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Contributions to the modeling, the analysis and the control of networked hybrid dynamical systems

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"A healthy man owes to the sick all that he can do for them,
An educated man owes to the ignorant all that he can do for them,
A free man owes to the world's slaves all that he can do for them,
And what is to be done is more, much more, than good words."
"Moral Indebtedness" - Orson Welles, Free World, October 1943

Summary

From smart grids and social and biological networks to fleets of drones, networked systems pervade our daily life. In each one of these systems, we can identify some recurring basic features: elementary dynamical units, called agents, locally mutually interact via a graph topology using local information, and give rise to a globally coherent and collective behavior. In numerous instances, networked systems exhibit continuous-time dynamics that are subject to sudden, instantaneous changes that may naturally arise, as for neurons, or may be enforced by design, as in the case of smart grids, where the control actions happen via switching devices. In both scenarios, it is effective to model the overall system as a so-called *networked hybrid dynamical system*.

The objective of this thesis revolves around demonstrating the strengths of hybrid theoretical tools to model and control in a distributed way important classes of networked systems. We first show how hybrid techniques can be used to model the evolution of opinions in a social network where the interactions between individuals depend on both their past and current opinions. This is a reasonable assumption when every individual knows the identity of the other members of the network. We thus present a model of opinion dynamics where each agent has active or inactive pairwise interactions depending on auxiliary state variables filtering the instantaneous opinions, thereby taking their past values into account. When an interaction is (de)activated, a jump occurs, leading to a networked hybrid dynamical model. The stability properties of this hybrid networked system are then analyzed and we establish that the opinions of the agents converge to local agreements/clusters as time grows, which is typical in the opinion dynamics literature.

In the second case study, we demonstrate how hybrid techniques can be used to overcome fundamental limitations of continuous-time coupling to synchronize a network of oscillators. In particular, we envision the engineering scenario where the goal is to design the coupling rules for heterogeneous oscillators to globally and uniformly synchronize to a common phase. Each oscillator has its own time-varying natural frequency taking values in a compact set. This problem is historically addressed in the literature by resorting to the well-known Kuramoto model whose original formulation comes from biological and physical networks. However, the Kuramoto model exhibits major shortcomings for engineering applications, namely the lack of uniform synchronization and phase-locking outside the synchronization set. To overcome these challenges, a communication layer is set in place to allow the oscillators to exchange synchronizing coupling actions through a tree-like leaderless network. The proposed couplings can recover locally the behavior of Kuramoto oscillators while ensuring the uniform global practical or asymptotic stability of the synchronization set, which is impossible with Kuramoto models. We further show that the synchronization set can be made uniformly globally prescribed finite-time stable by selecting the coupling function to be discontinuous at the origin. The effectiveness of the approach is illustrated in simulations where we apply our synchronizing hybrid coupling rules to models of power grids previously used in the literature. Novel mathematical tools on non-pathological functions and set-valued Lie derivatives are developed to carry out the stability analysis.

It appears that this last set of mathematical tools has broader applicability than the considered hybrid network of oscillators. We thus finally exploit these tools to analyze the stability properties of Lur'e systems with piecewise continuous nonlinearities, thanks to the underlying similarities between Lur'e systems and the continuous-time dynamics describing the synchronizing oscillators. We first extend a result from the literature by establishing the global asymptotic stability of the origin under more general sector conditions. We then present criteria under which Lur'e systems with piecewise continuous nonlinearities enjoy output and state finite-time stability properties. Moreover, we provide algebraic proofs of the results, which represents a novelty by itself. We show the relevance of the tools provided, by studying the stability properties of two engineering systems of known interest: cellular neural networks and mechanical systems affected by friction.

Résumé

Des réseaux intelligents aux flottes de drones, en passant par les réseaux sociaux et les réseaux biologiques, les systèmes en réseau sont omniprésents dans notre vie quotidienne. Dans chacun de ces systèmes, nous pouvons identifier certaines caractéristiques de base récurrentes : des unités dynamiques élémentaires, appelées agents, interagissent localement et mutuellement via une topologie de graphe utilisant des informations locales, et donnent lieu à un comportement global collectif cohérent. Dans de nombreux cas, les systèmes en réseau présentent une dynamique en temps continu soumise à des changements soudains et instantanés qui peuvent survenir naturellement, comme dans le cas des neurones, ou être imposés par la synthèse, comme dans le cas des réseaux intelligents, où les actions de contrôle se font par le biais de dispositifs de commutation. Dans les deux scénarios, il est natural de modéliser le système global comme un système dynamique hybride en réseau.

L'objectif de cette thèse est de démontrer la pertinence des outils théoriques hybrides pour modéliser et contrôler de manière distribuée des classes importantes de systèmes en réseau. Nous montrons d'abord comment les techniques hybrides peuvent être utilisées pour modéliser l'évolution des opinions dans un réseau social où les interactions entre les individus dépendent à la fois de leurs opinions passées et actuelles. Il s'agit d'une hypothèse raisonnable lorsque chaque individu connaît l'identité des autres membres du réseau. Nous présentons donc un modèle de dynamique d'opinions où chaque agent interagit ou non avec d'autres selon les valeurs actuelles et passée de leur différence d'opinion. Lorsqu'une interaction est (dés)activée, un saut se produit, conduisant à un modèle dynamique hybride en réseau. Les propriétés de stabilité de ce système hybride en réseau sont ensuite analysées et nous établissons que les opinions des agents convergent vers des accords/foyers locaux au fur et à mesure que le temps augmente, ce qui est un comportement caractéristique des modèles de dynamique d'opinions de la littérature.

Dans le deuxième cas d'étude, nous démontrons comment les techniques hybrides peuvent être utilisées pour surmonter des limitations fondamentales du couplage en temps continu pour synchroniser un réseau d'oscillateurs. En particulier, nous envisageons un scénario d'ingénierie où l'objectif est de concevoir les règles de couplage pour des oscillateurs hétérogènes afin de les synchroniser globalement et uniformément sur une phase commune. Chaque oscillateur a sa propre fréquence naturelle variant dans le temps et prenant des valeurs dans un ensemble compact. Ce problème est historiquement abordé dans la littérature en recourant au célèbre modèle de Kuramoto dont l'inspiration originale provient des réseaux biologiques et physiques. Cependant, le modèle de Kuramoto présente des inconvénients majeurs pour les applications en ingénierie, à savoir l'absence de synchronisation uniforme et la possibilité de verrouillage de phase en dehors de l'ensemble de synchronisation. Pour surmonter ces difficultés, les oscillateurs sont conçus pour être interconnectés via un réseau donné par un arbre sans leader par une classe de règles de couplage hybrides. Les couplages proposés peuvent reproduire localement le comportement des oscillateurs de Kuramoto tout en assurant la stabilité pratique ou asymptotique globale uniforme de l'ensemble de synchronisation, ce qui est impossible avec les modèles de Kuramoto. Nous montrons en outre que l'ensemble de synchronisation peut être rendu stable en temps fini de manière uniforme et globale en choisissant la fonction de couplage discontinue à l'origine. De nouveaux outils mathématiques sur les fonctions non pathologiques et les dérivées de Lie à valeurs dans un ensemble sont développés pour mener à bien l'analyse de stabilité.

Il apparaît que ce dernier ensemble d'outils mathématiques a une applicabilité plus large que le réseau hybride d'oscillateurs considéré. Nous exploitons donc finalement ces outils pour analyser les propriétés de stabilité des systèmes de Lur'e avec des non-linéarités continues par morceaux, grâce aux similarités entre les systèmes de Lur'e et la dynamique à temps continu décrivant les oscillateurs précédemment mentionnés. Nous étendons d'abord un résultat de la littérature en établissant la stabilité asymptotique globale de l'origine sous des conditions de secteur plus générales. Nous présentons ensuite des critères selon lesquels les systèmes de Lur'e avec des non-linéarités continues par morceaux bénéficient de propriétés de stabilité en temps fini de la sortie et de l'état. De plus, nous avons pu fournir des preuves algébriques des résultats que nous présentons, ce qui constitue une nouveauté en soi. Nous montrons la pertinence des outils fournis en étudiant les propriétés de stabilité de deux systèmes d'ingénierie dont l'intérêt est connu : les réseaux de neurones cellulaires et des systèmes mécaniques soumis aux frottements.

