25), we obtain the local in time solution $(\mathbf{u}, \phi, \sigma)$ of (3.5.1).

Part 2 : Global existence

Clearly, the fixed point (ϕ, σ) and the corresponding $\mathbf{u}$ constructed in the last part is a local in time solution for (3.5.1) with (3.5.22). In order to obtain a global in time solution, we must prove that

$$\|\mathbf{u}\|_{C([0,\hat{T}];H(\Omega))} + \|\sigma\|_{C([0,\hat{T}];C^2(\bar{\Omega}))} + \|\phi\|_{C([0,\hat{T}];H_0^1(\Omega))} \tag{3.5.31}$$

does not blow up in finite time. However, it can be seen from (3.5.27), (3.5.28), (3.5.29) and (3.5.30) that, it is only necessary to show that

$$\|\phi\|_{L^4(0,\hat{T};H_0^1(\Omega))} + \|\sigma\|_{C([0,\hat{T}];L^2(\Omega))} \tag{3.5.32}$$

do not blow up in finite time $\hat{T} \leqslant T$.

Since the solution $(\mathbf{u}, \phi, \sigma)$ satisfy (3.5.22), we deduce from Proposition 3.5.2 that $(\mathbf{u}, \phi, \sigma)$ satisfy (3.5.3), which indicates, combined with (3.2.22), that $\|\phi\|_{L^\infty(0,\hat{T};H_0^1(\Omega))}$ does not blow up in finite time. It remains to prove (3.5.23) which implies that $\|\sigma\|_{C([0,\hat{T}];L^2(\Omega))}$ does not blow up in finite time $\hat{T} \leqslant T$ (by choosing in (3.5.23) $a = 0$ and $b = \|\sigma_0\|_{L^\infty(\Omega)}$). However, this is nothing but the celebrated Liouville's theorem (see [48, page 35]) which can be proved as follows :

Let $\beta \in C^1([0,\infty);\mathbb{R})$, using (3.5.22), it follows that $\beta(\sigma)$ satisfies

$$\frac{\partial \beta(\sigma)}{\partial t} + \nabla \cdot (\mathbf{u}\beta(\sigma)) = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \beta(\sigma) = \beta'(\sigma) \left\{\frac{\partial \sigma}{\partial t} + \mathbf{u} \cdot \nabla \sigma\right\} = 0.$$

Therefore, integrating over Ω, we find, using the boundary conditions, that

$$\frac{d}{dt} \int_\Omega \beta(\sigma) \mathrm{d}x = 0,$$

or equivalently that $\int_\Omega \beta(\sigma)dx$ is independent of t. In particular, choosing $\beta = g_n \in C^1([0,\infty),\mathbb{R})$, $1 \geqslant g_n \geqslant 0$ such that $g_n = 0$ if $t \notin [a,b]$ (where $0 \leqslant a < b < \infty$ are fixed) and $g_n(t) = 1$ if $t \in [a+\frac{1}{n}, b-\frac{1}{n}]$ (take $n \geqslant \frac{2}{b-a}$), we deduce (3.5.23) from the preceding fact upon letting $n \to \infty$. In conclusion, we show that (3.5.32) and thus (3.5.31) does not blow up in finite time, which implies the existence of global solution of (3.5.1) with regularity (3.5.22). □

3.6 Proof of the main result

In this section, we give the proof of Theorem 3.1.1.

Construction of approximate solutions. The first step consists in proving the existence of approximate solutions for system (3.1.1), (3.1.2). As it can be seen in the formal derivation of the energy estimate (Subsection 3.2.1), this approximation has to be chosen carefully in order to preserve an energy estimates which permits to pass to the limit. Moreover, in order to handle σ which is transported by u, we need that the approximation of u is regular enough. To avoid modifying the right-hand side of (3.1.1a), i.e. the term

$$\nabla \cdot \left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla \phi|^2 + \epsilon}} \right) \nabla \phi \otimes \nabla \phi \right), \tag{3.6.1}$$

we prefer to add some viscosity in (3.1.1a). This method can be used to prove the existence of the classical Navier-Stokes system (see, for instance, [46, pp.97–98]). More precisely, we consider the following system

