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Sémantique algébrique des ressources pour la logique classique

THÈSE

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To my parents, Dragica and Momčilo.

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1 Introduction

La logique mathématique moderne est une discipline mathématique mûre, avec des racines profondes dans plus de deux mille ans d'études philosophiques du raisonnement. Depuis plus de cent ans de travaux de de Morgan, Boole, Peirce, Russel, Frege, Hilbert, Gödel et tant d'autres, elle s'étend actuellement sur un large éventail de sujets réunis autour de ses quatre piliers - théorie des ensembles, théorie des modèles, théorie de la calculabilité (récursion) et théorie de la démonstration (ou de la preuve). Même si le développement historique de cette discipline est extrêmement dynamique, il reste quand même certaines questions fondamentales à résoudre d'une manière satisfaisante. Si l'on devait choisir une de ces questions dans une de ces sous-disciplines, dans le cas de la théorie de la démonstration, il s'agirait probablement du problème de la sémantique de la logique classique, et celui de l'identité des preuves.

Lorsqu'on essaie de définir la sémantique, on s'efforce de fournir essentiellement un "sens mathématique" aux preuves formelles. Ces tentatives sont presque aussi vieilles que la théorie de la démonstration elle-même et leur importance va même au-delà. Ce qui est central pour la théorie de la démonstration, c'est la notion de preuve formelle et on peut affirmer que la théorie (structurelle) de la démonstration est l'étude de l'élimination des coupures dans ces formalismes. De grands succès dans la définition des systèmes de dérivation, de la formalisation de l'inférence logique, ont été obtenus par Frege, Hilbert et Gentzen au début du XXème siècle. Depuis ce temps, nous avons appris quelles sont les limites de la théorie de la démonstration, en quoi elle est utile, quelle est la signification informatique des démonstrations, et bien d'autres choses. Lorsqu'un objet mathématique comme une preuve formelle est défini, la définition est généralement suivie d'une critère indiquant quand deux de ces objets mathématiques nouvellement définis, doivent être considérés les mêmes. Dans le cas de preuves formelles de type Gentzen pour la logique classique, LA Logique, un tel critère d'égalité n'a pu être encore formulé de manière satisfaisante. Le succès dans la définition de la sémantique des preuves classiques se traduirait par une théorie de l'identité des preuves, par quoi deux preuves formelles seraient considérées comme égales si leur était affectée la même structure mathématique (en supposant que l'identité dans la sémantique est résolue). Inversement, une bonne notion d'identité

des preuvse devrait donner lieu à une sémantique où deux preuves assimilées représentent le même objet mathématique.

L'expérience que nous avons avec la sémantique des théories structurelles de la démonstration de différentes logiques, nous a enseigné que lorsqu'il y a une sémantique vraiment bonne pour la logique, elle peut être exprimée dans le langage de la théorie des catégories. Un exemple bien connu d'une telle sémantique est une observation, faite par Lawvere et Lambek, que les preuves en logique intuitionniste, c'est-à-dire le λ-calcul simplement typé modulo les identifications β et η, via l'isomorphisme de Curry-Howard, peuvent être entièrement interprétées en termes de catégories cartésiennes fermées (avec coproduits). Les diagrammes commutatifs dans la catégorie du modèle sont précisément les interprétations des formes normales des λ-termes, c'est-à-dire que le λ-calcul est le langage interne des catégories cartésiennes fermées. Ce qui dans cette ligne se rapproche le plus du travail qui doit être effectué ici, c'est établir la connexion entre catégories de modèles pour les preuves de la logique linéaire multiplicative et les catégories *-autonomes de Barr [Bar79]. Dans les deux cas, les catégories en question ont des formules propositionnelles comme des objets, et un morphisme $f : A \to B$ dans la catégorie représente la classe de preuves sans coupures dans un système de preuves que nous considérons comme égales, tandis que la composition dans la catégorie correspond à l'élimination des coupures en logique. Cette démarche remonte aux origines mêmes du traitement catégorique de la logique et au travail de Lawvere [Law63] et de Lambek [Lam68, Lam69].

Il y a une réponse possible mais banale et pas très intéressante à la question "Quand deux preuves classiques sont-elles égales ?", qui est : quand elles ont la même prémisse et la même conclusion. En acceptant cette réponse on ne parle plus de la théorie de la démonstration, mais plutôt de la prouvabilité dans une logique. Tout modèle catégorique de la logique s'effondre alors sur un pré-ordre. En utilisant l'analogie, nous pouvons formuler le problème de recherche de la sémantique (catégorique) de la logique classique comme le rapport

$$\begin{array}{rcl} \text{Cartesian-Closed Category} & : & \text{Heyting Algebra} \\ ??? & : & \text{Boolean Aalgebra} \end{array}$$

qui se lit comme suit : on veut une définition de la notion d'une "catégorie booléenne", la catégorie qui est à l'algèbre de Boole ce qui la catégorie cartésienne fermée est à l'algèbre de Heyting.

Il devrait être noté que la question d'identité des preuves peut également être approchée par l'angle syntaxique ; on peut essayer de s'éloigner plus ou moins loin de la tradition de Gentzen des systèmes de preuves structurales et proposer une notion de preuve dont la syntaxe doit être robuste par rapport à des manipulations triviales, qui ne changent pas "l'essence de preuve". Quelques exemples de ces dernières sont la formulation originale de Girard des réseaux de démonstration de la logique linéaire [Gir87], le travail sur l'inférence profonde, ou le calcul des Cirquents [Jap08].

Il n'y a pas si longtemps, le résultat négatif de Joyal d'effondrement de toute catégorie cartésienne fermée avec la négation involutive des algèbre de Boole donnait peu d'espoir

de faire le moindre progrès vers la formulation d'une catégorie Booléenne [LS86, Gir91]. Pour aggraver la situation, il a été démontré que les catégories cartésiennes fermées sont maximales, dans le sens où toute addition d'une équation non triviale sur les morphismes, produit l'effondrement de la catégorie sur un préordre [Sim95, DP04].

Cet effondrement peut être contourné en arrivant à la logique classique par le biais de logique intuitionniste : une série de suggestions sémantiques sont apparues suite à la découverte [Fil89, Gri90] que les opérateurs de contrôle dans un programme fonctionnel correspondent à l'ajout de constantes correspondant à la loi de Peirce au lambda calcul. Ceci a été généralisé au $\lambda-\mu$-calcul de Parigot [Par92], dont les différentes axiomatisations catégoriques sont [Sel01], basées sur [SR98] ; et [Ong96].

Girard lui-même a formulé le système LC pour la logique classique et en a offert deux modèles différents [Gir91].

Dans le premier modèle la négation double ne produit pas un objet isomorphe, tandis que la disjonction n'est pas bifonctorielle. La composition dans un des modèles de Girard n'est pas associative, et chacun d'eux rompt d'une manière ou d'autre des symétries inhérentes à la logique classique.

Durant la dernière décennie quelques propositions pour une sémantique symétriques de la logique classique sont apparues, [FP05, Lam07, LS05a, Str07, DP04, BHRC06] et nous avons une meilleure compréhension de ces questions, mais il y a encore beaucoup à faire.

Donc, quel est le problème de la logique classique ? Il se trouve qu'il y a une façon concise pour décrire ce problème du point de vue de l'élimination des coupures dans la syntaxe et elle est réduite à ces deux cas : affaiblissement-contre-affaiblissement et contraction-contre-contraction, dans l'élimination des coupures dans LK de Gentzen.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash \Sigma\end{array}}{\Gamma \vdash \Sigma, A}\ \textit{Weak} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Delta \vdash B, \Pi\end{array}}{A, \Delta \vdash B, \Pi}\ \textit{Weak}}{\Gamma, \Delta \vdash \Sigma, \Pi}\ \textit{Cut} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash \Sigma, A, A\end{array}}{\Gamma \vdash \Sigma, A}\ \textit{Contr} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ A, A, \Delta \vdash B, \Pi\end{array}}{A, \Delta \vdash B, \Pi}\ \textit{Contr}}{\Gamma, \Delta \vdash \Sigma, \Pi}\ \textit{Cut.}$$

Une sémantique vraiment symétrique ne ferait pas de préférence entre π_1 ou π_2 dans l'opération d'élimination de la coupure. Alors que le cas affaiblissement-contre-affaiblissement peut être résolu en incluant dans le calcul la règle du Mix, qui correspond à un morphisme naturel $A \vee B \to A \wedge B$ dans la sémantique, le cas contraction-contre-contraction reste paradigmatique des problèmes de la théorie des preuves formelles pour la logique classique.

La recherche présentée dans cette thèse a commencé comme une exploitation de l'interaction féconde entre la sémantique dénotationnelle et la syntaxe, une direction de recherche qui a commencé avec le travail de Dana Scott sur le lambda-calcul non typé, et inclut l'invention de la logique linéaire au travers de l'observation de la catégorie des espaces de cohérence de Girard et des morphismes linéaires. Notre recherche initiale de

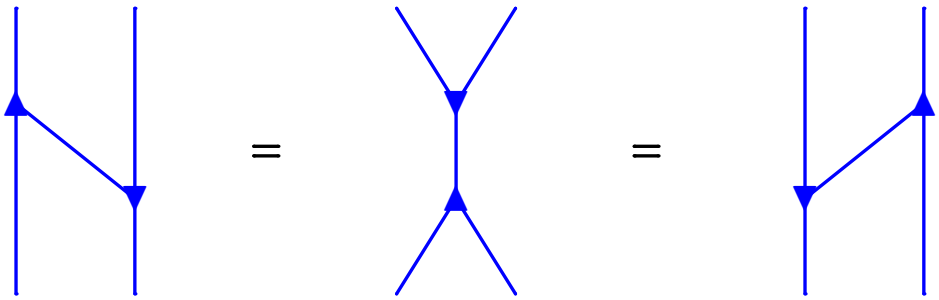
sémantique dénotationnelle pour la logique classique, présentée dans le chapitre 1, a été réalisée dans la catégorie des ensembles ordonnés et bimodules, suite aux travaux dans [Lam07].

Nous avons suivi les leçons apprises au cours des dernières années - en supprimant les règles structurelles classiques, on peut obtenir la véritable négation involutive des catégories *-autonomes, interprétant ainsi un calcul pour la logique linéaire multiplicative. Nos interprétations dénotationnelles sont au départ des catégories *-autonomes dégénérées où les deux connecteurs logiques sont équivalents (donc, nous avons en fait travaillé avec des catégories compactes fermées).

Pour réintroduire les règles structurelles classiques, nous avons donné à chaque objet une structure de monoïde commutatif. Il est bien connu que ces structures de conoïdes ne peuvent être naturelles, la naturalité introduirait des produits et nous renverrait au paradoxe de Joyal.

Les monoïdes commutatifs sur des objets concrets dans nos interprétations dénotationnelles, en présence d'une négation involutive contravariante, conduit à la fois un monoïde commutatif et une structure d'un comonoide cocomutatif sur un objet et les deux forment ensemble une structure d'algèbre de Frobenius sur un objet.

Une algèbre de Frobenius est précisément la structure mentionnée d'un monoïde commutatif et d'un comonoïde cocomutatif sur un objet, couplés dans le diagramme



Ce diagramme a son équivalent dans la logique. En fait, le diagramme central ci-dessus est le célèbre cas de contraction-contraction dans l'élimination des coupures. Les algèbres de Frobenius offrent une nouvelle approche pour gérer cette situation, par exemple, elles identifient deux preuves dans un calcul des séquents unilatère pour la logique classique :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{a}, a}\,Ax \quad \dfrac{}{\vdash \bar{a}, a}\,Ax}{\vdash \bar{a}, a, \bar{a}, a}\,Mix}{\vdash \bar{a}, \bar{a}, a}\,Contr \quad \dfrac{\dfrac{\dfrac{}{\vdash \bar{a}, a}\,Ax \quad \dfrac{}{\vdash \bar{a}, a}\,Ax}{\vdash \bar{a}, a, \bar{a}, a}\,Mix}{\vdash \bar{a}, \bar{a}, a}\,Contr}{\vdash \bar{a}, \bar{a}, a, a}\,Cut \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{a}, a}\,Ax \quad \dfrac{}{\vdash \bar{a}, a}\,Ax}{\vdash \bar{a}, a, \bar{a}, a}\,Mix}{\vdash \bar{a}, a, \bar{a}}\,Contr \quad \dfrac{}{\vdash \bar{a}, a}\,Ax}{\vdash \bar{a}, a, \bar{a}, \bar{a}, a}\,Mix}{\vdash \bar{a}, a, \bar{a}, a}\,Contr$$

Donc, notre motivation pour ce travail vient de la sémantique ; une interprétation dénotationnelle nous a conduit à une théorie algébrique pertinent pour la logique. Cette thèse peut être considérée comme une analyse de l'interaction entre les deux, qui seront vues comme un concept de ressources dans la logique classique.

2 Prélude : Sur les interprétations concrètes des preuves classiques

Dans ce chapitre nous interprétons une variation mineure du calcul des séquents unilatère de Gentzen pour la logique classique (obtenu en ajoutant la règle Mix de la logique linéaire à LK) dans une certaine catégorie dont les objets sont des ensembles ordonnés et les morphisme sont des relations entre eux. Nous allons montrer que les interprétations des preuves contiennent des informations significatives sur ces preuves et que ces informations sont étroitement liées à une sorte de réseau de preuves qui est présenté dans [LS05b, LS05a]. Quelques calculs que nous allons faire vont nous permettre de conclure que ces interprétations sont des représentations des preuves qui ne peuvent pas être des exemples de la correspondance de Curry-Howard.

Le point de départ de notre travail est une observation faite par plusieurs personnes, de manière indépendante, que l'interprétation bien connue de la logique linéaire dans les ensembles et les relations pouvait être étendue à une autre où les ensembles sont généralisés pour devenir des ensembles ordonnés ; donc il existe une notion de "relation entre ensembles ordonnés", que Lambek a appelé une fois *comparaisons* [Lam94].

Soient $(M, \leq), (N, \leq)$ des ensembles ordonnés. Une *comparaison* $f : M \to N$ est un sous-ensemble $f \subseteq M \times N$ qui est fermé en bas à gauche, et fermé en haut à droite, ou bien

$$\begin{array}{lll} m\ f\ n, & m' \leq m & \text{implies} \quad m'\ f\ n \\ m\ f\ n, & m \leq n' & \text{implies} \quad m\ f\ n'. \end{array}$$

La composition de ces morphismes est la composition ordinaire des relations : étant donné $F : M \to N$ et $g : N \to P$

$$m\ gf\ p, \quad \text{if} \quad (\exists n \in N)\ m\ f\ n,\ n\ g\ p'.$$

L'interprétation des formules se déroule :

- Comme d'habitude, les unités $\mathbf{1}$ et $\bot$ sont interprétées comme l'ensemble à un seul élément $\{*\}$
- chaque formule atomique a est interprétée comme un ensemble ordonné fixé a ;
- étant données les deux formules A, B, leur tenseur est le produit cartésien, $\mathrm{A} \times \mathrm{B}$,
- Si A interprète la formule A, alors l'interprétation de $\mathrm{A}^{\bot}$ inverse l'ordre, à savoir $A^{\bot} = A^{op}$. Ceci est aussi fonctoriel, mais cette fois de façon contravariante : étant donnée $f \colon A \to B$, on a $f^{\bot} \colon B^{\bot} \to A^{\bot}$.
- par conséquent, le par est calculé comme suit : l'interprétation de $A \parr B$ est la même que celle de $(\mathrm{A}^{\bot} \otimes \mathrm{B}^{\bot})^{\bot}$, c'est-à-dire

  $$(A^{op} \times B^{op})^{op} = A \times B = A \otimes B.$$

  En d'autres termes, le tenseur et le par sont interprétés des produits cartésiens. Il y a une certaine dégénérescence, mais nous sommes intéressés par l'interprétation des *preuves*, et non la *prouvabilité*.

– Le lecteur doit vérifier que nous avons effectivement la correspondance (naturelle) bijective

$$\frac{A \otimes B \to C}{A \to B^{\perp} \parr C.}$$

qui définit une adjonction. En lisant cette "règle inversible" de haut en bas on parle de currying, et de bas en haut on dit uncurrying. Nous avons montré aux lecteurs qui connaissent bien ce sujet que **Cmp** est une catégorie **-autonome* [Bar79].

Le lecteur devrait consulter la partie principale de la thèse pour la convention sur les notations et pour la représentation des morphismes dans un calcul unilatère.

L'interprétation du calcul des séquents unilatère pour la logique linéaire multiplicative (de fait, nous avons aussi besoin de la règle Mix) est dicté par les définitions que nous venons de donner.

$$\frac{}{\vdash \mathrm{a}^{\perp}, \mathrm{a}} \quad \rightsquigarrow \quad \mathrm{Id}_a = \{(x, y) \in a \times a \mid x \leq y\}$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \parr \mathrm{B}} \parr \quad \rightsquigarrow \quad \text{ne changer rien.}$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{A} \otimes \mathrm{B}, \Sigma} \otimes \quad \rightsquigarrow \quad \text{Compte tenu de } f \text{ pour } \Gamma \times A \text{ et } g \text{ pour } B \times \Sigma, \text{ prendre } f \times g \text{ pour } \Gamma \times A \times B \times \Sigma$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{A}^{\perp}, \Sigma}{\vdash \Gamma, \Sigma} \mathit{Cut} \quad \rightsquigarrow \quad \text{Compte tenu de } f \text{ pour } \Gamma \times A \text{ et } g \text{ pour } A^{\perp} \times \Sigma, \text{ prendre } \{(\gamma, \delta) \mid \exists x \in A : (\gamma, x) \in f, (x, \delta) \in g\} \text{ pour } \Gamma \times \Sigma$$

$$\frac{\vdash \Gamma \quad \vdash \Sigma}{\vdash \Gamma, \Sigma} \mathit{Mix} \quad \rightsquigarrow \quad \text{Compte tenu de } f \text{ pour } \Gamma \text{ et } g \text{ pour } \Sigma, \text{ prendre } f \times g \text{ pour } \Gamma \times \Sigma.$$

Nous allons utiliser $\mathcal{A}t(\mathrm{X})$ et $\mathcal{A}t(\Gamma)$ pour désigner l'ensemble des occurrences d'atomes et des négatomes dans une formule ou dans un séquent.

Proposition 0.2.1. *L'interprétation de tout séquent* Γ *est* $(\prod_{\mathrm{a} \in \mathcal{A}t(\Gamma)} a)$.

Passons maintenant à la logique classique. Le calcul LC que nous allons utiliser est le même que dans [LS05b]. Cela peut être considéré même comme la version unilatère du LK de Gentzen avec le Mix ajouté, ou comme MLL+Mix + Affaiblissement + Contraction.

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}}{\vdash \Gamma, \mathrm{A}} \mathit{Contr} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}} \mathit{Weak}$$

Le fait d'avoir une Contraction dans le calcul exige clairement la présence d'un morphisme $\nabla_A : A \parr A \to A$. En ce qui concerne l'Affaiblissement, le morphisme $\amalg_A : \perp \to A$ va clairement faire le travail. Nous ajoutons la condition que ces morphismes doivent obéir

à la même forme d'associativité, de commutativité et d'identité que nous associons aux monoïdes commutatifs.

Ainsi, dans notre modèle de la logique classique toute interprétation d'un type sera équipée d'une structure de $\parr$-monoïde commutatif. Comme les preuves sont définies par induction, il suffira de définir la structure de monoïde pour les atomes $\mathrm{a}, \mathrm{b} \ldots$ et négatomes (qui, pour signaler le fait que nous entrons dans la logique classique, seront notés $\bar{\mathrm{a}}, \bar{\mathrm{b}}, \ldots$) et de donner un moyen de construire une nouvelle structure de monoïde après l'application des connecteurs. Ë cause des règles linéaires que nous avons déjà, l'ensemble ordonné pour $A \wedge B$ sera évidemment $A \otimes B = A \times B$ et celui pour $A \vee B$ sera $A \parr B = A \times B$.

Compte tenu de la structure de monoïde sur A et B, nous définissons $\nabla_{A \vee B} : (A \parr B) \parr (A \parr B) \to A \parr B$ comme la composition :

$$(A \parr B) \parr (A \parr B) \xrightarrow{\sim} (A \otimes A) \parr (B \parr B) \xrightarrow{\nabla_A \parr \nabla_A} A \parr B.$$

L'isomorphisme est ici l'isomorphisme de permutation des produits cartésiens. Quant à $\Pi_{A \vee B} : \bot \to A \vee B$, on prend :

$$\bot \xrightarrow{\sim} \bot \parr \bot \xrightarrow{\Pi_A \parr \Pi_B} A \parr B.$$

Nous faisons la même chose pour la conjonction, car ici et le tenseur et le par sont interprétés par le produit cartésien. Mais, naturellement, en général, ce n'est pas le cas et puis il n'y a pas de moyen garanti de passer de $(A \otimes B) \parr (A \otimes B)$ à $A \otimes B$. Ceci est résolu en exigeant la présence d'un morphisme naturel $(A \otimes B) \parr (A \otimes B) \to (A \parr A) \otimes (B \parr B)$, qui est appelé *Médial*. Il est naturel d'inclure dans la logique et il est d'abord apparu comme une règle de déduction pour les calculs d'Inférence Profonde de la logique classique [Brü03]. Pour le traitement catégorique de ce morphisme, indépendamment de la logique classique, le lecteur peut regarder [Lam07]. Nous savons déjà comment interpréter la négation des atomes. Pour une formule composite, la structure de comonoïde de sa négation est obtenue de façon récursive, en utilisant la dualité de de Morgan : $\overline{A \wedge B} = \overline{A} \vee \overline{B}$ et $\overline{A \vee B} = \overline{A} \wedge \overline{B}$.

Il y a un certain avantage à considérer les deux structures de monoïde combinées $A, \overline{A}$ comme une structure algébrique *unique* sur l'objet A. On note qu'un morphisme $\nabla : \overline{A} \parr \overline{A} \to \overline{A}$ peut aussi être vu, par dualité, comme un morphisme $\Delta : A \to A \otimes A$, et c'est de même pour $\Pi : \bot \to \overline{A}$, qui devient $\Pi : A \to \mathbf{1}$. Les lois *d'associativité, de commutativité et d'unité* peuvent être traduites en cet contexte "inversé" par la dualité ; pendant des décennies algébristes ont appelé ces lois *coassociativité, cocoummutativité et co-unité*, tandis que la structure qui en résulte (A, Δ, Π) est appelé *comonoïde commutatif* et la chose entière $(A, \Delta, \Pi, \nabla, \Pi)$ est appelé *bimonoïde (commutatif, cocommutatif)* avec Δ la *diagonale*, ∇ la *co-diagonale*, Π la *co-unité* ou la *projection* et Π *l'unité* ou la *co-projection*.

Ainsi, étant donné un objet X dans une interprétation de la logique linéaire, en le transformant dans une interprétation d'une formule classique, équivaut à trouver une structure de bimonoïde sur cet objet.

Nous pouvons enfin définir l'interprétation de l'Affaiblissement et de la Contraction (le lecteur doit tenir compte de notre convention d'identifier les preuves avec les sous-ensembles fermés par le haut) :

$$\frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}}\ \mathit{Weak} \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \text{ take}\\ \{(\gamma_1, \ldots, \gamma_n, \epsilon) \mid (\gamma_1, \ldots, \gamma_n) \in f \text{ and } \epsilon \in \Pi_A\} \text{ for } \Gamma \times A\end{array}$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}}{\vdash \Gamma, \mathrm{A}}\ \mathit{Contr} \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \times A \times A, \text{ take}\\ \{(\gamma, x) \mid \exists x_1, x_2 \in A : (\gamma, x_1, x_2) \in f \text{ and } (x_1, x_2)\ \nabla_A\ x\}\\ \text{for } \Gamma \times A\end{array}$$

Schématiquement, les morphismes définis pour la Contraction et l'Affaiblissement peut être vus comme des compositions

$$\mathbf{1} \xrightarrow{f} \Gamma \xrightarrow{\sim} \Gamma \times \bot \xrightarrow{\mathrm{Id}_\Gamma \times \Pi_A} \Gamma \times A \quad \text{and} \quad \mathbf{1} \xrightarrow{f} \Gamma \times A \times A \xrightarrow{\mathrm{Id}_\Gamma \times \nabla_A} \Gamma \times A.$$

L'interprétation de la règle de Coupure, que nous avons donnée avant, correspond à la composition

$$\mathbf{1} \xrightarrow{f \times g} \Gamma \times A \times A^\perp \times \Sigma \xrightarrow{\mathrm{Id}_\Gamma \times C_A \times \mathrm{Id}_\Sigma} \Gamma \times \Sigma$$

où $C_A : A \otimes A^\perp \to \bot$ est le morphisme $\{(x, y, *) \mid x \leq y\}$ qui est dual de l'identité $\mathbf{1} \to A^\perp \parr A$.

Regardez la dérivation suivante :

$$\cfrac{\cfrac{\vdash \bar{\mathrm{a}}, \mathrm{a} \quad \vdash \bar{\mathrm{a}}, \mathrm{a}}{\vdash \bar{\mathrm{a}}, \mathrm{a}, \bar{\mathrm{a}}, \mathrm{a}}\ \mathit{Mix}}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\ 2 \times \mathit{Cont.}$$

Le morphisme affecté à cette preuve est est le composé $\nabla_a \circ \Delta_a$. Il est assez facile de construire une sémantique pour laquelle ce morphisme est l'identité, mais il est beaucoup plus difficile d'obtenir celles pour lesquelles il ne l'est pas. Les premiers résultats à ce sujet sont dans [Lam07]. Le fait est que nous voulons suivre les ressources, car cette dérivation peut être vue comme la superposition de deux liens d'axiome.

Notre objectif ici est de construire telles interprétations.

Rappelons qu'une partition d'un ensemble donné Q peut être considérée soit comme une relation d'équivalence sur Q ou comme un ensemble $\mathcal{R} \subseteq \mathcal{P}(X)$ de sous-ensembles de Q, dont les éléments sont appelés classes. Soit Γ un séquent, et q une dénotation de preuve. Par induction sur la preuve, nous définissons ci-dessous une partition $\mathrm{Prt}_q(\Gamma)$ sur l'ensemble $\mathcal{A}t(\Gamma)$. Dans cette définition, nous ne prenons pas le soin de suivre explicitement q :

– si Γ est $\vdash \bar{\mathrm{a}}, \mathrm{a}$, alors $\mathrm{Prt}(\Gamma) = \{\{\bar{\mathrm{a}}, \mathrm{a}\}\}$

– si Γ a été obtenu à travers l'application du tenseur ou la règle Mix, $\mathrm{Prt}(\Gamma)$ est la partition évidente de "somme" sur l'union disjointe $\mathcal{A}t(\Sigma_1) \uplus \mathcal{A}t(\Sigma_2)$, où Σ_1, Σ_2 sont les séquents prémisses de l'application de la règle .
– à supposer que A a été ajouté à $\vdash \Gamma$ par un Affaiblissement, $\mathrm{Prt}(\Gamma, \mathrm{A}) = \mathrm{Prt}(\Gamma) \uplus \big\{\{\mathrm{a}\} \mid \mathrm{a} \in \mathcal{A}t(\mathrm{A})\big\}$, plus explicitement, nous ajoutons chaque (neg)atome de A comme une classe singleton. [1].
– en supposant que $\vdash \Gamma, \mathrm{A}$ a été obtenu par contraction, $\vdash \Gamma, \mathrm{A}^1, \mathrm{A}^2$ soit

$$p\colon \mathcal{A}t(\Gamma, \mathrm{A}^1, \mathrm{A}^2) \longrightarrow \mathcal{A}t(\Gamma, \mathrm{A})$$

la surjection évidente qui identifie les paires d'atomes correspondantes de deux occurrences de A. On prend $\mathrm{Prt}(\Gamma, \mathrm{A})$ pour la plus petite partition engendrée par les ensembles d'image directe $\{\, p(U) \subseteq \mathcal{A}t(\Gamma, \mathrm{A}) \mid U \in \mathrm{Prt}(\Gamma, \mathrm{A}^1, \mathrm{A}^2) \,\}$. Il est facile de voir que si l'on définit la relation binaire $p(U) \frown p(U')$ comme $p(U) \cap p(U') \neq \emptyset$, alors les classes en $\mathrm{Prt}(\Gamma, \mathrm{A})$ sont en correspondance bijective avec les composants connexes du graphe de $\frown$ et chaque classe est l'union d'ensembles dans son composant correspondant.
– si $\vdash \Gamma, \Sigma$ a été obtenue en appliquant la Coupure sur $\vdash \Gamma, \mathrm{A}$ et $\vdash \overline{\mathrm{A}}, \Sigma$, nous définissons $\mathrm{Prt}(\Gamma, \Sigma)$ comme suit. D'abord on définit une relation d'équivalence $\sim$ comme la fermeture transitive symétrique de l'union de trois relations binaires R, S, T, où $\mathcal{A}t(\Gamma, \mathrm{A}) \uplus \mathcal{A}t(\overline{A}, \Sigma)$
– R, S sont des relations d'équivalence associées aux partitions $\mathrm{Prt}(\Gamma), \mathrm{Prt}(\Sigma)$ (respectivement)
– xTy lorsque $x = \mathrm{a}$ est un (neg)atome en A et $y = \bar{\mathrm{a}}$ sa négation correspondante en $\overline{A}$. $\mathrm{Prt}(\Gamma, \Sigma)$ est alors défini comme la restriction de la partition déterminée par $\sim$ sur le sous-ensemble $\mathcal{A}t(\Gamma, \Sigma) \subseteq \mathcal{A}t(\Gamma, \mathrm{A}) \uplus \mathcal{A}t(\overline{A}, \Sigma)$.

Cette définition peut être expliquée en termes des réseaux de preuves [LS05b], où les liens d'axiome sont *superposés* quand la Contraction est appliquée, donnant lieu à une relation sur $\mathcal{A}t(\Gamma)$ qui n'est pas le couplage habituel des réseaux de preuves linéaires, mais un type de relation beaucoup plus général. Supposons d'abord que ni l'Affaiblissement, ni la Coupure n'ont été utilisés dans une preuve de Γ, donc la partition $\mathrm{Prt}(\Gamma)$ correspond exactement à l'ensemble de composantes connexes du graphe du réseau de démonstration, associé à cette preuve. Si une preuve contient des Affaiblissements, dans notre cas, les atomes sont ajoutés comme des singletons dans le graphe, alors qu'ils seraient tout simplement ignorés dans un réseau de preuves. Il est plus simple pour nous de définir $\mathrm{Prt}(\Gamma)$ de cette manière, plutôt que comme une sous-partition (partition d'un sous-ensemble) de $\mathcal{A}t(\Gamma)$.

Proposition 0.2.2. *Soit q une preuve d'un séquent $\vdash \Gamma$. Alors si $f \subseteq \Gamma = \prod_{\mathrm{a} \in \mathcal{A}t(\Gamma)} \mathrm{a}$ est l'interprétation de q, il existe une famille $(f_U)_{U \in \mathrm{Prt}(\Gamma)}$ telle que f se décompose comme un produit :*

$$f = \prod_{U \in \mathrm{Prt}(\Gamma)} f_U \quad \text{où} \quad f_U \subseteq \prod_{\mathrm{a} \in U} a\,.$$

1. Notez le légère abus de la notation où a est utilisé pour désigner négatomes ainsi qu'atomes.

La preuve est une induction élémentaire.

La signification de ceci est que la décomposition associée à des composantes connexes pour un réseau de preuves de CL a un équivalent sémantique simple, comme une décomposition en produit au lieu d'une décomposition en somme. Ce sont des informations syntaxiques qui sont obtenues par la sémantique.

Le résultat bien connu qui suit (la preuve est dans [Lam07]) va se révéler utile pour la construction des classes de bimonoïdes.

Proposition 0.2.3. *Soit $(M, \leq, \cdot, e)$ un ensemble ordonné qui est équipé d'une structure de monoïde commutatif $(\cdot, e)$, telle que $\cdot$ est $\leq$ monotone. (Un théoricien de catégorie dirait : soit M un monoïde en* Poset*) Si nous définissons $\nabla : M \times M \to M$ et $\amalg : \{*\} \to M$ en* **Cmp** *par*

$$(m, n)^{\nabla} p \text{ si } \; m \cdot n \leq p, \qquad *^{\amalg} m \text{ si } e \leq m$$

nous obtenons un $\times$-monoïde commutatif. Par dualité, si nous définissons $\Delta : M \to M \times M$ et $\Pi : M \to \{*\}$ comme

$$m^{\Delta}(n, p) \text{ si } m \leq n \cdot p, \qquad m^{\Pi} * \text{ si } m \leq e$$

nous obtenons un $\times$-comonoïde cocommutatif.

Pour conclure, une manière de construire bimonoïdes en **Cmp** est de trouver des ensemble ordonnés équipés de deux structures monotones de monoïdes.

2.1 Une interprétation fondée sur $\mathbb{Z}$.

L'ensemble d'entiers $\mathbb{Z}$ est équipé avec deux structures qui nous intéressent : ordre $(\mathbb{Z}, \leqslant)$ et monoïde commutatif $(\mathbb{Z}, +, 0)$.

Alors nous attribuons l'ensemble ordonné $a = \mathbb{Z}$ à chaque type atomique.

Nous devons chercher d'autres structures de monoïdes. Peut-on modifier l'addition standard le moins possible, de manière à conserver des calculs simples ? Oui, car il est bien connu que pour tout nombre c l'opération $(x, y) \mapsto x + y - c$ définit une structure de monoïde sur $\mathbb{Z}$, dont l'unité est c. En fait, le choix de cette unité est tout ce qui est nécessaire pour définir le reste : regarder la traduction d'ordre isomorphe $x \mapsto x + c$, dont inverse est $x \mapsto x - c$. Nous avons seulement défini notre nouvelle opération (appelons-la $+_c$) en transportant l'addition le long du premier iso, à savoir

$$x +_c y = \big((x - c) + (y - c)\big) + c\,.$$

Mais cela montre que notre nouveau monoïde $(\mathbb{Z}, +_c, c)$ est isomorphe à celui avec lequel nous avons commencé. Il semble que nous n'avons pas progressé beaucoup. Mais souvenez-vous, nous sommes à la recherche d'une *nouvelle* structure de monoïde, puisque nous en avons déjà un !

Nous allons définir un *bimonoïde ordonné* de cette manière : c'est un sextuple $(M, \leqslant, +_1, e_1, +_2, e_2)$ où $(M, \leqslant)$ est un ensemble ordonné et $(+_i, e_i)$ sont deux structures de monoïde monotones sur M. Il est évident ce qu'un isomorphisme de bimonoïdes ordonnés devrait être : un isomorphisme d'ensembles ordonnés qui est également un isomorphisme de conoïdes autant pour $(+_1, e_1)$ que pour $(+_2, e_2)$. Passons maintenant aux bimonoïdes ordonnés de la forme $(\mathbb{Z}, \leqslant, +_{e_1}, e_1, +_{e_2}, e_2)$, pour tout choix de e_1, e_2. Nous les appellerons *bimonoïdes- translation*. Ce qui suit est très facile à montrer :

Remarque 0.2.4. Deux bimonoïdes-translation, $(\mathbb{Z}, \leqslant, +_{a_1}, a_1, +_{a_2}, a_2)$ et $(\mathbb{Z}, \leqslant, +_{b_1}, b_1, +_{b_2}, b_2)$, ils sont isomorphes si $a_2 - a_1 = b_2 - b_1$.

Maintenant que nous disposons de cette information, nous pouvons décider de ne considérer que les bimonoïdes de la forme $(\mathbb{Z}, \leqslant, +, 0, +_c, c)$, car ils nous donnent toutes les classes isomorphes de bimonoïdes-translation en variant c.

Ce qui nous intéressait en premier lieu, c'étaient les bimonoïdes dans **Cmp**.

Proposition 0.2.5. *Si* a *est un bimonoïde- translation, alors* ā *lui est isomorphe.*

Par conséquent, nous nous retrouvons avec une interprétation où les atomes et négatomes seront isomorphes (comme dans le modèle relationnel). Cependant, ce qui est intéressant à noter c'est que l'isomorphisme ne peut pas être une identité, sauf si $c = 0$.

Récapitulons. On choisit c, et on définit

$$\begin{array}{llll} (j,k)^{\nabla_a} i & \text{ssi} & j + k \leq i + c; & *^{\Pi_a} i \quad \text{ssi} \quad c \leq i. \\ i^{\Delta_a}(j,k) & \text{ssi} & i \leq j + k; & i^{\Pi_a} * \quad \text{ssi} \quad i \leq 0; \end{array} \tag{1}$$

Et maintenant $\mathcal{D}_a = \nabla \circ \Delta = \{(x,y) \mid \exists i, j : x \leq i+j,\ i+j \leq y+c\} = \{(x,y) \mid x \leq y+c\}$. Il est alors facile de voir que $\mathcal{D}_a^n = \{(x,y) \mid x \leq y + n \cdot c\}$ et donc la composition du morphisme de doublage $\mathcal{D}_a$ avec lui-même ne se stablise ; $\mathcal{D}_a^m \neq \mathcal{D}_a^n$ pour $n \neq m$.

==========

2.2 Quelques calculs de plus

Donc, le calcul de preuves dans cette interprétation est plutôt facile. Encore une fois, choisissez un a atome avec une interprétation $a = (\mathbb{Z}, \leqslant, +, 0, +_c, c)$.

Nous allons utiliser le fait que la relation $x \leq y$ dans a est équivalente à $(-x) + y \geq 0$ et que $x \leq y$ dans a^{op} est équivalent à $(-x) \leq (-y)$ dans a. Cela nous permet de travailler avec une structure d'ordre unique, la structure standard dans $\mathbb{Z}$, et de ne pas devoir faire face à sa duale, ce qui est nécessaire en général, et rend les choses très déroutantes quand, comme ici, l'ordre et son dual sont isomorphes.

Théorème 0.2.6. Soit f une preuve d'un séquent $\vdash \Gamma$. Alors chaque facteur f_j de la décomposition

$$f = f_1 \times f_2 \times \ldots \times f_N,$$

de la Proposition 1.1.4 est de la forme

$$f_j = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{Z}^{n+m} \mid (-x_1) + \cdots + (-x_n) + y_1 + \ldots + y_m \geq M_j c\}$$

où $\mathrm{a}^1, \bar{\mathrm{a}}^2, \ldots, \bar{\mathrm{a}}^n, \mathrm{a}^1, \ldots, \mathrm{a}^m$ est une énumération des (Nég)atomes de la classe $U_j \in \mathrm{Prt}(\Gamma)$ et M_j est une entier.

Cet entier M est lié au nombre de contractions qui ont été effectuées dans la preuve q et au nombre d'atomes qui sont vu le jour par Affaiblissement. Nous permettons de mentionner que les éléments singletons de $\mathrm{Prt}(\Gamma)$ sont soit de la forme $\{y \geqslant c\}$ pour les atomes positifs ou de la forme $\{-x \geqslant 0\}$ pour les négatifs.

Essayons-nous maintenant à calculer les entiers de Church. Ils correspondent aux preuves constructives de $(\mathrm{a} \Rightarrow \mathrm{a}) \Rightarrow (\mathrm{a} \Rightarrow \mathrm{a})$.

$$n \times \wedge \left\{ \begin{array}{l} \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \\ \\ \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \end{array} \right.$$

encode le nombre n. Un calcul simple montre que cet entier est interprété par

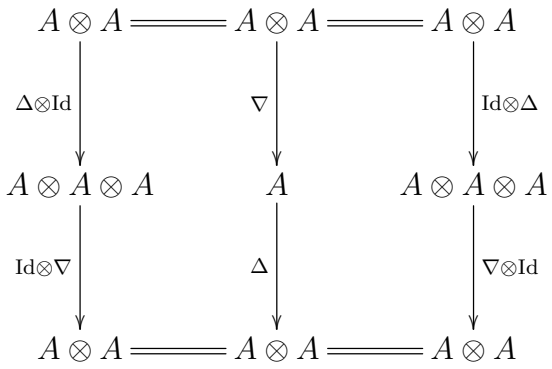
$$\{-x^0 - x^{2n-1} + y^{2n-2} + y^{2n-1} \geq -(n-1)c\}.$$

Rappelons que si deux entiers de Church sont considérés comme des morphismes $n, m \colon (\mathrm{a} \Rightarrow \mathrm{a}) \to (\mathrm{a} \Rightarrow \mathrm{a})$, les multiplier ensuite équivaut à la composition de ces deux morphismes, c'est-à-dire, leur appliquer une coupure. Un calcul simple montre que cette opération va donner le nombre $-(n+m-1)$. C'est la preuve assurée que l'interprétation que nous avons construite n'a rien à voir avec la correspondance de Curry-Howard. Une multiplication qui agit de manière additive, comme c'est le cas ici, montre que l'interprétation des termes ne peut pas correspondre à des programmes fonctionnels, puisque l'interprétation de Curry-Howard de l'entier n de Church est l'itérateur fonctionnel, "compose une endofunction n fois avec elle-même", et la composition des itérateurs pour n et m ne peut que donner l'itérateur pour nm.

3 Interprétations et algèbres de Frobenius

Dans le chapitre prélude, dans la section sur l'interprétation basée sur $\mathbb{Z}$, nous avons rencontré des structures qui étaient sous la forme de "paquets" - des entiers affectés aux classes des partitions des littéraux. L'interprétation, comme il a été auparavant remarqué, a introduit des structures sur les objets qui sont des algèbres de Frobenius. Avant de donner une définition à cette notion, nous rappelons au lecteur que dans le cadre le plus général, nous travaillons dans une catégorie *-autonome où chaque objet est équipé d'une structure de $\parr$-monoïde (donc aussi une structure de $\otimes$-comonoïde). Toutefois, pour la plupart du travail qui suit, nous n'allons faire aucune distinction entre les connecteurs / bifonncteurs, comme nous avons fait dans le cas de l'interprétation basée sur $\mathbb{Z}$. Ainsi, nous allons travailler dans des catégories compact-fermées où $(A \otimes B)^\perp$ et $A^\perp \otimes B^\perp$ sont identifiés.

Définition 0.3.1 (Algèbre de Frobenius)**.** Soit $(\mathbf{C}, \otimes, \mathbf{1})$ une catégorie monoïdale symétrique (SMC), et A un de ses objets. Une *algèbre de Frobenius* est un sextuple $(A, \Delta, \Pi, \nabla, \amalg)$ où $(A, \nabla, \amalg)$ est un monoïde commutatif, (A, Δ, Π) un comonoïde co-commutatif, où le diagramme suivant commute :



Une algèbre de Frobenius est dite *mince* si pour tout $k \geq 0$

$$\Pi \circ \underbrace{(\nabla \circ \Delta) \circ \cdots \circ (\nabla \circ \Delta)}_{k} \circ \amalg$$

est l'identité de Id.

Ce qui suit est bien connu.

Proposition 0.3.2. *Le tenseur de deux algèbres de Frobenius est aussi une algèbre de Frobenius, où les opérations de monoïde et comonoïde sont définies comme d'habitude dans une SMC. Il est mince si les deux facteurs le sont.*

Définition 0.3.3. Une catégorie de Frobenius $\mathbf{C}$ est une catégorie monoïdale symétrique où chaque objet A est muni d'une structure mince d'algèbre de Frobenius $(A, \nabla_A, \Pi_A, \Delta_A, \amalg)$, telle que l'algèbre sur le tenseur des deux objets est l'algèbre tensorielle habituelle, comme ci-dessus.

Les algèbres de Frobenius ont gagné beaucoup d'attention, après que l'on ait découvert qu'elles étaient étroitement liées aux Topological Quantum Field Theories (TQFTs) bidimensionnelles. Le résultat principal a été obtenu par plusieurs personnes indépendamment [Dij89, Koc04], et peut être énoncé comme suit. Nous présentons une version légèrement modifiée du résultat standard, qui s'adapte mieux à nos besoins et en est un simple corollaire.

Théorème 0.3.4. La catégorie de Frobenius mince libre **FrThFrob** sur un objet générateur est équivalente aux deux catégories suivantes.

1. On prend des unions disjointes finies de m cercles l'objet qu'on note m. Un morphisme $m \to n$ est une surface de Riemann (avec bords) dont les bords sont la somme disjointe $m + n$ (et qui serait orientables si les cercles étaient étendues à des disques), de telle sorte que chaque composante connexe a un bord non vide, où deux surfaces sont identifiées modulo homéomorphisme. La composition de deux morphismes $m \to n, n \to p$ est le recollement, en oubliant les frontières au milieu et en abandonnant les composants qui ne touchent pas les limites résultantes $m + p$.
2. On prend des ensembles finis $[m] = \{0, 1, \ldots, m-1\}$ comme des objets, considérés comme des espaces topologiques discrets. Le morphisme $[m] \to [n]$ est un graphe topologique G (c'est-à-dire un CW-complexe d'une dimension), équipé d'une fonction injective $[m + n] \to G$ telle que chaque composants connexes de G est à l'image de cette fonction, avec deux graphs identifiés s'ils sont équivalents modulo homologie. La composition est également le collage et l'abandon des composants qui sont exclus de l'ensemble résultant de critères.

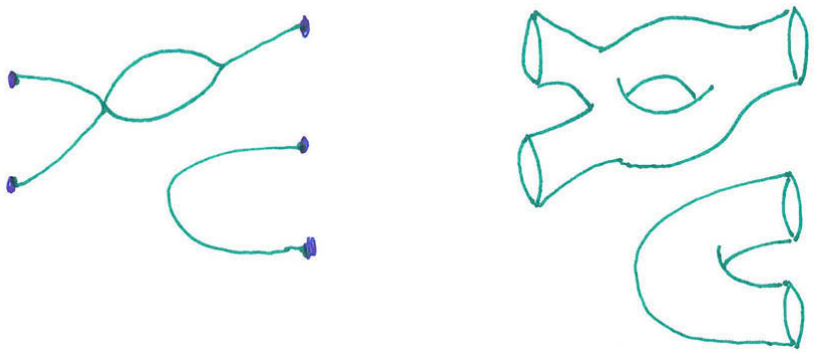


FIGURE 1 – Deux caractérisations équivalentes des algèbres libres de Frobenius. Les objets sont des end-points distingués à gauche ou des cercles à droite. Deux d'entre eux sont dans le domaine, trois dans le codomaine. Un morphisme est l'homologie de "câblage" vers la gauche ou la surface-définie classe homéomorphe de Riemann qui fixe la frontière. Un des composants connexes est de genre 1, l'autre de genre 0. Dans les deux cas, le morphisme est déterminé par le regroupement des littéraux dans une partition et par une affectation des genres aux classes de la partition.

Une catégorie libre de Frobenius est définie seulement modulo l'équivalence de catégories, avec la propriété standard universelle associée à cette situation. Les deux caractérisations dans le Théorème 3.1.12 sont en fait des catégories squelettique et sont isomorphes. Notre notion non-standard de la catégorie de Frobenius exige la mincesse ; morphismes dans la norme, non-mince catégorie libre de Frobenius peut contenir plusieurs composants "flottant" qui ne touchent pas la frontière.

Puisque l'homologie est beaucoup plus technique que l'homotopie, on préfère remplacer le deuxième résultat ci-dessus avec :

2'. Les objets sont des ensembles de la forme $[m]$*. Un morphisme* $[m] \to [n]$ *est un graphe* G *topologique muni d'une fonction injective* $[m+n] \to G$ *qui touche toues les composantes connexes de* G*, où deux telles sont identifiées si elles sont équivalents homotopiques dans la catégorie co-tranche* $(m+n)/Top$*, où les homotopies sont constantes sur la base* $[m+n]$*.*

Cela permet un traitement qui est en même temps bien formalisé et accessible à beaucoup plus de lecteurs.

Convention 0.3.5. Le résultat de caractérisation nous permet d'utiliser [2 '] comme une *définition* de catégorie **FrThFrob**.

Donc, pour nous la catégorie **FrThFrob**. est la catégorie dont les objets sont des ensembles finis $[m]$, vus comme des espaces discrets.

Le morphisme $[m] \to [n]$ est la paire (G, a) composé du graphe topologique G muni d'une fonction injective $a : [m] + [n] \hookrightarrow V$ de l'espace de domaine / codomaine aux sommets de G, qui est aussi surjectif sur les composants connexes. La composition est le collage d' espaces le long des sommets, avec le rejet d'éléments qui ne se croisent pas avec l'image de a.

Théorème 0.3.6. Chaque morphisme en **FrThFrob** peut être représenté par un graphe G de la forme suivante, où chaque composant connexe est un "star" dont le point nodal a n boucles attachés, avec $n \geq 0$.

Définition 0.3.7. Les objets de la catégorie **FrThFrob**($\mathcal{AT}$yp) sont des couples$([m], lab_m)$ constituées d'un ensemble fini $[m]$, vus comme un espace discret, et une fonction $lab_m : [m] \to \mathcal{AT}$yp, étiquetant des points en $[m]$ par les éléments de $\mathcal{AT}$yp.

Un morphisme $([m], lab_m) \to ([n], lab_n)$ est, une paire (G, a) composé d'un graphe G topologique muni d'une fonction injective $a : [m] + [n] \hookrightarrow V$ de l'espace de domaine / codomaine aux sommets de G, tel que les étiquettes d'éléments de $[m] + [n]$, qui sont mappés à un même composant connexe de G, sont de la même *sorte*. Également l'inclusion $a : [m] + [n] \hookrightarrow V$, de l'espace de domaine / codomaine aux sommets d'un graphe topologique G est surjective sur des composants connexes.

Composition, comme dans le cas **FrThFrob** est le collage d' espaces avec le rejet des composants qui ne se croisent pas avec l'image de a.

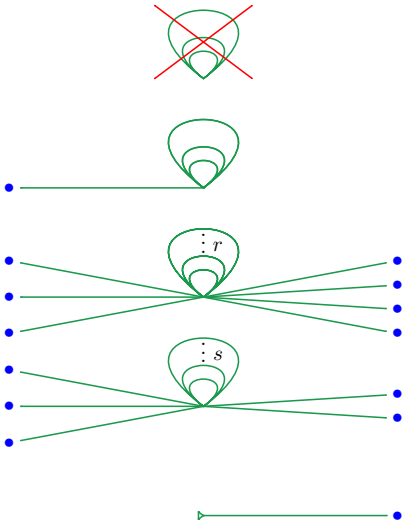


FIGURE 2 – Une "forme normale" de morphismes de **FrThFrob**. Les composants qui ne contiennent pas d'objets générateurs sont omis puisqu'ils se présentent comme l'identité d'unité monoïdale.

Par une légère généralisation des résultats de [Dij89, Koc04], la catégorie est la catégorie libre de Frobenius générée par un ensemble de générateurs $\mathcal{AT}\mathrm{yp}$.

Cela incite à la définition suivante.

Définition 0.3.8 (Liage)**.** Nous définissons *liage* comme un triple

$$P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$$

où

- P est un ensemble fini
- $\mathcal{C}omp_P$ est l'ensemble de classes d'une *partition* de l'ensemble P. Ses éléments sont appelés *composants*.
- la fonction $\mathcal{G}en_P : \mathcal{C}omp_P \to \mathbb{N}$ (appelée *genre*) attribue un entier naturel à chaque composant de $\mathcal{C}omp_P$.

C'est à noter l'abus de notation, où une seule lettre P peut être la truc complète ci-dessus ou juste son ensemble sous-jacent.

Il devrait être évident que le morphisme $m \to n$ en **FrThFrob**($\mathcal{AT}\mathrm{yp}$) peut être décrit comme une liaison de l'ensemble $m + n$. Évidemment, une définition formelle de la composition en termes des liages est un peu plus compliquée.

Proposition 0.3.9. *La catégorie* **FrThFrob**($\mathcal{AT}\mathrm{yp}$) *est compact-fermée, un objet étant lui-même son dual.*

C'est facile à voir, puisque si le morphisme $m \to n$ est donné, les choses en m peuvent être transférées vers le côté droit par une manipulation purement formelle, et vice-versa. Plus généralement, toute catégorie de Frobenius est compact-fermée.

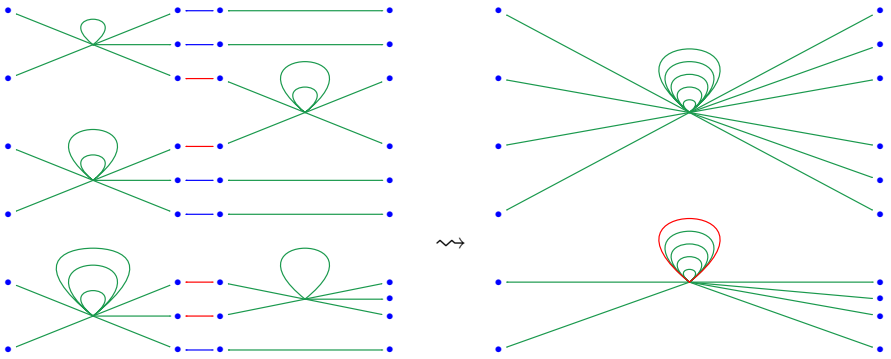


FIGURE 3 – Les morphismes dans une catégorie libre de Frobenius (dessinés horizontalement) perçus comme des graphes topologiques avec des générateurs d'objets pour les nœuds et le bouquet de cercles qui détermine le genre. La composition de morphismes équivaut au collage des graphiques le long des nœuds, et est déterminée par le type d'homotopie du nouveau graphe représenté.

La pertinence des "équations de Frobenius" pour la théorie de la preuve est due au fait qu'elle aborde le cas de contraction-contre-contraction dans l'élimination de coupures, comme expliqué dans le chapitre d'introduction. Ce cas de l'élimination de coupures résume les problèmes que l'on a avec la normalisation de la logique classique.

La façon la plus adaptée de représenter une catégorie de formules logiques et de preuves est par le biais de réseaux de preuves, où les objets sont des formules ordinaires, mais les morphismes appartiennent à une catégorie de **FrThFrob**($\mathcal{AT}$yp), où certains objets logiquement distincts ont été identifiés (par exemple, conjonction et disjonction). La structure originale des formules logiques introduit des contraintes sur ces morphismes, donc il y a un foncteur fidèle de la catégorie de réseaux de preuves **FrThFrob**($\mathcal{AT}$yp), mais qui n'est ni complet ni injectif sur les objets.

Nous introduisons un langage standard pour la logique classique propositionnelle, avec des atomes $\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots$, négatomes $\overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{c}}, \ldots$ et conjonction $\wedge$, disjonction $\vee$. Nous appelons une truc qui est soit un atome ou un négatome un *littéral*. La négation d'une formule composée est définie par la dualité de Morgan. Les séquents sont définis comme d'habitude, et étant donnée une formule A ou un Γ séquents on indique par $\mathcal{L}(A)$, leurs ensembles d'occurrences de littéraux.

Définition 0.3.10 (F-préréseau)**.** Nous définissons un F-préréseau comme une paire

$$P \triangleright \Gamma$$

composée d'un séquent Γ et d'un liage $(P, \mathcal{C}omp_P, \mathcal{G}en_P)$ où l'ensamble sous-jacent P est $\mathcal{L}(\Gamma)$ et chaque classe dans $\mathcal{C}omp_P$ contient des atomes seulement du même type et leur négation.

Quand nous disons que P *est* l'ensemble des occurrences de littéraux de Γ, nous voulons dire en fait que P est un ensemble arbitraire, équipé d'une bijection avec les réelles occurrences littérales dans Γ. Le point est que cette bijection ne doit jamais être rendue explicite dans la pratique, alors que le travail direct avec les occurrences d'atomes imposerait de mauvais contorsions.

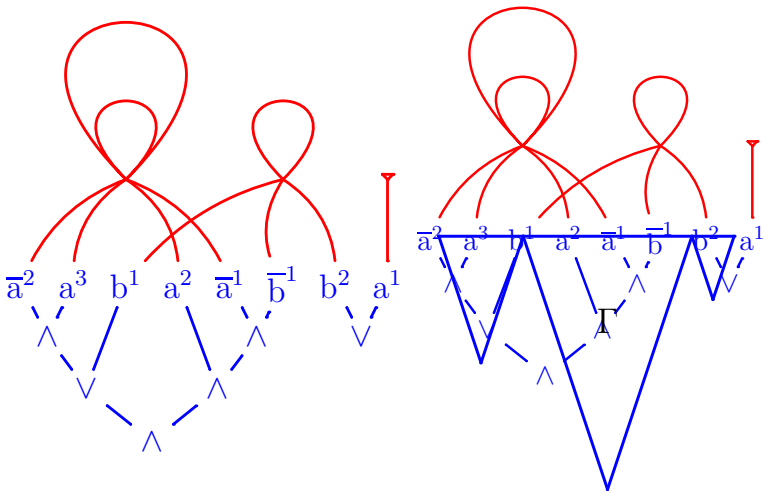


FIGURE 4 – Représentation graphique d'un F-préréseau (à gauche) et la représentation générique "glaçon/stalactite" d'une forêt de séquents (à droite)

Plusieurs systèmes déductifs peuvent être utilisés avec les F-préréseaux. Le premier est simplement le calcul ordinaire unilatère de séquents pour la logique classique, avec la règle Mix (de la logique linéaire). Il est présenté en détail en [LS05b] sous le nom de CL. En général, un calcul des séquents peut être utilisé pour définir une théorie des réseaux de preuves est chaque règle n-aire d'introduction du calcul

$$\frac{\vdash \Gamma_1 \quad \vdash \Gamma_2 \quad \cdots \quad \vdash \Gamma_n}{\vdash \Gamma}$$

peut être transformé en une famille de n morphismes $P_i \triangleright \Gamma_i \to Q \triangleright \Gamma$ dans la *catégorie syntaxique* suivante.

Définition 0.3.11 (Syntactic Category)**.** Soit $\mathcal{F}$**Synt** la catégorie dont les objects sont les F-préréseaux et un morphisme

$$f : P \triangleright \Gamma \to Q \triangleright \Delta$$

est donné par une fonction ordinaire sur l'ensemble sous-jacent de littéraux

$$f : P \to Q \qquad \big(= \mathcal{L}(\Gamma) \to \mathcal{L}(\Delta)\big)$$

tel que

1. pour toute formule A, f transfère $\mathcal{L}(A)$ à un sous-ensemble de $\mathcal{L}(\Delta)$ qui définit une sous-formule d'une formule en Δ, tout en préservant l'ordre syntaxique gauche-droite pour les littéraux.

2. pour chaque $C \in \mathcal{C}omp_P$, on a un $f(C) \subseteq \mathcal{L}(\Delta)$ qui est contenu dans un composant $C' \in \mathcal{C}omp_Q$, avec $\mathcal{G}en_P(C) \leq \mathcal{G}en_Q(C')$.

La procédure pour obtenir un F-préréseau $P \triangleright \Gamma$ à partir d'une preuve d'un séquent $\vdash \Gamma$ est absolument simple. Les cas qui méritent d'être mentionnés spécifiquement sont l'Affaiblissement et la Contraction. En supposant que nous avons construit $P \triangleright \Gamma$ à partir d'une preuve, puis en ajoutant la formule A par le biais d'affaiblissement, cela nous donne un liage sur l'union disjointe $P \uplus \mathcal{L}(\mathrm{A})$ où chaque composant ajouté est un singleton avec genre 0 associé. Pour la contraction, si les deux occurrences visibles de A dans $P \triangleright \Gamma, \mathrm{A}, \mathrm{A}$ sont contractées, on obtient un F-préréseau $P(\mathrm{A} \curlyvee \mathrm{A}) \triangleright \Gamma, \mathrm{A}$ en reliant le i-ème littéral de la première instance de A et le i-ème littéral dans la deuxième instance à un seul "terminale".

Cet opération $(- \curlyvee -)$ peut être réitérée, et peut être appliquée à sous-formules et sous-séquents ainsi que les formules. Dans ce qui suit nous utilisons les exposants pour désambiguër les occurrences lorsque nous estimons qu'il est utile de le faire.

De la même façon, étant donnés les deux occurrences visibles de A et $\overline{\mathrm{A}}$ en $P \triangleright \Gamma, \mathrm{A}, \overline{\mathrm{A}}$, le F-préréseau $P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma$ est obtenu en reliant le i-ème littéral de A et le i-ème littéral en $\overline{\mathrm{A}}$ par un fil ordinaire.

Définition 0.3.12. Dans la catégorie $\mathcal{F}\mathbf{Synt}$, nous définissons les familles de cospans *Mix* et $\wedge$ comme suit :

$$\begin{array}{ccc} P_l \triangleright \Gamma & & P_r \triangleright \Gamma \\ & \searrow^{Mix:l} \quad \swarrow_{Mix:r} & \\ & P_l \uplus P_r(\Gamma \curlyvee \Gamma) \triangleright \Gamma & \end{array}$$

et

$$\begin{array}{ccc} P_l \triangleright \Gamma, \mathrm{A}^1 \wedge \mathrm{B}^1, \mathrm{A}^2 & & P_r \triangleright \mathrm{B}^2, \mathrm{A}^3 \wedge \mathrm{B}^3, \Gamma \\ & \searrow^{\wedge:l} \quad \swarrow_{\wedge:r} & \\ & Q \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B} & \end{array}$$

où Q is $P_l \uplus P_r\big(\Gamma \curlyvee \Gamma, (\mathrm{A}^1 \curlyvee \mathrm{A}^2) \curlyvee \mathrm{A}^3, (\mathrm{B}^1 \curlyvee \mathrm{B}^2) \curlyvee \mathrm{B}^3\big)$.

Définition 0.3.13. Un morphisme *anodin* $\hat{\mathrm{E}}\ P \triangleright \Gamma \multimap Q \triangleright \Delta$ est un morphisme syntaxique qui peut être décomposé

$$P \triangleright \Gamma \xrightarrow{\sim} Q \triangleright \Delta_1 \xrightarrow{\vee} \cdots \xrightarrow{\vee} Q \triangleright \Delta_n = \Delta$$

comme un isomorphisme suivi par une séquence de morphismes d'introduction (qui n'affecte pas la liaison, mais uniquement le séquent).

Il y a un morphisme anodin important, ce qui correspond à l'élimination de toutes les disjonctions extérieures. Nous écrivons :

$$[P \triangleright \Gamma] \multimap P \triangleright \Gamma$$

pour désigner le morphisme anodin dont le domaine est le séquent où toutes les disjonctions extérieures ont été supprimées.

Définition 0.3.14 (F-réseaux corrects)**.** Un F-préréseau $P \triangleright \Gamma$ est un *F-réseau* CL-*correct*, (ou simplement un F-réseau) s'il est à la racine d'un *diagramme de justesse* $\mathcal{T} \to \mathcal{F}\mathbf{Synt}$, ce qui signifie un diagramme pour lequel :

1. $\mathcal{T}$est un ensemble ordonné qui est un arbre renversé (c'est à dire la racine est la partie supérieure, les feuilles sont minimales), avec $P \triangleright \Gamma$ à sa racine ;
2. morphismes de diagramme $\mathcal{T}$ sont soit anodins soit appartiennent à $\wedge$-où *Mix*-cospan ;
3. les branchements ne sont que $\wedge$-où *Mix*-cospans ;
4. chaque feuille de l'arbre est un F-préréseau $Q \triangleright \Delta$ avec $\mathcal{C}omp_Q = \{\{\mathrm{a}, \overline{\mathrm{a}}\}, \{x_1\}, \ldots, \{x_m\}\}$ et un $\mathcal{G}en_Q$ morphisme qui est partout 0, c'est à dire un axiome avec affaiblissements.

Théorème 2.17 Etant donné un F-préréseau, sa CL-correction (CL-sequentializabilité) peut être vérifiée en temps fini, c'est-à-dire, le critère de correction CL donne une procédure de décision pour corriger CL-F-réseaux. Nous avons des preuves solides qu'en réalité la procédure est NP-complète.

Cela peut être renforcé en obligeant les morphismes anodins d'être toujours $\Box$-morphismes et d'avoir une alternance entre ces derniers et morphismes de cospans. Nous montrons :

Théorème 0.3.15 (Séquentialisation)**.** Les F-réseaux corrects sont précisément ceux F-préréseaux qui viennent de CL.

Étant donné un P reliant, que $|P|$ soit la taille de son ensemble sous-jacent, $|\mathcal{C}omp_P|$ le nombre de composants en P et $|\mathcal{G}en_P|$ la somme de tous les genre en P, c'est-à-dire $|\mathcal{G}en_P| = \sum_{C \in \mathcal{C}omp_P} \mathcal{G}en_P(C)$. L'observation suivante est essentielle pour la preuve :

Lemma 0.3.16 (Compte des liens d'axioms dans un F-préréseau)**.** *Si un F-préréseau* $P \triangleright \Gamma$ *correspond à un preuve* CL*, donc*

$$|Ax| = |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|,$$

où $|Ax|$ *est le nombre d'axiomes de la preuve (corollaire : tout diagramme de justesse pour cette preuve aura le même nombre de feuilles).*

Ce lemme, avec une analyse complémentaire de preuves garantit finitude de l'espace de recherche :

Théorème 0.3.17. Etant donné un F-préréseau, sa CL-correction (CL-sequentializabilité) peut être vérifiée en temps fini, c'est-à-dire, le critère de correction CL donne une procédure de décision pour corriger CL-F-réseaux.

Nous avons des preuves solides qu'en réalité la procédure est NP-complète.

4 F-réseaux et Élimination des coupures

Quand la Coupure entre en jeu, les choses changent un peu. Tout d'abord, nous définissons une *formule coupée* comme $A \diamondsuit \overline{A}$, où $- \diamondsuit -$ est un nouveau connecteur binaire qui est seulement autorisé à apparaître comme racine dans un séquent.

Ceci nécessite une modification de la procédure de correction pour F-préréseaux pour accueillir le nouveau connecteur et nous renvoyons le lecteur au texte complet.

Notre objectif initial est de normaliser ces préréseaux avec coupures au moyen de la composition dans **FrThFrob**($\mathcal{AT}$yp) (ne pas oublier que c'est compact-fermé). Cette utilisation des algèbres de Frobenius dans la logique classique est assez différente de celle proposée par Hyland [Hyl04]. Ceci ressemble plus au travail en [LS05b], où l'équivalent de la catégorie **FrThFrob**($\mathcal{AT}$yp) est obtenu à partir d'une construction de "catégorie d'interaction" [Hyl04, Section 3] sur des ensembles et des relations, où la composition est définie par les moyens d'un opérateur de traces.

L'élimination des coupures définie de cette manière provoque immédiatement des problèmes. Regardez la partie droite de la figure. Pour que le résultant F-préréseau vienne d'une preuve, nous avons besoin d'un composant singleton venant d'un affaiblissement, mais cela ne peut pas se produire selon notre interprétation puisque son genre est > 0.

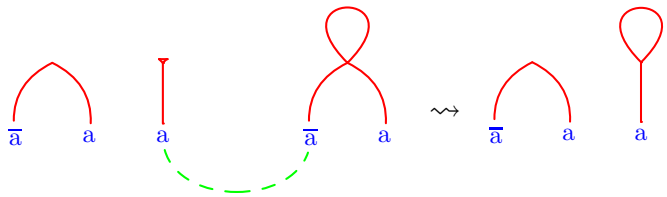


FIGURE 5 – Élimination des coupures effectuée sur deux corrects CL-F-réseaux résultant en un F-préréseau qui ne correspondent pas à un preuve CL.

Ces questions peuvent être traitées par la modification du système déductif et nous définissons un nouveau calcul coérent et complet pour la logique classique, FL.

Le but du bénitier est de mettre en évidence la partie du séquent qui provient à coup sûr de l'affaiblissement et aussi de permettre l'introduction des configurations arbitraires de liens au moyen de l'affaiblissement. C'est parce que *MulWeak* est interprété par adjonction d'un ensemble $\{a, a, \ldots, a, \overline{a}, \overline{a}, \ldots, \overline{a}\}$, au liage, qui contient un *seul* component de genre zéro.

La définition de la correction dans FL tient compte de ce nouveau connecteur de coupure et nous introduisons un autre "cospan" dans la catégorie syntaxique des F-préréseaux $\mathcal{F}$**Synt**. Nous assouplissons également la définition de morphisme anodin pour permettre à des fonctions qui sont injectives mais pas nécessairement bijectives de tenir compte de la règle d'Affaiblissement. Avec ces modifications, les Theorèmes 0.3.15 et 0.3.17 peuvent être reformulés, avec une différence prononcée : cette fois, pour un FL-réseau correct, nous

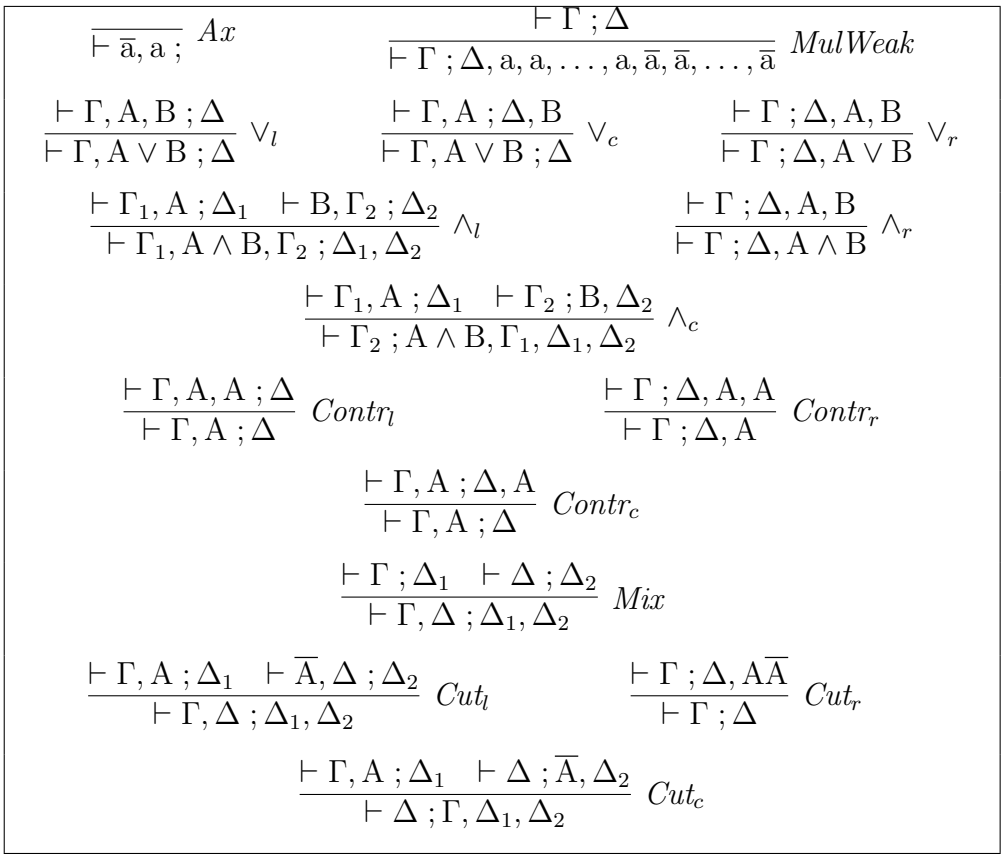


FIGURE 6 – Système FL.

avons $|Ax| \leq |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|$.

Même si les problèmes comme le contre-exemple ci-dessus sont résolus, en général, nous ne pouvons toujours pas éliminer les coupures sur un réseau FL-correct et en obtenir un qui est aussi FL-correct. Donc, nous n'avons pas encore une catégorie. Cela exige un peu plus d'analyse.

Jetons maintenant un autre coup d'œil sur l'élimination des coupures dans les F-réseaux. En suivant l'idéologie d'élimination-comme-composition, nous avons

$$P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}} \;\rightarrowtail\; P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma.$$

Ce qui suit, provient trivialement de la structure catégorique sous-jacente et de la définition de transformation $\smile$.

Observation 0.4.1. L'élimination des coupures dans les F-préréseaux a les propriétés de confluence et de terminaison, et donc produit une forme normale unique pour les F-préréseaux avec coupures.

Nous savons maintenant que l'élimination des coupures pour un F-préréseau donne une forme normale, c'est à dire, étant donné un F-préréseau avec des coupures, qu'il y a un unique F-préréseau sans coupures auquel le premier se réduit. Toutefois, la question importante est : est-ce que le même principe s'applique à F-réseaux FL-corrects ? Plus précisément, est-ce que, partant d'un F-réseau avec coupures, la procédure d'élimination de celles-ci aboutit à un autre F-réseau FL-correct ? La réponse directe à la version la plus ambitieuse de cette question est, malheureusement, négative. Les contre-exemples sont nombreux, et l'un est donné à la Figure 4.4.

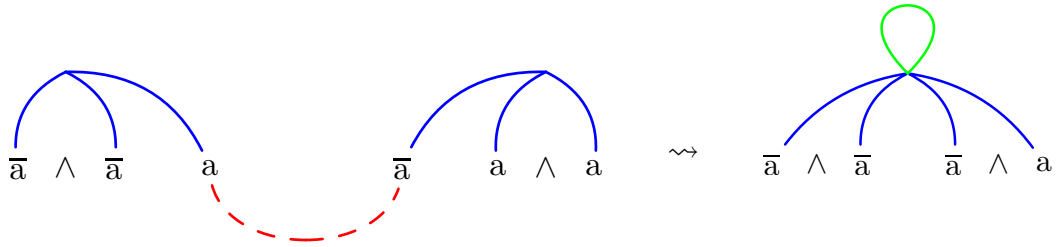


FIGURE 7 – Élimination des coupures sur des FL-correct F-réseaux ne produit pas un FL-correct F-réseau. La boucle verte qui est ajoutée est nécessaire pour obtenir un FL-correct net, mais elle n'est pas obtenue à travers de la composition.

Il y a une observation que nous pouvons faire sur ce contre-exemple :
– si nous ajoutions une boucle sur la seule composante du F-préréseau obtenu, tout en conservant la même partition des littéraux le résultat serait FL-correct.
Nous effectuons une analyse plus détaillée basée sur cette observation.

Définition 0.4.2. Etant donné un F-préréseau $P \triangleright \Gamma$, nous définissons un "switching" conjonctif comme le choix d'un sous-arbre immédiat pour chaque nœud $\wedge$ dans l'arbre syntaxique de Γ. Étant donné un switching conjonctif, soit Γ' l'arbre où tous les sous-arbres choisis par le switching dans Γ sont effacés, et soit $\mathcal{L}(\Gamma')$ l'ensemble des feuilles de cet arbre. (Notez que Γ' n'est pas une séquence de formule propositionnelles bien formées, c'est plutôt un graphe avec des nœuds décorés par des connecteurs et des littéraux). L'application d'un switching à un préréseau $P \triangleright \Gamma$ est la paire $P|_{\mathcal{L}(\Gamma')} \triangleright \Gamma'$.

Nous disons qu'un F-préréseau $P \triangleright \Gamma$ est *solide* (en anglais : *sound*) si pour tout switching Γ' le liage $P|_{\mathcal{L}(\Gamma')}$ contient une composante avec des atomes de polarité opposée. Nous disons que ces atomes de polarité opposée sont des *témoins de solidité* pour le switching.

Solidité d'un F-préréseau avec des formules de coupures est définie de la même manière, où, en outre, chaque switching supprime un sous-arbre pour chaque connecteur $\diamondsuit$, ou bien en traitant le symbole de coupure comme conjonction.

Le lecteur peut reconnaître la condition de Lamarche-Strassburger des $\mathbb{B}$-nets figurants dans [LS05b], dont le prédécesseur possible est la condition Danos-Regnier pour les réseaux de preuves pour la logique linéaire multiplicative, apparaissant en [DR89].

Proposition 0.4.3. *Une F-réseau (*CL- */* FL-*) correct est solide.*

Dans ce qui suit, quand nous écrivons Γ, nous supposons que Γ peut contenir des formules coupées.

Proposition 0.4.4. *Étant donné un F-préréseau solide $P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}}$, en exécutant*

$$P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}} \twoheadrightarrow P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma$$

, l'étape d'élimination des coupures conserve sa solidité.

Il convient de rappeler au lecteur de la structure catégorique (sans coupures) pour laquelle F-préréseaux sont présents ici.

Définition 0.4.5 (Catégorie de F-préréseaux)**.** La *catégorie de F-préréseaux*, écrit **F-prenet**, a pour ses objets des formules propositionnelles.

Le morphisme $\mathrm{A} \to \mathrm{B}$ est un F-préréseau sans coupures $P \triangleright \overline{\mathrm{A}}, \mathrm{B}$. L'identité de F-préréseau pour A est le F-préréseau $\mathrm{Id}_{\mathrm{A}} \triangleright \overline{\mathrm{A}}, \mathrm{A}$ où Id_{A} est le liage formée de composants à deux éléments qui relient i littéral (comptés, disons, de gauche à droite) en A avec le i en $\overline{\mathrm{A}}$, partout de genre 0.

Etant donné $P \triangleright \overline{\mathrm{A}}, \mathrm{B} :\ \mathrm{A} \to \mathrm{B}$ et $Q \triangleright \overline{\mathrm{B}}, \mathrm{C} :\ \mathrm{B} \to \mathrm{C}$, la composition de morphisme est le F-préréseau

$$P \uplus Q \left(\mathrm{B} \smile \overline{\mathrm{B}}\right) \triangleright \overline{\mathrm{A}}, \mathrm{C}.$$

Chaque F-préréseau $P \triangleright \Gamma, \Sigma$ pour toute partition et toute permutation des formules de $\Gamma \cup \Sigma$ en Γ, Σ détermine un morphisme $\overline{\Gamma} \to \bigvee \Sigma$ en **F-prenet**, où $\overline{(-)}$ est la négation étendue à séquents par $\overline{\mathrm{A}, \mathrm{B}} := \overline{\mathrm{A}} \wedge \overline{\mathrm{B}}$, et où $\bigvee(-)$ est une transformation d'un séquent dans une formule disjonctive qui remplace les symboles de virgule avec $'\vee'$.

Un morphisme de **F-prenet** issu d'un F-préréseau (sans coupures) $P \triangleright \Gamma, \Sigma$ détermine aussi un morphisme dans la catégorie librement engendrée de Frobenius $\widehat{\overline{\Gamma}} \to \widehat{\bigvee \Sigma}$ (Γ est nié), où $\widehat{(-)}$ est la transformation qui envoie une formule pour un objet de la catégorie de Frobenius en remplaçant les connecteurs propositionnels avec le symbole bifonctoriel monoïdal $\otimes$.

Par conséquent, **F-prenet** est équivalent à **FrThFrob**($\mathcal{AT}$yp).

La proposition précédente montre maintenant que la catégorie **F-prenet** a une sous-catégorie d'un solide F-préréseaux, **sF-prenet**. Revenant à la définition d'un solide F-préréseau, le lecteur pourra constater que la notion de la solidité ne dépend que des composants d'un liage et pas des genres.

Jetons un coup d'œil sur le foncteur d'oubli de **FrThFrob**($\mathcal{AT}$yp) à la catégorie que nous appelons **BFrThFrob**($\mathcal{AT}$yp), qui ignore les genres. En d'autres termes, **BFrThFrob**($\mathcal{AT}$yp) est **FrThFrob**($\mathcal{AT}$yp) privé d'informations sur genres. De la même façon **FrThFrob**($\mathcal{AT}$yp) donne lieu à la catégorie F-préréseau, la catégorie **BFrThFrob**($\mathcal{AT}$yp) donne lieu à la catégorie des préréseaux assignés avec liages représentant les morphismes de l'unité monoïdale en **BFrThFrob**($\mathcal{AT}$yp). Nous les appelons FB-préréseaux et leur catégorie est désignée par FB-préréseau. Nous nous référons à

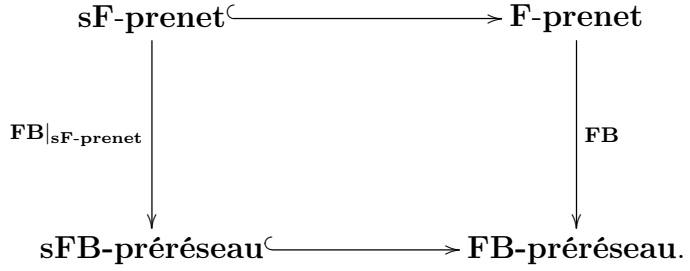
$$\mathbf{FB} : \{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Gamma \mapsto \{C_1, \ldots, C_k\} \triangleright \Gamma$$

comme un foncteur d'oubli de F-préréseau à FB-préréseau.

La notion d'un préréseau solide peut être répétée pour FB-préréseaux, c'est-à-dire qu'elle est stable sous le foncteur d'oubli que nous venons de définir. Par conséquent, les solide FB-préréseaux constituent une sous-catégorie **sFB-préréseau** de **FB-préréseau**. (Nous rappelons au lecteur que la composition en **FB-préréseau** diffère de la composition de **F-prenet**, car aucune information sur le genre est présente).

Nous avons déjà établi que les FL-correct F-réseaux sont solides, et donc, l'image sous le **FB** en **FB-préréseau** entre dans la sous-catégorie **sFB-préréseau**, qui est précisément l'affirmation de la proposition précédente.

Pour résumer, le diagramme suivant commue :



Nous continuons notre recherche sur l'élimination des coupures en introduisant plus de concepts.

Définition 0.4.6. On dit qu'une catégorie est *enrichie dans les ordres* s'il y a un ordre partiel définie sur chaque hom-set, de manière que s'il y a trois objets donnés X, Y, Z, la fonction de composition $\circ : Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$ est monotone dans les deux variables.

En utilisant la terminologie de la théorie des catégories enrichies, une catégorie ordonnée est enrichie dans la catégorie monoïdale $(\mathbf{Poset}, \times, \emptyset)$ des ordres partiels et des fonctions monotones.

La proposition suivante est un exercice de compréhension de la définition ci-dessus et la définition de la composition en **F-prenet**.

Proposition 0.4.7. *Soit $\leq$ un ordre partiel défini sur l'ensemble de tous les F-préréseaux de sorte que :*

1. $P \triangleright \Gamma \leq P' \triangleright \Gamma'$ *implique* $\Gamma = \Gamma'$,
2. *donnée sans coupures* $P \triangleright \Gamma, \mathrm{A} \leq P' \triangleright \Gamma', \mathrm{A}$ *et* $Q \triangleright \overline{\mathrm{A}}, \Sigma$, *on a*

$$P \uplus Q(\overline{\mathrm{A}} \frown \mathrm{A}) \triangleright \Gamma, \Sigma \ \leq \ P \uplus Q(\overline{\mathrm{A}} \frown \mathrm{A}) \triangleright \Gamma', \Sigma.$$

Puis $\leq$ induit un enrichissement dans les ordres pour le **F-prenet**.

La proposition précédente nous permet de généraliser la notion d'un enrichissement pour l'ensemble de tous F-préréseaux, pas seulement sur les deux-formules F-préréseaux de **F-prenet**. Donc, par "l'enrichissement sur l'ensemble des F-préréseaux", nous entendons la présence d'un ordre satisfaisant les conditions de la proposition précédente.

Un enrichissement de ce genre est donné dans la définition suivante.

Définition 0.4.8. Étant donnés deux F-préréseaux $P \triangleright \Gamma$ et $Q \triangleright \Gamma$ on écrit

$$P \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$$

quand

1. composants dans les partitions de littéraux de Γ in P and Q coïncident
2. pour chaque composant (C, G) de P, le composant correspondant (C, G') de Q est tel que $G \leq G'$.

Pour montrer que l'ordre ci-dessus est en effet un enrichissement, nous montrons

Proposition 0.4.9. *Pour tous les trois F-préréseaux$P \triangleright \Gamma, \mathrm{A}$; $Q \triangleright \Gamma, \mathrm{A}$; et $R \triangleright \overline{\mathrm{A}}, \Delta$;*

$$P \triangleright \Gamma, \mathrm{A} \ \leq_{\mathcal{G}} \ Q \triangleright \Gamma, \mathrm{A}$$

implique

$$P \uplus R(\mathrm{A} \frown \overline{\mathrm{A}}) \triangleright \Gamma, \Delta \ \leq_{\mathcal{G}} \ Q \uplus R(\mathrm{A} \frown \overline{\mathrm{A}}) \triangleright \Gamma, \Delta.$$

Le texte suivant est une conséquence directe de la proposition précédente.