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Notation

$\mathbb{C}$	The set of the complex numbers.
$\mathbb{R}$	The set of the real numbers.
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers.
$\mathbb{R}_{>0}$	The set of the strictly positive real numbers.
$\mathbb{Z}_{\geq 0}$	The set of the non-negative integers.
$\mathbb{Z}_{>0}$	The set of the strictly positive integers.
$\mathbb{R}^n$	The n-dimensional Euclidean space with $n \in \mathbb{Z}_{>0}$.
$\langle x, y \rangle$	The *Euclidean scalar product* in $\mathbb{R}^n$ between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$x^\top$	The transpose of $x \in \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\lvert x \rvert$	The *Euclidean norm* of $x \in \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\lvert x \rvert_1$	The 1—*norm* of $x \in \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\mathbf{0}_n$	The vector whose $n \in \mathbb{Z}_{>0}$ elements are all equal to 0.
$\mathbf{1}_n$	The vector whose $n \in \mathbb{Z}_{>0}$ elements are all equal to 1.
(x, y)	The vector equal to $[x^\top, y^\top]^\top$ with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $n, m \in \mathbb{Z}_{>0}$.
x_ℓ	The ℓ-th element of $x \in \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\lvert x \rvert_{\mathscr{A}}$	*Euclidean distance* between $x \in \mathbb{R}^n$ and the non-empty set $\mathscr{A} \subset \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$, that is $\lvert x \rvert_{\mathscr{A}} := \inf\{\lvert y - x \rvert, y \in \mathscr{A}\}$.
$\{x\}^k$	The *sequence of* $k \in \mathbb{Z}_{>0}$ *points in* $S \subseteq \mathbb{R}^n$ *with* $n \in \mathbb{Z}_{>0}$.
I_n	the identity matrix of dimension $n \times n$ with $n \in \mathbb{Z}_{>0}$
O_n	the null matrix of dimension $n \times n$ with $n \in \mathbb{Z}_{>0}$.
$[A]_\ell$	The ℓ-th column of $A \in \mathbb{R}^{n \times m}$ with $n, m \in \mathbb{Z}_{>0}$.

$\|A\|$	The spectral norm of $A \in \mathbb{R}^{n\times m}$ with $n, m \in \mathbb{Z}_{>0}$.
$(A)_s$	The s-th row of $A \in \mathbb{R}^{n\times m}$ with $n, m \in \mathbb{Z}_{>0}$.
$\mathrm{diag}(x_1, \ldots, x_n)$	The diagonal matrix of $\mathbb{R}^{n\times n}$ whose $n \in \mathbb{Z}_{>0}$ diagonal elements are $x_1, \ldots, x_n \in \mathbb{R}$
$\ker(A)$	The kernel of the linear map represented by matrix $A \in \mathbb{R}^{n\times m}$ with $n, m \in \mathbb{Z}_{>0}$.
$\mathrm{int}(\mathscr{A})$	The *interior* of set $\mathscr{A} \subseteq \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\partial(\mathscr{A})$	The *boundary* of set $\mathscr{A} \subseteq \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\mathrm{cl}(\mathscr{A})$	The *closure* of set $\mathscr{A} \subseteq \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\mathrm{co}(\mathscr{A})$	The *convex hull* of set $\mathscr{A} \subset \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\overline{\mathrm{co}}(\mathscr{A})$	The *closure of the convex hull* of set $\mathscr{A} \subset \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
e_i	The i-th element of the natural base of $\mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$\mathbb{B}_n$	The notation denotes the *closed unit ball* of $\mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$ centered at the origin. We write $\mathbb{B}$ when its dimension is clear from the context.
$\emptyset$	The *empty set*.
$\|\mathscr{S}\|$	The *cardinal number* of the finite set $\mathscr{S} \subset \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$.
$T : X \rightrightarrows Y$	The *set-valued map* T from X to Y, with X and Y two non-empty sets.
$\mathrm{uni}([a, b])$	The continuous uniform distribution over the compact interval $[a, b]$ with $a < b \in \mathbb{R}$.

Acronyms

1-DoF	One degree-of-freedom
FJ	Friedkin-Johnsen
GAS	Globally Asymptotically Stable
HK	Hegselmann-Krause
HBC	Hybrid Basic Conditions
OFTS	Output Finite-Time Stable
oGAS	output Globally Asymptotically Stable
SFTS	State Finite-Time Stable
SIoLAS	State-Independent output Locally Asymptotically Stable
UGAS	Uniformly Globally Asymptotically Stable
VC	Viability Condition

Chapter 1

Introduction

In this chapter, we introduce the problems and concepts investigated in this thesis. In particular, we discuss the main class of dynamical systems we study, namely networked hybrid dynamical systems using the formalism of [64] in Sections 1.1 and 1.2. The contributions we bring to the field are summarized in Section 1.3, where we also outline the structure of the thesis. The associated list of communications (publications, presentations) is given in Section 1.4.

1.1 Hybrid dynamical systems

When modeling physical evolutions, we often classify a system as a continuous-time dynamical system, as, for example, mechanical systems possibly controlled and monitored via analog electronic and electric apparatus, or as a discrete-time dynamical one (digital systems on all). However, systems that escape this dichotomy are ubiquitous: any physical system whose dynamics is affected by logical variables, such as mechanical systems controlled by digital computers, impulsive systems where the almost-instantaneous changes affecting the continuous dynamics can be modeled by discrete events (see the notorious example of the bouncing ball in [64]), or any biological systems where the continuous-time dynamics is affected by a reset mechanism, as for the dynamics describing the synchronization of flashing fireflies [64, Ch. 2] or neurons [69, Ch. 8]. The well-known switching systems are another important class of systems escaping the aforementioned dichotomy, being continuous-time systems where the differential equations describing their dynamics are selected from a finite or countable set of possible differential equations and are changed according to a switching rule. Interestingly enough, the richness of the behavior of the aforementioned systems cannot be captured using only differential equations (or differential inclusions) or difference equations (or difference inclusions), and thus, they cannot be described as solely continuous-time or discrete-time dynamical systems. Only a well-thought combination of these two dynamics leads to complete and exhaustive representations of these systems, leading to the so-called *hybrid dynamical systems*.

In the literature, different frameworks were proposed to model hybrid dynamics including the framework for switching systems in [77], for impulsive dynamical systems in [65], for hybrid automata in [81] or for complementarity systems in [22].

In this thesis, we choose to adopt another formalism to model hybrid dynamical systems: the one pioneered in [64] by Andrew Teel, Rafal Goebel, and Ricardo Sanfelice. Namely, given two sets $C, D \subseteq \mathbb{R}^n$ and two set-valued maps $F : \operatorname{dom} F \rightrightarrows \mathbb{R}^n$ and $G : \operatorname{dom} G \rightrightarrows \mathbb{R}^n$ with $n \in \mathbb{Z}_{>0}$, such that $C \subseteq \operatorname{dom} F$ and $D \subseteq \operatorname{dom} G$, a hybrid dynamical system, or shortly, hybrid system, $\mathscr{H} := (C, D, F, G)$ is defined in [64] by

$$\mathscr{H}: \begin{cases} \dot{x} \in F(x), & x \in C, \\ x^+ \in G(x), & x \in D. \end{cases} \tag{1.1}$$

Intuitively speaking systems $\mathscr{H}$ may evolve according to continuous-time and discrete-time dynamics, possibly alternating these behaviors. The choice between these evolutions is ruled by conditions defining two (possibly overlapping partially or totally) regions of the state space. In the most general setting, the continuous and discrete dynamics are represented by differential and difference inclusions. More explanations about model (1.1) are provided in Chapter 2.3 where the used terminology and notation is defined. Therefore, we choose to adopt the formalism in [64] by the following reasons:

(i) the generality of this modeling framework, which allows capturing a rich class of hybrid dynamical systems including switched systems, impulsive systems, and hybrid automata among others;

(ii) the careful, proposed notion of solutions, which generalizes existing ones for differential/difference inclusions;

(iii) the solid Lyapunov stability theory associated with this framework, which consistently covers existing results for continuous-time and discrete-time systems as special cases;

(iv) the way the issue of robustness is handled, which is extremely challenging when dealing with such discontinuous systems.

Regarding the last item, the authors of [64] have thus identified a set of general conditions, under which a hybrid dynamical system is said to be well-posed, the so-called hybrid basic conditions recalled in Chapter 2.3, thereby implying that any sequence of (graphically) converging solutions do converge to a solution. The latter property appears to be essential in establishing robustness properties.