$$\begin{aligned} &\int_\Omega \mathbf{u}_0(x) \cdot \mathbf{v}(0,x)dx - \int_{Q^T} \mathbf{u}_n \cdot \mathbf{v}_t \mathrm{d}x - \int_{Q^T} (\mathbf{u}_n \otimes \mathbf{u}_n - \nabla \mathbf{u}_n) : \nabla \mathbf{v} \mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{n} \int_{Q^T} \Delta^3 \mathbf{u}_n \cdot \Delta^3 \mathbf{v} \mathrm{d}x\mathrm{d}t + \int_{Q^T} \nabla \mathbf{v} : \left((\lambda + \frac{\sigma_n}{\sqrt{|\nabla \phi_n|^2 + \epsilon}}) \nabla \phi_n \otimes \nabla \phi_n \right) \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.6.2}$$

for all $\mathbf{v} \in C^1([0,T];V^6(\Omega)), \mathbf{v}(T,\cdot) = 0$,

$$\begin{aligned} &\int_\Omega \phi_0(x)\psi(0,x)\mathrm{d}x - \int_{Q^T} \phi_n \psi_t \mathrm{d}x\mathrm{d}t + \int_{Q^T} \left((\lambda + \frac{\sigma_n}{\sqrt{|\nabla \phi_n|^2 + \epsilon}}) \nabla \phi_n \right) \cdot \nabla \psi \mathrm{d}x\mathrm{d}t \\ &\qquad = - \int_{Q^T} W'(\phi_n)\psi \mathrm{d}x\mathrm{d}t + \int_{Q^T} \phi_n \mathbf{u}_n \cdot \nabla \psi \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.6.3}$$

for all $\psi \in C^1([0,T];V^3(\Omega)), \psi(T,\cdot) = 0$,

$$\int_\Omega \sigma_0^n(x)\eta(0,x)\mathrm{d}x - \int_{Q^T} \sigma_n \eta_t \mathrm{d}x\mathrm{d}t = \int_{Q^T} \phi_n \mathbf{u}_n \cdot \nabla \eta \mathrm{d}x\mathrm{d}t \tag{3.6.4}$$

for all $\eta \in C^1([0,T];V^6(\Omega)), \eta(T,\cdot) = 0$.

One of the differences with the original system consists in the dissipative term in (3.6.2) which can be written in a strong form as $-\frac{1}{n}\Delta^6\mathbf{u}_n$. Using the elliptic regularity, this term will allow to handle the term (3.6.1) which belongs to $L^2(0,T;V^{-3}(\Omega))$ with the regularity (3.2.24). The other difference is the regularization of the initial condition for σ_0. We have $\sigma_0^n \geqslant 0$ in Ω, $\sigma_0^n \in C^2(\overline{\Omega})$, and

$$\sigma_0^n \to \sigma_0 \quad \text{in} \quad L^\infty(\Omega). \tag{3.6.5}$$

Some uniform bounds for the approximate solutions By Proposition 3.5.6, we can prove the existence of a weak solution $(\mathbf{u}_n, \phi_n, \sigma_n)$ of (3.6.2), (3.6.3) and (3.6.4) with the energy estimates

$$\begin{aligned}\int\limits_\Omega \left(|\mathbf{u}_n(t)|^2 + \frac{\lambda}{2}|\nabla\phi_n(t)|^2 + W(\phi_n(t)) + \sigma_n\sqrt{|\nabla\phi_n(t)|^2+\epsilon}\right)\mathrm{d}x \\ + \int\limits_0^t\int\limits_\Omega |\nabla\mathbf{u}_n|^2 + \left|W'(\phi_n) + A_\sigma(\phi_n)\right|^2 \mathrm{d}x\mathrm{d}s + \frac{1}{n}\int\limits_0^t\int\limits_\Omega |\Delta^3\mathbf{u}_n|^2\mathrm{d}x\mathrm{d}s \\ \leqslant \int\limits_\Omega \left(|\mathbf{u}_0|^2 + \frac{\lambda}{2}|\nabla\phi_0|^2 + W(\phi_0) + \sigma_0^n\sqrt{|\nabla\phi_0|^2+\epsilon}\right)\mathrm{d}x,\end{aligned} \tag{3.6.6}$$

and with

$$\sigma_n \geqslant 0 \quad \text{in} \quad Q^T, \quad \|\sigma_n(t)\|_{L^\infty(\Omega)} = \|\sigma_0^n\|_{L^\infty(\Omega)} \leqslant \|\sigma_0\|_{L^\infty(\Omega)}, \quad t \in (0,T). \tag{3.6.7}$$