Théorème 0.4.10. L'ordre $\leq_{\mathcal{G}}$ sur F-penets en **F-prenet** définit un enrichissement dans les ordres de la catégorie.

L'enrichissement que nous venons de définir se comporte bien par rapport à FL-correction.

Lemma 0.4.11. *Si un F-préréseau $P \triangleright \Gamma$ vient d'un* FL *prueve classique qui, s'il est* FL*-correct, alors est ainsi tout F-préréseau $Q \triangleright \Gamma$, pour lequel $P \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$.*

Le lemme précédent montre que l'ensemble des FL-correct F-réseaux est up-closed pour la relation $\leq_{\mathcal{G}}$. Le même ensemble sera aussi fermé sous la transformation suivante.

Lemma 0.4.12. *Si une F-préréseau $\{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Gamma$ est* FL*-correct, alors il est aussi tout F-préréseau obtenu en connectant plusieurs composants, à savoir tout*

$$\{(C_1, G_1), \ldots, (C_{i-1}, G_{i-1}), (C_i \cup C_{i+1}, G_i + G_{i+1}), (C_{i+2}, G_{i+2}), \ldots, (C_k, G_k)\} \triangleright \Gamma$$

est également FL-correct.

Le lemme précédent motive la définition suivante.

Définition 0.4.13. Étant donnés deux F-préréseaux $P \triangleright \Gamma$ et $Q \triangleright \Gamma$ nous définissons

$$P \triangleright \Gamma \preceq Q \triangleright \Gamma$$

si

1. . la partition des littéraux en Γ est plus fine dans P que dans Q, c'est-à-dire que chaque composant de Q peut être obtenu en connectant plusieurs composants de P
2. fpour chaque composant (G, C) de Q pour lequel $C = C_1 \cup \cdots \cup C_k$, pour composants $(G_1, C_1), \ldots, (G_k, C_k)$ de P, nous avons

$$G \geq \sum_1^k G_k.$$

Notez que si $P \triangleright \Gamma \preceq Q \triangleright \Gamma$, alors $\mathcal{C}har(P \triangleright \Gamma) \leq \mathcal{C}har(Q \triangleright \Gamma)$, mais l'inverse n'est pas vrai en général.

Le morphisme de gauche a une caractéristique plus petite, mais les deux morphismes sont incomparables par rapport 'a l'ordre $\preceq$.

Théorème 0.4.14. L'ordre $\preceq$ sur F-préréseaux un ordre d'enrichissement sur le **F-prenet**.

Théorème 0.4.15. Pour chaque F-préréseau solide $P \triangleright \Gamma$ il y a un F-préréseau $Q \triangleright \Gamma$ pour lequel

$$P \triangleright \Gamma \ \leq_{\mathcal{G}} \ Q \triangleright \Gamma$$

et qui est FL-correct.

Réfléchissons un moment sur les dernières observations que nous avons faites. è cette fin, on considère les points suivants : que Γ soit un séquent avec des formules de coupure. Nous définissons $|Coupure|$ comme

$$|\,Cut\,| = \sum_{\mathrm{A}^k \diamondsuit \overline{\mathrm{A}^k} \text{ in } \Gamma} |\mathcal{L}(\mathrm{A}^k)|.$$

L'effet de l'élimination des coupures sur un liage se reflète dans l'inégalité :

$$|Ax_1\,| \leq |P_1| - |\mathcal{C}omp_{P_1}| + |\mathcal{G}en_{P_1}| = \mathcal{C}har(P_1 \triangleright \Gamma_1).$$

Chaque fois qu'un $\smile$ est réalisé sur une paire de formules, le côté gauche de l'inégalité diminue avec le nombre de paires de littéraux éliminées, donc, si $P_2 \triangleright \Gamma_2$ est le F-préréseau obtenu après l'élimination des coupures, on a

$$|Ax_1\,| \leq |P_1| - |\mathcal{C}omp_{P_1}| + |\mathcal{G}en_{P_1}| - |\,Cut\,| \leq |P_2| - |\mathcal{C}omp_{P_2}| + |\mathcal{G}en_{P_2}| = \mathcal{C}har(P_2 \triangleright \Gamma_2).$$

Le premier $\leq$ est une conséquence du fait que la caractéristique d'un F-préréseau est une limite supérieure dans le cas de FL, comme il est déjà établi, tandis que le second $\leq$ est au lieu de l'équation parce que l'élimination des coupures peut supprimer les composants complètement en raison de la minceur de la catégorie sous-jacente. Cette situation est illustrée dans la Figure 4.6

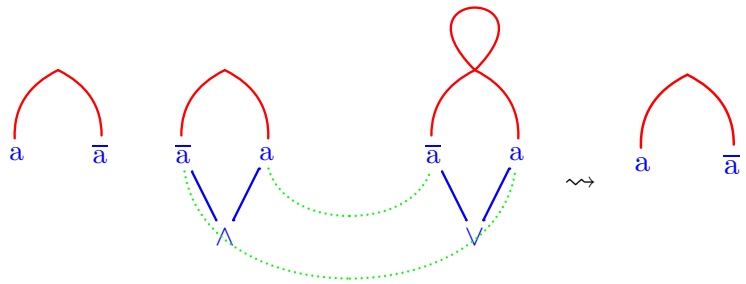


FIGURE 8 – Élimination des coupures peut supprimer les composants tout à fait. Cela correspond à la propriété de minceur des algèbres de Frobenius.

Notre point de départ dans les considérations sémantiques étaient catégories *-autonomes, qui sont des modèles pour la logique linéaire multiplicative. La théorie de la démonstration de la logique linéaire multiplicative est sensible aux ressources, entendue comme le

nombre d'applications de la règle d'axiome dans une preuve. Nous avons décidé de suivre cette méthode et essayé de concevoir des modèles qui diffèrent parmi les preuves par le nombre d'axiomes utilisés. L'analyse nous a menés à la notion de la catégorie de Frobenius et les F-(pre)nets, où nous avons vu que les interprétations des preuves ont un invariant - la caractéristique est augmentée de 1 par une seule instance de la règle d'axiome. Cette caractéristique représente une limite inférieure sur le nombre d'axiomes utilisés dans un FL correspondant au F-préréseau, si une telle preuve existe. L'élimination de coupures sur une paire de littéraux, en revacnhe, diminue la caractéristique de 1, doublement de l'axiome. Par conséquent, il est logique de constater

la caractéristique d'un F-préréseau garde la trace des *ressources* utilisées dans la preuve.

En effet, le premier lemme des deux derniers montrent que si nous avons réussi à construire une preuve avec des pièces offertes par un F- préréseau, nous sommes sûrs de réussir avec un F-préréseau avec de plus grands genres. Similaires au lemme qui suit. Comme l'accroissement des genres signifie augmentation de la caractéristique, juste comme la connexion de deux composants mutuellement déconnectés, ces deux lemmes peuvent être reformulée pour dire que si une preuve peut être construite en utilisant les ressources provenant d'un F-préréseau, donc, naturellement , une preuve peut être construite en utilisant plus de ressources. Un axiome introduit une unité de ressources, alors que l'élimination de coupures consomme des ressources. Enfin, le théorème précédent peut être reformulé, avec une légère généralisation.

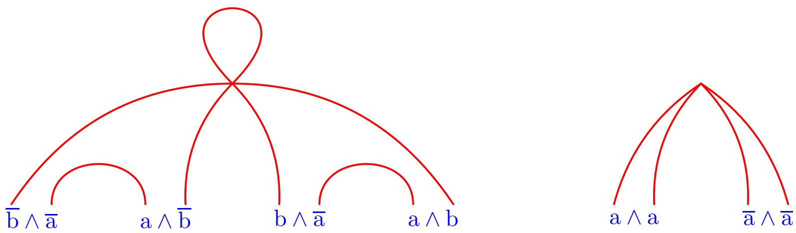
Theorem 4.3.17., restated. Un solide F-préréseau peut être transformé en un FL-correct F-réseau en ajoutant des ressources.

Notez que l'on peut parler de trois dimensions d'une ressource (nombre) de composants, de genres, et (nombre) de littéraux, dont le dernier est constant pour toutes les preuves du séquent donné. Ceci, bien sûr, concerne les F-préréseaux sans coupures.

Revenant au **FB-préréseau**, la catégorie des F-préréseaux sans les informations de genre et son sous-catégorie **sFB-préréseau** des préréseaux solides des **FB-préréseau**, le théorème précédent déclare précisément que le **sFB-préréseau** donne une interprétation des preuves classiques pour lesquelles détient la propriété *d'intégralité pleine.*

Nous avons déjà établi que la notion de justesse pour les F-préréseaux est fortement dépendante du système déductif en question. Dans le cas de CL, nous avons rencontré F-préréseaux pour lesquels il n'y a pas de dérivation dans le calcul, mais pour lesquels il existe des dérivations dans un système d'inférence profonde. Le même est pour le cas de la version FL d'inférence profonde.

Example 0.4.16.



Le F-préréseau à droite a la preuve suivante KS (notez que nous ne font aucun effort pour réduire la bureaucratie de preuve) :

$$
\begin{array}{cl}
\mathsf{T} & \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (\bar{b} \vee b) \wedge (b \vee \bar{b}) \wedge (\bar{b} \vee b) \wedge (\bar{b} \vee b) & 5 \times \mathsf{ai}\downarrow \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (\bar{b} \vee b) \wedge (b \vee \bar{b}) \wedge (\bar{b} \vee (b \wedge (\bar{b} \vee b))) & \mathsf{s} \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (\bar{b} \vee b) \wedge (b \vee \bar{b}) \wedge (\bar{b} \vee \bar{b} \vee (b \wedge b)) & \mathsf{s} \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (\bar{b} \vee b) \wedge (b \vee \bar{b}) \wedge (\bar{b} \vee (b \wedge b)) & \mathsf{ac}\downarrow \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (\bar{b} \vee (b \wedge (b \vee \bar{b}))) \wedge (\bar{b} \vee (b \wedge b)) & \mathsf{s} \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (b \vee b \vee (\bar{b} \wedge \bar{b})) \wedge (\bar{b} \vee (b \wedge b)) & \mathsf{s} \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge (b \vee (\bar{b} \wedge \bar{b})) \wedge (\bar{b} \vee (b \wedge b)) & \mathsf{ac}\downarrow \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge ((\bar{b} \wedge \bar{b}) \vee (b \wedge (\bar{b} \vee (b \wedge b)))) & \mathsf{s} \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge ((\bar{b} \wedge \bar{b}) \vee (b \wedge \bar{b}) \vee (b \wedge b)) & \mathsf{s} \\
\hline
(\bar{a} \vee a) \wedge (\bar{a} \vee a) \wedge ((b \wedge \bar{b}) \vee (b \wedge b)) & \mathsf{ac}\downarrow \\
\hline
(\bar{b} \wedge \bar{a}) \vee (a \wedge \bar{b}) \vee (b \wedge \bar{a}) \vee (a \wedge b) & 6 \times \mathsf{s}
\end{array}
$$

Pour voir que le même F-préréseau ne correspond à aucune preuve de $\mathsf{CL}/\ \mathsf{FL}$, on peut tenter de construire un arbre de correction pour le F-préréseau. Tout arbre de ce type doit commencer par une conjonction cospan, puisque un cospan Mix devrait avoir un F-préréseau non-solide comme le domaine d'une de ses jambes. Une conjonction cospan produit F-préréseaux dans les domaines où tous les deux doivent avoir un seul composant $\{\bar{a}, a\}$, ce qui garantit que celui qui contient la formule a (où $\bar{a}$) n'est pas solide

Comme pour le F-préréseau à droite, de manière similaire comme dans le cas précédent, nous nous rendons compte que tout Mix ou conjonction cospan qui commencerait un arbre hypothétique de correction, doit avoir non-solides F-préréseaux comme les domaines de ses jambes. Ainsi, aucun CL/FL (superficiel) est possible. Toute preuve profonde attribuée à la F-préréseau de droite doit avoir au maximum 3 $\mathsf{ai}\downarrow$ règles. Pour faire les trois composants créés par applications de règle dans le liage nécessaire, on effectue soit deux contractions sur les littéraux ou une seule contraction sur deux conjonctions de deux littéraux. Dans les deux cas, avec l'obtention d'un seul composant à travers la conjonction, il y aura un littéral dans un contexte disjonctif.

Un point de vue possible de la notion de ressources comme nous les avons définies, c'est qu'ils fournissent un compte sémantique d'efficacité relative des systèmes déductifs. Plus précisément, comme nous l'avons montré, certains systèmes déductifs, comme ceux d'inférence profonde, peuvent fournir des preuves pour certaines configurations de liages, alors que les séquents standards ont besoin des ressources supplémentaires. Il pourrait y

avoir certain potentiel pour cette ligne d'investigation et quelques idées relativement liées à ce sujet qui peuvent être trouvés dans le travail de Carboni [Car99b]. (plus d'informations en relation à ce travail peuvent être trouvées plus tard dans le texte dans la section sur autres approches pertinentes).

***

Résumons maintenant. Ce que que nous avons jusqu'ici est une catégorie de F-prréseaux, **F-prenet** et sa sous-catégorie de solides F-prréseaux **sF-prenet**. L'image de SF-prréseau sous le foncteur d'oubli, qui oublie les genres, définit la catégorie de réseaux de preuves classiques, dont détient l'exhaustivité complète. Cette catégorie contient moins d'informations sur les preuves que la catégorie des $\mathbb{B}$-nets de [LS05b]. En effet, un liage dans un FB-prréseau solide est une fermeture transitive d'un liage dans un B-net, donc on perd des informations sur les liens axiome dans cette catégorie (voir la section sur autres approches pertinentes).

FL-correct F-réseaux eux-mêmes ne forment pas une catégorie (au moins pas de façon standard de composer les nets en se connectant par liages), mais nous savons que nous pouvons construire un FL-correct F-réseau d'un net solide en ajoutant les genres. En outre, les FL-correct F-réseaux sont fermés sous cette addition de ressources, donc la question est quelle est la quantité minimale de ressources qu'il faut ajouter pour obtenir un FL-correct F-réseau. Nous n'avons pas une réponse générique à cette question (mais nous allons traiter à nouveau de cette question dans le texte), nous savons quand même comment calculer le montant minimal de ressources pour chaque F-prréseau solide séparément. En effet, en raison de la procédure de décision pour les FL-correct F-réseaux, en ajoutant progressivement des genres, nous pouvons trouver un minimal $\leq_{\mathcal{G}}$ FL-correct F-réseau

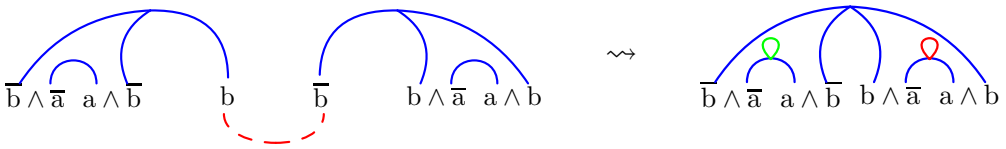


FIGURE 9 – Le solide F-prréseau sans la boucle verte et sans celle rouge n'est pas un FL-correct. Si une d'entre elles est ajoutée, la verte ou la rouge, le F-prréseau résultant est FL-correct.

On pourrait être tenté de commencer par FL-correct F-réseaux et ensuite pour chaque F-prréseau solide, obtenu par composition / élimination de coupures, et d'établir une politique d'addition de ressources pour rester dans le domaine des FL-correct F-réseaux. Malheureusement, la simple tentative s'avère défectueuse, puisque composition /élimination de coupures de $\leq_{\mathcal{G}}$-minimal FL-correct F-réseau s'avère être non-associative, comme dans les F-prréseaux de la Figure 10.

En outre, on peut espérer que le F-prréseau solide obtenu par élimination des coupures sur un FL-correct F-réseau est infimum de tous les $\preceq$ FL-correct nets, mais ce n'est

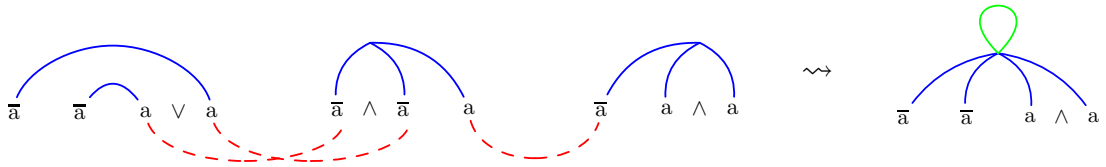


FIGURE 10 – Selon l'ordre d'élimination des coupures, il faut ajouter ou pas des ressources. Si le coupure à droite est le premier éliminé sur deux FL-correct F-réseaux, le genre a besoin d'être augmenté de 1 pour obtenir un FL-correct F-réseau. Si le coupure à gauche est éliminé en premier, la boucle verte ne doit pas être ajoutée.

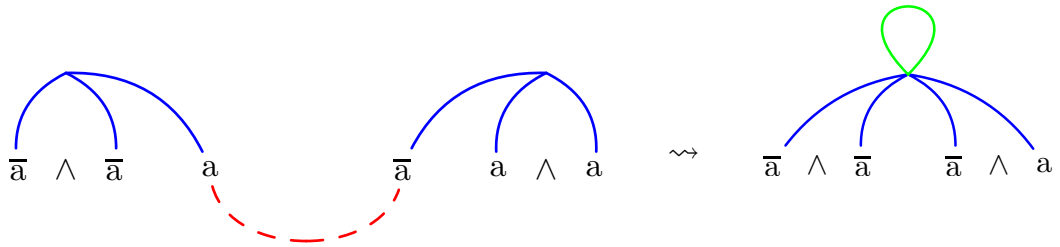


FIGURE 11 – Le F-préréseau obtenu par composition /élimination de coupures n'est pas infimum de tous les FL-correct F-réseaux, il est plus petit. Infimum est le F-réseau sans la boucle verte ajoutée, ce qui n'est pas correct.

pas le cas non plus, comme indiqué dans la Figure 4.9.

4.1 L'élimination des coupures pour les F-réseaux

è ce point de notre théorie de réseaux de preuves pour la logique classique, elle suggère que les formes normales que nous attribuons aux preuves avec des coupures, c'est à dire les F-préréseaux solides obtenus via coupure- normalisation de la composition de F-préréseaux, ne correspondent à aucune des preuves qui peuvent être obtenues par une procédure effective d'élimination de coupures *au niveau des preuves*. Nous avons déjà rencontré cette situation dans le chapitre Prélude où les formes normales de preuves pour les numéraux composés via coupure diffèrent de ce qui devrait être obtenu par une procédure efficace d'élimination des coupures, si nous en avions un à notre disposition. Nous avons décrit cela en disant que nos interprétations n'ont rien à voir avec Curry-Howard En [Hyl04] Hyland soutient que la théorie classique de la preuve nécessite ce type d'approche à l'élimination de coupures et à l'identification de preuves.

Pour ces raisons, nous sommes intéressés à une possibilité de définir une façon d'éliminer coupure dans un correct F-réseau qui garantit de donner un autre correct F-réseau. Naturellement, nous voulons que l'élimination de coupures soit associative et qu'ait une unité. Cela nécessite la définition d'une nouvelle composition dans la catégorie des F-

préréseaux.

Nous rappelons au lecteur que le centre de notre intérêt ici sont les étapes d'élimination des coupures effectuées sur des coupures simples découlant de paires de F-préréseaux sans coupures. De même façon, on peut supposer que dans ce qui suit nous limitons la logique de sorte qu'une fois une règle Coupure est appliquée, la seule règle qui peut être utilisée postfaces est Coupure. C'est l'utilisation standard de Coupure dans les interprétations catégoriques.

Définition 0.4.17. A bonus B est une fonction qui est définie sur toutes les paires $(P \triangleright \Gamma, C)$ where $P \triangleright \Gamma$ iest un F-préréseau solide et C un composant de P et dont la valeur $\mathsf{B}(P \triangleright \Gamma, C)$ est un nombre naturel.

Quand le contexte est clair nous nous permettons d'abandonner un des composants.

Que $P \triangleright \Gamma$ soit un F-préréseau solide et que $((C_1, G_1)...(C_k, G_k))$ soit son liage. Nous définissons $\lceil P \rceil^{\mathsf{B}}$ comme liage

$$\left((C_1, G_1 + \mathsf{B}(P \triangleright \Gamma, C_1)), (C_2, G_2 + \mathsf{B}(P \triangleright \Gamma, C_2)), ..., (C_k, G_k + \mathsf{B}(P \triangleright \Gamma, C_k))\right)$$

et $\lceil P \triangleright \Gamma \rceil^{\mathsf{B}}$ comme son correspondant de F-préréseau avec $\lceil P \rceil^{\mathsf{B}}$ liage.

On dit qu'un bonus est *juste* s'il est constant sur les composants d'un F-préréseau solide donné, dans ce cas il ne dépend que de F-préréseau et nous écrivons $\mathsf{B}(P \triangleright \Gamma)$.

On dit que le bonus B est *gagnant* si $\lceil P \triangleright \Gamma \rceil^{\mathsf{B}}$ est toujours un FL-correct net.

Conformément au Lemma 4.3.12, il est raisonnable de supposer que, étant donné un F-préréseau solide, un bonus gagnant est strictement positif sur les composants du F-préréseau donné. Un tel bonus gagnant B avec des valeurs ≥ 1 sur les composants de chaque F-préréseau est dit être *strictement gagnant*

Outre la définition de $\lceil P \triangleright \Gamma \rceil^{\mathsf{B}}$, nous définissons $\lfloor P \triangleright \Gamma \rfloor_{\mathsf{B}}$ comme F-préréseau obtenu en soustrayant à $\mathsf{B}(P \triangleright \Gamma)$ de nombreuses boucles de chaque composant de $P \triangleright \Gamma$, si possible, autrement $P \triangleright \Gamma$.

Définition 0.4.18. Pour F-préréseaux $P \triangleright \Gamma, \mathrm{A}$ et $Q \triangleright \overline{\mathrm{A}}, \Sigma$ et un bonus gagnant B, nous définisson

$$(P \triangleright \Gamma, \mathrm{A}) \diamond (Q \triangleright \overline{\mathrm{A}}, \Sigma)$$

comme

$$(P \triangleright \Gamma, \mathrm{A}) \diamond (Q \triangleright \overline{\mathrm{A}}, \Sigma) = \begin{cases} (P \triangleright \Gamma, \mathrm{A}) \circ (Q \triangleright \overline{\mathrm{A}}, \Sigma), & \text{si} \\ & (P \triangleright \Gamma, \mathrm{A}) \text{ où } (Q \triangleright \overline{\mathrm{A}}, \Sigma) \\ & \text{si est l'identite F-préréseau} \\ \lceil \lfloor P \triangleright \Gamma, \mathrm{A} \rfloor_{\mathsf{B}} \circ \lfloor Q \triangleright \overline{\mathrm{A}}, \Sigma \rfloor_{\mathsf{B}} \rceil^{\mathsf{B}}, & \text{autrement} \end{cases}$$

où le $\circ$ est la composition standard de Frobenius.

Ainsi le diamant "mange" la dernière formule du séquent gauche et la première du séquent droit, en laissant une concaténation de ce qui est resté. Cette notation n'est pas complètement satisfaisante en général, mais assez bonne pour les buts de ce travail.

Ce qui suit est évidente.

Proposition 0.4.19. *Étant donné un bonus gagnant* B*, sa* $\diamond$ *élimination de coupures appliquée à deux* FL*-correct F-réseaux produit un net correct.*

Théorème 0.4.20. Étant donné un bonus B strictement gagnant, le $\diamond$ élimination de coupures sur les FL-correct F-réseaux est associative. En outre, l'élimination de coupures $\diamond$ sur un F-préréseau et une identité F-préréseau donne l'ancien.

Naturellement, nous sommes intéressés à des exemples concrets des bonus gagnants qui définissent les $\diamond$ éliminations de coupures.

Définition 0.4.21. Pour un séquent (ou une formule) Γ, que $|_\Gamma \wedge |$ désigne le nombre de conjonctions en Γ et que $|\Gamma|$ désigne le nombre de littéraux en Γ. Pour un F-préréseau $P \triangleright \Gamma$ nous définissons un bonus Bon tcomme la valeur

$$\mathsf{Bon}(P \triangleright \Gamma) = (|_\Gamma \wedge | + 1) \cdot |\Gamma| \cdot \frac{3^{|_\Gamma \wedge |+1} - 1}{2}.$$

Que $\lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$ soit le F-préréseau obtenu en ajoutant à $\mathsf{Bon}(P \triangleright \Gamma)$de nombreuses boucles à chaque composant connexe de $P \triangleright \Gamma$, et que $\lfloor P \triangleright \Gamma \rfloor_{\mathsf{Bon}}$ soit le F-préréseau obtenu en soustrayant de $\mathsf{Bon}(P \triangleright \Gamma)$ de nombreuses boucles de chaque composant de $P \triangleright \Gamma$, si possible, autrement $P \triangleright \Gamma$.

Où cela ne provoque pas de confusion et où le séquent d'un F-préréseau peut être déduit du contexte, nous introduisons la notion$\lfloor P \rfloor^{\mathsf{Bon}}$ et $\lfloor P \rfloor_{\mathsf{Bon}}$ sur les liages avec la même signification que ci-dessus.

Théorème 0.4.22. Le bonus Bon est strictement gagnant.

Le théorème précédent donne un exemple concret d'une élimination des coupures associative $\diamond$ sur l'ensemble de F-préréseaux. Il y a aussi une identité concernant l'élimination des coupures assignée à tout F-préréseau. Donc, si on ne considère que F-préréseaux avec des séquents à deux formules, l'élimination des coupures comporte une composition dans la catégorie de ces préréseaux, de la même façon que l' élimination des coupures $\smile$ donne lieu aux F-préréseaux. La catégorie de **F-prenet** à deux formules et la composition concrète $\diamond$ induite par Bon est représenté par $\mathsf{CorrFrob}$. La catégorie $\mathsf{CorrFrob}$, selon ce qui

est dit précédemment, jouit d'une forme faible de la propriété de complétude, c'est-à-dire, tout morphisme de CorrFrob est assigné à une preuve.

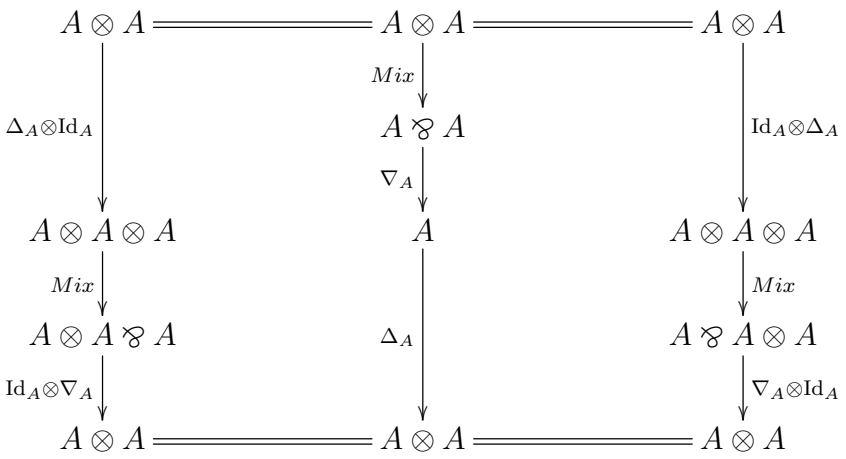
Il y a, cependant, peu de choses que nous pouvons dire sur la structure catégorielle de CorrFrob. Nous savons qu'elle n'est pas *-autonome, comme par exemple un morphisme $P \triangleright \overline{(\mathrm{A} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}}, \mathrm{C}$ avec le même liage comme $P \triangleright \overline{\mathrm{A} \vee \mathrm{C}}, \mathrm{B} \vee \mathrm{C}$ n'est pas en général le résultat de l'élimination des coupures

$$\left(P \uplus Q \triangleright \overline{(\mathrm{A} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}}, (\mathrm{B} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}\right) \diamond \left(R \uplus S \triangleright \overline{(\mathrm{B} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}}, \mathrm{C}\right),$$

où il est clair ce que sont Q, R, S, ils sont liages d'une certaine identité de F-prréseaux. La composition ici, en général, nécessite l'addition des bonus sur les composants avec des littéraux de A et de C. En outre, CorrFrob n'est même pas monoïdale. Dans une catégorie monoïdale la fonctorialité du tenseur (qui est ici l'union disjointe) se traduit par la "localité" de la composition, à savoir une composition devrait modifier uniquement les composants qui sont connectés par elle. Ici, un composant qui ne joue pas un rôle dans l'enchaînement de morphismes peut changer son genre à cause de l'équité de Bon. Un indice possible qui suggère que la même situation doit se produire pour toute définition d'une composition via un bonus gagnant, est fourni dans l'exemple de la Figure 9.

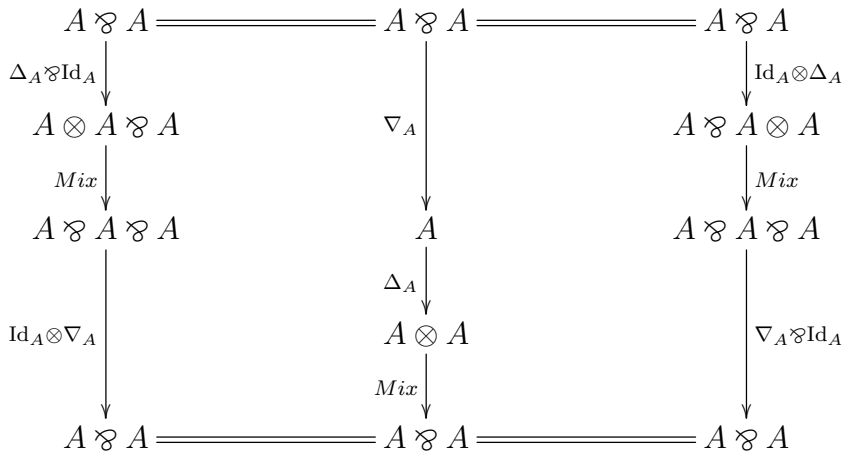
Nous concluons ce chapitre en rappelant que les algèbres de Frobenius sont définies pour des objets ayant structure de $\otimes$-bimonoïde, qui est un monoïde $\otimes$ et une structure$\otimes$-comonoïde. Comme nous interprétons les preuves classiques, il est naturel de se demander s'il est possible de définir une structure de Frobenius pour les interprétations avec les deux connecteurs étant différents, qui est – avec les objets équipés d'une structure de $\parr$-monoïde et $\otimes$-comonoïde.

Nous remarquons qu'il est logique de définir l'équation de Frobenius, en présence de Mix dans setting comme la commutativité des diagrammes suivants (comme d'habitude, nous omettons l'associativité isos) :



(1)

et



(2)

Notez que le second diagramme est duale du premier.

Il y a une quantité limitée de choses que nous sommes en mesure de dire ici. Malheureusement, nous n'avons pas encore d'exemples concrets de ces derniers, et la recherche de ceux-ci est une autre tâche à entreprendre.

5 Cmp Revu

5.1 Algèbres de Frobenius dans Cmp

Dans ce chapitre, nous revenons sur l'interprétation dénotationelle concrète basée sur $\mathbb{Z}$ et la catégorie **Cmp**, le point de départ dans notre étude de la structure algébrique qui est à la base des interprétations concrètes.

Nous commençons par analyser ce que signifie d'avoir une structure d'une algèbre de Frobenius dans une catégorie compacte-fermée, et nous prouvons la proposition suivante qui figure en [Dij89, Koc04]. Là, les auteurs ont montré le résultat dans la catégorie des espaces vectoriels sur un champ, mais ce qui importe c'est la structure compacte-fermée de la catégorie, qui est ce que nous faisons ici.

Proposition 0.5.1. *Proposition 1 Soit* $(\mathbf{C}, \otimes, \mathbf{1})$ *une catégorie compacte-fermée et* A *un objet de* $\mathbf{C}$*. Les éléments suivants définissent la même structure en* $\mathbf{C}$ *:*

1. *une algèbre de Frobenius* $(A, \Delta, \Pi, \nabla, \amalg)$ *;*
2. *une structure* $(A, \nabla, \amalg, \Pi)$*, où* $(A, \nabla, \amalg)$ *est un monoïde commutatif et* $\Pi : A \to \mathbf{1}$*, tel que le morphisme*

$$\Phi : \ A \longrightarrow A^{\perp}$$

obtenu par transposition

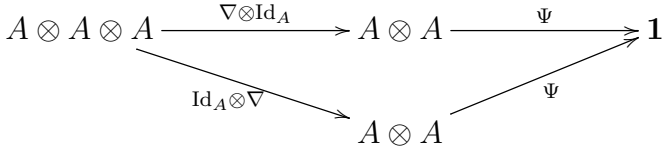
$$A \otimes A \xrightarrow{\ \nabla\ } A \xrightarrow{\ \Pi\ } \mathbf{1}$$

est un isomorphisme ;

3. *une structure* $(A, \nabla, \amalg, \Pi)$*, où* $(A, \nabla, \amalg)$ *est un monoïde commutatif*

$$\Psi : A \otimes A \longrightarrow \mathbf{1}$$

obéit à



et dont la transposition $\phi : A \to A^{\perp}$ *est un isomorphisme.*

4.1 Algèbres de Frobenius dans Cmp

Remark 0.5.2. Les formulations précédentes de structures équivalentes à une algèbre de Frobenius peuvent être trouvés dans la littérature dans un contexte plus général des *monades de Frobenius* en [Str04].

La plupart de notre travail jusqu'à présent a consisté à aller de catégories compact-fermées des catégories de Frobenius libres. La première catégorie concrète **Cmp**, la première catégorie compacte-fermée, dans laquelle nous avons formulé des interprétations de la logique classique, donc nous sommes naturellement intéressés à explorer les algèbres de Frobenius dans **Cmp**. Comme il s'avère que le cas général est trop général pour les buts de cette thèse, nous nous contenterons à examiner les algèbres de Frobenius qui proviennent d'un bimonoïde dans Poset, comme dans le premier chapitre.

Dans le chapitre 1 nous avons défini avec des notations différentes.

Définition 0.5.3. Un ensemble ordonné bimonoidal est un sextuple $(A, \leq, \cdot, 1, \star, e)$, où $(A, \leq)$ est un ensemble ordonné et $(\cdot, 1)$, $(\star, e)$ sont ses deux structures de monoïde monotone commutatif.

Un ensemble ordonné monoïdal induit un monoïde en **Cmp** ; pour les éléments x, y, z du ensemble ordonné A, on définit :

$$\begin{array}{llll} x \; {}^{\Delta_A}\; (y,z) & \text{ssi} & x \leq y \star z; & \qquad x \; {}^{\amalg_A}\; * \quad \text{ssi} \quad x \leq e; \\ (y,z) \; {}^{\nabla_A}\; x & \text{ssi} & y \cdot z \leq x; & \qquad * \; {}^{\amalg_A}\; x \quad \text{ssi} \quad 1 \leq x. \end{array} \tag{1}$$

La négation involutive détermine la structure sur un objet dual, et il est convenable de choisir et de fixer une polarité pour chaque type et d'exprimer l'opposé en termes du type choisi, par exemple

$$x \; {}^{\Delta_{\overline{A}}}\; (y,z) \;\; \text{ssi } y \cdot z \leq_A x.$$

Laissez-nous maintenant calculer ce que cela signifie pour un objet de **Cmp** d'avoir une structure d'une algèbre de Forbenius ; pour un ensemble ordonné donné$A = (A, \leq)$:

1. $\Delta \circ \nabla$ est la relation :

$$\begin{aligned}\{(d,(z,w)) \in A^3 \mid d \leq z \star w\} \ \circ \ \{((x,y),h) \in A^3 \mid x \cdot y \leq h\} \\ = \{((x,y),(z,w)) \in A^4 \mid \exists d:\ x \cdot y \leq d \leq z \star w\} \\ = \{x \cdot y \leq z * w\}\end{aligned}$$

2. $(\mathrm{Id}_A \otimes \nabla_A) \circ (\Delta_A \otimes \mathrm{Id}_A)$:

$$\begin{aligned}\{x \leq r \star s,\ y \leq t\} \circ \{((i,j,k),(z,w) \in A^5 \mid i \leq z,\ j \cdot k \leq w\} \\ = \{\exists r,s,d:\ x \leq r \star s,\ y \leq t, r \leq z,\ s \cdot t \leq w\} \\ = \{\exists s:\ x \leq z \star s,\ s \cdot y \leq w\}\end{aligned}$$

3. $(\nabla_A \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \Delta_A)$, de manière analogue :

$$\{\exists s:\ x \cdot s \leq z,\ y \leq s \star w\}.$$

Par conséquent, toute algèbre de Frobenius dans **Cmp** comme celle dessus doit obéir :

$$\{x \cdot y \leq z * w\} \ = \ \{\exists s:\ x \leq z \star s,\ s \cdot y \leq w\} \ = \ \{\exists s:\ x \cdot s \leq z,\ y \leq s \star w\}. \qquad (2)$$

La formulation au point 2. de la Proposition 5.1.1 propose une définition alternative d'une algèbre de Frobenius qui est particulièrement utile. Supposons, selon (5.1), que nous avons un monoïde $(A, \cdot, e)$ dans les ensembles ordonnés. Suivant 2. de la proposition précédente, pour obtenir une structure d'une algèbre de Frobenius sur le ensemble ordonné, nous avons besoin d'une constante

$$e \in A$$

pour laquelle

$$\{i \in A \mid i \leq e\}$$

définit un morphisme

$$A \longrightarrow \mathbf{1}$$

ce qui donne lieu à une transposition d'un iso en **Cmp**. Dans le détail, cela signifie

$$\{z \leq e\} \circ \{x \cdot y \leq z\} \ = \ \{x \cdot y \leq e\}$$

est une transposition d'un iso

$$\sigma:\ A \longrightarrow A^{op}$$

en **Cmp**. Un isomorphisme $A \to B$ en **Cmp**, comme montré en [Lam07], est toujours obtenu d'une bijection d'ensembles $\alpha : A \to B$ comme c'est aussi la fermeture gauche-bas-droite-haut, à savoir

$$\alpha \updownarrow := \{(i,j) \in A \times B \mid \exists k \in A:\ i \leq k;\ \alpha(k) \leq j\}.$$

Par conséquent, σ est une fermeture gauche-bas-droite-haut d'une bijection contrevariante (appelée aussi σ) $A \cong A^{op}$.

Maintenant, la transposition dans le point 2. de la proposition précédente, en utilisant contravariance et bijectivité de σ, est reformulée ainsi à dire que :

$$\begin{array}{rcl} \{(x,y) \in A^2 \mid \exists z \in A :\ x \leq z;\ y \geq^{op} \sigma(z)\} & = & \\ & = & \{(x,y) \in A^2 \mid\ \sigma(x) \geq \sigma(z);\ \sigma(z) \leq y\} \\ & = & \{(x,y) \in A^2 \mid\ y \leq \sigma(x)\} \end{array} \tag{3}$$

est le même que

$$\{(x,y) \in A^2 \mid x \cdot y \leq e\}. \tag{4}$$

De même, à partir de l'autre transposition du même morphisme $\Pi \circ \nabla$ du point 2, on obtient :

$$\begin{array}{rcl} \{\exists z \in A :\ y \leq z;\ x \geq^{op} \sigma(z)\} & = & \\ & = & \{x \leq \sigma(y)\}. \end{array} \tag{5}$$

Cette transposition doit être le même morphisme que(5.2) par commutativité de $(-) \cdot (-)$ (= commutativité de ∇). Par suite, dans l'nsemble ordonné $(A, \leq)$ les expressions (5.2),(5.3), (5.4) donnent

$$x \leq \sigma(y) \quad \text{ssi} \quad x \cdot y \leq e \quad \text{ssi} \quad y \leq \sigma(x).$$

Le lecteur peut remarquer que le point 3. de la Proposition 5.1.1, peut être vu comme une généralisation de 2. Dans la preuve de l'implication [3. $\Rightarrow$ 1.], nous avons donné une recette pour récupérer le comonoïde cocommutatif d'une algèbre de Frobenius étant donnés le monoïde et le morphisme counité. En outre, la comultiplication obtenue du comonoïde dans une algèbre de Frobenius est uniquement déterminée par les données fournies comme une conséquence de la cohérence en $\mathbf{FrThFrob}(\mathcal{AT}\text{yp})$.

Dans ce cas, cela signifie que donné $(A, \leq, \cdot, 1)$ et la constante e définissant le contravariant σ, nous pouvons définir

$$\begin{array}{rcl} x \star y & := & \sigma(\sigma(x) \cdot \sigma(y)) \\ e & := & \sigma(1) \end{array} \tag{6}$$

pour obtenir le ensemble ordonné bimonoïd, détenu par (2).

Pour la même structure, le monoïde de ensemble ordonné monotone $(A, \leq, \cdot, 1)$ t le contravariant σ, nous avons l'équivalence des inégalités suivantes :

$$\dfrac{\dfrac{\dfrac{\dfrac{x \cdot y \leq z}{(x \cdot y) \cdot \sigma(z) \leq e}}{x \cdot (y \cdot \sigma(z)) \leq e}}{x \leq \sigma(y \cdot \sigma(z))}}{x \leq \sigma(y) \star z.}$$

La dérivation réversible ci-dessus définit une structure familière sur un ensemble ordonné considéré comme une catégorie. Combinée avec la conclusion précédente, nous pouvons déclarer ce qui suit :

Proposition 0.5.4. *Soit $(A, \leq, \cdot, 1)$ un monoïde monotone sur un ensemble ordonné, et σ un antiautomorphisme $\sigma : (A, \leq) \to (A, \leq^{op})$ tel que* (5.2)*,* (5.3) *et* (5.4) *représentent la même relation. Alors A est un ensemble ordonné *-autonome, et vice versa, chaque ensemble ordonné *-autonome est une telle structure.*

En utilisant le résultat précédent sur **Cmp** via (5.1) nous obtenons :

Théorème 0.5.5. Soit $(A, \Delta, \Pi, \nabla, \amalg)$ une structure d'algèbre de Frobenius sur un ensemble ordonné $A \in \mathbf{Cmp}$ et que les morphismes bimonoïdes sont définies par deux ordre-monotone ensemble ordonnés monoïdes comme en (5.1). Alors, A est un ensemble ordonné *-autonome.

Inversement, soit $(A, \cdot, 1)$ un ensemble ordonné *-autonome avec l'involution contravariante σ. Puis, (5.7) et(5.1) définissent une structure d'une algèbre de Frobenius en $A \in \mathbf{Cmp}$.

Example 0.5.6. Des exemples immédiats de ces structures peuvent être obtenus en remarquant qu'un groupe abélien $(G, +, 0)$ arbitraire muni c'un ordre linéaire, définit un ensemble ordonné monoïdal. Pour obtenir l'involution contravariante qui manque, il suffit de choisir $e \in G$ et définir $\sigma : x \mapsto e - x$. Un exemple concret est la structure sur $\mathbb{Z}$ du chapitre 1. En effet, nous avions là :

$$\begin{array}{lcl} (G, +, \mathbf{0}) & \rightsquigarrow & (\mathbb{Z}, +, 0) \\ \sigma & \rightsquigarrow & x \mapsto c - x. \end{array}$$

(Notez que le modèle $\mathbb{Z}$, que nous avons défini $(\star, e)$ et σ plutôt que $(\cdot, 1)$ et σ pour obtenir la structure requise. Bien sûr, un monoïde induit l'autre.)

L'autre opération de monoïde induite dans le $\mathbb{Z}$ ensemble ordonné bimonoïde est

$$x +_c y \;:=\; x + y - c$$

dont l'unité est c.

D'autres exemples d'ensemble ordonnés *-autonomes (finis) peuvent être trouvés par exemple en [Lam95].

Un calcul important, où nous nous mettons à utiliser les notations introduites, est le calcul du morphisme de doublage $\mathcal{D}_A^k$.

$$\begin{aligned} x \mathcal{D} y \quad &\text{ssi} \quad \exists w, z \in A : \quad x \leq w \star z; \quad w \cdot z \leq y \\ &\text{ssi} \quad \exists w, z \in A : \quad \sigma(w) \cdot x \leq z; \quad z \leq \sigma(w) \star y \\ &\text{ssi} \quad \exists w \in A : \quad \sigma(w) \cdot x \leq \sigma(w) \star y \\ &\text{ssi} \quad \exists w \in A : \quad (w \cdot \sigma(w)) \cdot x \leq y. \end{aligned}$$

La dernière inégalité peut être facilement répétée pour 'obtenir

$$x \mathcal{D}^k y \quad \text{ssi} \quad \exists w_1, \ldots, w_k \in A : \quad (w_1 \cdot \sigma(w_1)) \cdot \cdots \cdot (w_k \cdot \sigma(w_k)) \cdot x \leq y,$$

ou, en termes de deux monoïdes en ensemble ordonné comme en (5.1)

$$x \mathcal{D}^k y \quad \text{ssi} \quad \exists w \in A : \quad w_1 \cdot \ldots \cdot w_k \cdot x \leq w_1 \star \ldots \star w_k \star y.$$

Il est particulièrement convenable de considérer les ensemble ordonnés *-autonomes où $x \cdot \sigma(x)$ est une valeur constante pour tout x. En mettant $x = 1$, nous concluons que la valeur doit être e. Ces ensemble ordonnés *-autonomes permettent un calcul plus facile des morphismes. Par exemple, selon le résultat précédent, le morphisme itéré de doublage devient :

$$x \mathcal{D}^k y \quad \text{ssi} \quad e^k \cdot x \leq y,$$

où

$$e^k := \underbrace{(e \cdot \ldots \cdot e)}_{k}.$$

Notre prototype de bimonoïdes $\mathbb{Z}$ sont les ensembles ordonnés *-autonomes ayant la propriété ci-dessus. Dans le cas de ceux bimonoïdes, comme mentionné, nous avons $\sigma : i \mapsto c - i$ pour une constante c, et pour chaque i d'un ensemble ordonné :

$$i \cdot -i = c.$$

Enfin, nous jetons un coup d'oeil sur le concept important de la minceur pour les catégories de Frobenius en **Cmp**.

Proposition 0.5.7. *Une algèbre de Frobenius sur un ensemble ordonné $A \in$ **Cmp** by ensemble ordonné monoïdes d'ordre monotone $(\cdot, 1)$ and $(\star, e)$ comme en (5.1) st mince si et seulement si*

$$1 \leq e.$$

6 Travaux futurs

Ce travail ouvre plusieurs possibles questions de recherche pour être étudiées. Sans aucun ordre particulier :

- Notre définition de la procédure de correction est formulée comme une sorte de procédure de recherche de preuve et caractérisée par un énoncé existentiel. Même si nous donnons une motivation convaincante pour que l'exponentialité de tout critère qui pourrait être conçu pour ces réseaux, il y a toujours une possibilité de modifier la définition pour obtenir des types de nouveaux F-préréseaux qui permettraient de réduire considérablement la complexité des calculs du problème de la correction.
- Les questions de complexité des calculs d'algorithmes existants pour la correction ne sont pas traitées dans cette thèse. Nous avons de forts indicateurs de leur appartenance à des classes de complexité bien définies et de plus, que les algorithmes sont *complets* pour ces classes, mais cela reste à être élaboré.
- Nous avons démontré qu'il existe un moyen de définir l'élimination des coupures sur les F-réseaux qui est associative, pour laquelle il y a l'unité de F-réseaux. Cela a été fait par au moyen des bonus. Une sujet potentielle d'étude est la possibilité d'autres définitions des bonus ou une théorie générale de ceux-ci. En outre, l'élimination des coupures que nous avons obtenue produit une catégorie, mais comme nous l'avons démontré dans plusieurs exemples, la catégorie qui en résulte ne semble même pas être monoïdale. La polycategorie/structade induite par les F-préréseaux, d'autre part, est sûrement plus intéressante et est probablement le type de structure optimal pour étudier ces objets.
- Dans sa thèse doctorale [McK06], McKinley étend les "catégories classiques" de Führmann-Pym à la sémantique de la logique du premier ordre. Il pourrait s'avérer intéressant de vérifier la robustesse de l'extension par rapport au travail de cette thèse, c'est-à-dire de voir qu'il y a une extension significative de notre travail au cas de premier ordre.
- Nous avons montré que l'équation de Frobenius peut être formulée dans le contexte d'une catégorie *- autonome où $\otimes \neq \parr$, mais sont reliés par Mix. Sans doute la question la plus importante de recherche ouverte est une étude supplémentaire de ces structures catégoriques et leur utilisation dans l'obtention de nouvelles sémantiques.

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Introduction

The modern-day mathematical logic is a mature mathematical discipline with deep roots in more that two thousand years of philosophical study of reasoning. More than hundred years from the work of de Morgan, Boole, Peirce, Russel, Frege, Hilbert, Gödel and others, it now spans over a wide range of topics gathered around its four pillars – set theory, model theory, recursion theory and proof theory. While the historical development of the discipline was nothing if not dynamic, certain fundamental questions lying in the very core of the discipline are yet to be resolved in a satisfactory manner. If one is to select a single such question in a sub-discipline of mathematical logic, in the case of proof theory it would probably be the issue of semantics for classical logic and the problem of identity of proofs.

Proof theory as a logical discipline has emerged as a result of Hilbert's efforts of providing finitary foundations of mathematics [HB34, HB39]. While he was the first to provide a fully formal framework of logical reasoning, it was the work of a fellow Göttingen researcher Gerhard Gentzen that has given rise to the currently most widely used notions of a formal proof [Gen34, Gen35]. The two systems Gentzen has formulated in the early 20th century – the natural deduction and the sequent calculi LK and LJ – define a formal proof as a well-defined graph-like mathematical object syntactical in its nature. Moreover, if we focus on the sequent systems – and we do so in this thesis – for a given propositional formula, construction one of these objects is strait-forward and can be done in a fully automatic way.

The whole story of formal proofs could pretty much end here if it wasn't for one particular rule of the Gentzen's sequent system – the cut rule. In essence, the rule allows auxiliary claims (lemmas) to be used in a mathematical proof, and as such is immediately of high importance. An attempt to construct a proof of a given formula in a system where cut is allowed involves guessing which kind of lemmas one needs to prove a certain proposition. On the other hand, just as one can replace every usage of a lemma in a mathematical textbook by the proof of the lemma in a systematic way, there is a way to get rid of the cut rule in a proof. Gentzen himself has provided us with one way of eliminating cuts in a given proof through an effective step-by-step procedure of pushing the rules to the leaves of the proof tree [Gen35]. These types of procedures lie in the core of the modern (general) proof theory and in the last 80 years of the development in the area we have learned of the profound consequences of the cut elimination and how it teaches us of e.g. limitations to proof theory, what it is good for, what is the computational

$$\frac{}{\mathrm{A} \vdash \mathrm{A}}\ Ax \qquad \frac{\Gamma, \mathrm{B}, \mathrm{A}, \Sigma \vdash \Delta}{\Gamma, \mathrm{A}, \mathrm{B}, \Sigma \vdash \Delta}\ Ex \qquad \frac{\Gamma \vdash \Delta, \mathrm{A} \quad \mathrm{A}, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}\ Cut$$

$$\frac{\Gamma, \mathrm{A} \vdash \Delta}{\Gamma, \mathrm{A} \wedge \mathrm{B} \vdash \Delta}\ \wedge L_1 \qquad \frac{\Gamma, \mathrm{B} \vdash \Delta}{\Gamma, \mathrm{A} \wedge \mathrm{B} \vdash \Delta}\ \wedge L_2 \qquad \frac{\Gamma \vdash \mathrm{A}, \Delta \quad \Sigma \vdash \mathrm{B}, \Pi}{\Gamma, \Sigma \vdash \mathrm{A} \wedge \mathrm{B}, \Delta, \Pi}\ \wedge R$$

$$\frac{\Gamma, \mathrm{A} \vdash \Delta \quad \Sigma, \mathrm{B} \vdash \Pi}{\Gamma, \Sigma, \mathrm{A} \vee \mathrm{B} \vdash \Delta, \Pi}\ \vee L \qquad \frac{\Gamma \vdash \mathrm{A}, \Delta}{\Gamma \vdash \mathrm{A} \vee \mathrm{B}, \Delta}\ \vee R_1 \qquad \frac{\Gamma \vdash \mathrm{B}, \Delta}{\Gamma \vdash \mathrm{A} \vee \mathrm{B}, \Delta}\ \vee R_2$$

$$\frac{\Gamma \vdash \mathrm{A}, \Delta}{\Gamma, \neg \mathrm{A} \vdash \Delta}\ \neg L \qquad \frac{\Gamma, \mathrm{A} \vdash \Delta}{\Gamma \vdash \Delta, \neg \mathrm{A}}\ \neg R$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \mathrm{A} \vdash \Delta}\ WeakL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathrm{A}}\ WeakR$$

$$\frac{\Gamma, \mathrm{A}, \mathrm{A} \vdash \Delta}{\Gamma, \mathrm{A} \vdash \Delta}\ ContrL \qquad \frac{\Gamma \vdash \Delta, \mathrm{A}, \mathrm{A}}{\Gamma \vdash \Delta, \mathrm{A}}\ ContrR$$

FIGURE 1 – System LK

significance of proofs and much more. In fact, one may state that the (general) proof theory *is* the study of cut elimination.

When a mathematical object such as a formal proof is defined, the definition is usually followed by a statement stating when two such newly defined mathematical objects are considered to be the same. With the formal proofs of Gentzen type we do have a valid definition of the concept of a (propositional) proof, but for classical logic ('the' logic), such an equality criterion on proofs has successfully avoided being formulated in a satisfactory way for quite some time.

There is a trivial and uninteresting possible answer to the question 'when are two classical proofs equal ?' : whenever they have the same premise and the same conclusion. By accepting this answer we do not speak of proof theory anymore, but rather of provability in a logic as there is always at most one proof of a conclusion given a premise. Propositional provability is an issue that has been resolved already with the work of Boole in the classical case and Heyting in the intuitionistic one. If one is to consider the order defined on formulas which states that A is smaller than B whenever there is a proof of $A \vdash B$, one ends up with a Boolean or a Heyting lattice (a poset) induced by the respective variety. We consider these theories of provability to be collapsed cases of proof theories ; we want to analyze the *multitude* of possible proofs of $A \vdash B$, their mutual relation and structure, not only their existence. Consequently, the equational criterion we are looking for needs to be much more fine-grained.

One possible answer to the identity of proofs question, in a context slightly different from sequent calculi, was proposed by Prawitz [Pra65, Pra71]. He proposed that one should equate precisely those proofs whose 'normal forms' are equal. The idea behind the 'normal form' concept is the requirement that cut elimination should not affect the essence of a proof, i.e. every step in a cut elimination procedure can produce only proofs

that are considered equal to the starting one. Thus, the normal form for a sequent proof should be the result of a cut elimination procedure applied to the proof, obviously modulo some syntactic 'trivial' rule permutations that do not affect the essence of a proof. Unfortunately, the one procedure that came with the notion of a sequent proof, the procedure Gentzen has formulated for classical logic not robust w.r.t. any reasonable notion of 'trivial' rule permutations.

One may argue that it is the deficiency of the sequent calculus that it obscures the true essence of a proof by insisting on a rigid interpretation of the logical inference rules, and that the problems with defining a good notion of 'trivial' rule permutations are intrinsic to the sequent calculus. It is due to Girard that we call these 'trivial' rule permutations *proof bureaucracy*. Therefore, one may opt for what can be called a *syntactic approach* to proof identity ; one may attempt to more-or-less step away from the Gentzen tradition of structural proof systems and come up with a notion of a proof whose syntax should be robust w.r.t. proof bureaucracy. Girard's original formulation of proof nets for linear logic [Gir87], the work on deep inference [2], or the cirquent calculus [Jap08] are some examples of these.

An alternative approach is a semantical one. When trying to define semantics, one strives at providing essentially 'mathematical meaning' to the formal proofs. Exploration of the fruitful interaction between denotational semantics and syntax is a direction of research which began with the work of Dana Scott on the untyped lambda calculus which has seen the invention of linear logic through Girard's observation of the category of coherence spaces and linear maps. Essentially, one assigns sets (with structure) to formulas, and a proof with a premise and a conclusion is a function (respecting the structures in a certain sense) from one set to another. A success in defining semantics for classical proofs would result in an account of proof identity, as two formal proofs would be considered equal if they are assigned the same mathematical structure (assuming that the identity in the semantics is resolved). Conversely, a good notion of proof identity should give rise to a semantics where two equated proofs represent the same mathematical object.

So, what is the problem with classical logic ? As it turns, there is a concise way to describe it from the point of view of the cut elimination in syntax, and it is reduced to the two cases, weakening-meets-weakening and contraction-meets contraction, in cut elimination in Gentzen's LK

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash \Sigma\end{array}}{\Gamma \vdash \Sigma, A}\ \textit{Weak} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Delta \vdash \Pi\end{array}}{A, \Delta \vdash \Pi}\ \textit{Weak}}{\Gamma, \Delta \vdash \Sigma, \Pi}\ \textit{Cut} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash \Sigma, A, A\end{array}}{\Gamma \vdash \Sigma, A}\ \textit{Contr} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ A, A, \Delta \vdash \Pi\end{array}}{A, \Delta \vdash \Pi}\ \textit{Contr}}{\Gamma, \Delta \vdash \Sigma, \Pi}\ \textit{Cut.}$$

A truly symmetric semantics would make no preference to π_1 or π_2 in the cut elimination step. While the weakening-meets-weakening case can be resolved by including into the calculus the rule Mix, (corresponding to a natural map $A \vee B \to A \wedge B$ in the semantics),

2. The deep inference paradigm will be addressed in following chapters

contraction against a contraction case in cut remains encapsulating all of the problems with the proof theory for classical logic.

We have seen that the proof theory we are looking for should be the one where cut elimination yields normal forms, and more, in the classical case, it should respect inherent symmetries of classical logic. In particular, the order in which we compose two consecutive cuts should not matter, as in the case below.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_1 \vdash \Sigma_1, \mathrm{A}\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \mathrm{A}, \Gamma_2 \vdash \Sigma_2, \mathrm{B}\end{array}}{\Gamma_1, \Gamma_2 \vdash \Sigma_1, \Sigma_2, \mathrm{B}}\ \mathit{Cut} \quad \begin{array}{c}\pi_3\\ \vdots\\ \mathrm{B}, \Gamma_3 \vdash \Sigma_3\end{array}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Sigma_1, \Sigma_2, \Sigma_3}\ \mathit{Cut} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_1 \vdash \Sigma_1, \mathrm{A}\end{array} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \mathrm{A}, \Gamma_2 \vdash \Sigma_2, \mathrm{B}\end{array} \quad \begin{array}{c}\pi_3\\ \vdots\\ \mathrm{B}, \Gamma_3 \vdash \Sigma_3\end{array}}{\mathrm{A}, \Gamma_2, \Gamma_3 \vdash \Sigma_2, \Sigma_3}\ \mathit{Cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Sigma_1, \Sigma_2, \Sigma_3}\ \mathit{Cut}$$

This associativity of cut, which is to be considered as associativity of composition of proofs, brings us to the category theory. In fact, the experience we have with structural proof theories of various logics has taught us that when a truly good notion of a proof theory for a logic exists it can be stated in the language of category theory. A prominent example of this is an observation made by Lawvere and Lambek that proofs in intuitionistic logic, i.e. simply-typed λ-calculus modulo $\beta - \eta$ identifications via the Curry-Howard correspondence, can be fully interpreted in cartesian-closed categories (with coproducts). Commutative diagrams in the model category are precisely interpretations of normal forms of λ terms, i.e. the λ-calculus is the internal language of cartesian closed categories. Closer to the work to be carried out here is the identification of model categories for proofs in unit-free multiplicative linear logic as Barr's *-autonomous categories [Bar79]. In both cases, the categories in question have propositional formulas as objects, and a map $f : A \to B$ in the category stands for the class of cut-free proofs in a proof system we consider to be equal, while composition in the category corresponds to the cut elimination in logic. This approach goes back to the very origins of the categorical treatment of logic and the work of Lawvere [Law63] and Lambek [Lam68, Lam69].

The success in formulation of the proof theory of intuitionistic logic marks the desired course one wants to take with classical logic. Using analogy, we may state the problem of searching for (categorical) semantics for classical logic as the proportion

$$\begin{array}{rcl} \text{Cartesian-Closed Category} & : & \text{Heyting Algebra} \\ ??? & : & \text{Boolean Algebra} \end{array}$$

which is read as follows : one wants a definition of a notion of a 'Boolean category', the category which is to Boolean algebras what Cartesian closed category is to a Heyting algebra.

Not so long ago, Joyal's negative result of a collapse of a cartesian-closed category with the involutive negation to a Boolean algebra has provided little hope in possibility of making any progress towards the formulation of a Boolean category at all [LS86, Gir91]. To make things worse, it was shown that the cartesian-closed categories are maximal, in

the sense that any addition of a non-trivial equation on maps yields a collapse of the category to a poset [Sim95, DP04].

Circumventing the collapse while going from the intuitionostic to classical logic, a series of semantics proposals have emerged following the realization [Fil89, Gri90] that having control operators in a functional program stands for adding constants corresponding to the Peirce's law to the lambda calculus. This was generalized to the Parigot's $\lambda - \mu$-calculus [Par92], whose different categorical axiomatizations are [Sel01], based on [SR98], and [Ong96].

Girard himself has formulated the system LC for classical logic and has offered two different models for it in [Gir91].

In the first group of semantics double negation not isomorphic to an object, while the disjunction is not bifunctorial, one of Girard's models is not associative, and all of them in one way or the other break the inherent symmetries of classical logic. It is fair to say that it became apparent that all of the desirable features of the classical logic cannot be present in the axiomatizations *in the same time.*

In the last decade few symmetric proposals for the categorical axiomatization of the classical proof theory have emerged, [FP05, Lam07, LS05a, Str07, DP04, BHRC06]. Each of the proposals is different to the others, and all of the proposed axiomatizations mutually differ in the choice of the particular feature of classical proofs that may or may not be present in the axiomatization, e.g. naturality of the structural rules or the monoidal closedness [3]. The state of affairs is sometimes described as going from having no answers at all to the question of what is a 'Boolean category' to having too many of them. The presented thesis does not solve the problem of defining *'the'* categorical notion of a classical proof theory. In a certain sense, this thesis makes the current situation worse; what we present (among other things) is a concrete category that interprets classical proofs, which does not obey any of the proposed axiomatizations.
[More on relation to some of the listed axiomatizations will be given in the later chapters.]

The research presented in this thesis has begun as an exploitation of denotational semantics for classical logic. In Chapter 1, we investigated the category of posets and bimodules, furthering the work in [Lam07].

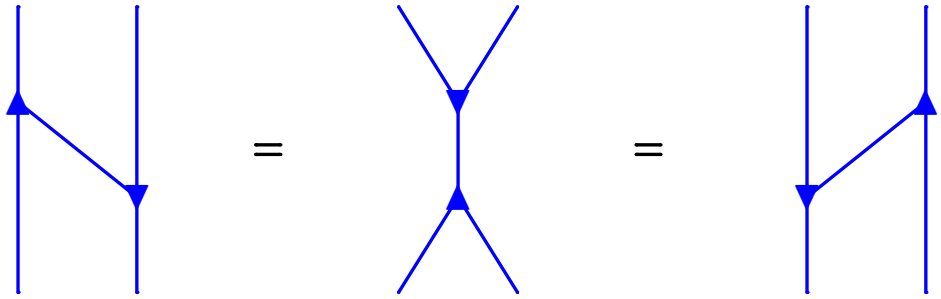
We have followed the lessons learned in the recent years – by removing classical structural rules, one may get the truly involutive negation of *-autonomous categories interpreting a calculus for multiplicative linear logic. Our denotational interpretations started as degenerate *-autonomous categories, equating the two logical connectives (thus, we have effectively dealt with compact closed categories).

To reintroduce the classical structural rules, we endowed each object with the structure of a commutative monoid. What is known is that these monoid structures cannot be natural in objects, the naturality would introduce products, and send us back to Joyal's paradox.

3. In Chapter 6 we give an overview of the relevant existing work

The commutative monoids on objects in our concrete denotational interpretations in the presence of a contravariant involutive negation entailed both a commutative monoid and a cocommutative comonoid structure on an object, and the two together have formed a structure of a Frobenius algebra on an object.

A Frobenius algebra is precisely the mentioned commutative monoid and a cocommutative comonoid structure on an object, coupled with the equation



This diagramatic equation has its counterpart in logic. In fact, the central diagram above is the notorious contraction-contraction case in cut elimination. Frobenius algebras provide a new approach to handling this situation, e.g. they equate two proofs in a one-sided sequent calculus for classical logic :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, \overline{a}, a}\,Contr \quad \dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, a}\,Contr}{\vdash \overline{a}, \overline{a}, a, a}\,Cut$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, \overline{a}}\,Contr \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, \overline{a}, a}\,Contr$$

Our motivation for this work, thus, comes from semantics; a denotational interpretation has lead us to an algebraic theory of relevance to logic, and this thesis can be regarded as an analysis of the interaction between the two that will eventually lead us to a concept of resources in classical logic.

Outline of the Thesis

The thesis is organized as follows :

– Chapter 1 is a prelude to the thesis. It contains the announced initial explorations of denotational semantics of classical logic in the category of posets and bimodules (= partially order sets and relations preserving the orders in a certain sense). The language of the chapter, however, purposefully tries to avoid categorical terminology. The reason for choosing this kind of presentation is twofold – first, the chapter presents itself as a self-contained introduction and a motivation for the rest of the thesis, and second, it may be of use to the readers with interest in proof theory and the problem of proof identity, not particularly experienced with the category theory.

– Chapter 2 contains definitions, notation and propositions used throughout the thesis. It recalls some fundamental concepts of logic, algebra and category theory. While most of its content is quite well known, certain definitions and claims do not appear in the literature as such.
– Chapters 3 and 4 are central for the thesis. Motivated by the concrete interpretations of Chapter 1, we undertake the task of analyzing the suitable algebraic structures our examples live in. The first of the two concerns with the structure of a Frobenius algebra and a suitable notion of a free category whose every object is equipped with such a structure. This gives rise to a theory of proof nets, which are defined and whose correctness is addressed in Chapter 3.
– Cut elimination on these nets is analyzed in Chapter 4. Its properties require a definition of a new sound and complete calculus for classical logic, and we argue that the semantical accounts we deal with capture a notion of resources for classical logic. As a consequence we are able to define several categories offering semantics with (weak) forms of full-completeness property.
– In Chapter 5 we go back to the category of posets and bimodules of Chapter 1. We give a characterization of the objects in that category which have a Frobenius algebra structure defined on them, and we show that these objects are precisely *-autonomous posets. Also, we define a subcategory which offers a faithful representation of the freely generated categories with Frobenius algebra structures on every object.
– Chapter 6 is dedicated to an overview of relevant existing results and their relation to our work.

Parts of Chapter 1 and Chapter 2 appear in the proceedings of the workshop on Structures and Deductions held in Bordeaux in 2009, while various versions and sections of later chapters were communicated at the 2010 Category Theory Conference in Genova and at the 2011 conference Topology, Algebra, and Categories in Logic held in Marseilles.

1

Prelude : On Concrete Interpretations of Classical Proofs

Contents

In this chapter we will interpret a minor variation of the Gentzen sequent calculus for classical logic (obtained by adding the Mix rule of linear logic to LK) into a certain category whose objects are posets and whose morphisms relations between them. Although they are very similar these two interpretations have quite different good and bad points, that we will discuss. We will show that the interpretations of proofs they allow contain meaningful information about these proofs, and that this information is closely related to the kind of proof net which is presented in [LS05b, LS05a]. Some computations we will make will alow us to conclude that these interpretation are representations of proofs that cannot be examples of the Curry-Howard correspondence.

We have taken pains to make our presentation very elementary ; for instance, although some terminology of category theory is used (and the whole of the paper is based on a category-theoretical methodology), someone who has only the barest knowledge of what a category is should be able to read this.

1.1 The general framework

The starting point for our work is an observation made by several people independently, that the well-known interpretation of linear logic in sets and relations could be extended to one where the sets are generalized to posets ; thus there exists a notion of "relation between posets", that Lambek once called *comparisons* [Lam94].

Let $(M, \leq), (N, \leq)$ be posets. A *comparison* $f : M \to N$ is a subset $f \subseteq M \times N$ which is down-closed to the left, and up-closed to the right, i.e.,

$$\begin{array}{llll} m\ f\ n, & m' \leq m & \text{implies} & m'\ f\ n \\ m\ f\ n, & m \leq n' & \text{implies} & m\ f\ n'. \end{array}$$

Composition of these maps is the ordinary composition of relations : given $F : M \to N$ and $g : N \to P$

$$m\ gf\ p, \quad \text{if} \quad (\exists n \in N)\ m\ f\ n,\ n\ g\ p'.$$

The reader should check that this defines a category, that we will denote **Cmp**. In other words that composition of comparisons is associative and that every poset M is equipped with an identity comparison, that acts as a left- and right- unit, namely $\mathrm{Id}_M = \{\, (m, m') \mid m \leq m' \,\}$—this definition is surprising at first, but then the "ordinary" identity relation (diagonal) is not a comparison ! Or the reader can look at [Lam07]. In the present paper, we restrict our attention to the multiplicative fragment of linear logic, since it suffices for our purposes.

We will follow the convention of using the roman typeface for syntactical objects, say $\mathrm{a}, \mathrm{A}, \Gamma \ldots$ and ordinary math italics for their semantical counterparts (here : posets), like $a, A, \Gamma \ldots$ As for the connectives/operations, we do not make distinctions between syntax and semantics in notation. This convention on notation will suffice for purposes of distinguishing between syntactical and semantical objects, since the only types we will be dealing with are obtained from primitive types for variables and the operations for connectives.

So the interpretation of formulas goes as :

- as usual, units **1** and $\bot$ are interpreted as a ("the") one-element set $\{*\}$
- each atomic a is interpreted as a chosen poset a ;
- for two formulas A, B, tensoring is taking the cartesian product, i.e., the interpretation of $\mathrm{A} \otimes \mathrm{B}$ is $A \times B$, which naturally we will also sometimes write $A \otimes B$. Notice that this operation is (bi)functorial, in other words, given comparisons $f \colon A \to A'$ and $g \colon B \to B'$ there is a well-defined $f \otimes g \colon A \otimes B \to A' \otimes B'$ obtained by taking the cartesian product $f \times g$ (and permuting the order of the posets a little in the product).
- If A interprets the formula A then the interpretation of $\mathrm{A}^\bot$ is inverting the order, i.e. $A^\bot = A^{op}$. This is also functorial, but contravariantly so this time : given $f \colon A \to B$ there is $f^\bot \colon B^\bot \to A^\bot$
- par is, therefore, computed as follows : the interpretation of $A \parr B$ is the same as that of $(\mathrm{A}^\bot \otimes \mathrm{B}^\bot)^\bot$, which is

  $$(A^{op} \times B^{op})^{op} = A \times B = A \otimes B.$$

  In other words, both tensor and par are cartesian product. There is some degeneracy, but we are interested in the interpretation of *proofs,* not *provability.*
- the reader should check that we do indeed have the standard (natural) bijection

  $$\frac{A \otimes B \to C}{A \to B^\bot \parr C}.$$

which defines an adjunction. Going down this "invertible rule" is called *currying*, and going up *uncurrying*. Readers who are knowledgeable about this know we have shown that **Cmp** is a $*$-*autonomous category* [Bar79].

Remark 1.1.1. Naturally a proof of a formula A is interpreted by a map $\mathbf{1} \to A$, which is a certain subset of $\{*\} \times A$. But it is only natural to drop the first factor, and simply think of the denotation of a proof as an up-closed subset of A. *We will do this all the time.* Notice that in this view, a proof of $\mathrm{A}^{\perp} \parr \mathrm{B}$ is just an up-close subset of $A^{op} \times B$, which is the same as a map $A \to B$ according to our definition.

Let us use these definitions to interpret the one-sided sequent calculus for multiplicative linear logic (as a matter of fact we also need the Mix rule). Everything in what follows is *dictated* by the definitions we just gave, provided that we think of a map $A \to B$ as a proof of $\vdash \mathrm{A}^{\perp}, \mathrm{B}$.

$$\frac{}{\vdash \mathrm{a}^{\perp}, \mathrm{a}} \quad \rightsquigarrow \quad \mathrm{Id}_a = \{(x, y) \in a \times a \mid x \leq y\}$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \parr \mathrm{B}} \parr \quad \rightsquigarrow \quad \text{do nothing}$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{A} \otimes \mathrm{B}, \Sigma} \otimes \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \times A \text{ and } g \text{ for } B \times \Sigma, \text{ take } f \times g \\ \text{for } \Gamma \times A \times B \times \Sigma\end{array}$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{A}^{\perp}, \Sigma}{\vdash \Gamma, \Sigma} \mathit{Cut} \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \times A \text{ and } g \text{ for } A^{\perp} \times \Sigma, \text{ take} \\ \{(\gamma, \delta) \mid \exists x \in A : (\gamma, x) \in f, (x, \delta) \in g\} \text{ for } \Gamma \times \Sigma\end{array}$$

$$\frac{\vdash \Gamma \quad \vdash \Sigma}{\vdash \Gamma, \Sigma} \mathit{Mix} \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \text{ and } g \text{ for } \Sigma, \text{ take } f \times g \\ \text{for } \Gamma \times \Sigma.\end{array}$$

Notice that in the definition for Cut, the variable x is seen as belonging both to the poset a and its opposite a^{op}, and the same goes for the Axiom rule. Also notice that the Mix rule would be presented in the style of the previous section as a map $A \otimes B \to A \parr B$. Naturally since here the tensor coincides with the par, it is just identity for us.

Some remarks : We will work under the implicit assumption of associativity of cartesian product, i.e., we will not distinguish between things like $(x, (y, z))$ and $((x, y), z)$, even though the corresponding types would actually differ. Consequently, the associativity of the connectives and of the comma in sequents is implicit. Also, the permutations of the formulas within a sequent, sometimes made explicit through the Exchange rule of the calculus will not be subject of our concern. The reason is the fact that we always have at our disposal the obvious natural isomorphism that permutes formulas within a sequent and commutes the arguments of a connective. As is usual in semantics, Cuts are automatically eliminated.

Some notation : we will be using $\mathcal{A}t(\mathrm{X})$ and $\mathcal{A}t(\Gamma)$ to denote set of occurrences of atoms and negatoms in a formula or a sequent, respectively. When we need to denote

different occurrences of a same atom or formula, we use superscripts, e.g. $\mathrm{a}^1, \mathrm{a}^2, \ldots$ while subscripts will be used in the usual way.

The following is easy and will be left to the reader :

Proposition 1.1.2. *The interpretation of any sequent* Γ *is* $(\prod a)_{\mathrm{a}\in\mathcal{A}t(\Gamma)}$.

Let us now turn to classical logic. The calculus LC we will use is the same as in [LS05b] ; it can be seen either as the one-sided version of Gentzen's LK with Mix added, or as MLL + Mix + Weakening + Contraction.

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}}{\vdash \Gamma, \mathrm{A}}\ \mathit{Contr} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}}\ \mathit{Weak}$$

Having Contraction in the calculus clearly requires the presence of a map $\nabla_A : A \parr A \to A$, with the interpretation of the derivation above just post-composition with that map. As for Weakening a map $\amalg_A : \bot \to A$ will clearly do the job. We add the standard condition that these maps have to obey the same kind of of associativity, commutativity and identity that we associate to commutative monoids. To illustrate what this means, look at the two derivations :

$$\dfrac{\dfrac{\vdash \Gamma, \mathrm{a}^1, \mathrm{a}^2, \mathrm{a}^3}{\vdash \Gamma, \mathrm{a}^4, \mathrm{a}^3}\ \mathit{Contr}_{\mathrm{a}^1,\mathrm{a}^2}}{\vdash \Gamma, \mathrm{a}^5}\ \mathit{Contr}_{\mathrm{a}^4,\mathrm{a}^3} \qquad \dfrac{\dfrac{\vdash \Gamma, \mathrm{a}^1, \mathrm{a}^2, \mathrm{a}^3}{\vdash \Gamma, \mathrm{a}^3, \mathrm{a}^4}\ \mathit{Contr}_{\mathrm{a}^2,\mathrm{a}^3}}{\vdash \Gamma, \mathrm{a}^5}\ \mathit{Contr}_{\mathrm{a}^1,\mathrm{a}^4}$$

where the $\mathrm{a}^1, \mathrm{a}^2, \mathrm{a}^3$ are different occurrences of the same atom. The two derivations differ in the order the contraction rules have been applied, and it is natural to want these two proofs to have the same interpretation—what's the point of semantics if we can't get rid of that ugly bureaucracy ? This means, we want to have $\nabla \circ (\nabla \parr \mathrm{Id}) = \nabla \circ (\mathrm{Id} \parr \nabla)$ for every type. Notice that strictly speaking the types do not agree exactly here ; we are applying our rule of not bothering with the issues related to the not-really-strict associativity of the cartesian product.

Suppose that M is an ordinary set and that $m\colon M \times M \to M$ is an ordinary binary operation. Instead of the usual equation in x, y and z we could always write the associativity of m as $m \circ (m \times \mathrm{Id}) = m \circ (\mathrm{Id} \times m)$. The advantage of doing so is that *we can define a monoid using greatly generalized notions of Cartesian product, to suit our needs,* for example replacing ordinary Cartesian product by $\parr$ (in the present case it seems that $\parr$ is still the Cartesian product, but actually this is not strictly true because the *maps* are different in **Cmp**). Thus we can define a notion of commutative $\parr$-monoid, using well-chosen equations between maps. They are usually presented as commutative diagrams, and the reader can consult [FP05, FP04, LS05a] and countless other papers for a full display of these diagrams.

Thus in our model of classical logic every interpretation of a type will come equipped with a commutative $\parr$-monoid structure. Since proofs are defined by induction, it will suffice to define the monoid structure for the atomics $\mathrm{a}, \mathrm{b} \ldots$ and negatomics (which, to signal the fact that we are entering classical logic, will be denoted $\bar{\mathrm{a}}, \bar{\mathrm{b}}, \ldots$) and to give a

way to construct a new monoid structure after applying the connectives. Because of the linear rules we already have, the poset for $\mathrm{A} \wedge \mathrm{B}$ will obviously be $A \otimes B = A \times B$ and the one for $\mathrm{A} \vee \mathrm{B}$ will be $A \parr B = A \times B$.

Given the $\parr$-monoid structure on A and B, we define $\nabla_{A \vee B} : (A \parr B) \parr (A \parr B) \to A \parr B$ as the composition :

$$(A \parr B) \parr (A \parr B) \xrightarrow{\sim} (A \parr A) \parr (B \parr B) \xrightarrow{\nabla_A \parr \nabla_A} A \parr B.$$

The isomorphism here is the obvious associativity/permutation isomorphism of Cartesian products. As for $\amalg_{A \vee B} : \bot \to A \vee B$, we take :

$$\bot \xrightarrow{\sim} \bot \parr \bot \xrightarrow{\amalg_A \parr \amalg_B} A \parr B.$$

We do the same for conjunction, since here both tensor and par are interpreted by the Cartesian product. But naturally in general this is not the case, and then there is no ready-made way to go from $(A \otimes B) \parr (A \otimes B)$ to $A \otimes B$. This is solved by requiring the presence of a natural map $(A \otimes B) \parr (A \otimes B) \to (A \parr A) \otimes (B \parr B)$, which is called *Medial*. It is natural to make it explicitly part of the logic, and it actually first appeared as a deduction rule for Deep Inference calculi in classical logic [Brü03]. For the category-theoretical treatment of the map, independently of classical logic, the reader can look at [Lam07, Str07].

Our interpretation already has negation for atomics, by definition. For a composite formula the $\parr$-monoid structure on its negation is obtained recursively, using de Morgan duality : $\overline{A \wedge B} = \overline{A} \vee \overline{B}$ and $\overline{A \vee B} = \overline{A} \wedge \overline{B}$.

There is some advantage to considering the combined monoid structures on $A, \overline{A}$ as a *single* algebraic structure on the object A. Notice that a map $\nabla \colon \overline{A} \parr \overline{A} \to \overline{A}$ can also be seen, by duality, as a map $\Delta \colon A \to A \otimes A$, and the same goes for $\amalg \colon \bot \to \overline{A}$, which becomes $\amalg \colon A \to \mathbf{1}$. The associativity, commutativity and unit laws can be translated in this "reversed" contexts by duality ; for decades algebraists have called these laws *coassociativity, cocoummutativity and co-unit,* while the resulting structure $(A, \Delta, \amalg)$ is called a *(cocommutative) comonoid.*, and the full thing $(A, \Delta, \amalg, \nabla, \amalg)$ is called a *(commutative, cocommutative) bimonoid* with Δ being the *diagonal*, ∇ the *co-diagonal*, $\amalg$ the *co-unit* or the *projection*, $\amalg$ the *unit* or the *co-projection.*

Thus, given an object X in an interpretation of linear logic, turning it into an interpretation of a classical formula amounts to finding a bimonoid structure on it.

We can finally define the interpretation of Weakening and Contraction (the reader should keep in mind our convention to identify the proofs with up-closed subsets) :

$$\frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}}\ \textit{Weak} \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \text{ take} \\ \{(\gamma_1, \ldots, \gamma_n, \epsilon) \mid (\gamma_1, \ldots, \gamma_n) \in f \text{ and } \epsilon \in \amalg_A\} \text{ for } \Gamma \times A\end{array}$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}}{\vdash \Gamma, \mathrm{A}}\ \textit{Contr} \quad \rightsquigarrow \quad \begin{array}{l}\text{given } f \text{ for } \Gamma \times A \times A, \text{ take} \\ \{(\gamma, x) \mid \exists x_1, x_2 \in A : (\gamma, x_1, x_2) \in f \text{ and } (x_1, x_2)\ \nabla_A\ x\} \\ \text{for } \Gamma \times A\end{array}$$

Diagrammatically, the defined maps for Contraction and Weakening can be seen as compositions

$$\mathbf{1} \xrightarrow{f} \Gamma \xrightarrow{\sim} \Gamma \times \bot \xrightarrow{\mathrm{Id}_\Gamma \times \Pi_A} \Gamma \times A \quad \text{and} \quad \mathbf{1} \xrightarrow{f} \Gamma \times A \times A \xrightarrow{\mathrm{Id}_\Gamma \times \nabla_A} \Gamma \times A.$$

Before we state the following, easy to prove proposition, notice that the interpretation for the Cut rule we gave before corresponds to the composite

$$\mathbf{1} \xrightarrow{f \times g} \Gamma \times A \times A^\perp \times \Sigma \xrightarrow{\mathrm{Id}_\Gamma \times C_A \times \mathrm{Id}_\Sigma} \Gamma \times \Sigma$$

where $C_A : A \otimes A^\perp \to \bot$ is the map $\{(x, y, *) \mid x \leq y\}$ which is dual to the identity $\mathbf{1} \to A^\perp \parr A$.

Proposition 1.1.3. *Let* X *be a formula and* $\{a_1, \ldots, a_n\} = \mathcal{A}t(\mathrm{X})$. *Then the following diagrams commute :*

$$\begin{array}{ccc} X^1 \times X^2 & \xrightarrow{\nabla_X} & (a_1^3 \times \cdots \times a_n^3) = X \\ \| & & \uparrow{\scriptstyle \nabla_{a_1} \times \cdots \times \nabla_{a_n}} \\ (a_1^1 \times \cdots \times a_n^1) \times (a_1^2 \times \cdots \times a_n^2) & \xrightarrow{\sim} & (a_1^1 \times a_1^2) \times \cdots \times (a_n^1 \times a_n^2). \end{array}$$

$$\begin{array}{ccc} \Gamma \times X \times \overline{X} \times \Sigma & \xrightarrow{C_X} & \Gamma \times \Sigma \\ \| & & \uparrow{\scriptstyle C_{a_1} \times \cdots \times C_{a_n}} \\ \Gamma \times (a_1 \times \cdots \times a_n) \times (\overline{a}_1 \times \cdots \times \overline{a}_n) \times \Sigma & \xrightarrow{\sim} & \Gamma \times (a_1 \times \overline{a}_1) \times \cdots \times (a_n \times \overline{a}_n) \times \Sigma. \end{array}$$

Notice that the last diagram commutes modulo obvious isomorphisms that we choose to omit.