Numerous results from continuous-time and discrete-time systems have been extended to hybrid dynamical systems in the formalism in [64]. In [116] and [115], LaSalle's invariance principles and a Matrosov's theorem for hybrid dynamical systems are provided. The characterization of input-to-state stability properties for hybrid dynamical systems is given in [28], while in [92], the authors portray the finite-gain $\mathscr{L}_2$ stability properties for hybrid dynamical systems while also providing additional results on homogeneous hybrid dynamical systems. The results in [78] provide Lyapunov-based small-gain theorems for hybrid dynamical systems to certify input-to-state stability (ISS) of hybrid systems with inputs and global asymptotic stability (GAS) of hybrid dynamical systems without inputs. In [94] the authors extend Gronwall's inequality, a well-known and useful result both for continuous-time and discrete-time signals, to hybrid dynamical signals. The recent contribution to the field made in [113] aims instead to generalize the results about designing controllers for continuous-time and discrete-time dynamical systems to hybrid ones.

In this thesis, we aim to demonstrate the strengths of hybrid theoretical tools to model and control important classes of *networked* systems leading to *networked hybrid dynamical systems*.

1.2 Networked hybrid dynamical systems

Social networks, industry 4.0, smart grids, and many other networked dynamical systems pervade today's world. These systems can be represented similarly as networks of agents evolving toward a common global goal while constrained to local information and communications. The behavior of these systems as a whole is often described by continuous-time dynamics affected by possible instantaneous changes, that may occur naturally or may be enforced to achieve a desired network behavior. The superposition of these two dynamics generates the so-call *networked hybrid dynamical system*.

In this context, we can identify three main phenomena causing instantaneous changes in the systems (which are, in general, not mutually exclusive):

(i) the intrinsic dynamics of each node, as for the flashing fireflies in [64, Ch. 2] or the neurons dynamics in [102];

(ii) the intrinsic degrees of freedom of the control actions like in power converters within smart grids [17,135], or in networked systems subject to computation and communication constraints, as described in [95] for the event-triggered control for multi-agent systems and [40] where the coordination of a network of agents in a cyber-physical environment is addressed via Lyapunov redesign;

(iii) the creation/loss of links or the addition/removal of links or the addition/removal of nodes as in models of opinion dynamics [58].

In this thesis, we concentrate on item (iii), and our objective is to demonstrate the strengths of hybrid tools to model, analyze and control networked systems via the study of two case studies. The hybrid formalism has already demonstrated to be successful to generalize and improve existing works in the purely continuous-time or discrete-time framework, as done, for example, in [93], where the authors go beyond the limits of the continuous-time reset control systems by providing less conservative and more general stability results by using the hybrid formalism in [64], or in [100], where, by the same means, the authors improve the estimation performance of a given nonlinear observer. We are convinced that hybrid tools will also play an important role for networked systems.

In the literature, there exist already results that study networked systems via the hybrid formalism in [64]. In his visionary work [131], the author proposes to exploit dissipativity notions to establish asymptotic stability for large-scale hybrid systems, thereby generalizing results for continuous-time systems in [90], and dissipativity tools are known to be powerful tools to analyze collective behavior in networked systems [10]. In [112] and [78], which provide small gain theorems for analyzing interconnections of hybrid dynamical systems are provided, which are very useful tools to establish stability guarantees for interconnected systems. Still, much needs to be done to exploit the potential of hybrid techniques to model, analyze and control networked systems.

Thus, the objective of this thesis is to develop hybrid theoretical tools to model, analyze and sometimes control important classes of networked systems, namely opinion dynamics and networks of synchronizing oscillators, as explained in more detail in the next section.

1.3 Contributions and outline of the thesis

In this thesis, we aim to show the effectiveness of the hybrid theoretical tools to model and control in a distributed way important classes of networked systems. We first demonstrate how the hybrid formalism in [64] can be exploited to model the evolution of opinions in a social network, in a scenario where the interactions between individuals depend on both their past and current opinions. This is a reasonable assumption when every individual knows the identity of the other members of the network. To do so, we merge the features of the Friedkin-Johnsen (FJ) model [59] (where importance is given to the past, the initial conditions) and those of the Hegselmann-Krause (HK) model [67] (the bounded confidence mechanism based on current opinions). We account for the past by filtering the neighbors' opinion mismatch. We then exploit these filtered data together with the current opinion mismatch to decide whether a link between two agents needs to be (de)activated, leading to a jump. A new model is then developed in the formalism of [64]. We then analyze the stability properties of the model using a new Lyapunov function, which guarantees a suitable $\mathscr{K}\mathscr{L}$-stability property ensuring the asymptotic convergence of the opinions to clusters/agreement. In addition, solutions are proven not to generate Zeno phenomenon and to stop jumping in finite-time. We illustrate numerically the impact of the memory dynamics and the design parameters over the consensus achieved by the agents. This work thus illustrates how hybrid tools can be used to model and analyze opinion dynamics.

In the second case study, we overcome the fundamental limitations of continuous-time coupling to synchronize a network of oscillators by exploiting hybrid techniques. Indeed, this problem is historically addressed in the literature by resorting to the well-known Kuramoto model whose original inspiration comes from biological and physical networks. However, Kuramoto model exhibits major shortcomings for engineering applications, namely the lack of uniform synchronization and phase-locking outside the synchronization set [120]. In particular, we envision the engineering scenario where the goal is to design the coupling rules to globally and uniformly synchronize the phases of heterogeneous oscillators. Each oscillator has a time-varying natural frequency taking values in a compact set. Therefore, to achieve uniform, global, synchronization, a communication layer is set in place to allow the oscillators to exchange synchronizing coupling actions through a tree-like leaderless network, which can always be derived from generic connected graphs in a distributed way by using the algorithms surveyed in [98]. Furthermore, we introduce suitable resets (a 2π-unwinding mechanisms) of the oscillators phases coordinates, so that they are unwrapped to evolve in a compact set, which includes $[-\pi, \pi]$ consistently with their angular nature. In this setting, we present novel hybrid coupling rules for which a Lyapunov-based analysis ensures uniform global (practical or asymptotic) phase synchronization. Curiously, being the set of admissible coupling functions quite general, there exist selections of coupling rules that allow recovering the behavior of the original Kuramoto model when the oscillators are near phase synchronization. Furthermore, due to the mild properties that we require for our hybrid coupling function, discontinuous selections are allowed, like in [35]. When the discontinuity is at the origin, we prove finite-time stability properties. In particular, exact synchronization can be reached in a prescribed finite-time [121], and convergence is thus independent of the initial conditions. Compared to the related works in [84] and [141], the finite-time stability property we ensure is global and the convergence time can be arbitrarily prescribed, respectively. Due to the possible presence of discontinuities in the coupling function, the stability analysis is carried out by focusing on the Krasovskii regularization of the continuous-time dynamics. We were able to prove that solutions to the regularized dynamics enjoy an average-dwell time property and that the maximal ones are complete. Therefore, because these two properties imply that maximal solutions are $t-$complete, we certify the absence of the Zeno phenomenon. Furthermore, the effectiveness of the approach is illustrated in simulations where we apply our synchronizing hybrid coupling rules to models of power grids previously used in the literature. Finally, we provide ancillary results on nonpathological functions and set-valued Lie derivatives, motivated by the necessity to deploy novel and effective tools to study the stability properties of hybrid dynamical systems with set-valued continuous dynamics that are locally bounded and outer semicontinuous but not inner semicontinuous (as typical for the Krasovskii regularization of discontinuous single-valued piecewise-continuous functions), while using nonsmooth Lyapunov functions.

Due to the similarity between Lur'e-Postinkov Lyapunov functions and the Lyapunov functions used in Chapter 4, and due to the similar nature of the studied nonlinearities, we exploit the aforementioned mathematical tools in Chapter 5, to analyze the stability of Lur'e systems with piecewise continuous nonlinearities. Because Lur'e systems are ubiquitously used in many engineering domains, a conspicuous amount of publications studying their stability properties are available in the literature, both in general settings, i.e. [143], then for specific applications, i.e, [26]. However, the literature is still lacking behind in terms of characterizing the state and finite-time stability properties of Lur'e systems with piecewise continuous nonlinearities, whose relevance is clear from [144] and [105]. Therefore, by exploiting the notion of set-valued Lie derivative we define conditions to guarantee the output and state finite-time stability properties for Lur'e systems with piecewise continuous nonlinearities. Furthermore, we also generalize an existing result on the asymptotic stability of the origin for this class of systems using the same tools. We illustrate the usefulness of our results to certify the output and state stability properties in two engineering applications, which are respectively considered in [26, 55] and that can be modeled as Lur'e systems. Indeed, we establish output finite-time and state-independent local asymptotic stability properties for mechanical systems subject to friction, which is a novelty compared to [26]. Furthermore, we certify that the cellular neural networks modeled as in [55] are state finite-time stable, thus retrieving the results in [55, Thm. 4] while coping with a more general class of Lur'e systems.