The two above equations imply the following bounds :

$$\{\phi_n\} \quad \text{is bounded in} \quad L^\infty([0,T];H_0^1(\Omega)), \tag{3.6.8}$$

$$\{\mathbf{u}_n\} \quad \text{is bounded in} \quad L^2(0,T;V^1(\Omega)) \cap L^\infty([0,T];V^0(\Omega)), \tag{3.6.9}$$

$$\left\{\frac{1}{\sqrt{n}}\Delta^3\mathbf{u}_n\right\} \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), \tag{3.6.10}$$

$$\{W'(\phi_n)\} \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), \tag{3.6.11}$$

$$\{A_{\sigma_n}(\phi_n)\} \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)). \tag{3.6.12}$$

Note that (3.6.11) is a consequence of (3.3.4) and (3.6.8), whereas, (3.6.12) is obtained by using (3.6.11) and (3.6.6).

In order to obtain some compactness in the later part, we also need some bounds on $\{\mathbf{u}_{n,t}\}$ and on $\{\phi_{n,t}\}$. Equations (3.6.3) and (3.5.22) imply that

$$\phi_{n,t} = -A_{\sigma_n}(\phi_n) - W'(\phi_n) - \mathbf{u}_n\cdot\nabla\phi_n \quad \text{in} \quad a.e.\ (t,x) \in (0,T)\times\Omega,$$

and thus, from (3.6.8)-(3.6.12), we deduce

$$\{\phi_{n,t}\} \quad \text{is bounded in} \quad L^2(0,T;L^1(\Omega)). \tag{3.6.13}$$

The above estimate, (3.6.8) and Lemma 3.3.2 imply that, up to a subsequence,

$$\phi_n \to \phi \quad \text{strongly in} \quad C([0,T]; L^4(\Omega)), \tag{3.6.14}$$

with $\phi \in C([0,T]; L^4(\Omega))$.

On the other hand, from (3.6.7) and (3.6.8) and by applying Proposition 3.3.1, we deduce that

$$\{\mathbf{F}(\sigma_n, \phi_n)\} \quad \text{is bounded in} \quad L^2(0,T; V^{-3}(\Omega)). \tag{3.6.15}$$

Combining (3.6.15) with Proposition 3.4.1 yields

$$\{\mathbf{u}_{n,t}\} \quad \text{is bounded in} \quad L^2(0,T; V^{-6}(\Omega)). \tag{3.6.16}$$

The above estimate combined with (3.6.9) and Lemma 3.3.3 indicates that

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in} \quad L^2(0,T; V^0(\Omega)). \tag{3.6.17}$$

for some $\mathbf{u} \in L^2(0,T; V^0(\Omega))$.

Pass to the limits. By the weak compact properties of Banach spaces, we deduce from (3.6.8)-(3.6.12), (3.6.14), (3.6.17) and (3.6.20) that, up to the extraction of a subsequence,

$$\phi_n \to \phi \quad \text{weak-star in} \quad L^\infty(0,T; H_0^1(\Omega)), \tag{3.6.18a}$$
$$\phi_n(T) \to \xi \quad \text{weakly in} \quad H_0^1(\Omega), \tag{3.6.18b}$$
$$A_{\sigma_n}(\phi_n) \to \chi \quad \text{weakly in} \quad L^2(0,T; V^0(\Omega)), \tag{3.6.18c}$$
$$\frac{1}{\sqrt{n}}\Delta^3 \mathbf{u}_n \to \Phi \quad \text{weakly in} \quad L^2(0,T; V^0(\Omega)), \tag{3.6.18d}$$
$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in} \quad L^2(0,T; V(\Omega)), \tag{3.6.18e}$$

and

$$\phi_n \to \phi \quad \text{strongly in} \quad C([0,T]; L^4(\Omega)), \tag{3.6.19a}$$
$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in} \quad L^2(0,T; V^0(\Omega)), \tag{3.6.19b}$$
$$\sigma_n \to \sigma \quad \text{strongly in} \quad C([0,T]; L^p(\Omega)), \quad 1 \leqslant p < \infty. \tag{3.6.19c}$$

By (3.6.4), (3.6.7) and (3.6.18e), we can apply Proposition 3.4.10, which induces that for every $1 \leqslant p < \infty$,

$$\sigma_n \to \sigma \quad \text{strongly in} \quad C([0,T]; L^p(\Omega)), \tag{3.6.20}$$

where σ is the unique weak solution of transport equation (see (3.2.27))

$$\sigma_t + \nabla \cdot (\sigma \mathbf{u}) = 0, \quad \sigma(0,x) = \sigma_0.$$