Look at the following derivation :

$$\dfrac{\dfrac{\vdash \overline{\mathrm{a}}, \mathrm{a} \quad \vdash \overline{\mathrm{a}}, \mathrm{a}}{\vdash \overline{\mathrm{a}}, \mathrm{a}, \overline{\mathrm{a}}, \mathrm{a}}\, Mix}{\vdash \overline{\mathrm{a}}, \mathrm{a}}\, 2 \times Cont.$$

Seen as a map, and using the equality of tensor and par, this derivation is the composite $\nabla_a \circ \Delta_a$. It is rather easy to construct semantics for which this map is identity, but much more difficult to get ones for which it is not. The first partially successful results on this are found in [Lam07]. The point is that we want to keep track of resources, since this derivation can be seen as the superposition of two axiom links, not just a single axiom link. In the aforementioned paper, for any number n a bimonoid is constructed such that superpositions of $1, 2, \ldots, n$ axiom links (by iteration of the derivation just above) can be distinguished, but since the bimonoids are finite, there is a ceiling where the count saturates. One aim of this chapter is to construct interpretations where there is no saturation.

Recall that a partition on a given set Q can be seen either as an equivalence relation on Q or as a set $\mathcal{R} \subseteq \mathcal{P}(X)$ of subsets of Q, whose elements are called classes. Let Γ be a sequent, along with a proof q of it. By induction on the proof, we define below a partition $\mathrm{Prt}_q(\Gamma)$ on the set $\mathcal{A}t(\Gamma)$. In that definition we do not bother with keeping track explicitly of q :

- if Γ is $\vdash \bar{\mathrm{a}}, \mathrm{a}$, then $\mathrm{Prt}(\Gamma) = \{\{\bar{\mathrm{a}}, \mathrm{a}\}\}$
- if Γ has been obtained by application of the tensor or the Mix rule, $\mathrm{Prt}(\Gamma)$ is the obvious "sum" partition on the disjoint union $\mathcal{A}t(\Sigma_1) \uplus \mathcal{A}t(\Sigma_2)$, where Σ_1, Σ_2 are the premiss sequents of the rule application.
- supposing that A has been added to $\vdash \Gamma$ by Weakening, $\mathrm{Prt}(\Gamma, \mathrm{A}) = \mathrm{Prt}(\Gamma) \uplus \big\{\{\mathrm{a}\} \mid \mathrm{a} \in \mathcal{A}t(\mathrm{A})\big\}$, i.e., we add every (neg)atom of A as a singleton class. [4]
- supposing that $\vdash \Gamma, \mathrm{A}$ has been obtained by contraction on $\vdash \Gamma, \mathrm{A}^1, \mathrm{A}^2$, let

$$p \colon \mathcal{A}t(\Gamma, \mathrm{A}^1, \mathrm{A}^2) \longrightarrow \mathcal{A}t(\Gamma, \mathrm{A})$$

  be the obvious surjection that identifies pairs of corresponding atoms in the two occurences of A. Take $\mathrm{Prt}(\Gamma, \mathrm{A})$ to be the smallest partition generated by the direct image sets $\{\, p(U) \subseteq \mathcal{A}t(\Gamma, \mathrm{A}) \mid U \in \mathrm{Prt}(\Gamma, \mathrm{A}^1, \mathrm{A}^2) \,\}$. It is easy to see that if we define the binary relation $p(U) \frown p(U')$ as $p(U) \cap p(U') \neq \emptyset$ then the classes in $\mathrm{Prt}(\Gamma, \mathrm{A})$ are in bijective correspondence with the connected components of the graph of $\frown$ and every class is the union of the sets in its corresponding component.
- if $\vdash \Gamma, \Sigma$ has been obtained by applying Cut on $\vdash \Gamma, \mathrm{A}$ and $\vdash \overline{\mathrm{A}}, \Sigma$, we define $\mathrm{Prt}(\Gamma, \Sigma)$ as follows. First define an equivalence relation $\sim$ on $\mathcal{A}t(\Gamma, \mathrm{A}) \uplus \mathcal{A}t(\overline{A}, \Sigma)$ as the symmetric-transitive closure of the union of three binary relations R, S, T, where
  - R, S are the equivalence relations associated with the partitions $\mathrm{Prt}(\Gamma), \mathrm{Prt}(\Sigma)$ respectively,
  - xTy when $x = \mathrm{a}$ is a (neg)atom in A and $y = \bar{\mathrm{a}}$ its corresponding negation in $\overline{A}$.

  $\mathrm{Prt}(\Gamma, \Sigma)$ is then defined as the restriction of the partition determined by $\sim$ on the subset $\mathcal{A}t(\Gamma, \Sigma) \subseteq \mathcal{A}t(\Gamma, \mathrm{A}) \uplus \mathcal{A}t(\overline{A}, \Sigma)$

This definition can be explained in terms of the proof nets given in [LS05b], where axiom links are *superposed* when Contraction is applied, giving rise to a relation on $\mathcal{A}t(\Gamma)$ which is not the usual coupling of ordinary linear proof nets, but a much more general kind of relation. Supposing first that neither Weakening nor Cut has been used in a proof of Γ, then the partition $\mathrm{Prt}(\Gamma)$ corresponds exactly to the set of *connected components* of the proof net graph associated to that proof. If a proof contains Weakenings, in our case the atoms are added as singletons in the graph, while they would simply be ignored in a proof net. It is simpler for us to define $\mathrm{Prt}(\Gamma)$ that way, instead of as a subpartition (partition of a subset) of $\mathcal{A}t(\Gamma)$.

Proposition 1.1.4. *Let q be a proof of a sequent $\vdash \Gamma$. Then if $f \subseteq \Gamma = \prod_{\mathrm{a} \in \mathcal{A}t(\Gamma)}$ is the interpretation of q, there is a family $(f_U)_{U \in \mathrm{Prt}(\Gamma)}$ such that f decomposes as a product of factors*

$$f = \prod_{U \in \mathrm{Prt}(\Gamma)} f_U \quad \text{where} \quad f_U \subseteq \prod_{\mathrm{a} \in U} a\,.$$

4. Notice the slight abuse of notation where a is used to denote negatomics as well as atomics.

The proof is a quite straightforward induction, making use of Proposition 1.1.3.

The meaning of this is that the decomposition associated with the connected components for a proof net in CL has a simple semantical counterpart, as a product decomposition instead of a sum decomposition. This is syntactical information which is retained by the semantics.

The following well-known result (a proof is in [Lam07]) will turn out to be useful for constructing classes of bimonoids.

Proposition 1.1.5. *Let $(M, \leq, \cdot, e)$ be a poset which is equipped with a commutative monoid structure $(\cdot, e)$ such that binary $\cdot$ is $\leq$-monotone. (A category theorist would say : let M be a monoid in* **Poset**.*) Then if we define $\nabla : M \times M \to M$ and $\amalg : \{*\} \to M$ in* **Cmp** *as*

$$(m,n)^{\nabla} p \text{ if } \ m \cdot n \leq p, \qquad *^{\amalg} m \text{ if } e \leq m$$

we get a commutative $\times$-monoid (which is the same, remember, as a $\parr$-monoid). Dually, if we define $\Delta : M \to M \times M$ and $\Pi : M \to \{*\}$ as

$$m^{\Delta}(n,p) \text{ if } m \leq n \cdot p, \qquad m^{\Pi} * \text{ if } m \leq e$$

we get a cocommutative $\times$-comonoid, i.e., a cocommutative $\otimes$-comonoid.

As a conclusion, a way to construct bimonoids in **Cmp** is to find posets equipped with two monotone monoid structures.

1.2 An interpretation based on $\mathbb{Z}$

The set $\mathbb{Z}$ of integers comes equipped with the two structures we are interested in : poset $(\mathbb{Z}, \leqslant)$ and commutative monoid $(\mathbb{Z}, +, 0)$.

So

we assign the poset $\mathbb{Z}$ to every atomic type a and thus $a = \mathbb{Z}$.

We have to look for other monoid structures. Can we modify standard addition as little as possible, so as to keep calculations simple ? Yes, since it is well known that for any number c the operation $(x, y) \mapsto x + y - c$ defines a monoid structure on $\mathbb{Z}$, whose unit is c. As a matter of fact the choice of that unit is all which is needed to define the rest : look at the order-isomorphic translation $x \mapsto x + c$, which has inverse $x \mapsto x - c$. We have only defined our new operation (let us call it $+_c$) by transporting addition along the first iso, i.e.

$$x +_c y = \big((x - c) + (y - c)\big) + c\,.$$

But this shows that our new monoid $(\mathbb{Z}, +_c, c)$ is isomorphic to the one we started with. It seems we haven't progressed very much. But remember, we are looking for *a new* monoid structure, as we already had one !

Let us define a *poset-bimonoid* to be just that : a sextuple $(M, \leqslant, +_1, e_1, +_2, e_2)$ where $(M, \leqslant)$ is a poset and $(+_i, e_i)$ two monotone monoid structures for it. It is obvious what an isomorphism of poset-bimonoids should be : an isomorphism of posets which is also a monoid isomorphism for both $(+_1, e_1)$ and $(+_2, e_2)$. Now let us look at poset-bimonoids of the form $(\mathbb{Z}, \leqslant, +_{e_1}, e_1, +_{e_2}, e_2)$, for any choice of e_1, e_2. We will call these *translation bimonoids*. The following is very easy to show :

Fact 1.2.1. Given two translation bimonoids $(\mathbb{Z}, \leqslant, +_{a_1}, a_1, +_{a_2}, a_2)$ and $(\mathbb{Z}, \leqslant, +_{b_1}, b_1, +_{b_2}, b_2)$, then they are isomorphic iff $a_2 - a_1 = b_2 - b_1$.

Now that we have this information, we can decide to look only at poset-bimonoids of the form $(\mathbb{Z}, \leqslant, +, 0, +_c, c)$ since it gives us all the isomorphism classes of translation bimonoids by varying c.

What we were interested in in the first place was bimonoids in **Cmp**.

Proposition 1.2.2. *If a is a translation bimonoid, then $\bar{a}$ is isomorphic to it.*

Démonstration. Suppose for simplicity that c is positive. Then it is used for the co-unit of the comonoid structure in the negation $\overline{\mathbb{Z}}$, which has reverse order $\mathbb{Z}^{op}$, which makes c the lesser of the two units in the poset-bimonoid, and if we map it to zero by a translation the whole of the poset-bimonoid structure will be respected. □

Thus we end up with an interpretation where atomics and negatomics will be isomorphic (as in the relational model). But, interestingly, that isomorphism cannot be an identity unless $c = 0$.

Let us recapitulate. Choose c, and define

$$\begin{array}{llll} (j,k)^{\nabla_a} i & \text{iff} & j + k \leq i + c; & \qquad *^{\Pi_a} i \quad \text{iff} \quad c \leq i. \\ i^{\Delta_a}(j,k) & \text{iff} & i \leq j + k; & \qquad i^{\Pi_a} * \quad \text{iff} \quad i \leq 0; \end{array} \tag{1.1}$$

And now $\mathcal{D}_a = \nabla \circ \Delta = \{(x, y) \mid \exists i, j : x \leq i + j,\ i + j \leq y + c\} = \{(x, y) \mid x \leq y + c\}$. Then it is easy to see that $\mathcal{D}_a^n = \{(x, y) \mid x \leq y + n \cdot c\}$, and therefore the composition of the doubling map with itself will never stabilize.

1.2.1 More computations

So computing proofs in this interpretation is rather easy. Again, choose one atomic a along with an interpretation $a = (\mathbb{Z}, \leqslant, +, 0, +_c, c)$.

We will make use of the fact that the relation $x \leq y$ in a is equivalent to $(-x) + y \geq 0$ and that $x \leq y$ in a^{op} is equivalent to $(-x) \leq (-y)$ in a. This allows us to work with a single order structure, the standard one in $\mathbb{Z}$, and not have to deal with its opposite, which is required in general, and makes things very confusing when, as here, the ordering and its opposite are isomorphic.

Theorem 1.2.3. *Let f be a proof of a sequent $\vdash \Gamma$. Then every factor f_j of the decomposition*

$$f = f_1 \times f_2 \times \ldots \times f_N,$$

of Proposition 1.1.4 is of the form

$$f_j = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in a^{n+m} \mid (-x_1) + \cdots + (-x_n) + y_1 + \ldots + y_m \geq M_j c\}$$

where $\mathrm{a}^1, \bar{\mathrm{a}}^2, \ldots, \bar{\mathrm{a}}^n, \mathrm{a}^1, \ldots, \mathrm{a}^m$ is an enumeration of the (neg)atomics of the class $U_j \in \mathrm{Prt}(\Gamma)$ and M_j an integer.

That integer M has to do with the number of contractions that were performed in the proof q and the number of atoms that came into existence through Weakening. A more detailed discussion of the relationship between M and q will be provided in the later chapters ; right now we will content ourselves with examples. But let us mention that singleton elements of $\mathrm{Prt}(\Gamma)$ are either of the form $\{y \geqslant c\}$ for positive atoms or of the form $\{-x \geqslant 0\}$ for negative ones.

Let us now try our hand at Church numerals. First, write the type $(\mathrm{a} \Rightarrow \mathrm{a}) \Rightarrow (\mathrm{a} \Rightarrow \mathrm{a})$ as the sequent $\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}$, where there cannot be any ambiguity about the way the atomics are reordered.

The proofs we are dealing with are of the form

$$n \times \wedge \left\{ \begin{array}{l} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \\ \\ \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,Contr \quad \dfrac{}{\vdash \bar{\mathrm{a}}, \mathrm{a}}\,Ax}{\vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}}\,\wedge \\ \\ \vdash \bar{\mathrm{a}}, \mathrm{a} \wedge \bar{\mathrm{a}}, \mathrm{a}\,Contr \end{array} \right.$$

for encoding the numeral n.

Let us see what this gives in our interpretation.

Starting with a concrete numeral, say, $n = 3$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{\mathrm{a}}^0, \mathrm{a}^0}\,Ax \quad \dfrac{}{\vdash \bar{\mathrm{a}}^1, \mathrm{a}^1}\,Ax}{\vdash \bar{\mathrm{a}}^0, \mathrm{a}^0 \wedge \bar{\mathrm{a}}^1, \mathrm{a}^1}\,\wedge \quad \dfrac{}{\vdash \bar{\mathrm{a}}^2, \mathrm{a}^2}\,Ax}{\vdash \bar{\mathrm{a}}^0, \mathrm{a}^0 \wedge \bar{\mathrm{a}}^1, \mathrm{a}^1 \wedge \bar{\mathrm{a}}^2, \mathrm{a}^2}\,\wedge}{\vdash \bar{\mathrm{a}}^0, \mathrm{a}^3 \wedge \bar{\mathrm{a}}^3, \mathrm{a}^2}\,Contr \quad \dfrac{}{\vdash \bar{\mathrm{a}}^4, \mathrm{a}^4}\,Ax}{\vdash \bar{\mathrm{a}}^0, \mathrm{a}^3 \wedge \bar{\mathrm{a}}^3, \mathrm{a}^2 \wedge \bar{\mathrm{a}}^4, \mathrm{a}^4}\,\wedge}{\vdash \bar{\mathrm{a}}^0, \mathrm{a}^5 \wedge \bar{\mathrm{a}}^5, \mathrm{a}^4}\,Contr$$

its proof is computed as follows, following the leftmost branch in the proof tree :

$$\{-x^0 + y^0 \geq 0\}$$
$$\blacktriangledown$$
$$\{-x^0 + y^0 \geq 0\} \times \{-x^1 + y^1 \geq 0\}$$
$$\blacktriangledown$$
$$\{-x^0 + y^0 \geq 0\} \times \{-x^1 + y^1 \geq 0\} \times \{-x^2 + y^2 \geq 0\}$$
$$\blacktriangledown$$
$$\{-x^0 - x^3 + y^2 + y^3 \geq -c\}$$
$$\blacktriangledown$$
$$\{-x^0 - x^3 + y^2 + y^3 \geq -c\} \times \{-x^4 + y^4 \ \geqslant 0\}$$
$$\blacktriangledown$$
$$\{-x^0 - x^5 + y^4 + y^5 \geq -2c\}$$

The third and the fifth transition require some comments. When contracting $\mathrm{a}^0 \wedge \overline{\mathrm{a}}^1, \mathrm{a}^1 \wedge \overline{\mathrm{a}}^2$ two atomic contractions are conducted, one on the positive atoms $\mathrm{a}^0, \mathrm{a}^1$, which is the one that adds c, and the other one on the negatoms $\overline{\mathrm{a}}^1, \overline{\mathrm{a}}^2$ whose contraction monoid is plain addition in $\mathbb{Z}$, and thus does not modify the constants M.

It is not difficult to conclude now that in general, given a *positive* n its Church numeral is (with a notation for occurences that allows all the numerals to be comparable) :

$$\{-x^0 - x^{2n-1} + y^{2n-2} + y^{2n-1} \geq -(n-1)c\}.$$

Recall that if two Church numerals are seen as maps $n, m \colon (\mathrm{a} \Rightarrow \mathrm{a}) \to (\mathrm{a} \Rightarrow \mathrm{a})$ then multiplying them amounts to *composing* these two maps, i.e., is obtained through a Cut. A simple computation will show that this operation will give the numeral $-(n + m - 1)$.

This is definite evidence that the interpretation we have built has nothing to do with the Curry-Howard correspondence. A multiplication which acts additively as it does here shows that the interpretation of terms cannot correspond to functional programs, since the Curry-Howard interpretation of a Church numeral n is the iterator functional "compose an endofunction n times with itself", and composing the iterators for n and m can only give the iterator for nm.

1.2.2 Discussion

We had seen that in general an interpretation of a classical proof in **Cmp** could be decomposed as a Cartesian product sets, each of which corresponds to a connected component U_j of the graph of axiom links of the associated proof net. We now see that the information contained in our $\mathbb{Z}$-interpretation consists in that decomposition, along with a natural number M_j which is associated to every connected component U_j, related to the Contractions and Weakenings that were effected, *and nothing else.* This suggests a class of proof nets where the binary relation determined by axiom links would be replaced by an undifferentiated "blob", just a set of atomics and negatomics of the same type,

with a number associated to it. This possibility of defining a new notion of a proof net is suggested by the model that we demonstrated, but one may ask if there is deeper reason for having such a notion of a proof net. For instance, one may if the numerical invariant associated to *blobs* is a direct consequence of the fact that the bimonoid we have constructed is actually a Frobenius algebra. Frobenius algebra are a specific class of bimonoids that usually live in categories of vector spaces, but nothing prevents them from being defined in **Cmp**, in the same way we did for bimonoids. We continue our investigation along these lines later in this thesis.

It is also interesting to notice the incontrovertible evidence that there are valid, normalizing semantics of proofs that have nothing to do with the Curry-Howard correspondence.

2

Conceptual Apparatus

Contents

This chapter contains definitions, terminology and notation of accounts used throughout this thesis. Much of the content of this chapter is widely known and standard, but in several places we take slightly non-standard views of well-known notions that are crucial for discussion in the rest of the text. In a few cases the formulations do not appear elsewhere in the literature as such. An effort has been made to make the notation appearing here is as standard as possible.

2.1 Classical Logic. Multiplicative Linear Logic. Proof Systems.

The logic of interest in this thesis is classical propositional logic.

We assume an infinite set (an enumerable set is enough) of *propositional letters*. Propositional *formulas* are made of propositional letters with binary connectives :

$$\wedge,\ \vee,\ \Rightarrow$$

and a unary connective

$$\neg$$

More formally, given the enumerable set $\mathcal{P}$ of propositional letters, the set of propositional formulas $\mathcal{F}orm$ is the smallest language over the alphabet $\mathcal{P} \cup \{\neg, \wedge, \vee, \Rightarrow\} \cup \{(,)\}$, such that

1) if $\mathrm{x} \in \mathcal{P}$, then $x \in \mathcal{F}orm$,
2) if $\mathrm{x}, \mathrm{y} \in \mathcal{F}orm$, then $(\neg \mathrm{x}), (\mathrm{x} \wedge \mathrm{y}), (\mathrm{x} \vee \mathrm{y}), (\mathrm{x} \Rightarrow \mathrm{y}) \in \mathcal{F}orm$.

With priority of connectives being

$$priority(\neg) < priority(\wedge) = priority(\vee) < priority(\Rightarrow),$$

certain bracket symbols become superfluous and are standardly omitted.

Convention 2.1.1. There is certain advantage in taking somewhat different approach to propositional logic. Namely, we think of a the two logical connectives $\wedge$ and $\vee$ as the only primitive connectives, while

$$\mathrm{x} \Rightarrow \mathrm{y}$$

is a shorthand for

$$\neg \mathrm{x} \vee \mathrm{y},$$

and

$$\neg(\mathrm{x} \wedge \mathrm{y}) \text{ and } \neg(\mathrm{x} \vee \mathrm{y})$$

stand for

$$\neg \mathrm{x} \vee \neg \mathrm{y} \text{ and } \neg \mathrm{x} \wedge \neg \mathrm{y},$$

respectively. In the 'standard' view of classical propositional logic, this corresponds to negation-normal forms of formulas, and each formula can be transformed into its negation normal form by a simple linear-time transformation. Since negation is pushed to propositional letters, we also prefer to think of a formula

$$\neg \mathrm{a}$$

as a simply another propositional letter

$$\overline{\mathrm{a}}.$$

The previous convention prompts

Convention 2.1.2. A formula as defined over a set of *distinct elements* which are *labeled* by propositional symbols. These symbols come from the set of *atomic types* $\mathcal{AT}$yp which is a product of the form $S \times P$, where

- S is a set of *sorts* (corresponding to letters)
- P is a set of polarities, usually $\{+, -\}$ (but could be a singleton).

We refer to each out of these labeled elements as to an *atom* or a *negatom*, depending on the polarity of the label.

The atomic types with negative polarity are written as $\overline{\mathrm{a}}$ which is of the *opposite polarity* to a and, as previously mentioned, is a shorthand for negation $\neg$a. The relation 'opposite polarity' is symmetric, i.e. a is also of the opposite polarity to $\overline{\mathrm{a}}$. Thus, a formula

is built over these letters with the two binary connectives for conjunction and disjunction, $\wedge$ and $\vee$.

When we want to speak about a particular labeled element in a formula, we refer to it as a (neg)atom *occurrence*. Sometimes, when the intended meaning is clear, we will omit the word 'occurrence' when speaking of a particular labeled element.

The word 'type' will sometimes also be used as a synonym for the word 'formula' in light of the well known proofs-of-formulas-as-terms-inhabiting-types connections [GLT89].

If we speak of an atom with of a fixed sort regardless of its polarity, we sometimes refer to it as a *literal*, e.g. both a and $\overline{\mathrm{a}}$ are literals of sort a.

In this work we are not particularly interested in classical semantics at the level of validity and provability, boolean algebras, satisfiability etc. We are focus our attention on semantics of *proofs*, seen as formal mathematical objects. More concretely, we investigate Gentzen-style proof theory.

In the thirties of the 20th century, in his seminal work, Gerhard Gentzen [Gen34] has developed a way to perform logical reasoning in classical and intuitionistic logic by means of purely syntactic transformations. The systems, if one restricts its attention to propositional formulas, decide the logical consequence relation between formulas by explicit construction of a graph, a *proof tree* whose nodes are *derivation rules* in a *deductive system*.

Definition 2.1.3. A *sequent* is an expression of the form

$$\mathrm{A}_1, \ldots, \mathrm{A}_m \vdash \mathrm{B}_1, \ldots \mathrm{B}_n$$

where A_is and B_is are propositional formulas of classical logic. In this work, sequents will usually be one sided, i.e. of the form

$$\vdash \mathrm{B}_1, \ldots \mathrm{B}_n.$$

Therefore, where it does not confusion, by a sequent we may also understand the sequence of formulas itself, i.e. $\Gamma = \mathrm{A}_1, \ldots, \mathrm{A}_m$ and $\Sigma = \mathrm{B}_1, \ldots, \mathrm{B}_n$ will also be called sequents. Sequents are usually denoted by capital Greek letters. [5]

A *derivation rule* [6] is an expression of the form

$$\frac{\vdash \Delta_1 \quad \ldots \quad \vdash \Delta_k}{\vdash \Sigma}\ .$$

All of $\Sigma, \Delta_i s$ are sequences of formulas. Sequents $\vdash \Delta_1, \ldots, \vdash \Delta_k$ form the *premise* and $\vdash \Sigma$ is the *conclusion* of the rule.

5. Sometimes in the literature a sequent is defined as a multiset, i.e. a set of formulas coupled with positive integers denoting the number of occurrences of a formula. Such an approach is unsuitable for categorical approaches to proof theory, as pursued here.

6. in this thesis, all the calculi are one-sided

A *rule system* or *calculus* is a finite set of rules. Propositional letters appearing in a rule of a proof system are thought of as meta variables that can be substituted by formulas, obtaining in such way *instances* of rules. Rules of a rule system are sometimes referred to as *rule schemata*.

A *proof (= derivation, proof tree)* of a sequent $\vdash \Sigma$ in a calculus $\mathcal{K}$ is a tree whose nodes are labeled by instances of rules of $\mathcal{K}$, such that
- root node is labeled by a rule instance whose conclusion is $\vdash \Sigma$,
- leaves of the tree are instances of rules with no premisses,
- combined conclusions in labels of child nodes form the premise in the label of the parent node.

The same way we speak of an occurrence of a (neg)atom in a formula, we also speak of an occurrence of a (neg)atom in a proof. This further extends to the notion of an occurrence of a formula in a sequent or in a proof, and to the notion of an occurrence of a sequent in a proof.

Notation 2.1.4. We use roman fonts for propositional syntax. Thus, $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{X}, \mathrm{Y}, ...$ will denote formulas, $\Gamma, \Sigma, ...$ will denote sequents. Lowercase roman letters are reserved for atoms, $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{l}, \mathrm{m}, \mathrm{n}, ...$ In all three cases, we are allowed to use indices to denote mutually *distinct* formulas/sequences/atoms, $\mathrm{a}_1, \mathrm{a}_2, ...\mathrm{B}_1, \mathrm{B}_2$. Atomic types that come in two opposite polarities, a and $\overline{\mathrm{a}}$. When we want to refer to a literal of type a, regardless of polarity, we use bold typeface $\mathbf{a}$. Still, we will use this notation sparingly, more often we will use $\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots$ to speak of an atom regardless of its label or polarity.

Superscripts are reserved for different *occurrences* of a *same* type (atom, formula or a sequent), $\mathrm{A}^1, \mathrm{A}^2, ...$

Gentzen's deductive system for classical logic, LK is the starting point for our considerations, while the central theorem of cut-elimination, Gentzen's Hauptsatz, will make the central role in this work as well. The one place where our prototype calculus differs from LK, having our convention on structure of the propositional formulas in mind, is in that our calculus is one-sided. Partially, this is enabled by the convention that negation exists only at the level of literals.

Definition 2.1.5 (CL)**.** The calculus CL is the one-sided sequent calculus[7] whose rules are given in Figure 2.1.

The last rule, Mix, is not a standard part of LK, but we include it into our calculus for classical logic.

It is clear that CL is sound and complete for classical logic since LK is, while Mix is obviously a sound rule.

7. the calculus appears under this name in [LS05b]

$$\frac{}{\vdash \overline{a}, a}\ Ax$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Sigma}{\vdash \Gamma, A \wedge B, \Sigma}\ \wedge$$

$$\frac{\vdash \Gamma, A \quad \vdash \overline{A}, \Sigma}{\vdash \Gamma, \Sigma}\ Cut$$

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ Contr \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ Weak$$

$$\frac{\vdash \Gamma \quad \vdash \Sigma}{\vdash \Gamma, \Sigma}\ Mix$$

FIGURE 2.1 – System CL

Also, a standard part of the CL calculus and all of the claculi we will ever consider is the Exchange rule

$$\frac{\vdash \Gamma, A, B, \Sigma}{\vdash \Gamma, B, A, \Sigma}\ Exch$$

which allows for permutation of formulas within a sequent. As a convention, we choose not to list or treat this rule explicitly, but always assume its presence. [8]

2.1.1 Multiplicative Linear Logic

Proof-theoretically, linear logic of Jean-Yves Girard [Gir87] derives from an analysis of classical sequent calculus LK in the absence of the structural rules of weakening and contraction. Contraction and weakening are preformed in a controlled fashion in presence of modalities, the exponentials ? and !. In a classical sequent system, due to intrinsic presence of the contraction and weakening, one can formulate logical rules additively or multiplicatively, or as a mix of the two. In linear logic contraction and weakening are tamed, so one is forced to separate the two cases, which gives rise to the separation between the connectives, depending on the treatment of the context in the corresponding rule. As a consequence, one has pairs of additive and multiplicative connectives in the logic, as well as the neutrals for the logical operations.

We will restrain ourself of going into full generality of linear logic, since for our needs it suffice to look at multiplicative (unit free) fragment of the linear logic. As was shown in the previous chapter, multiplicative linear logic is a potential starting point in the analysis of semantics of classical proofs.

8. The equational systems and the interpretations/semantics we will encounter will always be symmetric, i.e. we will always have a suitable algebraic means at our disposal to capture the rule.

$$\frac{}{\vdash \mathrm{a}^{\perp}, \mathrm{a}}\ Ax$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \parr \mathrm{B}}\ \parr \qquad \frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{A} \otimes \mathrm{B}, \Sigma}\ \otimes$$

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{A}^{\perp}, \Sigma}{\vdash \Gamma, \Sigma}\ Cut$$

FIGURE 2.2 – System MLL

Formulas of multiplicative linear logic are, as usual, built over the set of atoms and negatoms, with $\mathrm{a}^{\perp}$ instead of $\overline{\mathrm{a}}$ as notation for negative polarity. Binary connectives are tensor and par, $\otimes, \parr$, while units are denoted by $\mathbf{1}$ and $\perp$, respectively.

Theorems of multiplicative linear logic are derivable formulas in the calculus MLL.

Definition 2.1.6 (MLL)**.** The calculus MLL is the one-sided sequent calculus for multiplicative linear logic whose rules are given in Firure 2.2

2.1.2 Deep Inference

Recent development of the paradigm of deep inference in proof theory has proven itself as a meaningful effort towards the problem of identity of proofs. It is argued (Guglielmi et.al.) that deep inference manages

- to shorten proofs, even by an exponential factor, compared to shallow inference, with no loss of analyticity ;
- to express logics that cannot be expressed in shallow-inference formalisms ;
- to make the proof theory of a really vast range of logics very regular and modular ;
- to design very simple deductive systems for logics that only have awkward ones in shallow inference ;
- to design deductive systems whose inference rules are all local, which is usually impossible in shallow inference ;
- to recover a De Morgan-like premiss-conclusion symmetry that is usually lost in shallow inference ;
- to obtain new notions of normalisation for proofs, in addition to cut elimination ;
- to inspire a new generation of proof nets and semantics of proofs ;
- to attack the problem of identity of proofs.

The systems we have considered so far, LK, CL, MLL are characterized by *shallow* rules, i.e. a formula in the conclusion of a rule is either present in the premise, or the subformulas obtained by removing the outermost connective are present in the premise. In a deep inference deductive system rules are applied deep inside a formula.

Associativity

$$(S \vee P) \vee Q = S \vee (P \vee Q)$$
$$(S \wedge P) \wedge Q = S \wedge (P \wedge Q)$$

Commutativity

$$S \vee P = P \vee S$$
$$S \wedge P = P \wedge S$$

Units

$$\top \wedge \top = \top \qquad \top \wedge S = S$$
$$\bot \vee \bot = \bot \qquad \bot \vee S = S$$

Negation

$$\overline{\bot} = \top$$
$$\overline{\top} = \bot$$

Context closure

$$\text{if } S = P \text{ then } \begin{array}{l} R\{S\} = R\{P\} \\ \overline{S} = \overline{P} \end{array}$$

$$\overline{(S \vee P)} = \overline{S} \wedge \overline{P}$$
$$\overline{(S \wedge P)} = \overline{S} \vee \overline{P}$$
$$\overline{\overline{S}} = S.$$

Formula contexts, denoted by S{ }, are formulas with one occurrence of { }, the empty context or hole, that does not appear in the scope of a negation. S{R} denotes the formula obtained by filling the hole in S{ } with R. Formulas are (syntactically) equivalent modulo the smallest equivalence relation induced by syntactic associativity and symmetry in the calculus. As usual, formulas are in negation normal form if negation occurs only on propositional variables. Units for disjunction and conjunction are made implicit through the equational system at the level of formulas.

The dual of an inference rule is obtained by exchanging premise and conclusion and replacing each connective by its De Morgan dual. In a deep inference system one considers derivations that are derivation trees with a single branch, whose leaves need not be rule instances without premisses. In full generality one usually allows both, a rule and its dual.

Definition 2.1.7 (KS/SKS)**.** The deep inference symmetric derivation system [9] SKS is a derivation system whose rules are given in Figure 2.3.

By removing the $\uparrow$ rules, and keeping only their duals, $\downarrow$ rules, and by asking that a derivation begins by a rule without a premise, one obtains a (cut-free) proof system KS.

A body of work in deep inference is substantial by now, whose overview is kept and maintained by Guglielmi [10]. Brünnler's Ph.D. thesis [Brü03] is the reference for classical deep inference systems.

2.2 Categorical Concepts

This work assumes the reader's familiarity with some of the basic concepts of category theory, functors, natural transformations, adjunctions, etc. The definitions we give in this

9. the calculus has appeared for the first time in [Brü03]
10. http ://alessio.guglielmi.name/res/cos/#Papers

$$\frac{\mathrm{S}\{\top\}}{\mathrm{S}\{\overline{\mathrm{a}} \vee \mathrm{a}\}}\ \mathsf{ai}\downarrow \qquad \frac{\mathrm{S}\{\overline{\mathrm{a}} \wedge \mathrm{a}\}}{\mathrm{S}\{\bot\}}\ \mathsf{ai}\uparrow$$

$$\frac{\mathrm{S}\{(\mathrm{A} \vee \mathrm{B}) \wedge \mathrm{C}\}}{\mathrm{S}\{(\mathrm{A} \wedge \mathrm{C}) \vee \mathrm{B}\}}\ \mathsf{s}$$

$$\frac{\mathrm{S}\{(\mathrm{A} \wedge \mathrm{B}) \vee (\mathrm{C} \wedge \mathrm{D})\}}{\mathrm{S}\{(\mathrm{A} \vee \mathrm{C}) \wedge (\mathrm{B} \vee \mathrm{D})\}}\ \mathsf{m}$$

$$\frac{\mathrm{S}\{\bot\}}{\mathrm{S}\{\mathrm{a}\}}\ \mathsf{aw}\downarrow \qquad \frac{\mathrm{S}\{\mathrm{a}\}}{\mathrm{S}\{\top\}}\ \mathsf{aw}\uparrow$$

$$\frac{\mathrm{S}\{\mathrm{a} \vee \mathrm{a}\}}{\mathrm{S}\{\mathrm{a}\}}\ \mathsf{ac}\downarrow \qquad \frac{\mathrm{S}\{\mathrm{a}\}}{\mathrm{S}\{\mathrm{a} \wedge \mathrm{a}\}}\ \mathsf{ac}\uparrow$$

FIGURE 2.3 – System SKS

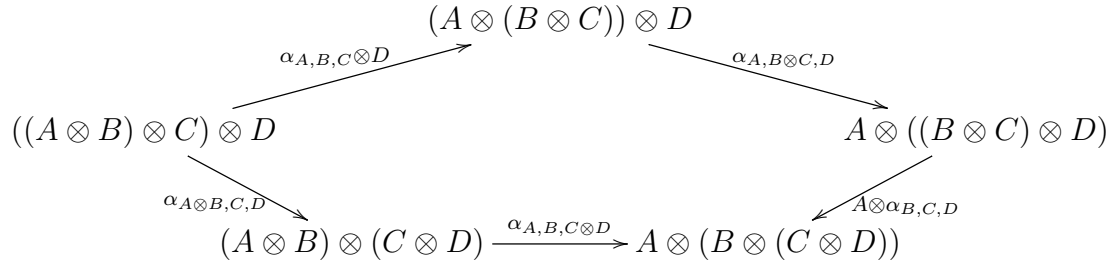
section repeat only few of the many standard notions relevant for this work with the benefit of fixing the notation. For other basic concepts of category theory, the reader is referred to the canonical reference, Mac Lane's classical text [Mac71].

This section contains proofs of some of the properties of the structures we consider. While most of the properties are well-known, providing references to their proofs is surprisingly difficult. Finding a reference containing *all* or *most* of those in one place is even more challenging, if not impossible. That is why we devise our own proofs of the claims we make here.

Definition 2.2.1. A *monoidal category* is a category $\mathbf{C}$ equipped with

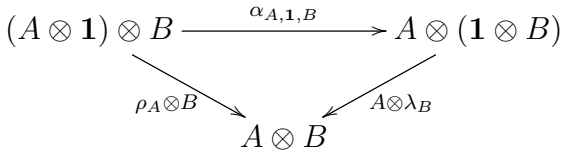
- a bifunctor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called the *tensor product* or *monoidal product*,
- an object $\mathbf{1}$ called the *unit object* or *identity object*,
- three natural isomorphisms subject to certain coherence conditions expressing the fact that the tensor operation
  - · is associative : there is a natural isomorphism α, called *associator*, with components $\alpha_{A,B,C}\colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$,
  - · has $\mathbf{1}$ as left and right identity : there are two natural isomorphisms λ and ρ, respectively called *left and right unitor*, with components $\lambda_A\colon \mathbf{1} \otimes A \cong A$ and $\rho_A\colon A \otimes \mathbf{1} \cong A$.

  The coherence conditions for these natural transformations are :
- for all A, B, C and D in $\mathbf{C}$, the diagram



commutes ;

- for all A and B in $\mathbf{C}$, the diagram



commutes.

The three conditions imply that any diagram whose morphisms are built using α, λ, ρ, identities and tensor product, i.e. any map which is denoted by a well-formed term built out of these symbols, commutes. This is Mac Lane's *coherence theorem* [Mac71], frequently formulated in contexts where a categorical axiomatization is given, stated simply as 'all diagrams commute'.

Definition 2.2.2. Let $\mathbf{C}$ be a monoidal category and $A_1, \ldots, A_n$ be an ordered list of n objects of $\mathbf{C}$. A *bracketing* β of this family is a formula built with $(-) \otimes (-)$ and $\mathbf{1}$, in which every one of the A_i appears as a formal variable exactly once, and in the order given by their numbering.

Obviously the monoidal structure on $\mathbf{C}$ defines by induction a single object of $\mathbf{C}$ which we denote $\beta(A_1, \ldots, A_n)$, obtained by substitution of the real object for the formal variables.

Also, the associativity iso can be seen as a map between (the objects defined by) two bracketings of the 3-element list A, B, C. The two unit isos are seen as maps between (the objects defined by) two bracketings of A. Thus, by induction, any combination of the monoidal isos by the usual functorial operations as well as ordinary composition can be seen as a map between two bracketings of the same list of objects of $\mathbf{C}$.

The Mac Lane coherence theorem for monoidal categories now formulated as to say :

Theorem 2.2.3 (Coherence in Monoidal Categories)**.** *Given an ordered list $A_1, \ldots, A_n$ and two bracketings β, β' of it, there always exist a* unique *map $\beta(A_1, \ldots, A_n) \to \beta(A_1, \ldots, A_n)$ in $\mathbf{C}$, whatever combination of monoidal isos is used to construct it.*

A *strict monoidal category* is one for which the natural isomorphisms α, λ and ρ are identities. Every monoidal category is monoidally equivalent to a strict monoidal category.

A *symmetric monoidal* category is a monoidal category with a natural transformation $\sigma_{A,B} : A \otimes B \cong B \otimes A$ such that

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\sigma_{A,B}} & A \otimes B \\ & \searrow^{\mathrm{Id}_{A\otimes B}} & \downarrow{\scriptstyle \sigma_{A,B}} \\ & & A \otimes B \end{array}$$

$$\begin{array}{ccccc} \mathbf{1} \otimes A & & \xrightarrow{\sigma_{\mathbf{1},A}} & & A \otimes \mathbf{1} \\ & \searrow^{\lambda_A} & & \swarrow_{\rho_A} & \\ & & A & & \end{array}$$

$$\begin{array}{ccccc} (C \otimes A) \otimes B & \xrightarrow{\sigma_{C,A}\otimes \mathrm{Id}_B} & (A \otimes C) \otimes B & \xrightarrow{\alpha_{A,C,B}} & A \otimes (C \otimes B) \\ \downarrow{\scriptstyle \alpha_{C,A,B}} & & & & \downarrow{\scriptstyle \mathrm{Id}_A \otimes \sigma_{C,D}} \\ C \otimes (A \otimes B) & \xrightarrow{\sigma_{C,A\otimes B}} & (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) \end{array}$$

Coherence theorem for symmetric monoidal categories is also due to Mac Lane (in fact, it was the first ever coherence theorem proved, in [Lan63]). To state it for the symmetric case, we have to add the concept of a bracketing with permutation.

Definition 2.2.4. Let $A_1, \dots A_n$ be n objects of $\mathbf{C}$. A *bracketing with permutation* (σ, β) is given by a permutation σ of the set $\{1, \dots, n\}$ and a bracketing β of the list

$$A_{\sigma(1)}, A_{\sigma(2)}, \dots, A_{\sigma(n)}.$$

The notation used to denote this list is $(A_1, A_2, \dots, A_n)^\sigma$.

The symmetry iso can be seen as a map

$$(- \otimes -)(A, B) \to (- \otimes -)(A, B)^{\tau_{12}}$$

where $(- \otimes -)$ is the trivial bracketing for a 2-element list of formal variables and τ_{12} is standard notation for the transposition that exchanges the first and the second element of the list. Moreover, we see by induction that any iso constructed with the symmetric monoidal natural isos induces an expression of the form

$$\beta(A_1, \dots, A_n) \to \beta'(A_1, \dots, A_n)^\sigma$$

where $A_1, \dots, A_n$ is a list of objects, β a bracketing of it and (σ, β') a bracketing with permutations of it.

Theorem 2.2.5 (Coherence in Symmetric Monoidal Categories)**.** *Given a family of objects $A_1, \dots, A_n$ in a symmetric monoidal category $\mathbf{C}$, two bracketings β, β' and a permutation σ, there exist a combination of symmetric monoidal isos that induces*

$$\beta(A_1, \dots, A_n) \to \beta'(A_1, \dots, A_n)^\sigma$$

and it defines a unique *morphism of $\mathbf{C}$.*

The categories we consider will be symmetric monoidal, and we stick to the convention of systematic omitting of explicit mention of the symmetry, except where necessary.

Definition 2.2.6. Let $(\mathbf{C}, \otimes, \mathbf{1_C})$ and $(\mathbf{D}, \odot, \mathbf{1_D})$ be monoidal categories. A *monoidal structure preserving functor*, from $\mathbf{C}$ to $\mathbf{D}$ consists of a functor $F : \mathbf{C} \to \mathbf{D}$ together with a natural isomorphism

$$\phi_{A,B} : F(A \otimes B) \to FA \odot FB$$

and an isomorphism

$$\phi : F\mathbf{1_C} \to \mathbf{1_D}$$

called the coherence maps or structure morphisms, which are such that for every three objects A, B and C of $\mathbf{C}$ the diagrams

$$\begin{array}{ccc} F((A \otimes B) \otimes C) & \xrightarrow{F\alpha_{\mathbf{C}}} & F(A \otimes (B \otimes C)) \\ \downarrow{\scriptstyle \phi_{A\otimes B,C}} & & \downarrow{\scriptstyle \phi_{A,B\otimes C}} \\ F(A \otimes B) \odot FC & & FA \odot F(B \otimes C) \\ \downarrow{\scriptstyle \phi_{A,B}\odot \mathrm{Id}_{FC}} & & \downarrow{\scriptstyle \mathrm{Id}_{FA}\odot\phi_{B,C}} \\ (FA \odot FB) \odot FC & \xrightarrow{\alpha_{\mathbf{D}}} & FA \odot (FB \odot FC) \end{array}$$

$$\begin{array}{ccc} F(A \otimes \mathbf{1_C}) & \xrightarrow{F\rho_{\mathbf{C}}} & FA \\ \downarrow{\scriptstyle \phi_{A,\mathbf{1_C}}} & & \uparrow{\scriptstyle \rho_{\mathbf{D}}} \\ FA \odot F\mathbf{1_C} & \xrightarrow{\mathrm{Id}_{FA}\odot\phi} & FA \odot \mathbf{1_D} \end{array} \quad \text{and} \quad \begin{array}{ccc} F(\mathbf{1_C} \otimes B) & \xrightarrow{F\lambda_{\mathbf{C}}} & FB \\ \downarrow{\scriptstyle \phi_{\mathbf{1_C},B}} & & \uparrow{\scriptstyle \lambda_{\mathbf{D}}} \\ F\mathbf{1_C} \odot FB & \xrightarrow{\phi\odot\mathrm{Id}_{FB}} & \mathbf{1_D} \odot FB \end{array}$$

commute.

Note : this terminology is slightly nonstandard but particularly clear and dispenses us from dealing with the lax (monoidal) and oplax (comonoidal) functors which we don't need anyway.

A *symmetric monoidal structure preserving functor*, is a monoidal structure preserving functor from $\mathbf{C}$ to $\mathbf{D}$ where, in addition,

$$\begin{array}{ccc} F(A \otimes B) & \xrightarrow{F\gamma_{A,B}} & F(B \otimes A) \\ \downarrow{\scriptstyle \phi_{A,B}} & & \downarrow{\scriptstyle \phi_{B,A}} \\ FA \odot FB & \xrightarrow{\gamma_{FA,FB}} & FB \odot FA \end{array}$$

commutes for every two objects A, B of $\mathbf{C}$.

Definition 2.2.7. A covariant or a contravariant endofunctor F on a category $\mathbf{C}$ is *involutive* if for every object A there is a natural isomorphism

$$A \cong F(F(A)).$$

In this work all involutive functors are contravariant.

Proposition 2.2.8. *An involutive functor is self-adjoint.*

Démonstration. Assume the case of a contravariant functor $F : \mathbf{C}^{op} \to \mathbf{C}$, the covariant case is similar. Given isos $i_A : F(F(A)) \to A$ and $i_B : F(F(B)) \to B$ in $\mathbf{C}$, one defines a map Ψ in $\mathbf{Set}$, $\Psi : Hom_{\mathbf{C}}(F(A), B) \to Hom_{\mathbf{C}^{op}}(A, F(B))$ to be $\Psi : f \mapsto i_A \circ F(f)$. Inverse of Ψ is now, up to an automorpism of $Hom(F(A), B)$, given by $\Psi^{-1} : f \mapsto i_B \circ F(f)$, since $\Psi^{-1} \circ \Psi(g)$ is $i_B \circ F(F(g)) \circ F(i_A)$. Naturality of bijection follows from functoriality of F and naturality of the family i. □

Definition 2.2.9. A **-autonomous category* is a symmetric monoidal category $\mathbf{C}$ together with an involutive functor $(-)^\perp : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ such that for every three objects A, B and C there is a natural bijection

$$\mathrm{Hom}(A \otimes B, C^\perp) \cong \mathrm{Hom}(A, (B \otimes C)^\perp).$$

In a *-autonomous category one defines biendofunctor $\multimap$ by

$$A \multimap B \ := \ (A \otimes B^\perp)^\perp.$$

Involutivity of the functor $(-)^\perp$ allows an equivalent formulation of the natural bijection of the hom sets :

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, B \multimap C).$$

This condition precisely states that a *-autonomous category is *monoidal closed*, with $\multimap$ as the *internal hom functor*. As it is customary, we introduce notation

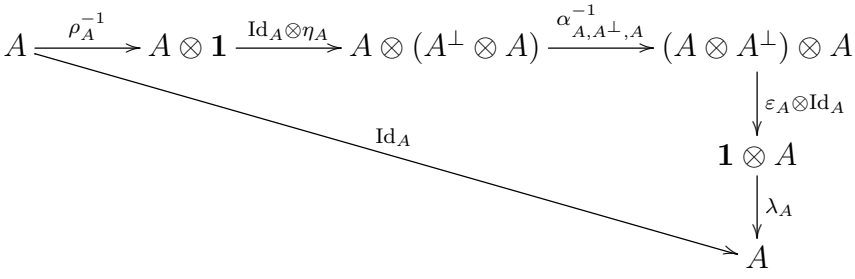
$$A \parr B \ := \ (A^\perp \otimes B^\perp)^\perp$$

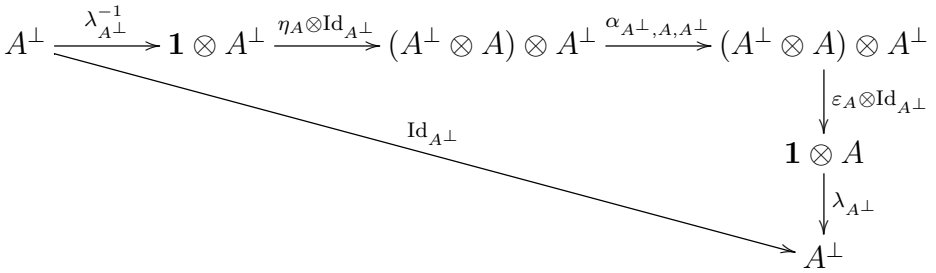
and

$$\perp := \mathbf{1}^\perp.$$

Notice that a *-autonomous category is defined to be $\otimes$-monoidal, and the previous definitions entail that it is also $\parr$-monoidal.

Definition 2.2.10. An object $A^\perp$ of a symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$ is called the *dual* of A if it is equipped with two morphisms called the *unit* $\eta_A : \mathbf{1} \to A^\perp \otimes A$ and the *counit* $\varepsilon_A : A \otimes A^\perp \to \mathbf{1}$, for which the diagrams below commute :





Definition 2.2.11. A symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$ is *compact closed* if every object $A \in \mathbf{C}$ has a dual object.

The following claims can be found in [KL80]. The language used there differs slightly from ours, and some of the proofs are omitted, which is the reason for proving the claims here ourselves.

Proposition 2.2.12. *Let* $\mathbf{C}$ *be a compact closed category. The following holds :*

1. *the dual object of an object in* $\mathbf{C}$ *is unique up to a canonical isomorphism*
2. *there is a canonical involutive functor* $(-)^\perp : \mathbf{C}^{op} \to \mathbf{C}$ *which sends an object to its dual*
3. *for every two objects* A, B *of* $\mathbf{C}$ *one has isomorphism*

$$(A \otimes B)^\perp \cong A^\perp \otimes B^\perp$$

   which is natural in A, B

4. $\mathbf{C}$ *is *-autonomous*

Démonstration. We show the claim in point 3. (without naturality in A, B) prior to showing the point 2. as the statement in 3. it is used in the proof of 2. The naturality is shown when showing 2.

1. Assume two duals A^* and A^+ of A in $\mathbf{C}$, with η, ε and η', ε' two different unit/counit pairs, respectively. Now $\xi : A^* \to A^+$ defined as the composition

$$A^* \xrightarrow{\lambda^{-1}_{A^*}} \mathbf{1} \otimes A^* \xrightarrow{\eta' \otimes \mathrm{Id}_{A^*}} (A^+ \otimes A) \otimes A^* \xrightarrow{\alpha} A^+ \otimes (A \otimes A^*) \xrightarrow{\mathrm{Id}_{A^+} \otimes \varepsilon} A^+ \otimes \mathbf{1} \xrightarrow{\rho_{A^+}} A^+$$

   is the canonical iso between the duals whose inverse is

$$A^+ \xrightarrow{\lambda^{-1}_{A^+}} \mathbf{1} \otimes A^+ \xrightarrow{\eta \otimes \mathrm{Id}_{A^+}} (A^* \otimes A) \otimes A^+ \xrightarrow{\alpha} A^* \otimes (A \otimes A^+) \xrightarrow{\mathrm{Id}_{A^*} \otimes \varepsilon'} A^* \otimes \mathbf{1} \xrightarrow{\rho_{A^*}} A^*.$$

   To check that this is indeed the case, because of the symmetry of the problem, it suffice to show that the first map post-composed with the second one gives the identity on A^*. Using the functoriality of tensor and the Mac Lane's triangle, the composition of the two rewrites to

$$A^* \xrightarrow{\lambda_{A^*}^{-1}} \mathbf{1} \otimes A^* \xrightarrow{\eta \otimes \mathrm{Id}_{A^*}} (A^* \otimes A) \otimes A^* \xrightarrow{(\mathrm{Id}_{A^*} \otimes \rho_A^{-1}) \otimes \mathrm{Id}_{A^*}}$$
$$(A^* \otimes (A \otimes \mathbf{1})) \otimes A^* \xrightarrow{(\mathrm{Id}_{A^*} \otimes (\mathrm{Id}_A \otimes \eta')) \otimes \mathrm{Id}_{A^*}} (A^* \otimes (A \otimes (A^+ \otimes A))) \otimes A^* \xrightarrow{(\mathrm{Id}_{A^*} \otimes (\mathrm{Id}_A \otimes \alpha^{-1})) \otimes \mathrm{Id}_{A^*}}$$
$$(A^* \otimes ((A \otimes A^+) \otimes A)) \otimes A^* \xrightarrow{(\mathrm{Id}_{A^*} \otimes (\mathrm{Id}_A \otimes \varepsilon')) \otimes \mathrm{Id}_{A^*}} (A^* \otimes (\mathbf{1} \otimes A)) \otimes A^* \xrightarrow{(\mathrm{Id}_{A^*} \otimes \lambda_A) \otimes \mathrm{Id}_{A^*}}$$
$$(A^* \otimes A) \otimes A^* \xrightarrow{\alpha} A^* \otimes (A \otimes A^*) \xrightarrow{\mathrm{Id}_{A^*} \otimes \varepsilon} A^* \otimes \mathbf{1} \xrightarrow{\rho_{A^*}} A^*$$

which composes to the identity by definition of the dual object applied on A, A^+ and subsequently on A, A^*.

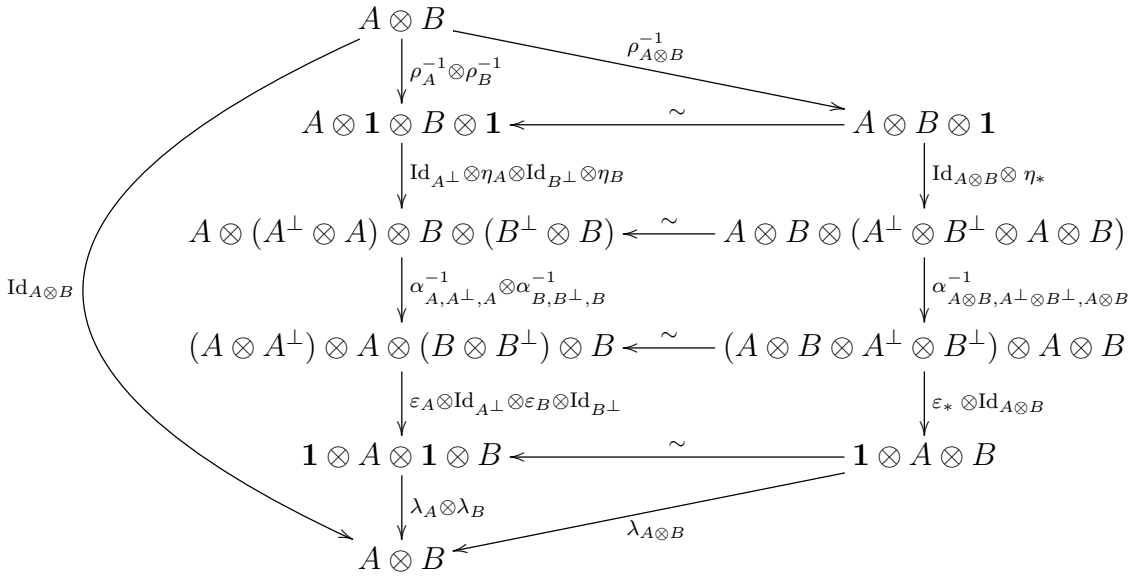
3. Define $\eta_* : \mathbf{1} \to A^\perp \otimes B^\perp \otimes A \otimes B$ as

$$\eta_* = \mathrm{Id}_{A^\perp} \otimes \sigma_{A,B^\perp} \otimes \mathrm{Id}_B \circ \eta_A \otimes \eta_B \circ \rho_{\mathbf{1}}^{-1},$$

and $\varepsilon_* : A \otimes B \otimes A^\perp \otimes B^\perp \to \mathbf{1}$ to be

$$\varepsilon_* = \lambda_{\mathbf{1}} \circ \varepsilon_A \otimes \varepsilon_B \circ \mathrm{Id}_A \otimes \sigma_{B,A^\perp} \otimes \mathrm{Id}_{B^\perp}.$$

Observe now the diagram



The isos written generically are the combinations of maps of a monoidal category, easily deduced from the domains/codomains. The left part of the diagram commutes from definition of duals for A, B and bifunctoriality of $\otimes$, the triangles to the right and the middle square commute from the general coherence result in a monoidal category of Mac Lane. The upper and the lower squares commute from the definitions of η_* and ε_*. The diagram shows that $A^\perp \otimes B^\perp$ is also a dual of $A \otimes B$, so the claim follows from the point 1.

2. Define $(A)^\perp = A^\perp$, and for $f : A \to B$, $(f)^\perp$ to be the composition

$$f^\perp : B^\perp \to A^\perp = \rho_{A_\perp} \circ \mathrm{Id}_{A^\perp} \otimes \varepsilon_B \circ \alpha_{A^\perp,B,B^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes f) \otimes \mathrm{Id}_{B^\perp} \circ \eta_A \otimes \mathrm{Id}_{B^\perp} \circ \lambda_{B^\perp}^{-1}.$$

Taking f to be Id_A, the definition of $(\mathrm{Id}_A)^\perp$ reduces to the second equation in definition of a dual object and is equal to $\mathrm{Id}_{A^\perp}$. For two maps $f : A \to B$ and $g : B \to C$ consider now $(f)^\perp \circ (g)^\perp$. By definition, this is

$$\begin{aligned}
&\rho_{A_\perp} \circ \mathrm{Id}_{A^\perp} \otimes \varepsilon_B \circ \alpha_{A^\perp,B,B^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes f) \otimes \mathrm{Id}_{B^\perp} \circ \eta_A \otimes \mathrm{Id}_{B^\perp} \circ \lambda^{-1}_{B^\perp} \circ \\
&\circ \rho_{B_\perp} \circ \mathrm{Id}_{B^\perp} \otimes \varepsilon_C \circ \alpha_{B^\perp,C,C^\perp} \circ (\mathrm{Id}_{B^\perp} \otimes f) \otimes \mathrm{Id}_{C^\perp} \circ \eta_B \otimes \mathrm{Id}_{C^\perp} \circ \lambda^{-1}_{C^\perp} \\
&= \lambda_{A^\perp} \circ \mathrm{Id}_{A^\perp} \otimes \varepsilon_C \circ \lambda_{A^\perp,C,C^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes g) \otimes \mathrm{Id}_{C^\perp} \circ \\
&\circ (\mathrm{Id}_{A^\perp} \otimes \lambda_B) \otimes \mathrm{Id}_{C^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes (\varepsilon_B \otimes \mathrm{Id}_B)) \otimes \mathrm{Id}_{C^\perp} \circ \\
&\circ (\mathrm{Id}_{A^\perp} \otimes \alpha_{B,B^\perp,B}) \otimes \mathrm{Id}_{C^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes \mathrm{Id}_B \otimes \eta_B) \otimes \mathrm{Id}_{C^\perp} \circ \\
&\circ (\mathrm{Id}_{A^\perp} \otimes \rho_B^{-1}) \otimes \mathrm{Id}_{C^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes f) \otimes \mathrm{Id}_{C^\perp} \circ \eta_A \otimes \mathrm{Id}_{C^\perp} \circ \lambda^{-1}_{C^\perp} \\
&= \rho_{A_\perp} \circ \mathrm{Id}_{A^\perp} \otimes \varepsilon_C \circ \alpha_{A^\perp,C,C^\perp} \circ (\mathrm{Id}_{A^\perp} \otimes (g \circ f)) \otimes \mathrm{Id}_{C^\perp} \circ \eta_A \otimes \mathrm{Id}_{C^\perp} \circ \lambda^{-1}_{C^\perp} \\
&= (g \circ f)^\perp
\end{aligned}$$

The first equation follows from functoriality of $\otimes$, naturality of monoidal isos and the Mac Lane triangle in the definition of a monoidal category. The second one is definition of a dual object for $B^\perp$. This shows that $(-)^\perp$ indeed defines a functor. The involutivity of the functor reduces to the issue of naturality of the isomorphism $A \cong ((A)^\perp)^\perp$, since the involutivity isomorphism itself is established by the uniqueness of the dual for $A^\perp$. Naturality follows from functoriality of $(-)^\perp$.

Notice that, as a consequence, the iso in the previous point of this proposition is natural in A, B.

4. From the point 2. we have the involutive functor, what is left to show is the *-autonomous adjunction in a compact closed category. So, let $f : A \otimes B \to C^\perp$ be a map in $\mathbf{C}$. To this map we assign $\Psi(f)$, the composition

$$A \xrightarrow{\lambda_A^{-1}} \mathbf{1} \otimes A \xrightarrow{\eta_B \otimes \mathrm{Id}_A} (B^\perp \otimes B) \otimes A \xrightarrow{\sim} B^\perp \otimes (A \otimes B) \xrightarrow{\mathrm{Id}_{B^\perp} \otimes f} B^\perp \otimes C^\perp \xrightarrow{\sim} (B \otimes C)^\perp.$$

The middle iso is a composition of associativity and symmetry isos,

$$(B^\perp \otimes B) \otimes A \xrightarrow{\alpha_{B^\perp,B,A}} B^\perp \otimes (B \otimes A) \xrightarrow{\mathrm{Id}_B \otimes \sigma_{B,A}} B^\perp \otimes (A \otimes B),$$

while the last iso is established in points 2. and 3. Given $h : A \to (B \otimes C)^\perp$, we define $\Psi^{-1}(h)$ as the composition

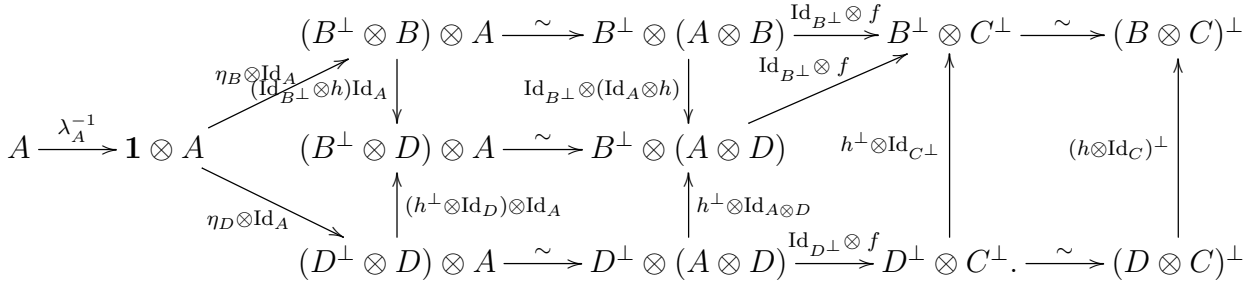
$$A \otimes B \xrightarrow{h \otimes \mathrm{Id}_B} (B \otimes C)^\perp \otimes B \xrightarrow{\sim} (B^\perp \otimes C^\perp) \otimes B \xrightarrow{\sim} (B \otimes B^\perp) \otimes C^\perp \xrightarrow{\varepsilon_B \otimes \mathrm{Id}_{C^\perp}} \mathbf{1} \otimes C^\perp \xrightarrow{\lambda} C,$$

where the first generically notated iso is the inverse of the last iso in the definition of $\Psi(f)$, while the second one is

$$(B^\perp \otimes C^\perp) \otimes B \xrightarrow{\sigma} B \otimes (B^\perp \otimes C^\perp) \xrightarrow{\alpha^{-1}} (B \otimes B^\perp) \otimes C^\perp.$$

That Ψ^{-1} is an inverse to Ψ is easily verifiable from the functoriality of tensor and definition of dual objects. Naturality of the bijection Ψ in A, C follows from

naturality of λ, α, σ and the last iso, as established in the previous points. To show naturality in B, one has to show that given $h : B \to D$ and $f : A \otimes D \to C$ on has $\Psi(f \circ \mathrm{Id}_A \otimes h) = (h \otimes \mathrm{Id}_C)^\perp \circ \Psi(f)$. This is shown by chasing



Commutativity of the rightmost square is the naturality established in previous points, the triangle commutes from the assumption on f, h, the quadrilateral underneath commutes from the bifunctoriality of $\otimes$, while commutativity of the two squares to the left follows from naturality of associativity and symmetry. The fact that the leftmost quadrilateral commutes is a consequence of the the equation

$$\begin{aligned} h^\perp \otimes \mathrm{Id}_D \circ \eta_D &= (\mathrm{Id}_{B^\perp} \otimes \varepsilon_D \circ \alpha^{-1}_{B^\perp, D, D^\perp} \circ (\mathrm{Id}_{B^\perp} \otimes h) \otimes \mathrm{Id}_{D^\perp} \\ &\qquad \circ \eta_B \otimes \mathrm{Id}_{D^\perp} \circ \lambda^{-1}_{D^\perp}) \otimes \mathrm{Id}_D \circ \eta_D \\ &= \mathrm{Id}_{B^\perp} \otimes h \circ \eta_B \end{aligned}$$

which is definition of the functor $(-)^\perp$ combined with functoriality of $\otimes$ and definition of the dual for D.

□

From the previous proposition it follows that a compact closed category is monoidal closed, and one can establish a canonical isomorphism

$$A \multimap \perp = = = (A \otimes \perp^\perp)^\perp \xrightarrow{\sim} (A \otimes \mathbf{1})^\perp \xrightarrow{\sim} A^\perp.$$

between the dual $A^\perp$ of an object A and the object $A \multimap \perp$. The presence of the isomorphism is above is usually formulated in saying that $\perp$ is the *dualizing object.*

Mix

Definition 2.2.13. In a *-autonomous category one has a family of maps called *Switch* (some authors call this family weak associativity, or disassociativity), which is natural in all A, B, C

$$s_{A,B,C} : A \otimes (B \parr C) \to (A \otimes B) \parr C, \tag{2.1}$$

obtained by transposing the identity $B \parr C \to B \parr C$ to obtain $C^\perp \otimes (B \parr C) \to B$, combining it with identity $A \to A$ on A by the $- \otimes -$ bifunctor to obtain $A \otimes C^\perp \otimes (B \parr C) \to A \otimes B$. The map is, finally, defined by transposing $C^\perp$ to the right.

In a similar fashion one can obtain maps

$$(A \parr B) \otimes C \to A \parr (B \otimes C), \qquad A \otimes (B \parr C) \to B \parr (A \otimes C), \qquad (A \parr B) \otimes C \to (A \otimes C) \parr B.$$

Alternatively, these could be obtained by pre- or post- composing $s_{A,B,C}$ by symmetries for $\otimes$- and $\parr$- monoidal structures. Since no confusion can occur, we refer to all of these as to *Switch maps*.

Switch is usually said to be self-dual, as its image under the $(-)^{\perp}$ functor is (up to the *-autonomous isos of the domain and codomain) another Switch, i.e. given $s_{A,B,C} : A \otimes (B \parr C) \to (A \otimes B) \parr C$, $s^{\perp}_{A,B,C}$ is the Switch map,

$$((A \otimes B) \parr C)^{\perp} \xrightarrow{\sim} (A^{\perp} \parr B^{\perp}) \otimes C^{\perp} \xrightarrow{s} A^{\perp} \parr (B^{\perp} \otimes C^{\perp}) \xrightarrow{\sim} (A \otimes (B \parr C))^{\perp}.$$

Definition 2.2.14. A *-autonomous category *has Mix* if it has a fixed map

$$Mix_{\perp} : \perp \to \mathbf{1}$$

and for every pair A, B of $\mathbf{C}$ there exist a family of maps

$$Mix_{A,B} : \ A \otimes B \to A \parr B,$$

natural in A and B, such that

$$\begin{array}{ccc} A \otimes B & \xrightarrow{Mix_{A,B}} & A \parr B \\ {\scriptstyle \sigma_{A,B}}\downarrow & & \downarrow{\scriptstyle \sigma_{B,A}} \\ B \otimes A & \xrightarrow[Mix_{B,A}]{} & B \parr A \end{array}$$

and

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\mathrm{Id}_A \otimes Mix_{B,C}} & A \otimes (B \parr C) & \xrightarrow{Mix_{A,B \parr C}} & A \parr (B \parr C) \\ {\scriptstyle \alpha_{A,B,C}}\downarrow & & \downarrow{\scriptstyle s} & & {\scriptstyle \alpha_{A,B,C}}\downarrow \\ (A \otimes B) \otimes C & \xrightarrow{Mix_{A \otimes B,C}} & (A \otimes B) \parr C & \xrightarrow{Mix_{A,B} \parr \mathrm{Id}_C} & (A \parr B) \parr C \end{array}$$

commute.

The following, in a slightly different formulation, can be found in [Lam07, Str07, Str11].

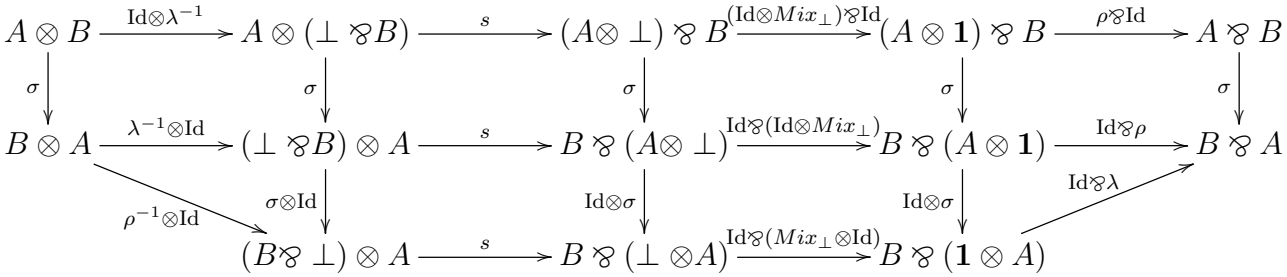
Proposition 2.2.15. *To have Mix in a *-autonomous category it suffice to choose a map $Mix_{\perp} : \perp \to \mathbf{1}$ for which the diagram below commutes. $Mix_{A,B}$ is defined as the diagonal of the diagram.*

$$\begin{array}{ccccc} A \otimes B & \xrightarrow{\mathrm{Id} \otimes \lambda^{-1}} & A \otimes (\perp \parr B) & \xrightarrow{s} & (A \otimes \perp) \parr B \\ {\scriptstyle \rho^{-1} \otimes \mathrm{Id}}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{Id} \otimes Mix_{\perp}) \parr \mathrm{Id}} \\ (A \parr \perp) \otimes B & & & & (A \otimes \mathbf{1}) \parr B \\ {\scriptstyle s}\downarrow & & & & \downarrow{\scriptstyle \rho \parr \mathrm{Id}} \\ A \parr (\perp \otimes B) & \xrightarrow{\mathrm{Id} \otimes (Mix_{\perp} \parr \mathrm{Id})} & A \parr (\mathbf{1} \otimes B) & \xrightarrow{\mathrm{Id} \parr \lambda} & A \parr B \end{array}$$

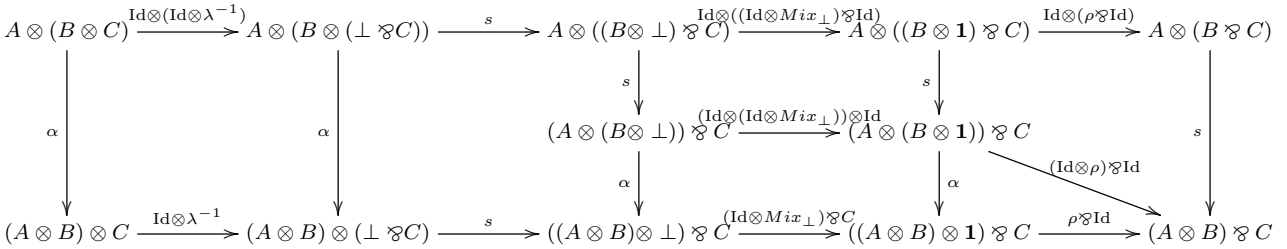
(In the following diagrams, where it does not cause confusion, we omit subscripts in natural families of maps to improve readability.)

Démonstration. Let $Mix_\perp$ be chosen so that the diagram above commutes. The fact that $Mix_{A,B}$ is natural in A, B follows from naturality of $\mathrm{Id}, \alpha, s, \rho$ and bifunctoriality of $\otimes, \parr$.

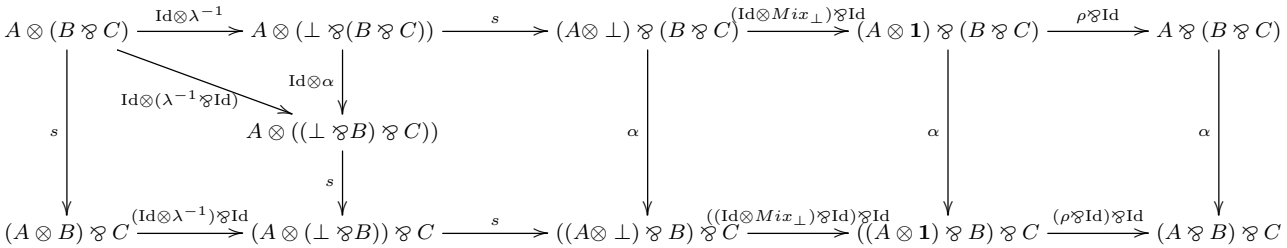
The fact that Mix defined in this way respects follows from the definition of Mix, naturality of σ and definition of Switch that make the diagram below commute.



That Mix defined as above agrees with associativity can be seen by chasing



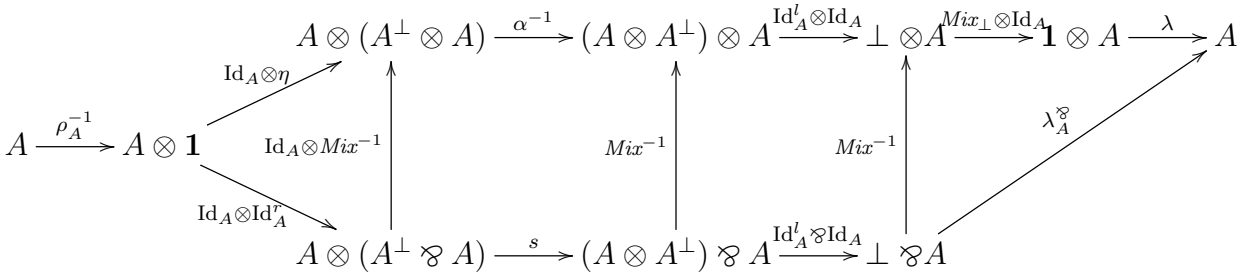
and superposing it along the rightmost vertical map with



The squares involving Switch and $Mix_\perp, \rho, \lambda$ commute because of naturality of Switch, the squares with $Mix_\perp$ and α commute because of naturality of α, while the rest are either commutative squares and triangles in a symmetric monoidal category, or pentagons describing the property of Switch - agreeing with associativity. □

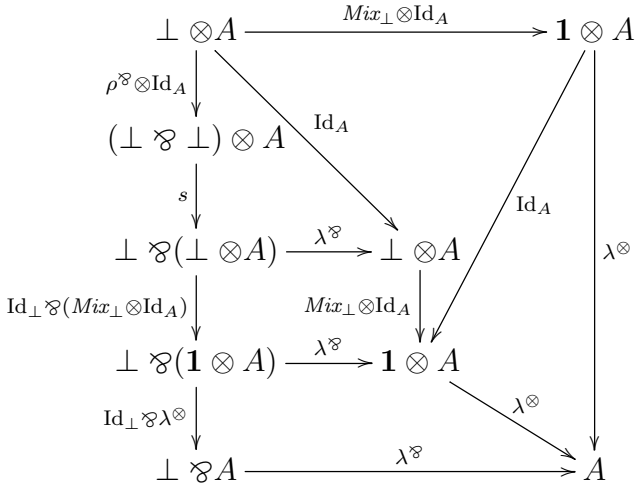
Proposition 2.2.16. *A *-autonomous category with Mix is compact closed if, for every pair of objects $A, B \in \mathbf{C}$, $Mix_{A,B} : A\otimes B \to A\parr B$ is an iso. Conversely, a compact-closed category is a *-autonomous category with the identity as Mix.*

Démonstration. What we show is that for every object A of a *-autonomous category with Mix being an isomorphism, there is a pair of maps ε_A, η_A in the category, witnessing the fact that $A^\perp$, the image of A under the involutive functor of the *-autonomous category, is the dual of A. To that purpose, let $\mathrm{Id}_A^l : A \otimes A^\perp \to \perp$ be one transpose of the identity of A under the *-autonomous adjunction, and let $\mathrm{Id}_A^r : \mathbf{1} \to A^\perp \parr A$ be the other one. Now, define η to be $Mix^{-1} \circ \mathrm{Id}_A^r$ and ε as $Mix_\perp \circ \mathrm{Id}_A^l$. The following diagram commutes.



(2.2)

The map $\lambda_A^\parr$ is the left unitor for the $\parr$-symmetric monoidal structure. The triangle to the right is definition of η, the square next to it commutes from the definition of *Mix*, the one adjacent to it is naturality of *Mix*. The see that the rightmost quadrilateral commutes is suffice to chase



where the quadrilateral involving Switch is a property of the Switch family of maps -- – agreeing with the $\parr$-monoidal unit (cf. [Lam07], Proposition 2.2). The lower composition in 2.2 composes to the identity of A in any *-autonomous category (see e.g. [Hug05, DP06]), which shows one of the two equations required to show that $A^\perp$ is the dual of A, the other one is shown analogously.

To show the converse part of the proposition, the only thing left is to show that a map satisfying the equations for *Mix* is present in a compact-closed category. Thus, define

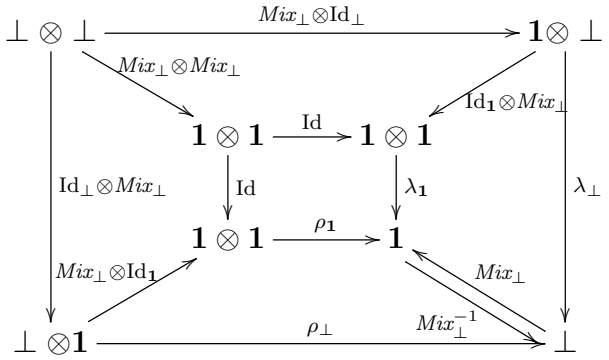
$Mix_\perp$ to be the composition

$$\perp \xrightarrow{\lambda_\perp^{-1}} \mathbf{1}\otimes \perp \xrightarrow{\rho_{\mathbf{1}\otimes\perp}^{-1}} (\mathbf{1}\otimes \perp)\otimes \mathbf{1} \xrightarrow{\varepsilon\otimes\mathbf{1}} \mathbf{1}\otimes\mathbf{1} \xrightarrow{\rho} \mathbf{1}$$

which, using duality of $\mathbf{1}$ and $\perp$, allows definition of the inverse $Mix_\perp^{-1}$ as

$$\mathbf{1} \xrightarrow{\lambda^{-1}} \mathbf{1}\otimes\mathbf{1} \xrightarrow{\mathrm{Id}_\mathbf{1}\otimes\eta} \mathbf{1}\otimes(\perp\otimes\mathbf{1}) \xrightarrow{\alpha} (\mathbf{1}\otimes\perp)\otimes\mathbf{1} \xrightarrow{\rho_{(\mathbf{1}\otimes\perp)}} \mathbf{1}\otimes\perp \xrightarrow{\lambda_\perp} \perp .$$

The fastest way to see that we have defined $Mix_\perp$ for which the diagram of Proposition 2.2.15 commutes is by using an equivalent claim which says that the diagram of Proposition 2.2.15 commutes if and only if the outer square of the diagram below commutes (see e.g. [Str07], Theorem 4.1).



The diagram, however, commutes from functoriality of tensor, naturality of ρ, λ and the fact that in a symmetric monoidal category $\lambda_\mathbf{1} = \rho_\mathbf{1}$, which completes the proof.

□

Since compact-closed categories are *-autonomous they have Switch maps. In the compact-closed case, from the definition of Switch follows that *in a compact-closed category Switch is the associativity map.*

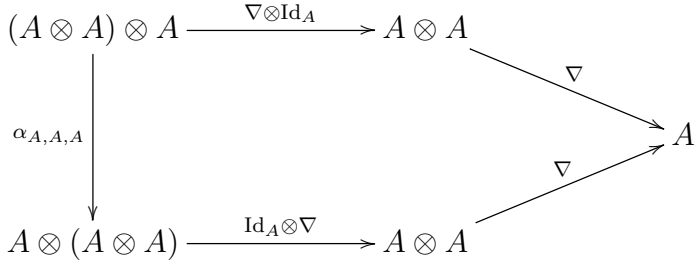
Monoids and comonoids

Definition 2.2.17. A *monoid* (or *monoid object*) $(A, \nabla, \amalg)$ in a monoidal category $\mathbf{C}$ is an object A together with two morphisms [11]

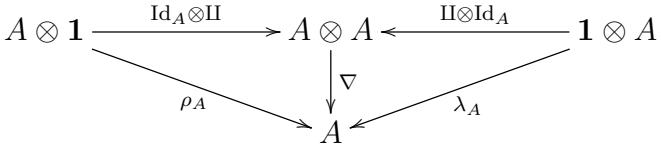
* $\nabla : A\otimes A \to A$ called *multiplication*,
* and $\amalg : \mathbf{1} \to A$ called *unit*,

11. the choice of notation for the monoid structure will be clear soon

such that the diagrams



and



commute. In the above notations, $\mathbf{1}$ is the unit element and α, λ and ρ are respectively the associativity, the left identity and the right identity of the monoidal category $\mathbf{C}$.

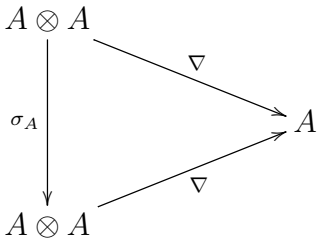
Dually, a *comonoid* (A, Δ, Π) in a monoidal category $\mathbf{C}$ is a monoid in the dual category $\mathbf{C}^{\mathrm{op}}$, with

$$\Delta : A \longrightarrow A \otimes A \qquad \text{and} \qquad \Pi : A \longrightarrow \mathbf{1}.$$

Terminology used is *comultiplication* for Δ and Π is *counit* of the comonoid.

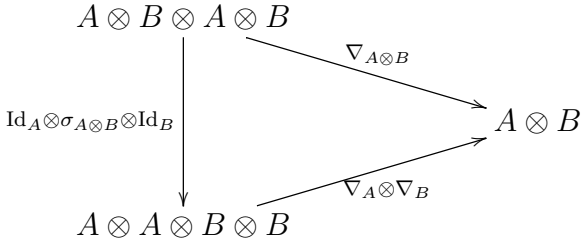
An alternative terminology for the (co)monoid maps, which we adopt throughout this text is *diagonal* for Δ and *co-diagonal* for ∇, *projection* for the counit Π and *co-projection* for the unit $\amalg$.

Suppose that the monoidal category $\mathbf{C}$ has a symmetry σ. A monoid A in $\mathbf{C}$ is *commutative* when

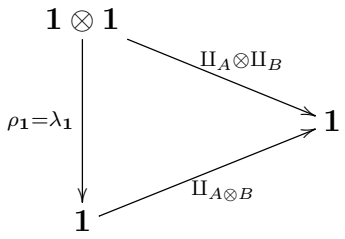


Cocommutativity of a comonoid is defined as commutativity of a monoid in the dual category.