The rest of the thesis is structured as follows.

— In **Chapter 2** we provide the mathematical preliminaries by presenting the fundamental notions used in this thesis on differential inclusions and hybrid dynamical systems. Furthermore, we added for the reader's convenience some additional mathematical background including some of graph theory, which is used in two core chapters of the thesis.

— In **Chapter 3** we present our first case study: the hybrid model of opinion dynamics with memory-based connectivity. We start by presenting the memory-less model given in [58], to which we add the memory mechanism subsequently. We then proceed to prove that the opinions of the agents converge asymptotically to clusters/agreements, results that are illustrated numerically together with the impact of the memory-based filters and their parameters.

— In **Chapter 4**, we investigate the problem of synchronizing heterogeneous oscillators via hybrid couplings. We start presenting the model and its Krasovskii regularization. We then proceed in proving the property of its solutions and detailing the stability properties of the synchronization set, along with ancillary results on nonsmooth Lyapunov analysis. We finally provide numerical illustrations of the theoretical results.

— In **Chapter 5**, we investigate the state and output stability properties of continuous-time Lur'e systems with piecewise continuous nonlinearities. We introduce the original and regularized dynamics at the beginning of the chapter. We perform the state and output stability analysis for the aforementioned class of systems by exploiting the set-valued Lie derivative in the later sections, where we also illustrate numerically the shortcomings arising when adopting Clarke's generalized directional derivatives in the mathematical derivations. We finally provide two examples of engineering systems whose stability analysis benefits from our results: cellular neural networks and mechanical systems affected by friction.

— We provide conclusions summarizing the work presented in this thesis in **Chapter 6**, where we present also possible research directions we think are worth exploring.

1.4 Publications

In this section, we provide a short list, detailed by categories, of scientific production and dissemination.

1.4.1 Journal papers

1. S. Mariano, I.C. Morarescu, R. Postoyan and L. Zaccarian. *A hybrid model of opinion dynamics with memory-based connectivity*, IEEE Control Systems Letters, vol. 4, no. 3, pp. 644-649, July 2020, doi: 10.1109/LCSYS.2020.2989077.

1.4.2 Pre-prints

1. S. Mariano, R. Postoyan, L. Zaccarian. *Finite-time stability properties of Lur'e systems with piecewise continuous nonlinearities,* to be submitted for journal publication.

2. S. Mariano, R. Bertollo, R. Postoyan, L. Zaccarian. *Hybrid coupling rules for leaderless heterogeneous oscillators: uniform global asymptotic and finite-time synchronization,* submitted for journal publication.

1.4.3 Conferences (without proceedings)

1. S. Mariano. *A hybrid model of opinion dynamics with memory-based connectivity,* CDC (IEEE Conference on Decision and Control), Jeju Island: South Korea, 2020.
2. S. Mariano. *Global synchronization of a tree-like network of Kuramoto oscillators*, MTNS (International Symposium on Mathematical Theory of Networks and Systems), Bayreuth: Germany, 2022.

1.4.4 Seminars

1. S. Mariano. *A hybrid model of opinion dynamics with memory-based connectivity,* ANR HANDY Webinar, December 2020.
2. S. Mariano. *A hybrid model of opinion dynamics with memory-based connectivity,* "Journées Nationales Automatique de la SAGIP et demi-journée GDR MACS", September 2020.
3. S. Mariano. *Uniform global asymptotic synchronization of a network of Kuramoto oscillators via hybrid coupling,* "2e Journée virtuelle SAGIP", September 2021.
4. S. Mariano. *Uniform global asymptotic synchronization of a network of Kuramoto oscillators via hybrid coupling,* 10ème Biennale SMAI (SMAI "Biennale Française de Mathématiques Appliquées et Industrielles"), Montpelier: France, June 2021.
5. S. Mariano. *Hybrid coupling rules for leaderless heterogeneous oscillators: uniform global asymptotic and finite-time synchronization*, ANR HANDY Workshop, Toulouse: France, June 2022.

Chapter 2

Mathematical Preliminaries

In this chapter, we present some mathematical preliminaries, which are used in the next chapters. The material is borrowed from [11, 33, 64, 66, 73, 110].

2.1 Single-valued functions

In this section, we provide some basic definitions related to single-valued functions.

Let $f:\mathbb{R}^n\to\mathbb{R}$ with $n\in\mathbb{Z}_{>0}$ and let $r\in\mathbb{R}$. We denote by $f(r)^{-1}$ the set $\{x\in\mathbb{R}^n : f(x)=r\}$, which may be empty. Let $f:\mathbb{R}\to\mathbb{R}$ and $s_\circ\in\mathbb{R}$, then $f'(s_\circ):=\lim_{s\to s_\circ}(f(s)-f(s_\circ))/(s-s_\circ)$, when it exists.

The function $f: X\subseteq\mathbb{R}^n\to Y\subseteq\mathbb{R}^m$ with $n,m\in\mathbb{Z}_{>0}$ is:

— *continuous at some point* $x\in X$ if $\lim\limits_{s\to x} f(s)=f(x)$;

— *continuous* if it is continuous at every point $x\in X$;

— *globally Lipschitz-continuous*, or simply *Lipschitz-continuous* if there exists a constant $M\in\mathbb{R}_{>0}$ such that $|f(x)-f(z)|\le M|x-z|$ for any $x,z\in X$;

— *locally Lipschitz-continuous*, or simply *locally Lipschitz* if for every $x\in X$ there exist a neighborhood $V\subset X$ of x and a constant $M\in\mathbb{R}_{>0}$ such that, for any $z,z'\in V$, $|f(z)-f(z')|\le M|z-z'|$.

A function $f:\mathbb{R}\to\mathbb{R}$ is:

— *piecewise continuous* if for any given interval $[a,b]$, with $a<b\in\mathbb{R}$, there exist a finite number of points $a\le x_0<x_1<x_2<\cdots<x_{k-1}<x_k\le b$, with $k\in\mathbb{Z}_{\ge0}$ such that f is continuous on (x_{i-1},x_i) for any $i\in\{1,\dots,k\}$ and its one-sided limits exist as finite numbers;

— *piecewise continuously differentiable* if f is continuous and for any given interval $[a,b]$, with $a<b\in\mathbb{R}$, there exists a finite number of points $a\le x_0<x_1<x_2<\cdots<x_{k-1}<x_k\le b$, with $k\in\mathbb{Z}_{\ge0}$ such that f is continuously differentiable on (x_{i-1},x_i) for any $i\in\{1,\dots,k\}$ and the one-sided limits $\lim_{s\to x_{i-1}^+} f'(s)$ and $\lim_{s\to x_i^-} f'(s)$ exist for any $i\in\{1,\dots,k\}$.

Given a compact interval $I:[a,b]\subset\mathbb{R}$, $f:I\to\mathbb{R}^n$ is *absolutely continuous on I* if f is differentiable almost everywhere in I, its derivative f' is Lebesgue integrable and $f(x)=f(a)+\int_a^x f'(s)ds$

for all $x \in I$. Consider a right-open interval $I := [0, b)$, possibly with $b = \infty$, $f : I \to \mathbb{R}^n$ is *locally absolutely continuous*, or simply, *absolutely continuous* if it is absolutely continuous on any compact interval of I.

A function $f : \mathbb{R}^n \to \mathbb{R}$ with $n \in \mathbb{Z}_{>0}$ is *radially unbounded* if $f(x) \to \infty$ as $|x| \to \infty$. A function $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is *positive definite* with respect to the set $\mathscr{A} \subseteq \mathbb{R}^n$ if $\alpha(x) = 0$ for any $x \in \mathscr{A}$ and $\alpha(x) > 0$ for all $x \in \mathbb{R}^n \setminus \mathscr{A}$.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is:

— of *class* $\mathscr{K}$ ($\alpha \in \mathscr{K}$) if it is continuous, strictly increasing, and $\alpha(0) = 0$.

— of *class* $\mathscr{K}_\infty$ ($\alpha \in \mathscr{K}_\infty$) if $\alpha \in \mathscr{K}$ and it is unbounded.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathscr{KL}$ ($\beta \in \mathscr{KL}$) if it is continuous, nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r,s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r,s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathscr{KLL}$ ($\beta \in \mathscr{KLL}$) if it is continuous, nondecreasing in its first argument, nonincreasing in its second and third arguments, $\lim_{r \to 0^+} \beta(r,s,t) = 0$ for each $s, t \in \mathbb{R}_{\geq 0}$, $\lim_{s \to \infty} \beta(r,s,t) = 0$ for each $r, t \in \mathbb{R}_{\geq 0}$, and $\lim_{t \to \infty} \beta(r,s,t) = 0$ for each $r, s \in \mathbb{R}_{\geq 0}$.