The last argument to pass to the limit in (3.6.2)–(3.6.4), is to obtain a strong convergence of $\nabla\phi_n$:

$$\nabla\phi_n \to \nabla\phi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{3.6.21}$$

Indeed, by the regularity result of ϕ_n (see (3.5.22)), ϕ_n are also strong solutions, which implies that for almost every $(t,x) \in (0,T)\times\Omega$:

$$\dot{\phi}_n + A_{\sigma_n}(\phi_n) + W'(\phi_n) + \mathbf{u}_n \cdot \nabla\phi_n = 0. \tag{3.6.22}$$

The last equation induces :

$$\frac{1}{2}\int\limits_{\Omega} |\phi_n(T)|^2 \mathrm{d}x + \int\limits_0^T \langle A_{\sigma_n}(\phi_n), \phi_n\rangle \mathrm{d}t + \int\limits_{Q^T} \phi_n W'(\phi_n)\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int\limits_{\Omega} |\phi_n(0)|^2 \mathrm{d}x. \tag{3.6.23}$$

Using (3.6.18) and (3.6.19), we can pass to the limits in the above equation and obtain

$$\frac{1}{2}\int\limits_{\Omega} |\phi(T)|^2 \mathrm{d}x + \int\limits_0^T \langle \chi, \phi\rangle \mathrm{d}t + \int\limits_{Q^T} \phi W'(\phi)\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int\limits_{\Omega} |\phi(0)|^2 \mathrm{d}x. \tag{3.6.24}$$

It can be seen from (3.6.18), (3.6.19), (3.6.23) and (3.6.24) that, the conditions for applying Proposition 3.4.6 are satisfied, which implies (3.6.21).

We end this section by passing to the limits in (3.6.2)-(3.6.4). However, it can be seen that, the only problem in taking $n \to \infty$ is the terms on the right hand side of (3.6.2). By (3.6.18d), we have

$$\frac{1}{\sqrt{n}}\int\limits_{Q^T} \frac{1}{\sqrt{n}} \Delta^3\mathbf{u} : \Delta^3\mathbf{v}\mathrm{d}x\mathrm{d}t \mapsto 0.$$

Combining (3.6.21), (3.6.19c) with Lebesgue's Dominated convergence theorem, it follows that

$$\int\limits_{Q^T} \nabla\mathbf{v} : \left((\lambda + \frac{\sigma_n}{\sqrt{|\nabla\phi_n|^2+\epsilon}})\nabla\phi_n \otimes \nabla\phi_n \right) \mathrm{d}x\mathrm{d}t$$
$$\mapsto \int\limits_{Q^T} \nabla\mathbf{v} : \left((\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \otimes \nabla\phi \right) \mathrm{d}x\mathrm{d}t.$$

So we end the proof of the main theorem.

Bibliographie

[1] Helmut Abels. On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. *Arch. Ration. Mech. Anal.*, 194(2) :463–506, 2009.

[2] Helmut Abels and Matthias Röger. Existence of weak solutions for a non-classical sharp interface model for a two-phase flow of viscous, incompressible fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(6) :2403–2424, 2009.

[3] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1) :319–361, 1966.

[4] V. I. Arnol′d. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[5] Alain Bensoussan, Giuseppe Da Prato, Michel C. Delfour, and Sanjoy K. Mitter. *Representation and control of infinite-dimensional systems. Vol. 1.* Systems & Control : Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1992.

[6] T. Biben and C. Misbah. Tumbling of vesicles under shear flow within an advected-field approach. *Phys. Rev. E*, 67(3) :031908, Mar 2003.

[7] Thierry Biben, Klaus Kassner, and Chaouqi Misbah. Phase-field approach to three-dimensional vesicle dynamics. *Phys. Rev. E*, 72(4) :041921, Oct 2005.

[8] Muriel Boulakia. Existence of weak solutions for the motion of an elastic structure in an incompressible viscous fluid. *C. R. Math. Acad. Sci. Paris*, 336(12) :985–990, 2003.

[9] Muriel Boulakia. Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid. *J. Math. Fluid Mech.*, 9(2) :262–294, 2007.

[10] Muriel Boulakia. Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid. *J. Math. Fluid Mech.*, 9(2) :262–294, 2007.

[11] Muriel Boulakia and Axel Osses. Local null controllability of a two-dimensional fluid-structure interaction problem. *ESAIM Control Optim. Calc. Var.*, 14(1) :1–42, 2008.

[12] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6) :771–831, 1982.

[13] Gunduz Caginalp. An analysis of a phase field model of a free boundary. *Arch. Rational Mech. Anal.*, 92(3) :205–245, 1986.