Definition 2.2.18. A symmetric monoidal category $\mathbf{C}$ *has monoids* if there is a chosen symmetric monoid $(A, \nabla_A, \amalg_A)$ for every object A, compatible with the symmetric monoidal structure in the following sense :



(obvious associativity isos are left-out)



$$\amalg_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}} : \mathbf{1} \longrightarrow \mathbf{1}$$

A symmetric monoidal category *has comonoids* if its dual has monoids.

It is worth stressing out that having (co)monoids is not a property of a category, but rather additional structure on it.

It will sometimes be the case that a category $\mathbf{C}$ is equipped with two different symmetric monoidal structures at the same time (as in a *-autonomous category), $(\otimes, \mathbf{1})$ and $(\parr, \bot)$. In this situation, monoids/comonoids can be present in the category w.r.t. these two distinct symmetric monoidal structures, and in that case we also index their names by their respective bifunctors, i.e. we speak of $\otimes$-(co)monoids and $\parr$-(co)monoids.

Notice that in a *-autonomous category, due to the presence of the involutive functor, one has that the category has $\parr$-monoids if and only if it has $\otimes$-comonoids.

Definition 2.2.19. In a *-autonomous category with Mix, which has $\parr$-monoids (thus also $\otimes$-comonoids), given two maps $f, g : A \to B$, one defines the map

$$f + g : A \to B$$

to be the following composition

$$\begin{array}{ccccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{Mix} & A \parr A & & \\ & & \downarrow{\scriptstyle f \otimes g} & & \downarrow{\scriptstyle f \parr g} & & \\ & & B \otimes B & \xrightarrow[Mix]{} & B \parr B & \xrightarrow{\nabla} & B. \end{array}$$

Commutativity of the square is naturality of Mix.

Medial

Definition 2.2.20. A *-autonomous category $\mathbf{C}$ *has (binary) Medial* if there exist a family of maps

$$M_{A,B;C,D} : (A \otimes B) \parr (C \otimes D) \to (A \parr C) \otimes (B \parr D)$$

for every four objects A, B, C and D of $\mathbf{C}$. In addition, the following equations are required to hold :

– $M_{A,B;C,D}$ is natural in all four of A, B, C, D ;

– Medial is self-dual, i.e.

$$\begin{array}{ccc} \overline{(A \parr C) \otimes (B \parr D)} & \xrightarrow{\overline{M}} & (A \otimes B) \parr (C \otimes D) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ (\overline{D} \otimes \overline{B}) \parr (\overline{C} \otimes \overline{A}) & \xrightarrow[M]{} & (\overline{D} \parr \overline{C}) \otimes (\overline{B} \parr \overline{A}) \end{array}$$

– M-α :

$$\begin{array}{ccc} (A \otimes (B \otimes C)) \parr (D \otimes (E \otimes F)) & \xrightarrow{\alpha \parr \alpha} & ((A \otimes B) \otimes C) \parr ((D \otimes E) \otimes F) \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} \\ (A \parr D) \otimes ((B \otimes C) \parr (E \otimes F)) & & ((A \otimes B) \parr (D \otimes E)) \otimes (C \parr F) \\ \downarrow{\scriptstyle \mathrm{Id} \otimes M} & & \downarrow{\scriptstyle M \otimes \mathrm{Id}} \\ (A \parr D) \otimes ((B \parr E) \otimes (C \parr F)) & \xrightarrow[\alpha]{} & ((A \parr D) \otimes (B \parr E)) \otimes (C \parr F) \end{array}$$

– M-σ :

$$\begin{array}{ccc} (A \otimes B) \parr (C \otimes D) & \xrightarrow{\sigma \parr \sigma} & (B \otimes A) \parr (D \otimes C) \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} \\ (A \parr C) \otimes (B \parr D) & \xrightarrow[\sigma]{} & (B \parr D) \otimes (A \parr C) \end{array}$$

– M-s :

$$\begin{array}{ccc} ((A \otimes B) \parr (C \otimes D)) \otimes E & \xrightarrow{s} & (A \otimes B) \parr (D \otimes C \otimes E) \\ \downarrow{\scriptstyle M \otimes \mathrm{Id}} & & \downarrow{\scriptstyle M} \\ (A \parr C) \otimes (B \parr D) \otimes E & \xrightarrow[\mathrm{Id} \otimes s]{} & (A \parr C) \otimes (B \parr (D \otimes E)) \end{array}$$

A *-autonomous category $\mathbf{C}$ *has nullary Medial* if there is a map

$$0M : \mathbf{1} \parr \mathbf{1} \to \mathbf{1}$$

such that for all objects A, B, the following holds

$$\begin{array}{ccccc} & & A \otimes B & & \\ & {\scriptstyle \Pi_A \parr \Pi_B} \swarrow & & \searrow {\scriptstyle \Pi_{A \otimes B}} & \\ \mathbf{1} \parr \mathbf{1} & & \xrightarrow{\quad 0M \quad} & & \mathbf{1}. \end{array}$$

The way we define Medial suits our purposes. Other authors do not require immediately for Medial to agree with the associativity, symmetry and Switch.

Detailed categorical treatment of the Medial transformation is the subject of [Lam07], where the notion of the nullary and binary Medial is generalized to the notion of n, m-ary Medial, defined from the nullary and binary cases. In the work [BFSV03] the family of maps appears in somewhat more general context of n-fold monoidal categories and iterated loop spaces, while [DP07] concerns with coherence in the presence of the transformation, using the name Intermutation for it (in work of Došen and Petrić, sticking to the tradition form universal algebra, Medial is used to denominate a transformation we call Medial with the two bifunctors $\otimes$ and $\parr$ coinciding). We sum-up some of the results of [Lam07] to state :

Proposition 2.2.21. *If a *-autonomous category has (nullary and binary) Medial, then $- \otimes -$ is an oplax-monoidal bifunctor over the $\parr$-monoidal structure.*

Proposition 2.2.22. *Every compact-closed category has (binary and nullary) Medial.*

Démonstration. Defining $M_{A,B;C,D} : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$ to be the composition

$$(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} A \otimes (B \otimes (C \otimes D)) \xrightarrow{\mathrm{Id}_A \otimes \alpha^{-1}} A \otimes ((B \otimes C) \otimes D) \xrightarrow{\mathrm{Id}_A \otimes (\sigma \otimes \mathrm{Id}_D)}$$

$$A \otimes ((C \otimes B) \otimes D) \xrightarrow{\mathrm{Id}_A \otimes \alpha} A \otimes (C \otimes (B \otimes D)) \xrightarrow{\alpha^{-1}} (A \otimes C) \otimes (B \otimes D)$$

one trivially sees that the conditions of Definition 2.2.20 follow from the coherence result for a symmetric monoidal category and the definition of the involutive functor.

Nullary medial is defined by $0M = \lambda_{\mathbf{1}} = \rho_{\mathbf{1}}$. □

Definition 2.2.23. A *-autonomous category with Mix [12], $\parr$-monoids and Medial *has Medial-compatible $\parr$-monoids* if given two objects A and B

i) $\nabla_{A \wedge B} : (A \otimes B) \parr (A \otimes B) \to A \otimes B$ is equal to :

$$(A \otimes B) \parr (A \otimes B) \xrightarrow{M_{A,B;A,B}} (A \parr A) \otimes (B \parr B) \xrightarrow{\nabla_A \parr \nabla_B} A \parr B$$

ii) and $\amalg_{A \wedge B} : \bot \to A \otimes B$ equals

$$\bot \xrightarrow{\sim} \bot \otimes \bot \xrightarrow{\amalg_A \otimes \amalg_B} A \otimes B.$$

By duality, in a *-autonomous category with Medial-compatible $\parr$-monoids one also has $\otimes$-comonoids, and :

iii) $\Delta_{A \vee B} : A \parr B \to (A \parr B) \otimes (A \parr B)$ to be the composition :

$$A \parr B \xrightarrow{\Delta_A \parr \Delta_B} (A \otimes A) \parr (B \otimes B) \xrightarrow{M_{A,A;B,B}} (A \parr B) \otimes (A \parr B)$$

12. It can be shown that the category with Medial also contains Mix. The way we present the material makes the discussion more general, e.g. one may be interested in categories where Mix is present, but Medial is not.

iv) $\Pi_{A \vee B} : A \parr B \to \mathbf{1}$ is equal to the composition :

$$A \parr B \xrightarrow{\Pi_A \parr \Pi_B} \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}.$$

Notice that the previous definition does make sense as stated, since in a free *-autonomous category with Medial-compatible $\parr$-monoids, diagrams of Definition 2.2.18 commute.

Medial first appears in the syntactic context in the work on deep inference systems for classical logic [Brü03, BT01, BG03] as the rule m, and it is present in the systems SKS and KS . The purpose of the rule is the same as in semantics - to propagate structural rules via syntactic transformations to the literals.

We conclude this chapter by making an observation that the syntax for propositional classical logic as we have introduced it in the case of the calculus CL (not the deep inference systems) does not include propositional units. The interpretations of syntax, however, will have monoidal units present, as in the definition of a *-autonomous category. We will say a bit more on this in the chapter on relation to other approaches.

3

Interpretations and Frobenius Algebras

Contents

3.1 Frobenius Algebras

Convention 3.1.1. We will often omit bracketings in large formulas involving tensors. This can always be interpreted as saying we have chosen a "canonical" bracketing like $A \otimes (B \otimes (C \otimes (D \otimes \dots)))$, but more generally any object that can be obtained by a random bracketing can be used to interpret this formula, due to the coherence theorem for monoidal categories. In general we assume that the order of the factors stays the same.

Definition 3.1.2 (Frobenius algebra)**.** Let $(\mathbf{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category (SMC), and A an object of it. A *Frobenius algebra* is a sextuple $(A, \Delta, \Pi, \nabla, \amalg)$ where

- $(A, \nabla, \amalg)$ is a commutative monoid,
- $(A, \Delta, \amalg)$ a co-commutative comonoid,
- the following equations hold :

$$
\begin{array}{ccccc}
A \otimes A & = & A \otimes A & = & A \otimes A \\
{\scriptstyle \Delta \otimes \mathrm{Id}}\downarrow & & {\scriptstyle \nabla}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes \Delta} \\
A \otimes A \otimes A & & A & & A \otimes A \otimes A \\
{\scriptstyle \mathrm{Id} \otimes \nabla}\downarrow & & {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \nabla \otimes \mathrm{Id}} \\
A \otimes A & = & A \otimes A & = & A \otimes A.
\end{array}
$$

As is our custom we use the standard abuse of notation of using the same symbol for the underlying object and the whole Frobenius algebra structure.

It is common to represent the diagram equations using this diagrammatic notation as in Figure 3.1 [Tensoring is represented as the horizontal component of the diagram

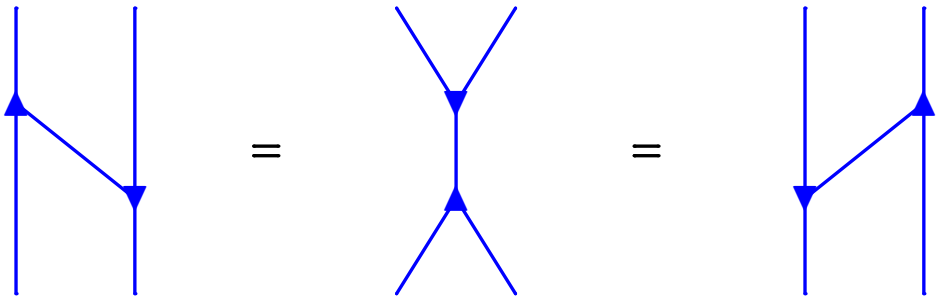


FIGURE 3.1 – String-diagram version of Frobenius equations

and arrow composition is read top-down. The literature on string diagrams is substantial. We refer the reader to a classical text [JS91], while an early amplification of the use of string-diagram notation as an alternative for the traditional index-calculus for tensors is due to Roger Penrose [Pen71]. In the diagram above, first equation is commutativity of the right square, the second one is commutativity of the hexagon.]

What we call a Frobenius algebra is called a symmetric Frobenius algebra in the literature. In this work all Frobenius algebras will be symmetric.

Proposition 3.1.3. *Bimonoids* $(\mathbb{Z}, \leqslant, +, 0, +_c, c)$ *of* $\mathbb{Z}$*-interpretation are Frobenius algebras in the category* $(\mathbf{Cmp}, \times, \{*\})$.

Démonstration. For an object $a = (\mathbb{Z}, \leqslant, +, 0, +_c, c)$, (co)multiplication maps are defined in 1.1. Composition $\Delta \circ \nabla$ is

$$\begin{aligned} \{(t, y_1, y_2) \in \mathbb{Z}^3 \mid\ t \leq y_1 + y_2\} \quad &\circ \quad \{(x_1, x_2, z) \in \mathbb{Z}^3 \mid\ x_1 + x_2 \leq z + C\} \\ &= \quad \{(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4 \mid \exists d \in \mathbb{Z} :\ x_1 + x_2 \leq d + C; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d \leq y_1 + y_2\} \end{aligned}$$

Since $d \in \mathbb{Z}$, one can always choose $d = x_1 + x_2 - C$, and the last relation can be seen as

$$\{(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4 \mid\ x_1 + x_2 \leq y_1 + y_2 + C\}.$$

The composition $\nabla \otimes \mathrm{Id} \circ \mathrm{Id} \otimes \Delta$ is by definition

$$\begin{aligned} \{(t_1, t_2, t_3, y_1, y_2) \in \mathbb{Z}^5 \quad &\mid \quad t_1 + t_2 \leq y_1 + C;\ t_3 \leq y_2\} \\ &\circ \quad \{(x_1, x_2, z_1, z_2, z_3) \in \mathbb{Z}^5 \mid\ x_1 \leq z_1;\ x_2 \leq z_2 + z_3\} \\ &= \quad \{(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4 \mid \exists d_1, d_2, d_3 \in \mathbb{Z} :\ x_1 \leq d_1;\ x_2 \leq d_2 + d_3; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d_1 + d_2 \leq y_1 + C;\ d_3 \leq y_2\} \end{aligned}$$

Again, one can always choose $d_1 = x_1$, $d_3 = y_2$ and d_2 to be $x_2 - d_3$, so the last relation can be seen as

$$\{(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4 \mid \ x_1 + x_2 \leq y_1 + y_2 + C\}.$$

The other equation follows from symmetry. □

Definition 3.1.4. A Frobenius algebra is called *thin* if for all $k \geq 0$

$$\Pi \circ \underbrace{(\nabla \circ \Delta) \circ \cdots \circ (\nabla \circ \Delta)}_{k} \circ \amalg$$

is the identity.

Proposition 3.1.5. *Bimonoids* $(\mathbb{Z}, \leqslant, +, 0, +_c, c)$ *of* $\mathbb{Z}$*-interpretation are* thin *Frobenius algebras in the category* $(\mathbf{Cmp}, \times, \{*\})$.

Démonstration. For the bimonoids in the category, $\Pi \circ \underbrace{(\nabla \circ \Delta) \circ \cdots \circ (\nabla \circ \Delta)}_{k} \circ \amalg$ is the comparison :

$$\begin{aligned}\{(n, *) \mid n \leq 0\} \circ \{(m_1, m_2, m_3) \mid m_1 + m_2 \leq m_3 + c\} \circ \{(l_1, l_2, l_3) \mid l_1 \leq l_2 + l_3\} \circ \ldots \\ \cdots \circ \{(k_1, k_2, k_3) \mid k_1 + k_2 \leq k_3 + c\} \circ \{(j_1, j_2, j_3) \mid j_1 \leq j_2 + j_3\} \circ \{(*, i) \mid c \leq i\}\end{aligned}$$

which is equal to

$$\begin{aligned}\{(*, *) \mid \exists s_1, s_2, s_3, s_4, s_5, s_6, s_7 \in \mathbb{Z} : \\ s_1 \leq 0;\ s_2 + s_3 \leq s_1 + c;\ s_4 \leq s_2 + s_3;\ s_5 + s_6 \leq s_4 + c;\ s_7 \leq s_5 + s_6;\ c \leq s_7\}\end{aligned}$$

while the last relation rewrites to the identity on the single-element set,

$$\{(*, *)\},$$

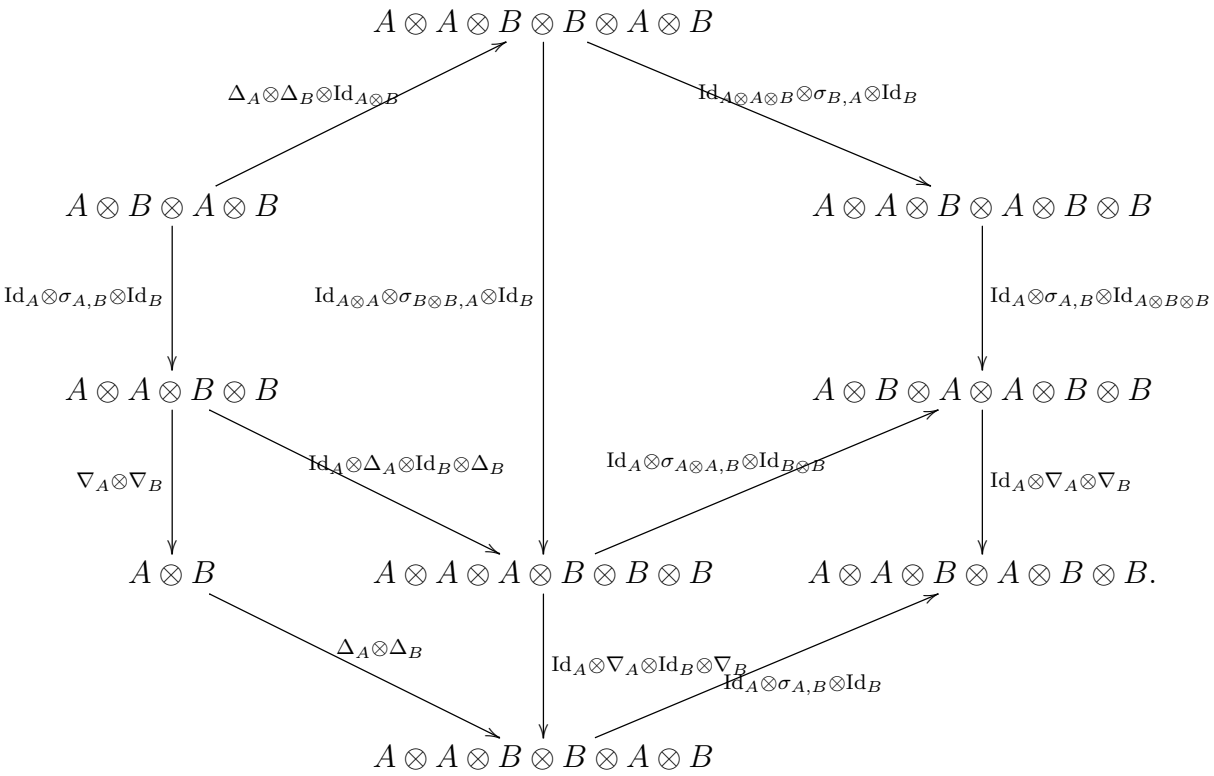
as one can always select e.g. $(s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (0, c, 0, c, c, c, c)$.

□

Proposition 3.1.6. *Let* $(A, \Delta_A, \Pi_A, \nabla_A, \amalg_A)$ *and* $(B, \Delta_B, \Pi_B, \nabla_B, \amalg_B)$ *be Frobenius algebras. Then the tensor* $A \otimes B$ *is a Frobenius algebra, with the operations defined as in Definition 2.2.23. If both the algebras are thin, then the result is thin as well.*

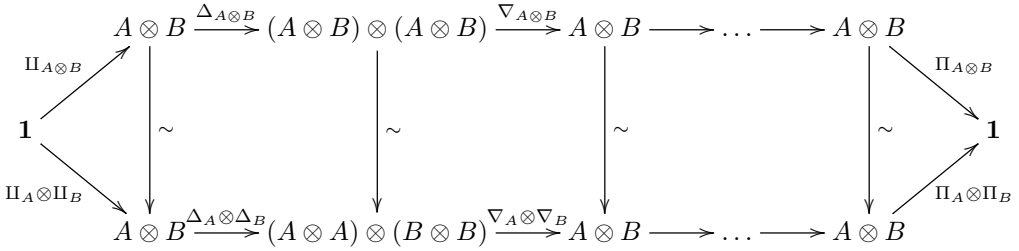
Démonstration. Diagramatically, the Frobenius law in the algebra generated by the com-

posite object follows from the diagram below.



The lower left quadrilateral is the Frobenius law, the upper left and the lower right ones are naturallity of symmetry, while the upper right one is involutivity of symmetry. This shows one of the equations, the other one follows from symmetry.

As for thinness, $\Pi_{A\otimes B} \circ (\nabla_{A\otimes B} \circ \Delta_{A\otimes B}) \circ \cdots \circ (\nabla_{A\otimes B} \circ \Delta_{A\otimes B}) \circ \amalg_{A\otimes B}$ is by the definition of $\Delta_{A\otimes B}, \Pi_{A\otimes B}, \nabla_{A\otimes B}, \amalg_{A\otimes B}$ and by the coherence theorem for symmetric monoidal categories, the diagram



commutes. Here, the vertical isos are the coherence isos in a symmetric monoidal category. □

One can show that the tensor algebra of two Frobenius algebras is Frobenius by giving a graphical proof that uses the string diagram for Frobenius equation for multiplication/comultiplication of

$$(A \otimes B, \Delta_{A\otimes B}, \Pi_{A\otimes B}, \nabla_{A\otimes B}, \amalg_{A\otimes B})$$

as depicted in Figure 3.2.

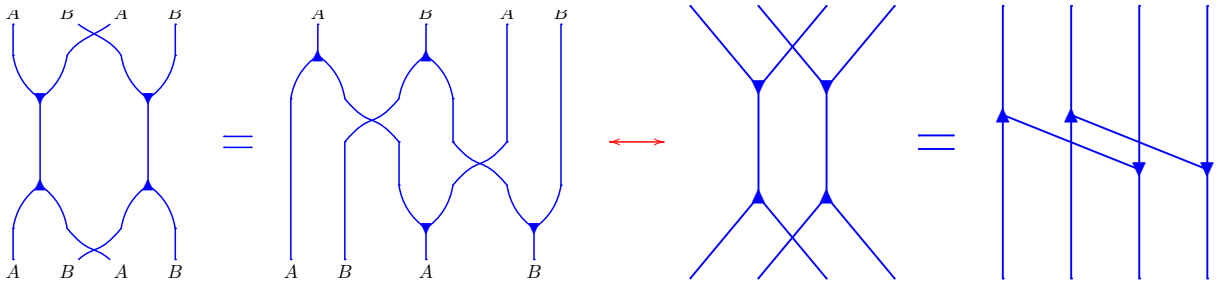


FIGURE 3.2 – String diagram of (one of) Frobenius equations for a tensor algebra

Definition 3.1.7. A *Frobenius category* **F** is a symmetrical monoidal category where *every* object A is equipped with a Frobenius algebra structure $(A, \Delta_A, \Pi_A, \nabla_A, \amalg_A)$ and such that the algebra structure on the tensor of two objects is obtained as in the previous proposition.

Definition 3.1.8. A Frobenius category is *intrinsic* if every iso between two objects $A \xrightarrow{\sim} B$ is an iso for the associated Frobenius structures $(A, \Delta_A, \Pi_A, \nabla_A, \amalg_A) \xrightarrow{\sim} (B, \Delta_B, \Pi_B, \nabla_B, \amalg_B)$.

The following is known, it appears in [Hyl04, Koc04] among other places, but here we give a proof consistent with our notation and definitions.

Proposition 3.1.9. *Every Frobenius category is compact-closed, with the involution functor identity on objects.*

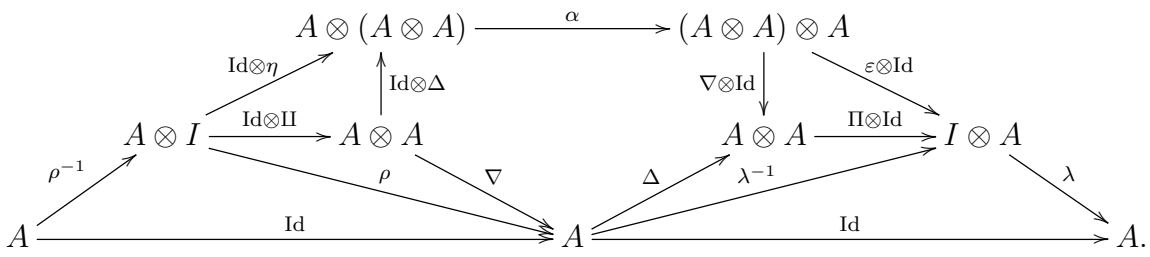
Démonstration. To see that this is the case, it suffice to define

$$\eta : I \to A \otimes A = \Delta_A \circ \amalg_A$$

and

$$\varepsilon : A \otimes A \to I = \Pi_A \circ \nabla_A,$$

for which the following diagram commutes



The pentagram in the diagram is the Frobenius law. The upper most composition is a map that is required to be identity, with the other being $\rho_A \circ (\mathrm{Id}_A \otimes \varepsilon) \circ \alpha_{A,A,A} \circ (\eta \otimes \mathrm{Id}_A) \circ \lambda_A^{-1}$, for which the result follows by duality. □

3.1.1 Constructing Frobenius Categories

Frobenius categories are not easily found in nature. In general, any equation on $-\otimes-$ objects may interfere with the Frobenius algebra structure in a possibly unpredictable and complex way. Therefore, one is interested in ways of constructing these.

To that purpose, let $(\mathbf{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category and let $\mathcal{S}$ be a set of Frobenius algebras in $\mathbf{C}$.

One defines a new category where an object is a sequence

$$(A_1, \ldots, A_n), \qquad \text{for } A_i \in \mathcal{S}.$$

A map $(A_1, \ldots, A_n) \to (B_1, \ldots, B_m)$ is just a map in $\mathbf{C}$:

$$(((A_1 \otimes A_2) \otimes A_3) \otimes \cdots \otimes A_n) \to (((B_1 \otimes B_2) \otimes B_3) \otimes \cdots \otimes B_m).$$

The symmetric monoidal structure in the new category is obtained by defining the tensor $(A_1, \ldots, A_n) \otimes (B_1, \ldots, B_m)$ to be the concatenation

$$(A_1, \ldots, A_n, B_1, \ldots, B_m).$$

This way the symmetric monoidal structure translates to the new category.

Now, a commutative monoid and a cocommutative comonoid structure on *every* object of the new category is obtained from respective structures in $\mathcal{S}$ in the following way :

$\nabla : (A_1, \ldots, A_n) \otimes (A_1, \ldots, A_n) \to (A_1, \ldots, A_n)$ is equal to the composition :

$$\begin{array}{c} ((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \otimes ((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \\ \Big\downarrow \sim \\ (((A_1 \otimes A_1) \otimes (A_2 \otimes A_2)) \otimes \cdots \otimes (A_n \otimes A_n)) \\ \Big\downarrow ((\nabla_{A_1} \otimes \nabla_{A_2}) \otimes \cdots \otimes \nabla_{A_n}) \\ ((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \end{array}$$

(the isomorphism here is the obvious associativity/symmetry isomorphism of the monoidal category).

$\amalg : \mathbf{1} \to (A_1, \ldots, A_n)$ is equal to the composition :

$$\mathbf{1} \xrightarrow{\quad\sim\quad} ((\mathbf{1} \otimes \mathbf{1}) \otimes \cdots \otimes \mathbf{1}) \xrightarrow{((\amalg_{A_1} \otimes \amalg_{A_1}) \otimes \cdots \otimes \amalg_{A_n})} ((A_1 \otimes A_2) \otimes \cdots \otimes A_n).$$

Analogously for the comonoid structure :

$\Delta : (A_1, \ldots, A_n) \to (A_1, \ldots, A_n) \otimes (A_1, \ldots, A_n)$ is defined as :

$$\begin{array}{c} ((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \\ \downarrow{\scriptstyle ((\Delta_{A_1} \otimes \Delta_{A_2}) \otimes \cdots \otimes \Delta_{A_n})} \\ (((A_1 \otimes A_1) \otimes (A_2 \otimes A_2)) \otimes \cdots \otimes (A_n \otimes A_n)) \\ \downarrow{\scriptstyle \sim} \\ ((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \otimes ((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \end{array}$$

$\Pi : (A_1, \ldots, A_n) \to \mathbf{1}$ is :

$$((A_1 \otimes A_2) \otimes \cdots \otimes A_n) \xrightarrow{((\Pi_{A_1} \otimes \Pi_{A_2}) \otimes \cdots \otimes \Pi_{A_n})} ((\mathbf{1} \otimes \mathbf{1}) \otimes \cdots \otimes \mathbf{1}) \xrightarrow{\sim} \mathbf{1}.$$

It is easy to verify that the above defines a commutative monoid and a cocommutative comonoid structure on every object of the category. Moreover, by Proposition 3.1.6 iteratively applied, the structures $((A_1, \ldots, A_n), \Delta, \Pi, \nabla, \amalg)$ are Frobenius algebras.

3.1.2 Free Frobenius Categories

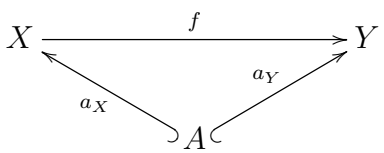
Frobenius algebras have recently gained a lot of attention following the identification of the free Frobenius algebras with 2-dimensional Quantum Field Theories (TQFT). We refer the reader to the historical overview of the development of the concept of Frobenius algebras in [Koc]. The result was achieved by several people independently, and one of the standard references is [Dij89]. A readable, self-contained treatment of the issue can be found in monograph [Koc04].

Definition 3.1.10. A *(finite) topological graph* G is a topological space that is obtained from a finite set V of points, called *vertices*, and a (finite) set A of functions $a : \{0, 1\} \to V$ as the quotient space of the disjoint union $V \uplus (A \times [0, 1])$ that is obtained by identifying $(a, 0)$ with $a(0)$ and $(a, 1)$ with $a(1)$. The images of the intervals $\{a\} \times [0, 1]$ are called *edges*.

The knowledgeable reader can verify that topological graphs are precisely 1-dimensional CW complexes.

By **TopGraphs** we denote the subcategory of topological graphs of the category **Top** of topological spaces and continuous maps.

Definition 3.1.11. Let A be a topological space. We define the category of *A-based spaces* $\mathbf{Top_A}$, to be the full subcategory of the coslice category $\mathbf{A/Top}$ in which the objects are pairs (X, a) where a is a subspace inclusion map $a_X : A \hookrightarrow X$. A morphism is this category is an $f : X \to Y$ that makes



commute.

The following appears in the existing literature on Frobenius algebras.

Theorem 3.1.12. *The following categories are isomorphic*

1. *The category of topological Riemann Surfaces. An objects $[n]$ is a finite disjoint union of n of circles. A map $[m] \to [n]$ is a Riemann surface (with boundary) whose boundary is the disjoint sum of circles $[m] + [n]$, where two surfaces are identified modulo homeomorphism. Composition is glueing, forgetting the boundaries in the middle.*
2. *The category of one dimensional finite topological graphs up to homology. An objects n is the finite n-element set $\{0, 1, \ldots, n-1\}$ of points seen as discrete topological spaces. A map $[m] \to [n]$ is an object in $([\mathbf{m}] + [\mathbf{n}])/\mathbf{TopGraphs}$, with two graphs being identified if they are equivalent modulo homology. Composition is also glueing.*

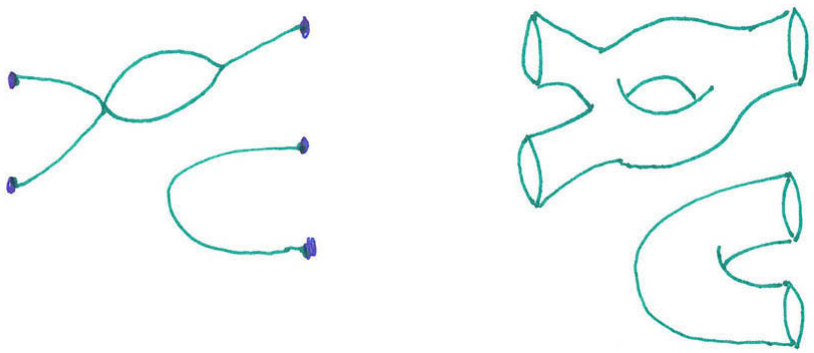


FIGURE 3.3 – Two equivalent characterizations of free Frobenius algebras. Objects are as distinguished end-points in the left or as circles to the right. Two of them are in the domain, three in the codomain. A map is the homology of the 'wiring' to the left or the Riemann surface-defined homeomorphism class that fixes the border. One of the connected components has genus 1, the other 0. In both cases the map is determined by grouping of the literals in a partition and an assignment of genera to the classes of the partition.

The general claim on the representation of Frobenius is usually stated at this point as saying that the free Frobenius category on one object generator is equivalent to either of the two categories of Theorem 3.1.12. To be able to give a more self contained treatment of the matter, we favor another approach where we replace the second category in the previous theorem by the homotopy version, that we present in the next section, which needs much less machinery to be defined and proved and yields an isomorphic category.

Theorem 3.1.13. *The category of one dimensional finite topological graphs up to homology of Theorem 3.1.12 is isomorphic to the category of one dimensional finite topological graphs up to relative homotopy, as defined in the next section.*

Démonstration. The two categories have the same objects (finite sets). The maps in either category are topological graphs, so what is left to show the isomorphism of the categories is to show that a topological graph is identified with an another in one category if and only if it is identified in the other.

It is well known that the homotopy equivalence of two spaces entails isomorphism of homology groups of the two spaces, [Spa66, Arm83, Hat01, May99, Mun84, Sti93]. So, if two topological graphs are identified modulo a relative homotopy that fixes some of the nodes, the homotopy entails isomorphism of the homology groups of the two graphs, i.e. The two maps are identified modulo homology.

Conversely, if two topological graphs have isomorphic homology groups, then they have the same number of connected components and the came number of cycles for each connected component (again, the reference is the set of textbooks in algebraic topology from above, [Spa66, Arm83, Hat01, May99, Mun84, Sti93]). This very same data defines the class of homotopy equivalent spaces relative to the object generator nodes. (We give a proof of this fact in 3.1.18). □

The Homotopy Characterization

The first two following definitions are a standard part of textbooks in algebraic topology, e.g. [Spa66, Arm83, Hat01, May99]).

Definition 3.1.14. A *homotopy* between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ such that, if $x \in X$ then $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Continuous functions f and g are said to be *homotopic* if and only if there is a homotopy H taking f to g. Being homotopic is an equivalence relation on the set of all continuous functions from X to Y.

The *homotopy category* **hTop** is the category whose objects are topological spaces, and whose morphisms are homotopy equivalence classes of continuous maps.

Two topological spaces X and Y are *homotopy-equivalent* if they are isomorphic in **hTop**. In more detail, X and Y are homotopy equivalent or of *the same homotopy type* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map Id_X and $f \circ g$ is homotopic to Id_Y.

The category **hTop** is obtained by a quotient construction on **Top**, the ordinary category of topological spaces and continuous maps. Of interest to us is the quotient category obtained from a slightly tweaked category **Top**.

For a pair of parallel maps $f, g : (X, a_X) \to (Y, a_Y)$ in $\mathbf{Top_A}$ we say that f and g are *homotopic relative to* A if there exists a homotopy $H : X \times [0,1] \to Y$ between f and g such that $H(a_X(a), t) = f(a_X(a)) = g(a_X(a))$ for all $a \in A$ and $t \in [0,1]$. Being homotopic relative to A is a congruence relation on $\mathbf{Top_A}((X, a_X), (Y, a_Y))$.

We define category $\mathbf{hTop_A}$ as the category $\mathbf{Top_A}$ whose maps are quotients under the relation of homotopy equivalence relative to A.

Two objects (X, a_X) and (Y, a_Y) of $\mathbf{Top_A}$ are *homotopy equivalent relative to* A or of *the same homotopy type relative to* A is they are isomorphic in $\mathbf{hTop_A}$.

Definition 3.1.15. The category **FrFrob** is defined to be the category whose objects $[n]$ are n element finite sets $\{0, 2, \ldots, n-1\}$, seen as discrete spaces.

A map $[m] \to [n]$ is apair (G, a) consisting of a topological graph G equipped with an injective function $a : [m] + [n] \hookrightarrow V$ from the domain/codomain spaces to the vertices of G, where two such spaces are identified if they are homotopy equivalent relative to $[m] + [n]$ (= isomorphic in $\mathbf{hTop_{(m+n)}}$).

To define composition of such maps, consider two representatives $(G, a) : [k] \to [m]$ and $(F, b) : [m] \to [n]$ of maps. Composition of the two maps is the class of homotopy equivalence relative to $[k] + [n]$ represented by (H, c), where H is the quotient space of the disjoint union $G \uplus F$ obtained by identifying $a(i)$ with $b(i)$, for all $i \in [m]$. Here, c is the injection which agrees with a on $[k]$ and with b on $[n]$.

To ensure correctness of the definition given by representatives, take any two parallel $G, G' : [k] \to [m]$ of the same homotopy type relative to $[k] + [m]$, any $F, F' : [m] \to [n]$ of the same homotopy type relative to $[m] + [n]$. Let also $u : G \to G'$, $u' : G' \to G$ be the continuos maps witnessing relative homotopy equivalence of G, G'; $H_1 : G \times [0, 1] \to G'$ and $H_1' : G' \times [0, 1] \to G$ the two homotopies between $u' \circ u$, Id_G, and $u \circ u'$, $\mathrm{Id}_{G'}$, respectfully. The corresponding data witnessing homotopy equivalence of F, F' is $v : F \to F'$, $v :' F' \to F$ and $H_2 : F \times [0, 1] \to F'$ and $H_2' : F' \times [0, 1] \to F$. Now, the extensions $u + v : G \cup F \to G' \cup F'$ and $u' + v' : G' \cup F' \to G \cup F$ of maps u, v and u', v' respectively are well defined since u, v and u', v' agree on $[m]$. Homotopies between $(u' + v') \circ (u + v)$ and $\mathrm{Id}_{G \cup F}$ as well as $(u + v) \circ (u' + v')$ and $\mathrm{Id}_{G' \cup F'}$ are now easily obtained by extension

$$H_1 \cup H_2 : \ G \cup F \times [0, 1] \to G' \cup F'$$

$$H_1' \cup H_2' : \ G' \cup F' \times [0, 1] \to G \cup F$$

which is well defined as u, v and u', v' agree on $[m]$.

The reader is reminded that in general, one cannot compose ordinary homotopy equivalent spaces this way. Homologically equivalent spaces, however (and the spaces here are homologically equivalent), can be composed because of the Mayer-Vietoris property, and the properties of long exact sequences [Vie30, Spa66].

To gloss the previous definition a bit, category **FrFrob** can be thought of as the category of homotopy types of topological graphs where homotopies fix certain vertices (comprising domain/codomain of maps), and where composition is the homotopy type defined by concatenation of graphs.

Definition 3.1.16. For a graph G which is a representative of a map $f : [m] \to [n]$ in **FrFrob** we say that it is in *normal form* if

i) its set of vertices V is $[m] + [k] + [n]$,

ii) each point of $[m] + [n]$ is an endpoint of precisely one edge, the other one being a point of $[k]$,

iii) each point of $[k]$ is an endpoint of an edge whose other endpoint is either in $[m] + [n]$, or the point itself.

The following appears in the literature.

Theorem 3.1.17 ([Dij89, Koc04])**.** *Every graph G representing a map $f : [m] \to [n]$ in* **FrFrob** *is of the same* homology *as a topological graph in normal form.*

Here we show

Proposition 3.1.18. *A graph G representing a map $f : [m] \to [n]$ in* **FrFrob** *is of the same* homotopy type relative to $[m] + [n]$ *to a topological graph in normal form.*

Démonstration. See Appendix A. □

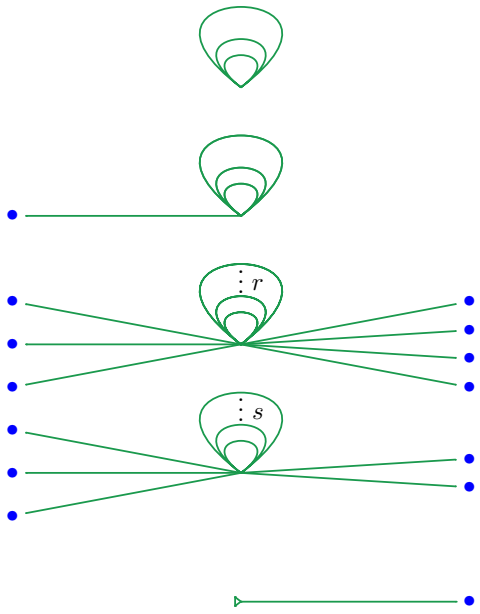


FIGURE 3.4 – A map in the category **FrFrob** as a collection of wires between endpoints with loops, depicted horizontally. Notice that it is possible for a component not to be connected to any endpoint and notice how we use the 'terminator' symbol to denote the base-point with no loops, connected to a single endpoint.

The previous proposition prompts the following convention :

we depict of maps in the category **FrFrob** as one dimensional wires with marked endpoints. One of (possibly several) connected components is collection of wires going from a base point to the endpoints, with several wire loops sharing the same base point (a rose). The number of loops of a component is one of the invariants of the homotopy type of the map which, by convention, we refer to as *genus* (as in Figure 3.4).

The notion of genus we have just defined should not be confused with the standard notion of the genus of a graph, which is the minimal number of handles n for which the graph can be embedded into n-fold torus so that edges of the graph do not overlap.

The following is trivial.

Proposition 3.1.19. *The category* $(\mathbf{FrFrob}, \uplus, \emptyset)$ *is a strict symmetric monoidal category with disjoint union for a tensor and the empty set for the monoidal unit.*

We have shown in Theorem 3.1.18 that every map in **FrFrob** is represented by a topological graph in normal form. These, in turn, can be obtained by the set of generators depicted in Figure 3.5.

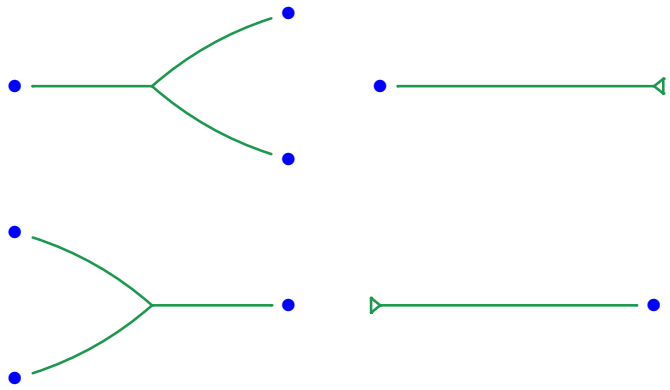


FIGURE 3.5 – Generators in the category **FrFrob** – up : comonoid comultiplication Δ (diagonal) and counit $\amalg$ (projection) ; down : monoid multiplication ∇ (co-diagonal) and unit $\amalg$ (co-projection). Notice the use of a 'terminator' triangle symbol in (co)unit generators to denote the single graph vertex that is not an object generator.

Let $\mathbf{C}$ be a symmetric monoidal category, and let us see what a monoidal structure preserving functor $F : \mathbf{FrFrob} \to \mathbf{C}$ looks like. Let $A = F[1]$. By definition there are isos $\phi_{1,1} : F[2] = F([1] + [1]) \to (A) \otimes (A)$ and $\phi_0 : F[0] \to \mathbf{1}$.

For every n and for every bracketing β of the constant family $\underbrace{A, A, A, \ldots, A}_{n}$ we can use these to construct by induction an isomorphism

$$\phi_\beta : \; F[n] \to \beta(\underbrace{A, A, \ldots, A}_{n})$$

and moreover, given two bracketings β, β' the coherence theorem ensures that the following triangle commutes.

$$\begin{array}{ccc} & F[n] & \\ {\scriptstyle \phi_\beta}\swarrow & & \searrow{\scriptstyle \phi_{\beta'}} \\ \beta(A, A, \ldots, A) & \xrightarrow{\quad\sim\quad} & \beta'(A, A, \ldots, A). \end{array} \tag{3.1}$$

(Notice, the diagram is a triangle and not a square because in **FrFrob** all monoidal isos are identities).

Theorem 3.1.20 ([Dij89, Koc04])**.** *Let* $\mathbf{C}$ *be a symmetric monoidal category, let* $F : \mathbf{FrFrob} \to \mathbf{C}$ *be a symmetric monoidal structure preserving functor, and let* $A = F[1]$*. Then the following*

$$\nabla_A : \quad A \otimes A \xrightarrow{\phi_{1,1}^{-1}} F[2] \xrightarrow{\nabla} F[1] = A; \qquad \amalg_A : \quad \mathbf{1} \xrightarrow{\phi_0^{-1}} F[0] \xrightarrow{\amalg} F[1] = A$$

$$\Delta_A : \quad A = F[1] \xrightarrow{\Delta} F[2] \xrightarrow{\phi_{1,1}} A \otimes A; \qquad \Pi_A : \quad A = F[1] \xrightarrow{\Pi} F[0] \xrightarrow{\phi_0} \mathbf{1}; \tag{3.2}$$

define a Frobenius algebra structure on A*.*

Conversely, let $(A, \Delta_A, \Pi_A, \nabla_A, \amalg_A)$ *be a Frobenius algebra. For every* n *choose an object* $F[n]$*, a bracketing* β_n *of* $(A, A, A, \ldots, A)$*, and an isomorphism*

$$\phi_n : F[n] \to \beta(A, A, A, \ldots, A).$$

Then there is a unique symmetric monoidal structure respecting functor

$$F : \mathbf{FrFrob} \to \mathbf{C}$$

that maps every $[n]$ *to* $F[n]$ *and* $\Delta, \Pi, \nabla, \amalg$ *in* **FrFrob** *to*

$$F[2] \xrightarrow{\phi_{1,1}} A \otimes A \xrightarrow{\nabla} A = F[1]; \qquad F[0] \xrightarrow{\phi_0} \mathbf{1} \xrightarrow{\amalg} A = F[1];$$

$$F[1] = A \xrightarrow{\Delta} A \otimes A \xrightarrow{\phi_{1,1}^{-1}} F[2]; \qquad F[1] = A \xrightarrow{\Pi} \mathbf{1} \xrightarrow{\phi_0^{-1}} F[0]; \tag{3.3}$$

respectively.

Obviously, given two families of choices $(F[n], \beta_n, \phi_n)_n$ and $(F'[n], \beta'_n, \phi'_n)_n$ they will define the same functor iff $F[n] = F'[n]$ for every n and the triangle (3.1) commutes for every n.

The previous theorem allows us to say that **FrFrob** is the free Frobenius category with one object generator. In the ordinary world of groups, rings, etc. a free construction

is defined up to a unique isomorphism. Here we are in the world of categories and the free thing is defined up to a *unique equivalence*. Right now our category is skeletal, but any symmetric monoidal category which is equivalent to it could be considered as the free Frobenius category on one object generator.

For example, we could decide that the objects are all formulas built with the constants $\mathbf{1}, a$, and the binary operator $\otimes$. Such a formula can be seen as a finite set whose elements are the occurrences of a. Then maps are defined just as before, where occurrences replace $0, 1, 2, \ldots$ This is actually the free Frobenius category, when a symmetric monoidal category is taken as an ordinary algebraic structure, not one in the world of categories.

We could also use all formulas built with only the constant a and the binary operator $\otimes$, with an additional "empty" formula, for the tensor unit/empty set.

Definition 3.1.21. FrThFrob is the category whose objects are finite sets $\{0, \ldots, n-1\}$, denoted by $[n]$, viewed as discrete topological spaces. A map $[m] \to [n]$ is a pair (G, a) consisting of a topological graph G equipped with an injective function $a : [m] + [n] \hookrightarrow V$ from the domain/codomain spaces to the vertices of G, where two such spaces are identified if they are homotopy equivalent relative to $[m] + [n]$ ($=$ isomorphic in $\mathbf{hTop}_{(\mathbf{m+n})}$). In addition, we ask that the inclusion $a : [m]+[n] \hookrightarrow V$ is surjective on connected components of G. Composition in the category is defined as in **FrFrob**, where in addition, whenever a composition yields a connected component not containing a point in the image of a, that component is discarded.

There is an obvious full functor $Thin$: **FrFrob** $\to$ **FrThFrob** which maps every "floating component" to the empty graph.

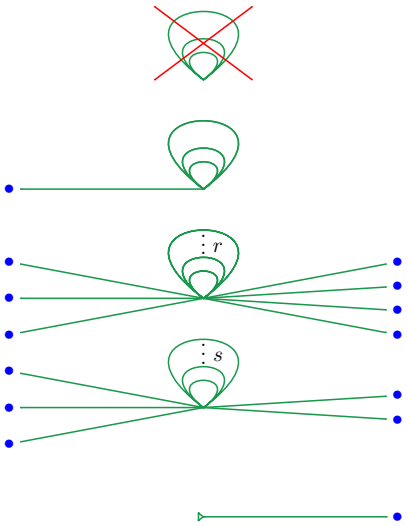


FIGURE 3.6 – In **FrThFrob** the components which do not contain object generators are omitted as they stand for the identity on the monoidal unit.

Theorem 3.1.22. *The category* **FrThFrob** *is the* free thin Frobenius category *in the sense of Theorem 3.1.20 and the comment that follows it.*

Démonstration. Let $F : \mathbf{FrThFrob} \to \mathbf{C}$ be a structure preserving monoidal functor to a monoidal category $\mathbf{C}$, and let $F[1] = A$. Let $\nabla_A, \amalg_A, \Delta_A, \amalg_A$ be the bimonoid operations defined as in (3.2). We need to show that $(A, \nabla_A, \amalg_A, \Delta_A, \amalg_A)$ is a *thin* Frobenius algebra structure.

From Theorem 3.1.20 we know that the structure is a Frobenius algebra. What is left is to show thinness.

It is easy to see from (3.2) that $(\nabla_A \circ \Delta_A)^k$:

$$A = F[1] \xrightarrow{\Delta} F[2] \xrightarrow{\phi_{1,1}} A \otimes A \xrightarrow{\phi_{1,1}^{-1}} F[2] \xrightarrow{\nabla} F[1] \longrightarrow$$

$$\ldots \longrightarrow F[1] \xrightarrow{\Delta} F[2] \xrightarrow{\phi_{1,1}} A \otimes A \xrightarrow{\phi_{1,1}^{-1}} F[2] \xrightarrow{\nabla} F[1] = A$$

which is simply image of the $(\nabla \circ \Delta)^k$ in **FrThFrob**. Now, $\amalg \circ (\nabla_A \circ \Delta_A)^k \circ \amalg_A$ is

$$\phi_0 \circ \amalg \circ (\nabla \circ \Delta)^k \circ \amalg \circ \phi_0^{-1}$$

and this is by thinness if **FrThFrob** $\phi_0 \circ \phi_0^{-1} = \mathrm{Id}_{\mathbf{1}}$.

For the converse direction, let $(A, \nabla_A, \amalg_A, \Delta_A, \amalg_A)$ be a thin Frobenius algebra in $\mathbf{C}$, $(F[n], \beta_n, \phi_n)_n$ a choice of $\phi : F[n] \to \beta_n(A, \ldots, A)$, i.e. an object of $\mathbf{C}$ an isomorphism and a bracketing, for every n, and let the image of map generators in **FrThFrob** be given by (3.3).

By Theorem 3.1.20, there is a uniquely defined functor $F' :$ **FrFrob** $\to \mathbf{C}$ by the previous data. But, as the Frobenius algebra on A is thin, the F' sends every $\amalg\circ(\nabla\circ\Delta)^k\circ\amalg$ to the identity. Now, F' defines a functor $F :$ **FrThFrob** $\to \mathbf{C}$ agreeing with F on objects and maps. Moreover, any other functor $G :$ **FrThFrob** $\to \mathbf{C}$, agreeing with F on the above data yields a functor $G' :$ **FrFrob** $\to \mathbf{C}$ which agrees on objects and maps with G, and in addition sends every $\amalg \circ (\nabla \circ \Delta)^k \circ \amalg$ to identity. By uniqueness of F', it has to be $F' = G'$, and thus $F = G$.

□

From what we have shown, and from the Theorem 3.1.18, it is easy to see that a map in **FrThFrob** is defined by a partition of the disjoint union of the domain and codomain generators and an assignment of integers to the classes in the partition. Also, each map $f : [m] \to [n]$ in **FrThFrob** can be transposed to a map $f^r : \mathbf{1} \to [m] + [n]$, whose domain is the monoidal unit, so to analyze maps of **FrThFrob** is to analyze these.

Definition 3.1.23. In the category **FrFrob**, the map $\mathcal{D} : [1] \to [1]$ defined as

$$\mathcal{D} = \nabla \circ \Delta$$

is called the *doubling map*. The composition $\mathcal{D} \cdots \circ \mathcal{D} \circ \mathcal{D}$ of k doubling maps is denoted $\mathcal{D}^k$.

FIGURE 3.7 – Two presentations of $\mathcal{D}^k$

Proposition 3.1.24. *Every map $f : [m] \to [n]$ in* **FrThFrob** *is a composition of the form*

$$[m] \xrightarrow{\sim} [m] \xrightarrow{f'} [n] \xrightarrow{\sim} [n],$$

where the two isos are coherent isos of a symmetric monoidal category, and

$$f' = f_1 \otimes f_2 \otimes \ldots \otimes f_k,$$

where f_i, $1 \leq i \leq k$ is of the form

$$[m_i] \xrightarrow{\nabla^{m_i}} [1] \xrightarrow{\mathcal{D}^k} [1] \xrightarrow{\Delta^{n_i}} [n_i]. \tag{3.4}$$

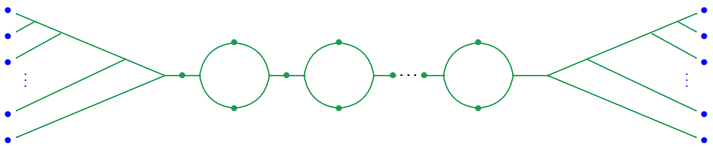
Here,

$$\nabla^0 = \amalg, \quad \nabla^1 = \mathrm{Id}, \quad \nabla^2 = \nabla, \quad \nabla^3 = \nabla \circ (\nabla \otimes \mathrm{Id}), \quad \ldots$$

and

$$\Delta^0 = \amalg, \quad \Delta^1 = \mathrm{Id}, \quad \Delta^2 = \Delta, \quad \Delta^3 = (\Delta \otimes \mathrm{Id}) \circ \Delta, \quad \ldots$$

The proof of the proposition is left to the reader, as it is an easy consequence of the Proposition 3.1.18, as illustrated by the representations of the doubling map in diagrams of the form :



The previous conclusions induce the following definition :

Definition 3.1.25 (Linking)**.** Let P be a set. With the obvious abuse of notation, we define *linking* to be the triple

$$P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$$

where

- P is previously fixed set
- $\mathcal{C}omp_P$ is *partition* of the set P, and we refer to each class $C \in \mathcal{C}omp_P$, as a *component*
- $\mathcal{G}en_P : \mathcal{C}omp_P \to \mathbb{N}$ is a function that assignees a natural number to each component of the class $\mathcal{C}omp_P$

There are several reasons for choosing to think of maps as one-dimensional spaces up to homotopy equivalence. The choice allows for uniform representation of maps in the category that draws analogy with string diagrams, which is an advantage to the Riemann-surface characterization. Also, we believe that working with the homotopies is probably closer to intuition of the wider circle of readers than working with homologies.

The Frobenius Category FrFrob($\mathcal{AT}$yp)

For the purposes of interpreting proofs, of interest to us are categories generated by objects corresponding to different atoms and negatoms. The central category of our interest will be the category **FrThFrob**($\mathcal{AT}$yp), the thin Frobenius category generated by a set of literals $\mathcal{AT}$yp. We recall Convention 2.1.2 that each element of $\mathcal{AT}$yp is assigned a sort and a polarity.

Definition 3.1.26. **FrFrob**($\mathcal{AT}$yp) is the category whose objects are all formulas built with the atoms and negatoms of $\mathcal{AT}$yp, the binary operator $\otimes$, and an additional symbol $\emptyset$ for the tensor unit.

A map $\mathrm{A} \to \mathrm{B}$ is a pair (G, a) consisting of a topological graph G equipped with an injective function $a : \mathcal{L}(\mathrm{A}) + \mathcal{L}(\mathrm{B}) \hookrightarrow V$ from the literals of the domain and codomain the vertices of G, such that literals which are mapped to a same connected component of G are of the same *sort*.

Two such spaces are identified if they are homotopy equivalent relative to $a(\mathcal{L}(\mathrm{A}) + \mathcal{L}(\mathrm{B}))$.

Composition of maps is defined as composition in **FrFrob**, as it is guaranteed to respect the labelings.

The category **FrThFrob**($\mathcal{AT}$yp) is defined analogously to **FrThFrob**; it is the category with the same objects as **FrFrob**($\mathcal{AT}$yp), while a map $\mathrm{A} \to \mathrm{B}$ is a map in **FrFrob**($\mathcal{AT}$yp) such that the inclusion $a : \mathcal{L}(\mathrm{A}) + \mathcal{L}(\mathrm{B}) \hookrightarrow V$ from the domain/codomain spaces to the vertices of a topological graph G is surjective on connected components. Composition, as in the **FrThFrob** case, is the one of **FrFrob**($\mathcal{AT}$yp) with discarding of the components who do not intersect with the image of a.

There is an obvious full functor $F : \mathbf{FrFrob}(\mathcal{AT}\mathrm{yp}) \to \mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})$.

Notice that there is also an obvious functor $U : \mathbf{FrFrob}(\mathcal{AT}\mathrm{yp}) \to \mathbf{FrFrob}$ that maps a formula A with n literals to $[n]$.

Also, notice that the tensor in **FrThFrob**($\mathcal{AT}$yp) is defined by cases; tensor of two non-empty formulas is the ordinary tensor, while a tensor of a formula with an $\emptyset$ is the formula itself.

Since we have shown that a Frobenius algebra of a tensor is the standard tensor of Frobenius algebras, much of the analysis reduces to the case of a single object generator. Similarly to the case of **FrFrob**, maps on in **FrFrob**($\mathcal{AT}$yp) can be represented by topological graphs in normal form, the only condition now is that the endpoints are labeled by literals so that a connected component is labeled by those of the same *sort*. For each sort in $\mathcal{AT}$yp, a minimal set of generators is depicted in Figure 3.8.

In what follows we focus on the thin Frobenius category **FrThFrob**($\mathcal{AT}$yp).

Theorem 3.1.27. *Let* **C** *be a Frobenius category every one of whose Frobenius algebra*

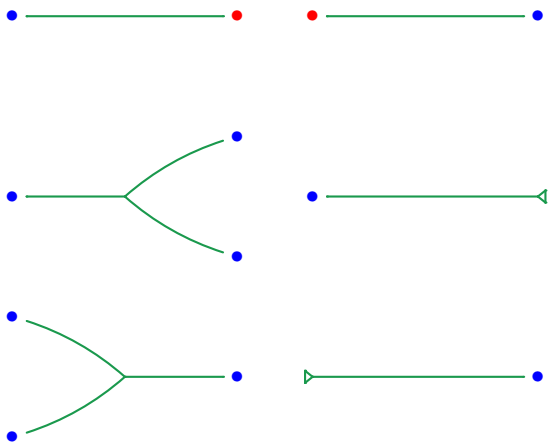


FIGURE 3.8 – A minimal family of generators indexed by *atoms* – $\Delta_{\mathrm{a}}, \Pi_{\mathrm{a}}, \nabla_{\mathrm{a}}, \amalg_{\mathrm{a}}$. Blue and red dots represent object generators of the same sort but opposite *polarity*.

on an object is thin. For every $\mathrm{a}, \overline{\mathrm{a}}$ *in* $\mathcal{AT}\mathrm{yp}$ *let* $F\mathrm{a}, F\overline{\mathrm{a}}$ *be objects of* $\mathbf{C}$ *and* σ_{a} *an iso between them.*

Then there exists a unique monoidal structure preserving functor

$$F : \mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp}) \to \mathbf{C},$$

such that

1. *literals* a *and* $\overline{\mathrm{a}}$ *in* $\mathcal{AT}\mathrm{yp}$ *are mapped to the preselected* $F\mathrm{a}, F\overline{\mathrm{a}}$ *;*
2. *for every* $\mathrm{a} \in \mathcal{AT}\mathrm{yp}$*, every one of the six diagrams* $\sigma_{\mathrm{a}}, \sigma_{\mathrm{a}}^{-1}, \Delta_{\mathrm{a}}, \Pi_{\mathrm{a}}, \nabla_{\mathrm{a}}, \amalg_{\mathrm{a}}$ *in* $\mathbf{FrFrob}(\mathcal{AT}\mathrm{yp})$ *in Figure 3.8 is mapped to*

$$F\mathrm{a} \xrightarrow{\sigma_{\mathrm{a}}} F\overline{\mathrm{a}}; \qquad F\overline{\mathrm{a}} \xrightarrow{\sigma_{F\mathrm{a}}^{-1}} F\mathrm{a};$$

$$F(\mathrm{a} \otimes \mathrm{a}) \xrightarrow{\phi_{\mathrm{a},\mathrm{a}}} F\mathrm{a} \otimes F\mathrm{a} \xrightarrow{\nabla} F\mathrm{a}; \qquad F\emptyset \xrightarrow{\phi} \mathbf{1} \xrightarrow{\amalg} F\mathrm{a};$$

$$F\mathrm{a} \xrightarrow{\Delta} F\mathrm{a} \otimes F\mathrm{a} \xrightarrow{\phi_{\mathrm{a},\mathrm{a}}^{-1}} F(\mathrm{a} \otimes \mathrm{a}); \qquad F(\mathrm{a}) \xrightarrow{\Pi} \mathbf{1} \xrightarrow{\phi^{-1}} F\emptyset;$$

   respectively ;
3. *the empty formula* $\emptyset$ *is mapped to the tensor unit* $\mathbf{1}$ *and* ϕ *is the identity ;*
4. *the map* $\phi_{A,B}$ *is identity when neither* A, B *is the empty formula ;*
5. $\phi_{A,\emptyset}$ *and* $\phi_{\emptyset,B}$ *are the unit isos in* $\mathbf{C}$*.*

The theorem follows from the second part of the theorem for **FrThFrob**. For that, the following easy corollary of Proposition 3.1.18 is useful and we will not give a complete proof of it. It is only used once, at the end of Chapter 5, and is not crucial to this thesis.

Proposition 3.1.28. *Every map $f : A \to B$ in* **FrThFrob**($\mathcal{AT}$yp) *is a composition of the form*

$$A \xrightarrow{\sim} A' \xrightarrow{f'} B' \xrightarrow{\sim} B,$$

where

1. *the two isos are coherent isos of a symmetric monoidal category,*
2. *the map f' is a tensor*[13]
$$f' = f_1 \otimes f_2 \otimes \ldots \otimes f_k,$$
where f_i, $1 \leq i \leq k$, for some $a \in \mathcal{AT}$yp, is of the form

$$\begin{array}{l} (a \otimes \cdots \otimes a) \otimes (\overline{a} \otimes \cdots \otimes \overline{a}) \\ \quad \Big\downarrow {\scriptstyle \mathrm{Id}\otimes\cdots\otimes\mathrm{Id}\otimes\sigma^{-1}\otimes\cdots\otimes\sigma^{-1}} \\ a \otimes \otimes a \cdots \otimes a \xrightarrow{\nabla^{n+n'}} a \xrightarrow{\mathcal{D}^p} a \xrightarrow{\Delta^{m+m'}} a \otimes a \otimes \cdots \otimes a \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow {\scriptstyle \mathrm{Id}\otimes\cdots\otimes\mathrm{Id}\otimes\sigma\otimes\cdots\otimes\sigma} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a \otimes \cdots \otimes a) \otimes (\overline{a} \otimes \cdots \otimes \overline{a}) \end{array}$$

Here, there are n positive, n' negative literals in the two factors to the left, and m, m' to the right, respectively. Also, as usual,

$$\nabla^0 = \Pi, \quad \nabla^1 = \mathrm{Id}, \quad \nabla^2 = \nabla, \quad \nabla^3 = \nabla \circ (\nabla \otimes \mathrm{Id}), \quad \ldots$$

and

$$\Delta^0 = \Pi, \quad \Delta^1 = \mathrm{Id}, \quad \Delta^2 = \Delta, \quad \Delta^3 = (\Delta \otimes \mathrm{Id}) \circ \Delta, \quad \ldots$$

while

$$\sigma_a : a \to \overline{a}$$

is the canonical iso, and $\sigma_a^{-1} : \overline{a} \to a$, its inverse.

Now the proof of the Theorem 3.1.27 is a reduced to the second part of the theorem for **FrThFrob** by noticing that the existence and the uniqueness of the functor F *once the choice for all the images of $a, \overline{a} \in \mathcal{AT}$yp is made*, follows form the existence and uniqueness of the functors from **FrThFrob** which map an f_i of the previous proposition to **C**.

In the category we have just defined, we give two different names to isomorphic generators (those that differ only in the polarity). This isomorphism can be visualized in Figure 3.8 as the identity map whose vertices are labeled by an atom and a negatom. Reasons for not identifying isomorphic objects have to do with the way this category will be used with syntax, and we will come back to this point.

As a consequence, **FrThFrob**($\mathcal{AT}$yp) is not freely generated by the set $\mathcal{AT}$yp, as its elements two-by-two represent the isomorphic copies of the same generator. Up to this notational nuance reflected in the *a priori* choice of the canonical iso between the two, **FrThFrob**($\mathcal{AT}$yp) is free in the sense of Theorem 3.1.27.

13. The tensor is presented without a bracketing, but one should be chosen, say all to the left. The same applies to other tensors in the proposition.

By changing the set of generators in the category, the notion of a linking needs to be changed accordingly.

Definition 3.1.29 (Linking, general case)**.** Let P be a set whose elements are labeled by (possibly different) symbols for atoms and negatoms. With the obvious abuse of notation, we define *linking* to be the triple

$$P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$$

where

- P is previously fixed set whose elements are simply referred to as (neg)atoms or literals (if the polarity is irrelevant)
- $\mathcal{C}omp_P$ is *partition* of the set P of literals such that each class $C \in \mathcal{C}omp_P$, to which we refer as a *component*, contains only atoms and negatoms of the same type
- $$\mathcal{G}en_P : \mathcal{C}omp_P \to \mathbb{N}$$
  is a function that assignees a natural number to each component of the class $\mathcal{C}omp_P$.

Computing Composition

See Appendix B.

3.1.3 Proof interpretations

The relevance of the Frobenius algebras for proof theory becomes obvious once the reader notice that central diagram in Figure 3.1 is in fact notorious contraction-contraction case in cut elimination we encountered in the proof to the right in Figure 3.10. Together with the weakening-weakening case in cut elimination, this is known to encapsulate all of the problems in classical proof theory, seen through the non-confluence of cut elimination in standard Genzen's systems.

The usual way of dealing with the weakening-weakening case of classical cut-elimination is by introduction of Mix rule (which is the reason why we include it in the calculus). As for the contraction-contraction case, one possibility is to push cut-elimination to atoms and negatoms (as we do). Usually, this is the point where analysis of cut elimination would stop. Alternatively, the equation $\Delta_A \circ \nabla_A = \nabla_A \otimes \nabla_A \circ \Delta_A \otimes \Delta_A$, commonly referred to as *bialgebra equation* is exploited, most recently in the work on atomic flows [GG08, GGS10] and before that in [Str05, Str09, Str07], depicted on Figure 3.9 using string diagrams

From the algebraic point of view, the bialgebra equation comes rather naturally, since its presence means that bimonoid operations respect bimonoid structure. The problem with imposing bialgebra equations is that, given sufficient amount of usual equations (those that guarantee (co)monoid operations being propagated to subformulas), they force idempotent interpretations where one cannot count axiom links, where e.g. the doubling

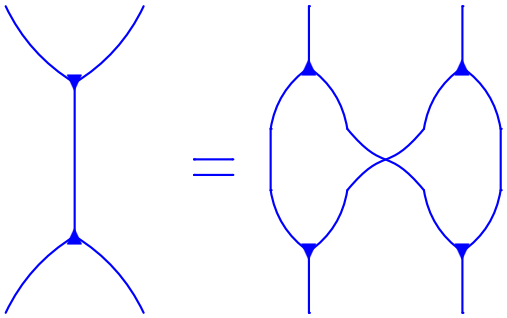


FIGURE 3.9 – Bialgebra equation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, \overline{a}, a}\,Contr \quad \dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, a}\,Contr}{\vdash \overline{a}, \overline{a}, a, a}\,Cut \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, a, \overline{a}}\,Contr \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, a, a}\,Contr$$

FIGURE 3.10 – Two proofs identified by Frobenius equations

map is idempotent (cf. [Str05, Str09]). Frobenius axioms provide a novel way to reason in this situation.

To illustrate this, let us take a look at two proofs on Figure 3.10. The reader can try to see that the proofs above have the same interpretation if Frobenius equations are assumed. The proof to the left induces the rightmost diagram in Figure 3.1, wheres the proof to the right stands for the the middle one form Figure 3.1. This will be an easy exercise very soon.

To the best of our knowledge, the first time the structure of a Frobenius algebra was used to provide interpretations of proofs was in Hyland's paper [Hyl04] and the subsequent work of Richard Garner [Gar05] on construction of models for multiplicative linear logic. Though it is based on the same idea of exploiting the Frobenius equations in proof identification, our discussion is different to the one of Hyland. Here we choose to stay on the degenerate level where the two connectives are identified. The idea behind this decision is to try to explore invariants of proofs stemming from the structural rules alone, which is in accordance with the recent work on atomic flows (cf. [GG08], for instance).

By identifying the connectives, we start from a compact closed category instead of a *-autonomous one, as noted before. We remind the reader that the special case we have encountered so far, the case of $\mathbb{Z}$ interpretation suffers from degeneracy where negation of an object is isomorphic to the object itself. We have showed that there is a deeper reason for that, namely that the Frobenius structure enforces dual objects to be isomorphic. Therefore, Frobenius categories come with inherit degeneracy. Still, for the purposes of interpreting proofs, it is useful to consider interpretations where the negation is not an identity, but does yield an isomorphic structure, as it is done in the definition of *the*

Frobenius category. Reasons for this lay in the possibilities

- of putting additional structure on Frobenius algebras since that would eliminate the isomorphism. This reintroduction of duality was already carried out in slightly different context in [Gar05] where Frobenius algebras are endowed with so called orthogonality structure.
- interpreting proofs is defining a functor from the syntactic category of a deductive system to the interpretation category. This functor need not (and will not) be full, and there is hope that the unwanted maps will cease to be isos in the image of the interpretation functor.

3.2 F-nets

The cleanest way of presenting an interpretation of proofs is by means of proof nets, which is what we intend to do here. In what follows we define an account of proof nets for classical proofs utilizing the established concepts of freely generated Frobenius categories.

We begin by giving some preliminary definitions. Notation and terminology we use here tries to follow the one of [LS05b].

The proof system we will be focusing on is the system CL with explicit Exchange rule added, to state the definition in full generality we need.

We remind the reader of the conventions on syntax and notation.

Convention 3.2.1. We think of a sequent Γ as a sequence of formulas defined over a set of *distinct* elements which are labeled by atom or negatom symbols. We refer to each out of these labeled elements as to an atom or a negatom *occurrence*, depending on the label.

When the intended meaning is clear, we will omit the word 'occurrence' when speaking of (neg)atom occurrences.

Notation 3.2.2. In the remainder of this chapter, we will use roman fonts for syntax, and *italic* for semantics (interpretation of syntax). Thus, $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{X}, \mathrm{Y}, ...$ will denote formulas, $\Gamma, \Sigma, ...$ will denote sequents. Lowercase roman letters are reserved for (positive) atoms, $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{l}, \mathrm{m}, \mathrm{n}, ...$ In all three cases, we are allowed to use indices, $\mathrm{a}_1, \mathrm{a}_2, ...\mathrm{B}_1, \mathrm{B}_2$. Sometimes we will refer to literals, formulas and sequents as *types*. A labeled element of a set which s assigned a sort and a polarity, e.g. a and $\overline{\mathrm{b}}$, is an atom. Atoms a and $\overline{\mathrm{a}}$ are of the same sort and opposite polarity. We also refer to the former as an *atom* and to the latter as a *negatom*. Atoms or negatoms are *literals*. We will use $\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots$ to speak of an atom without the information on its sort or polarity.

Superscripts are reserved for different *occurrences* of a *same* type (literal, formula or a sequent), $\mathrm{A}^1, \mathrm{A}^2, ...$

Definition 3.2.3 (F-prenet)**.** Let $P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$ be a linking (see Definition 3.1.29).

We define a *F-prenet* to be the pair

$$P \triangleright \Gamma$$

consisting of a linking P and a sequent Γ, where there is a bijective, type-preserving correspondence between the literal occurrences in Γ and the elements of P. There is no need to have a name for that bijection, since it will always be clear what it is.

There is a notational alternative to the previous definition of an F-prenet, obviously equivalent to it.

Notation 3.2.4.

- A *component with genus* K is a pair $K = (C, G)$, where G is a natural number, and C is a non-empty set of literals of the same type (set of (neg)atomic symbols decorated with elements in N). We refer to G as *genus*, and to C as a *component* of K. Thus, C is of the form $\{\mathrm{a}^k, \ldots, \mathrm{a}^l, \overline{\mathrm{a}}^m, \ldots \overline{\mathrm{a}}^n\}$. Sometimes we will refer to a component as a *blob*, which is a colloquial term we have used so far.
- A *linking* P is a finite set $P = \{K_1, \ldots, K_s\}$ of components with genera $K_1 = (C_1, G_1), \ldots, K_s = (C_k, G_k)$ where sets $C_1, \ldots, C_k$ are pairwise disjoint.
- A *F-prenet* is a pair $P \triangleright \Gamma$ where P is a linking and Γ is a sequent such that union $C_1 \cup \cdots \cup C_s$ of all of the components of P is in a bijective correspondence with the set of literals in Γ. Again, by convention we do not name the bijection as it is always obvious and present.

It is worth stressing out few points in the definitions above. First of all, given an F-prenet $P \triangleright \Gamma$ and the notation above, we have that $\mathcal{C}omp_P$ can be seen an equivalence relation on the set of literals P, where each $C_i \in \mathcal{C}omp_P$ is an equivalence class. Also, the partition means

$$P = \biguplus_{C_i \in \mathcal{C}omp_P} C_i,$$

(symbol $\uplus$ stands for disjoint union), while $\mathcal{G}en_P : P/\mathcal{C}omp_P \to \mathbb{N}$ maps classes of $\mathcal{C}omp_P$ to naturals, with

$$K_i = (C_i, \mathcal{G}en_P(C_i)),$$

for all $C_i \in \mathcal{C}omp_P$.

We will give no preference to any of the equivalent notations, and will use the one that facilitates easier notation in a given situation.

Graphical representation of F-prenets is the following : the sequent is represented as its syntactic forest with (neg)atoms as leaves. Each literal is connected to a base point representing a component of the linking, while the number assigned to the component is represented by number of loops sharing the same base point.

As mentioned before, an F-prenet is nothing more than association to a sequent of a map in the free Frobenius category generated by the literals of that sequent.

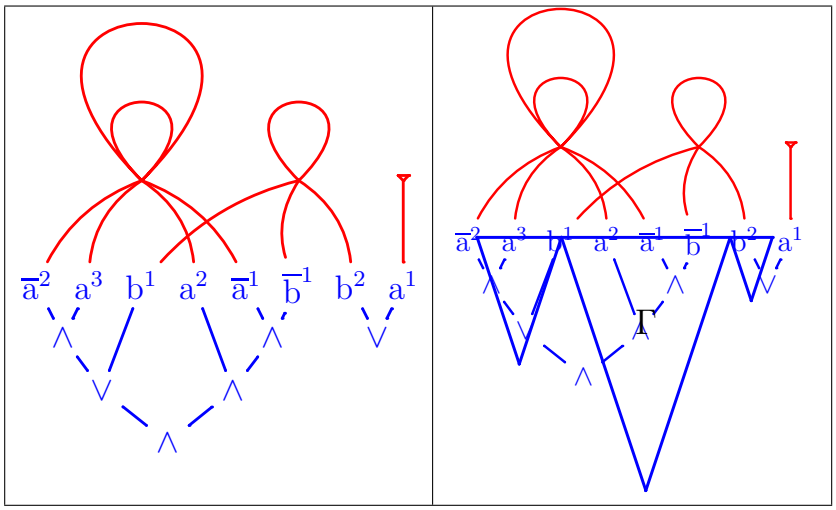


FIGURE 3.11 – Graphical representation of an F-prenet (left) and a generic 'icicle' representation of a sequent forest (right)

Remark 3.2.5. We have just defined the notion of an F-prenet, a well-defined mathematical structure. Equality of these structures is induced by their definition : two F-prenets are equal if both components are equal - the linkings and the sequents. Two sequents are equal if they are syntactically identical. The identification of two sequents induces a bijection between corresponding literal occurrences of the sequents we identify. Two linkings, thought of as triples, are equal when the components are equal, modulo the identification of the literals of the two sequents - if the underlaying sets of literals are equal modulo the identification, the partitions, and the corresponding functions. Notice that, as a consequence, we distinguish between F-prenets where where formulas are merely permuted within the sequent. This also means that we deal explicitly with permutations of formulas within a sequent at the level of syntax.

It is clear that the the notion of F-prenet captures our intuition on 'blobs assigned to sequents'. The following definitions describe operations that can be performed on linkings of F-prenets.

The following definitions introduce transformations on linkings. They do not carry a lot of meaning at this point and they are essentially of a notational nature. Their true purpose is to facilitate formulations of transformations appearing later in the text.

We begin by defining a transformation corresponding to ∇ operation of bimonoids.

Definition 3.2.6 ($\curlyvee$ transformation)**.**

- Let $P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$ be a linking and x^i, x^j two distinct (occurrences of) literals from a *same* component in $\mathcal{C}omp_P$, i.e. $(x^i, x^j) \in \mathcal{C}omp_P$.
  We define linking $P(x^i \curlyvee_s x^j)$ as follows (the s in $\curlyvee_s$ stresses the fact that the literals are from the same component) :
  - the set of literals $P(x^i \curlyvee_s x^j)$ is obtained from P by identifying the two literals $x^i$and x^j and giving them a fresh superscript for the literal x, x^{i-j}.

– $\mathcal{C}omp_{P(\mathrm{x}^i \curlyvee_s \mathrm{x}^j)}$ is partition of $P(\mathrm{x}^i \curlyvee_s \mathrm{x}^j)$ obtained from $\mathcal{C}omp_P$ by replacing $\mathrm{x}^i$and x^j by x^{i-j} in the component they both belong to
– $\mathcal{G}en_{P(\mathrm{x}^i \curlyvee_s \mathrm{x}^j)}$ is defined as follows : for a component $C \in \mathcal{C}omp_{P(\mathrm{x}^i \curlyvee_s \mathrm{x}^j)}$,

$$\mathcal{G}en_{P(\mathrm{x}^i \curlyvee_s \mathrm{x}^j)}(C) \quad = \quad \begin{cases} \mathcal{G}en_P(C), & C \text{ is not altered by } \mathrm{x}^i \curlyvee_s \mathrm{x}^j \\ \mathcal{G}en_P(C)+1, & \text{otherwise.} \end{cases}$$

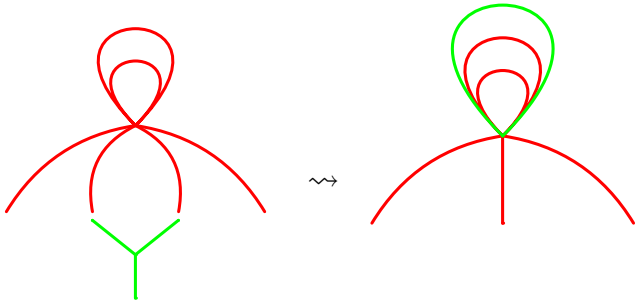


FIGURE 3.12 – $\curlyvee_s$ transformation on a linking

◦ Let now $\mathrm{x}^i, \mathrm{x}^j$ be two literals from *different* components in $\mathcal{C}omp_P$, i.e. $(\mathrm{x}^i, \mathrm{x}^j) \notin \mathcal{C}omp_P$. We define linking $P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)$ as follows (the d in $\curlyvee_d$ stresses the fact that the literals are from different components) :
– the set of literals $P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)$ is obtained from P by identifying the two literals $\mathrm{x}^i$and x^j and giving them a fresh superscript for the literal x, x^{i-j}.
– $\mathcal{C}omp_{P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)}$ is partition of $P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)$ obtained from $\mathcal{C}omp_P$ by joining the two classes $\mathrm{x}^i$and x^j belong to and replacing both $\mathrm{x}^i$and x^j by x^{i-j}
– $\mathcal{G}en_{P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)}$ is defined as follows : for a component $C \in \mathcal{C}omp_{P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)}$:

$$\mathcal{G}en_{P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j)}(C) = \begin{cases} \mathcal{G}en_P(C), & C \text{ is not affected by } \mathrm{x}^i \curlyvee_d \mathrm{x}^j \\ \mathcal{G}en_P(C_1) + \mathcal{G}en_P(C_2), & C \text{ is obtained by merging } C_1, C_2. \\ & \text{and replacing } \mathrm{x}^i, \mathrm{x}^j \text{ by } \mathrm{x}^{i-j}. \end{cases}$$

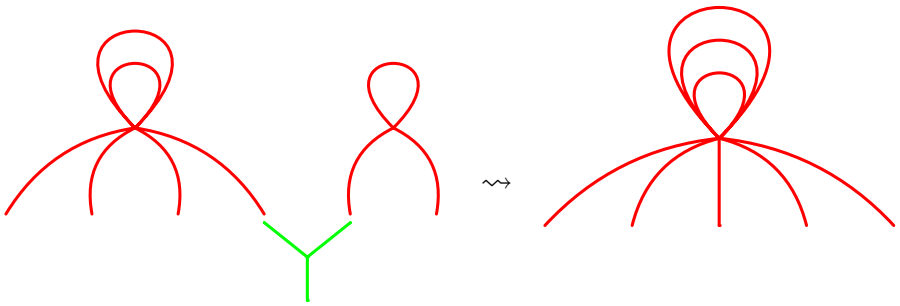


FIGURE 3.13 – $\curlyvee_d$ transformation on a linking

◦ One defines $P(\mathrm{x}^i \curlyvee \mathrm{x}^j)$ to be

$$P(\mathrm{x}^i \curlyvee \mathrm{x}^j) \quad := \quad \begin{cases} P(\mathrm{x}^i \curlyvee_s \mathrm{x}^j) & \mathrm{x}^i \text{ and } \mathrm{x}^j \text{ are from the same component in } P \\ P(\mathrm{x}^i \curlyvee_d \mathrm{x}^j) & \text{otherwise .} \end{cases}$$

- $P(\mathrm{x}^i \curlyvee \mathrm{x}^j, \ldots, \mathrm{y}^r \curlyvee \mathrm{y}^s)$ is defined as $P(\mathrm{x}^i \curlyvee \mathrm{x}^j) \ldots (\mathrm{y}^r \curlyvee \mathrm{y}^s)$. It is not difficult to see that the order is irrelevant here, i.e.

$$P(\mathrm{x}^i \curlyvee \mathrm{x}^j, \mathrm{y}^r \curlyvee \mathrm{y}^s) = P(\mathrm{x}^i \curlyvee \mathrm{x}^j)(\mathrm{y}^r \curlyvee \mathrm{y}^s) = P(\mathrm{y}^r \curlyvee \mathrm{y}^s)(\mathrm{x}^i \curlyvee \mathrm{x}^j).$$

  It suffices to think of linkings as maps in a suitable Frobenius category and observe that the result follows from functoriality of tensor. Indeed, as the $\curlyvee$ stands for the monoid multiplication, in that category $\nabla_x \otimes \nabla_y = \mathrm{Id}_x \otimes \nabla_y \circ \nabla_x \otimes \mathrm{Id}_y = \nabla_x \otimes \mathrm{Id}_y \circ \mathrm{Id}_x \otimes \nabla_y$.
- If $\mathrm{x}^i, \ldots, \mathrm{y}^r$ are all the literals in A^1 and $\mathrm{x}^j, \ldots, \mathrm{y}^s$ are all the literals in A^2, we define

$$P(\mathrm{A}^1 \curlyvee \mathrm{A}^2) := P(\mathrm{x}^i \curlyvee \mathrm{x}^j, \ldots, \mathrm{y}^r \curlyvee \mathrm{y}^s).$$

- If $\Gamma^1 = \mathrm{A}^i, \ldots, \mathrm{B}^r =$ and $\Gamma^2 = \mathrm{A}^j, \ldots, \mathrm{B}^s$ are two sequents, we define

$$P(\Gamma^1 \curlyvee \Gamma^2) := P(\mathrm{A}^i \curlyvee \mathrm{A}^j, \ldots, \mathrm{B}^r \curlyvee \mathrm{B}^s).$$

Definition 3.2.7 ($\smile$ transformation)**.** Let $P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$ be a linking and $\mathrm{x}^i, \mathrm{x}^j$ two literals from a *same* component in $\mathcal{C}omp_P$, i.e. $(\mathrm{x}^i, \mathrm{x}^j) \in \mathcal{C}omp_P$.