2.2 Set-valued maps and differential inclusions

In this section, we present basic properties of set-valued maps and differential inclusions introduced in [11, 110], thereby providing the necessary preliminaries for studying the systems proposed in Chapters 2, 3, 4, and 5.

2.2.1 Preliminaries on set-valued maps

A *set-valued map* $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a function that at each point $x \in \operatorname{dom} F$ associates a set $F(x) \subseteq \mathbb{R}^m$. The set $\operatorname{dom} F := \{x \in \mathbb{R}^n | F(x) \neq \emptyset\}$ is the domain of F. Consider the set-valued map $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. As a first step, we provide the definition of *outer semicontinuity* and *inner semicontinuity* of F at x [110, Def. 5.4], [110, Eq. 5.1]. We will mostly rely on the concept of outer semicontinuity throughout this thesis.

Definition 2.1. *Consider a set-valued map $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Then F is* outer semicontinous *at $x \in \mathbb{R}^n$ if*

$$F(x) \supseteq \limsup_{y \to x} F(y)$$

where

$$\limsup_{y \to x} F(y) := \{f \in \mathbb{R}^m | \exists \{x\}^k \to x, \exists \{f\}^k \in F(x_k) \text{ such that } \{f\}^k \to f \text{ for } k \to \infty\}.$$

Given a set $C \subseteq \mathbb{R}^n$, F is outer semicontinous *(on C) if the set-valued map $\widetilde{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $F(x)$ for $x \in C$ and $\emptyset$ for $x \notin C$ is outer semicontinous at each $x \in C$.* □

Definition 2.2. *Consider a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Then F is* inner semicontinous *at $x \in \mathbb{R}^n$ if*

$$F(x) \subseteq \liminf_{y \to x} F(y)$$

where

$$\liminf_{y \to x} F(y) := \{f \in \mathbb{R}^m | \forall \{x\}^k \to x, \exists \{f\}^k \in F(x_k) \text{ such that } \{f\}^k \to f \text{ for } k \to \infty\}.$$

Given a set $C \subseteq \mathbb{R}^n$, F is inner semicontinous *(on C) if the set-valued map $\widetilde{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $F(x)$ for $x \in C$ and $\mathbb{R}^m$ for $x \notin C$ is inner semicontinous at each $x \in C$.* □

The outer semicontinuity of $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ relative to $C \subseteq \operatorname{dom} F \subseteq \mathbb{R}^n$ simply means that for each $x \in C$ and each sequence of points $x_i \in C$ convergent to x and each sequence of points $y_i \in F(x_i)$ convergent to y, $y \in F(x)$, from which it follows that the graphs of an outer semicontinuous set-valued map is closed [64, Lem. 5.10]. We recall that a *graph* of a set-valued map $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the set $\operatorname{gph}(F) := \{(x, y) \in \operatorname{dom} F \times \mathbb{R}^m : y \in F(x)\}$. Finally we provide the definition of *local boundedness relative to C* for F in (2.1).

Definition 2.3. *A set-valued map $F : dom\, F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is* locally bounded at $x \in C \subseteq \operatorname{dom} F$ relative to C *if there exists a neighborhood U of x such that the set $F(U \cap C) \subset \mathbb{R}^m$ is bounded. It is* locally bounded *(on C) if this holds at every $x \in C$.* □

2.2.2 Existence of solutions for constrained differential inclusions

Given a set $C \subseteq \mathbb{R}^n$ and a set-valued map $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $C \subseteq \operatorname{dom} F$ closed, we define a *constrained differential inclusion* as

$$\dot{x}(t) \in F(x(t)), \quad x(t) \in C. \tag{2.1}$$

Next, we provide the notion of solution to (2.1).

Definition 2.4. *Consider the constrained differential inclusion in* (2.1). *We define a* solution *to* (2.1) *to be an absolutely continuous function $\phi : dom\, \phi \to C$, with $dom\, \phi := [0, T)$ with $T > 0$ and possibly $T = \infty$, such that $\dot{\phi}(t) \in F(\phi(t))$ for almost all $t \in dom\, \phi$ and $\phi(t) \in C$ for any $t \in dom\, \phi$.* □

We give next the definition of *maximal* solutions. Namely, a solution is *maximal* if it cannot be extended forward in time.

Definition 2.5. *A solution is* maximal *if there exists no solution ψ such that $dom\, \phi$ is a proper subset of $dom\, \psi$ and $\phi(t) = \psi(t)$ for all $t \in dom\, \phi$.* □

We now define the concept of *forward complete* differential inclusion.

Definition 2.6. *A differential inclusion is said* forward complete *if all maximal solutions have an unbounded domain.* □

We next give the definition of a tangent cone to a set $S \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$.

Definition 2.7. *The tangent cone to a set $S \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted as $T_S(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist sequences $x_i \in S$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \searrow 0$, and $w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$.* □

We are now ready to recall sufficient existence results for constrained differential inclusions given in [64, 5.26].

Proposition 2.1. *Consider a closed set $C \subset \mathbb{R}^n$ and a set-valued map $F : \operatorname{dom} F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $C \subseteq \operatorname{dom} F$, locally bounded in C, outer semicontinuous on C and with closed convex values. Then, given $\xi \in C$, if there exists a neighborhood U of ξ such that for all $x \in U \cap C$, $F(x) \cap T_C(x) \neq \emptyset$ (viability condition VC), then there exists $T > 0$ and a solution $z : I = [0, T] \to \mathbb{R}^n$ to (2.1) with $z(0) = \xi$.* □

Notice that, in view of [64, Lemma 5.15], the sufficient conditions we gave in Proposition 2.1 coincide with the ones presented in Theorem 3 in [11, Ch.2] when $C = \operatorname{dom} F = \mathbb{R}^n$.

2.2.3 Regularization of differential equations with a discontinuous right-hand side

We will encounter scenarios in Chapters 4 and 5 where the original continuous-time system is given by an ordinary (constrained) differential equation with a discontinuous vector field. To define what we mean by a solution, we will resort to Kravoskii regularization, leading to a (constrained) difference inclusion, as explained in this section.

Consider a single-valued function $f : \operatorname{dom} f \subseteq \mathbb{R}^n \to \mathbb{R}^n$, locally bounded and possibly discontinuous, and let us define the dynamics

$$\dot{x}(t) = f(x(t)). \tag{2.2}$$

System (2.2) may have a discontinuous right-hand side. Therefore, when we refer to the solutions to system (2.2), we consider its so-called *(generalized) Krasovskii or Filippov solutions*, which are generated by *the Krasovskii or Filippov regularization* of (2.2).

Definition 2.8. *Consider function f in (2.2), we define:*

— $\boldsymbol{F}(x) := \bigcap_{s>0} \overline{co}\, \boldsymbol{f}(x + s\mathbb{B})$ *for any $x \in \operatorname{dom} f$, as its* Krasovskii regularization.

— $\overline{\boldsymbol{F}}(x) := \bigcap_{s>0} \overline{co}\, \boldsymbol{f}((x + s\mathbb{B}) \setminus Z)$ *for any $x \in \operatorname{dom} f$ and with Z being a set of Lebesgue measure 0, as its* Filippov regularization. □

We next recall the notion of *Krasovskii and Filippov solutions*.

Definition 2.9. *We call an absolutely continuous function $\phi : \operatorname{dom}\phi \to C$ with $C \subseteq \operatorname{dom} f$ and with $\operatorname{dom}\phi := [0, T)$ with $T > 0$ and possibly $T = \infty$:*

— *a* Filippov solution *of (2.2) if $\dot{\phi}(t) \in \overline{\boldsymbol{F}}(\phi(t))$, for almost all $t \in \operatorname{dom}\phi$.*

— *a* Krasovskii solution *of (2.2) if $\dot{\phi}(t) \in \boldsymbol{F}(\phi(t))$, for almost all $t \in \operatorname{dom}\phi$.* □

In the following chapters, we always start by defining the continuous-time dynamics as differential equations with a discontinuous right-hand side, and then we proceed to analyze its Krasovskii regularization. We choose to study the Krasovskii regularization instead of the Filippov ones because this last regularization ignores the behavior of f in (2.2) on sets of measure zero, and hence is problematic for hybrid systems, or even for constrained differential equations [64, Ch. 4.6]. Notice that, in view of [64, Lemma 5.16] and [11, page 101-103], any set-valued map that is the Krasovskii regularization or the Filippov regularization of a piecewise-continuous function is outer semicontinuous. On the other hand, any Krasovskii regularization of a discontinuous piecewise-continuous function is not inner semicontinuous, as shown in the next example. This fact has important consequences when studying the stability properties of (constrained) differential inclusions, as discussed in more detail in [44].