[14] Peter Constantin and Ciprian Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[15] B. Desjardins and M. J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Ration. Mech. Anal.*, 146(1) :59–71, 1999.

[16] B. Desjardins and M. J. Esteban. On weak solutions for fluid-rigid structure interaction : compressible and incompressible models. *Comm. Partial Differential Equations*, 25(7-8) :1399–1413, 2000.

[17] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3) :511–547, 1989.

[18] Szymon Dolecki and David L. Russell. A general theory of observation and control. *SIAM J. Control Optimization*, 15(2) :185–220, 1977.

[19] Anna Doubova and Enrique Fernández-Cara. Some control results for simplified one-dimensional models of fluid-solid interaction. *Math. Models Methods Appl. Sci.*, 15(5) :783–824, 2005.

[20] Qiang Du, Manlin Li, and Chun Liu. Analysis of a phase field Navier-Stokes vesicle-fluid interaction model. *Discrete Contin. Dyn. Syst. Ser. B*, 8(3) :539–556 (electronic), 2007.

[21] Qiang Du, Chun Liu, Rolf Ryham, and Xiaoqiang Wang. Energetic variational approaches in modeling vesicle and fluid interactions. *Phys. D*, 238(9-10) :923–930, 2009.

[22] Qiang Du, Chun Liu, and Xiaoqiang Wang. A phase field approach in the numerical study of the elastic bending energy for vesicle membranes. *J. Comput. Phys.*, 198(2) :450–468, 2004.

[23] Qiang Du, Chun Liu, and Xiaoqiang Wang. Retrieving topological information for phase field models. *SIAM J. Appl. Math.*, 65(6) :1913–1932 (electronic), 2005.

[24] David G. Ebin and Jerrold Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92 :102–163, 1970.

[25] E.Maitre, P.Peyla C.Misbah, and A.Raoult. Comparison between advected-field and level-set methods in the study of vesicle dynamics. *Phys. Rev. E*, 72(4) :041921, Oct 2005.

[26] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[27] H. O. Fattorini and D. L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43 :272–292, 1971.

[28] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of Evolution Equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[29] Giovanni P. Galdi. On the steady self-propelled motion of a body in a viscous incompressible fluid. *Arch. Ration. Mech. Anal.*, 148(1) :53–88, 1999.

[30] Fausto Gozzi and Paola Loreti. Regularity of the minimum time function and minimum energy problems : the linear case. *SIAM J. Control Optim.*, 37(4) :1195–1221 (electronic), 1999.

[31] Max D. Gunzburger, Hyung-Chun Lee, and Gregory A. Seregin. Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *J. Math. Fluid Mech.*, 2(3) :219–266, 2000.

[32] K.-H. Hoffmann and V. N. Starovoitov. On a motion of a solid body in a viscous fluid. Two-dimensional case. *Adv. Math. Sci. Appl.*, 9(2) :633–648, 1999.

[33] K.-H. Hoffmann and V. N. Starovoitov. On a motion of a solid body in a viscous fluid. Two-dimensional case. *Adv. Math. Sci. Appl.*, 9(2) :633–648, 1999.

[34] Oleg Imanuvilov and Takéo Takahashi. Exact controllability of a fluid-rigid body system. *J. Math. Pures Appl. (9)*, 87(4) :408–437, 2007.

[35] Oleg Yu. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6 :39–72 (electronic), 2001.

[36] D. Jamet and C. Misbah. Towards a thermodynamically consistent picture of the phase-field model of vesicles : Local membrane incompressibility. *Phys. Rev. E*, 76(5) :051907, Nov 2007.

[37] D. Jamet and C. Misbah. Thermodynamically consistent picture of the phase-field model of vesicles : Elimination of the surface tension. *Phys. Rev. E*, 78(4) :041903, Oct 2008.

[38] D. Jamet and C. Misbah. Toward a thermodynamically consistent picture of the phase-field model of vesicles : Curvature energy. *Phys. Rev. E*, 78(3) :031902, Sep 2008.

[39] N. V. Judakov. The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami) :249–253, 255, 1974.

[40] K.H. de Haas, C. Bloom, D. van den Ende, M.H.G. Duits, and J. Mellema. Deformation of giant lipid bilayer vesicles in shear flow. *Physical Review Letters*, 56(6) :7132–7137, 1997.

[41] C. Lartigue, M. Abkarian, and A. Viallat. Tank treading and unbinding of deformable vesicles in shear flow : Determination of the lift force. *Physical Review Letters*, 88(6), 2002.