- Let $\mathrm{x}^i, \mathrm{x}^j$ be two occurrences of literals from a *same* component in $\mathcal{C}omp_P$, i.e. $(\mathrm{x}^i, \mathrm{x}^j) \notin \mathcal{C}omp_P$. We define linking $P(\mathrm{x}^i \smile_s \mathrm{x}^j)$ as follows (the s in $\smile_s$ stresses the fact that the literal occurrences are from the same component) :
  - the set of literals $P(\mathrm{x}^i \smile_s \mathrm{x}^j)$ is obtained from P by removing the two literals $\mathrm{x}^i$and x^j,
  - $\mathcal{C}omp_{P(\mathrm{x}^i \smile_s \mathrm{x}^j)}$ is partition of $P(\mathrm{x}^i \smile_s \mathrm{x}^j)$ obtained from $\mathcal{C}omp_P$ by removing $\mathrm{x}^i$and x^j from the component they both belong to
  - $\mathcal{G}en_{P(\mathrm{x}^i \smile_s \mathrm{x}^j)}$ is defined as follows : for a component $C \in \mathcal{C}omp_{P(\mathrm{x}^i \smile_s \mathrm{x}^j)}$,

$$\mathcal{G}en_{P(\mathrm{x}^i \smile_s \mathrm{x}^j)}(C) \quad = \quad \begin{cases} \mathcal{G}en_P(C) & C \text{ is not altered by } \mathrm{x}^i \smile_s \mathrm{x}^j \\ \mathcal{G}en_P(C) + 1, & \text{otherwise} \end{cases}$$

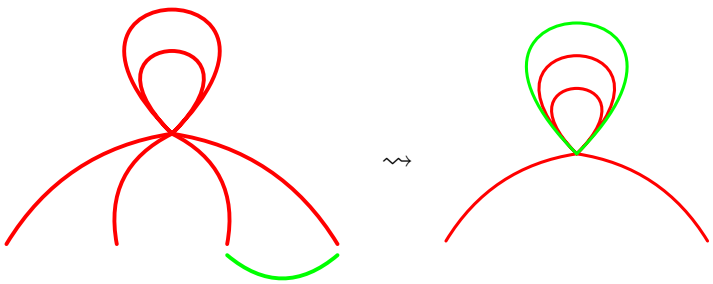


FIGURE 3.14 – $\smile_s$ transformation on a linking

- Let now $\mathrm{x}^i, \mathrm{x}^j$ be two literals from *different* components in $\mathcal{C}omp_P$, i.e. $(\mathrm{x}^i, \mathrm{x}^j) \notin \mathcal{C}omp_P$. We define linking $P(\mathrm{x}^i \smile_d \mathrm{x}^j)$ as follows (the d in $\smile_d$ stresses the fact that the literals are from different components) :
  - the set of literals $P(\mathrm{x}^i \smile_d \mathrm{x}^j)$ is obtained from P by removing the two literals $\mathrm{x}^i$and x^j,

– $\mathcal{C}omp_{P(\mathrm{x}^i \smile_d \mathrm{x}^j)}$ is partition of $P(\mathrm{x}^i \smile_d \mathrm{x}^j)$ obtained from $\mathcal{C}omp_P$ by joining the two classes $\mathrm{x}^i$and x^j belong to and removing both $\mathrm{x}^i$and x^j from the newly formed class
– $\mathcal{G}en_{P(\mathrm{x}^i \smile_d \mathrm{x}^j)}$ is defined as follows : for a component $C \in \mathcal{C}omp_{P(\mathrm{x}^i \smile_d \mathrm{x}^j)}$:

$$\mathcal{G}en_{P(\mathrm{x}^i \smile_d \mathrm{x}^j)}(C) = \begin{cases} \mathcal{G}en_P(C) & C \text{ is not affected by } \mathrm{x}^i \smile_d \mathrm{x}^j \\ \mathcal{G}en_P(C_1) + \mathcal{G}en_P(C_2), & C \text{ is obtained by merging } C_1, C_2 \\ & \text{and removing } \mathrm{x}^i, \mathrm{x}^j. \end{cases}$$

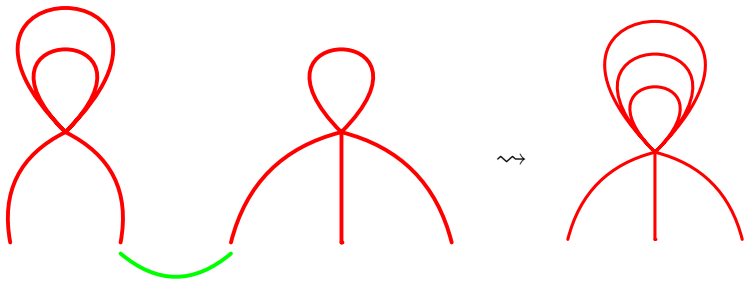


FIGURE 3.15 – $\smile_d$ transformation on a linking

∘ One defines $P(\mathrm{x}^i \smile \mathrm{x}^j)$ to be

$$P(\mathrm{x}^i \smile \mathrm{x}^j) := \begin{cases} P(\mathrm{x}^i \smile_s \mathrm{x}^j) & \mathrm{x}^i \text{ and } \mathrm{x}^j \text{ are from the same component in } P \\ P(\mathrm{x}^i \smile_d \mathrm{x}^j) & \text{otherwise .} \end{cases}$$

∘ $P(\mathrm{x}^i \smile \mathrm{x}^j, \ldots, \mathrm{y}^r \smile \mathrm{y}^s)$ is defined as $P(\mathrm{x}^i \smile \mathrm{x}^j) \ldots (\mathrm{y}^r \smile \mathrm{y}^s)$. Again, the order is irrelevant, i.e.

$$P(\mathrm{x}^i \smile \mathrm{x}^j, \mathrm{y}^r \smile \mathrm{y}^s) = P(\mathrm{x}^i \smile \mathrm{x}^j)(\mathrm{y}^r \smile \mathrm{y}^s) = P(\mathrm{y}^r \smile \mathrm{y}^s)(\mathrm{x}^i \smile \mathrm{x}^j).$$

and we choose to omit an easy proof-by-cases.
∘ If $\mathrm{x}^i, \ldots, \mathrm{y}^r$ are all of the literals in A^1 and $\mathrm{x}^j, \ldots, \mathrm{y}^s$ are all of the respective literals in A^2, we define [14]

$$P(A^1 \smile A^2) := P(\mathrm{x}^i \smile \mathrm{x}^j, \ldots, \mathrm{y}^r \smile \mathrm{y}^s).$$

Definition 3.2.8 ($\pitchfork$ transformation)**.** Let $P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$ be a linking and $\mathrm{x}^i, \ldots, \mathrm{y}^j$ fresh literals. One defines :

$$P \pitchfork \{\mathrm{x}^i, \ldots, \mathrm{y}^j\} := P \cup \{(\{\mathrm{x}^i\}, 0), \ldots, (\{\mathrm{y}^j\}, 0)\}.$$

If $\mathrm{x}^i, \ldots, \mathrm{y}^r$ are all literals in A, we define

$$P \pitchfork A := P \pitchfork \{\mathrm{x}^i, \ldots, \mathrm{y}^r\}.$$

Also, if $\Gamma = \mathrm{A}, \ldots, \mathrm{B}$, we define

$$P \pitchfork \Gamma := P \pitchfork \mathrm{A} \cdots \pitchfork \mathrm{B}.$$

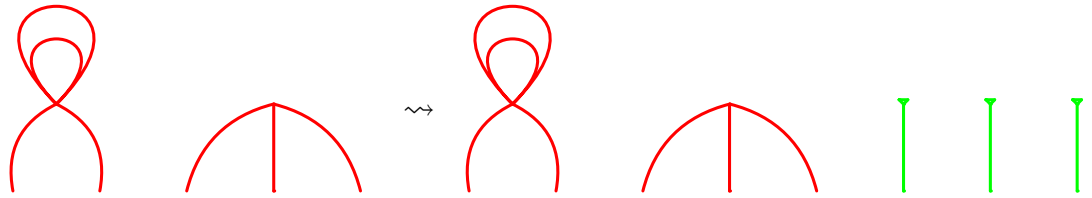


FIGURE 3.16 – ⅋ transformation on a linking

Definition 3.2.9 (Restriction)**.** Let $P = \{(C_1, G_1), \dots (C_k, G_k)\} = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$ be a linking, let Q be an arbitrary subset of the set P of literals. One defines restriction of P to Q, denoted by $P|_Q$ as :

$$P|_Q := \bigcup_{C_i \cap Q \neq \emptyset} \{(C_i \cap Q, G_i)\}.$$

In particular, given a F-net $P \triangleright \Gamma$, and an arbitrary subforest Γ' of the syntactic forest of Γ induced by choice of subset Q of the set of literals in Γ, we define $P|_{\Gamma'}$ to be $P|_Q$. Notice that Γ' need not (and it frequently won't) be a sequent itself.

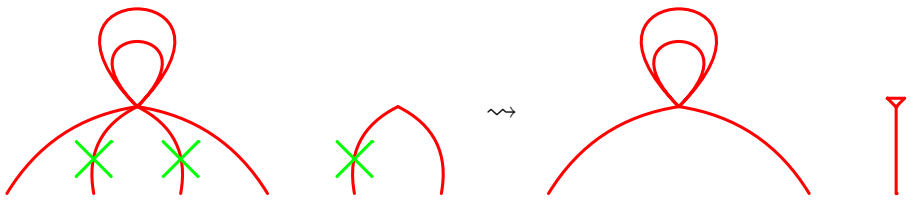


FIGURE 3.17 – Restriction of a linking

The following is an easy consequence of the definitions of transformations on linkings, but it is useful to stress it out

Proposition 3.2.10. *Let P be a linking. Then*

1. $P(\mathrm{x}^i \curlyvee \mathrm{x}^j)$ *is equal to* $P(\mathrm{x}^j \curlyvee \mathrm{x}^i)$ *(⅄ is symmetric);*
2. $(P(\mathrm{x}^i \curlyvee \mathrm{x}^j))(\mathrm{x}^{i-j} \curlyvee \mathrm{x}^k)$ *is equal to* $(P(\mathrm{x}^j \curlyvee \mathrm{x}^k))(\mathrm{x}^i \curlyvee \mathrm{x}^{j-k})$ *(⅄ is associative);*
3. $(P \parr \{\mathrm{x}^j\})(\mathrm{x}^i \curlyvee \mathrm{x}^j)$ *is equal to* P *(⅋ is neutral to ⅄).*

The reader can check that the definitions we have given above closely relate to the map generators and compositions of maps in the Frobenius category generated by a set of atoms.

Notation 3.2.11. We remind the reader that if $P \triangleright \Gamma$ is a F-net corresponding to a CL proof π, then, the linking P defines a map in a free Frobenius category from $P : I \to F(\Gamma)$,

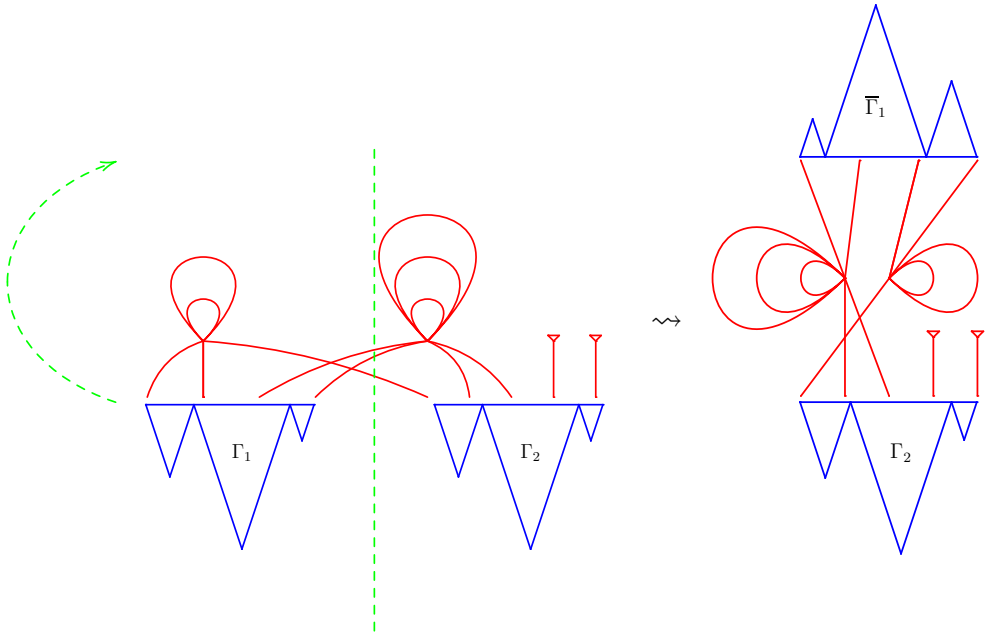


FIGURE 3.18 – From F-nets to Frobenius categories - a F-net corresponds to a map $P : \mathbf{1} \to F(\Gamma)$ in the free Frobenius category generated by the literals of Γ. The self-adjunction in the category can be visualized by a rotation/negation transformation of the subnets.

the image under the ('functor-like') interpretation F of the proof syntax, which acts on objects by replacing $\wedge, \vee$ and ", " by $\otimes$.

The propositional negation $\overline{(-)}$ is extended to sequents by defining $\overline{A, B} := \overline{A} \wedge \overline{B}$. Notice that by transposing maps in the free Frobenius category, the linking P of an F-prenet $P \triangleright \Gamma$, for Γ of the form Γ_1, Γ_2, corresponds to several maps $P : F(\overline{\Gamma}_1) \to F(\Gamma_2)$, for any choice of Γ_1, Γ_2, as in Figure 3.18.

Alternatively, a F-prenet $P \triangleright \Sigma$ corresponds to a class of maps in the free Frobenius category whose every object A is is labeled by a propositional sequent Γ, so that $A = F(\Gamma)$. We say that these maps are *induced maps by the F-prenet* $P \triangleright \Sigma$ and call the free Frobenius category with objects labeled by the sequents simply *the category of F-prenets*.

Syntactic category of F-prenets

F-prenets are also *objects* in a category.

Definition 3.2.12. We define the *syntactic category of F-prenets* $\mathcal{F}$**Synt**, where
– an object is an F-prenet

14. again, the definition is nothing more than a notation, and there is little sense in the definition at this point

– a map

$$f : P \triangleright \Gamma \to Q \triangleright \Sigma$$

is a function between sets

$$f : P \to Q \qquad (= \mathcal{L}(\Gamma) \to \mathcal{L}(\Sigma))$$

such that

1. for every formula A, f maps $\mathcal{L}(A)$ to a subset of $\mathcal{L}(\Delta)$ which is the set of literals of (and thus defines) a subformula in Δ, and the restriction of f to $\mathcal{L}$(A) *preserves the syntactical left-to-right order* of the literals on these formulas ;
2. for every component (C, m) in the linking P there is a (C', n) in Q such that the direct image $f(C) \subseteq C'$ and $m \leq n$.

Example 3.2.13. If Γ has only one formula, there can be at most one syntactic map $P \triangleright \Gamma \to Q \triangleright \Gamma$, because the map $P \to Q$ is necessarily determined by the identity map on Γ.

Example 3.2.14 (Isomorphisms in $\mathcal{F}$**Synt**)**.** Consider an iso f in $\mathcal{F}$**Synt**. Property 2. in the definition of $\mathcal{F}$**Synt** requires f restricted to a component to be a bijection, i.e. for an iso $\subseteq$ is an equality. Similarly, $\leq$ is equality, hence f preserves linkings. Since literals of a formula are mapped to literals defining a subformula, formulas are mapped to formulas. Therefore, *an iso in* $\mathcal{F}$**Synt** *is a map that permutes formulas of an F-prenet.*

On the other hand, all Exchange-induced maps are isos in $\mathcal{F}$**Synt**, as mentioned before. Thus isos in $\mathcal{F}$**Synt** are precisely maps induced by Exchange.

Since in our definition of linking $(P, \mathcal{C}omp_P, \mathcal{G}en_P)$ the set P is arbitrary, for any F-prenet $P \triangleright \Gamma$ there are many other prenets $P' \triangleright \Gamma$ where the function $P \to P'$ that corresponds to the identity on Γ is an isomorphism of linkings. Thus there are many copies lying around of "the same" F-Prenet. This situation is useful technically. This can be related to the following observation : given a natural number n, in some circumstances it is preferable to have a single totally ordered set with n elements (usually the ordinal $[n]$) but in other circumstances (like here) it is better to have arbitrary totally ordered sets, whose elements can have a range of names. As with total orderings the presence of many copies of an F-Prenet isomorphism type is not a problem because the isomorphism that relates two of them is *uniquely defined.*

From CL proofs to F-prenets

Definitions of transformations on linkings we have just given facilitate correspondence between F-prenets and the sequent system CL we have considered so far.

In what follows, every unary rule

$$\frac{\vdash \Gamma'}{\vdash \Gamma}$$

will give rise to a syntactic map $P' \triangleright \Gamma' \to P \triangleright \Gamma$

and every binary rule

$$\frac{\vdash \Gamma' \quad \vdash \Gamma''}{\vdash \Gamma'}$$

will give rise to two syntactic maps $P' \triangleright \Gamma' \to P \triangleright \Gamma$ and $P'' \triangleright \Gamma'' \to P \triangleright \Gamma$. We name these maps by the rule they are assigned to, i.e. we speak of a $\vee$-map, of $\wedge$-maps.

1. A proof consisting of a single application of axiom rule corresponds to an F-prenet as follows :

$$\overline{\vdash \overline{\mathrm{a}}, \mathrm{a}} \qquad \rightsquigarrow \qquad \overline{\{(\{\overline{\mathrm{a}}, \mathrm{a}\}, 0)\} \triangleright \overline{\mathrm{a}}, \mathrm{a}}$$

An axiom link is graphically presented as in Figure 3.19
Intuitively, this is consistent with traditional presentation of an axiom link – as an

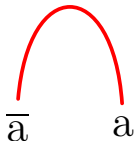


FIGURE 3.19 – F-prenet for the Ax rule

arc with two literals as the boundary. The ark, of course, does not create any loops.

2. The Exchange rule affects, naturally, the syntactic forest of the sequent, and permutes wires in graphical representation of the linking, but does not change the linking itself (!).

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{B}, \mathrm{A}, \Sigma}\ \mathit{Exch} \qquad \rightsquigarrow \qquad \frac{P \triangleright \Gamma, \mathrm{A}, \mathrm{B}}{P \triangleright \Gamma, \mathrm{B}, \mathrm{A}}\ .$$

As shown in 3.2.14, *Exch*-maps are precisely isos in $\mathcal{F}$**Synt**.

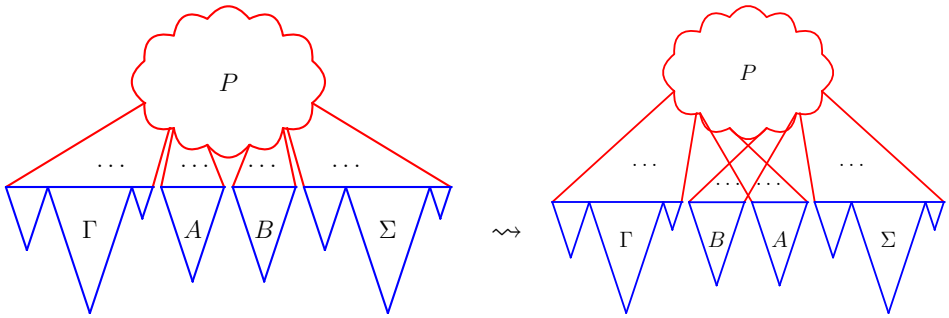


FIGURE 3.20 – F-prenet for the *Exch* rule

3. Logical rules affect, naturally, the syntactic tree, but do not essentially change the linking. For the disjunction rule, this is literally the case :

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}}\ \vee \qquad \rightsquigarrow \qquad \frac{P \triangleright \Gamma, \mathrm{A}, \mathrm{B}}{P \triangleright \Gamma, \mathrm{A} \vee \mathrm{B}}\ .$$

Conjunction rule, in accordance with the discussion on Frobenius algebras generated by composite types, amounts to 'putting things side-by-side'.

$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \mathrm{B}, \Sigma}{\vdash \Gamma, \mathrm{A} \wedge \mathrm{B}, \Sigma}\ \wedge \qquad \rightsquigarrow \qquad \frac{P' \triangleright \Gamma, \mathrm{A} \quad P'' \triangleright \mathrm{B}, \Sigma}{P' \uplus P'' \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B}, \Sigma}\ ,$$

where $\uplus$ stands for disjoint union. In both cases, the induced $\vee$-map and $\wedge$-maps are uniquely determined by definition.

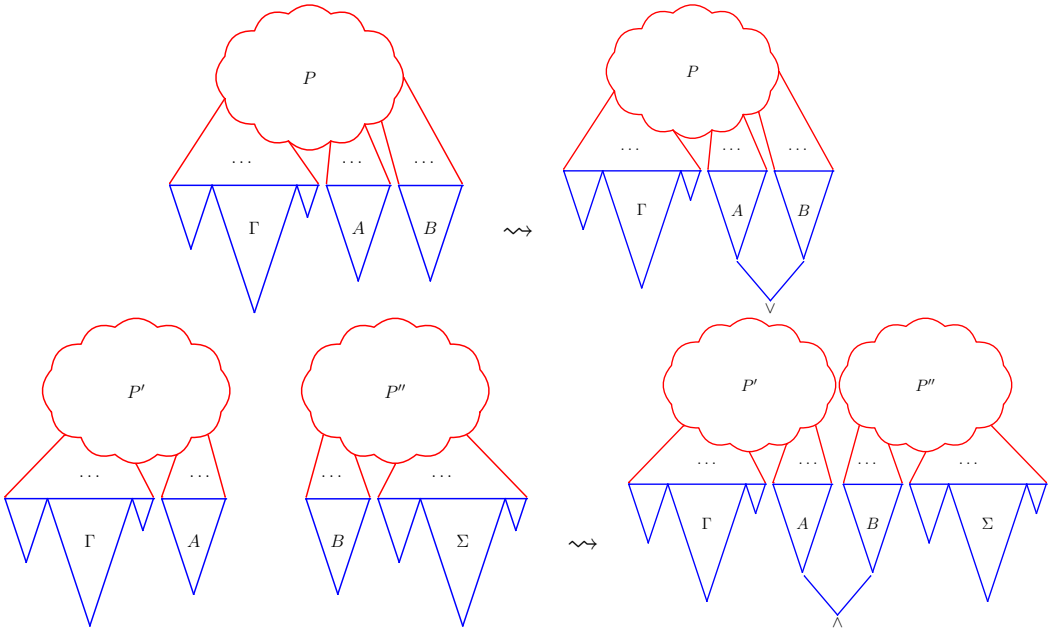


FIGURE 3.21 – F-prenet for the logical rules

4. The Mix rule is treated the same way as conjunction :

$$\frac{\vdash \Gamma \quad \vdash \Sigma}{\vdash \Gamma, \Sigma}\ \mathit{Mix} \qquad \rightsquigarrow \qquad \frac{P' \triangleright \Gamma \quad P'' \triangleright \Sigma}{P' \uplus P'' \triangleright \Gamma, \Sigma}\ .$$

The analogy with the conjunction rule is extending to the induced map in $\mathcal{F}$**Synt**.

5. Weakening :

$$\frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}}\ \mathit{Weak} \qquad \rightsquigarrow \qquad \frac{P \triangleright \Gamma}{P \upharpoonright \mathrm{A} \triangleright \Gamma, \mathrm{A}}\ .$$

One should think of weakening as introducing literals of a formula 'out of thin air'. This is geometrically presented as a semi-closed interval whose boundary is the corresponding literal. Again, the induced map in $\mathcal{F}$**Synt** is defined uniquely, and is and embedding.

6. Contraction rule is also mapped to F-prenets using one of the previously defined operations :

$$\frac{\vdash \Gamma, \mathrm{A}^1, \mathrm{A}^2}{\vdash \Gamma, \mathrm{A}^{1-2}}\ \mathit{Contr} \qquad \rightsquigarrow \qquad \frac{P \triangleright \Gamma, \mathrm{A}^1, \mathrm{A}^2}{P(\mathrm{A}^1 \curlyvee \mathrm{A}^2) \triangleright \Gamma, \mathrm{A}^{1-2}}\ .$$

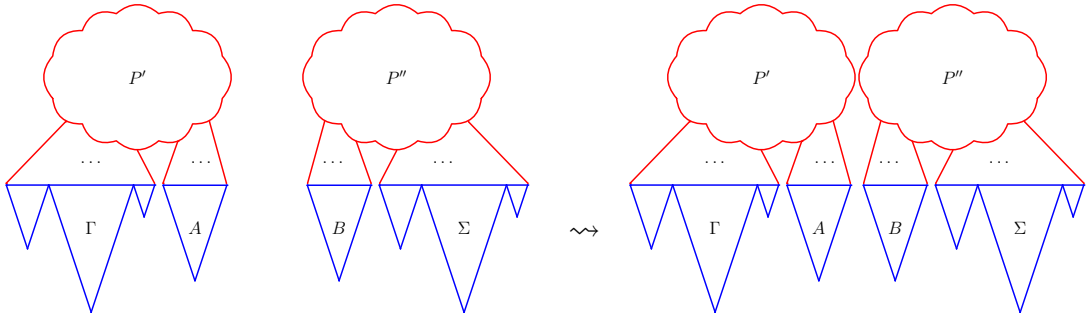


FIGURE 3.22 – F-prenet for the *Mix* rule. Notice how this rule is 'invisible' from the point of view of F-prenet representations.

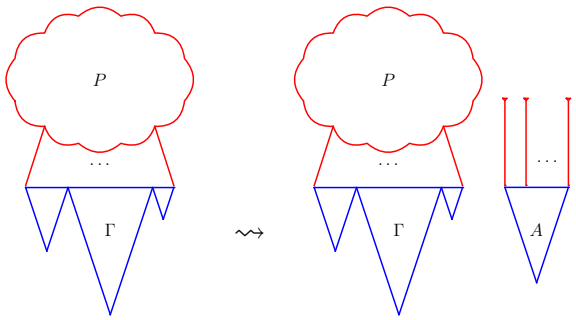


FIGURE 3.23 – F-prenet for the *Weak* rule

The $1-2$ style of superscript notation is useful in denoting formulas obtained by contraction for keeping track of the formulas being contracted. We extend this to the case where contraction is performed on a sequent / iteratively on a sequence of formulas belonging to two sequents – it is useful to have Γ^{r-s} denoting the sequence of formulas $\mathrm{A}^{r-s}, \ldots, B^{r-s}$, each of which obeys the superscript convention for contractions.

What we have just defined is a construction that assigns a well defined F-prenet to a given CL proof. The inverse, the problem of existence of a CL proof for given an F-prenet is more challenging, and we will address it soon, making the use of the syntactic category of F-prenets.

Let us have a look at an example of an F-prenet corresponding to a proof. The F-prenet corresponding to our benchmark proof is shown in Figure 3.24. It is worth noticing that had we iterated the rule application in the proof (composed doubling map with itself), the corresponding F-prenet would be almost the same, with the difference of genus of the single component being equal to the number of iterations. This implies that F-prenets are capable of differentiating between such proofs, i.e. F-prenets are can count applications of the axiom rule.

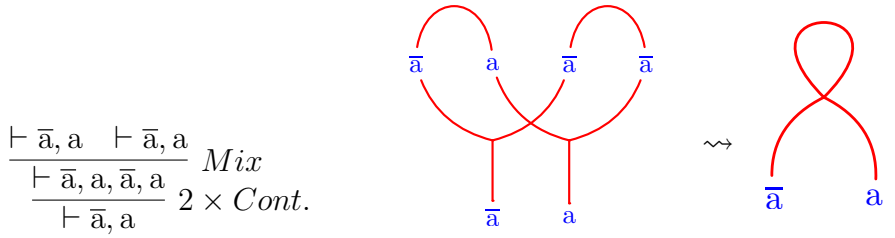


FIGURE 3.24 – F-prenet of the proof corresponding to doubling map

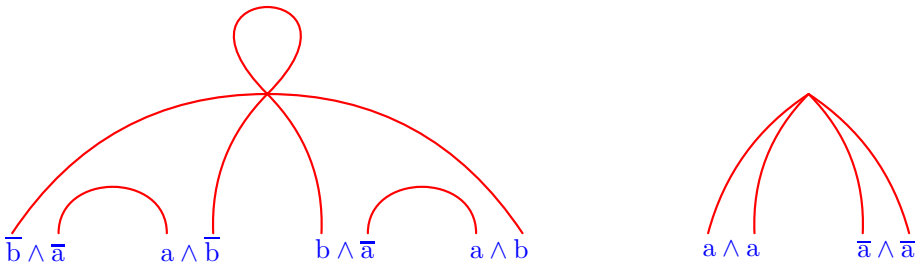


FIGURE 3.25 – A F-prenet that corresponds to no proof in CL, but it does to one in SK (left), and a one for which there is no proof in the systems (right). Notice that the both sequents are provable.

3.2.1 Correctness for F-prenets

The question 'when does an F-prenet correspond to a proof?' seems to be a difficult one. In fact, as argued in [LS05b] for $\mathbb{N}$-nets, one should not hope for an easy correctness criterion. As argued by the authors in the case of $\mathbb{N}$-nets, ability to count axiom links of these nets, i.e. to keep track of resources, makes the problem close to the NP=coNP ? problem (cf. [CR79]). The same is bound to happen here, since we have seen that the F-prenets are also capable of counting axiom links. We elaborate on on this and other connections to complexity issues in Section 4.4.

Example 3.2.15. To illustrate how delicate the issue of correctness is, consider the F-prenets in Figure 3.25. The figure to the left corresponds to no CL proof, but it corresponds to a proof in a deep inference system SK from [Brü03], while there is neither CL nor SK proof for the F-prenet to the right.

We show the claims on existence/non-existence of proofs in respective systems for the given F-prenets in Example 4.3.18.

As the previous example shows, any correctness criterion one may devise needs to be tailored to a specific deductive system. Therefore, in what immediately follows, we restrict our attention to the correctness of F-prenets for the one-sided classical sequent system CL we have considered so far.

Before we are able to formulate the correctness procedure, we need some more notation.

Notation 3.2.16. Let $P \triangleright \Gamma$ be an F-prenet and let π be a CL proof whose F-prenet is $P \triangleright \Gamma$, with $P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$.

i) We denote the number of literal occurrences in Γ (= elements of P) by $|\mathcal{L}(\Gamma)| = |P|$;

ii) $|\mathcal{C}omp_P|$ stands for number of components of the linking P;

iii) the sum of all genera in P is denoted by $|\mathcal{G}en_P|$, i.e.

$$|\mathcal{G}en_P| = \sum_{(C_i,G_i)\in P} G_i = \sum_{(C_i,G_i)\in P} \mathcal{G}en_P(C_i).$$

iv) (a) given the CL proof π, the number of applications of axioms in π is denoted by $|Ax|$;

(b) let π be a CL derivation consisting of a single application of a weakening,

$$\frac{\vdash \Gamma}{\vdash \Gamma, \mathrm{A}}\ \mathit{Weak}.$$

For such π we define $|\,\wr\,|$ to be $|\mathcal{L}(\mathrm{A})|$. For π an arbitrary proof in CL, we define $|\,\wr\,|$ to be sum of of these, i.e.

$$|\,\wr\,| = \sum_{\substack{\mathrm{A} \text{ is introduced in the} \\ \text{proof by weakening}}} |\mathcal{L}(\mathrm{A})|.$$

(c) let π be a CL derivation consisting of a single application of contraction

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}}{\vdash \Gamma, \mathrm{A}}\ \mathit{Contr}.$$

For such π we define $|\mathsf{Y}|$ to be $|\mathcal{L}(\mathrm{A})|$. For an arbitrary CL proof π, $|\mathsf{Y}|$ is defined as sum of these over all contractions in π, i.e.

$$|\mathsf{Y}| = \sum_{\substack{\mathrm{A}^{l-m} \text{ obtained by} \\ \text{appying } \mathit{Contr} \text{ to } \mathrm{A}^l, \mathrm{A}^m}} |\mathcal{L}(\mathrm{A}^{l-m})|.$$

We have defined contractions of literals as combination of two kinds of contractions, a genus-increasing one, Y_s, and component-connecting one, Y_d, thus we denote accordingly the number of corresponding rule applications, i.e. we have $|\mathsf{Y}| = |\mathsf{Y}_s| + |\mathsf{Y}_d|$.

Figure 3.26 represents the action of individual rule applications on a linking (we ignore the transformation $\smile$ for the moment). The effect of a rule application on a linking P is reflected in a combination of increase/decrease of the sizes of the three components of a linking, $|P|$, $|\mathcal{C}omp_P|$, $|\mathcal{G}en_P|$, that is, number of literals in the linking, number of components, and/or genus. Notice that the table in Figure 3.26 encompasses some, but not all of the data of the transformations of linkings we consider.

The following observation is easy to deduce from Figure 3.26.

	$\lvert P\rvert$	$\lvert \mathcal{C}omp_P\rvert$	$\lvert \mathcal{G}en_P\rvert$
$\lvert Ax\,\rvert$	$+2$	$+1$	0
$\lvert \,\wp\,\rvert$	$+1$	$+1$	0
$\lvert \mathsf{Y}_s\rvert$	-1	0	$+1$
$\lvert \mathsf{Y}_d\rvert$	-1	-1	0
$\lvert \smile_s \rvert$	-2	0	$+1$
$\lvert \smile_d \rvert$	-2	-1	0

FIGURE 3.26 – Action of the CL rules and transformations on linkings

Lemma 3.2.17 (Counting axiom links in an F-prenet)**.** *If an F-prenet $P \triangleright \Gamma$ corresponds to a CL proof, then*

$$\lvert Ax\,\rvert = \lvert P\rvert - \lvert \mathcal{C}omp_P\rvert + \lvert \mathcal{G}en_P\rvert.$$

So we see that given an F-prenet $P \triangleright \Gamma$, the value $\lvert Ax\,\rvert = \lvert P\rvert - \lvert \mathcal{C}omp_P\rvert + \lvert \mathcal{G}en_P\rvert$ can be read-out immediately from the prenet, and represents an intrinsic feature of an F-prenet. This justifies the following

Definition 3.2.18 (Characteristic of an F-prenet)**.** Given an F-prenet $P \triangleright \Gamma$, we call the value

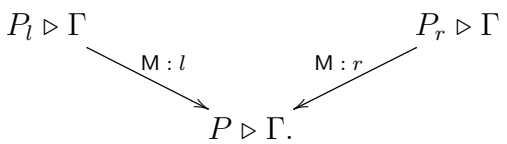
$$\mathcal{C}har(P \triangleright \Gamma) = \lvert P\rvert - \lvert \mathcal{C}omp_P\rvert + \lvert \mathcal{G}en_P\rvert$$

the *characteristic* of the prenet.

Thus, given a (cut-free) F-prenet, one can conclude immediately the number of applications of the axiom rule in the corresponding CL proof, if such exists.

Some more work needs to be done to be able to formulate the correctness procedure. To that purpose, we look at $\mathcal{F}$**Synt**.

Definition 3.2.19. a) (M) : In the category $\mathcal{F}$**Synt**, we define a family of *elementary Mix cospans* or *elementary* M *cospans*, i.e. diagrams of the form $Y \to X \leftarrow Z$ to be



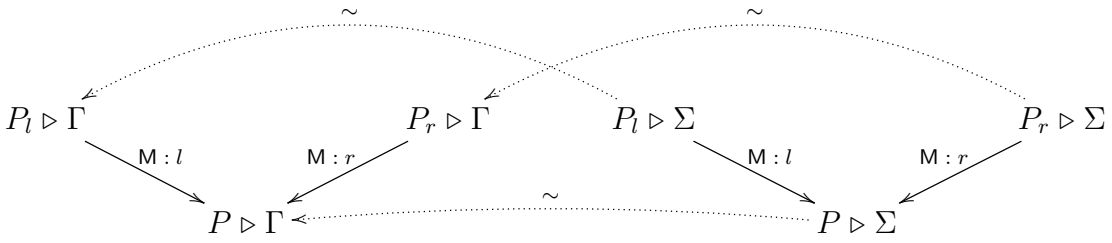
where each cospan in the family is defined by the maps

$$P_l \triangleright \Gamma \xrightarrow{\mathsf{M}\,:\,l} P \triangleright \Gamma \qquad \text{and} \qquad P_r \triangleright \Gamma \xrightarrow{\mathsf{M}\,:\,r} P \triangleright \Gamma$$

for which

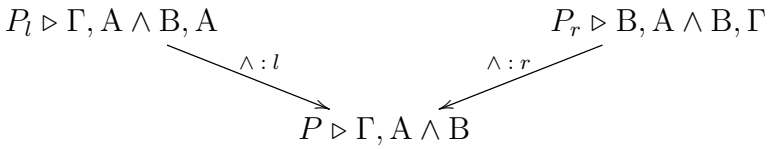
$$P = P_l \uplus P_r(\Gamma \curlyvee \Gamma).$$

A *general* M-*cospan*, abbreviated to just M-*cospan*, denoted also by M, is any cospan of $\mathcal{F}$**Synt** isomorphic to an elementary M-cospan, i.e. a cospan to the right of the following commutative diagram



where the cospan to the left is an elementary M.

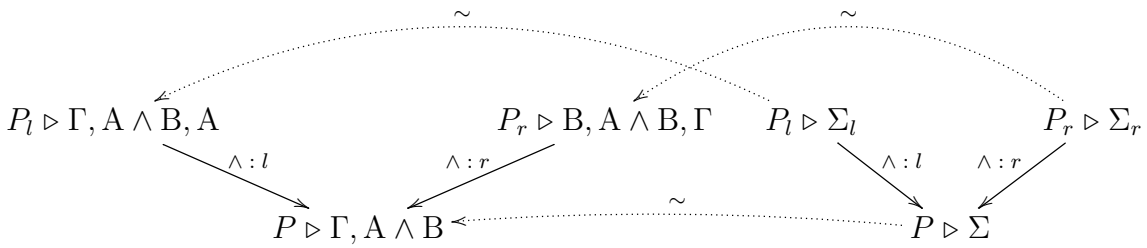
b) ($\wedge$) Further, we define *elementary family of* $\wedge$-*cospans* to be with legs $\xrightarrow{\wedge : l}$ and $\xrightarrow{\wedge : r}$ between the objects in of $\mathcal{F}$**Synt** – $P \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B}$, $P_l \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B}, \mathrm{A}$ and $P_r \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B}, \mathrm{B}$:



where one has

$$P = P_l \uplus P_r(\Gamma \curlyvee \Gamma, \mathrm{A} \curlyvee \mathrm{A} \curlyvee \mathrm{A}, \mathrm{B} \curlyvee \mathrm{B} \curlyvee \mathrm{B}).$$

Again a *generalized* $\wedge$-*cospan,* simply a $\wedge$-*cospan* is a cospan in $\mathcal{F}$**Synt** isomorphic to an elementary one. Thus, the commutative diagram looks like this, where the elementary $\wedge$-cospan is the one to the left.



The clear intention when defining the cospans we have just introduced in the category $\mathcal{F}$**Synt** is to handle places where a classical proof branches, i.e. the $\wedge$-rule and the *Mix* rule cases. It also implicitly accounts for contractions that are permitted in restricted cases where the proof branchings occur.

Notice that given an M-cospan, permuting the left and right side will also yield an M-cospan. This is not the case for a $\wedge$-cospan because $\mathrm{A} \wedge \mathrm{B}$ is not equal to $\mathrm{B} \wedge \mathrm{A}$ in general

Another thing worth noticing is the fact that the maps, of a cospan, both M- and $\wedge$-cospan 'add up' domain F-prenets to obtain the codomain F-prenet in a sense that will be make precise later. However, the domain F-prenets in a cospan cannot be seen as unique complements of each other, e.g., one part of a M-cospan, say $P_l \triangleright \Gamma \xrightarrow{\wedge : l} P \triangleright \Gamma$ does not uniquely determine the other part such that $P_r \triangleright \Gamma^r \xrightarrow{\mathsf{M} : r} P \triangleright \Gamma$. An example of a (portion of) M-cospan is given in Figure 3.27, a similar can be constructed for a $\wedge$-cospan.

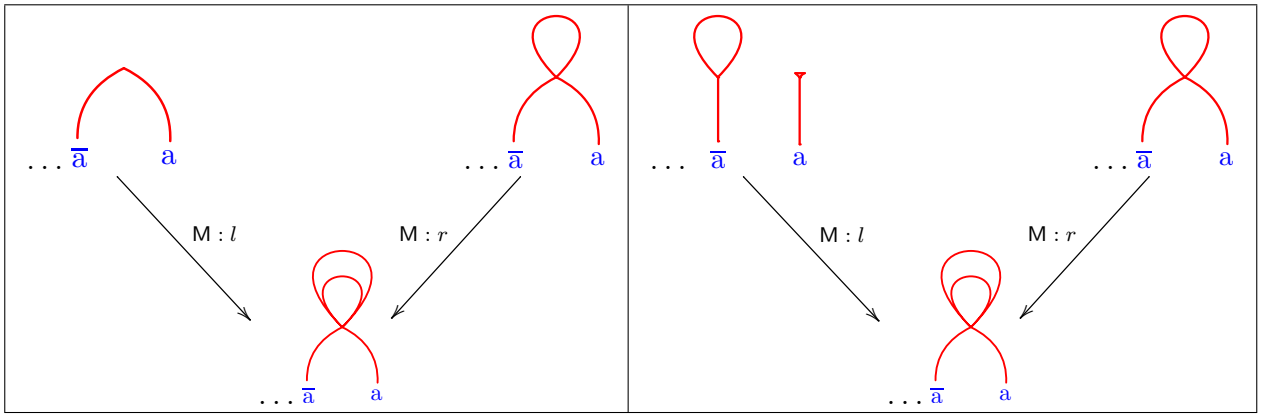


FIGURE 3.27 – Two different (elementary) M-cospans with a common leg.

Definition 3.2.20. An *anodyne map*

$$P \triangleright \Gamma \multimap P \triangleright \Delta$$

is a syntactic map that can be decomposed

$$P \triangleright \Gamma \xrightarrow{\sim} P \triangleright \Delta_1 \xrightarrow{\vee} \cdots \xrightarrow{\vee} P \triangleright \Delta_n = \Delta$$

as an isomorphism followed by a sequence of $\vee$-introduction maps (which do not affect the linking, only the sequent).

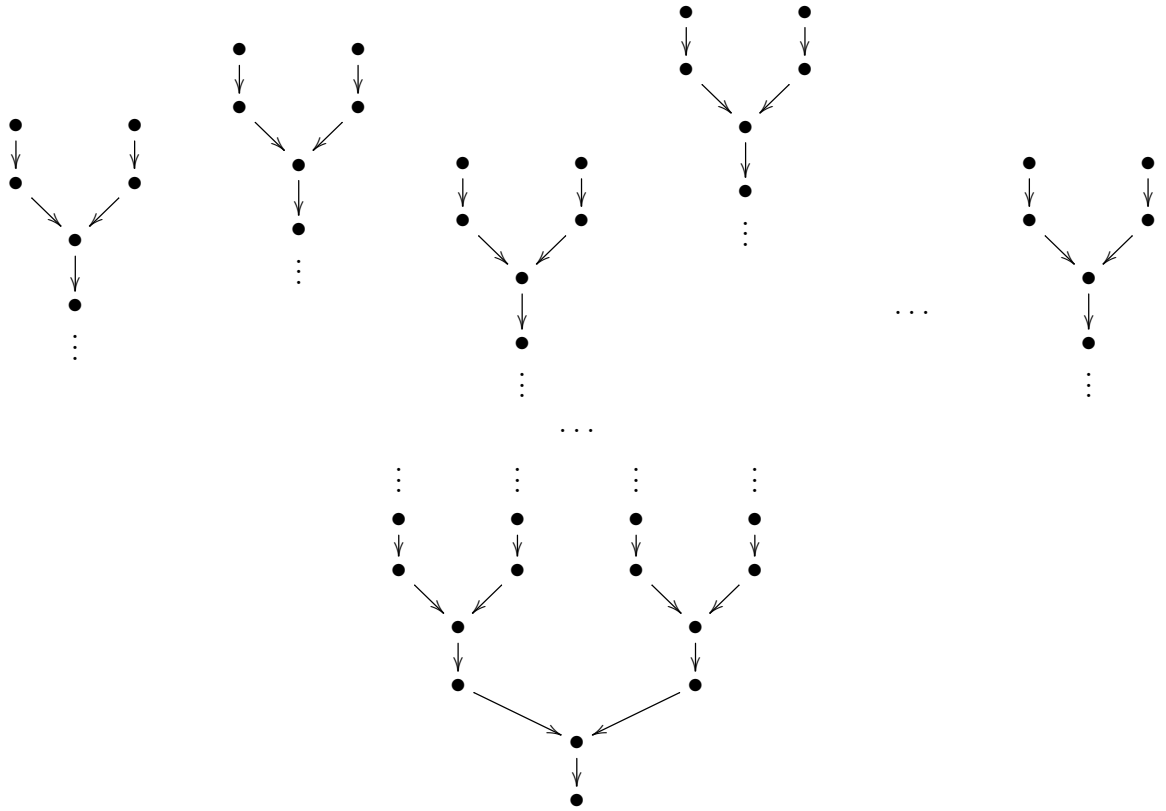
There is an important anodyne map, which corresponds to an iso followed by the removal of *all* outer disjunctions in a sequent ; we write

$$[P \triangleright \Gamma] = P \triangleright [\Gamma] \longrightarrow\!\!\!\!\!\!\square\!\!\!\!\!\!\longrightarrow P \triangleright \Gamma$$

to denote the anodyne map whose domain is the sequent where all outer disjunctions have been removed.

All the necessary ingredients are present to formulate the notion of correctness for F-prenets.

Definition 3.2.21 (Correct F-nets)**.** A *correctness tree* or *correctness diagram* $\mathcal{T}$ is a diagram (functor) $\mathcal{T} : \mathsf{T} \to \mathcal{F}\mathbf{Synt}$ where T is of the form



(3.1)

and every branching is mapped to either a $\wedge$- or a M-cospan and every vertical arrow is mapped to an anodyne map.

In addition, each leaf of the tree is an F-prenet of the form

$$\{(\{\mathrm{a}, \overline{\mathrm{a}}\}, 0), (\{y\}, 0), \ldots, (\{x\}, 0)\} \triangleright \mathrm{a}, \overline{\mathrm{a}}, \Sigma.$$

A F-prenet $P \triangleright \Gamma$ is a CL-*correct F-net*, (or simply a F-net) if it is the root of some *correctness diagram* $\mathcal{T}$.

The previous definition introduces notion of a correctness diagram involving anodyne maps. It is obvious that a correct F-net has a correctness diagram if and only if it has one where all anodyne maps are of the $\multimap\!\!\boxplus\!\!\rightarrow$ kind. Indeed, it amounts to removing disjunctions sooner than later in a sequent, and this is always possible since both types of cospans preserve disjunction formulas. To simplify discussion, from now on we will always assume that the anodyne maps are of the $\multimap\!\!\boxplus\!\!\rightarrow$ type.

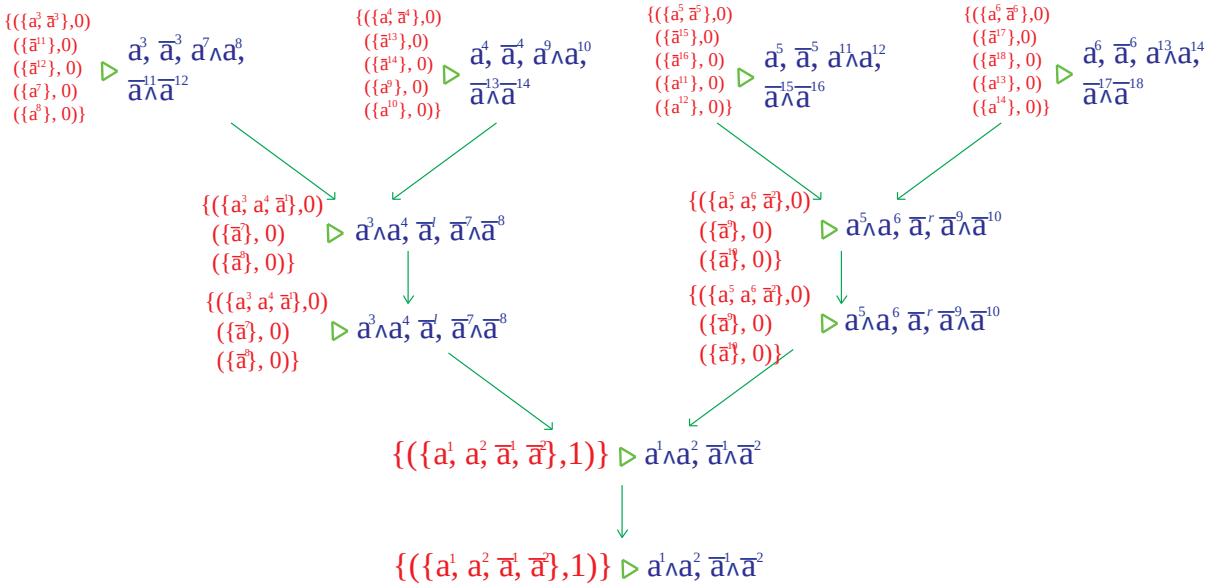
Proposition 3.2.22. *Let $\mathcal{T}$ be a correctness diagram for a F-net $P \triangleright \Gamma$. The diagram $\mathcal{T}$ has exactly $\mathcal{C}har(P \triangleright \Gamma)$ leaves.*

Démonstration. By induction on the size of $\mathcal{T}$. If $\mathcal{T}$ is a leaf or a single $\multimap\!\!\boxplus\!\!\rightarrow$ branch, P has all 0 genus components, one with two elements and the rest with a single element, thus

$\mathcal{C}har(P \triangleright \Gamma) = 1$. For the induction step, notice that for both types of cospans the sum of the characteristics of domain F-prenets is equal to the characteristic of the codomain F-prenet. This follows from Lemma 3.2.17, as Y does not change the characteristic of a F-prenet. □

Example 3.2.23. The image below shows a correctness diagram for the F-prenet

$$\{(\{a^1, a^2, \overline{a}^1, \overline{a}^2\}, 1)\} \triangleright a^1 \wedge a^2, \overline{a}^1 \wedge \overline{a}^2$$



Observation 3.2.24. If an F-prenet is correct, then so is any F-prenet occurring in its correctness diagram.

The definition we just gave should be understood as follows : an F-prenet is a correct F-net (as will be shown soon, this is same as saying 'comes from a proof'), if one can construct a canonical proof directed by the F-prenet. In a certain sense the definition of correctness tree corresponds to a logic where the only rules are axiom with weakening applied only in the beginning of a proof, $\wedge$ and *Mix* coupled with contractions according to the cospan definitions, and $\vee$-introduction, all of them applied in a specific order.

Since these rules are derivable in CL, it follows immediately that a correct net is provable in CL, and the details can be found in the proof of the following theorem. The converse is the proof that these rules are sufficient and just as powerful as CL and more, that the *proof theory* of the restricted logic is the same.

Theorem 3.2.25 (Sequentialization)**.** *For every F-net there is a* CL *proof that corresponds to that F-net. The converse also holds, i.e. given a* CL *proof, its F-prenet is in fact a* CL*-correct F-net.*

Démonstration. Let us first show the existence of a CL proof for a F-net. Therefore, let $P \triangleright \Gamma$ be a F-net and let $\mathcal{T}$ be its correctness diagram from Definition 3.2.21. We will use $\mathcal{T}$ to construct a CL proof whose F-prenet is $P \triangleright \Gamma$.

Notice first that each leaf of $\mathcal{T}$ has a CL proof that corresponds to it. Indeed, each leaf F-prenet is of the form : $\{(\{\mathrm{a}, \overline{\mathrm{a}}\}, 0)\} \uplus P' \triangleright \mathrm{a}, \overline{\mathrm{a}}, \Gamma'$, where components of P' are one-element sets, with assigned genus of 0. A corresponding CL proof can be read-out from the leaf now ; it consists of an instance of the the Axiom rule, followed by weakenings, for every formula of Γ'.

Now, what is left is to check that the proof corresponding to $P \triangleright \Gamma$ can be recovered from the proofs we have just constructed. The guidance is provided now from the correctness diagram $\mathcal{T}$. First, we replace the leaves of $\mathcal{T}$ by the corresponding CL proofs we have already constructed. Further, each $\xrightarrow{\wedge : l}$ / $\xrightarrow{\wedge : r}$ branching can be replaced by an instance of $\wedge$ rule followed by sequence of contractions that eliminate duplicates in the context. The similar with the $\xrightarrow{\mathsf{M} : l}$ / $\xrightarrow{\mathsf{M} : r}$ branching where the *Mix* is followed by sequence of contractions of duplicates. Finally, the $\multimap\!\!\boxminus\!\!\rightarrow$ is to be replaced by the sufficient number of instances of $\vee$-rule. What we end up is a CL proof whose F-prenet is $P \triangleright \Gamma$, and which utilizes all of the intermediate proofs as in the example on Figure 3.28.

For the opposite direction, we give a proof by induction on length of the CL proof π whose F-prenet we want to show to be a correct F-net. Base of the induction, the case where π is instance of the Axiom rule is trivial. Consider now different cases depending on the last rule applied in π.

– *Weak* : in the last rule application, $\vdash \Gamma, \mathrm{A}$ was concluded from $\vdash \Gamma$.

$$\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \mathrm{A}}\ \mathit{Weak}.$$

By induction hypothesis, there is a correctness diagram $\mathcal{T}$ for the F-prenet $P \triangleright \Gamma$ corresponding to the proof of $\vdash \Gamma$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by replacing the root node by $P \parr \mathrm{A} \triangleright \Gamma, \mathrm{A}$, and each other node $Q \triangleright \Sigma$ of $\mathcal{T}$ by $Q \parr \mathrm{A} \triangleright \Sigma, [\mathrm{A}]$. The claim follows from the virtue of $\parr$ being the neutral for Y.

– $\vee$-rule : $\vdash \Gamma, \mathrm{A} \vee \mathrm{B}$ was concluded from $\vdash \Gamma, \mathrm{A}, \mathrm{B}$. By induction hypothesis, there is a tree $\mathcal{T}$ from the definition of F-nets. Since $[P \triangleright \Gamma, \mathrm{A} \vee \mathrm{B}] = [P \triangleright \Gamma, \mathrm{A}, \mathrm{B}]$, the correctness diagram for $P \triangleright \Gamma, \mathrm{A} \vee \mathrm{B}$, is obtained from $\mathcal{T}$ simply by replacing the root $P \triangleright \Gamma, \mathrm{A}, \mathrm{B}$ by $\vdash \Gamma, \mathrm{A} \vee \mathrm{B}$.

– $\wedge$-rule : $\vdash \Gamma, \mathrm{A} \wedge \mathrm{B}, \Sigma$ was concluded from $\vdash \Gamma, \mathrm{A}$ and $\vdash \mathrm{B}, \Sigma$. Again, by induction hypothesis, there are two correctness diagrams $\mathcal{T}'$ and $\mathcal{T}''$ from the definition of F-nets for $L \triangleright \Gamma, \mathrm{A}$ and $R \triangleright \mathrm{B}, \Sigma$ for proofs of $\vdash \Gamma, \mathrm{A}$ and $\vdash \mathrm{B}, \Sigma$, respectively. We already showed in the *Weak* case that

$$L \parr \Sigma \parr (\mathrm{A} \wedge \mathrm{B}) \triangleright \Gamma, \Sigma, \mathrm{A} \wedge \mathrm{B}, \mathrm{A} \quad \text{and} \quad R \parr \Gamma \parr (\mathrm{A} \wedge \mathrm{B}) \triangleright \mathrm{B}, \mathrm{A} \wedge \mathrm{B}, \Gamma, \Sigma$$

are also correct with

$$L \parr \Sigma \triangleright \Gamma, \Sigma, \mathrm{A} \wedge \mathrm{B}, \mathrm{A} \ \multimap\!\!\boxminus\!\!\rightarrow\ \mathcal{T}_l$$

and

$$R \parr \Gamma \triangleright \mathrm{B}, \mathrm{A} \wedge \mathrm{B}, \Gamma, \Sigma \ \multimap\!\!\boxminus\!\!\rightarrow\ \mathcal{T}_r$$

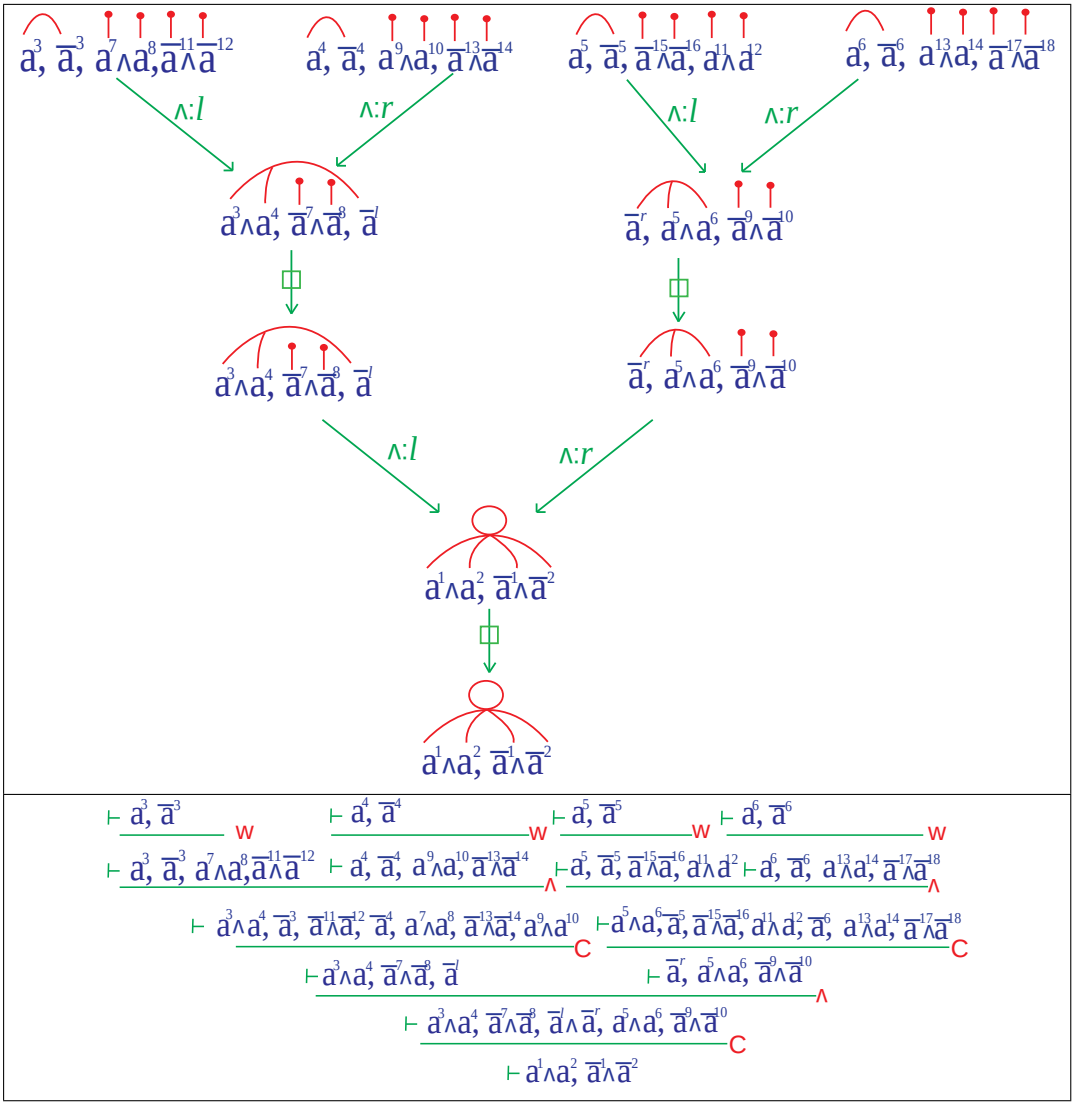


FIGURE 3.28 – From correctness diagrams to CL proofs.

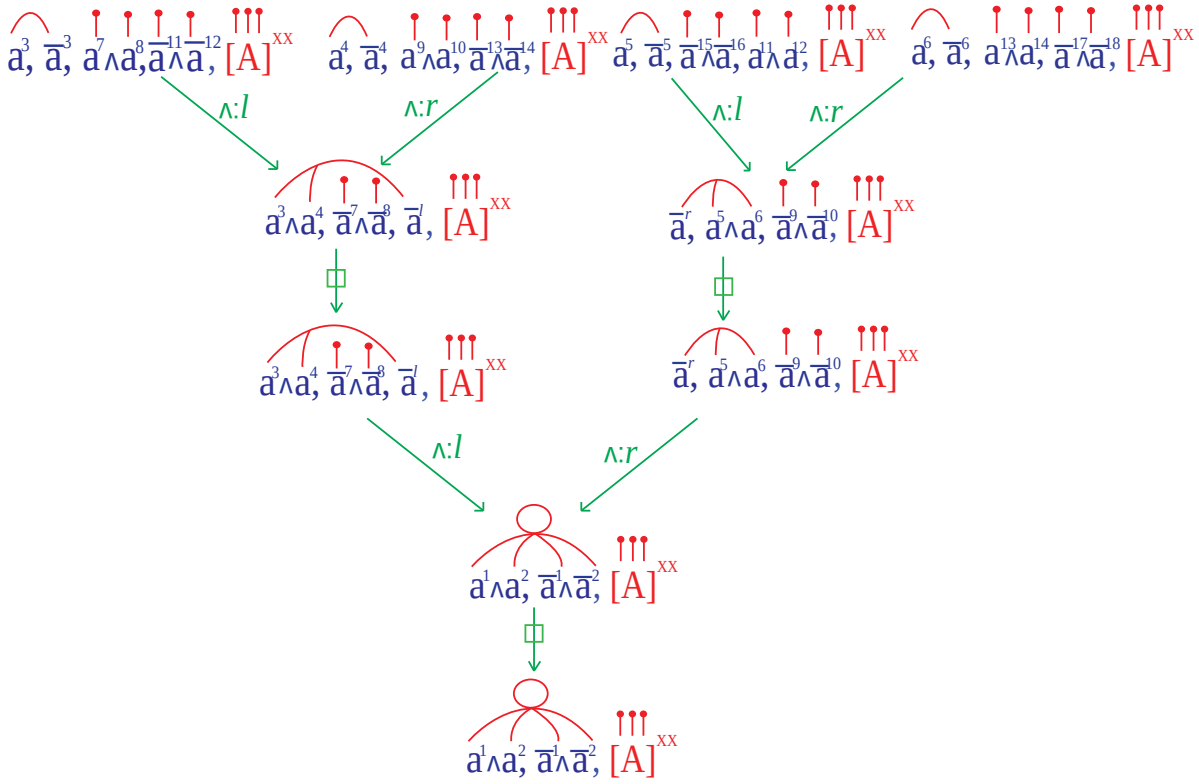


FIGURE 3.29 – Weakening respects correctness

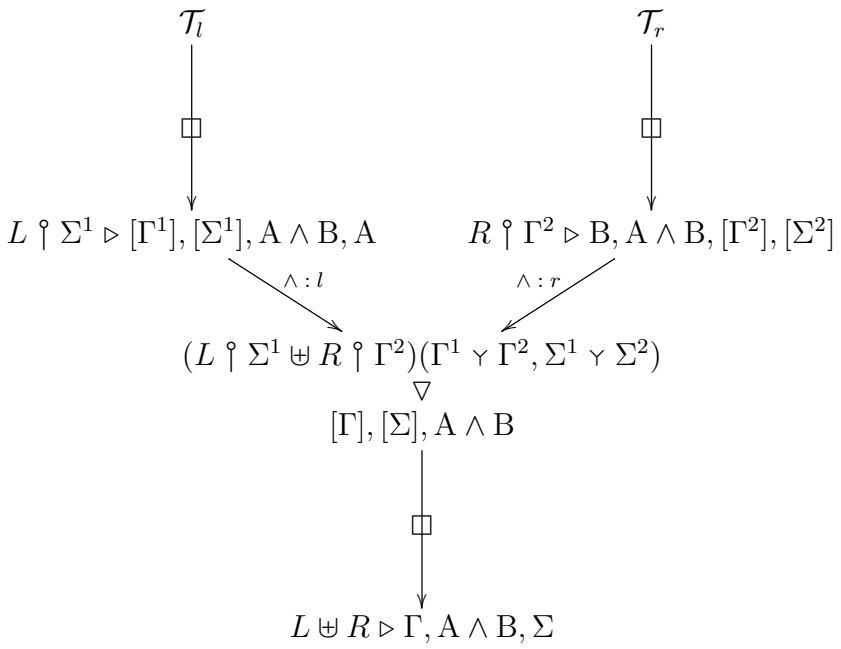
as the correctness diagrams. Now,

$$L \parr \Sigma \triangleright [\Gamma], [\Sigma], A \wedge B, A \quad \leftarrow\!\boxminus\!- \quad \mathcal{T}_l \qquad \text{and}$$

$$R \parr \Gamma \triangleright B, A \wedge B, [\Gamma], [\Sigma] \quad \leftarrow\!\boxminus\!- \quad \mathcal{T}_r$$

are obviously also correctness diagrams, and the correctness diagram for $L \uplus R \triangleright$

$\Gamma, \mathrm{A} \wedge \mathrm{B}, \Sigma$ can now be obtained as follows



Here, $\Gamma, \mathrm{A}, \mathrm{B}, \Sigma$ in the lower two F-prenets are obtained by appropriate contractions. The claim follows from the fact that, as in the $\wr$-case, the weakened formulas will not affect the linkings when contracted.

– *Mix* : $\vdash \Gamma, \Sigma$ was concluded from $\vdash \Gamma$ and $\vdash \Sigma$. This case is not that different to the conjunction case, due to the uniform nature of the definitions. By induction hypothesis, there are two correctness diagrams $\mathcal{T}'$ and $\mathcal{T}''$ from the definition of F-nets for $L \triangleright \Gamma$ and $R \triangleright \Sigma$ for proofs of $\vdash \Gamma$ and $\vdash \Sigma$, respectively. Again, from the *Weak* case,

$$L \wr \Sigma \triangleright \Gamma, \Sigma \text{ and } \quad R \wr \Gamma \triangleright \Gamma, \Sigma$$

need to be correct with

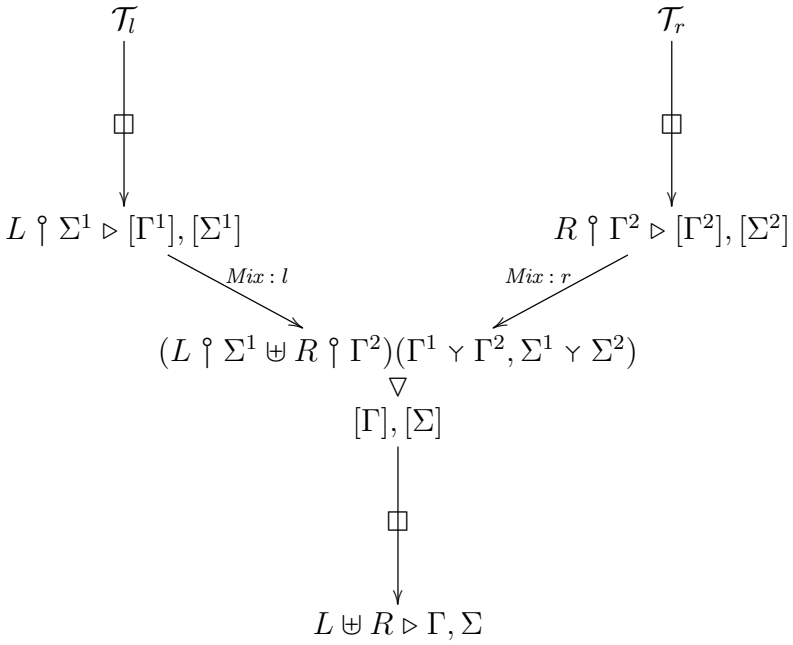
$$L \wr \Sigma \triangleright \Gamma, \Sigma \ \prec\!\boxminus\!\!- \ \mathcal{T}_l \quad \text{and} \quad R \wr \Gamma \triangleright \Gamma, \Sigma \ \prec\!\boxminus\!\!- \ \mathcal{T}_r$$

as the correctness diagrams. Here, Γ, Σ are copies of Γ, Σ with fresh occurrences of literals. Now,

$$L \wr \Sigma \triangleright [\Gamma], [\Sigma], \ \prec\!\boxminus\!\!- \ \mathcal{T}_l \quad \text{and} \quad R \wr \Gamma \triangleright [\Gamma], [\Sigma] \ \prec\!\boxminus\!\!- \ \mathcal{T}_r$$

are obviously also correctness diagrams, and the correctness diagram for $L \uplus R \triangleright \Gamma, \Sigma$

can now be obtained as follows



– *Contr* : Suppose $\vdash \Gamma, \mathrm{A}^{1-2}$ is inferred from $\vdash \Gamma, \mathrm{A}^1, \mathrm{A}^2$. This case is a delicate one, and it should be obvious that if we prove the following

Lemma 3.2.26. *If $P \triangleright \Gamma, A, A$ is correct, then $P(A \curlyvee A) \triangleright \Gamma, A$ is correct too.*

then we have proved the induction step for *Contr*.

This will be done by induction on the number $\mathcal{C}har(P \triangleright \Gamma, \mathrm{A}, \mathrm{A})$ of the leaves in the underlying graph of the correctness diagram $\mathcal{T}$ of $P \triangleright \Gamma, \mathrm{A}, \mathrm{A}$.

For the induction base one has $\mathcal{T}$ to be

$$P \triangleright \Gamma, \mathrm{A}^1, \mathrm{A}^2 \leftarrow\!\square\!- \; P \triangleright [\Gamma, \mathrm{A}^1, \mathrm{A}^2].$$

But then, either A^1 or A^2, say A^2 is obtained by weakening and subsequent disjunctions. Thus,

$$P|_{\Gamma,\mathrm{A}^1} \triangleright \Gamma, \mathrm{A}^1 \leftarrow\!\square\!- \; P|_{\Gamma,\mathrm{A}^1} \triangleright [\Gamma, \mathrm{A}^1]$$

is the correctness diagram for $P(\mathrm{A} \curlyvee \mathrm{A}) \triangleright \Gamma, \mathrm{A}$.

Further, we show the induction step. Let $P \triangleright \Gamma, \mathrm{A}, \mathrm{A}$ be a F-net whose correctness diagram $\mathcal{T}$ has n leaves. Two cases can be distinguished, depending on the nature of the branching of $[P \triangleright \Gamma, \mathrm{A}, \mathrm{A}]$ in $\mathcal{T}$.

i) (M) - the reduction is of M type.

$$\begin{array}{ccc} \vdots & & \vdots \\ L \triangleright [\Gamma^l], [\mathrm{A}]^{1l}, [\mathrm{A}]^{2l} & & R \triangleright [\Gamma^r], [\mathrm{A}]^{1r}, [\mathrm{A}]^{2r} \\ & \searrow^{\mathsf{M}:l} \quad \swarrow_{\mathsf{M}:r} & \\ & P \triangleright [\Gamma]^{l-r}, [\mathrm{A}]^{1l-1r}, [\mathrm{A}]^{2l-2r} & \end{array}$$

Here,

$$P = (L \uplus R)(\Gamma^l \curlyvee \Gamma^r, [\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{1r}, [\mathrm{A}]^{2l} \curlyvee [\mathrm{A}]^{2r}).$$

The two subtrees with $L \triangleright [\Gamma^l], [\mathrm{A}]^{1l}, [\mathrm{A}]^{2l}$ and $R \triangleright [\Gamma^r], [\mathrm{A}]^{1r}, [\mathrm{A}]^{2r}$ as roots are both correctness diagrams with strictly smaller number of leaves than $\mathcal{T}$. By induction hypothesis, $L([\mathrm{A}] \curlyvee [\mathrm{A}]) \triangleright [\Gamma], [\mathrm{A}]$ and $R([\mathrm{A}] \curlyvee [\mathrm{A}]) \triangleright [\Gamma], [\mathrm{A}]$ are correct F-nets, and we have the correctness diagram

$$\begin{array}{ccc} \vdots & & \vdots \\ L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \triangleright [\Gamma^l], [\mathrm{A}]^{1l-2l} & & R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r}) \triangleright [\Gamma^r], [\mathrm{A}]^{1r-2r} \\ & \searrow^{\mathsf{M}:l} \quad \swarrow_{\mathsf{M}:r} & \\ \big(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \uplus R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r})\big)\ (\Gamma^l \curlyvee \Gamma^r, [\mathrm{A}]^{1l-2l} \curlyvee [\mathrm{A}]^{1r-2r}) & & \\ \triangledown & & \\ [\Gamma]^{l-r}, [\mathrm{A}]^{1l-2l-1r-2r} & & \\ \Big\downarrow \square & & \\ \big(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \uplus R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r})\big)\ \big([\mathrm{A}]^{1l-2l} \curlyvee [\mathrm{A}]^{1r-2r}\big) & & \\ \triangledown & & \\ \Gamma^{l-r}, \mathrm{A}^{1l-2l-1r-2r} & & \end{array}$$

What is left to show is that

$$\big(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \uplus R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r})\big)\ \big(\Gamma^l \curlyvee \Gamma^r,\ [\mathrm{A}]^{1l-2l} \curlyvee [\mathrm{A}]^{1r-2r}\big)$$

is equal to

$$\big((L \uplus R)([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{1r},\ [\mathrm{A}]^{2l} \curlyvee [\mathrm{A}]^{2r})\big)\ \big(\Gamma^l \curlyvee \Gamma^r, [\mathrm{A}]^{1l-1r} \curlyvee [\mathrm{A}]^{2l-2rl}\big)\,,$$

but this follows from symmetry and associativity of Y, as shown in Proposition 3.2.10.

ii) ($\wedge$ first case) - the reduction is of $\wedge$ type, but the active conjunction formula is neither in $[\mathrm{A}]^1$ nor $[\mathrm{A}]^2$

$$\begin{array}{ccc} \vdots & & \vdots \\ L \triangleright [\mathrm{A}]^{1l}, [\mathrm{A}]^{2l}, [\Gamma_l] & & R \triangleright [\Gamma_r], [\mathrm{A}]^{1r}, [\mathrm{A}]^{2r} \\ & {\scriptstyle \wedge : l} \searrow \quad \swarrow {\scriptstyle \wedge : r} & \\ & P \triangleright [\mathrm{A}]^1, [\mathrm{A}]^2, [\Gamma] & \end{array}$$

This case is very similar to the M case, with the difference being that $[\Gamma_l]$ and Γ_r are not two occurrences of the same sequent, rather they differ in light of the $\wedge$ cospan family. Consequently, $[\Gamma]$ is not obtained by applying Y to the two, it is obtained by conjunction and subsequent contractions, as presented in the $\wedge$-type cospan definition. The notation we use for the linkings is $(\Gamma_l \wedge\!\curlyvee \Gamma_r)$, rather than $(\Gamma_l \curlyvee \Gamma_r)$. Thus, we have that in the diagram above

$$P = (L \uplus R)\left(\Gamma_l \wedge\!\curlyvee \Gamma_r,\ [\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{1r},\ [\mathrm{A}]^{2l} \curlyvee [\mathrm{A}]^{2r}\right).$$

In this case, the induction hypothesis, states that $L([\mathrm{A}] \curlyvee [\mathrm{A}]) \triangleright [\mathrm{A}], [\Gamma_l]$ and $R([\mathrm{A}] \curlyvee [\mathrm{A}]) \triangleright [\Gamma_r], [\mathrm{A}]$ are correct F-nets, and we have the correctness diagram

$$\begin{array}{ccc} \vdots & & \vdots \\ L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \triangleright [\mathrm{A}]^{1l-2l}, [\Gamma_l] & & R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r}) \triangleright [\Gamma_r], [\mathrm{A}]^{1r-2r} \\ & {\scriptstyle \wedge : l} \searrow \quad \swarrow {\scriptstyle \wedge : r} & \\ & \left(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \uplus R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r})\right) & \\ & \left(\Gamma_l \wedge\!\curlyvee \Gamma_r,\ [\mathrm{A}]^{1l-2l} \curlyvee [\mathrm{A}]^{1r-2r}\right) & \\ & \triangledown & \\ & [\mathrm{A}]^{1l-2l-1r-2r}, [\Gamma] & \\ & \Big\downarrow \square & \\ & \left(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \uplus R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r})\right) & \\ & \left(\Gamma_l \wedge\!\curlyvee \Gamma_r,\ [\mathrm{A}]^{1l-2l} \curlyvee [\mathrm{A}]^{1r-2r}\right) & \\ & \triangledown & \\ & \Gamma, \mathrm{A}^{1l-2l-1r-2r} & \end{array}$$

The claim follows from symmetry and associativity of Y, since

$$\left(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}) \uplus R([\mathrm{A}]^{1r} \curlyvee [\mathrm{A}]^{2r})\right)\left(\Gamma_l \wedge\!\curlyvee \Gamma_r,\ [\mathrm{A}]^{1l} \curlyvee ([\mathrm{A}]^{1r}\right)$$

is equal to

$$\left(L([\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{1r}) \uplus R([\mathrm{A}]^{2l} \curlyvee [\mathrm{A}]^{2r})\right)\left(\Gamma_l \wedge\!\curlyvee \Gamma_r,\ [\mathrm{A}]^{1l} \curlyvee [\mathrm{A}]^{2l}\right).$$

iii) ($\wedge$, second case) - the reduction is of $\wedge$ type, and the active conjunction formula is in, say $[\mathrm{A}^2]$, i.e. $[\mathrm{A}^2]$ is of the form $[\mathrm{B}]^2, \mathrm{C}^2 \wedge \mathrm{D}^2$ and

$$\begin{array}{ccc}
 & & \vdots \\
 & & R \triangleright \mathrm{D}^{2r}, \mathrm{C}^{2rr} \wedge \mathrm{D}^{2rr}, [\Gamma^r], [\mathrm{B}]^{1r}, \mathrm{C}^{1r} \wedge \mathrm{D}^{1r}, [\mathrm{B}]^{2r} \\
 & & \Big\downarrow \wedge : r \\
\vdots & & \\
L \triangleright [\Gamma^l], [\mathrm{B}]^{1l}, \mathrm{C}^{1l} \wedge \mathrm{D}^{1l}, \mathrm{B}^{2l}, \mathrm{C}^{2ll} \wedge \mathrm{D}^{2ll}, \mathrm{C}^{2l} & & \\
 \searrow \wedge : l & & \\
 & & P \triangleright [\Gamma]^{l-r}, [\mathrm{B}]^{1l-1r}, \mathrm{C}^{1l-1r} \wedge \mathrm{D}^{1l-1r}, [\mathrm{B}]^{2l-2r}, \\
 & & \mathrm{C}^{2l-2ll-2rr} \wedge \mathrm{D}^{2ll-2r-2rr}
\end{array}$$

Here,

$$\begin{aligned} P = \big((L \uplus R)([\Gamma^l], [\mathrm{B}]^{1l}, \mathrm{C}^{1l} \wedge \mathrm{D}^{1l}, \mathrm{B}^{2l} \curlyvee [\Gamma^r], [\mathrm{B}]^{1r}, \mathrm{C}^{1r} \wedge \mathrm{D}^{1r}, [\mathrm{B}]^{2r}) \\ (\mathrm{C}^{2ll} \wedge \mathrm{D}^{2ll} \curlyvee \mathrm{C}^{2rr} \wedge \mathrm{D}^{2rr})\big)(\mathrm{C}^{2l} \curlyvee \mathrm{C}^{2ll-2rr})(\mathrm{D}^{2ll-2rr} \curlyvee \mathrm{D}^{2r}). \end{aligned}$$

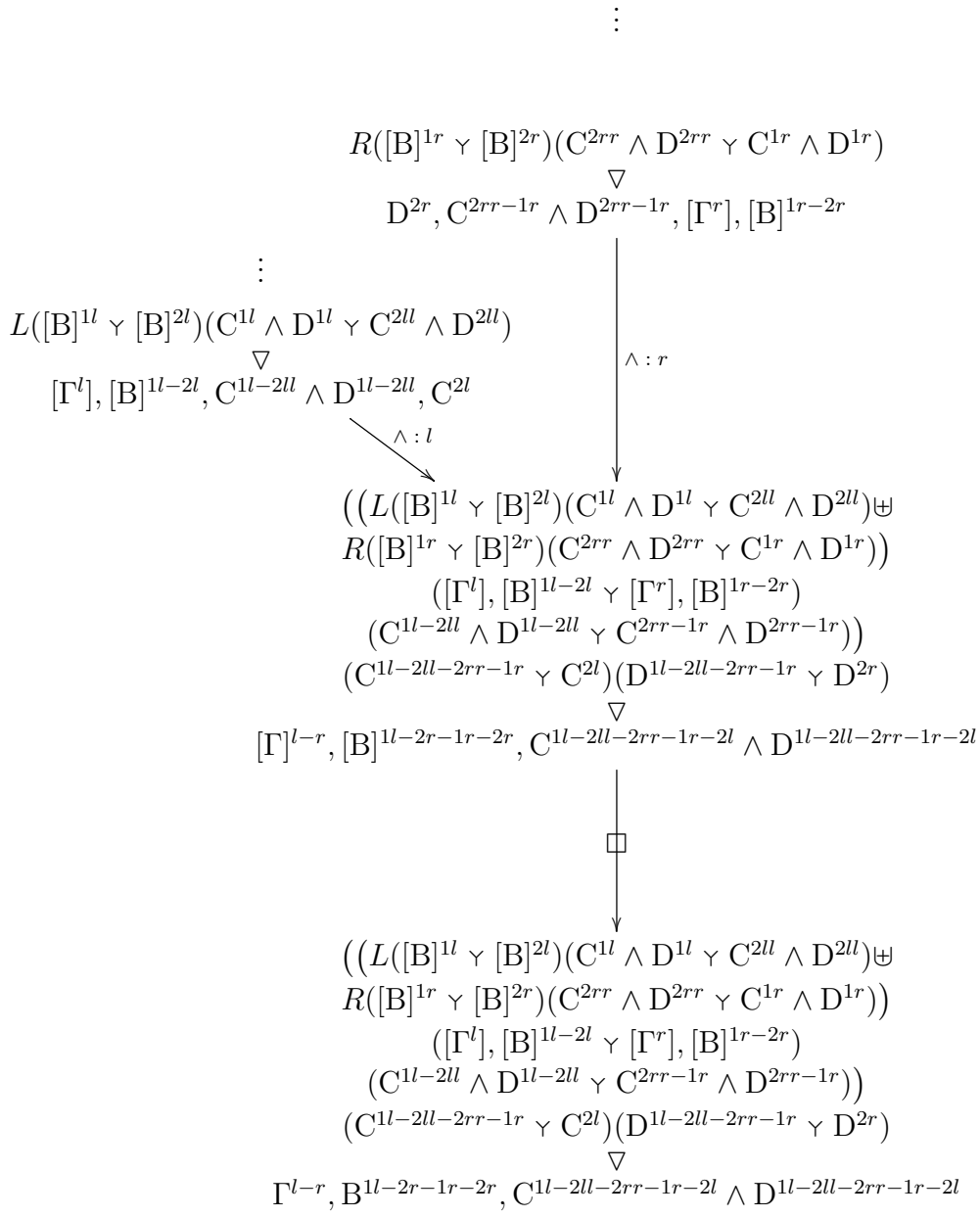
The induction hypothesis yields correctness of

$$L([\mathrm{B}] \curlyvee [\mathrm{B}])(\mathrm{C} \wedge \mathrm{D} \curlyvee \mathrm{C} \wedge \mathrm{D}) \triangleright [\Gamma], [\mathrm{B}], \mathrm{C} \wedge \mathrm{D}, \mathrm{C}$$

and

$$R([\mathrm{B}] \curlyvee [\mathrm{B}])(\mathrm{C} \wedge \mathrm{D} \curlyvee \mathrm{C} \wedge \mathrm{D}) \triangleright \mathrm{D}, \mathrm{C} \wedge \mathrm{D}, [\Gamma], [\mathrm{B}],$$

and we have the correctness diagram



The claim now follows from symmetry and associativity of Y, since

$$\begin{aligned}\big(\big(L([\mathrm{B}]^{1l} \curlyvee [\mathrm{B}]^{2l})(\mathrm{C}^{1l} \wedge \mathrm{D}^{1l} \curlyvee \mathrm{C}^{2ll} \wedge \mathrm{D}^{2ll}) \uplus R([\mathrm{B}]^{1r} \curlyvee [\mathrm{B}]^{2r})\\ (\mathrm{C}^{2rr} \wedge \mathrm{D}^{2rr} \curlyvee \mathrm{C}^{1r} \wedge \mathrm{D}^{1r})\big)([\Gamma^l], [\mathrm{B}]^{1l-2l} \curlyvee [\Gamma^r], [\mathrm{B}]^{1r-2r})\\ (\mathrm{C}^{1l-2ll} \wedge \mathrm{D}^{1l-2ll} \curlyvee \mathrm{C}^{2rr-1r} \wedge \mathrm{D}^{2rr-1r})\big)\\ (\mathrm{C}^{1l-2ll-2rr-1r} \curlyvee \mathrm{C}^{2l})(\mathrm{D}^{1l-2ll-2rr-1r} \curlyvee \mathrm{D}^{2r})\end{aligned}$$

is equal to

$$\begin{aligned}P = \big((L \uplus R)([\Gamma^l], [\mathrm{B}]^{1l}, \mathrm{C}^{1l} \wedge \mathrm{D}^{1l}, \mathrm{B}^{2l} \curlyvee [\Gamma^r], [\mathrm{B}]^{1r}, \mathrm{C}^{1r} \wedge \mathrm{D}^{1r}, [\mathrm{B}]^{2r})\\ (\mathrm{C}^{2ll} \wedge \mathrm{D}^{2ll} \curlyvee \mathrm{C}^{2rr} \wedge \mathrm{D}^{2rr})\big)(\mathrm{C}^{2l} \curlyvee \mathrm{C}^{2ll-2rr})(\mathrm{D}^{2ll-2rr} \curlyvee \mathrm{D}^{2r}).\end{aligned}$$

□

The proof above reflects the intention of the correctness procedure - it amounts to having a proof net-guided proof search procedure. The key point of the procedure is when linkings are split in to facilitate branchings of $\wedge$ and *Mix*. The branchings are the only places where contractions are allowed, and they are performed in a uniform way. This is possible due to the fact that if we 'overduplicate', the excess formulas can be eliminated by weakenings. Also, the correctness diagram represents a canonical proof for a given F-prenet. If such a proof does not exist, than no proof corresponds to the F-prenet.