Example 2.1. *Consider the set-valued map $F : \mathbb{R}^2 \rightrightarrows [-1,1]$, being the Krasovskii regularization of $f : \mathbb{R} \to [-1,1]$, defined as $f(s) = 1$ if $s > 0$, $f(s) = -1$ if $s < 0$, and $f(s) = 0$ if $s = 0$; hence $F(0) = [-1,1]$. Therefore, we have that $F(0) = [-1,1] \not\subseteq \liminf_{y\to 0} F(y) = \emptyset$.* □

2.3 Hybrid dynamical systems

This section is devoted to presenting the hybrid formalism introduced in [64]. Consider system (1.1), which we recall next

$$\mathcal{H}: \begin{cases} \dot{x} \in F(x), & x \in C, \\ x^+ \in G(x), & x \in D, \end{cases} \tag{1.1}$$

where $C, D \subseteq \mathbb{R}^n$, $F : \operatorname{dom} F \rightrightarrows \mathbb{R}^n$ and $G : \operatorname{dom} G \rightrightarrows \mathbb{R}^n$, such that $C \subseteq \operatorname{dom} F$ and $D \subseteq \operatorname{dom} G$. In this thesis and as customarily done in the hybrid dynamical system literature, when dealing with hybrid dynamical systems, we will refer to C and D as the *flow* and *jump sets* while we identify F and G as the *flow* and *jump maps*. We assume that the hybrid dynamical systems we study satisfy the following set of conditions, known as the *hybrid basic conditions* (HBC) [64, Ass. 6.5].

Assumption 2.1. *Consider a system represented as in* (1.1)*. The following holds.*

- *C and D are closed subsets of $\mathbb{R}^n$.*
- *$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to C, $C \subseteq \operatorname{dom} F$, and $F(x)$ is convex for every $x \in C$.*
- *$G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to D, $D \subseteq \operatorname{dom} G$.* □

In the following, we will discuss the advantages of studying hybrid dynamical systems satisfying Assumption 2.1.

2.3.1 Solution concept

In this subsection, we introduce the notion of solutions to a hybrid dynamical system along with the required concept to define it. We start introducing the concept of a *hybrid time domain* given in

[64, Def. 2.3].

Definition 2.10. *A subset* $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ *is a compact hybrid time domain if*

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j) \tag{2.3}$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. *It is a hybrid time domain if for all* $(T, J) \in E$, $E \cap ([0, T] \times \{0, \ldots, J\})$ *is a compact hybrid domain.* □

Roughly speaking, E is a *hybrid time domain* if it is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$ with the last one, if it exists, of the shape $[t_j, T)$, with T possibly finite or $T = \infty$. Given a hybrid time domain E, we have that:

- $\sup_t E := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{Z}_{\geq 0} \text{ such that } (t, j) \in E\}$.
- $\sup_j E := \sup\{j \in \mathbb{Z}_{\geq 0} : \exists t \in \mathbb{R}_{\geq 0} \text{ such that } (t, j) \in E\}$.
- $\sup E := (\sup_t E, \sup_j E)$.
- $\text{length}(E) := \sup_t E + \sup_j E$.

In the following definition [64, Def. 2.4], we recall the required property for a function ϕ, with $\text{dom}\,\phi$ being a hybrid time domain, to be a solution candidate for a hybrid dynamical system.

Definition 2.11. *A function* $\phi : \text{dom}\,\phi \to \mathbb{R}^n$ *is a hybrid arc if the following holds:*

- *$\text{dom}\,\phi$ is a hybrid time domain;*
- *For each* $j \in \mathbb{Z}_{\geq 0}$, *the function* $t \to \phi(t, j)$ *is locally absolutely continuous on the interval* $I^j := \{t \in \mathbb{R}_{\geq 0} : (t, j) \in \text{dom}\,\phi\}$. □

The next definition characterizes properties of hybrid arcs [64, Def. 2.5].

Definition 2.12. *A hybrid arc* ϕ *is called:*

- nontrivial *if dom* ϕ *contains at least two points;*
- complete *if dom* ϕ *is unbounded, i.e., if* $\text{length}(\text{dom}\,\phi) = \infty$*;*
- Zeno *if it is complete and* $\sup_t \text{dom}\,\phi < \infty$*;*
- eventually discrete *if* $T = \sup_t \text{dom}\,\phi < \infty$ *and* $\text{dom}\,\phi \cap (\{T\} \times \mathbb{Z}_{\geq 0})$ *contains at least two points;*
- discrete *if nontrivial and* $\text{dom}\,\phi \subset \{0\} \times \mathbb{Z}_{\geq 0}$*;*
- eventually continuous *if* $J = \sup_j \text{dom}\,\phi < \infty$ *and* $\text{dom}\,\phi \cap (\{J\} \times \mathbb{R}_{\geq 0})$ *contains at least two points;*
- continuous *if nontrivial and* $\text{dom}\,\phi \subset \{0\} \times \mathbb{R}_{\geq 0}$. □

We are now ready to define a solution of system (1.1) [64, Def. 2.6]

Definition 2.13. *A hybrid arc $\phi : \mathrm{dom}\,\phi \to \mathbb{R}^n$ is a solution to the hybrid dynamical system* (1.1) *if the following holds.*

- $\phi(0,0) \in C \cup D$.
- *For all $j \in \mathbb{Z}_{\geq 0}$ such that $I^j := \{t \in \mathbb{R}_{\geq 0} : (t,j) \in \mathrm{dom}\,\phi\}$ has non-empty interior, we have $\phi(t,j) \in C$, for all $t \in I^j$ and $\phi(t,j) \in F(\phi(t,j))$ for almost all $t \in I^j$.*
- *For all $(t,j) \in \mathrm{dom}\,\phi$ such that $(t,j+1) \in \mathrm{dom}\,\phi$, we have $\phi(t,j) \in D$ and $\phi(t,j+1) \in G(\phi(t,j))$.* □

Finally, we give the definition of *maximal* solutions to (1.1) [64, Def. 2.7]; roughly speaking, a solution to (1.1) is *maximal* if it cannot be extended.

Definition 2.14. *A solution ϕ to system* (1.1) *is* maximal *if there does not exist another solution ψ to* (1.1) *such that $\mathrm{dom}\,\phi \subset \mathrm{dom}\,\psi$ and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \mathrm{dom}\,\phi$.* □

2.3.2 Existence of solutions

In this subsection, we present basic results on the existence of maximal complete solutions, which we use later in the thesis. We start recalling [64, Prop. 6.10].

Proposition 2.2. *Let $\mathcal{H} = (C,F,D,G)$ satisfy Assumption 2.1. Take an arbitrary $\xi \in C \cup D$. If $\xi \in D$ or there exists a neighborood U of ξ such that for every $x \in U \cap C$,*

$$F(x) \cap T_C(x) \neq \emptyset, \tag{2.4}$$

then there exists a nontrivial solution ϕ to $\mathcal{H}$ with $\phi(0,0) = \xi$. If (2.4) *holds for every $\xi \in C \setminus D$, then there exists a nontrivial solution to $\mathcal{H}$ from every initial point in $C \cup D$, and every maximal solution ϕ to $\mathcal{H}$ satisfies exactly one of the following conditions:*

- *ϕ is complete;*
- *$\mathrm{dom}\,\phi$ is bounded and the interval I^J, where $J = \sup_j \mathrm{dom}\,\phi$, has nonempty interior and $t \mapsto \phi(t,J)$ is a maximal solution to $\dot{z} \in F(z)$, in fact $\lim_{t\to T} |\phi(t,J)| = \infty$, where $T = \sup_t \mathrm{dom}\,\phi$;*
- *$\phi(T,J) \notin C \cup D$, where $(T,J) = \sup \mathrm{dom}\,\phi$.* □

We note that the last item of Proposition 2.2 does not occur when $G(D) \subseteq C \cup D$. For a more thorough discussion on the problem of the existence of solutions to hybrid dynamical systems, see [64, Ch. 6].