[42] Gilles Lebeau and Luc Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2) :335–356, 1995.

[43] Fang-Hua Lin. Nonlinear theory of defects in nematic liquid crystals ; phase transition and flow phenomena. *Comm. Pure Appl. Math.*, 42(6) :789–814, 1989.

[44] Fang-Hua Lin and Chun Liu. Nonparabolic dissipative systems modeling the flow of liquid crystals. *Comm. Pure Appl. Math.*, 48(5) :501–537, 1995.

[45] Fang-Hua Lin and Chun Liu. Partial regularity of the dynamic system modeling the flow of liquid crystals. *Discrete Contin. Dynam. Systems*, 2(1) :1–22, 1996.

[46] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

[47] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[48] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[49] Luc Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4) :1465–1485, 2010.

[50] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[51] Jorge Alonso San Martín, Victor Starovoitov, and Marius Tucsnak. Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, 161(2) :113–147, 2002.

[52] Jean-Claude Saut and Bruno Scheurer. Unique continuation for some evolution equations. *Journal of Differential Equations*, 66(1) :118 – 139, 1987.

[53] Denis Serre. Chute libre d'un solide dans un fluide visqueux incompressible. Existence. *Japan J. Appl. Math.*, 4(1) :99–110, 1987.

[54] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146 :65–96, 1987.

[55] Takéo Takahashi. Analyse des équations modelisant le mouvement de système couplant des solides rigides et des fluides visqueux. *thèse*, 2002.

[56] Takéo Takahashi. Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. *Adv. Differential Equations*, 8(12) :1499–1532, 2003.

[57] G. Tenenbaum and M. Tucsnak. New blow-up rates for fast controls of Schrödinger and heat equations. *J. Differential Equations*, 243(1) :70–100, 2007.

[58] Gérald Tenenbaum and Marius Tucsnak. On the null-controllability of diffusion equations. *ESAIM COCV*.

[59] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[60] Juan Luis Vázquez and Enrique Zuazua. Large time behavior for a simplified 1D model of fluid-solid interaction. *Comm. Partial Differential Equations*, 28(9-10) :1705–1738, 2003.

[61] Juan Luis Vázquez and Enrique Zuazua. Lack of collision in a simplified 1D model for fluid-solid interaction. *Math. Models Methods Appl. Sci.*, 16(5) :637–678, 2006.

[62] Xiaoqiang Wang and Qiang Du. Modelling and simulations of multi-component lipid membranes and open membranes via diffuse interface approaches. *J. Math. Biol.*, 56(3) :347–371, 2008.

[63] Eberhard Zeidler. *Nonlinear functional analysis and its applications. III*. Springer-Verlag, New York, 1985. Variational methods and optimization, Translated from the German by Leo F. Boron.

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D.F.D.Mathématiques

Analyse et Contrôle de Quelques Problèmes d'interaction Fluide-Structures

Dans cette thèse, on s'intéresse au caractère bien posé et à la contrôlabilité de quelques systèmes d'interaction fluide-structure. Plus précisément, on considère le système constitué de structures déformables ou indéformables et d'un fluide visqueux incompressible. On suppose que le fluide satisfait les équations non linéaires de Navier-Stokes en dimension 2 ou 3 et de Burgers visqueux en dimension 1. Les équations du mouvement des structures sont obtenues en minimisant une énergie du système (principe de D'Alembert) ou en appliquant le principe fondamental de la dynamique (lois de Newton). Les principaux résultats de cette thèse concernent l'existence des solutions (faibles ou fortes) dans le cas déformable, et la contrôlabilité à zéro dans le cas indéformable.

Analysis and Control of Some Problems of Fluid-Structures interaction.

In this thesis, we consider the well-posedness and controllability of some systems of fluid-structure interaction. More precisely, we consider the system consisted of deformable or non-deformable structure and of a viscous incompressible fluid. We suppose that the fluid satisfy the Navier-Stokes equation in 2 or 3 dimensions and the viscous Burger equation in 1-d. The equations for the structures are obtained by minimizing certain energy of the system (D'Alembert principle) or by applying the fundamental principle of dynamics (Newton's laws). The principal results of this thesis are: the existence of solutions (strong or weak) in the deformable case and the null-controllability in the non-deformable case.

Mots-clefs : interaction fluide-structure; fluides complexes; Navier-Stokes, équations de; Problèmes aux limites non linéaires; commande, théorie de la.

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