The reader may argue that what we have just proposed is little different to saying 'a F-net corresponds to a proof if there is a proof whose F-net is the given one'. However, this is not the case for the simple reason - if one would like to establish whether a F-net corresponds to a proof by taking one proof after another and checking the equality of the F-nets, the obvious problem is the infinite search space; the number of proofs of a sequent is infinite. Even once we know the number of axioms is finite, there has to be a way to control the lengths of branches in a proof. More concretely, one needs to be able to control the number of weakenings in a proof, which is potentially infinite. Once one tries to control these, one is bound to go back to something akin to the correctness criterion.

It may appear that the proof we have just given is surprisingly unproblematic, even though there are some technical intricacies involved, and one needs to be pedantic with the nested inductions. However, as it happens often in categorical proof theory and in category theory in general, it is the proper definitions that are essential, those which facilitate proofs. In this case, had we chosen only slightly different way to construct the correctness diagram, e.g. had we chosen to make seemingly innocent modifications of the way we handle conjunctions :

$$\begin{array}{ccc} P_l \triangleright \Gamma, \mathrm{A} & & P_r \triangleright \mathrm{B}, \Gamma \\ & {\scriptstyle \wedge : l} \searrow \quad \swarrow {\scriptstyle \wedge : r} & \\ & P \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B} & \end{array}$$

that is, without two repetitions of the conjunction formula of the codomain in the domains of the cospan maps, the induction step of the proof could not be shown.

The procedure relies on ability to construct a correctness diagram given an F-prenet. The number of branchings/size of a correctness diagram is finite and is read-out from the F-net directly. We start by the F-prenet itself and its child node and proceed by expanding the tree. Having an intermediate tree, the choice for expansion is finite. Indeed, first we select a candidate leaf we want to expand, out of finitely many (fewer than the characteristic of the F-prenet). Then, we choose to expand either via $\wedge$ or M. In the first case, one has to decide which conjunction formula, out of finitely many to expand next. In both cases, however, the choice for the child nodes is finite. To see this, we call upon the crucial observation of Lemma 3.2.17 that connects a characteristic of an F-prenet with number of axioms of its proof, if such exist. As a consequence, in order to have a correctness diagram, each F-prenet in the tree has to have a positive characteristic (it has at least

one axiom in its proof), and the characteristic of father node is sum of characteristics of child nodes. There is finite number of ways to represent a positive characteristic as sum of two positive characteristics of child nodes. Knowing the characteristic, there is finite number of configurations of possible linkings in the child nodes, given the fact that their sequents are known. The $\vee$ step in the expansion of the tree is deterministic, while the permutation step is not, but the choices for a permutation are finite. Thus, the $\multimap$ expansion has always finitely many possibilities.

The construction stops if it is successful, and that is checked if the tree is properly expanded, has the number of leaves equal to the characteristic, and the leaves are of the correct form, which is checkable in finite time. If the current tree cannot be expanded, we backtrack and choose different expansion out of the previous ones. We also backtrack once we did not reach a correctness diagram, but did expand as many times as is the characteristic of the F-prenet whose correctness we want to establish. Once we exhaust all of the options, out of finitely many, we conclude that the F-prenet in question is not correct.

Checking does require extensive backtracking, i.e., there will be a large, but finite search space for the collection of individual choices to be made for establishing that a given F-prenet is a F-net.

The algorithm for deciding correctness is given below :

Procedure DecideCorrectness

Input : F-prenet $P \triangleright \Gamma$ whose correctness (= sequentiability) is decided
Output : 'Yes!' if the input F-prenet is correct, 'No!', otherwise

Begin
  If createDiagram($P \triangleright \Gamma$) is nonempty
    Return 'Yes!';
  Else Return 'No!';
End

Function createDiagram
Parameter : F-prenet $Q \triangleright \Sigma$
Returns : a correctness diagram, if such exist, empty set, otherwise

Begin
  expand $Q \triangleright \Sigma$ by $\prec\!\boxminus\!\!-^{-1}$;
  If $[Q \triangleright \Sigma]$ has the form of a leaf of a correctness diagram
    Return : $Q \triangleright \Sigma \prec\!\boxminus\!\!- [Q \triangleright \Sigma]$;
  For all conjunction formulas of $[Q \triangleright \Sigma]$
    create list of $\wedge$ child candidates;
  add M child candidates to the list of child candidates;
  For all pairs of nodes (left, right) from the list of child candidates
    If createDiagram([left]) is nonempty
      If createDiagram([right]) is nonempty
        Return $Q \triangleright \Sigma \prec\!\boxminus\!\!- [Q \triangleright \Sigma] \prec\!\!-^{-}$ createDiagram(left)
        $\prec\!\!-^{-}$ createDiagram(left)
  Return : empty;
End

The discussion above is summed-up in the following :

Theorem 3.2.27. *Given an F-prenet, its* CL*-correctness (*CL*-sequentializability) can be checked in finite time, i.e. the* CL*-correctness procedure is a decision procedure for* CL*-correct F-nets.*

Notice how we choose to speak of a correctness *procedure*, rather than a *criterion*. This choice of terminology acknowledges the fact that the procedure is not formulated as a universally quantified sentence, the way correctness criteria are usually presented. Another reason for the decision is the issue of complexity of the procedure which is not polynomial. We return to the issue of sequentializability and correctness in the following chapter, while in Section 4.4 we re-address the topic from the complexity point of view.

4

F-nets and Cut Elimination

Contents

4.1 F-nets with cut.

The previous chapter introduces F-nets where the linkings stand for maps in a freely generated Frobenius category. Maps in the category come with well defined composition, which suggests a canonical way to eliminate cut at the level of F-prenets. These are the issues we deal with in this chapter.

We begin with introduction of cut symbol as a syntactic connective.

Using notation adopted in [LS05b], we define *cut formula* to be $A \diamondsuit \overline{A}$, where $\overline{(-)}$ is, as before, a function defined on formulas as follows :

$$\overline{a} = \overline{a};\ \overline{\overline{a}} = a;\ \overline{(A \wedge B)} = \overline{A} \vee \overline{B};\ \overline{(A \vee B)} = \overline{A} \wedge \overline{B}.$$

An F-prenet whose sequent contains cut formulas is an F-prenet with cuts. Here, $- \diamondsuit -$ is a new binary connective that is only allowed to appear in as a root in a sequent, and

if a sequent Γ contains a cut formula $\mathrm{A}^{\diamondsuit}\overline{\mathrm{A}}$, literals of both $\mathrm{A}, \overline{\mathrm{A}}$ are formally considered to be in $\mathcal{L}(\Gamma)$

We can finally complete interpretation of the system CL with interpretation of the *Cut* rule :

8.
$$\frac{\vdash \Gamma, \mathrm{A} \quad \vdash \overline{\mathrm{A}}, \Sigma}{\vdash \Gamma, \Sigma}\ \mathit{Cut} \qquad \rightsquigarrow \qquad \frac{P_1 \triangleright \Gamma, \mathrm{A} \quad P_2 \triangleright \overline{\mathrm{A}}, \Sigma}{P_1 \uplus P_2 \triangleright \Gamma, \mathrm{A}^{\diamondsuit}\overline{\mathrm{A}}, \Sigma}\ ,$$

Thus, we have extended a definition of an F-net to the case with cut.

Some of the results and definitions for F-prenets without cut can be reformulated to the case with cut.

Formulation of the correctness criterion for proof nets with cut usually follows from the treatment of the cut rule as a kind of conjunction. Here it will not strictly be the case, since we have defined the conjunction cospan to have three contractions on components of the active conjunction formula, while a cut formula cannot be contracted. The treatment of the cut in the correctness procedure thus needs to be slightly different.

To accommodate the new connective, we introduce another cospan in the category $\mathcal{F}\mathbf{Synt}$, and we need to revise the previously given definitions to indicate the presence of cut formulas, due to the impossibility of uniform contractions.

Definition 4.1.1. Let $\Sigma^{\diamondsuit}$ denote a sequence of cut formulas.

a) (M) : In $\mathcal{F}\mathbf{Synt}$ where the cut connective $\diamondsuit$ has been added to the language with the extended language on formulas by the cut connective $\diamondsuit$, we define the elementary M cospan to be

$$P_l \triangleright \Gamma, \Sigma_1^{\diamondsuit} \xrightarrow{\mathsf{M}:l} P \triangleright \Gamma, \Sigma_1^{\diamondsuit}, \Sigma_2^{\diamondsuit} \xleftarrow{\mathsf{M}:r} P_r \triangleright \Gamma, \Sigma_2^{\diamondsuit}.$$

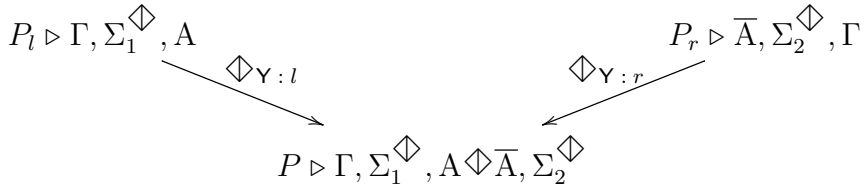
where one has
$$P = P_l \uplus P_r(\Gamma \curlyvee \Gamma).$$

b) $(\wedge)$ Further, we define the elementary $\wedge$ cospan :

$$P_l \triangleright \Gamma, \Sigma_1^{\diamondsuit}, \mathrm{A} \wedge \mathrm{B}, \mathrm{A} \xrightarrow{\wedge:l} P \triangleright \Gamma, \Sigma_1^{\diamondsuit}, \mathrm{A} \wedge \mathrm{B}, \Sigma_2^{\diamondsuit} \xleftarrow{\wedge:r} P_r \triangleright \mathrm{B}, \mathrm{A} \wedge \mathrm{B}, \Sigma_2^{\diamondsuit}, \Gamma$$

with
$$P = P_l \uplus P_r(\Gamma \curlyvee \Gamma, \mathrm{A} \curlyvee \mathrm{A} \curlyvee \mathrm{A}, \mathrm{B} \curlyvee \mathrm{B} \curlyvee \mathrm{B}).$$

c) ($\diamondsuit\mathsf{Y}$) We define the elementary $\diamondsuit\mathsf{Y}$ cospan :



where one has

$$P = P_l \uplus P_r(\Gamma \curlyvee \Gamma).$$

The definition of the generalized M and $\wedge$ cospans translates word-by-word to the case of newly introduced elementary cospans, while the generalized $\diamondsuit$ cospan is a cospan in $\mathcal{F}\mathbf{Synt}$ isomorphic to an elementary one.

Also, the definition of anodyne maps need not be changed when the language is extended by the $\diamondsuit$ symbol.

Formulation of the correctness procedure is no different to the case without cut.

Definition 4.1.2 (Correct F-nets)**.** An F-prenet with cut $\Phi = P \triangleright \Gamma$ is a *correct F-net with cuts* if if it is the root of some *correctness diagram* $\mathcal{T}$. A correctness diagram for $P \triangleright \Gamma$ with cuts is a diagram of the type (3.1) of Definition 3.2.21, with the difference that branchings can also be $\diamondsuit$-cospans.

Notice that the definition of correct F-nets with cut also requires the number of leaves in the underlying graph of the correctness diagram to be equal to the characteristic of the F-net, i.e. Proposition 3.2.22 is repeated for the case of F-prenets with cut. In particular, all the literals in the cut formula $\mathrm{A}\diamondsuit\overline{\mathrm{A}}$ are still literals of the linking, i.e. no elimination of literals is performed at this point yet.

Theorem 4.1.3 (Sequentialization)**.** *For every F-net with cuts there is a* CL *proof (with cuts) that corresponds to that F-net. The converse also holds, i.e. given a* CL *proof with cuts, its F-prenet with cuts is in fact a correct F-net with cuts.*

Démonstration. The proof follows the one for the case without cut, Theorem 3.2.25. Indeed, having the splitting of a $\Sigma^{\diamondsuit}$ context in cospans is unproblematic as the cospans allowed in building of the correctness diagram do not apply $\curlyvee$ on cut formulas. Indeed, one may observe that Lemma 3.2.26 can be restated for the case with cut formulas, when the contexts with cut formulas are added. The proof of the lemma concerns with four cases of branchings in a proof and in each of the cases the inductive steps allow contexts with cut formulas to be split.

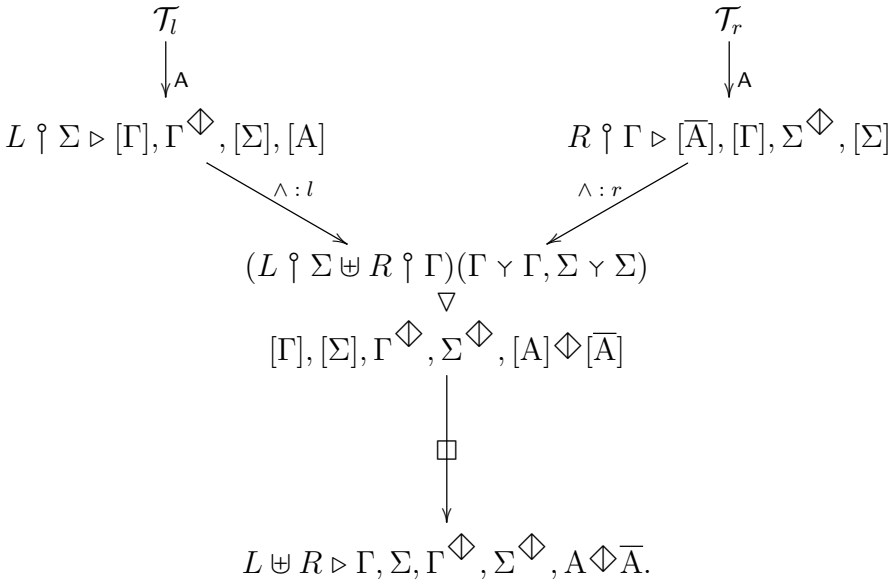
Thus, the proof of the sequentialization theorem translates directly to the case with cut. Look e.g. at the potentially concerning case when a contraction is the last rule applied in a proof for which we want to show existence of an F-net. Therefore, assume $\vdash \Gamma, \mathrm{A}$ is

inferred from $\vdash \Gamma, \mathrm{A}, \mathrm{A}$. By the induction hypothesis, an F-net with cuts $P \triangleright \Gamma, \mathrm{A}^1, \mathrm{A}^2, \Sigma^{\diamondsuit}$ is assigned to the proof of $\vdash \Gamma, \mathrm{A}^1, \mathrm{A}^2$. The induction step follows from Lemma 3.2.26 restated for the case with cut formulas which guarantees that $P(\mathrm{A} \curlyvee \mathrm{A}) \triangleright \Gamma, \mathrm{A}, \mathrm{A}, \Sigma^{\diamondsuit}$.

The case which is not covered when we want to show that for a CL proof one has a corresponding correct F-net is when the last rule of the proof is a cut, i.e.

$$\frac{\Gamma, \mathrm{A} \quad \overline{\mathrm{A}}, \Sigma}{\vdash \Gamma, \Sigma}\ \mathit{Cut}.$$

The induction hypothesis yields correctness diagrams $\mathcal{T}_l, \mathcal{T}_r$ for F-prenets assigned to proofs of both Γ, A and $\overline{\mathrm{A}}, \Sigma$, respectively. The correctness diagram for the F-prenet for the entire proof can be obtained from



□

The discussion on the finiteness of the procedure that checks for the correctness of an F-prenet without cut extends to the case with cut, since the case when one has a cut formula in a proof net, there is finite choice of those formulas to expand along the $\diamondsuit\mathsf{Y}$ cospan, and for every choice of an active cut formula one again needs to split the characteristic of the F-prenet into two non-negative characteristics for the F-prenets in domains of cospan maps, which can be done in finitely many ways, each of which yields finitely many possible configurations for linkings.

In fact, the treatment of the cut connective is akin to the one of $\wedge$ connective, which is common when constructing proof net theories. The marked difference is, of course, inability to perform contractions on these formulas.

Theorem 4.1.4. *Given an F-prenet with cuts, its* CL*-correctness (*CL*-sequentializability) can be checked in finite time, i.e. the* CL*-correctness criterion yields a decision procedure for* CL*-correct F-nets.*

4.2 Eliminating cuts. FL - Calculus of resources.

The real challenge in dealing with cut is when it is eliminated. In the case of F-nets, by cut elimination at the level of F-nets we understand a procedure of removing cut formulas with the simultaneous transformation of linkings standing for composition of maps in the underlying free Frobenius category. From geometric point of view, cut elimination acts on a linking by putting a wire between atoms of the opposite polarity, for every pair of literals in the cut formula, as discussed for the $\smile$ transformation of linkings. Cut elimination of a single cut formula in an F-prenet is done directly, at once, in accordance with the practice of propagating the structural rules to literals.

Definition 4.2.1. Reduction of an F-prenet with one cut formula, $\twoheadrightarrow$ acts on linkings :

$$P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}} \twoheadrightarrow P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma.$$

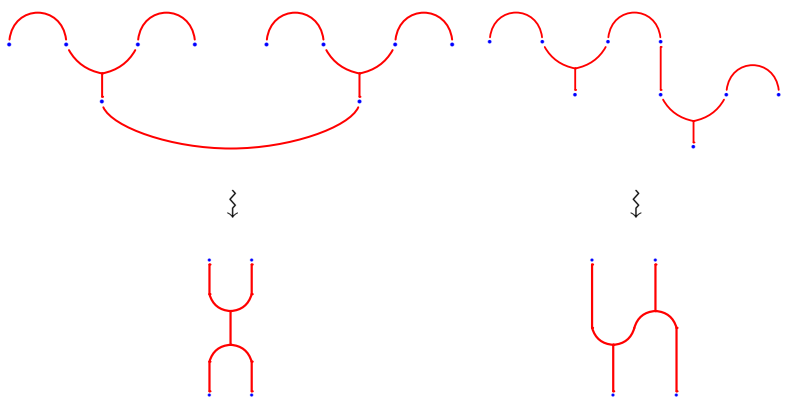


FIGURE 4.1 – F-nets for proofs on Figure 3.10 identified by the Frobenius axiom.

As an illustration of the treatment of the cut, look at the cut wires in the F-prenet on Figure 4.1, corresponding to classical proofs on Figure 3.10.

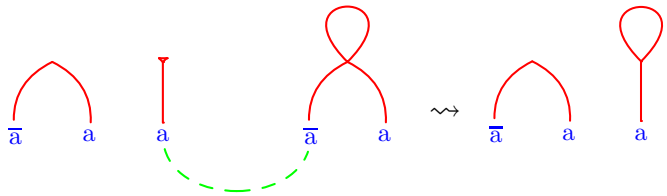


FIGURE 4.2 – Cut elimination performed on two CL-correct F-nets that results in an F-prenet not corresponding to a CL proof.

This is, however, where we first run into problems. Namely, the main condition on cut elimination is to be a closed operation in the set of correct F-nets. However, consider

the cut on nets of Figure 4.2. The resulting F-prenet cannot come from a proof, since it contains a single-atom class with a non-zero genus that should come from weakening, but it doesn't. Indeed,

Proposition 4.2.2. *In a cut free correct F-net each component in the linking contains at least one literal of each of the polarities, except for the single-literal components introduced by a weakening.*

Démonstration. By induction on the size of a proof corresponding to a CL-correct net. An easy verification for every rule of CL. □

In the previous example the resulting F-prenet fails to obey the proposition we have just shown and therefore cannot be correct.

A solution to this problem lies in different treatment of weakening. To that purpose, we go to the syntax and choose to formulate the calculus we interpret, Figure 4.3.

The deductive system FL we have just introduced results from a simple idea : we want to be able to weaken several literals at once, which will form *a single component* in the interpretation partition. Such weakened literals should form composite formulas, but the introduction rules for the connectives should be different from those on the 'ordinary' literals. This requires presence of the stoup symbol ' ; ' to separate the part of a sequent that comes from weakenings from the rest of the sequent.

Definition 4.2.3. Let Γ, Σ be a sequence of formulas of classical propositional logic, with the condition that the Γ is non-empty. We say that the sequence Γ, Σ is derived in the system FL, if there is a derivation in the system of the sequent

$$\vdash \Gamma \ ; \ \Sigma.$$

The following holds.

Proposition 4.2.4. FL *is a sound and complete calculus for classical propositional formulas. More precisely, if* $\vdash \Gamma \ ; \ \Sigma$ *is provable in* FL, $\bigvee \Gamma$ *is a valid propositional formula, as is* $\bigvee \Gamma, \Sigma$. *(*$\bigvee(-)$ *denotes the formula which is disjunction of the formulas in the argument). Conversely, for a valid propositional formula* A, $\vdash \mathrm{A} \ ;$ *is derivable in* FL.

Démonstration. Soundness of the calculus is straightforwardly checked by induction, by checking each rule separately, in the sense of Definition 4.2.3. Completeness is also unproblematic, since CL can be seen as a fragment of FL. Indeed, by forgetting the stoup, every CL rule other than weakening appears as a rule in FL. The weakening of CL can be viewed as a sequence of *MulWeak* rules introducing the literals, followed by a sequence of $-_r$ rules which introduce the connectives of the weakened formula. Since CL is complete for classical propositional logic, so is FL. □

The intended interpretation of rules largely coincides with the one for rules of CL, where one disregards the stoup symbol– the axiom rule is interpreted the same way, so

$$\frac{}{\vdash \overline{\mathrm{a}}, \mathrm{a}\ ;}\ Ax \qquad \frac{\vdash \Gamma\ ; \Delta}{\vdash \Gamma\ ; \Delta, \mathrm{a}, \mathrm{a}, \ldots, \mathrm{a}, \overline{\mathrm{a}}, \overline{\mathrm{a}}, \ldots, \overline{\mathrm{a}}}\ MulWeak$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}\ ; \Delta}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}\ ; \Delta}\ \vee_l \qquad \frac{\vdash \Gamma, \mathrm{A}\ ; \Delta, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}\ ; \Delta}\ \vee_c^1 \qquad \frac{\vdash \Gamma, \mathrm{A}\ ; \Delta, \mathrm{B}}{\vdash \Gamma, \mathrm{B} \vee \mathrm{A}\ ; \Delta}\ \vee_c^2 \qquad \frac{\vdash \Gamma\ ; \Delta, \mathrm{A}, \mathrm{B}}{\vdash \Gamma\ ; \Delta, \mathrm{A} \vee \mathrm{B}}\ \vee_r$$

$$\frac{\vdash \Gamma_1, \mathrm{A}\ ; \Delta_1 \quad \vdash \mathrm{B}, \Gamma_2\ ; \Delta_2}{\vdash \Gamma_1, \mathrm{A} \wedge \mathrm{B}, \Gamma_2\ ; \Delta_1, \Delta_2}\ \wedge_l \qquad \frac{\vdash \Gamma_1, \mathrm{A}\ ; \Delta_1 \quad \vdash \Gamma_2\ ; \mathrm{B}, \Delta_2}{\vdash \Gamma_2\ ; \mathrm{A} \wedge \mathrm{B}, \Gamma_1, \Delta_1, \Delta_2}\ \wedge_c^1$$

$$\frac{\vdash \Gamma_1, \mathrm{A}\ ; \Delta_1 \quad \vdash \Gamma_2\ ; \mathrm{B}, \Delta_2}{\vdash \Gamma_2\ ; \mathrm{B} \wedge \mathrm{A}, \Gamma_1, \Delta_1, \Delta_2}\ \wedge_c^2 \qquad \frac{\vdash \Gamma\ ; \Delta, \mathrm{A}, \mathrm{B}}{\vdash \Gamma\ ; \Delta, \mathrm{A} \wedge \mathrm{B}}\ \wedge_r$$

$$\frac{\vdash \Gamma\ ; \Delta_1 \quad \vdash \Sigma\ ; \Delta_2}{\vdash \Gamma, \Sigma\ ; \Delta_1, \Delta_2}\ Mix$$

$$\frac{\Gamma_1, \mathrm{A}, \mathrm{B}, \Gamma_2\ ; \Delta}{\vdash \Gamma_1, \mathrm{B}, \mathrm{A}, \Gamma_2\ ; \Delta}\ Exch_l \qquad \frac{\vdash \Gamma\ ; \Delta_1, \mathrm{A}, \mathrm{B}, \Delta_2}{\vdash \Gamma\ ; \Delta_1, \mathrm{B}, \mathrm{A}, \Delta_2}\ Exch_r$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}\ ; \Delta}{\vdash \Gamma, \mathrm{A}\ ; \Delta}\ Contr_l \qquad \frac{\vdash \Gamma, \mathrm{A}\ ; \Delta, \mathrm{A}}{\vdash \Gamma, \mathrm{A}\ ; \Delta}\ Contr_c \qquad \frac{\vdash \Gamma\ ; \Delta, \mathrm{A}, \mathrm{A}}{\vdash \Gamma\ ; \Delta, \mathrm{A}}\ Contr_r$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ; \Delta_1 \quad \vdash \overline{\mathrm{A}}, \Sigma\ ; \Delta_2}{\vdash \Gamma, \Sigma\ ; \Delta_1, \Delta_2}\ Cut_l \qquad \frac{\vdash \Gamma, \mathrm{A}\ ; \Delta_1 \quad \vdash \Sigma\ ; \overline{\mathrm{A}}, \Delta_2}{\vdash \Sigma\ ; \Gamma, \Delta_1, \Delta_2}\ Cut_c$$

$$\frac{\vdash \Gamma\ ; \Delta, \mathrm{A}, \overline{\mathrm{A}}}{\vdash \Gamma\ ; \Delta}\ Cut_r$$

FIGURE 4.3 – System FL.

are contractions, cut introduces a cut formula, while logical rules do not affect the linkings, only the sequents. The main difference is in the interpretation of the weakening :

$$\frac{\vdash \Gamma\,;\Delta}{\vdash \Gamma\,;\Delta,\mathrm{a},\mathrm{a},\ldots,\mathrm{a},\overline{\mathrm{a}},\overline{\mathrm{a}},\ldots,\overline{\mathrm{a}}}\ \mathit{MulWeak}$$

The weakening in the case of FL is interpreted as adding a single component of genus 0 consisting of the weakened literals

$$\frac{P \triangleright \Gamma\,;\Delta}{P \uplus (\{\mathrm{a},\mathrm{a},\ldots,\mathrm{a},\overline{\mathrm{a}},\overline{\mathrm{a}},\ldots,\overline{\mathrm{a}}\},0) \triangleright \Gamma\,;\Delta,\mathrm{a},\mathrm{a},\ldots,\mathrm{a},\overline{\mathrm{a}},\overline{\mathrm{a}},\ldots,\overline{\mathrm{a}}}\ .$$

Let CL+ be the calculus obtained from CL with the following rule instead of the one for weakening (with the obvious breach of syntactic independence of semantics) :

$$\frac{\vdash \Gamma}{Q \vdash \Gamma,\Sigma}\ \mathit{Weak+},$$

where Q is an arbitrary linking over the literals of Σ.

The intended interpretation of the rule is obviously :

$$\frac{P \triangleright \Gamma}{P \uplus Q \triangleright \Gamma,\Sigma}\ .$$

The proof theories of the two calculi are very similar. But, to be able to express the precise connection between the two, we need to introduce the following notation : given an FL proof of $\vdash \Gamma\,;\Sigma$ and its assigned F-prenet $P \triangleright \Gamma,\Sigma,\Pi^{\diamondsuit}$ for the sequence $\Pi^{\diamondsuit}$ of cut formulas, we split the cut formulas into the left- and right- ones, with suitable subscript notation, $\Pi^{\diamondsuit_l}$ and $\Pi^{\diamondsuit_r}$. These two groups are defined as follows : the cut formula created when Cut_l is interpret is a left-cut formula, the one created by Cut_r is a right-cut formula. Eg.

$$\frac{P_1 \triangleright \Gamma,\mathrm{A},\Delta_1,\Pi_1^{\diamondsuit_l},\Pi_2^{\diamondsuit_r} \qquad P_2 \triangleright \overline{\mathrm{A}},\Sigma,\Delta_2,\Pi_3^{\diamondsuit_l},\Pi_4^{\diamondsuit_r}}{P_1 \uplus P_2 \triangleright \Gamma,\Sigma,\Delta_1,\Delta_2,\mathrm{A}\diamondsuit_l\overline{\mathrm{A}},\Pi_1^{\diamondsuit_l},\Pi_2^{\diamondsuit_r},\Pi_3^{\diamondsuit_l},\Pi_4^{\diamondsuit_r}}\ \mathit{Cut}_l$$

A formula remains left or right, respectively, when an FL rule is interpreted, except in the case of $\wedge_c$ and Cut_c rules. In these cases, all left-cut formulas assigned to the branch of the proof of the left premise become right cut formulas. Eg.

$$\frac{P_1 \triangleright \Gamma_1,\mathrm{A},\Delta_1,\Pi_1^{\diamondsuit_l},\Pi_2^{\diamondsuit_r} \qquad P_2 \triangleright \Gamma_2,\mathrm{B},\Delta_2,\Pi_3^{\diamondsuit_l},\Pi_4^{\diamondsuit_r}}{P_1 \uplus P_2 \triangleright \Gamma_2,\mathrm{A}\wedge\mathrm{B},\Gamma,\Delta_1,\Delta_2,\Pi_1^{\diamondsuit_r},\Pi_2^{\diamondsuit_r},\Pi_3^{\diamondsuit_l},\Pi_4^{\diamondsuit_r}}\ \wedge_c^1$$

Notice that we are only discussing notation here that is going to be used in the following proposition, FL proofs are still assigned F-prenets in the usual sense.

Proposition 4.2.5. *Given a proof of a sequent* $\vdash \Gamma$ *in* CL+*, there exists a decomposition of* $\Gamma = \Gamma', \Gamma''$ *such that there is a proof of* $\vdash \Gamma' \,;\Gamma''$ *in* FL *that yields the same prenet.*

Conversely, let an FL *proof of* $\vdash \Gamma;\Sigma$ *be given and let* $P \triangleright \Gamma, \Sigma, \Delta^{\Diamond_l}, \Delta^{\Diamond_r}$ *be the assigned F-net, for sequences of left- and righ-cut formulas* $\Delta^{\Diamond_l}$ *and* $\Delta^{\Diamond_r}$*. Then, there is a* CL+ *proof of* $\vdash \Gamma$ *with the assigned F-net* $P|_{\Gamma,\Delta^{\Diamond_l}} \triangleright \Gamma, \Delta^{\Diamond_l}$*.*

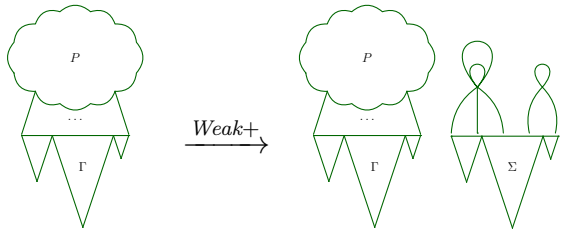
Démonstration. As usual, we omit explicit mention of Exchange in further text.

Let us show first how a proof of in CL+ can be seen as a proof in FL. To that purpose, notice that each rule of CL+ can be seen as a rule of FL when one disregards the stoup symbol, with the exception of *Weak+*. Indeed, one is allowed to introduce an arbitrary configuration in the linking over the weakened literals, wheres FL allows only a single component with the genus 0. However, the *Weak+* can be simulated in FL through a suitable combination of *MulWeak*, $Contr_r$ and logical $-_r$ rules. Indeed, the

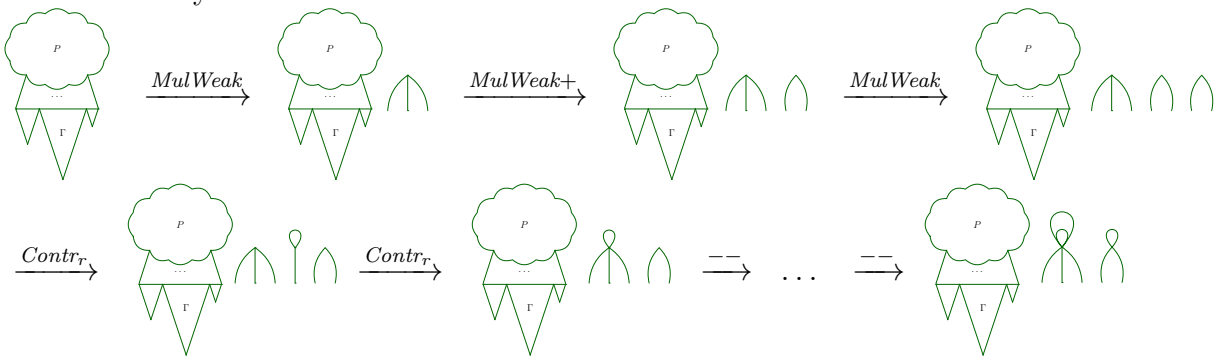
$$\frac{\vdash \Gamma}{\{(C_1, G_1), \ldots, (C_n, G_n)\} \vdash \Gamma, \Sigma}\ \textit{Weak+}$$

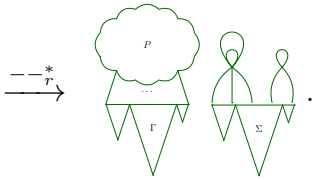
can be simulated by n instances of *MulWeak*, each of which introduces literals of a different component C_i. To get the proper genera, for a $\mathrm{a}^1 \in C_i$, one can introduce $(\{\mathrm{a}^2, \mathrm{a}^3\}, 0)$ by a *MulWeak*, which is followed by $Contr_r$ to get $(\{\mathrm{a}^{2-3}\}, 1)$, while another contraction with $\mathrm{a}^1 \in C_i$ will increase the genus by 1 for the component C_1. This is repeated G_i times, and for every i until the proper linking is achieved. Finally logical $-_r$ rules turn the literals into the sequent Σ.

We give here an example reflecting the general algorithm, where



is simulated by





First, one introduces a component with a genus 0 ; afterwards, genus is adjusted by a subsequent sequence of weakenings and contractions, and the process is repeated for all components introduced by the instance of *Weak+*.

The second part of the proposition can be shown by induction on an FL proof. The base of the induction, an instance of the axiom rule is immediate. As for the induction step, notice that every $-_l$ rule, can be replicated in $\mathsf{CL+}$. The induction step follows also immediate for the *MulWeak*, *Exch*, *Mix*, $\wedge_c^-$, Cut_c and all of the $\wedge_r, \vee_r$, Cut_r. For instance, in the case of Cut_c, the induction hypothesis for premisses gives $\mathsf{CL+}$ proofs of $\vdash \Gamma, \mathrm{A}$ and $\vdash \Sigma$ whose F-prenets are $P_1 \triangleright \Gamma, \mathrm{A}, \Pi_1^{\diamondsuit_l}$ and $P_2 \triangleright \Sigma, \Pi_3^{\diamondsuit_l}$. Now, by the above definition of left- and right-cuts, what we need is a $\mathsf{CL+}$ proof of $\vdash \Sigma$ with a F-prenet $P_2 \triangleright \Sigma, \Pi_3^{\diamondsuit_l}$, which entails keeping the one of subproofs we already have.

Therefore, what it remains to show is that $Contr_c$, $Contr_r$, $\vee_c$, can be simulated in $\mathsf{CL+}$ in the sense of the claim.

In the cases of $Contr_c$ and $Contr_r$, one is required to produce a $\mathsf{CL+}$ proof of the sequents $\vdash \Gamma, \mathrm{A}$ and $\vdash \Gamma$, respectively, form the given proofs of the same sequents, but whose assigned F-prenets are possibly more connected or with larger genus. How this can be done in the case of FL is shown in Lemma 4.3.12 and in Lemma 4.3.13 (the result presented there is independent of the current discussion and could be replicated here).

Finally, for e.g. $\vee_c^1$, one can simulate the FL derivation in the sense of the claim by extending the $\mathsf{CL+}$ proof of the induction hypothesis as below

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, \mathrm{A}}{- \vdash \Gamma, \mathrm{A}, \mathrm{B}, \Gamma}\,\textit{Weak+}}{\vdash \Gamma, \mathrm{A}, \mathrm{B}}\,\textit{Contr}}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}}\,\vee.$$

We also need to describe the linking in the *Weak+* step. For a every component of the F-net assigned to the FL proof connected to the literals of B we choose a literal of Γ. The linking of *Weak+* contains zero-genus components consisting of these literals of Γ and literals of B they are connected to. Every other literal of the introduced B, Γ is in its own single-element-zero-genus component. This completes the proof of the proposition.

□

With an appropriate use of *MulWeak* and perhaps subsequent contractions, one has :

Corollary 4.2.6. *For every cut-free proof of a sequent* $\vdash \Gamma\,;\Sigma$ *in* FL *there exists a cut-free proof of* $\vdash \Gamma, \Sigma$ *in* CL+ *that produces the same F-prenet.*

Conversely, given a cut-free proof of $\vdash \Gamma$ *in* CL+ *there exists a decomposition of the sequent* $\Gamma = \Gamma_1, \Gamma_2$ *such that there exists a proof of* $\vdash \Gamma_1\,;\Gamma_2$ *in* FL *that produces the same F-prenet.*

Notice that the two Γ_1 *and* Γ_2 *have disjoint linkings.*

We see that the systems FL and CL+ are not exactly equivalent in terms of the F-prenets they produce, but in practice they are. This is because the extra stuff that appears in a prenet that comes from a proof in FL is disjoint from the rest, and has no real logical contents whatsoever, since it comes from a succession of $-_r$ introduction rules applied to weakened formulas. Thus something logically equivalent could be produced without using the Cut_r formula, making for a prenet that is much easier to deal with.

One can design a version of FL where this discrepancy would disappear, but at the cost of much added complexity.

Thus the system FL should actually be seen as a syntactic justification for CL+. The latter cannot be seen as a formal system in the usual sense, but it is what we need in practice to prove the important results that follow. We will see that what will be called FL-correctnes (soon to be defined) could perhaps be better called CL+-correctness.

Once we have introduced the new calculus FL, whenever we speak of correctness for an F-prenet, we are should specify what is the proof system the correctness refers to, i.e. we speak of CL-correctness and FL-correctness.

By changing the calculus, the issue of correctness for (cut-free) F-nets also needs to be readdressed. The main difference is that this time whole subnets of F-prenets can come from weakenings, and thus can be disregarded in the net-directed proof search. Also, in this case, the characteristic of an F-prenet is increased by an axiom by 1, it is still not changed by contractions, but is increased by a weakening by ≥ 0. Indeed, a weakening adds a component with at least one atom in the component of genus 0, which consequently increases the characteristic $|P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|$ of an F-prenet by $\geq 1 - 1 + 0$.

Proposition 4.2.7. *If an F-prenet* $P \triangleright \Gamma$ *corresponds to an* FL *proof, then*

$$|Ax| \leq \mathcal{C}har(P \triangleright \Gamma).$$

This time, however, we need to address Weakening. A way to obtain correctness for the new calculus is to reuse the old approach, where one is allowed, in addition to removing leading disjunction connectives, to discard a piece of an F-prenet. This is precisely inverse to the interpretation of the weakening rule in CL+.

Definition 4.2.8. In $\mathcal{F}$**Synt**, we define an *extension* map to be of the form :

$$\begin{array}{c} P \vartriangleright \Gamma \\ \big\downarrow \\ P \uplus Q \vartriangleright \Gamma, \Sigma. \end{array}$$

It should be clear that the extension maps in $\mathcal{F}$**Synt** are induced by the calculus, and correspond to the rule *Weak+* of $\mathsf{CL}+$, as explained.

Definition of the notion of a correct F-net remains the same if one changes the notion of an anodyne map to incorporate the extension.

Definition 4.2.9. An *extension-anodyne map* in $\mathcal{F}$**Synt** is defined to be the composition of an extension map followed by an anodyne map

$$P \vartriangleright \Gamma \hookrightarrow P \uplus Q \vartriangleright \Gamma, \Sigma \xrightarrow{\ominus} P \uplus Q \vartriangleright \Delta$$

We also write :

$$P \vartriangleright \Gamma \xhookrightarrow{\ominus} P \uplus Q \vartriangleright \Delta$$

for an extension-anodyne map, and

$$P \vartriangleright [\Delta] \xhookrightarrow{\boxminus} P \uplus Q \vartriangleright \Delta$$

for an extension-anodyne map where $[\Delta]$ contains no formula with outermost disjunction.

Definition 4.2.10. An F-prenet $P \vartriangleright \Gamma, \Sigma$ is an FL-correct F-net with cuts (both Γ, Σ can contain cut formulas) if $P|_\Gamma \vartriangleright \Gamma$ is a root of a correctness tree of the Definition 4.1.2, for some choice of Γ and Σ. This time, however, all the non-cospan maps are extension-anodyne. Also, each leaf of the tree is an F-prenet of the form

$$(P_1 \uplus P_2)(\Gamma \curlyvee \Gamma) \vartriangleright \Gamma$$

such that $P_1 \vartriangleright \Gamma$ and $P_2 \vartriangleright \Gamma$ are F-prenets without cut formulas, with $P_1 \vartriangleright \Gamma$ being of the form

$$\{(\{\mathrm{a}, \overline{\mathrm{a}}\}, 0), (\{y\}, 0), \ldots, (\{x\}, 0)\} \vartriangleright \mathrm{a}, \overline{\mathrm{a}}, \Delta,$$

for some cut-free Δ.

Notice that this definition is based on system $\mathsf{CL}+$ and not on FL. There is no stoup information for prenets, and the shape of the leaf F-prenets corresponds to something that comes from an ordinary axiom that has beeen contracted with the result of applying *Weak+*.

An important difference to the CL case is the size of a correctness diagram for a correct F-net. In the CL case Lemma 3.2.17 has shown that the number of leaves in the correctness diagram is read-out directly from the F-prenet. In the case of the system FL the same value is an upper bound on the size of the corresponding correctness diagram.

Proposition 4.2.11. *Let* $\mathcal{T}$ *be an* FL*-correctness diagram for an F-net* $P \triangleright \Gamma$*. The diagram* $\mathcal{T}$ *has* not more than $\mathcal{C}har(P \triangleright \Gamma)$ *leaves.*

Démonstration. By induction on the size of $\mathcal{T}$. If $\mathcal{T}$ is a leaf or a single $\multimap\!\!\boxminus\!\!\rightarrow$ branch, P has a 0-genus component with two elements and possibly some additional components with positive genera, which gives an overall characteristic not smaller than 1. For the induction step, it suffice to notice that form Lemma 3.2.17 sum of the characteristics of domain F-prenets in a cospan is equal to the characteristic of the codomain F-prenet, while the $\multimap\!\!\boxminus\!\!\rightarrow$ map never decreases characteristic. □

The sequentialization theorem for F-nets and FL is reduced to the one for CL+ by Proposition 4.2.5, and the latter follows from the definition of extension-anodyne maps, Corollary 4.2.6 and the proof in the case of CL, up to the fact that the size of the correctness diagram is not longer that the prenet characteristic, in contrast to the cut-free CL case. In this case, the characteristic of an F-prenet is a semi-invariant, rather than the invariant of the bottom-up traversal of the diagram.

Also, the leaves of the correctness diagram are not as before F-prenets that consist of an axiom part and (optionally) a weakening part for the reasons explained in the definition. Consequently, discussion in the proof of Theorem 3.2.25 in the case where the last rule applied in a CL+ proof is *Contr* needs to be addressed, and the correctness tree for the premise of the rule is a leaf in the correctness diagram. In this case the claim follows from the associativity of Y.

The rest of the proof for the FL (CL+) sequentialization of follows the one for CL sequentialization.

Theorem 4.2.12 (Sequentialization)**.** *For every F-net there is an* FL *proof that corresponds to that F-net. The converse also holds, i.e. given an* FL *proof, its F-prenet is in fact a correct F-net.*

4.3 Cut elimination and FL- correct F-nets.

Let us have another look now at cut elimination on F-nets. Following the cut elimination-as-composition ideology we follow, we have

$$P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}} \twoheadrightarrow P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma.$$

The following follows trivially from the underlying categorical structure and the definition of the $\smile$ transformation.

Observation 4.3.1. Cut elimination for F-prenets is terminating and confluent and yields a unique normal form for an F-prenet with cuts.

We know now that cut elimination for an F-prenet yields a normal form, i.e., given an F-prenet with cuts, there is unique F-prenet without cuts that the original one reduces to. However, the important question is : does the same apply to FL-correct F-nets ? More concretely, starting from an F-net with cuts, does the cut elimination procedure result in another FL-correct F-net ? The direct answer to the most ambitious version of this question is, somewhat disappointingly, negative. The counter-examples are many, and one is given in 4.4.

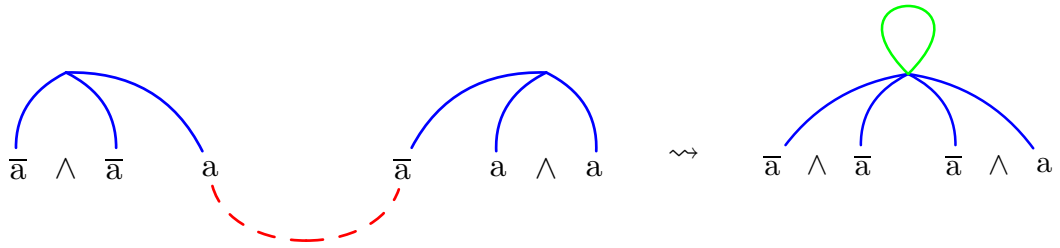


FIGURE 4.4 – Cut elimination on FL-correct F-nets does not yield an FL-correct F-net. The green loop that is added is required to get an FL-correct net, but is not obtained from the composition.

The F-prenet obtained after eliminating the cut does not come from a proof, neither in the FL system, nor in a (non-symmetric) deep inference version of FL we could formulate. There is, however, an observation we can make about the counter-example :

– had we added a loop on the single component of the resulting F-prenet while keeping the partition of the literals in the linking the same, it would be FL-correct.

Let us perform further analysis along these lines. But before doing so, we some need additional conceptual tools.

Definition 4.3.2. Given an F-prenet $P \triangleright \Gamma$, by *conjunctive switching* we understand a choice of an immediate subtree for every $\wedge$ node in the syntactic tree of Γ. Given a conjunctive switching, let Γ' be the tree where all of the subtrees chosen by the switching in Γ are erased, and let $\mathcal{L}(\Gamma')$ denote the set of leaves of the tree. (Notice that Γ' is not a sequence of well-formed propositional formulas, it is rather a graph with nodes decorated by connectives and literals.) By *switching of the F-prenet* $P \triangleright \Gamma$ we understand the pair $P|_{\mathcal{L}(\Gamma')} \triangleright \Gamma'$, for the choice of conjunctive switching of Γ.

We say an F-prenet $P \triangleright \Gamma$ is *sound* if for any switching Γ' the linking $P|_{\mathcal{L}(\Gamma')}$ contains a component with atoms of opposite polarity. We refer to these atoms of opposite polarity as *soundness witnesses* for the switching.

Soundness of an F-prenet with cut formulas is defined in the same way, where, in addition, each switching deletes a subtree for every $\diamondsuit$ connective, i.e. by treating the cut symbol as conjunction.

The reader can recognize the Lamarche-Straßburger condition on $\mathbb{B}$-nets appearing in [LS05b], whose possible predecessor is the Danos-Regnier condition for proof nets for multiplicative linear logic, appearing in [DR89].

Proposition 4.3.3. *A (*CL- */* FL-*) correct F-net is sound.*

Démonstration. By induction on the correctness diagram. Leaves of the correctness diagram always contain a component with two atoms of the opposite polarity that are present in every switching. As for the induction step, notice that Anodyne maps are unproblematic - disjunction introduction or a formula permutation does not affect switchings, wheres every switching of an F-prenet after extension by induction gets its witness from the restricted F-prenet in the domain of an extension. As for cospans, we consider them as a rule application, Mix and a conjunction, respectively, followed by a series of contractions. By applying a Mix one notices that switching of the codomain F-prenet is obtained by pairing switchings of the premisses, with components that are soundness witnesses coming from either of the witnesses of the premisses.
When encountering a conjunction rule, each switching of the F-prenet in the conclusion is a switching of one of the premisses, restricted to the corresponding F-prenet, and thus the soundness witness for the switching will come from one of the premisses. As for contractions,

$$\frac{P \triangleright \Gamma, \mathrm{A}, \mathrm{A}}{P(\mathrm{A} \curlyvee \mathrm{A}) \triangleright \Gamma, \mathrm{A}} \ ,$$

a switching Γ', A' of the conclusion generates a switching in the premise $\Gamma', \mathrm{A}', \mathrm{A}'$, by duplicating the switching of the contracted formula. Then, for the switching of the conclusion witnessing components may come from the premise directly, if soundness witness includes literals of Γ' of opposite polarity. Otherwise, if soundness witnesses for the switching are literals of one of the A' in the premise, the same literals after contraction will provide a witness for the prenet in conclusion, since the literals will belong to the same component after switching. □

In the following, when we write Γ, we assume that Γ may contain cut formulas.

Proposition 4.3.4. *Given a sound F-prenet $P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}}$, performing the*

$$P \triangleright \Gamma, \mathrm{A} \diamondsuit \overline{\mathrm{A}} \ \twoheadrightarrow \ P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma$$

cut elimination step preserves its soundness.

Démonstration. We show that

$$P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma$$

is sound by induction on the complexity of A.

- $\mathrm{A} = \mathrm{a}$, the cut is on literals. Let Γ' be a switching of Γ. The switching is extended to the switching of $\Gamma, \mathrm{a} \diamondsuit \overline{\mathrm{a}}$ by choice of either a or $\overline{\mathrm{a}}$. For each of the two choices there is a pair of literals that are witnesses for soundness of the sequent given the

switching. If it happens that a pair of witnesses is from Γ, it is a pair of witnesses for the switching Γ' in $P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma$. Alternatively, for the switching with a, there is an atom x of Γ' in the same component as a, with whom it witnesses the soundness for the switching. The switching with $\overline{\mathrm{a}}$ similarly gives a witness y in the same component as $\overline{\mathrm{a}}$. Finally, x and y are in the same component in $P(\mathrm{A} \smile \overline{\mathrm{A}})$, and are witnesses for soundness for the switching Γ';

– $\mathrm{A} = \mathrm{C} \wedge \mathrm{D}$ (the case $\mathrm{A} = \mathrm{C} \vee \mathrm{D}$ is the same one, since if $\mathrm{A} = \mathrm{C} \wedge \mathrm{D}$, then $\overline{\mathrm{A}} = \overline{\mathrm{C}} \vee \overline{\mathrm{D}}$).
Let again Γ' be a switching of Γ. By assumption, $P \triangleright \Gamma, (\mathrm{C} \wedge \mathrm{D}) \diamondsuit (\overline{\mathrm{C}} \vee \overline{\mathrm{D}})$ is sound. The switching Γ' is extended to a switching of $\Gamma, (\mathrm{C} \wedge \mathrm{D}) \diamondsuit (\overline{\mathrm{C}} \vee \overline{\mathrm{D}})$ by a choice of exactly one of the following : C, D or both $\overline{\mathrm{C}}$ and $\overline{\mathrm{D}}$. Once the choice is made, it suffice to choose a switching for the chosen formula(s), one of the $\mathrm{C}', \mathrm{D}', \overline{\mathrm{C}}', \overline{\mathrm{D}}'$ for the respective formulas.
Consider now $P \triangleright \Gamma, \mathrm{C} \diamondsuit \overline{\mathrm{C}}, \mathrm{D} \diamondsuit \overline{\mathrm{D}}$. A switching for $\Gamma, \mathrm{C} \diamondsuit \overline{\mathrm{C}}, \mathrm{D} \diamondsuit \overline{\mathrm{D}}$ consists of a switching Γ', a choice of either, C or $\overline{\mathrm{C}}$ and either D or $\overline{\mathrm{D}}$, and a subsequent switchings of the chosen formulas. When compared to the switchings of $\Gamma, (\mathrm{C} \wedge \mathrm{D}) \diamondsuit (\overline{\mathrm{C}} \vee \overline{\mathrm{D}})$, one sees readily that each switching of $\Gamma, \mathrm{C} \diamondsuit \overline{\mathrm{C}}, \mathrm{D} \diamondsuit \overline{\mathrm{D}}$ yields a set of literals that is a superset of the set of literals in a switching of $\Gamma, (\mathrm{C} \wedge \mathrm{D}) \diamondsuit (\overline{\mathrm{C}} \vee \overline{\mathrm{D}})$. Thus, if $P \triangleright \Gamma, (\mathrm{C} \wedge \mathrm{D}) \diamondsuit (\overline{\mathrm{C}} \vee \overline{\mathrm{D}})$ is sound, so must be $P \triangleright \Gamma, \mathrm{C} \diamondsuit \overline{\mathrm{C}}, \mathrm{D} \diamondsuit \overline{\mathrm{D}}$.
Now, the induction hypothesis on the complexity of cut formula applied twice shows that

$$P(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma \;\text{ is }\; P(\mathrm{C} \wedge \mathrm{D} \smile \overline{\mathrm{C}} \vee \overline{\mathrm{D}}) \triangleright \Gamma \;\text{ which is }\; P(\mathrm{C} \smile \overline{\mathrm{C}})(\mathrm{D} \smile \overline{\mathrm{D}}) \triangleright \Gamma.$$

□

Convention 4.3.5. As we have established, there is a canonical way to perform cut elimination on F-prenets which yields normal forms that are attainable in a single step. Our main interest in the remaining sections of this chapter is to investigate the conditions under which FL-*correct* F-prenets with cuts yield cut-free FL-correct F-nets via cut elimination. For this purpose it suffice to assume that the cut elimination is done on a pair of two cut-free F-prenets, i.e. it suffice to consider

$$P \uplus Q \triangleright \Gamma, \Sigma, \overline{\mathrm{A}} \diamondsuit \mathrm{A} \;\twoheadrightarrow\; P \uplus Q(\overline{\mathrm{A}} \smile \mathrm{A}) \triangleright \Gamma, \Sigma,$$

where $P \triangleright \Gamma, \overline{\mathrm{A}}$ and $Q \triangleright \mathrm{A}, \Sigma$ are without cut formulas. This is slightly less general than the cut elimination applied on an F-prenet with several cut formulas, but for the purposes stated here the assumption does not cause loss of generality.

Indeed, assume we know how (under which conditions) cut elimination on a pair of two FL-correct *cut-free* F-nets yields another FL-correct F-net. An FL-correct F-prenet *with several cuts* yields a correctness tree for that F-prenet which contains $\diamondsuit$-cospans. Going from the leaves of the correctness tree towards the root, we may replace each its subtree which starts with a $\diamondsuit$-cospan by a correctness tree of another FL-correct F-net. The FL-correct net in question is the one obtained by performing a cut elimination (on a single formula) on the pair of the FL-correct cut-free F-nets in the domains of the $\diamondsuit$-cospan. This can be done since we know under which conditions a cut can be eliminated on a pair of

cut-free formulas. The cut formula is than erased from the rest of the tree and the process is repeated until the root is reached. This construction yields a correctness tree without $\diamondsuit$-cospans and therefore also gives the conditions under which the cut elimination on a general FL-correct F-net with cuts results in the FL-correct F-prenet where the multiple cut formulas are eliminated.

It is worth reminding the reader of the categorical structure (cut-free) F-prenets stand for here.

Definition 4.3.6 (Category of F-prenets)**.** The *category of F-prenets*, written **F-prenet**, has propositional formulas as its objects.

A map $\mathrm{A} \to \mathrm{B}$ is a cut-free F-prenet $P \triangleright \overline{\mathrm{A}}, \mathrm{B}$.

The identity F-prenet for A is the F-prenet $\mathrm{Id}_{\mathrm{A}} \triangleright \overline{\mathrm{A}}, \mathrm{A}$, where Id_{A} is the linking consisting of two-element components connecting the i-th literal (counted, say from left to right) in A with the i-th in $\overline{\mathrm{A}}$, with genus 0 everywhere.

Given $P \triangleright \overline{\mathrm{A}}, \mathrm{B} :\ \mathrm{A} \to \mathrm{B}$ and $Q \triangleright \overline{\mathrm{B}}, \mathrm{C} :\ \mathrm{B} \to \mathrm{C}$, the composition map is the F-prenet

$$P \uplus Q \left(\mathrm{B} \smile \overline{\mathrm{B}}\right) \triangleright \overline{\mathrm{A}}, \mathrm{C}.$$

The reader should be aware by now that every F-prenet $P \triangleright \Gamma, \Sigma$ for any partition and any permutation of the formulas in $\Gamma \cup \Sigma$ into Γ, Σ, determines a map $\overline{\Gamma} \to \bigvee \Sigma$ in **F-prenet**, where $\overline{(-)}$ is the negation extended to sequents by $\overline{\mathrm{A}, \mathrm{B}} := \overline{\mathrm{A}} \wedge \overline{\mathrm{B}}$, and where $\bigvee(-)$ is a transformation of a sequent into a disjunction formula replacing comma symbols by $'\vee'$.

A map in **F-prenet** stemming from a (cut-free) F-prenet $P \triangleright \Gamma, \Sigma$ also determines a map in the freely generated Frobenius category $\widehat{\overline{\Gamma}} \to \widehat{\bigvee \Sigma}$ (Γ is negated), where $\widehat{(-)}$ is the transformation that sends a formula to an object of the Frobenius category by replacing the propositional connectives with the monoidal bifunctor symbol $\otimes$. This connection is depicted in 3.18.

Therefore **F-prenet** is equivalent to **FrThFrob**($\mathcal{AT}$yp) [15].

The previous proposition now shows that the category **F-prenet** has a subcategory of sound F-prenets, **sF-prenet**.

Going back to the definition of a sound F-prenet, the reader may notice that the notion of the soundness depends only on components of a linking, but not on the genera.

Let us have a look at the forgetful functor from **FrThFrob**($\mathcal{AT}$yp) to the category we call **BFrThFrob**($\mathcal{AT}$yp), which forgets about genera. In other words, **BFrThFrob**($\mathcal{AT}$yp) is **FrThFrob**($\mathcal{AT}$yp) deprived of the genus information. The same way **FrThFrob**($\mathcal{AT}$yp) gives rise to the category **F-prenet**, the category **BFrThFrob**($\mathcal{AT}$yp) gives rise to the

15. we accept that formulas of a F-prenet in **F-prenet** can be empty formulas

category of the assigned prenets with linkings standing for maps from the monoidal unit in **BFrThFrob**($\mathcal{AT}$yp). We call these *FB-prenets*, and their category is denoted by **FB-prenet**. We refer to

$$\mathbf{FB} : \{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Gamma \mapsto \{C_1, \ldots, C_k\} \triangleright \Gamma$$

as to a forgetful functor from **F-prenet** to **FB-prenet**.

The notion of a sound prenet can be repeated for FB-prenets, i.e. it is stable under the forgetful functor we have just defined. Consequently, the sound FB-prenets constitute a subcategory **sFB-prenet** of **FB-prenet**. (We remind the reader that the composition in **FB-prenet** differs from the composition in **F-prenet** as no information on genus is present).

We have already established that FL-correct F-nets are sound, and therefore, the image under the functor **FB** into **FB-prenet** falls into the subcategory **sFB-prenet**, which is precisely the claim of the previous proposition.

To sum up, the following diagram commutes :

$$\begin{array}{ccc} \textbf{sF-prenet} & \hookrightarrow & \textbf{F-prenet} \\ \Big\downarrow{\scriptstyle \mathbf{FB}|_{\textbf{sF-prenet}}} & & \Big\downarrow{\scriptstyle \mathbf{FB}} \\ \textbf{sFB-prenet} & \hookrightarrow & \textbf{FB-prenet}. \end{array}$$

We continue our investigation of cut elimination by introducing some more concepts.

Definition 4.3.7. We say that a category is *order-enriched* or *ordered* if there is a partial order defined on every hom-set, such that given three objects X, Y, Z the composition function $\circ : Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$ is monotone in both variables.

Using the terminology of the enriched category theory, an ordered category is enriched over the monoidal category (**Poset**, $\times, \emptyset$) of partial orders and monotone functions.

The following proposition is an exercise in understanding of the above definition and the definition of composition in **F-prenet**.

Proposition 4.3.8. *Let $\leq$ be a partial order defined on the set of all F-prenets such that*

1. *$P \triangleright \Gamma \leq P' \triangleright \Gamma'$ implies $\Gamma = \Gamma'$,*
2. *given cut-free $P \triangleright \Gamma, \mathrm{A} \leq P' \triangleright \Gamma', \mathrm{A}$ and $Q \triangleright \overline{\mathrm{A}}, \Sigma$, one has*

$$P \uplus Q(\overline{\mathrm{A}} \smile \mathrm{A}) \triangleright \Gamma, \Sigma \;\leq\; P \uplus Q(\overline{\mathrm{A}} \smile \mathrm{A}) \triangleright \Gamma', \Sigma.$$

Then $\leq$ *induces an order enrichment on* **F-prenet**.

The previous proposition enables us to generalize the notion of an order enrichment to the set of all F-prenets, not only on the two-formula F-prenets of **F-prenet**. Therefore, by "enrichment on the set of F-prenets" we mean the presence of an order satisfying the conditions of the previous proposition.

One such concrete enrichment is given in the following definition.

Definition 4.3.9. Given two F-prenets $P \triangleright \Gamma$ and $Q \triangleright \Gamma$ we write

$$P \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$$

when

1. components in the partitions of literals of Γ in P and Q coincide
2. for every component (C, G) of P, the corresponding component (C, G') of Q is such that that $G \leq G'$.

To show that the above order is indeed an enrichment, we show

Proposition 4.3.10. *For every three F-prenets* $P \triangleright \Gamma, \mathrm{A}$*;* $Q \triangleright \Gamma, \mathrm{A}$; *and* $R \triangleright \overline{\mathrm{A}}, \Delta$;

$$P \triangleright \Gamma, \mathrm{A} \ \leq_{\mathcal{G}} \ Q \triangleright \Gamma, \mathrm{A}$$

implies

$$P \uplus R(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma, \Delta \ \leq_{\mathcal{G}} \ Q \uplus R(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma, \Delta.$$

Démonstration. By the definition of $\smile$, the partitions of literals after the both $\smile$ transformations remain the same, the genera created by the composition are the same, wheres those coming from the factors in the composition are added. The claim thus follows from the monotonicity of addition. □

The following is a direct consequence of the previous proposition.

Theorem 4.3.11. *The order* $\leq_{\mathcal{G}}$ *on F-penets in* **F-prenet** *defines an order-enrichment of the category.*

The enrichment we have just defined behaves well with respect to the FL-correctness.

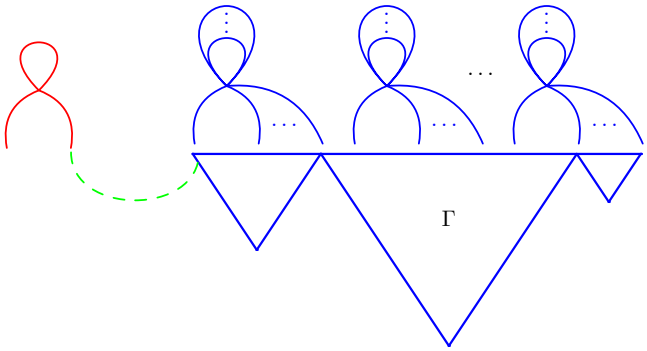
Lemma 4.3.12. *If an F-prenet* $P \triangleright \Gamma$ *comes from a classical* FL *proof, that is, if it is* FL*-correct, then so is any F-prenet* $Q \triangleright \Gamma$*, for which* $P \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$.

Démonstration. Let P be $\{(C_1, G_1), \ldots, (C_k, G_k)\}$ and Q be $\{(C_1, G'_1), \ldots, (C_k, G'_k)\}$, with $G_i \leq G'_i$. It suffice to show the claim for the case $G'_1 = G_1 + 1$, $G'_i = G_i$ for $i > 1$.

Given

$$\{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Gamma,$$

one can consider an FL proof that corresponds to it (out of many). Now, for (C_1, G_1), one can choose an axiom (out of possibly several), such that one of the literals of the rule is a domain of a proof-induced map in $\mathcal{F}\mathbf{Synt}$ which has the codomain form the set C_1. In other words, one can pick an instance of the axiom rule that introduced an atom which is an ancestor in the history of the proof of an atom in C_1. Now, the proof whose F-prenet is $\{(C_1, G'_1), \ldots, (C_k, G'_k)\} \triangleright \Gamma$ is obtained by replacing the axiom by the proof corresponding to the doubling map from Figure 3.24, which yields correctness of the F-net $\{(C_1, G_1 + 1), \ldots, (C_k, G_k)\} \triangleright \Gamma$.



□

The previous lemma shows that the set of FL-correct F-nets is up-closed for the $\leq_{\mathcal{G}}$ relation. The same set will also be closed under the following transformation.

Lemma 4.3.13. *If an F-prenet* $\{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Gamma$ *is* FL*-correct, then so is any F-prenet obtained by connecting several components , i.e. any*

$$\{(C_1, G_1), \ldots, (C_{i-1}, G_{i-1}), (C_i \cup C_{i+1}, G_i + G_{i+1}), (C_{i+2}, G_{i+2}), \ldots, (C_k, G_k)\} \triangleright \Gamma$$

is also FL*-correct.*

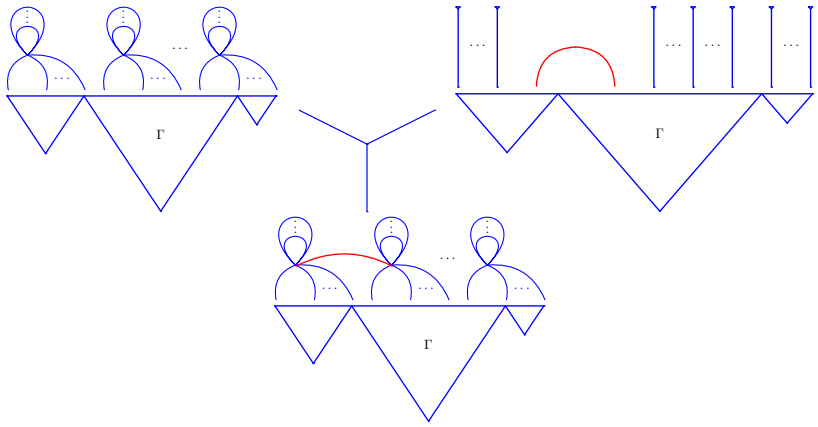
Démonstration. Given $\{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Gamma$, consider a corresponding FL proof of the sequent $\vdash \Gamma$. We know that for this proof there is one in $\mathsf{CL+}$ with the same F-net. Now, extend the $\mathsf{CL+}$ proof by the weakening :

$$\frac{\vdash \Gamma}{\{(0, \{a_1\}), \ldots, (0, \{a_{k-1}\}), (0, \{a_k, a_{k+1}\}), (0, \{a_{k+2}\}), \ldots, (0, \{a_n\})\} \vdash \Gamma, \Gamma} \; \textit{Weak+}$$

where $\mathrm{a}_1, \ldots, \mathrm{a}_m = \mathcal{L}(\Gamma^2)$, and a_k is a copy of an atom form C_i, while a_{k+1} is a copy of an atom form C_{i+1}. Now, for this $\mathsf{CL+}$ proof there is an FL proof with the same F-net. Finally, by performing contractions :

$$\frac{\vdash \Gamma, \Gamma}{\vdash \Gamma} \; \textit{Contr}$$

one obtains an FL proof whose F-net is the desired one.



□

Corollary 4.3.14. *The rules Cut_c and Cut_r of* FL *can be simulated in* CL+ *is such a way as to produce the same F-prenets.*

We kindly remind the reader here that FL-correctness is actually CL+ correctness.

The Lemma 4.3.13 motivates the following definition.

Definition 4.3.15. Given two F-prenets $P \triangleright \Gamma$ and $Q \triangleright \Gamma$ we define

$$P \triangleright \Gamma \preceq Q \triangleright \Gamma$$

if

1. the partition of literals in Γ is finer in P than in Q, i.e. each component of Q can be obtained by connecting several components of P
2. for every component (G, C) of Q for which $C = C_1 \cup \cdots \cup C_k$, for components $(G_1, C_1), \ldots, (G_k, C_k)$ of P, we have

$$G \geq \sum_1^k G_k.$$

Notice that if $P \triangleright \Gamma \preceq Q \triangleright \Gamma$, then $\mathcal{C}har(P \triangleright \Gamma) \leq \mathcal{C}har(Q \triangleright \Gamma)$, but converse is not generally true.

The map to the left has the smaller characteristic, but the two maps are incomparable w.r.t. the $\preceq$ order.

Theorem 4.3.16. *The order $\preceq$ on F-prenets an order-enrichment on* **F-prenet**.

Démonstration. What needs to be shown, by Proposition 4.3.8, is that the composition in **F-prenet** is monotone w.r.t. $\preceq$. Indeed, let $P \triangleright \Gamma, \mathrm{A} \preceq Q \triangleright \Gamma, \mathrm{A}$. Then, after composing with $R \triangleright \overline{\mathrm{A}}, \Delta$, finer partitions of literals in of P remain finer in $P \uplus R(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma, \Delta$, and the increase of genus due to the composition is larger with $Q \uplus R(\mathrm{A} \smile \overline{\mathrm{A}}) \triangleright \Gamma, \Delta$, since there is fewer cases where composition connects different components (see Figure 4.5). In the case the partitions of P, Q coincide, Q contributes, by definition, with more genera to the composition with $R \triangleright \overline{\mathrm{A}}, \Delta$.

FIGURE 4.5 – If $P \prec Q$ (both are over the same sequent) and Q is *more connected*, by composition, P can become *as connected*, but in that case Q will keep larger genera. In this example : $m \geq l + m$, and after composition $m + 1 > l + m$.

□

Theorem 4.3.17. *For every sound F-prenet $P \triangleright \Gamma$ there is an F-prenet $Q \triangleright \Gamma$ for which*

$$P \triangleright \Gamma \ \leq_{\mathcal{G}} \ Q \triangleright \Gamma$$

and which is FL*-correct.*

Démonstration. By induction on number of connectives in Γ.

Let P be $\{(C_1, G_1), \ldots, (C_k, G_k)\}$ and let Q be $\{(C_1, G'_1), \ldots, (C_k, G'_k)\}$, with $G_i \leq G'_i$.

- there is a disjunction in Γ, i.e. Γ is of the form $\Sigma, \mathrm{A} \vee \mathrm{B}$. Then, $\{(C_1, G_1), \ldots, (C_k, G_k)\} \triangleright \Sigma, \mathrm{A}, \mathrm{B}$ is also FL-correct, since the switching of the two F-prenets coincide. By induction, there is $\{(C_1, G'_1), \ldots, (C_k, G'_k)\} \triangleright \Sigma, \mathrm{A}, \mathrm{B}$, with $G_i \leq G'_i$, which is an FL-correct F-net, and then so is $\{(C_1, G'_1), \ldots, (C_k, G'_k)\} \triangleright \Sigma, \mathrm{A} \vee \mathrm{B}$, by the disjunction rule.
- there is a conjunction in Γ, i.e. Γ is of the form $\Sigma, \mathrm{A} \wedge \mathrm{B}$. This time, one can notice, form a simple arguments on switchings that

$$\{(C_1^1, G_1), \ldots, (C_k^1, G_k)\}|_{\Sigma^1, \mathrm{A}^1} \triangleright \Sigma^1, \mathrm{A}^1;$$

$$\{(C_1^2, G_1), \ldots, (C_k^2, G_k)\}|_{\Sigma^2, \mathrm{B}^2} \triangleright \Sigma^2, \mathrm{B}^2;$$

$$\{(C_1^3, G_1), \ldots, (C_k^3, G_k)\} \triangleright \Sigma^3, \mathrm{A}^3, \mathrm{B}^3$$

are also FL-correct. By induction, there are

$$\{(C_1^1, H_1), \ldots, (C_k^1, H_k)\}|_{\Sigma^1, \mathrm{A}^1} \triangleright \Sigma^1, \mathrm{A}^1;$$

$$\{(C_1^2, H_{k+1}), \ldots, (C_k^2, H_{2k})\}|_{\Sigma^2, \mathrm{B}^2} \triangleright \Sigma^2, \mathrm{B}^2;$$
$$\{(C_1^3, H_{2k+1}), \ldots, (C_k^3, H_{3k})\} \triangleright \Sigma^3, \mathrm{A}^3, \mathrm{B}^3$$

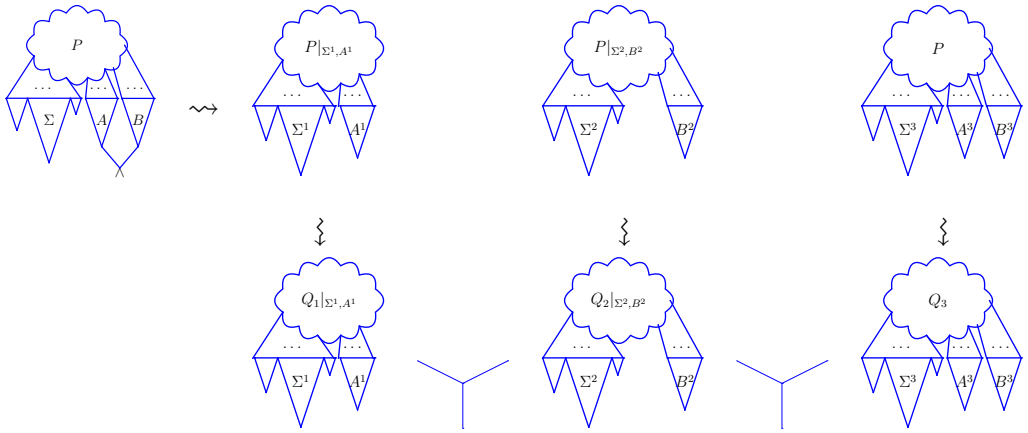
with $G_i \leq H_{m \cdot k+i}$, which are FL-correct F-nets. By performing two $\wedge$s and contractions on Σ's, A's and B's in an underlying FL proof, one obtains a linking of an FL-correct F-net

$$\left(\{(C_1^1, H_1), \ldots, (C_k^3, H_{3k})\}|_{\Sigma^1, \mathrm{A}^1, \Sigma^2, \mathrm{B}^2, \Sigma^3, \mathrm{A}^3, \mathrm{B}^3}\right) (\Sigma^1 \curlyvee \Sigma^2 \curlyvee \Sigma^3, \mathrm{A}^1 \wedge \mathrm{B}^3 \curlyvee \mathrm{A}^3 \wedge \mathrm{B}^2).$$

The crucial observation here is that the contractions respect the linking partition with genera augmented, i.e. the result can be seen as

$$\{(C_1, F_1), \ldots, (C_k, F_k)\},$$

which shows that $\{(C_1, F_1), \ldots, (C_k, F_k)\} \triangleright \Sigma, \mathrm{A} \wedge \mathrm{B}$ is FL-correct, with $G_i \leq F_i$.



– if Γ is a set of literals, then one of the components, say (C_1, G_1) is of the form

$$(\{\mathrm{a}^1, \ldots, \mathrm{a}^m, \overline{\mathrm{a}}^1, \ldots, \overline{\mathrm{a}}^n\}, G_1),$$

with, say, $m \geq n > 0$. Now, by taking $2m$ instances of the axiom rule, performing Mix $2m-1$ times, contracting positive atoms $2m-1$ times to obtain a single component with m occurrences of a and $2m$ occurrences of $\overline{\mathrm{a}}$, of genus 0, and finally, contracting $\overline{\mathrm{a}}$ occurrences $2m-n$ times, one obtains a proof whose FL-correct F-prenet is

$$\{(\{\mathrm{a}^1, \ldots, \mathrm{a}^m, \overline{\mathrm{a}}^1, \ldots, \overline{\mathrm{a}}^n\}, 2m-n)\} \triangleright \mathrm{a}^1, \ldots, \mathrm{a}^m, \overline{\mathrm{a}}^1, \ldots, \overline{\mathrm{a}}^n.$$

By what we have previously shown,

$$\{(\{\mathrm{a}^1, \ldots, \mathrm{a}^m, \overline{\mathrm{a}}^1, \ldots, \overline{\mathrm{a}}^n\}, \max(2m-n, G_1))\} \triangleright \mathrm{a}^1, \ldots, \mathrm{a}^m, \overline{\mathrm{a}}^1, \ldots, \overline{\mathrm{a}}^n$$

is an FL-correct F-net. Finally,

$$\{(C_1, \max(2m-n, G_1)), \ldots, (C_k, G_k)\} \triangleright \Gamma$$

is also FL-correct, since it comes from a *Weak+* rule of CL+ applied in the end.

□

Let us reflect for a while on the last observations we made. To that purpose, consider the following : let Γ be a sequent with cut formulas. We define $|Cut|$ to be

$$|Cut| = \sum_{A^k \diamondsuit \overline{A^k} \text{ in } \Gamma} |\mathcal{L}(A^k)|.$$

Figure 3.26 shows the effect of operations on linkings, and in particular, the last two rows show the effect of cut elimination. More concretely, if $P_1 \triangleright \Gamma_1$ is an FL-correct F-net with cuts, according to Proposition 4.2.7 we have

$$|Ax_1| \leq |P_1| - |\mathcal{C}omp_{P_1}| + |\mathcal{G}en_{P_1}| = \mathcal{C}har(P_1 \triangleright \Gamma_1).$$

Each time a $\smile$ is performed on a pair of formulas, the left side of the inequality decreases by the number of eliminated pairs of literals, thus, if $P_2 \triangleright \Gamma_2$ is the F-prenet obtained after cut elimination, one has

$$|Ax_1| \leq |P_1| - |\mathcal{C}omp_{P_1}| + |\mathcal{G}en_{P_1}| - |Cut| \leq |P_2| - |\mathcal{C}omp_{P_2}| + |\mathcal{G}en_{P_2}| = \mathcal{C}har(P_2 \triangleright \Gamma_2).$$

The first $\leq$ is a consequence of the fact that characteristic of an F-prenet is an upper bound in the case of FL, as already established, while the second $\leq$ stands instead of the equation because cut elimination can remove components altogether due to the thinness of the underlying category. This situation is depicted in Figure 4.6

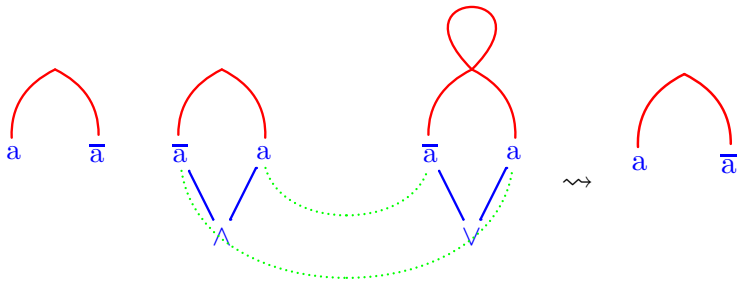


FIGURE 4.6 – Cut elimination can remove components altogether. This corresponds to the thinness property of Frobenius algebras.

Our starting point in semantical considerations were *-autonomous categories, which are models for multiplicative linear logic. The proof theory of multiplicative linear logic is sensitive to resources, understood as number of applications of the axiom rule in a proof. We have decided to follow this approach and tried do devise models that differentiate between proofs that differ in number of axioms used. The analysis has taken us to the notion of the Frobenius category and the F-(pre)nets, where we have seen that the interpretations of proofs have an invariant – the characteristic is increased by 1 by a single instance

of the axiom rule. This characteristic represents a lower-bound on the number of axioms used in an FL proof corresponding to the F-prenet, if such a proof exists. Cut elimination on a pair of literals, on the other hand, decreases the characteristic by 1, dually to the axiom. Therefore, it makes sense to state

the characteristic of an F-prenet keeps track of *resources* used in a proof.

Indeed, the first lemma out of the last two shows that if we have succeeded in building a proof out of pieces offered by an F-prenet, we are sure to succeed with an F-prenet with larger genera. Similar with the lemma that follows. Since increase of the genera means increase of the characteristic, just as connecting two mutually disconnected components, these two lemmas can be rephrased to saying that if a proof can be constructed using resources coming from an F-prenet, then, naturally, a proof can be constructed using more resources. An axiom introduces a unit of resources, wheres cut elimination is consuming resources. Finally, the previous theorem can be rephrased, with slight generalization

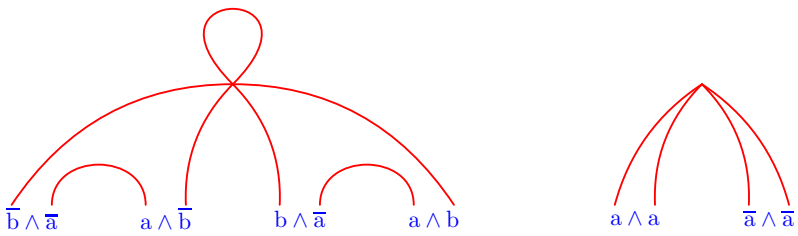
Theorem 4.3.17., restated. A sound F-prenet can be turned into an FL-correct F-net by adding resources.