Before concluding this section, we want to recall that a hybrid dynamical system satisfying the hybrid basic conditions is also well-posed [64, Def. 6.27], meaning that the sequences of graphically convergent solutions of the system with vanishing state perturbations converge to solutions of the nominal system (sequential compactness of the solutions). This property allows ensuring robustness properties for globally asymptotically stable compact attractors. See [64, Ch. 7.2- 7.4] for further insight and clarifications on the subject.

2.4 Clarke's generalized directional derivative and set-valued Lie derivative

In this section, we briefly recall the notion of directional derivative and gradient introduced in [33] and [136] and that we use throughout the thesis.

The stability analyses presented in this thesis all rely on the study of Lyapunov functions. These functions appear not to be continuously differentiable but locally Lipschitz and thus differentiable almost everywhere. Because, we investigate the stability properties of dynamical systems which are described by differential inclusions that are locally bounded, outer semicontinuous, but and not inner semicontinuous, we cannot use the "almost everywhere" results in [44]. As a consequence, we will use the *Clarke's generalized directional derivative of a function* V *at* $x \in \mathrm{dom}\, V$ *in the direction* f, whose definition is given next.

Definition 2.15. *Consider a locally Lipschitz function* $V : \mathrm{dom}\, V \subseteq \mathbb{R}^n \to \mathbb{R}$ *with* $n \in \mathbb{Z}^n$*, then the* Clarke's generalized directional derivative of V *for each direction* $f \in \mathbb{R}^n$ *at each* $x \in \mathrm{dom}\, V$ *is defined as [33, page 11]*

$$V^{\circ}(x; f) := \max\{\langle v, f\rangle : v \in \partial V(x)\},$$

where $\partial V(x)$ *denotes the* Clarke's generalized gradient of V *at* x *given by*

$$\partial V(x) := co\{\lim_{i\to\infty} \nabla V(x_i) : x_i \to x,\ x_i \notin \mathscr{X},\ x_i \notin \Omega_u\},$$

where Ω_u *is the set (of Lebesgue measure zero) where* V *is not differentiable, and* $\mathscr{X}$ *is any other set of Lebesgue measure zero.* □

Notice that in view of V being locally Lipschitz and by the definition of Clarke's generalized gradient of V at x, $\partial V(x)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$ for each $x \in \mathrm{dom}\, V$. When this last notion of derivative appears to be too conservative to establish the asymptotic (or finite-time) stability of the origin for the systems, as we will see in Chapters 4 and 5, we will resort to the notion of *set-valued Lie derivative of* V *with respect to a set-valued map* F *at* $x \in \mathrm{dom}\, V$, which is given next.

Definition 2.16. *Consider a locally Lipschitz function* $V : \mathrm{dom}\, V \subseteq \mathbb{R}^n \to \mathbb{R}$ *and the set-valued map* $F : \mathrm{dom}\, F \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *with* $n \in \mathbb{Z}_{>0}$*, then the* set-valued Lie derivative of V with respect to F at $x \in C$ *with* $C \subset \mathrm{dom}\, V$ *and* $C \subset \mathrm{dom}\, F$ *is defined as [12]*

$$\dot{\overline{V}}_F(x) := \{a \in \mathbb{R} |\ \exists f \in F(x) : \langle v, f\rangle = a,\ \forall v \in \partial V(x)\}. \tag{2.5}$$

□

Whenever function V is nonpathological (according to the definition given next), [43, Lemma 2.23] ensures that $\frac{d}{dt}V(\phi(t)) \in \dot{\overline{V}}_F(\phi(t))$ for almost all t in the domain of ϕ.

Definition 2.17. *[136] A locally Lipschitz function $V : \mathrm{dom}\, V \subseteq \mathbb{R} \to \mathbb{R}$ is* non-pathological *if, given any absolutely continuous function $\phi : \mathbb{R}_{\geq 0} \to \mathrm{dom}\, V$, we have that for almost every $\mathrm{t} \in \mathbb{R}_{\geq 0}$ there exists $a_{\mathrm{t}} \in \mathbb{R}$ satisfying*

$$\langle w, \dot{\phi}(\mathrm{t}) \rangle = a_{\mathrm{t}}, \quad \forall w \in \partial W(\phi(\mathrm{t})). \tag{2.6}$$

In other words, for almost every $\mathrm{t} \in \mathbb{R}_{\geq 0}$, $\partial V(\phi(\mathrm{t}))$ is a subset of an affine subspace orthogonal to $\dot{\phi}(\mathrm{t})$. □

Roughly speaking, when considering a continuous-time dynamical system whose dynamics is defined by a set-valued map F, for some $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$ there may exist a selection $f_{\mathrm{bad}} \in F(x)$ that is never viable for any solution and such that for a locally Lipschitz Lyapunov function V it holds that $V^{\circ}(x, f_{\mathrm{bad}}) > 0$, while a direct inspection shows that V strictly decreases along all solutions outside the attractor set. In the thesis, we provide results on set-valued Lie derivatives and nonpathological functions, which will be used in the stability analysis to overcome this aforementioned limitation.

2.5 Graph theory

We conclude this chapter with basic background on graph theory.

We denote an unweighted undirected graph as $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u)$, where $\mathcal{V}$ is the set of vertices or nodes, and $\mathcal{E}_u \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, or arcs, composed by unordered pairs of nodes. If a pair (i, j) of nodes belongs to $\mathcal{E}_u$, we say that those nodes are *adjacent* and that j is a *neighbour* of i and vice versa. Given two nodes x and y of an undirected graph $\mathcal{G}_u$, we define as *path* from x to y a set of vertices starting with x and ending with y, such that consecutive vertices are adjacent. If there is a path between any couple of nodes, the graph is called *connected*, otherwise it is called *disconnected*. We define as *subgraph* of $\mathcal{G}_u$ a graph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$, where $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E}_u$.

An induced subgraph of $\mathcal{G}_u$ that is maximal and connected, is called a *connected component* of $\mathcal{G}_u$. A *cycle* is a connected graph where every vertex has exactly two neighbours. An *acyclic* graph is a graph for which no subgraph is a cycle. A connected acyclic graph is called a *tree*. We denote an unweighted directed graph as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is composed of ordered pairs, therefore arcs have a specific direction. An arc going from node i to node j is denoted by $(i, j) \in \mathcal{E}$. If a directed graph $\mathcal{G}$ is obtained by choosing an arbitrary direction for the edges of an undirected graph $\mathcal{G}_u$, we call it an *oriented* graph, and we say that $\mathcal{G}$ is obtained from an orientation of $\mathcal{G}_u$.

If $(i, j) \in \mathcal{E}$, we say that i belongs to the set of *in-neighbors* $\mathcal{I}_j$ of j, while j belongs to the set of *out-neighbors* $\mathcal{O}_i$ of i. The union of $\mathcal{I}_i$ and $\mathcal{O}_i$ gives the more generic set of neighbors $\mathcal{V}_i := \mathcal{I}_i \cup \mathcal{O}_i$ of node i, containing all the nodes connected to it, in any direction.

With $B \in \mathbb{R}^{n \times m}$ we denote the *incidence matrix* of graph $\mathcal{G}$ such that each column $[B]_\ell$, $\ell \in \{1, \ldots, m\}$, is associated to an edge $(i, j) \in \mathcal{E}$, and all entries of $[B]_\ell$ are zero except for $b_{i\ell} = -1$ (the tail of edge ℓ) and $b_{j\ell} = 1$ (the head of edge ℓ), namely $[B]_\ell = e_j - e_i$.

Let $\mathcal{G}$ be an arbitrary orientation of a graph $\mathcal{G}_u$, and let B be the incidence matrix of $\mathcal{G}$. Then the *Laplacian matrix* is the matrix $L := BB^\top$ [63, page 279]. We recall that the Laplacian does not depend on the orientation of $\mathcal{G}$, thus being well-defined [63, Lem. 8.3.2]. Given an arbitrarily oriented graph $\mathcal{G}$, then $L_e := B^\top B$ is the *edge Laplacian matrix* of $\mathcal{G}$ [87, pages 22]. We recall that the set of non-zero eigenvalues of L is equal to the set of non-zero eigenvalues of L_e and the set of non-zero eigenvalues of L and L_e are equal to the square of the nonzero singular values of B [87, pages 23-24]. The *flow space* of the graph $\mathscr{G}$ consists of all the vectors x such that $Bx = 0$ [63, page 310].