Notice that one can speak of three dimensions of a resource - (number of) components, genera, and (number of) literals, with the last one being constant for all proofs of the given sequent. This, of course, concerns the cut free F-prenets/proofs.

Going back to **FB-prenet**, the category of F-prenets without the genus information, and its subcategory **sFB-prenet** of sound prenets from **FB-prenet**, the previous Theorem precisely states that **sFB-prenet** yields an interpretation of classical proofs for which the property of *full completeness* holds.

We have already established that the notion of correctness for F-prenets is heavily dependent on the deductive system in question. In the case of CL, we have encountered F-prenets for which there is no derivation in the calculus, but for which there exist derivations in a deep inference system. The same is for the case of FL and its deep inference version.

Example 4.3.18. We go back to the two F-prenets in Example 3.2.15.



The F-prenet to the right has the following KS proof (notice that we make no effort

towards reducing the proof bureaucracy) :

$$
\begin{array}{cl}
\mathsf{T} & \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee b)\wedge(b\vee\bar{b})\wedge(\bar{b}\vee b)\wedge(\bar{b}\vee b) & 5\times\mathsf{ai}\downarrow \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee b)\wedge(b\vee\bar{b})\wedge(\bar{b}\vee(b\wedge(\bar{b}\vee b))) & \mathsf{s} \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee b)\wedge(b\vee\bar{b})\wedge(\bar{b}\vee\bar{b}\vee(b\wedge b)) & \mathsf{s} \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee b)\wedge(b\vee\bar{b})\wedge(\bar{b}\vee(b\wedge b)) & \mathsf{ac}\downarrow \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee(b\wedge(b\vee\bar{b})))\wedge(\bar{b}\vee(b\wedge b)) & \mathsf{s} \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee\bar{b}\vee(b\wedge b))\wedge(\bar{b}\vee(b\wedge b)) & \mathsf{s} \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge(\bar{b}\vee(b\wedge b))\wedge(\bar{b}\vee(b\wedge b)) & \mathsf{ac}\downarrow \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge((b\wedge b)\vee(\bar{b}\wedge(\bar{b}\vee(b\wedge b)))) & \mathsf{s} \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge((b\wedge b)\vee(\bar{b}\wedge\bar{b})\vee(b\wedge b)) & \mathsf{s} \\ \hline
(\bar{a}\vee a)\wedge(\bar{a}\vee a)\wedge((\bar{b}\wedge\bar{b})\vee(b\wedge b)) & \mathsf{ac}\downarrow \\ \hline
(\bar{b}\wedge\bar{a})\vee(a\wedge\bar{b})\vee(b\wedge\bar{a})\vee(a\wedge b) & 6\times\mathsf{s}
\end{array}
$$

To see that the same F-prenet corresponds to no proof in CL/FL, one may attempt to construct a correctness tree for the F-prenet. Any such tree needs to begin with a conjunction cospan, since a Mix cospan would have to have a non-sound F-prenet as the domain of one of its legs. A conjunction cospan yields F-prenets in the domains that both have to have a single component $\{\bar{a}, a\}$, which guarantees that the one containing the formula a (or $\bar{a}$) is not sound.

As for the F-prenet to the right, in a similar way as in the previous case, we realize that any Mix or conjunction cospan that would start a hypothetical correctness tree has to have non-sound F-prenets as domains of its legs. Thus, no (shallow) CL/FL proof is possible. Any deep proof assigned to the F-prenet to the right has to have at most 3 $\mathsf{ai}\downarrow$ rules. To make the three components created by the rule applications into the required linking, one either performs two contractions on literals or a single contraction on two conjunctions of two literals. In both cases, obtaining a single component through conjunction there will leave a literal in a disjunctive context.

One possible view of the concept of resources as we have defined them is that they provide a semantical account of relative efficiency of deductive systems. More precisely, as we have shown, certain deductive systems, such as the deep inference ones, can provide proofs for certain configurations of linkings, wheres the standard sequent ones require additional resources. There could be certain potential for this line of investigation, and some remotely related ideas can be found in work of Carboni [Car99b] (more on relation to this work can be found later in the text in the section on other relevant approaches).

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Let us now summarize. What we have so far is a category of F-prenets **F-prenet** and its subcategory of sound F-prenets **sF-prenet**. Image of **sF-prenet** under the forgetful functor that forgets about genera defines a category of classical proof nets, for which full completeness holds. This category contains less information on proofs than the category of $\mathbb{B}$-nets from [LS05b]. In fact, a linking in a sound FB-prenet is a transitive closure of a

linking in a B-net, hence one looses information on axiom links in this category (see the section on other relevant approaches).

FL-correct F-nets themselves do not form a category (at least not in the standard way of composing nets by connecting linkings), but we know that we can construct an FL-correct F-net out of a sound one by adding genera. In addition, FL-correct F-nets are closed under this adding of resources, so the question is what is the minimal amount of resources one needs to add to obtain an FL-correct F-net. We do not have a generic answer to this question (but we re-address this issue further in the text), still we know how to compute the minimal amount of resources for every sound F-prenet separately. Indeed, due to the decision procedure for FL-correct F-nets, by incremental adding of genera, we can find minimal FL-correct F-nets w.r.t. $\leq_{\mathcal{G}}$ relation.

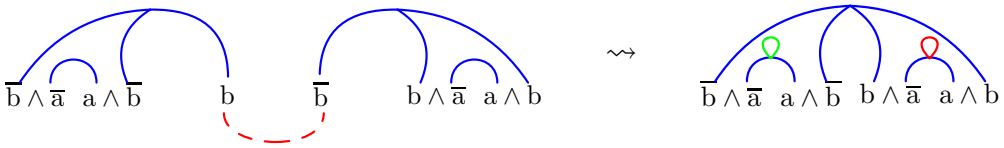


FIGURE 4.7 – The sound F-prenet without both the green and the red loop is not FL-correct. If either, the green one or the red one is added, resulting F-prenet is FL-correct.

These minimal FL-correct F-nets can be several, as shown in Figure 4.7.

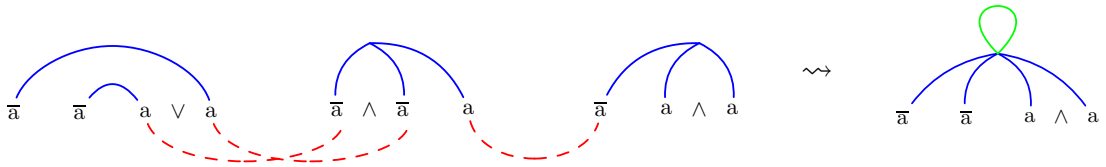


FIGURE 4.8 – Depending on the order of elimination of cuts, one has to add resources or not to. If the cut to the right is eliminated first on two FL-correct F-nets, genus needs to increased by 1 to get an FL-correct F-net. If the cut to the left is eliminated first, the green loop need not be added.

One may be tempted to start with FL-correct F-nets and then for every sound F-prenet that is obtained in the composition / cut elimination, to establish a policy of adding resources to it to remain in the realm of FL-correct F-nets. Unfortunately, the straightforward attempt proves to be faulty, since the composition/ cut elimination of $\leq_{\mathcal{G}}$-minimal FL-correct F-nets turns to be non-associative, as in the F-prenets of Figure 4.8

Also, one can hope that the sound F-prenet obtained from cut elimination on an FL-correct F-net is the infimum of all FL-correct nets it is $\preceq$-smaller than, but this is also not the case, as shown in the Figure 4.9

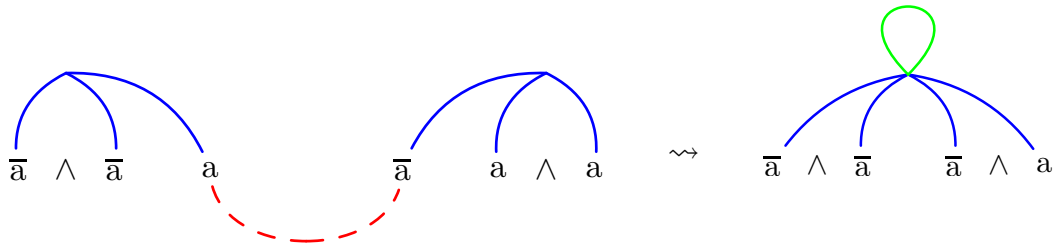


FIGURE 4.9 – The F-prenet obtained by composition/cut elimination is not the infimum of all FL-correct F-nets it is smaller than. The infimum is the F-net without the added green loop, which is not correct.

4.3.1 Cut elimination for F-nets

At this point our theory of proof nets for classical logic suggests that normal forms we assign to proofs with cuts, i.e. sound F-prenets obtained via cut-as composition normalization *of F-prenets*, do not correspond to any proofs that may be obtained by an effective cut elimination procedure *at the level of proofs*. We have already encountered this situation in the Prelude chapter where normal forms of proofs for numerals composed via cut differ from what should be obtained by an effective cut elimination procedure, had we had one at our disposal. We have described this by saying that our interpretations have nothing to do with Curry-Howard. In [Hyl04] Hyland argues that classical proof theory requires this kind of approach to cut elimination and proof identification.

For these reasons, we are interested in a possibility of defining a way to eliminate cut in a correct F-net that is guaranteed to yield another correct F-net. Naturally, we want the cut elimination to be associative, and to have a unit. This requires defining a new composition in the category of F-prenets.

We remind the reader here that the focus of our interest here are the cut elimination steps performed on single cuts arising from pairs of cut-free F-prenets. Equivalently, one may assume that in what follows we restrict the logic so that once a *Cut* rule is applied, the only rule that can be used afterwords is *Cut*. This is the standard use of *Cut* in categorical interpretations.

Definition 4.3.19. A *bonus* B is a function which is defined on all pairs $(P \triangleright \Gamma, C)$ where $P \triangleright \Gamma$ is a sound F-prenet and C a component of P, and whose value $\mathsf{B}(P \triangleright \Gamma, C)$ is a natural number.

When the context is clear we allow ourselves to drop one of the components.

Let $P \triangleright \Gamma$ be a sound F-prenet and $((C_1, G_1)...(C_k, G_k))$ be its linking. We define $\lceil P \rceil^{\mathsf{B}}$ to be the linking

$$\left((C_1, G_1 + \mathsf{B}(P \triangleright \Gamma, C_1)), (C_2, G_2 + \mathsf{B}(P \triangleright \Gamma, C_2)), ..., (C_k, G_k + \mathsf{B}(P \triangleright \Gamma, C_k))\right)$$

and $\lceil P \triangleright \Gamma \rceil^{\mathsf{B}}$ to be the corresponding sound F-prenet with the linking $\lceil P \rceil^{\mathsf{B}}$.

A bonus is said to be *fair* if it is constant on components for a given sound F-prenet, in which case it depends only on the F-prenet, and we write $\mathsf{B}(P \triangleright \Gamma)$.

A bonus B is said to be *winning* if $\lceil P \triangleright \Gamma \rceil^{\mathsf{B}}$ is always an FL-correct net.

According to Lemma 4.3.12, it is safe to assume that given a sound F-prenet a winning bonus is strictly positive on the components of the given F-prenet. Such a winning bonus B with values ≥ 1 on components of every F-prenet is said to be *strictly winning*.

In addition to the definition of $\lceil P \triangleright \Gamma \rceil^{\mathsf{B}}$ we define $\lfloor P \triangleright \Gamma \rfloor_{\mathsf{B}}$ be the F-prenet obtained by subtracting $\mathsf{B}(P \triangleright \Gamma)$ many loops from every component of $P \triangleright \Gamma$, if possible, $P \triangleright \Gamma$ otherwise.

Definition 4.3.20. For F-prenets $P \triangleright \Gamma, \mathrm{A}$ and $Q \triangleright \overline{\mathrm{A}}, \Sigma$ and a winning bonus B, we define

$$(P \triangleright \Gamma, \mathrm{A}) \diamond (Q \triangleright \overline{\mathrm{A}}, \Sigma)$$

to be

$$(P \triangleright \Gamma, \mathrm{A}) \diamond (Q \triangleright \overline{\mathrm{A}}, \Sigma) = \begin{cases} (P \triangleright \Gamma, \mathrm{A}) \circ (Q \triangleright \overline{\mathrm{A}}, \Sigma), & \text{if either} \\ & (P \triangleright \Gamma, \mathrm{A}) \text{ or } (Q \triangleright \overline{\mathrm{A}}, \Sigma) \\ & \text{is an identity F-prenet} \\ \left\lceil \lfloor P \triangleright \Gamma, \mathrm{A} \rfloor_{\mathsf{B}} \circ \lfloor Q \triangleright \overline{\mathrm{A}}, \Sigma \rfloor_{\mathsf{B}} \right\rceil^{\mathsf{B}}, & \text{otherwise} \end{cases}$$

where the $\circ$ is the standard Frobenius composition.

So the diamond "eats" the last formula of the left sequent and the first one of the right sequent, leaving a concatenation of what is left. this notation is not completely satisfactory in general, but good enough for the purposes of this work.

The following is obvious.

Proposition 4.3.21. *Given a winning bonus* B*, its* $\diamond$ *cut-elimination applied to two* FL*-correct F-nets yields a correct net.*

Theorem 4.3.22. *Given a strictly winning bonus* B*, the* $\diamond$ *cut-elimination on* FL*-correct F-nets is associative. Also, the* $\diamond$ *cut-elimination on an F-prenet and an identity F-prenet yields the former one.*

Démonstration. From the definition of the $\diamond$ cut-elimination, it is clear that the identity map in **F-prenet** is the identity with respect to the $\diamond$ cut-elimination as well. What is left to show is the associativity of the $\diamond$ cut-elimination.

So, let $P_3 \triangleright \Delta, \mathrm{B}$, $P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}$ and $P_1 \triangleright \overline{\mathrm{C}}, \Gamma$ be the F-prenets on which two $\diamond$ cut-eliminations are performed in different order. The first case to distinguish is the one where at least one of the $\diamond$ cut-eliminations in $(P_3 \triangleright \Delta, \mathrm{B}) \diamond \big((P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}) \diamond (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)\big)$ or $\diamond \big((P_3 \triangleright \Delta, \mathrm{B}) \diamond (P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C})\big) \diamond (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)$ is the upper case in the definition, i.e. one of the maps is the identity (which is the same for $\circ$ and $\diamond$). From the definition, this results in the Frobenius cut-elimination whose result is equal to the one of the maps composed. Whichever of the $(P_3 \triangleright \Delta, \mathrm{B}), (P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}), (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)$ is the identity, both $(P_3 \triangleright \Delta, \mathrm{B}) \diamond \big((P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}) \diamond (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)\big)$ and $\diamond \big((P_3 \triangleright \Delta, \mathrm{B}) \diamond (P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C})\big) \diamond (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)$ reduce to the cut-elimination of the remaining two maps, showing the equality of the two F-prenets obtained by the cut-eliminations.

Assume now that neither of the $(P_3 \triangleright \Delta, \mathrm{B}), (P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}), (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)$ is the identity. Then, the $\diamond$ cut-eliminations are always those that include $\lfloor - \rfloor_{\mathsf{B}}$ and $\lceil - \rceil^{\mathsf{B}}$. Indeed, it can never happen that a $\diamond$ cut-eliminations produces an identity F-prenet unless both F-prenets that are cut are identity F-prenets. If neither of the composing F-prenets is an identity F-prenet, by the definition of $\lceil - \rceil^{\mathsf{B}}$, every component in the resulting F-prenet has genera ≥ 1. This is the point where strictness of the winning bonus B comes into play. So, if neither of the $(P_3 \triangleright \Delta, \mathrm{B}), (P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}), (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)$ is the identity, the cut-elimination of performed in either order $(P_3 \triangleright \Delta, \mathrm{B}) \diamond \big((P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C}) \diamond (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)\big)$ or $\big((P_3 \triangleright \Delta, \mathrm{B}) \diamond (P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C})\big) \diamond (P_1 \triangleright \overline{\mathrm{C}}, \Gamma)$ can be seen as

$$\lceil \lfloor P_3 \triangleright \Delta, \mathrm{B} \rfloor_{\mathsf{B}} \circ \lfloor P_2 \triangleright \overline{\mathrm{B}}, \Sigma, \mathrm{C} \rfloor_{\mathsf{B}} \circ \lfloor P_1 \triangleright \overline{\mathrm{C}}, \Gamma \rfloor_{\mathsf{B}} \rceil^{\mathsf{B}}.$$

□

Naturally, we are interested in concrete examples of winning bonuses that define $\diamond$ cut eliminations.

Definition 4.3.23. For a sequent (or a formula) Γ, let $|_\Gamma \wedge |$ denote the number of conjunctions in Γ and let $|\Gamma|$ denote the number of literals in Γ. For an F-prenet $P \triangleright \Gamma$ we define a fair bonus Bon to be the value

$$\mathsf{Bon}(P \triangleright \Gamma) = |\Gamma| \cdot \left(2^{|_\Gamma \wedge|+1} - 1\right).$$

Let $\lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$ be the F-prenet obtained by adding $\mathsf{Bon}(P \triangleright \Gamma)$ many loops to every connected component of $P \triangleright \Gamma$, and let $\lfloor P \triangleright \Gamma \rfloor_{\mathsf{Bon}}$ be the F-prenet obtained by subtracting $\mathsf{Bon}(P \triangleright \Gamma)$ many loops from every component of $P \triangleright \Gamma$, if possible, $P \triangleright \Gamma$ otherwise.

Where it does not cause confusion and where the sequent of an F-prenet can be deduced from the context, we introduce the notion $\lfloor P \rfloor^{\mathsf{Bon}}$ and $\lfloor P \rfloor_{\mathsf{Bon}}$ on linkings with the same meaning as above.

Theorem 4.3.24. *The bonus* Bon *is strictly winning.*

Démonstration. What needs to be shown is that for any sound net $P \triangleright \Gamma$, $\lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$ is FL-correct. From the definition it is clear that Bon is fair and strictly positive.

Obviously, the most difficult case is the one where $P \triangleright \Gamma$ has genus 0 everywhere, so we will assume that each component in $\lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$ is of the genus $\mathsf{Bon}(P \triangleright \Gamma)$.

We show the lemma by induction on number of conjunctions in Γ.

Base of induction : For the case of no conjunctions, $P \triangleright \Gamma$ is by itself its own single conjunctive switching. As a sound net, it has contain a component with literals of opposite polarities. Thus, the F-prenet is assigned to a proof consisting of an axiom introducing the two literals of opposite polarities witnessing the soundness, followed by some weakenings and contractions that introduce other components and (possibly) connect the axiom component with others, and finally some disjunction introductions. Adding a non-zero amount of loops to an FL-correct F-net does not change correctness, so $\lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$ is FL-correct as well.

The Induction Step : Assume now that the claim of the theorem holds for all F-prenets with less than n conjunctions, let Γ be with n conjunctions, and let $P \triangleright \Sigma, \mathrm{A} \wedge \mathrm{B} \multimap P \triangleright \Gamma$.

Both F-prenets $P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}$ and $P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}$ are sound, any switching of either of the two can be extended to a switching of $P \triangleright \Sigma, \mathrm{A} \wedge \mathrm{B}$, which will provide a witness for the switching. The induction hypothesis on number of conjunctions in Γ yields FL-correctness of $\lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A} \rceil^{\mathsf{Bon}}$ and $\lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B} \rceil^{\mathsf{Bon}}$.

Let $Q \triangleright \Gamma$ be the F-prenet for which

$$Q \triangleright \Gamma \multimap \left(\lceil P|_{\Sigma,\mathrm{A}} \rceil^{\mathsf{Bon}} \uplus \lceil P|_{\Sigma,\mathrm{B}} \rceil^{\mathsf{Bon}} \right) (\Sigma \curlyvee \Sigma) \triangleright \Sigma, \mathrm{A} \wedge \mathrm{B}.$$

The F-prenet $Q \triangleright \Gamma$ is obviously FL-correct – an assigned proof is obtained by performing a conjunction followed by contractions and disjunction introductions on two proofs assigned to FL-correct F-nets $\lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A} \rceil^{\mathsf{Bon}}$ and $\lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B} \rceil^{\mathsf{Bon}}$.

If we manage to show $Q \preceq \lceil P \rceil^{\mathsf{Bon}}$, then the induction step will follow from the fact that $\preceq$ preserves FL-correctness.

Notice here that the set of literals in Q is obviously P. Also, notice that for every component in $\lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A} \rceil^{\mathsf{Bon}}$ there is a component of $\lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B} \rceil^{\mathsf{Bon}}$ such that the two are (disregarding the genus for the moment) obtained by restrictions of the same component in P. Consequently, components of the two F-nets agree on Σ. We show by case analysis that every component C of Q is a subcomponent of a component in P, i.e. the partition of literals in Q is finer than in P (which is the same as in $\lceil P \rceil^{\mathsf{Bon}}$).

So, let

1. $(C, G) \in Q$ be such that C contains at least one literal of Σ. In this case (C, G) is obtained by performing $(\Sigma \curlyvee \Sigma)$ on disjoint union of two components, $(C_1, G_1) \in \lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A} \rceil^{\mathsf{Bon}}$ and $(C_2, G_2) \in \lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B} \rceil^{\mathsf{Bon}}$. As noticed, the two components must be obtained as restrictions of a same component in P and they agree on literals in Σ. So, we conclude that C has exactly the same set of literals as the component in P whose restrictions are C_1 and C_2 ;

2. $(C, G) \in Q$ be such that C contains only literals of A and B. First of all, as $\lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}\rceil^{\mathsf{Bon}}$ contains no literals of B and $\lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}\rceil^{\mathsf{Bon}}$ contains no literals of A, C can contain literals from either A, or literals form B, but not form both in the same time. Moreover, C is equal to a component of one of the the two restriction linkings $\lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}\rceil^{\mathsf{Bon}}$ or $\lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}\rceil^{\mathsf{Bon}}$, and is thus a subset of a component in P.

We have shown that partition of literals in Q is finer than the one of P. What is left to show is that the relation on genera is the one required by the $\preceq$ order. To that purpose, we revisit the two cases above :

1. $(C, G) \in Q$ is such that C contains at least one literal of Σ. In this case we have shown that C is also a class in P. Let us compute G. First, we know genera of $\lceil P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}\rceil^{\mathsf{Bon}}$ and $\lceil P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}\rceil^{\mathsf{Bon}}$ (keep in mind, we have assumed genus in P to be 0 everywhere) :

$$\mathsf{Bon}(P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}) = (|\Gamma| - |\mathrm{A}|) \cdot \left(2^{|\Gamma\wedge|-|\mathrm{A}\wedge|} - 1\right)$$

and

$$\mathsf{Bon}(P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}) = (|\Gamma| - |\mathrm{B}|) \cdot \left(2^{|\Gamma\wedge|-|\mathrm{B}\wedge|} - 1\right).$$

Obviously, G is the the sum of the two values above, augmented by the number of loops created by composition. The latter is not greater than the number of the elements in C minus 1, so it can be bounded by $|\Gamma| - 1$. So, one has :

$$\begin{aligned}
&(|\Gamma| - |\mathrm{A}|) \cdot \left(2^{|\Gamma\wedge|-|\mathrm{A}\wedge|} - 1\right) \quad + \quad (|\Gamma| - |\mathrm{B}|) \cdot \left(2^{|\Gamma\wedge|-|\mathrm{B}\wedge|} - 1\right) \quad + \quad |\Gamma| - 1 \\
&\leq \quad (|\Gamma| - 1) \cdot \left(2^{|\Gamma\wedge|} - 1\right) \quad + \quad (|\Gamma| - 1) \cdot \left(2^{|\Gamma\wedge|} - 1\right) \quad + \quad |\Gamma| - 1 \\
&= \quad 2 \cdot |\Gamma| \cdot 2^{|\Gamma\wedge|} - 2 \cdot 2^{|\Gamma\wedge|} - |\Gamma| + 1, \\
&\leq \quad |\Gamma| \cdot \left(2^{|\Gamma\wedge|+1} - 1\right),
\end{aligned}$$

where the last inequality follows from

$$1 < 2 \cdot 2^{|\Gamma\wedge|}$$

for $|_\Gamma \wedge | \geq 1$.
Thus, the component with genus in $\lceil P\rceil^{\mathsf{Bon}}$ that corresponds to (C, G) has larger genus.

2. (C, G) is such that C contains only literals of A and B. As we have seen, this reduces to the case where C contains literals of only one of A and B, and these in turn give rise to two cases
   a) $C \subseteq \mathcal{L}(\mathrm{A})$ (symmetrically, $C \subseteq \mathcal{L}(\mathrm{B})$) and C is equal to a component in P. This case may arise when P contains a component from literals of A (resp. B) alone. In this case G is

$$\mathsf{Bon}(P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}) = (|\Gamma| - |\mathrm{B}|) \cdot \left(2^{|\Gamma\wedge|-|\mathrm{B}\wedge|} - 1\right).$$

which is obviously not larger than

$$|\Gamma| \cdot \left(2^{|\Gamma^\wedge|+1} - 1\right).$$

Again, the component with genus in $\lceil P \rceil^{\mathsf{Bon}}$ that corresponds to (C, G) has larger genus.

b) $C \subseteq \mathcal{L}(\mathrm{A})$ (symmetrically, $C \subseteq \mathcal{L}(\mathrm{B})$) and is a strict subset of a component in P. This case arises when P contains a component from literals of both A and B and those alone. Here, C is a restriction of that component to A (resp. B), and the component of P can be recovered by merging its two restrictions, the one A ($= C$) and the one on B. For the $\preceq$ order, this translates to the requirement on genera that

$$\mathsf{Bon}(P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}) + \mathsf{Bon}(P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}) \leq \mathsf{Bon}(P \triangleright \Gamma),$$

which follows from the case 1. above where we have proven a stronger claim :

$$\mathsf{Bon}(P|_{\Sigma,\mathrm{A}} \triangleright \Sigma, \mathrm{A}) + \mathsf{Bon}(P|_{\Sigma,\mathrm{B}} \triangleright \Sigma, \mathrm{B}) + |\Gamma| - 1 \leq \mathsf{Bon}(P \triangleright \Gamma),$$

This shows that $Q \triangleright \Gamma \preceq \lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$, which yields FL-correctness of $\lceil P \triangleright \Gamma \rceil^{\mathsf{Bon}}$ in the induction step for the number of conjunctions, which completes the proof of the theorem.

□

The previous theorem yields a concrete example of an associative cut-elimination $\diamond$ on the set of all F-prenets. There is also an identity with respect to the cut-elimination assigned to any F-prenet. So, if we consider only F-prenets with two-formula sequents, the cut-elimination entails a composition in the category of these prenets, the same way the $\smile$ cut-elimination gives rise to **F-prenet**. The category of two-formula F-prenets and the concrete $\diamond$ composition induced by Bon is denoted by CorrFrob. The category CorrFrob, according to the previous, enjoys a weak form of the full completeness property i.e. every map in CorrFrob is assigned to a proof.

There is, however, little we can say on the categorical structure of CorrFrob. We know that it is not *-autonomous, as e.g. a map $P \triangleright \overline{(\mathrm{A} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}}, \mathrm{C}$ with the same linking as $P \triangleright \overline{\mathrm{A} \vee \mathrm{C}}, \mathrm{B} \vee \mathrm{C}$ is not in general the result of the cut-elimination

$$\left(P \uplus Q \triangleright \overline{(\mathrm{A} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}}, (\mathrm{B} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}\right) \diamond \left(R \uplus S \triangleright \overline{(\mathrm{B} \vee \mathrm{C}) \wedge \overline{\mathrm{B}}}, \mathrm{C}\right),$$

where it is clear what the Q, R, S are, they are linkings of some identity F-penets. The composition here in general requires adding bonuses on components with literals from A and C. Moreover, CorrFrob is not even monoidal. In a monoidal category the functoriality of the tensor (which is here the disjoint union) translates to the 'locality' of the composition, i.e. a composition should alter only those components that are being connected by it. Here, a component that does not play a role in the concatenation of maps may have its genus changed, because of fairness of Bon. A possible clue suggesting that the same situation has to happen for any definition of a composition via a winning bonus is provided in the example of Figure 4.7.

So far, our algebraic analysis of the semantics has always dealt with categories as suitable structures reflecting the symmetries of classical logic. At this point, however, a more general algebraic setting proves to be instrumental.

Given a proof $f : \Gamma \vdash \Sigma$ in a two-sided derivation system, the categorical approach suggests that f should be a map in a category whose domain object is Γ and codomain Σ.

In the cases where Γ is a sequence of formulas, and Σ is a single formula, as in systems for intuitionistic logic, one may generalize the categorical approach by reverting to *multicategories*, where a map is a set of morphisms sharing a codomain (see e.g. [Lei04]). An alternative generalization is the concept of polycategories for truly two-sided derivation systems. There, one generalize an object to a list of objects, and a map becomes a map between the lists. The concept is originally due to Szabo (see [Sza75]), and the context of classical proof theory appears in the work of Robinson, Bellin, Hyland, and Urban [BHRC06].

Finally, in the case of one-sided sequent calculi, which is what we have here, the proper algebra for structural contexts is the notion of Lamarche *structads*, as described in [Lam05, Lam02].

Going back to the structure with bonuses, in all the models we have seen until now, the categorical structure in question was is sufficient to derive the generalizing structad, and vice versa. In the case here, due to the notion of the bonus being 'global', more structure is contained in the structad than in just the category.

Making use of this generalization is an important major task yet to be undertaken.

4.3.2 Sound F-prenets, revisited

There exist an alternative way to think of sound F-prenets. Namely, one can think of an F-prenet as corresponding to a *class* of deductive proofs whose F-nets are ($\preceq$- or $\leq_{\mathcal{G}}$-)larger than the given F-prenet. This makes sense, since we can effectively determine the set of these FL-correct F-nets for every F-prenet. The fact that the order $\preceq$ is an enrichment, cut elimination / composition is bound to respect the assignment. This defines, somewhat different notion of nets, that incorporates in itself certain degree of nondeterminism. The category of sound F-prenets can be seen as having an another weak form of the full completeness in the previous sense. Nondeterminism in itself is a phenomenon that is not unknown to the semantics of classical proofs.

Sound F-prenets and proof compression

We have seen that to go form a sound F-prenet to an FL-correct F-net it suffice to add resources thereby increasing the characteristic of an F-prentet. Here we address somewhat inverse problem – as composition of two FL-correct F-nets results in a sound F-prenet whose characteristic is strictly smaller than the sum of the characteristics of the two, is

it the case that *every* sound F-prenet is obtained by a cut-elimination-as-composition of FL-correct F-nets.

Definition 4.3.25. For an F-prenet $P \triangleright \Gamma$, we define its FL-*correct decomposition* to be a tree whose nodes are F-prenets, where every edge between two nodes $Q \triangleright \Sigma, R \triangleright \Delta$ is decorated by a pair of formulas, one in Σ, one in Δ. These formulas decorating an edge are negations of each other appear in no other node of the tree. The tree is also required to be such that $P \triangleright \Gamma$ is obtained by cutting every pair of edge formulas in the decomposition tree and eliminating all these cuts.

An F-prenet which has an FL-correct decomposition is said to be FL-*correctly decomposable.*

Proposition 4.3.26. *If an F-prenet has an* FL*-correct decomposition, then is has one of the form*

$$(P_n \triangleright A_n) \circ \left(\cdots \circ \left((P_2 \triangleright A_2) \circ \left((P_1 \triangleright A_1) \circ (P_0 \triangleright \overline{A_1}, \overline{A_2}, \ldots, \overline{A_n}, \Gamma)\right)\right)\right).$$

Démonstration. By induction on the size of a decomposition tree of an F-prenet. In the case of a single-node tree (i.e. an FL-correct F-net) there is nothing to show. Given a decomposition of size $n+1$, consider the tree obtained by removing a single leaf $S \triangleright \Sigma_1, \mathrm{A}$. By induction hypothesis the resulting decomposition tree shows FL-correct decomposability of an F-prenet $Q \triangleright \overline{A}, \Sigma_2$, whose FL-correct decomposition is also

$$(R_{n-1} \triangleright A_{n-1}) \circ \left(\cdots \circ \left((R_1 \triangleright A_1) \circ (R_0 \triangleright \overline{A_1}, \ldots, \overline{A_{n-1}}, \overline{A}, \Sigma_2)\right)\right).$$

The decomposition for $(S \triangleright \Sigma_1, A) \circ (Q \triangleright \overline{A}, \Sigma_2)$ is

$$(S \triangleright \bigvee \Sigma_1 \vee A) \circ$$
$$\circ \left((R_{n-1} \triangleright A_{n-1}) \circ \left(\cdots \circ \left((R_1 \triangleright A_1) \circ (R \triangleright \overline{A_1}, \ldots, \overline{A_{n-1}}, \overline{\bigvee \Sigma_1 \vee A}, \Sigma_2, \Sigma_1)\right)\right)\right)$$

where R is obtained from R_0 by adding two literal components with genus 0 of the matching literals of opposite polarity from $\Sigma_1, \overline{\Sigma}_1$.

Both $S \triangleright \bigvee \Sigma_1 \vee A$ and $R \triangleright \overline{A_1}, \ldots, \overline{A_{n-1}}, \overline{\bigvee \Sigma_1 \vee A}, \Sigma_2, \Sigma_1$ are trivially FL-correct as $S \triangleright \Sigma_1, A$ and $R_0 \triangleright \overline{A_1}, \ldots, \overline{A_{n-1}}, \overline{A}, \Sigma_2$ are. □

Lemma 4.3.27. *If $P \triangleright \Gamma \preceq Q \triangleright \Gamma$, and $P \triangleright \Gamma$ is* FL*-correct decomposable, then so is $Q \triangleright \Gamma$, i.e. the set of* FL*-decomposable F-prenets is up-closed with respect to the $\preceq$ order.*

Démonstration. It suffice to show the claim for the case where $Q \triangleright \Sigma$ is obtained by merging together two components of $P \triangleright \Gamma$ (and adding genera of the merged components), and the case where $P \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$. The general case reduces to these two by observing that the definition of $\preceq$ assumes existence of a sequence

$$P \triangleright \Gamma = P_0 \triangleright \Gamma, P_1 \triangleright \Gamma, P_2 \triangleright \Gamma, \ldots, P_k \triangleright \Gamma, Q \triangleright \Gamma$$

where each $P_i \triangleright \Gamma$, $i \geq 1$ is obtained by merging two components of $P_{i-1} \triangleright \Gamma$, and $P_k \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$.

For the case where $Q \triangleright \Gamma$ is obtained by merging two components of $P \triangleright \Gamma$, we know from Lemma4.3.13 that there is a derivation in FL corresponding to an FL-correct F-net

$$R \triangleright \overline{\Gamma}, \Gamma$$

which has the property that composed with an F-prenet $P \triangleright \Gamma$ it results in an F-prenet whose components are obtained by merging two components of $P \triangleright \Gamma$. This way any FL-correct decomposition can be, by the previous proposition, extended by $\cdots \circ (R \triangleright \overline{\Gamma}, \Gamma)$ to an FL-correct decomposition of $Q \triangleright \Gamma$.

If, on the other hand $P \triangleright \Gamma \leq_{\mathcal{G}} Q \triangleright \Gamma$, for every component with genus $(C, G) \in P$ for which there is $(C, G+k)$ in Q, $k \geq 1$, one can trace a component (C', G') of an FL-correct F-net $R \triangleright \Sigma$ in the FL-correct decomposition of $P \triangleright \Gamma$ which when composed with the other components in the decomposition results in (C, G). The F-net $R \triangleright \Sigma$ can be, by Lemma 4.3.12 replaced in the decomposition by an FL-correct F-net with the component $(C', G' + k)$, and agreeing with $R \triangleright \Sigma$ on all other components. By virtue of Frobenius composition, the added genus will propagate itself to the component $C \in P$, and the decomposition will result in an F-prenet $P' \triangleright \Gamma$ whose only difference to $P \triangleright \Gamma$ is the component $(C, G + k)$ instead of (C, G).

The process can be repeated for all components in P with a smaller genus to the respective in Q until a decomposition of $Q \triangleright \Gamma$ is reached. □

The following statement is given as a conjecture in this version of the text. We have strong evidence supporting the claim, but our arguments at the moment contain quite a complex, lengthy and thus not easily verifiable case-distinction.

Conjecture 4.3.28. Every sound F-net is FL-correct decomposable.

The previous conjecture suggests that the category **sF-prenet** of sound F-nets can be viewed as precisely the category of normal forms for FL-correct F-nets. Thus, we get yet another form of a weak completeness property – every map in **sF-prenet** is a *composition* of maps coming from proofs.

4.4 Complexity Issues

We end this chapter by a few considerations on complexity issues that arise from the accounts we have analyzed so far. The main purpose of this section is not to undertake a fully self-contained detailed investigation, as the complexity issues are not the main focus of the thesis. Rather, the purpose is to sketch a connection to the previous results and thereby reveal some of the deeper reasons behind the claims we make.

4.4.1 Complexity and Sound F-prenets

In the previous chapter we have defined the notion of a sound F-prenet and in this chapter we have also shown that a (FL-)correct F-net is a sound F-prenet. Further, we have shown that for any sound F-prenet there is a $\leq_{\mathcal{G}}$-larger FL-correct F-net. As FL is complete for classical propositional logic, we conclude that one can devise a validity checking algorithm for propositional classical logic by solving the problem :

Problem 1. *Given a sequent Γ, decide if there is a linking P such that $P \triangleright \Gamma$ is a sound F-prenet.*

When solving the previous problem, one can restrict the attention to the linkings where genus is 0 everywhere, as genus does not affect soundness. Moreover, we have shown that it suffice to consider the $\preceq$-largest such linkings, i.e. solving the above problem can be reduced to the problem of checking of the soundness of an F-prenet whose linking components are exactly classes the literals of the same sort and with genus 0.

On the other hand, given an F-prenet $P \triangleright \Gamma$, the linking P can contain several components with literals of the same sort. In such case, one can transform the F-prenet by replacing the labeling of literals while keeping the linking fixed. This is done in such a way that the polarity of the literals remains the same, while no two different components of P contain elements with the same sort. This way one relabels the literals and goes from $P \triangleright \Gamma$ to $P \triangleright \Gamma'$. Clearly, the transformed F-prenet is sound if and only if the original one is, while the last one is sound if and only if Γ' is valid.

From the complexity point of view, we have just reduced the prototype coNP complexity problem of propositional validity (cf. [CR79]) to the problem of checking the soundness of a linking. The reduction is polynomial as the size of an F-prenet is polynomial in the number of literals. Also, we have shown that the problem lies in coNP, thereby showing

Proposition 4.4.1. *Deciding soundness of an F-prenet is a* coNP *complete problem.*

4.4.2 Complexity and (FL-)Correct F-nets

We have seen that a FL-correct F-net is obtained by adding loops to a sound F-prenet with genus 0 everywhere. The natural question to ask is how many loops does one need to add to get a correct F-net ? Or, in terms of the concept of resources we have developed

Problem 2. *Given a sound F-prenet, what is the minimal amount of resources that need to be added to get a correct F-net ?*

As we have demonstrated, this question, as the notion of a correct F-net itself, is calculus-dependent.

Assume now that the minimal amount of loops that has to be added can be bounded by a polynomial function in the number of literals of the sequent of an F-prenet. Without

going into the formal definition of the complexity class NP via non-deterministic Turing machines, we remind the reader of one of the characterizations of the class. Namely, the complexity class NP characterizes precisely the problems whose solutions can be obtained by a non-deterministic listing of potential candidates for a solution, each of which can be checked in polynomial time. Now, with the assumption of the polynomial size of the added loops, one can reduce the validity problem of a sequent to the problem of finding a correctness diagram for a given F-prenet by saying that a sequent is valid if and only if there is a correctness tree for it whose size is bounded by a polynomial function of the size of the sequent. Given a candidate for a correctness tree, establishing that it really does obey the definition can obviously be done in polynomial time *in the size of the tree*, which is by assumption a polynomial function *in the size of the sequent*. Thus, we have shown

Proposition 4.4.2. *If the minimal number of resources needed to add to a sound F-prenet to obtain a correct one can be bounded by a polynomial in the number of literals then* coNP = NP*.*

By the same argumentation, as every correctness tree is of linear size to the size of an F-prenet, understood as size of the sequent plus the characteristic, checking of the correctness for a given F-prenet is in the class NP. This perhaps justifies the way we have defined the correctness procedure for F-nets. Namely, a correctness criterion for proof nets is usually formulated by an universally quantified sentence. A prominent example is the Danos-Reigner criterion for the nets for the (unit-free) multiplicative linear logic, cf. [DR89]. Another example is the Lamatche-Straßburger criterion for $\mathbb{B}$-nets appearing in [LS05b], which also features in this work as the criterion for what we call sound F-prenets here. In the first case deciding correctness is polynomial in the size of a proof net for the logic, as shown in [Gue99], which is also polynomial in the size of the sequent, since linkings of these nets are pairings of the literals. In the second case a polynomial procedure would imply P=coNP. The criterion for these nets begins with :

'For every conjunctive switching of the F-prenet...'

It is not surprising that the sentence defining a coNP-complete problem is a universally quantified one. Indeed, propositional validity is usually formulated by a similar sentence stating

'For every evaluation of the variables ...'.

Our correctness procedure for F-nets is an existentially quantified sentence *'There exists a correctness diagram...'*, again unsurprisingly for a problem in NP. Complete problems for the class are frequently formulated by these sentences –

'There exists an evaluation of variables/ a 3-coloring/ a path in a graph...'
for the satisfiability, 3-coloring, the traveling salesman problem, respectively. We acknowledge this fact by speaking of a correctness *procedure* rather than a *criterion* . In principle, whenever the criterion is such that a proof can be constructed from it in polynomial time, we are back to the situation where the polynomial size of proofs implies coNP=NP.

The likely culprit for the difficulties in trying to formulate a correctness criterion using an universal sentence is the intrinsic non-locality of the semantics / F-prenets. Much like in one of the standard NP-complete problems, a local modification in configuration of a problem instance dramatically affects the global solution. In the traveling salesmen problem an additional edge between two neighboring nodes affects the global paths in the graph. With F-prenets, local transformations that are parts of proof systems, a $\smile$ or $\curlyvee$ on two atoms which merge components or increase their genera, are propagated globally – they alter a component for all of the literals that belong to it.

4.4.3 Complexity and Cut Elimination

We have argued that a linking of an F-prenet captures the notion of resources used in a proof. The notion is very close to the notion of proof complexity the way it is usually defined in proof theory. A definition of the complexity of a proof commonly reaches to a some sort of syntactic measure of a proof, either the number of derivations, symbols used in a proof or similar. In our case, the linking of an F-prenet partially captures this information by, in the case of FL, providing an upper-bound on the syntactic size of the proofs read-out from correctness diagrams of a F-net, as it is a function of the characteristic of the F-net and the sequent in question. In the case of CL, the correctness diagram-assigned proofs are of the size which is precisely a function of the two.

By normalizing FL-correct F-nets via cut-elimination-as-Frobenius-composition, as we have shown, the characteristic of a F-net is lowered by the number of literals in the cut formula. This means that the normalized F-net has fewer resources available to build an FL proof, if such proof exist. This is to the contrary from the experience we have with the cut elimination in a proof system, which, in general, results in the exponential blowup in the proof size. From the complexity point of view, this reflects the difference we make between the normalization of F-prenets and the effective cut elimination at the level of proofs.

5

Cmp Revisited

Contents

5.1 Frobenius algebras in Cmp

In this chapter we go back to the introductory concrete interpretation based on $\mathbb{Z}$ and the category **Cmp** as the starting point in our investigation of the algebraic structure underlying the concrete interpretations.

We begin by analyzing what it means to have a structure of a Frobenius algebra in a compact-closed category, and we prove the following proposition appearing in [Dij89, Koc04]. There, the authors have shown the claim in the category of vector spaces over a field, but what matters is the compact-closed structure of the category, which is what we do here.

Proposition 5.1.1. *Let* $(\mathbf{C}, \otimes, \mathbf{1})$ *be a compact-closed category and* A *object of* $\mathbf{C}$*. The following define the same structure in* $\mathbf{C}$ *:*

1. *a Frobenius algebra* $(A, \Delta, \amalg, \nabla, \amalg)$ *;*
2. *a structure* $(A, \nabla, \amalg, \Pi)$*, where* $(A, \nabla, \amalg)$ *is a commutative monoid and* $\Pi : A \to \mathbf{1}$ *is such that the map*

$$\Phi : \ A \longrightarrow A^{\perp}$$

obtained by transposing

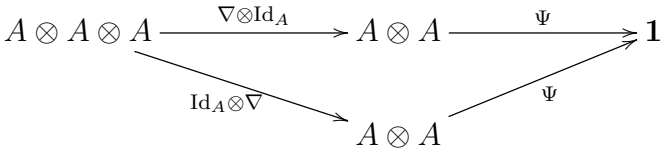
$$A \otimes A \xrightarrow{\ \nabla\ } A \xrightarrow{\ \Pi\ } \mathbf{1}$$

is an isomorphism[16] *;*

3. *a structure* $(A, \nabla, \amalg, \Psi)$*, where* $(A, \nabla, \amalg)$ *is a commutative monoid and,*

$$\Psi : \ A \otimes A \longrightarrow \mathbf{1}$$

obeys



and (either of) its transpose $\phi : A \to A^\perp$ *is an isomorphism.*

Démonstration.

1. $\Rightarrow$ 2. In a Frobenius algebra, the transpose of $\amalg \circ \nabla$

$$A \xrightarrow{\mathrm{Id} \otimes \eta} A \otimes A \otimes A^\perp \xrightarrow{(\amalg \circ \nabla) \otimes \mathrm{Id}} A^\perp.$$

has the inverse

$$A^\perp \xrightarrow{\mathrm{Id} \otimes (\Delta \circ \amalg)} A^\perp \otimes A \otimes A \xrightarrow{\varepsilon \otimes \mathrm{Id}} A$$

which follows from the coherence in **FrFrob** and the definition of a dual object ;

2. $\Rightarrow$ 3. This is obvious, by setting $\amalg \circ \nabla$ for Ψ, which is possible since the diagram in 3. becomes the associativity diagram for the monoid structure on A ;

3. $\Rightarrow$ 1. First, assume ϕ is the transpose (out of two) for which

$$\Psi = \mathrm{Id}^A_\perp \circ (\phi \otimes \mathrm{Id}_A),$$

and for such ϕ let

$$\Omega : \ \mathbf{1} \longrightarrow A \otimes A$$

be the transpose

$$\Omega = (\mathrm{Id}_A \otimes \phi^{-1}) \circ \mathrm{Id}^\perp_A$$

of ϕ^{-1}, where $\mathrm{Id}^A_\perp : \ A^\perp \otimes A \to \mathbf{1}$ and $\mathrm{Id}^\perp_A : \ \mathbf{1} \to A \otimes A^\perp$ are transposes of the identity of A. Obviously :

$$(\mathrm{Id}_A \otimes \Psi) \circ (\Omega \otimes \mathrm{Id}_A) = (\Psi \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \Omega) = \mathrm{Id}_A.$$

Had we taken the other transpose of Ψ for ϕ, we would have used defined Ω by the other transpose of ϕ^{-1}. Because of the symmetry of the problem, the further conclusions would translate to that case, i.e. if we manage to get a cocommutative comonoid with one choice of ϕ, Ω, the resulting structure will be the same as the one for the other choice of the two by cocommutativity.

16. there are two ways to transpose the map, but the both transposes are equal, by commutativity of ∇

To reconstruct the cocommutative comonoid we first notice :

$$\begin{aligned}(\mathrm{Id}_A \otimes \nabla) \circ (\Omega \otimes \mathrm{Id}_A)& \\ &= (\mathrm{Id}_A \otimes \Psi \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A \otimes \mathrm{Id}_A) \circ (\Omega \otimes \mathrm{Id}_A \otimes \Omega) \\ &= (\mathrm{Id}_A \otimes \Psi \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\Omega \otimes \mathrm{Id}_A \otimes \Omega) \\ &= (\nabla \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \Omega)\end{aligned}$$

which allows us to define
Δ :

$$A \xrightarrow{\Omega \otimes \mathrm{Id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{Id}_A \otimes \nabla} A \otimes A.$$

To define Π, notice

$$\begin{aligned}\Psi \circ (\amalg \otimes \mathrm{Id}_A)& \\ &= \Psi \circ (\nabla \otimes \mathrm{Id}_A) \circ (\amalg \otimes \mathrm{Id}_A \otimes \amalg) \\ &= \Psi \circ (\mathrm{Id}_A \otimes \nabla) \circ (\amalg \otimes \mathrm{Id}_A \otimes \amalg) \\ &= \Psi \circ (\mathrm{Id}_A \otimes \amalg),\end{aligned}$$

so we have
Π :

$$A \xrightarrow{\amalg \otimes \mathrm{Id}_A} A \otimes A \xrightarrow{\Psi} \mathbf{1}.$$

We need to verify that the defined maps form a cocommutative comonoid. Coassociativity if the Δ is result of associativity of ∇ :

$$\begin{aligned}(\Delta \otimes \mathrm{Id}_A) \circ \Delta = (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\Omega \otimes \mathrm{Id}_A \otimes \mathrm{Id}_A) \circ (\nabla \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \Omega)& \\ &= (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\Omega \mathrm{Id}_A \otimes \Omega) \\ &= (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\Omega \mathrm{Id}_A \otimes \Omega) \\ &= (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \mathrm{Id}_A \otimes \Omega) \circ (\nabla \otimes \mathrm{Id}_A) \circ (\Omega \otimes \mathrm{Id}_A).\end{aligned}$$

Similarly, the counit comonoid law is a result of the unit law for the monoid structure $(A, \nabla, \amalg)$:

$$\begin{aligned}\Psi \circ (\nabla \otimes \mathrm{Id}_A) \circ (\Omega \otimes \mathrm{Id}_A \otimes \amalg)& \\ &= \Psi \circ (\mathrm{Id}_A \otimes \nabla) \circ (\Omega \otimes \mathrm{Id}_A \otimes \amalg) \\ &= (\mathrm{Id}_A \otimes \Psi) \circ (\Omega \otimes \mathrm{Id}_A),\end{aligned}$$

the other counit law follows analogously.

Cocommutativity of the comonoid follows from the commutativity of ∇ – the map Δ transposed gives

$$A^\perp \otimes A \xrightarrow{\phi^{-1}\otimes \mathrm{Id}_A} A \otimes A \xrightarrow{\quad\nabla\quad} A,$$

equal to

$$\nabla \circ \sigma \circ (\mathrm{Id}_A \otimes \phi^{-1}) \circ \sigma$$

$$= \nabla \circ (\mathrm{Id}_A \otimes \phi^{-1}) \circ \sigma$$

which is $\sigma \circ \Delta$ transposed.
Finally, we are able to show the Frobenius equation in the newly defined bimonoid :

$$\begin{aligned}
&(\mathrm{Id}_A \otimes \nabla) \circ (\Delta \otimes \mathrm{Id}_A) \\
&\quad = (\mathrm{Id}_A \otimes \nabla) \circ (\mathrm{Id}_A \otimes \nabla \otimes \mathrm{Id}_A) \circ (\phi^{-1} \otimes \phi^{-1} \otimes \mathrm{Id}_A \otimes \mathrm{Id}_A) \circ (\Omega \otimes \mathrm{Id}_A \otimes \mathrm{Id}_A) \\
&\quad = (\mathrm{Id}_A \otimes \nabla) \circ (\phi^{-1} \otimes \phi^{-1} \otimes \nabla) \circ (\Omega \otimes \mathrm{Id}_A \otimes \mathrm{Id}_A) \\
&\quad = (\mathrm{Id}_A \otimes \nabla) \circ (\phi^{-1} \otimes \phi^{-1} \otimes \mathrm{Id}_A)(\Omega \otimes \mathrm{Id}_A) \circ \lambda^{-1} \circ \nabla \\
&\quad = \Delta \circ \nabla.
\end{aligned}$$

The other Frobenius equation, $(\nabla \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \Delta) = \Delta \circ \nabla$ is established analogously.

□

Remark 5.1.2. The previous formulations of structures equivalent to a Frobenius algebra can be found in literature in a more general context of *Frobenius monads* in [Str04].

Most of our work so far has involved going from compact-closed categories to (free) Frobenius ones. The category **Cmp** was the first concrete compact closed category in which we have formulated interpretations of classical logic, so we are naturally interested in investigating Frobenius algebras in **Cmp**. As it turns, the general case proves to be too general for the purposes of this thesis, so we are looking at those Frobenius algebras that come from a bimonoid in Poset, as in the first chapter.

In Chapter 1 we have defined with a different notation

Definition 5.1.3. A *poset-bimonoid* is a sextuple $(A, \leq, \cdot, 1, \star, e)$, where $(A, \leq)$ is a poset and $(\cdot, 1)$, $(\star, e)$ two monotone commutative monoid structures for it.

A poset monoid induces a monoid in **Cmp**; for elements x, y, z of a poset A, one defines :

$$\begin{aligned}
x \overset{\Delta}{} (y, z) \quad &\text{iff} \quad x \leq y \star z; \qquad & x \overset{\amalg}{} * \quad &\text{iff} \quad x \leq e; \\
(y, z) \overset{\nabla}{} x \quad &\text{iff} \quad y \cdot z \leq x; \qquad & * \overset{\amalg}{} x \quad &\text{iff} \quad 1 \leq x.
\end{aligned} \tag{5.1}$$

The formulation in the point 2. of Proposition 5.1.1 offers an alternative definition of a Frobenius algebra which is particularly useful. Assume, according to (5.1), we have a monoid $(A, \cdot, 1)$ in Poset. Following 2. of the previous proposition, to get a structure of a Frobenius algebra on the poset, we need a constant

$$e \in A$$

for which

$$\{i \in A \mid i \leq e\}$$

defines a map

$$A \longrightarrow \mathbf{1}$$

which gives rise to a transpose of an iso in **Cmp**. In more detail, this means

$$\{z \leq e\} \circ \{x \cdot y \leq z\} \;=\; \{x \cdot y \leq e\}$$

is a transpose of an iso

$$\sigma :\; A \longrightarrow A^{op}$$

in **Cmp**. An isomorphism $A \to B$ in **Cmp**, as shown in [Lam07], is always obtained from an iso in POSET $\alpha : A \to B$ as is left-down-right-up closure, i.e.

$$\alpha \updownarrow := \{(i,j) \in A \times B \mid \exists k \in A :\; i \leq k;\; \alpha(k) \leq j\}.$$

Therefore, σ is a left-down-right-up closure of a contravariant bijection (also called σ) $A \cong A^{op}$.

Now, the transposition in the point 2. of the previous proposition, using the contravariance and bijectivity of σ, is rephrased as to say that :

$$\begin{array}{rcl} \{(x,y) \in A^2 \mid \exists z \in A :\; x \leq z;\; y \geq^{op} \sigma(z)\} & = & \\ & = & \{(x,y) \in A^2 \mid\; \exists z :\; \sigma(x) \geq \sigma(z);\; \sigma(z) \leq y\} \\ & = & \{(x,y) \in A^2 \mid\; y \leq \sigma(x)\} \end{array} \tag{5.2}$$

is the same as

$$\{(x,y) \in A^2 \mid x \cdot y \leq e\}. \tag{5.3}$$

Similarly, starting from the other transpose of the same map $\Pi \circ \nabla$ from the point 2., one gets :

$$\begin{array}{rcl} \{\exists z \in A :\; y \leq z;\; x \geq^{op} \sigma(z)\} & = & \\ & = & \{x \leq \sigma(y)\}. \end{array} \tag{5.4}$$

This transposition has to be the same map as (5.2) by commutativity of $(-) \cdot (-)$ (= commutativity of ∇). Therefore, in the poset $(A, \leq)$ the expressions (5.2),(5.3), (5.4) yield

$$x \leq \sigma(y) \quad \text{iff} \quad x \cdot y \leq e \quad \text{iff} \quad y \leq \sigma(x). \tag{5.5}$$

Notice that σ is defined by the choice of e, and vice-versa ; by the above equation, the contravariant involutive σ determines the choice for e.

Also, by putting $x = e,\ y = 1$, we get

$$e \leq \sigma(1) \quad \text{iff} \quad e \cdot 1 \leq e \quad \text{iff} \quad 1 \leq \sigma(e),$$

and by putting $x = \sigma(1),\ y = 1$

$$\sigma(1) \leq \sigma(1) \quad \text{iff} \quad \sigma(1) \cdot 1 \leq e \quad \text{iff} \quad 1 \leq \sigma(\sigma(1)).$$

Thus,

$$e = \sigma(1).$$

Proposition 5.1.4. *Let $(A, \leq, \cdot, 1)$ be a monotone monoid on a poset, and σ an order-antiautomorphism $\sigma : (A, \leq) \to (A, \leq^{op})$ defined by the choice of $e \in A$ for which* (5.5) *holds. Then A is a *-autonomous poset, and vice versa, every *-autonomous poset is such a structure.*

Démonstration. For a monotone poset monoid $(A, \leq, \cdot, 1)$ and the contravariant σ for which (5.5) holds, it is useful to *define*

$$x \star y \ := \ \sigma(\sigma(x) \cdot \sigma(y)); \tag{5.6}$$

while we already have :

$$e \ = \ \sigma(1). \tag{5.7}$$

The notation will also be be used what follows outside this proof.

Now we have the sequence of bijective correspondences of the elements of A for which the following inequalities hold :

$$\dfrac{\dfrac{\dfrac{\dfrac{x \cdot y \leq z}{(x \cdot y) \cdot \sigma(z) \leq e}}{x \cdot (y \cdot \sigma(z)) \leq e}}{x \leq \sigma(y \cdot \sigma(z))}}{x \leq \sigma(y) \star z.}$$

which is natural by monotonicity of $(-)\cdot(-)$ and the contravariance of σ. This is precisely the data defining a *-autonomous poset (cf. the definition in Chapter 2). In the standard notation

$$x \otimes y := x \cdot y; \qquad\qquad (-)^{\perp} := \sigma(-).$$

Conversely, let the $(A, \leq, \otimes, (-)^{\perp})$ be a *-autonomous poset. Then, the invertible derivation above holds, with $\sigma(-)$ for $(-)^{\perp}$ and $(-)\cdot(-)$ for $-\otimes-$. By putting $z = \sigma^{-1}(1)$, (5.5) is recovered. □

Lifting the previous result to **Cmp** via (5.1) and the fact that by Proposition 5.1.1 we have derived we get :

Theorem 5.1.5. *Let* $(A, \Delta, \Pi, \nabla, \amalg)$ *be a Frobenius algebra structure over a poset* A *in* ***Cmp*** *such that the bimonoid maps are defined by two order-monotone poset monoids as in* (5.1)*. Then,* A *is a *-autonomous poset.*

Conversely, let $(A, \cdot, 1)$ *be a *-autonomous poset with the contravariant involution* σ*. Then,* (5.7) *and* (5.1) *define a structure of a Frobenius algebra in* ***Cmp*** *on* A*.*

The point 3. of the Proposition 5.1.1, can be seen as a generalization of 2. In the proof of the implication [3. $\Rightarrow$ 1.], we have given a recipe of how to recover the cocommutative comonoid of a Frobenius algebra given the monoid and the counit map. Moreover, the obtained comultiplication of the comonoid in a Frobenius algebra is uniquely determined by the given data as a consequence of coherence in **FrFrob**.

Example 5.1.6. Let $(G, \leq, +, 0)$ be a partially ordered Abelian group, and e an arbitrary element of G. That is, G is an ordered commutative monoid as before, but such that every element has an inverse for the monoid operation. It is well-know that the inverse operation $x \mapsto -x$ is contravariant (antimonotone).

Defining $\sigma(x) = e - x$ one has immediately

$$x \star y = e - ((e - x) + (e - y)) = x + y - e$$

and

$$\dfrac{\dfrac{\dfrac{x \cdot y \leq z}{x + y \leq z}}{x \leq z - y}}{\dfrac{x \leq (e - y) + z - e}{x \leq \sigma(y) \star x}}$$

which gives us a *-autonomous poset. Naturally, a special case of this was encountered in Chapter 1, with $G = \mathbb{Z}$ (and more generally, $G = \mathbb{Z}^n$).

But we are doing things a little differently here, in the sense that we take the group's addition as the *tensor* of the *-autonomous poset, because the tensor is usually thought of as the primary operation in a *-autonomous category, while the par is a derived one, obtained by adjunction.

Going back to Chapter 1, where the importance of *-autonomy was not taken into consideration, we see that (+,0) was used for the par structure, and a shifted version of addition for the tensor.

But this is only a difference of notation. Recall that we defined

$$x +_a y = x + y - a.$$

In Chapter 1 we were working with

$$x \cdot y = x +_c y \qquad\qquad x \star y = x +_0 y$$

and in this chapter we have

$$x \cdot y = x +_0 y \qquad\qquad x \star y = x +_e y$$

According to Fact 1.2.1, these structures are isomorphic when $0-c = e-0$, i.e. $e = -c$.

The paper [Lam95] contains further examples of *-autonomous (finite) posets. In particular, the paper contains several examples where $x \cdot \sigma(x)$ is not a constant value for every element of the poset.

An important computation where we put to use the introduced notation is the computation of the doubling map $\mathcal{D}_A^k$.

$$\begin{aligned} x\ \mathcal{D}\ y \quad \text{iff} \quad & \exists w, z \in A: \quad x \leq w \star z; \quad w \cdot z \leq y \\ \text{iff} \quad & \exists w, z \in A: \quad \sigma(w) \cdot x \leq z; \quad z \leq \sigma(w) \star y \\ \text{iff} \quad & \exists w \in A: \quad \sigma(w) \cdot x \leq \sigma(w) \star y \\ \text{iff} \quad & \exists w \in A: \quad (w \cdot \sigma(w)) \cdot x \leq y. \end{aligned}$$

The last inequality can be iterated easily to obtain

$$x\ \mathcal{D}^k\ y \quad \text{iff} \quad \exists w_1, \ldots, w_k \in A: \quad (w_1 \cdot \sigma(w_1)) \cdot \cdots \cdot (w_k \cdot \sigma(w_k)) \cdot x \leq y,$$

or,

$$x\ \mathcal{D}^k\ y \quad \text{iff} \quad \exists w \in A: \quad w_1 \cdot \ldots \cdot w_k \cdot x \leq w_1 \star \ldots \star w_k \star y.$$

It is particularly convenient to consider *-autonomous posets where $x \cdot \sigma(x)$ is constant value for every x in the poset. By putting $x = 1$, we conclude that the value has to be e. Such *-autonomous posets allow easier computation of maps. For instance, by the previous, the doubling map iterated becomes

$$x\ \mathcal{D}^k\ y \quad \text{iff} \qquad e^k \cdot x \leq y,$$

where

$$e^k := \underbrace{(e \cdot \ldots \cdot e)}_{k}.$$

Our prototype $\mathbb{Z}$ bimonoids are *-autonomous posets with the above property. In the case of those bimonoids, as mentioned, we have $\sigma : i \mapsto e - i$ for a chosen e, and for every i of a poset :

$$i \cdot (e - i) = i + (e - i) = e.$$

The next thing we want to take a look at is the important concept of thinness for Frobenius categories in **Cmp**.

Proposition 5.1.7. *A Frobenius algebra over a poset A in **Cmp** defined by order-monotone poset monoids $(\cdot, 1)$ and $(\star, e)$ as in (5.1) is thin if and only if*

$$1 \leq e.$$

Démonstration. Recall that the thinness property for A is the request that for every k $\Pi \circ {\mathcal{D}_{\mathcal{A}}}^k \circ \Pi$ is the identity. The composition is seen as

$$\begin{aligned}\{(y,*) \mid y \leq e\}& \\ &\circ\{(x,y)|\ \exists w_1,\ldots,w_k \in A: \quad x \cdot w_1 \cdot \ldots \cdot w_k \ \leq\ w_1 \star \ldots \star w_k \star y\} \\ &\circ\{(*,x) \mid 1 \leq x\}\end{aligned}$$

which is the same as (for the minimal choice of x and maximal of y)

$$\{(*,*)|\ \exists w_1,\ldots,w_k \in A: \quad w_1 \cdot \ldots \cdot w_k \ \leq\ w_1 \star \ldots \star w_k\}.$$

Thus, thinness of the algebra is a request that for every k the above set is non-empty. For $k = 0$ that implies $1 \leq e$.

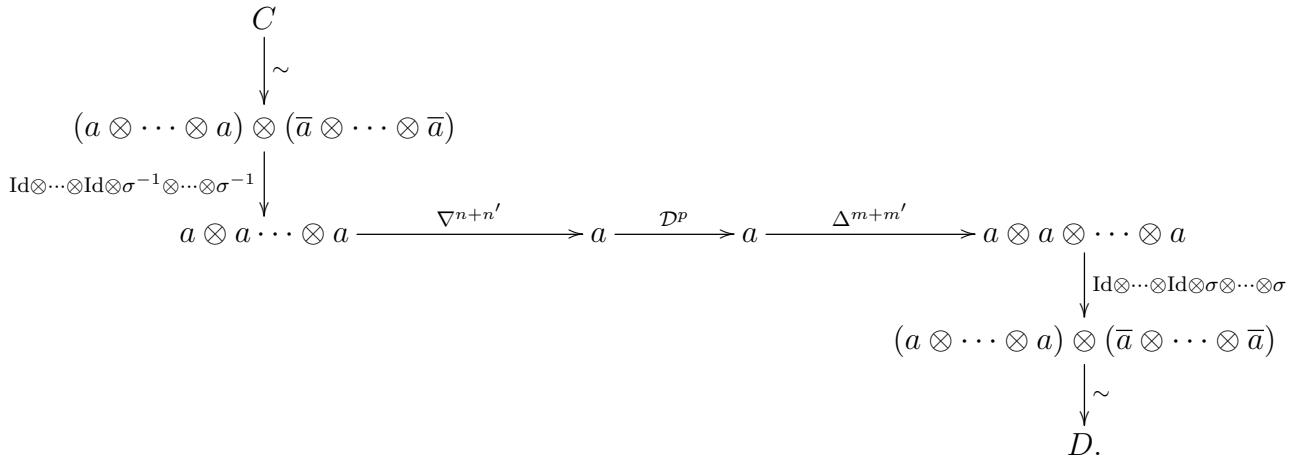
Conversely, the inequality in the relation above for any k will follow from the inequality on the units by monotonicity of the two operations. □

To a logician it may seem strange to ask $1 \leq e$, i.e. that there is a map in a *-autonomous category (poset here) $\mathbf{1} \to \bot$ from the neutral for the $\otimes$- to the neutral for the $\parr$-monoidal structure. Still, we remind the reader that these *-autonomous posets are not used to interpret logic directly.

Thus, we see that in Chapter 1 a $\mathbb{Z}$-bimonoid gives rise to a thin Frobenius algebra iff $c \leq 0$. Since we were defining the interpretations by induction on *classical proofs*, thinness was irrelevant there.

Let now $f : C \to D$ be a map of **FrThFrob**($\mathcal{AT}$yp) which has a single component.

We know from the Proposition 3.1.28 that it can be written as



Here, there are n positive, n' negative literals in the two factors to the left, and m, m' to the right, respectively. The isos at the extremities are coherent isos of a symmetric monoidal category.

Choosing a thin Frobenius algebra A of the kind we have described, we get that the non-coherent iso maps in succession are

$$\{x_1 \leq u_1\} \times \cdots \times \{x_n \leq u_n\} \times \{\sigma(x'_1) \leq u'_1\} \times \cdots \times \{\sigma(x'_{n'}) \leq u'_{n'}\}$$

$$\{u_1 \cdot u_2 \cdot \cdots \cdot u_n \cdot \sigma(u'_1) \cdot \sigma(u'_2) \cdot \cdots \cdot \sigma(u'_{n'}) \quad \leq \quad w\},$$

$$\{\exists d_1, \ldots, d_p : \qquad w \cdot d_1 \cdot \ldots \cdot d_p \quad \leq \quad z \star d_1 \star \ldots \star d_p\},$$

$$\{z \quad \leq \quad v_1 \star v_2 \star \ldots \star v_m \star \sigma(v'_1) \star \ldots \star (v'_{m'})\}$$

$$\{v_1 \leq y_1\} \times \cdots \times \{v_m \leq y_m\} \times \{\sigma(v'_1) \leq y'_1\} \times \cdots \times \{\sigma(v'_{m'}) \leq y'_{m'}\}.$$

putting these together, (replacing all u_is, u'_is, v_is, v'_is by x_is, x'_is,y_is, y'_is respectively in the inner three inequalities, and solving them) we get

$$\{\exists d_1, \ldots, d_p :$$

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot \sigma(x'_1) \cdot \sigma(x'_2) \cdot \ldots \cdot \sigma(x'_{n'}) \cdot d_1 \cdot \ldots \cdot d_p$$

$$\leq \quad y_1 \star y_2 \star \ldots \star y_m \star \sigma(y'_1) \star \ldots \star (y'_{m'}) \star d_1 \star \ldots \star d_p\}.$$

This can also be written as

$$\{\exists d_1, \ldots, d_p :$$

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot y'_1 \cdot y'_2 \cdot \ldots \cdot y'_{m'} \cdot d_1 \cdot \ldots \cdot d_p \tag{5.8}$$

$$\leq \quad y_1 \star y_2 \star \ldots \star y_m \star x'_1 \star \ldots \star x'_{n'} \star d_1 \star \ldots \star d_p\}.$$

If now our Frobenius algebra is of the form $(\mathbb{Z}, +, 0, e)$ this can be further simplified.

$$\{\exists d_1, \ldots, d_p :$$

$$x_1 + x_2 + \ldots + x_n + y'_1 + y'_2 + \ldots + y'_{m'} + d_1 + \ldots + d_p$$

$$\leq \quad y_1 +_e y_2 +_e \ldots +_e y_m +_e x'_1 +_e \ldots +_e x'_{n'} +_e d_1 +_e \ldots +_e d_p\}.$$

Replacing $(-) +_e (-)$ by $(-) + (-) - e$, and subtracting d_is from both sides, we get the familiar

$$\{x_1 + x_2 + \ldots + x_n + y'_1 + y'_2 + \ldots + y'_{m'}$$

$$\leq \quad y_1 + y_2 + \ldots + y_m + x'_1 + \ldots + x'_{n'} - (K + p)e\}.$$

where K is entirely determined by the number of factors, $K = m+n'-1$, and p corresponds to the genus of the one-component map.

So we get

Theorem 5.1.8. *Let $\mathcal{AT}$yp be a set of atomic types, and for every $a \in \mathcal{AT}$yp we choose a Frobenius algebra A_a of the form $(\mathbb{Z}, +, 0, e_a)$, where $e_a > 0$ always. This determines a functor*

$$F : \mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp}) \to \mathbf{C},$$

where $\mathbf{C}$ is the Frobenius category generated by the above, as in 3.1.1. Then this functor is faithful.

Démonstration. If $e_a > 0$, by the previous, the algebras are thin. In addition, if f, g are two distinct parallel maps that have exactly one component, we know that they can only differ by their genus. But the calculations above show that in this case $F(f) \neq F(g)$. The general result follows from the fact that every map can be decomposed as a product of one-component maps. □

So we have a faithful representation theory of *thin* Frobenius algebras.

Some work has been made to extract a subcategory that would make the functor full in addition, but the results are still preliminary in the moment of completing this version of the text.

5.1.1 The $\otimes \neq \parr$ case

Frobenius algebras are defined for objects having $\otimes$-bimonoid structure, that is a $\otimes$-monoid and a $\otimes$-comonoid structure. Since we interpret classical proofs, it is natural to ask if it is possible to define a Frobenius structure for interpretations with the two connectives being different, that is - with objects equipped with $\parr$-monoid and a $\otimes$-comonoid structure.

We notice that it makes sense to define the Frobenius equation in the presence of Mix in such setting as commutativity of the following diagrams (as usual, we omit the associativity isos) :

$$\begin{array}{ccccc}
A\otimes A & = & A\otimes A & = & A\otimes A \\
{\scriptstyle \Delta_A\otimes \mathrm{Id}_A}\downarrow & & {\scriptstyle Mix}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_A\otimes\Delta_A} \\
A\otimes A\otimes A & & A\parr A & & A\otimes A\otimes A \\
{\scriptstyle Mix}\downarrow & & {\scriptstyle \nabla_A}\downarrow & & \downarrow{\scriptstyle Mix} \\
A\otimes (A\parr A) & & A & & (A\parr A)\otimes A \\
{\scriptstyle \mathrm{Id}_A\otimes\nabla_A}\downarrow & & {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \nabla_A\otimes\mathrm{Id}_A} \\
A\otimes A & = & A\otimes A & = & A\otimes A
\end{array} \tag{5.9}$$

and

$$\begin{array}{ccccc}
A\parr A & = & A\parr A & = & A\parr A \\
{\scriptstyle \Delta_A\parr\mathrm{Id}_A}\downarrow & & {\scriptstyle \nabla_A}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_A\otimes\Delta_A} \\
(A\otimes A)\parr A & & A & & A\parr (A\otimes A) \\
{\scriptstyle Mix}\downarrow & & {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle Mix} \\
A\parr A\parr A & & A\otimes A & & A\parr A\parr A \\
{\scriptstyle \mathrm{Id}_A\otimes\nabla_A}\downarrow & & {\scriptstyle Mix}\downarrow & & \downarrow{\scriptstyle \nabla_A\parr\mathrm{Id}_A} \\
A\parr A & = & A\parr A & = & A\parr A
\end{array} \tag{5.10}$$

Notice that the second diagram is dual to the first one.

It is convenient to define :

$$\widetilde{\nabla}_A : \quad A\otimes A \xrightarrow{\ Mix\ } A\parr A \xrightarrow{\ \nabla_A\ } A, \tag{5.11}$$

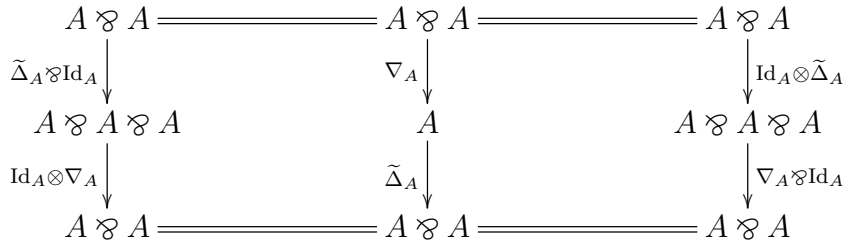
and

$$\widetilde{\Delta}_A : \quad A \xrightarrow{\ \Delta\ } A\otimes A \xrightarrow{\ Mix\ } A\parr A, \tag{5.12}$$

so that diagrams above turn into familiar

$$\begin{array}{ccccc}
A\otimes A & = & A\otimes A & = & A\otimes A \\
{\scriptstyle \Delta_A\otimes \mathrm{Id}_A}\downarrow & & {\scriptstyle \widetilde{\nabla}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_A\otimes\Delta_A} \\
A\otimes A\otimes A & & A & & A\otimes A\otimes A \\
{\scriptstyle \mathrm{Id}_A\otimes\widetilde{\nabla}_A}\downarrow & & {\scriptstyle \Delta_A}\downarrow & & \downarrow{\scriptstyle \widetilde{\nabla}_A\otimes\mathrm{Id}_A} \\
A\otimes A & = & A\otimes A & = & A\otimes A
\end{array}$$

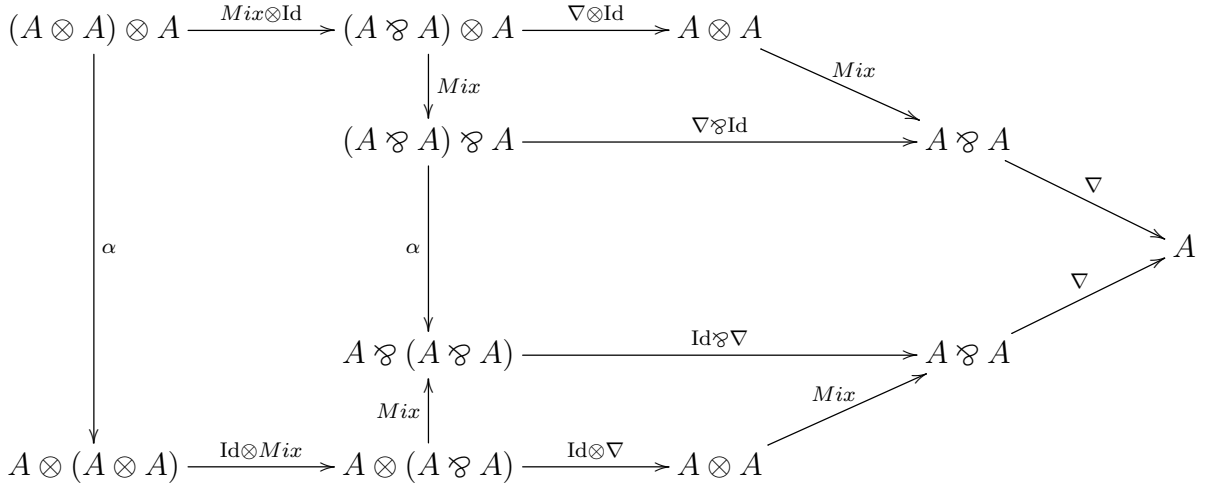
and



One cannot expect anymore to have $\parr$- or $\otimes$-Frobenius algebra structure on an object in the standard sense, due to the absence of the standard bimonoid structures. More concretely, the $\parr$-comonoid counit and the $\otimes$-monoid unit cannot be present in general, as using a structure with e.g. unit for a $\otimes$-monoid structure

$$\widetilde{\Pi}_A : \quad \mathbf{1} \xrightarrow{\Pi_{\mathbf{1}}} A$$

to interpret proofs would imply that any formula is provable. However, one can still show associativity of $\widetilde{\nabla}$



Hexagon to the left is agreeing of Mix with associativity of a *-autonomous category (cf. [Blu93, Lam07, Str07, Str11], chapter on apparatus), the two quadrilaterals are naturality of Mix, and the pentagon is associativity of ∇.
Dual diagram shows coassociativity of $\widetilde{\Delta}$.

We have shown :

Proposition 5.1.9. *Given a Medial-compatible bimonoid structure on an object,* $\widetilde{\nabla}$ *and* $\widetilde{\Delta}$ *defined as in* (5.11) *and* (5.12) *are associative and coassociative, respectively.*

Definition 5.1.10. A *weak free* $\otimes$*-Frobenius algebra* in a symmetric $\otimes$-monoidal category is a free algebra generated by an object equipped with a $\otimes$-comonoid structure and an associative, commutative operation $\widetilde{\nabla}_A : A \otimes A \to A$ for which the diagram 5.9 commutes.

Dually, one defines a *weak free* $\parr$*-Frobenius algebra* in a symmetric $\parr$-monoidal category as a free algebra generated by an object equipped with a $\parr$-monoid structure and

an coassociative, cocommutative operation $\widetilde{\Delta}_A : A \to A \parr A$ for which the diagram 5.10 commutes.

What seems to be in play when the two connectives are distinguished is the identification of maps coming from Frobenius equations coupled with a kind of 'phase transition'. As it turns, the generalized Frobenius algebras yield normal forms of maps obtained by interpreting structural rules in a conjunctive or a disjunctive context alone. So, a map in general can be viewed as a sequence of steppings from a conjunctive structural context into a disjunctive one, each of which is provided with a notion of a normal form.

Frobenius algebras do provide a new way to treat cut elimination in the problematic contraction-against-contraction case. Still, it seems that that additional, more fine-grained structures are needed to fully capture the classical cut elimination. In particular, it may be the case that in certain situations bialgebra-style resolutions are required in place of Frobenius ones, i.e. that the two are interchanged, depending on a situation. The generalized notion of a Frobenius algebra in this sense, is a step in right direction.

This thesis does pursue this line of investigation, but we acknowledge the importance this observation may have. Unfortunately, we still do not have concrete examples of these, and a search for those is another task to be undertaken.

6

F-nets and Related Approaches to Boolean Categories

6.1 Related Approaches

This chapter addresses some of the approaches to the problem of finding a suitable notion of semantics for classical logic. Admittingly, the attention is given to the work of most relevance to ours. Therefore, this chapter should by no means be understood as an objective overview of the state-of-the-art in the research area.

6.1.1 Non-symmetric Classical Semantics

In the early 90s, Joyal's negative result of the collapse of a Cartesian-closed category with the involutive negation to a Boolean algebra provided little hope in possibility of making any progress towards the formulation of a Boolean category at all [LS86, Gir91].

As the later development of our knowledge of classical semantics has shown, a Cartesian-closed category is not a necessary step towards a Boolean category (and this thesis is an example), even in the presence of products and coproducts, the category need not be closed as in [DP04]. But historically the first relaxing of the conditions of the Joyal's paradox was the abandonment of the involutive negation. The realization coming form computer science [Fil89, Gri90] that control operators in a functional program stand for constants corresponding to the Peirce's coupled with the 'pure' lambda calculus has provided a step from intuitionistic to classical logic. This realization has lead to the Parigot's formulation of well-behaved system for natural deduction giving rise to his $\lambda - \mu$-calculus [Par92]. Categorical axiomatizations have followed, Selinger's [Sel01], based on the work of Streicher and Reus [SR98], and later Ong's, and [Ong96]. Girard's treatment of classical logic has involved formulation of the system LC for classical logic and two different models for it in [Gir91].

In the first group of semantics double negation not isomorphic to an object, while the

disjunction is not bifunctorial, in one of Girard's models cut elimination is not associative, and all of them in one way or the other break the inherent symmetries of classical logic.

As our work insists on the classical symmetries, this chapter puts more focus on certain more closely related approaches to Boolean categories.

6.1.2 Došen, Petrić. Boolean Category

In their monograph [DP04] Došen and Petrić perform systematic analysis of coherence problem for various axiomatizations of categories linked to proof systems. The monumental work formulates deductive systems of increasing strength, seen as *syntactic categories* coupled with equational systems on arrow terms. As usual, the categories are with propositional formulas as objects and maps generated by certain primitive arrow types, using compositions and bifunctors $- \wedge -$ and $- \vee -$. Coherence is understood as existence of faithful functors from the syntactic categories to the category *Rel* of relations on finite ordinals, to which authors refer to as the *graphical* category.

The authors revoke the early work on categorical logic by Lambek, [Lam68, Lam69, Lam72, Lam73], where, among other things, he introduced a notion of *generality of a derivation*[17] contrasting the Prawitz's notion of identity of derivation through identity of normal forms [Pra65]. Roughly speaking, one considers generalizations of derivations that diversify variables without changing the rules of inference. Two derivations have the same generality when every generalization of one of them leads to a generalization of the other, so that in the two generalizations we have the same premise and conclusion. The notion of Generality induces an equivalence relation between derivations, and two derivations are equivalent if and only if they have the same generality. What is called 'Generality Conjecture' is formulated to state that two derivations represent the same proof if and only if they are equivalent in the new sense.

It turns out that the simplest way to understand generality is to use graphs whose vertices are occurrences of propositional letters in the premise and the conclusion of a derivation. An edge connects two occurrences of propositional letters that must remain occurrences of the same letter after generalizing, and do not connect those that may become occurrences of different letters. Thus, the generality of a derivation is such a graph.

This is precisely the rationale for choosing *Rel* for the coherence problem and the graphical category, with the other one being that the coherence in *Rel* is clearly decidable (the authors use the term 'manageable'). However, the authors do acknowledge possibility of alternative notions of coherence through a different choice for a graphical category. In [Str05, Str09], Straßburger argues that the notion of *Rel*-based coherence is too restricted for the approach to classical proof theory exhibited in the paper. Notice that our semantics (say, linkings of a F-net) can be seen as an alternative geometric category to *Rel*.

17. the term used by Lambek was 'scope' rather than 'generality'

The axiomatizations of syntactic categories given in the monograph are of increasing strength, with one of the strongest ones being the boolean category B is what is argued to be a category of classical proofs. Without going into full set of equations defining B, we point out that the most important differences to our approach being :

- the category B of classical proofs is bicartesian, that is, has finite products and coproduts
- to avoid collapse to a preorder under the Joyal's paradox, the category is not cartesian closed
- unlike our categories, the families of bimonoid maps assigned to objects in B enjoy naturality in the objects.

Another recent work of Došen and Petrić in immediate relation to the work of this thesis is a recent investigation [DP08] on Frobenius monads. The paper provides geometrical descriptions of the Frobenius monad freely generated by a single object where ordinals in epsilon zero play a prominent role. The authors also show how the notion of Frobenius algebra induces a collapse of the hierarchy of ordinals in epsilon zero.

6.1.3 Carbone. Proof Cycles

To the best of out knowledge, the first time cycles in graphs assigned to proofs were emphasized was in the work [Car97]. The important difference to our work is the fact that the graphs tied to sequent derivations of classical logic were more of invariants of proofs than the identity criteria. The work does not refer to to categories, and cut elimination procedure, for instance, may transform some of the cycles in the assigned graphs into spirals, cf. [Car97, Car99a, Car99a, Car00]. Notice that something like that does never happen in our case.

6.1.4 Hyland. Frobenius Invariants.

What can probably be considered as one of the direct precursors to out work and what is, to the best of our knowledge, the first time Frobenius algebras were employed in classical proof theory, was the Hyland's paper [Hyl04]. From the historical perspective, it should be mentioned, though, that the characterization of the maps in our initial $\mathbb{Z}$-interpretation was reached independently from the Hyland's work. After exploring the Frobenius structure underlying the interpretation and after noticing the relevance of Hyland's work and the general claims on equivalence between the category freely generated by a Frobenius algebra on a single object with the category of 2-cobordisms to our efforts, we have looked into a similar characterization in the category of comparisons we were working in. Though our two approaches stem from the same idea, there are considerable differences, and here we name some.

In his paper, Hyland starts with the observation that it should be clear how to interpret

classical proofs in the symmetric monoidal category generated by a Frobenius algebra. The guidance provided states that the interpretation maps all atomic propositions to a single generating Frobenius algebra, so atoms are not distinguished. Also, the interpretation is not sensitive to negation, i.e. negation is the identity, not merely an iso as in our case.

Still, the biggest point of difference between the approaches lies in continuation of the paper where possibility of having multiple connected components in an interpretation of a proof is abandoned, and a single-component interpretation is enforced through interpretation of logical connectives. More concretely, given two single-component interpretations p_1 and p_2 of proofs π_1 and π_2 of $\vdash \Gamma, \mathrm{A}$ and $\vdash \Sigma, \mathrm{B}$, respectively, the interpretation of π, for

$$\frac{\vdash^{\pi_1} \Gamma, \mathrm{A} \quad \vdash^{\pi_2} \Sigma, \mathrm{B}}{\vdash^{\pi} \Gamma, \mathrm{A} \wedge \mathrm{B}, \Sigma}$$

is given by a single component obtained by connecting p_1 and p_2. In other words, π is interpreted as a component with elements being those in union of p_1 and p_2, and genus being the sum of the two respective genera.

As for disjunction,

$$\frac{\vdash^{\pi_1} \Gamma, \mathrm{A}, \mathrm{B}, \Sigma}{\vdash^{\pi} \Gamma, \mathrm{A} \vee \mathrm{B}, \Sigma}$$

interpretation of π adds 1 to the genus of the single component (notice : regardless of the number of atoms in A, B).

In this setting, Hyland manages to derive meaningful invariants of proofs – when interpreting proofs in multiplicative linear logic, the interpretation counts for number of disjunctions in a proof, and points toward Danos-Regnier correctness criterion, while in the case of classical proofs, the interpretation records a mixture of structural and logical rule. The author notices, as we do in the $\mathbb{Z}$ interpretation, that by encoding Church numerals and the arithmetic operations, the interpretation violates the Curry-Howard isomorphism, which marks a detachment from the constructive nature of intuitionistic proof theory.

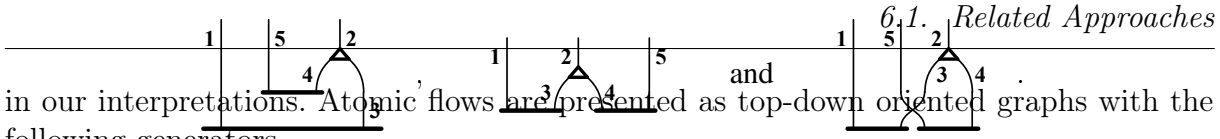
The choice for interpretation of classical proofs mixes structural and logical reasoning, and as such is necessarily quite different to our approach.

6.1.5 Gundersen, Guglielmi, Bruscoli, Straßburger, Parigot. Atomic Flows.

The clear separation of logical and structural dimensions of classical proofs found in our approach is also the driving idea of the recent work on atomic flows. The body of work on atomic flows consists of a series of papers [BGGP10, GGS10, GG08, Gun09] by Gundersen, Guglielmi, Bruscolli, Straßburger, Parigot.

Atomic flows share a lot of similarities with our linkings of F-nets. The idea is to track occurrences of atoms within a (deep inference) classical proof, much like it is done

in our interpretations. Atomic flows are presented as top-down oriented graphs with the following generators



It should be noted that atomic flows built from derivations have no directed cycles and bear a natural polarity assignment (corresponding to atoms versus negated atoms in the derivation), that is a mapping of each edge to an element of $\{-,+\}$, such that the two edges of each identity or cut vertex map to different values and the three edges of each contraction or cocontraction vertex map to the same value. We denote atomic flows by ϕ and ψ.

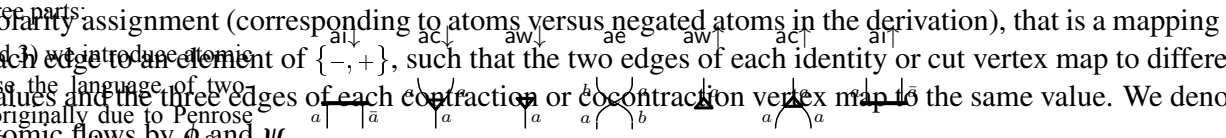
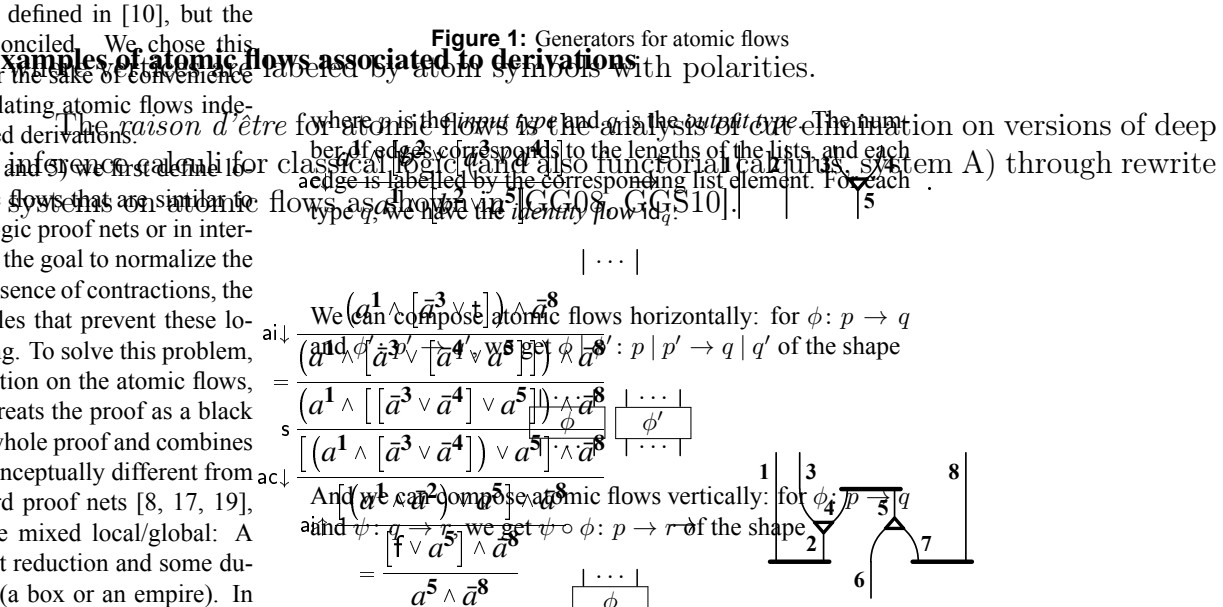


Figure 1: Generators for atomic flows

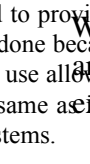
Examples of atomic flows associated to derivations

The *raison d'être* for atomic flows is the analysis of cut elimination on versions of deep inference calculi for classical logic and also functorial calculus, system A) through rewrite systems on atomic flows as shown in [GG08, GGS10].

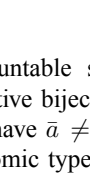


Abstract notation of atomic flows

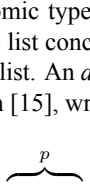
When certain details of a flow are not important, but only the vertex kinds and its upper and lower edges are, we can use boxes, labelled with all the vertex kinds that can appear in the flow they represent. For example, the generators for the identity



and the cut rule



are not transposes of identity under self-adjunction of negation



It may be tempting to see if the various normal forms of atomic flows that correspond to normal forms of derivations in SKS can be distinguished in the Frobenius category. The answer is, however obvious: simply by looking at them, by converting axioms and

cuts into ordinary wires, and by composing with (co)monoidal units, it is clear that these forms are indistinguishable the Frobenius category

which is invariant in the Frobenius category.

6.1.6 Lamarche, Straßburger. $\mathbb{B}$- and $\mathbb{N}$-nets.

In a series of papers [LS05b, LS05a, Lam07, Str07, Str11] that derive also from [SL04, LS06], Lamarche and Straßburger have developed an approach to classical proof theory that is, unsurprisingly, a direct predecessor of this work. The general framework is shared – objects of a *-autonomous category with mix and medial are equipped with bimonoid structures, with the marked difference being the fact that Frobenius equations are assumed here.

In the paper [LS05a], Lamarche and Straßburger provide semantics for multiplicative linear logic without units as the free *-autonomous category $\mathcal{C}$ where units are replaced by so called *virtual units* - functors from the category $\mathcal{C}$ to Set, that would be representable functors for units, provided those exist. These are subject to certain conditions, mainly those coming form the adjunction in the category and the presence of the internal Hom functor. In this work we have units explicitly present in the semantics, but not in the syntax. It is not unusual to do things this way, but, as the authors argue in [LS05a], the F-nets cannot be used to interpret themselves as a theory *per se*. Nevertheless, we decide to live with this shortcoming and not to pursue this investigation further here.

Further in [LS05a] and in subsequent papers [Lam07, Str07, Str11] the authors perform detailed analysis of what it means for a *-autonomous category to have Mix and

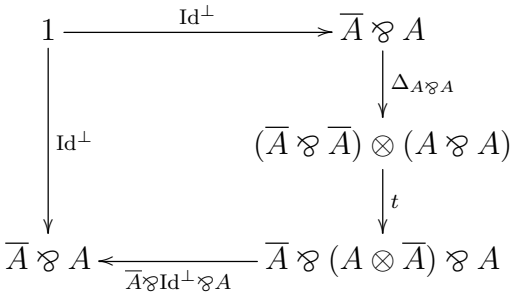


FIGURE 6.1 – The Loop killing axiom. Map t is the canonical composition of two switch maps

Medial, and give the taxonomy of categories of increasing number of axioms that result in variations of what could be the free boolean category. In fact, the main category we work with, **FrThFrob**($\mathcal{AT}$yp) incorporates some of the strongest axiomatizations which are introduced throughout this work (while displaying the $\otimes = \parr$ degeneracy). One of the definitions that cannot be found in this thesis is the notion of the *graphicality* (see [LS05a, Str07, Str11]), which basically states that a forgetful functor from a category to the one which identifies the connectives is faithful. Naturally, the category **FrThFrob**($\mathcal{AT}$yp) is such a category. Moreover, as we have shown in the Chapter 2, this axiom in its strengthened form, which is asking Mix to be an iso, turns a *-autonomous category into a compact-closed one.

Also, the authors propose an axiom that they call *loop killing* or *contractibility* (see Figure 6.1.6), which is well described by its name – for it to hold in **FrThFrob**($\mathcal{AT}$yp) would mean to identify certain maps with different genera but same components.

Another axiom generally not assumed in **FrFrob**($\mathcal{AT}$yp) or **FrThFrob**($\mathcal{AT}$yp) is the bialgebra axiom. As shown in [Str07, Str11] requesting the presence of the bialgebra equation 3.9, i.e. requesting that bimonoid maps respect bimonoid structure yields idempotent models, that is, models where doubling map $\mathcal{D}_a$ is the identity Id_a.

Papers [LS05b, LS05a] also offer concrete examples of the free categories that are axiomatized, $\mathbb{B}$ - and $\mathbb{N}$-nets. In fact, as the authors show, only the former class of proof nets forms a category, while the latter fails to provide confluent cut elimination. While $\mathbb{N}$ nets precisely keep record of the history of axiom links, they also fail to give a correctness procedure, which exists for the $\mathbb{B}$ nets.

Both $\mathbb{B}$ nets and $\mathbb{N}$ nets are not unrelated to our F-nets. In fact, one can define a partition in a linking of a correct F-net to be the transitive closure of the linking of a $\mathbb{B}$ net. Thus, linking of a $\mathbb{B}$ net carries more information than a partition of a F-net. Of course, F-nets are capable to count axiom links, and in that sense they carry strictly more information on a proof history than $\mathbb{B}$ nets. Precise relation between $\mathbb{N}$ nets and F-nets is not so clear, since both are capable of counting axiom links, $\mathbb{N}$ nets do it explicitly and appear to have more information knowing that they keep track of the atom whose ancestors in the proof history formed axiom links. However, there is no canonical way to

2. one of x and y is decorated by $\mathbf{t}$ and the other by $\mathbf{f}$.

A *W-pre-proof-net* (or shortly *W-prenet*) consists of a sequent Γ and a linking P for it. It will be denoted by $P \rhd \Gamma$. If $W = \mathbb{B}$ we say it is a *simple pre-proof-net* (or shortly *simple prenet*).

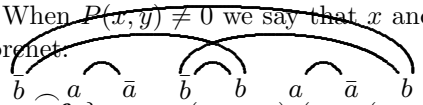
In what follows, we will simply say prenet, if no semiring W is specified. If we choose $W = \mathbb{B}$, we have that an *Boolean, Undirected* graph structure on $\mathscr{L}(\Gamma)$, with an edge between x and y when $P(x, y) = 1$. An example is

To save space that example can also be written as

$$\{\, \overset{1}{\overbrace{\bar{b}_1\ b_5}}\ ,\ \overset{2}{\overbrace{b_1\ b_8}}\ ,\ \overset{3}{\overbrace{b_4\ \bar{b}_5}}\ ,\ \overset{4}{\overbrace{b_4\ \bar{b}_8}}\ ,\ \overset{5}{\overbrace{a_2\ \bar{a}_3}}\ ,\ \overset{6}{\overbrace{a_6\ \bar{a}_7}}\ \} \rhd b_1 \wedge a_2, \bar{a}_3 \wedge b_4, b_5 \wedge a_6, \bar{a}_7 \wedge b_8\ ,$$

where the set of linked pairs is written explicitly in front of the sequent. Here we use the indices only to distinguish different atom occurrences (i.e., a_3 and a_6 are not different atoms but different occurrences of the same atom a).

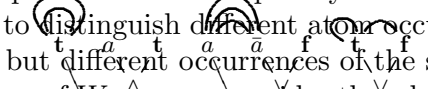
For more general cases of W, we can consider the edges to be decorated with elements of $W \setminus \{0\}$. When $P(x, y) \neq 0$ we say that x and y are *linked*. Here is an example of an $\mathbb{N}$-prenet:



(3)

$$\{\, \overset{2}{\overbrace{\mathbf{t}_1\ \mathbf{t}_1}}\ ,\ \overset{3}{\overbrace{a_4\ \bar{a}_5}}\ ,\ \overset{1}{\overbrace{\mathbf{t}_7\ \mathbf{t}_7}}\ ,\ \overset{1}{\overbrace{\mathbf{t}_7\ \mathbf{f}_8}}\ \} \rhd \mathbf{t}_1 \wedge (a_2 \wedge \mathbf{t}_3), (a_4 \vee (\bar{a}_5 \vee \mathbf{f}_6)) \wedge (\mathbf{t}_7 \vee \mathbf{f}_8)\ . \quad (4)$$

As before, no link means that the value is 0. Furthermore, we use the convention that a link without value means that the value of the linking is 1. When drawing $\mathbb{N}$-prenets as graphs (i.e., the sequent forest plus linking), we will draw n links between two leaves x and y, if $P(x, y) = n$. Although this might be a little cumbersome, we find it more intuitive. Example (4) is then written as



(5)

FIGURE 6.3 – A $\mathbb{N}$ net

[2] Now for some more notation: let $P \rhd \Gamma$ be a prenet and $L \subseteq \mathscr{L}(\Gamma)$ an arbitrary subset of leaves. Then it always $P(x, y) \in W$ which is obtained by restricting P on the smaller domain. But L also determines a subforest $\Gamma|_L$

[1] What we call *pre-proof-net* is in the literature often called a *proof structure*.

compose these nets, so a precise correspondence is tampered by this fact.

In [Str07, Str11], Strasburger proposes a fix to the non-confluence of cut elimination of $\mathbb{N}$ nets, by defining so called *extended nets*. These nets emerge when one allows for a limited cut elimination in $\mathbb{N}$ nets (cf. [Str05, Str09]). Precisely those problematic cases of cut elimination are blocked, and extended nets correspond to normal forms of classical proofs where certain cuts are not eliminated. Linkings in $\mathbb{N}$ nets are mappings of pairs of atoms of the different polarity into $\mathbb{N}$, which also exist in extended nets. In addition, the sets of atoms of a formula is extended by a finite set of so called *anchors* standing for the cuts which are not eliminated, and a linking is an undirected graph with atoms and anchors as nodes. A component of the graph can connect only the atoms of the same type and each anchor needs to have an input and an output degree of at least 2. Composition on these nets is defined as concatenation of paths that needs to preserve definition of a net, and these form a category.

[5] Now for some more notation: let $P \rhd \Gamma$ be a prenet and $L \subseteq \mathscr{L}(\Gamma)$ an arbitrary subset of leaves. Then it always $P|_L \in W$ which is obtained by restricting P on the smaller domain. But L also determines a subforest $\Gamma|_L$

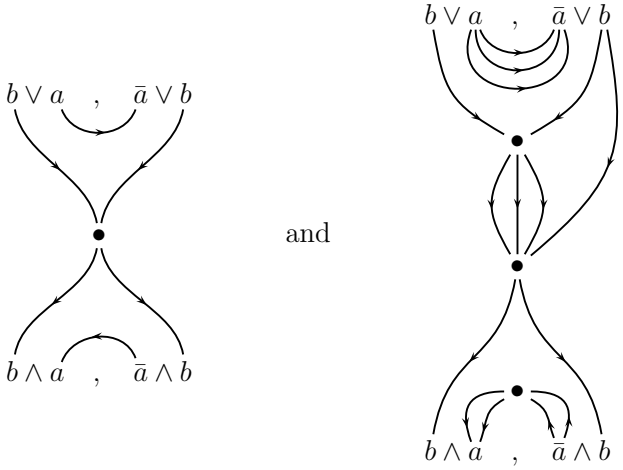
[4] What we call *pre-proof-net* is in the literature often called a *proof structure*.

It seems that extended nets contain more information on the history of proofs than F-nets, since they incorporate $\mathbb{N}$ nets and are able to count axiom links. Still, the main difference to F-nets is that these nets are loop-killing, that is, the cut elimination can eliminate some of the loops (= lower genera), and make identifications that F-nets do not make. When one forgets about orientation of the arcs in a graph, anchors can be seen as connected components of our F-nets (with genera!), added on top of the $\mathbb{N}$ nets.

6.1.7 Führmann, Pym. Order Enrichment.

In their work [FP05] Führmann and Pym consider semantics of classical propositional proof theory and give axiomatization of what they call the *classical category*. Their work starts, as in our case, by focusing attention on semantics of linear proofs. For classical

As before, every $\mathbf{t}$ has to be linked to itself. That we allow only one and not many such links is due to Lemma 5.11 (which is a consequence of having proper units, cf. [Str05]). But contrary to what we had before, we do now allow not only many links between two atoms but also "non-direct" links visiting anchors on their way. But each anchor can only serve links between atoms of the same name. Furthermore, an anchor must have at least two incoming and at least two outgoing links. Here are two examples of extended prenets:

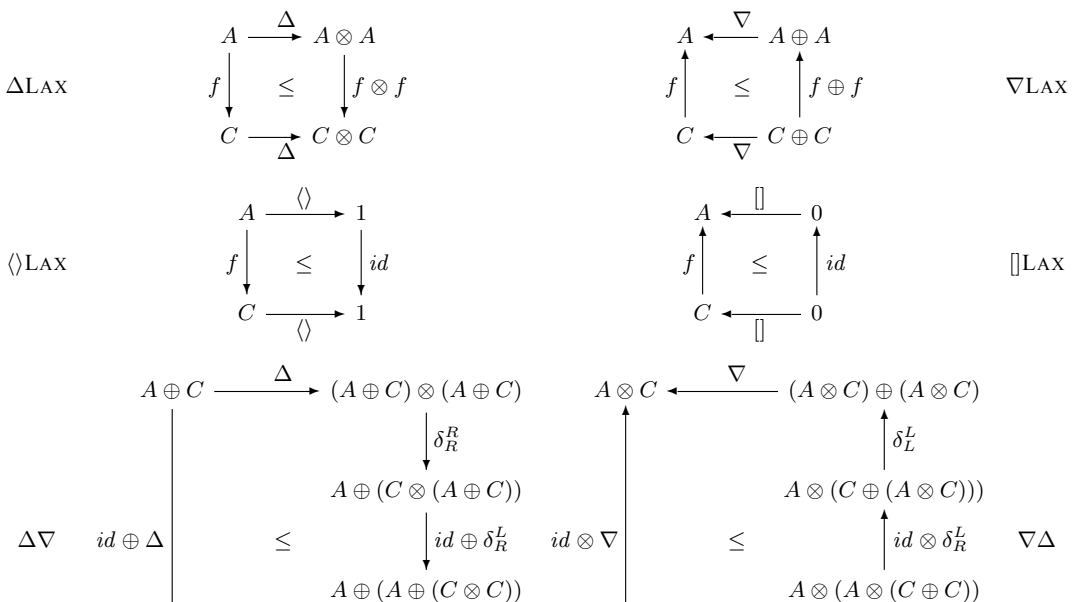


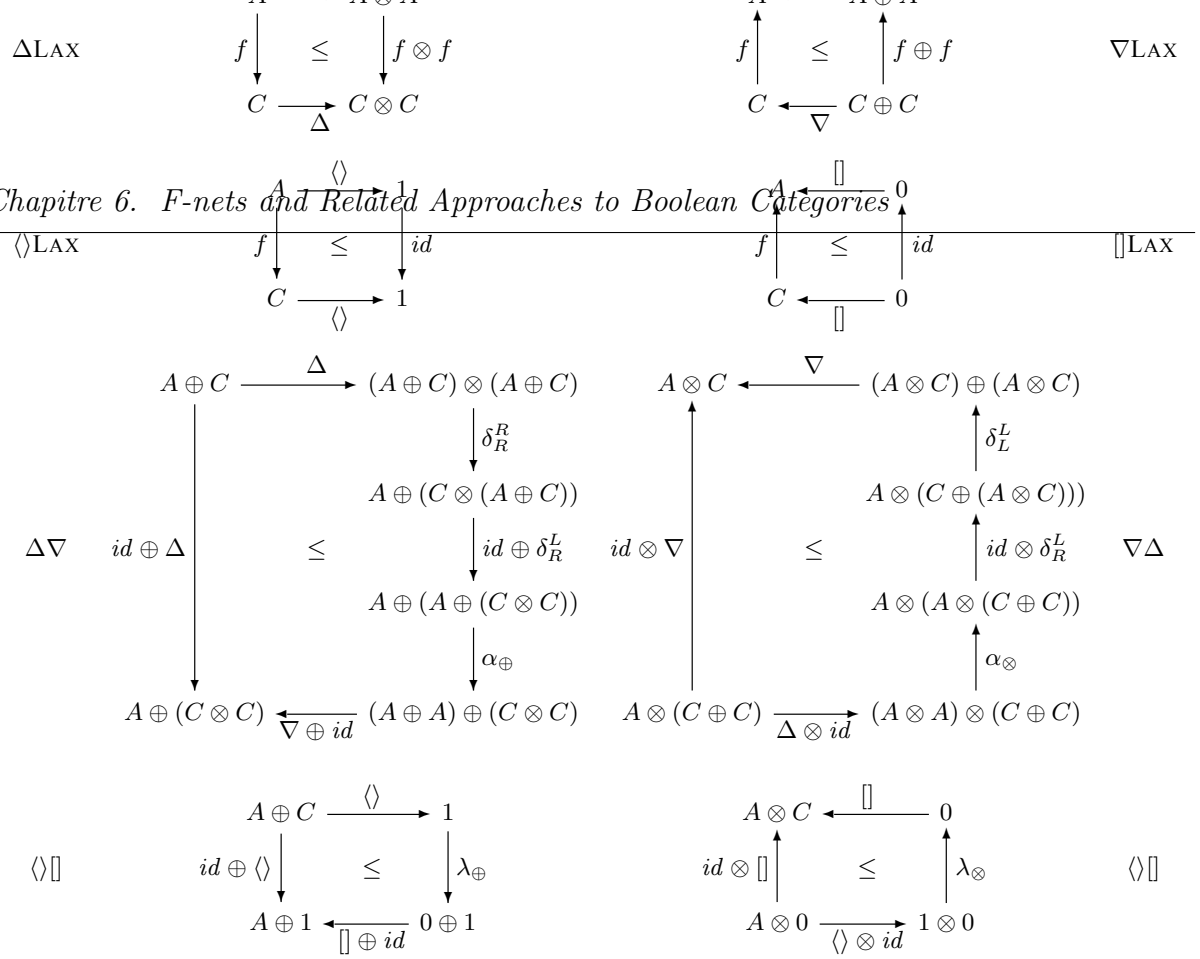
(55)

FIGURE 6.4 – Extended nets

logic they look at proof nets of Robinson as given in [Rob03] and analyze the two cases that encapsulate complications one has in classical cut elimination - the weakening-meets-weakening and the contraction-meets-contraction case. The both cases allow for multiple options and choices of how to proceed in each of the cases, the authors decide to introduce an order relation on nets (with cuts) to deal with these. Central part in the definition of the order is the requirement that a net is smaller than another one if the latter is a possible reduct in a cut elimination step of the former one. The sets of inequalities determined by these defines a *net theory*.

The underlying categorical axiomatization for the net theories is based on a free symmetric *-autonomous category with bimonoid structures on objects. The category need not have Medial, and to match the order on nets, an order on maps is defined, i.e. the category is enriched over the category of posets. The axioms defining the enrichment with slightly different notation : $\oplus$ for $\parr$, $\langle\rangle$ for Π, $[]$ for $\amalg$ and δ for switch are :





Example 4. **Rel**, where the order between morphism is the set-theoretic inclusion of morphisms. Also, every Boolean lattice, where the order between morphisms is trivial (because hom-spaces have at most one element).

Obviously, given two classical categories $\mathbf{C}$ and $\mathbf{C}'$, the product category $\mathbf{C} \times \mathbf{C}'$, which we already observed to be a linearly distributive category with monoids, co-monoids and negation, is again the classical category with

$$(f, f') \leq (g, g') \iff f \leq g \text{ and } f' \leq g'$$

In particular, we have

Example 5. $\mathbf{Rel} \times \mathbf{B}$ is a classical category for every Boolean lattice $\mathbf{B}$.

The use of the eight inequality laws will be explained precisely in the soundness and completeness proofs. However, we shall first explain these laws in a more intuitive way. The law ΔLax, which states that Δ is a "lax natural transformation", is essentially the left-to-right reduction in Equation 17. (As observed earlier, the converse does not generally hold.) This reduction is possible owing to Cut!C. But Cut!C is more powerful, in a subtle way: note that the net M in Equation 17 has only

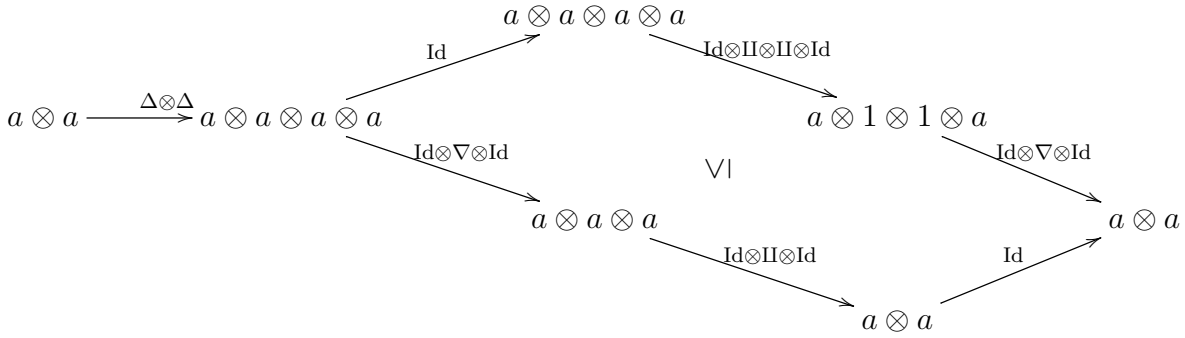
Definition 6.1.1. For two maps $f, g : A \to B$ in $\mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})$ we say that $f \sqsubseteq g$ if the induced partition of object generators in f is finer than the one induced by g.

Observation 6.1.2. Relation $\sqsubseteq$ is a preorder relation on hom-sets in $\mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})$ that is cruder than our order relation $\preceq$, since it does not take genera into account. Moreover, if we define an equivalence relation on hom-sets where two parallel maps are said to be equivalent if the induced partitions of object generators are the same (i.e. they are the same when we forget about the genera), the $\sqsubseteq$ relation gives rise to a relation in the quotient category, also denoted by $\sqsubseteq$.

Proposition 6.1.3. *Let $\leq$ be the preorder enrichment on* **FrThFrob**($\mathcal{AT}$yp) *defined from the inequality diagrams above. Then for parallel maps f, g*

$$f \leq g \qquad \textit{iff} \qquad g \sqsubseteq f.$$

Démonstration. Observe now the following diagram



which $\leq$-commutes as an instance of ∇Lax. Let $u, l : a \otimes a \to a \otimes a$ be the upper and the lower branch of the diagram.

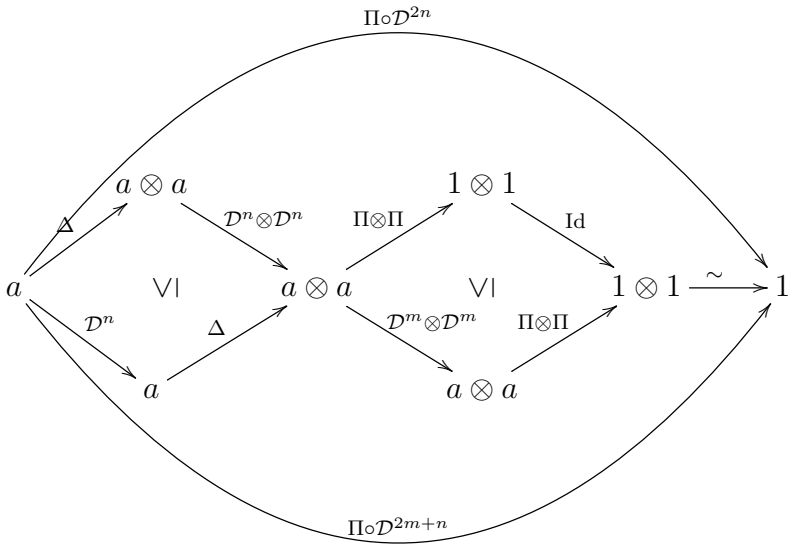
Let $f : A \to B$, be a map where B is of the form $B' \otimes a \otimes a \otimes B''$ so that the two generators a are in different components in f. Postcomposing f with $\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes u \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}$, the upper branch of the diagram (tensored with some identities) does not change f, while postcomposing with $\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes u \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}$ yields a map where the two components of f containing the two instances of a are merged. Mediated by coherent isos, this generalizes to the case where the two generators are not next to each other in the tensor of B, and the canonical iso $\sigma^{-1} : \overline{a} \to a$ generalizes this to the case where the generators are possibly negative. Finally, by taking a dual diagram, one shows that by precomposing with suitable maps one can also merge components of a map, getting a $\leq$ smaller map.

Thus, $f \leq g \Rightarrow g \sqsubseteq f$. To show the opposite, it suffice to show that the $\leq$-defining diagrams commute when $\leq$ is replaced by $\sqsupseteq$. Indeed, by identifying the connectives and the units in the $\leq$ diagrams, $\langle\rangle[]$ and $[]\langle\rangle$ $\sqsubseteq$-commute, as the smaller maps are less connected by (co)monoidal equations. The $\Delta\nabla$ and $\nabla\Delta$ $\sqsubseteq$-commute, since they reduce to $\mathrm{Id}_A \leq \mathcal{D}_A$. Diagrams ∇Lax and []Lax are simply ΔLax and $\langle\rangle$Lax repeated from the point of view of $\sqsubseteq$, i.e. if one pair commutes so will the other. A simple geometric argument shows that $\langle\rangle$Lax commutes, since f has the opportunity to connect some of the atoms, and cannot split a component. To show that ΔLax commutes, one can show that if x, y are in a same component of $f \otimes f \circ \Delta_C$, they are also in a same component in $\Delta_A \circ f$. There are three possibilities – either $x, y \in A$, or $x \in A$, $y \in C \otimes C$, or $x, y \in C \otimes C$. In all three cases it is f itself that needs to have a component both x and y belong to, and so they do in $\Delta_A \circ f$.

□

Proposition 6.1.4. *Let $\leq$ be the preorder enrichment on* **FrThFrob**($\mathcal{AT}$yp) *defined from the inequality diagrams above. If parallel maps f, g differ only by their genera (and share the partitions of object generators) then $f \leq\geq g$.*

Démonstration. First of all, observe that for an atom a and its doubling map $\mathcal{D}_a$ the following diagram commutes



for each m and n. The inequality square to the left is an instance of ΔLax, the one to the right is an instance of $\langle\rangle$Lax, while the diagrams with long arcs commute by coherence in $\mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})$. Thus, $\Pi\circ\mathcal{D}^{2n} \geq \Pi\circ\mathcal{D}^{2m+n}$ for all m, n. For $m = 0 : \Pi\circ\mathcal{D}^{2n} \geq \Pi\circ\mathcal{D}^{n}$, for $n = 0$, combined with the previous, $\Pi \circ \mathcal{D}^0 \geq \Pi \circ \mathcal{D}^{2m} \geq \Pi \circ \mathcal{D}^m$ for every m.

In the previous proof we concluded that $\Delta\nabla$ yields $\mathcal{D}^1 = \mathcal{D} \leq \mathrm{Id}$. Therefore, $\Pi \circ \mathcal{D}^1 \geq \Pi \circ \mathcal{D}^0$, and by precomposing with the previous, we get $\Pi \circ \mathcal{D}^k \geq \Pi \circ \mathcal{D}^0$ for every k. Combined with the previous paragraph,

$$\Pi \circ \mathcal{D}^m \geq \Pi \circ \mathcal{D}^n$$

for every $m, n \geq 0$.

Consequently

$$\mathcal{D}^m = ((\Pi \circ \mathcal{D}^m) \otimes \mathrm{Id}) \circ \Delta \leq\geq ((\Pi \circ \mathcal{D}^0) \otimes \mathrm{Id}) \circ \Delta = \mathcal{D}^0$$

for any atom/negatom a.

Let now $f : A \to B$ be a map in the category, and let $f' : A \to B$ be a map obtained from f by putting all genera to 0. Since f can be presented as :

$$h \circ f' \circ g,$$

with the previous observation, one has $f \leq\geq f'$, i.e. all maps in a hom-set that differ only in genera are both $\leq$ smaller and larger to each other. □

Going back to the equivalence relation on hom-sets in $\mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})$ where two maps are equivalent if they agree on components, disregarding the genera, the previous proposition states that the relation is precisely the $\leq\geq$. The quotient category $\mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})/\leq\geq$ is now *order* enriched by the $\leq$ order.

On the other hand, requesting that $\leq$ is an *order* enrichment in the first place is the requirement that $\leq\geq$ is equality of maps, and we have a collapse of $\mathbf{FrThFrob}(\mathcal{AT}\mathrm{yp})$ to the category $\mathbf{BFrThFrob}(\mathcal{AT}\mathrm{yp})$ we have already encountered in Chapter 4, which forgets about the information of the genera. As shown here, the category is enriched over em Poset by $\leq$.

In the chapter Apparatus, in 2.2.19 we have defined an operation $f + g$ for two maps $f, g : A \to B$ in a *-autonomous category with Mix and bimonoids. This operation appears with most of the authors we consider here, in [DP04, LS05a, Lam07, Str07, Str11], and it is noticed that this operation on maps defines an order by

$$f \leq^c g \quad \text{if} \quad f + g = g.$$

In [Str07, Str11], Straßburger shows that the $\leq^c$ order and the Führmann-Pym's $\leq$ coincide. As a consequence, our discussion translates to the order $\leq^c$ appearing in [DP04, LS05a, Lam07, Str07, FP05].

7

Conclusions and Future Work

We conclude the thesis by taking a brief review of some (but not all) of the work done in this thesis.

This dissertation presents an approach to semantics of classical propositional proofs. In Chapter 1 we have investigated concrete proof interpretations in the category **Cmp** of posets and comparisons, which is known to provide semantics for multiplicative linear logic (mll). Stepping into the realm of classical logic from models of mll was done by the well-known technique of equipping objects with a structure of a commutative monoid and a cocommutative comonoid. Naturality of these leads to a collapse to a poset, which demonstrates that the task undertaken in Chapter 1 is non-trivial. There, we have devised concrete subcategories of **Cmp** that have enabled us to, at least in one case, compute proofs, i.e. to assign them certain effectively computed relations. We have noticed that for the interpretation based on the poset of integers and two poset monoids assigned to it, a proof of a propositional sequent is an expression which was a function of a partition of the literals of the sequent and natural numbers assigned to every class in the partition. The numbers appearing in the interpretation were noticed to have to do with the history of a proof, and the computations have shown that what we assign to a proof with cuts differs from what is assigned to the proof where cuts are eliminated. This is demonstrated when we compute Church numerals and conclude that the interpretation does not respect the addition of these, showing that it has nothing to do with the Curry-Howard isomorphism. The interpretation based on integers suffers from the presence of undesirable isos not having their counterparts in logic. Moreover, the isos do not transport the monoid/comonoid structures on objects, contrary to the intended nature of these.

The structure generalizing the concrete interpretation based on integers of Chapter 1 is the structure of a Frobenius algebra. The structure is analyzed and extended to the notion of a Frobenius category freely generated by a set of generators in Chapter 3. We give a slight modification of the well-known coherence theorem for the category. The usual way maps in a freely (one object) generated Frobenius category are represented in the literature is by the means of one-dimensional topological spaces equated by their homologies. We have rephrased this in the language of relative homotopy equivalence, and

introduced isomorphic copies for every object to be able to interpret negation. Frobenius categories come with the identity-as-duality, and canonical isos between an object and its dual. Still, having an isomorphic copy of an object dedicated to its dual is better than *equating* the two, as we have argued. For the purposes of interpreting logic, another concept has proven to be very useful – the concept of thinness. This is the requirement that all the $\mathbf{1} \to \mathbf{1}$ maps obtained from the map generators are identities. Thus, a we have concluded that a map in the freely generated *thin* Frobenius category is determined by a partition of object generators into *non-empty* sets containing the same sort generators (but possibly different polarities), and an assignment of a natural number called genus to every class in the partition.

All this preparation allowed us to define the notion of a proof net for classical logic. An F-prenet was defined as a pair consisting of a sequent and a linking – a map from the monoidal unit to the interpretation of the sequent in the free Frobenius category generated by the literals of the sequent. This way of defining a (pre-) proof net for classical logic required thinness of the Frobenius algebras since they keep no track of non-identity $\mathbf{1} \to \mathbf{1}$ maps. The rest of the Chapter 3 was dedicated to the development of the theory of F-nets. After showing how to assign an F-prenet to a proof, the inverse question of correctness for F-prenets was addressed. We give a decision procedure for establishing correctness of a given (cut-free) F-prenet where the crucial role in showing its termination properties was the realization that a value akin to an Euler characteristic of a graph is an invariant of the procedure. This characteristic of an F-prenet is the sum of the number of literals of the sequent and the genera of the linking, minus the number of connected components and is read-out directly from an F-prenet.

While we have managed to define a decision procedure for the set of (CL-, cut-free-) correct F-nets, the procedure is not polynomial (and in the end of Chapter 4 we argue why), which means that F-nets as such shouldn't be thought of as an alternative syntax to proofs, which is what Girard had in mind when he defined proof nets for linear logic. They are closer to semantics and akin to $\mathbb{B}/\mathbb{N}-$nets of Lamarche/Straßburger.

Tracing an atom in a proof is quite a canonical thing to do in construction of a proof net (and not only proof nets, atomic flows are one such examples). A way to view F-prenets is to say that their linkings are these paths made by atoms in a proof, quotiened by homologies or relative homotopy equivalence of the paths.

The Chapter 4 dealt with the cut elimination on F-prenets. We have introduced a new binary connective for the formulas being cut, modified the correctness procedure for F-nets with cut to accommodate the new connective. There is a canonical way to eliminate a cut via the composition in the free thin Frobenius category. However, it had became quickly obvious that the treatment of the weakening needs to be changed, so we have formulated a new sound and complete calculus for classical logic, FL. The main nonstandard thing in the calculus is the treatment of the weakening, where an arbitrary subnet to be introduced. The number of axioms in a proof assigned to a F-prenet is this time only bounded from above by the characteristic of an F-prenet, if such proof exists.

Cut elimination on a correct F-net, even in the new calculus, is not guaranteed to yield

a correct F-net. However, it is sure to respect the soundness. Sound nets are precisely those satisfying a criterion of the Lamarche/Straßburger type. We have defined two order enrichments on F-prenets, and for the both the set of FL-correct F-nets is up-closed. Moreover, we have shown that *any* sound F-prenet can be found a larger one w.r.t. one of the orders, obtained by adding sufficient genera to get an FL correct F-net. It was argued that this going from a sound F-prenet to an FL-correct F-net, from the complexity point of view, is analogous to going from the theoremhood (validity) problem to the problem of proof construction, which partially justifies the way the correctness procedure was formulated. It was also argued that the genera, components in a linking and the number of literals in a component constitute (negatively, in the case of the number of components) *resources* offered by an F-prenet to build a proof out of them.

Sound nets can be transformed to FL-correct F-nets by adding resources. In fact, if we forget about the genera altogether, we get free categories and proof nets for which a cut elimination on correct nets yields a correct one (the full completeness holds in the induced categories). Cut elimination from the established viewpoint is consummation of resources, and we asked ourself if we can consume arbitrarily many resources by cut elimination. More precisely, we formulate a conjecture stating that the answer to the question 'is it the case that every sound net is obtained by cut-elimination on FL-correct nets ?' is positive.

In the opposite direction, we have shown that we can compute an amount of the resources which, when added to a sound F-prenet are guaranteed to yield an FL-correct F-net, and this was the concept of a bonus. We were also able to define an associative cut elimination on FL-correct F-nets for which there are also unit F-prenets, and for which a cut elimination on a pair of correct F-nets yields an FL-correct one. What we have obtained is a category whose every map is assigned to a proof, and whose composition models classical cut elimination. However, the 'global' nature of a bonus which takes into account the connectives in the object formula and the global partition of the literals into components, leaves little structure to that category, suggesting that the more general analysis in the world of Lamarche structads is in order.

In Chapter 5 we have revisited the category **Cmp** we begun with. After we have shown what it means to have a Frobenius algebra structure in the category,we have concluded that a poset in **Cmp** with a bimonoid structure defined by two poset monoids is a Frobenius algebra iff it is *-autonomous. Going back to the concrete interpretations of Chapter 1, we have showed that these provide a subcategory of **Cmp** to which there is a faithful functor from the freely generated thin Frobenius categories.

Finally, in the Chapter 6 we compared the work done to some of the those existing approaches to the topic that share common ground with what is done in this thesis. For instance, we have characterized the intersection of our approach with the one of Führmann and Pym. It was shown that assuming the partial order of Führman and Pym [FP05] on the maps in a free thin Frobenius category (or equivalently, imposing the fact that Mix is an iso, and the Frobenius equations hold in their 'classical category'), reduces the equality of maps in the category to the case where two maps are equal if the induced partitions of object generators are the same.

This works opens several possible research questions to be investigated. In no particular order :

- Our definition of the correctness procedure is formulated as a sort of a proof-net-guided search procedure and characterized by an existential sentence. Even though we give a possible motivation why this is likely to be the case in any criterion that may be devised for these nets, there is always a possibility to tweak the definition of an F-prenet that would significantly reduce the computational complexity of the correctness problem. We expect such a new notion to reflect the change in computational complexity through a new correctness criterion of the form 'for all ... exist', in the tradition of correctness criteria for a number proof nets for different logics.
- The computational complexity issues of the existing algorithms for correctness are not dealt with in this thesis. We do have strong indications of their membership to certain complexity classes and moreover, to completeness for the classes, but this remains to be worked out.
- We have demonstrated that there is a way to define cut elimination on F-nets which is associative, for which there are unit F-nets. This was done by means of bonuses. One potential thing to investigate is the possibility of alternative definitions of bonuses or the general theory of those. Also, the cut elimination we have obtained thorough bonuses does yield a category, but as we have demonstrated in several examples, the resulting category isn't even monoidal. The polycategory/structad induced by F-prenets, on the other hand, is likely to be the proper structure to study the concept, and it certainly is a topic of interest.
- In his Ph.D. thesis [McK06], McKinley extends the Führmann-Pym's 'classical categories' to the semantics of the first-order logic. A possibly interesting task is to check the robustness of the McKinley's first-order axiomatization w.r.t. the work of this thesis, i.e. to see if there is a meaningful extension of our work here to the first-order case.
- We have shown that the Frobenius equation can be formulated in the context of a *-autonomous category where $\otimes \neq \parr$, mediated by Mix. Perhaps the most important open research question is the further investigation of these categorical structures, their concrete examples, and their utilization in providing more suitable semantics.

A

Proofs of Certain Statements

This appendix contains the proof of

Proposition 3.1.18 A graph G representing a map $f : [m] \to [n]$ in **FrFrob** is of the same *homotopy type relative to $[m] + [n]$* to a topological graph in normal form.

Démonstration. Let G be a topological graph whose set of vertices V is $[m] + [n] + [k]$, and A the family of functions $a : \{0, 1\} \to V$, such that G is the space $V \uplus (A \times [0, 1])$ where for every $a \in A$ $(a, 0)$ is equated with $a(0)$ and $(a, 1)$ with $a(1)$.

- A subset CC of V defines a connected component in G if it has the property that it is a maximal set such that for every two $c_1, c_2 \in CC$, there are $b_1, \ldots, b_k$ for which $(c_1, b_1), (b_i, b_{i+1}), (b_k, c_2)$ are endpoints of some edges in G.
  For every connected component CC, we define the number of cycles in CC as the maximal number of edges with endpoints in CC that can be removed from G so that CC remains connected.
- For a connected component CC of G and an edge a with endpoints v_1, v_2, we define

$$G[v_1 \leftarrow_a v_2]$$

  to be the graph such that

  1. its set of vertices is $V \setminus \{v_2\}$
  2. its the smallest set of functions A' such that
     (a) if $a_i \in A$ and $a_i(0), a_i(1) \neq v_2$, $a_i \in A'$
     (b) for a map $a_i \in A \setminus \{a\}$ for which $a_i(t) = v_2$, $a_i(1 - t) \neq v_2$, there is a corresponding a'_i in A' such that $a'_i(t) = v_1$, $a'_i(1 - t) = a_i(1 - t)$
     (c) for a map $a_i \in A \setminus \{a\}$ for which $a_i(t) = a_i(1 - t) = v_2$, there is a corresponding a'_i in A' such that $a'_i(t) = a'_i(1 - t) = v_1$.

  The aim here is to simulate the continuous deformation of the graph where v_2 is moved towards v_1 along $a \times [0, 1]$ eventually equating the two.

Lemma A.0.1. *Let G be a topological graph representing a map $f : [m] \to [n]$ in* **FrFrob**, *with the set of vertices $V = [m] + [n] + [k]$, and let v_1, v_2 be vertices in $CC \cap [k]$ for a connected component CC of G. Then, the transformation*

$$G \mapsto G[v_1 \leftarrow_a v_2]$$

preserves the connected components on $[m] + [n]$, i.e. $x, y \in [m] \uplus [n]$ are in a same connected component in G iff they are in a same connected component in $G[v_1 \leftarrow_a v_2]$.

Also, the transformation preserves cycles, i.e. CC and the corresponding connected component in $G[v_1 \leftarrow_a v_2]$ have the same number of cycles.

Lemma A.0.2. *Let G be a topological graph representing a map $f : [m] \to [n]$ in* **FrFrob**, *with the set of vertices $V = [m] + [n] + [k]$, and let v_1, v_2 be vertices in $CC \cap [k]$ for a connected component CC of G. Then, there is an iso*

$$h : G \to G[v_1 \leftarrow_a v_2]$$

in **hTop**, *i.e. G and $G[v_1 \leftarrow_a v_2]$ are homotopy equivalent.*

– Finally, for the G representing $f : [m] \to [n]$ in **FrFrob**, we define $D(G)$ to be the graph obtained by introduction of $m + n$ 'clones' of vertices from $[m] + [n]$ in G, connecting a clone $D(v)$ with its original $v \in [m] + [n]$ by an edge, and redirecting all of the edges with v as an endpoint to its clone $D(v)$.
More formally, set of vertices of $D(G)$ is $V \uplus D([m] + [n])$, for a bijection $D : [m] + [n] \to D([m] + [n])$ (notice the abuse of notation). The set of edges of $D(G)$ is defined using the family of functions $A_{D(G)} : \{0, 1\} \to V \uplus D([m] + [n])$. $A_{D(G)}$ is the smallest set such that
  1. $a(0) = v$, $a(1) = D(v)$ is in $A_{D(G)}$, for $v \in [m] + [n]$;
  2. there is an embedding $D : A_G \to A_{D(G)}$. If $a(i) = v \in [m] + [n]$, (for $i = 0, 1$), then $D(a)(i) = D(v)$. Otherwise, $D(a)(i) = a(i)$.

Having the previous two lemmas, we have first that the transformation $G \mapsto D(G)$ is such that

$$D(G)[v_i \leftarrow D(v_i)]_{v_i \in V} = G$$

i.e. result of applying all of the $D(G)[- \leftarrow -]$ transformations on $D(G)$ for every pair consisting of a vertex from $[m] + [n]$ and its clone, gives back the original graph. By Lemma A.0.2, G and $D(G)$ are of the same homotopy type, and by Lemma A.0.1, components of G are subcomponents of $D(G)$, and they share the same cycles.

Now, as $D(G)$ with vertices $[m]+[k+m+n]+[n]$ represents the same map $f : [m] \to [n]$ as G, one can apply the $D(G)[w_1 \leftarrow_a w_2]$ transformation for $w_1, w_2 \in [k + m + n]$ and $a \in A_{D(G)}$ until no longer possible. By the previous lemmas, the resulting graph will be of the same homotpy type relative to $[m]+[n]$ as G. Further more, Lemma A.0.1 guarantees that the result will be independent of the strategy of choosing the edges a. It is the normal form of the graph G w.r.t the transformation, where every component is a star graph $[m'] + [1] + [n']$, $[m'] \subseteq [m], [n'] \subseteq [n]$, and genus of the component is determined by the number of loops sharing the element of $[1]$ as a base point. □

Proof of Lemma A.0.1: Let $x, y \in ([m] \uplus [n]) \cap CC$. If $x, y \notin CC$, the component both x, y belong to is unaffected by the transformation, so they remain in the same component. So, let $b_1, \ldots, b_r, v_1 = b_{r+1}, b_{r+2}, \ldots, b_s, v_2 = b_{s+1}, b_{s+2}, \ldots, b_k$ be vertices for which $(x, b_1), (b_i, b_{i+1}), (b_k, y)$ are edges of G. Then, $b_1, \ldots, b_r, v_1 = b_{r+1} = b_s, v_2 = b_{s+1}, b_{s+2}, \ldots, b_k$ are vertices of $G[v_1 \leftarrow_a v_2]$ such that (x, b_1), (b_i, b_{i+1}), for $i < r$ and $i > s$, and (b_k, y) are edges of $G[v_1 \leftarrow_a v_2]$. Thus, x, y are in the same component in the new graph.

Let $a_1 \times [0,1], \ldots, a_l \times [0,1]$ be the maximal set of edges of G with endpoints in CC that can be removed so vertices of CC remain connected. If neither of a_i is a, the same set $a_1 \times [0,1], \ldots, a_l \times [0,1]$ is the maximal set of edges with vertices in the connected component corresponding to CC in $G[v_1 \leftarrow_a v_2]$ whose removal leaves the component connected.

If $a = a_i$, then there exist a sequence $v_1 = d_1, d_2, \ldots, d_k = v_2$ of vertices such that (d_i, d_{i+1}) are endpoints of edges in G. Then, the edge with endpoints (d_j, d_{j+1}) for some j can replace a in the maximal set of edges that can be removed while saving the connected component. This reduces the problem to the previous case $a_i \neq a$, and we conclude that the number of cycles is preserved by $\leftarrow_a$.

Proof of Lemma A.0.2: Without loss of generality, assume $v_1 = a(0), v_2 = a(1)$.

Define $f : G \to G[v_1 \leftarrow_a v_2]$ to be the identity on $V \setminus \{v_2\}$ and on any edge $a_i \times [0,1]$ whose neither endpoint is v_2. For an edge $a_i \times [0,1]$ whose endpoint is v_2, one has $f(a_i, t) = (a_i', t)$, where a_i' is the map in A' assigned to a_i. Finally, f is constant on $a \times [0,1]$, i.e. $f(a, t) = v_1$.

The map $g : G[v_1 \leftarrow_a v_2] \to G$ is defined as the inclusion map on vertices, $g(a_i \times [0,1])$ is the identity on the edges that coincide in the two graphs. On those that don't, let a_i, a_i' be the corresponding maps of A, A'.

1. if $a_i(1) = v_2 \neq a_i(0)$:
$$g(a_i', t) = (a_i, 2t), \text{ for } t \leq 1/2,$$
and
$$g(a_i', t) = (a, 2 - 2t), \text{ for } t > 1/2;$$
2. if $a_i(0) = v_2 \neq a_i(1)$:
$$g(a_i', t) = (a_i, 2t), \text{ for } t \leq 1/2,$$
and
$$g(a_i', t) = (a, 2t - 1), \text{ for } t > 1/2;$$
3. if $a_i(0) = v_2 = a_i(1)$:
$$g(a_i', t) = (a_i, 2t), \text{ for } t \leq 1/2,$$
and
$$g(a_i', t) = (a, 2 - 2t), \text{ for } t > 1/2.$$

There is some easy checking involved into verify that the maps f, g are continuous. Homotopy equivalence of the two spaces is now verified by defining H_1 between Id_G and $g \circ f$ on an edge $a_i \times [0,1]$:

1. if $G, G[v_1 \leftarrow_a v_2]$ coincide on $a_i \times [0,1]$,
$$H_1((a_i, k), t) = (a_i, k);$$
2. if $a_i(1) = v_2 \neq a_i(0)$
$$H_1((a_i, k), t) = (a_i, k + t \cdot k), \text{ for } (1+t)k \leq 1,$$
and
$$H_1((a_i, k), t) = (a, 2 - k + t \cdot k), \text{ for } (1+t)k > 1;$$
3. if $a_i(0) = v_2 \neq a_i(1)$
$$H_1((a_i, k), t) = (a_i, k + t \cdot k), \text{ for } (1+t)k \leq 1,$$
and
$$H_1((a_i, k), t) = (a, k + t \cdot k - 1), \text{ for } (1+t)k > 1;$$
4. if $a_i(0) = v_2 = a_i(1)$
$$H_1((a_i, k), t) = (a_i, k + t \cdot k), \text{ for } (1+t)k \leq 1,$$
and
$$H_1((a_i, k), t) = (a, k + t \cdot k - 1), \text{ for } (1+t)k > 1.$$

Analogously, H_2 between $\mathrm{Id}_{G[v_1 \leftarrow_a v_2]}$ and $f \circ g$ on an edge $a'_i \times [0,1]$:

1. if $G, G[v_1 \leftarrow_a v_2]$ coincide on $a'_i \times [0,1]$,
$$H_2((a'_i, k), t) = (a'_i, k);$$
2. otherwise (a_i, a'_i are the corresponding maps of A, A')
$$H_2((a'_i, k), t) = (a'_i, k + t \cdot k), \text{ for } (1+t)k \leq 1,$$
and
$$H_2((a'_i, k), t) = v_2, \text{ for } (1+t)k > 1.$$

B

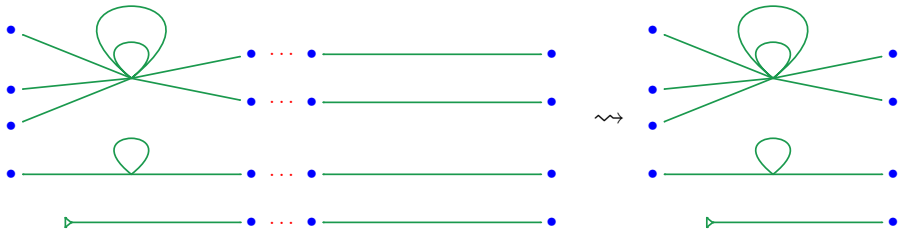
Computing Composition in FrFrob

Computing Composition

Composing maps in a free Frobenius category, as we have seen, amounts to glueing of two spaces along their boundaries (also, cf. [Dij89, Koc04]). Composition of maps the free Frobenius category over a Frobenius algebra is the class of homotopy equivalent spaces obtained by glueing of two representatives of their classes. While this fully defines the composition, there is need to provide an effective computation algorithm of these compositions, as they will facilitate interpretations of proofs where we always want things computed.

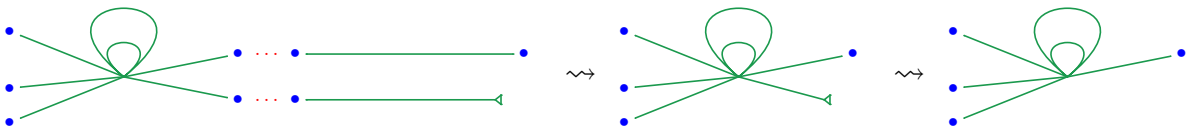
Let us consider first several instructive examples.

Example B.0.3. Composing with identity behaves as expected and is illustrated below.



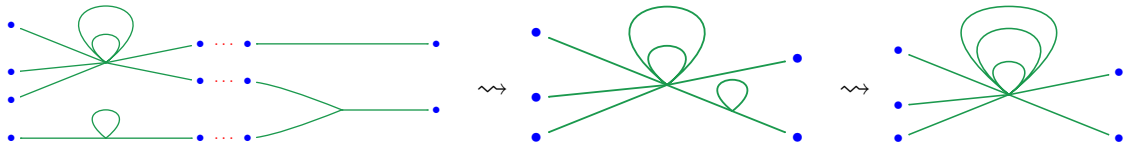
(B.1)

Unit law in a monoid is depicted. Homotopy equivalence to the identity space is obvious. In general, composition with the monoid unit map 'erases' endpoints.



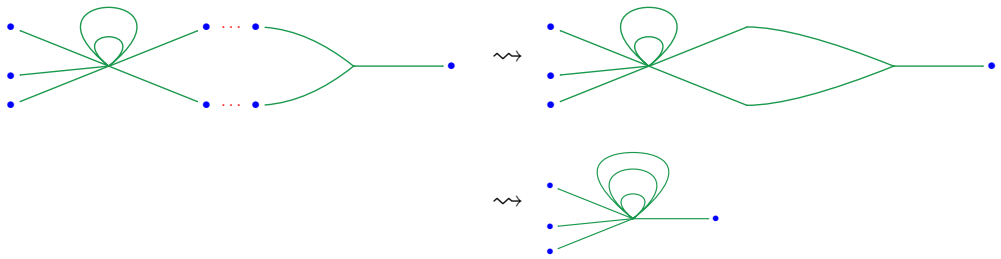
(B.2)

A space consisting of two disconnected components composed with monoid multiplication map results in a single connected component whose genus is sum of genera of the two components. The dual case is analogous. In a general, when two components are connected by composition, genera are added. This is easily visualized through homotopy type-preserving shrinkings and stretchings that move around the loops to a same base point.



(B.3)

Another feature of composition arises when composing a single connected component with the monoid multiplication map. This time, shrinking and stretching transformation shows that the resulting map is a homotopy type of the first space with a single loop added. In the general case, composition can create new loops through the same mechanism.



(B.4)

Ingredients

Let us look now into defining a method of effective computation of composition in the category **FrFrob**($\mathcal{AT}$yp). To simplify the presentation, we restrict our attention to the case of a single generator – the category **FrFrob**. Composition of maps in the general case, as already explained, reduces to the case of a single generator since components of a map always compose with components where domain/codomain object generators are of the same sort.

The real task in trying to give a computation of maps in **FrFrob** is to give a computation of connected components of the resulting map and the genera assigned. Given two topological graphs $G : [m] \to [n]$ and $F : [n] \to [k]$, representing a map in **FrFrob**, if G or F contain a connected component with no vertices in $[n]$, the graph after the composition will contain the component. It will not be altered by the composition, so we assume that the maps we compose do not have these factors.

Alternatively, an algorithm for computation of $F \circ G$ in the general case can be obtained from the one for the case where G contains no $G' : [m'] \to [0]$ connected component F contains no $[0] \to [n']$ connected component, by first stripping those from G, F, computing the composition on the remaining graphs, and the subsequent reintroduction of the components by $\uplus$.

So, let $f : [m] \to [n]$ be a map of **FrFrob** and $\mathcal{C}omp_f$ be the set of connected components of (any) graph that represents f. This is an invariant since any homotopy is an iso on these components.

Let **SplEqv** be the category whose objects are finite sets, and where a map $[m] \to [n]$ is an equivalence relation (or partition) on disjoint union of the two sets $[m] + [n]$. Some authors refer to these categories as *split equivalences* (cf. [DP09]). Maps are composed as follows : given $f : [m] \to [n]$ and $g : [n] \to [q]$ take the equivalence relation on $[m]+[n]+[q]$ generated by f, g and restrict it to $[m] + [q]$.

It should be obvious that

1. There is a functor
$$\mathcal{C}omp : \mathbf{FrFrob} \to \mathbf{SplEqv}$$
that takes a map $f : [m] \to [n]$, to a partition $\mathcal{C}omp_f$ of the set $[m] + [n]$. Two elements of $[m] + [n]$ are in the same class in $\mathcal{C}omp_f$ if they are embedded into a same connected component in (any) graph representing f.
2. Also, it should be obvious that the information missing to a map in **SplEqv** to recover a map of **FrFrob** can be seen as a function
$$\mathcal{G}en_f : \mathcal{C}omp_f \to \mathbb{N}$$
that associates to every component its genus.

Notice a slight abuse of notation in the previous definition of functor $\mathcal{C}omp$. Given a map f in **FrFrob**, we use $\mathcal{C}omp_f$ to denote the set of connected components of a representative graph for f in **FrFrob**, but also to denote the image of f under the functor.

The context where we use the notation will always be clear and there will not be any confusion.

Let $|\mathcal{C}omp_f|$ be the set $\{1, 2, \ldots, n\}$, of the same number of elements as the set (partition) $\mathcal{C}omp_f$, with $|-|$ being the bijection of the sets. *For the purposes of easier presentation alone,* we assume the standard linear order on the set. Now, it is easy to see that every map $\mathcal{C}omp_f : [m] \to [n]$ in **SplEqv** can be represented as a diagram

$$[m] \longrightarrow |\mathcal{C}omp_f| \longrightarrow [n].$$

in **FinRel**, category of finite sets and relations. Here, the first relation is a function and the second one an inverse of a function.

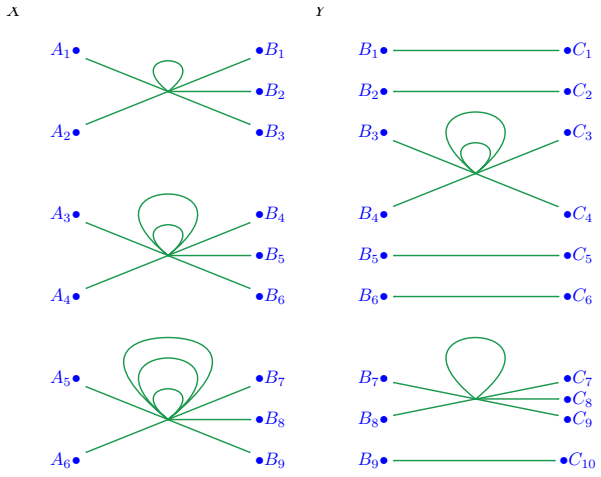
Let **Matrix**(**Q**) be the category of matrices over a semiring Q. It has linearly ordered finite sets $[m]$ as objects, a map $[m] \to [n]$ is a $m \times n$ matrix over the semiring Q, while the composition is standard composition of matrices over a semiring. It is easy to see that **Matrix**(**Q**) in the special case of the semiring being our Boolean semiring $\mathbb{B}$ is isomorphic to the category **FinRel**. Therefore, the two relations of the diagram

$$[m] \longrightarrow |\mathcal{C}omp_f| \longrightarrow [n]$$

are understood as matrices in **Matrix**($\mathbb{B}$).

We give an example to the concepts we define here.

Example B.0.4. Let the maps $x : A \to B$ and $y : B \to C$, with $A = A_1 \otimes \cdots \otimes A_6$, $B = B_1 \otimes \cdots \otimes B_9$ and $C = C_1 \otimes \cdots \otimes C_{10}$ be given as in the image below



(B.5)

Then, x and y are given by

$$[6] \xrightarrow{x_d} |\mathcal{C}omp_f| = [3] \xrightarrow{x_c} [9] \qquad \text{and} \qquad [9] \xrightarrow{y_d} |\mathcal{C}omp_g| = [7] \xrightarrow{y_c} [10].$$

With

$$x_d = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; x_c = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and}$$

$$\mathsf{G}enus_x = \{(1,1),(2,2),(3,3)\}.$$

$$y_d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; y_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$\mathsf{G}enus_y = \{(1,0),(2,0),(3,2),(4,0),(5,0),(6,1),(7,0)\}.$$

By composition / glueing of spaces along the endpoints, as in the example (B.4), one encounters situations where the base points / elements of $|\mathcal{C}omp_f|$ are connected by several paths, rather than one. Therefore, to keep track of these situations, we view maps in $[m] \longrightarrow\!\!\!\!\!\rightarrow |\mathcal{C}omp_f| \longrightarrow\!\!\!\!\!\rightarrow [n]$ as maps in **Matrix**$(\mathbb{N})$.

There is a morphism $\lceil - \rceil : \mathbb{N} \to \mathbb{B}$ which is the non-zero morphism in the category **SemiRing** form the standard semiring $(\mathbb{N}, +, \cdot)$ to the Boolean lattice $(\{0,1\}, \vee, \wedge)$ seen as a semiring, given by

$$\lceil 0 \rceil = 0, \qquad\qquad \lceil s \rceil = 1, \text{ for } s \geq 1.$$

The semiring homomorphism $\lceil - \rceil : \mathbb{N} \to \mathbb{B}$ can be lifted to the 'forgetful' functor $\lceil - \rceil : \mathbf{Matrix}(\mathbb{N}) \to \mathbf{Matrix}(\mathbb{B})$ which is identity on objects and acts on maps/matrices by acting on each entry of a matrix.

Another situation that happens when maps are composed is the identification of the base points which become connected by composition/concatenation. In the example (B.4), all four base points are identified and one ends up with a single endpoint. For this, we have to compute reflexive-symmetric-transitive closures of relations/maps in **Matrix**$(\mathbb{B})$.

A map $h : [m] \to [n]$ in **FinRel** (or, equivalently, $H_{m \times n}$ in **Matrix**$(\mathbb{B})$), determines a map in **SplEqv** as the reflexive-symmetric-transitive closure of h which can be computed

by considering the subcategory of endomorphisms in **Matrix**($\mathbb{B}$). In that category one considers an extension of the map $H_{m\times n}$,

$$ext(H_{m\times n}) = \left(\begin{array}{c|c} 0 & H_{m,n} \\ \hline 0 & 0 \end{array}\right)_{(m+n)\times(m+n)}$$

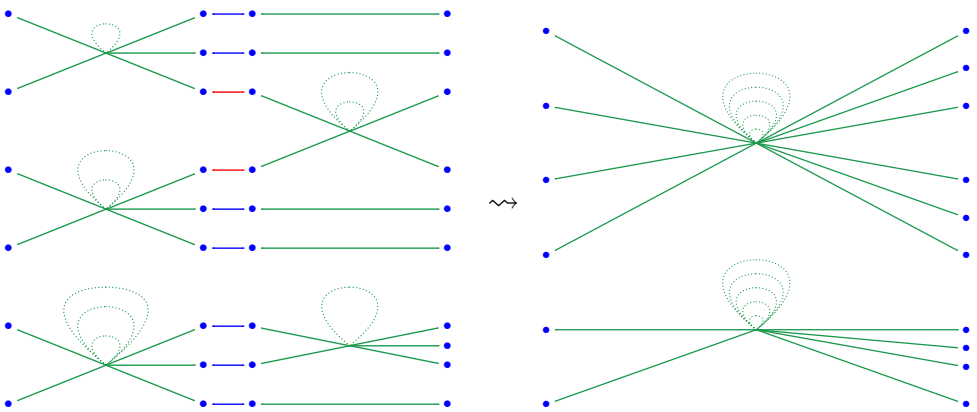
and applies to it iteratively the functor Φ :

$$\Phi((Z)_{k\times k}) := (Z + I + Z^T) \cdot \ldots \cdot (Z + I + Z^T)$$

which sends a matrix into the fixpoint of the composition above (whose length is clearly dominated by k^2). Here, I is the unit $k \times k$ matrix, $(-)^T$ stands for the transposition transformation of a matrix, and $+$ is the additive operation on matrices inherited from the additive operation in the underlying semiring $\mathbb{B}$. The closure of h in **SplEqv** is read-out from $\Phi(ext(H_{m\times n}))$.

Computation

Computation of composition consists of two parts, computation of the connected components and the computation of the assigned genera.



(B.6)

To compute the connected components, disregarding the genera, one sees that the continuos, homotopy type preserving deformation which contracts connected base points into a single one is the geometric dual to computing the symmetric-reflexive-transitive closure of the compositions of relations.

Computing genera requires a bit more work.

So, let the the two maps $f : [k] \to [m]$ and $g : [m] \to [n]$ be given by

$$[k] \xrightarrow{f_d} |\mathcal{C}omp_f| \xrightarrow{f_c} [m], \qquad [m] \xrightarrow{g_d} |\mathcal{C}omp_g| \xrightarrow{g_c} [n],$$

and $\mathcal{G}en_f$, $\mathcal{G}en_g$;

Step 1 : First, compute the composite in **Matrix**($\mathbb{B}$) :

$$|\mathcal{C}omp_f| \xrightarrow{f_c} [m] \xrightarrow{g_d} |\mathcal{C}omp_g|.$$

The composite is denoted by $\mathcal{C}omp_{g \bullet f}$.

Step 2 : Perform the symmetric-reflexive-transitive closure construction described above to obtain $\Phi(ext(\mathcal{C}omp_{g \bullet f}))$ and read-out $\mathcal{C}omp_{g \circ f}$ as distinct equivalence classes.
For a $r \in |\mathcal{C}omp_{f \circ g}|$, let $\mathcal{C}omp^r_f$ be the subset of $\mathcal{C}omp_f$ and $\mathcal{C}omp^r_g$ subset of $\mathcal{C}omp_g$ such that r is obtained by identifying elements of $\mathcal{C}omp^r_f$ and $\mathcal{C}omp^r_g$. In other words, r stands for an equivalence class in the reflexive-symmetric-transitive closure of

$$|\mathcal{C}omp_f| \longrightarrow [m] \longrightarrow |\mathcal{C}omp_g|.$$

Define $(y \circ x)_d : [k] \to \mathcal{C}omp_{g \circ f}$ as $(y \circ x)_d(i) = r$ if $f_d(i) \in \mathcal{C}omp^r_f$. Also, define $(y \circ x)_c : \mathcal{C}omp_{g \circ f} \to [n]$ as $(y \circ x)_c(r) = i$ if there exist $k \in \mathcal{C}omp^r_g$ such that $(k, i) \in g_c$.
At this point we have managed to calculate the connected components of a composition map

$$[k] \xrightarrow{(y \circ x)_d} |\mathcal{C}omp_{g \circ f}| \xrightarrow{(y \circ x)_c} [n].$$

In the our running Example B.5, disregarding the genera as in (B.6), we have $\mathcal{C}omp_{y \bullet x} = x_c \cdot y_d$ is the 3×7 matrix obtained by the multiplication of matrices / composition in **Matrix**($\mathbb{B}$),

$$\mathcal{C}omp_{y \bullet x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

while the reflexive-symmetric-transitive closure is read-out in

$$\Phi(ext(\mathcal{C}omp_{y \bullet x}))_{10,10} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

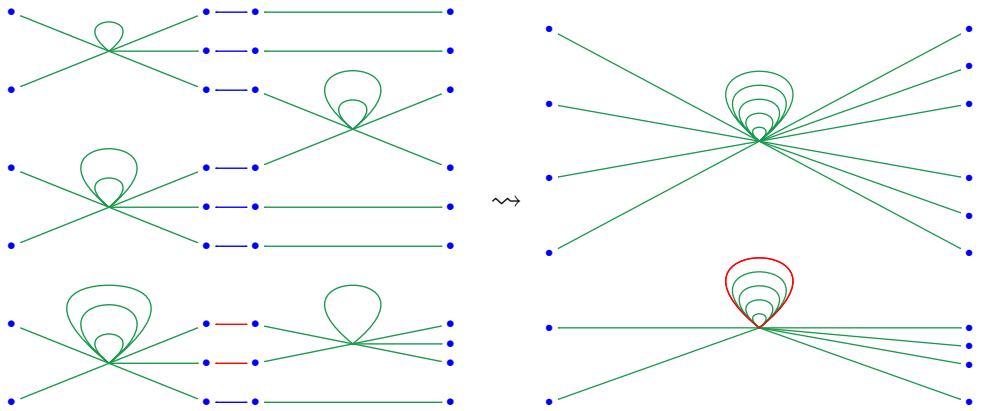
The number of mutually distinct rows in the matrix above defines the number of connected components, $|\mathcal{C}omp_{y\circ x}|$, while the maps in

$$[6] \xrightarrow{(y\circ x)_d} |\mathcal{C}omp_{y\circ x}| = [3] \xrightarrow{(y\circ x)_c} [10]$$

are read-out from the table in the following way : the function $[6] \xrightarrow{(y\circ x)_d} |\mathcal{C}omp_{y\circ x}|$ maps i to k, if the i-th row is the k-th one if we do not count identical rows. The co-function $|\mathcal{C}omp_{y\circ x}| = [3] \xrightarrow{(y\circ x)_c} [10]$ is the relation where k is related to all those j, where j-th row in the matrix is the k-th row, not counting identical rows.

The definition results in the connected components depicted in (B.6).

To compute the genera, we may apply the same principle of continuous contractions of paths between two base points that transform several connected base points into single one, as done so far. This transformation consequently adds-up all of the existing loops over connected base points to be over the same point. Also, new loops emerge – if there is more than one path between base points, one path is used to for continuos merging of the base points, while the others will be converted by the transformation into loops. Thus, the number of newly created loops is the number of 'nonessential paths', i.e. the maximal number of paths between base points that can be removed so that those base points that were connected, remain connected still.



(B.7)

In the example here, by removing any of the lower two paths with red connections, the classes of connected base points would remain the same, so the composition creates one loop, while the others are added. Hence,

Step 3 : Compute in **Matrix**($\mathbb{N}$)

$$|\mathcal{C}omp_f| \xrightarrow{f_c} [m] \xrightarrow{g_d} |\mathcal{C}omp_g|.$$

The composition is denoted by $g \diamond f$.

Step 4 : The standard total order $\leq$ in $\mathbb{N}$ lifts to an order enrichment in the category **Matrix**($\mathbb{N}$) by having $A_{m\times n} \leq B_{m\times n}$ if $A_{i,j} \leq B_{i,j}$ for all i, j. Now, define

$$CreatedGen_{|\mathcal{C}omp_f|\times|\mathcal{C}omp_g|}$$

to be a maximum (one out of possibly several) of

$$\{\delta \in Hom_{\mathbf{Matrix}(\mathbb{N})}(|\mathcal{C}omp_f|, |\mathcal{C}omp_g|) \mid \Phi(ext(\lceil \mathcal{C}omp_{g\diamond f}\rceil)) = \Phi(ext(\lceil \mathcal{C}omp_{g\diamond f} - \delta\rceil))\}.$$

This maximum represents a maximal amount of edges of the composition graph that can be removed without affecting the classes of connected base points. Notice the use of $\lceil - \rceil$ functor for going from **Matrix**($\mathbb{N}$) to **Matrix**($\mathbb{B}$) to be able to talk about matters of connected base points.

$y \diamond x = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$ while one can select $CreatedGen_{|\mathcal{C}omp_x|\times|\mathcal{C}omp_y|}$ to be

$$CreatedGen_{|\mathcal{C}omp_x|\times|\mathcal{C}omp_y|} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

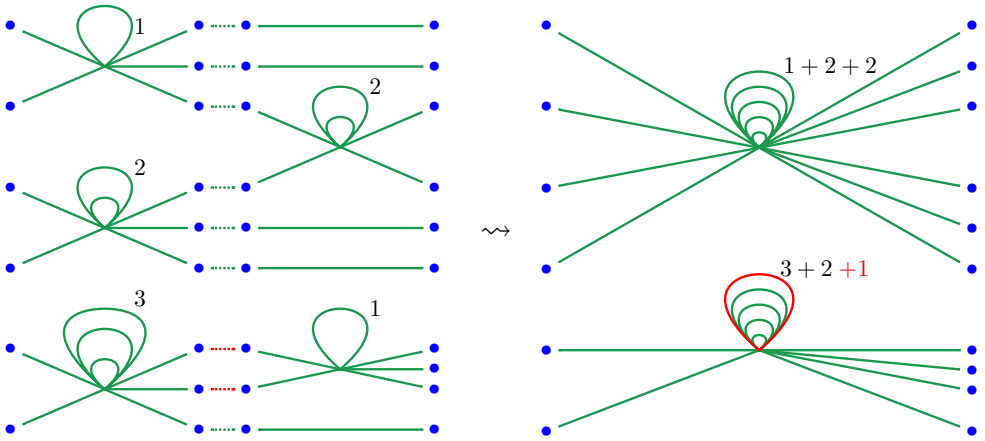
suggesting that the number of loops created by the composition is 1.

Step 5 : As before, for a $r \in |\mathcal{C}omp_{f\circ g}|$, let $\mathcal{C}omp_f^r$ be the subset of $\mathcal{C}omp_f$ and $\mathcal{C}omp_g^r$ subset of $\mathcal{C}omp_g$ such that r is obtained by identifying elements of $\mathcal{C}omp_f^r$ and $\mathcal{C}omp_g^r$. In other words, r stands for an equivalence class in the reflexive-symmetric-transitive closure of

$$|\mathcal{C}omp_f| \longrightarrow [m] \longrightarrow |\mathcal{C}omp_g|.$$

Now,

$$\begin{aligned} \mathcal{G}en_{g\circ f}(r) &= \sum_{s\in\mathcal{C}omp_f^r} \mathcal{G}en_f(s) \\ &+ \sum_{t\in\mathcal{C}omp_g^r} \mathcal{G}en_g(t) \\ &+ \sum_{\substack{u\in\mathcal{C}omp_f^r \\ v\in\mathcal{C}omp_g^r}} CreatedGen_{u,v}. \end{aligned}$$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AUTORISATION DE SOUTENANCE DE THESE
DU DOCTORAT DE L'INSTITUT NATIONAL
POLYTECHNIQUE DE LORRAINE

o0o

VU LES RAPPORTS ETABLIS PAR :
Monsieur Yves LAFONT, Professeur, Université de Marseille

Monsieur Lutz STRASSBURGER, Chargé de Recherche, INRIA Saclay, Palaiseau

Le Président de l'Institut National Polytechnique de Lorraine, autorise :

Monsieur NOVAKOVIC Novak

à soutenir devant un jury de l'INSTITUT NATIONAL POLYTECHNIQUE DE LORRAINE,
une thèse intitulée :

"Sémantique algébrique des ressources pour la logique classique. "

en vue de l'obtention du titre de :

DOCTEUR DE L'INSTITUT NATIONAL POLYTECHNIQUE DE LORRAINE

Intitulé du doctorat : « **Informatique** »

Fait à Vandoeuvre, le 25 octobre 2011
Le Président de l'I.N.P.L.,
F. LAURENT

Pour le Président par délégation
Le Directeur Général des Services
de l'INPL,

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Résumé

Le thème général de cette thèse est l'exploitation de l'interaction entre la sémantique dénotationnelle et la syntaxe. Des sémantiques satisfaisantes ont été découvertes pour les preuves en logique intuitionniste et linéaire, mais dans le cas de la logique classique, la solution du problème est connue pour être particulièrement difficile. Ce travail commence par l'étude d'une interprétation concrète des preuves classiques dans la catégorie des ensembles ordonnés et bimodules, qui mène à l'extraction d'invariants significatifs. Suit une généralisation de cette sémantique concrète, soit l'interprétation des preuves classiques dans une catégorie compacte fermée où chaque objet est doté d'une structure d'algèbre de Frobenius. Ceci nous mène à une définition de réseaux de démonstrations pour la logique classique. Le concept de correction, l'élimination des coupures et le problème de la "full completeness" sont abordés au moyen d'un enrichissement naturel dans les ordres sur la catégorie de Frobenius, produisant une catégorie pour lélimination des coupures et un concept de ressources pour la logique classique. Revenant sur notre première sémantique concrète, nous montrons que nous avons une représentation fidèle de la catégorie de Frobenius dans la catégorie des ensembles ordonnés et bimodules.

Mots-clés: sémantique, théorie de la démonstration, la logique classique, logique catégorique, la sémantique catégorique, l'algèbre de Frobenius.

Abstract

The general theme of this thesis is the exploitation of the fruitful interaction between denotational semantics and syntax. Satisfying semantics have been discovered for proofs in intuitionistic and certain linear logics, but for the classical case, solving the problem is notoriously difficult. This work begins with investigations of concrete interpretations of classical proofs in the category of posets and bimodules, resulting in the definition of meaningful invariants of proofs. Then, generalizing this concrete semantics, classical proofs are interpreted in a free symmetric compact closed category where each object is endowed with the structure of a Frobenius algebra. The generalization paves a way for a theory of proof nets for classical proofs. Correctness, cut elimination and the issue of full completeness are addressed through natural order enrichments defined on the Frobenius category, yielding a category with cut elimination and a concept of resources in classical logic. Revisiting our initial concrete semantics, we show we have a faithful representation of the Frobenius category in the category of posets and bimodules.

Keywords: semantics, proof theory, classical logic, categorical logic, categorical semantics, Frobenius algebra.