Chapter 3

A hybrid model of opinion dynamics with memory-based connectivity

Contents

Abstract - We present and analyze in this chapter a model of opinion dynamics where the connectivity among the individuals depends on both their current and past opinions. In a social network where the individuals know the identity of the other members, it is reasonable to assume that their interactions are not only based on the present states but also on their past relationships. A key question is then how to model these interactions accordingly. We propose for this purpose a multi-agent system with active or inactive pairwise interactions depending on auxiliary state variables filtering the instantaneous opinions, thereby taking the past experience into account. When an interaction is (de)activated, a jump occurs, leading to a hybrid dynamical model that we cast by using the formalism introduced in Chapter 2. The stability properties for the obtained interconnected hybrid dynamical system ensure that the opinions converge to local agreements/clusters as time grows. Simulation results are provided to illustrate the theoretical guarantees.

The results of this chapter have been published in [82].

3.1 Introduction

Motivated by the growing importance of digital social networks, opinion dynamics has received an increasing attention from the control community, e.g., [5, 30, 72, 89, 99]. The multi-agent systems formalism is well-suited for modelling these networks, as a node can model the individual's opinion and an edge describes the interaction between two given individuals, e.g., [32, 42, 60, 83]. Two main models provide convergence towards local agreement or disagreement patterns. One of them, the Friedkin-Johnsen model (FJ), [59] essentially filters the consensus dynamics by using the initial opinions of the agents. The idea is that, although individuals influence each other, a major role in the opinion update is played by their culture, belonging to a community (social class, political party, etc), principles and beliefs, as captured by the initial condition of each individual. The second one is the bounded confidence model, also known as the Hegselmann-Krause model (HK) and described in [67], which formalizes the idea that only individuals with similar opinions actually interact. Social psychologists agree that both the FJ and HK models are relevant, depending on the context, see [53] for a detailed survey. Nevertheless, as pointed out in [53], opinion dynamics in social networks is a complex phenomenon, whose key features cannot be completely captured by any of these models separately. This has motivated the development of other deterministic models, e.g., [5, 32, 57, 89, 99], as well as stochastic models, e.g., [41, 83, 137]. It is noteworthy that most of the aforementioned works provide either empirical or rigorous convergence results but stability is eluded in general. The recent work in [58] provides a Lyapunov analysis of the HK model and reveals that only an attractivity property can be guaranteed. This motivated developing in [58] a variant of the HK model where the connectivity depends on adaptive thresholds, instead of fixed ones as in [67]. In this way, a link is activated when the opinions mismatch between two given agents is small compared to the average opinion mismatch with their other neighbours. This strategy ensures stability of the emergent clusters.

The work presented in this chapter is a further step in the direction paved by [58]: we propose a model where connectivity also depends on the past history. With this, we merge the features of the FJ model (where importance is given to the past, the initial conditions) and those of the HK model (the bounded confidence mechanism based on current opinions). It is indeed reasonable to assume that the interactions within a social network do not only depend on the current state but also on the past relationships, when the members are aware of the identity of their neighbours.

To this end, we account for the past by linearly filtering the instantaneous (de)activation functions (de)activating a link only when both the adaptive threshold *and* its filtered version reach certain thresholds. Our model uses the hybrid formalism of [64]. A new Lyapunov function is constructed, which guarantees a suitable $\mathcal{KL}$-stability property ensuring asymptotic convergence to opinion clusters. In addition, solutions are proved not to generate Zeno phenomenon and to stop jumping in finite-time. Simulation results illustrate the behaviour of the model and the impact of the filters and their parameters.

The technical proofs of this work emerge from some interesting analogy between the adaptive threshold connectivity as in [58] and the event-triggered control technique of [128]. These two domains - a priori unrelated - have actually much in common in terms of modelling and analysis tools. As a result, the memory-based connectivity proposed in this chapter is inspired by the dynamic event-triggering control policy proposed in [62].

Background and problem statement are given in Section 3.2. The main results, including the new hybrid model and its stability analysis, are presented in Section 3.3. Illustrative simulations results are reported in Section 3.4 and Section 3.5 concludes the chapter.

3.2 Background and problem statement

Following [58], we consider a set of individuals $\mathcal{V} := \{1, \ldots, n\}$, also referred to as agents, connected through a social network. The opinions of individuals are modelled by a scalar variable $y_i \in \mathbb{R}$ for any $i \in \mathcal{V}$. The dynamics of opinion y_i with $i \in \mathcal{V}$, depends on the interactions of individual i with its neighbours. We define

$$\mathcal{E}^+ := \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i < j\} \tag{3.1}$$

and, for each $(i,j) \in \mathcal{E}^+$, variable a_{ij} defines whether agents i and j interact or not, i.e., whether or not they are neighbours. Thus, a_{ij} is the *connectivity variable* for link (i,j), satisfying

$$a_{ij} = a_{ji} := \begin{cases} 1 & \text{if } i \text{ and } j \text{ interact} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The graph is undirected as $a_{ij} = a_{ji}$. Interaction changes, if any, are described by a jump of the variable a_{ij}. The corresponding hybrid behavior is well represented with the formalism of [64]. Variables y_i and a_{ij} obey the next continuous-time dynamics between two successive jumps

$$\dot{y}_i = \textstyle\sum_{j=1}^n \varphi_{ij}(y_j - y_i), \quad \forall i \in \mathcal{V}, \tag{3.3}$$

$$\dot{a}_{ij} = 0, \qquad \forall (i,j) \in \mathcal{E}^+, \tag{3.4}$$

where $d_i := 1 + \sum_{j \neq i} a_{ij} \geq 1$ is the degree of agent i augmented by 1, $\varphi_{ij} := \dfrac{a_{ij}}{d_i d_j}$ when $i \neq j$ and $\varphi_{ii} := -\sum_{j \neq i} \varphi_{ij}$. We omit the dependence of d_i and φ_{ij} on the connectivity variables. By construction we have that $\varphi_{ij} = \varphi_{ji}$ and $\sum_{j=1}^n \varphi_{ij} = 0$ for any $i \in \mathcal{V}$. The variable φ_{ij} is such that $\Phi := [-\varphi_{ij}]_{(i,j) \in \mathcal{V} \times \mathcal{V}}$ defines a normalized Laplacian matrix [58]. The use of the normalized Laplacian matrix over the standard one serves the role of encompassing the intuition that, in a social network, agents with fewer neighbors tend to have more meaningful and stronger interactions with their peers [58]: indeed, for the agent i-th, having fewer active links correspond to a smaller d_i and thus larger non-zero φ_{ij}'s.

Dynamics (3.4) means that a_{ij} is constant between jumps (along flowing solutions) and that the time-derivative of y_i is given by the weighted average of the opinion mismatch between agent i and its neighbours.

When a jump occurs over the network, i.e., when one of the variables a_{ij} for some $(i,j) \in \mathcal{E}^+$ is updated, solutions obey the following discrete dynamics,

$$\begin{aligned} y_h^+ &= y_h, & \forall h \in \mathcal{V} \\ a_{hk}^+ &= \begin{cases} a_{hk} & \text{if } (h,k) \neq (i,j) \\ 1 - a_{hk} & \text{if } (h,k) = (i,j), \end{cases} & \forall (h,k) \in \mathcal{E}^+ \end{aligned} \tag{3.5}$$

Dynamics (3.5) states that the opinions y_i do not change across jumps and that the connectivity variable a_{ij} toggles between 0 and 1 according to (de)activation. It simplifies notation to write the second equation of system (3.5) as

$$a^+ \;=\; g_{ij}(y,a), \tag{3.6}$$

where $y := (y_1, \ldots, y_n) \in \mathbb{R}^n$ is the opinions vector, and

$$a := (a_{12}, \ldots, a_{1n}, a_{23}, \ldots, a_{n-2,n}, a_{n-1,n}) \in \{0,1\}^{\frac{n(n-1)}{2}}$$

is the connectivity variables vector.

To complete the model, we present the *memoryless* (de)activation criterion (jump dynamics) for each link between two agents, as proposed in [58, Section 4]. The adaptive thresholds idea of [58] is that two agents interact when their opinions are close relative to their respective neighbours' opinions (an alternative to the fixed threshold HK model [67]). Roughly speaking, given $(i,j) \in \mathcal{E}^+$:

• *Deactivation*. If $a_{ij} = 1$, link (i,j) is active. Deactivation is then enabled when $\Gamma_{ij}^{\text{off}}(y,a) \leq -\varepsilon$, where $\varepsilon > 0$ is a regularization parameter and $\eta > 0$ is a connectivity parameter, while Γ_{ij}^{off} is defined in (3.7) at the top of the next page. This means that link (i,j) is cut when y_i and y_j are too far apart, as compared to other neighbours' opinions. Parameter[1] $\varepsilon > 0$ rules out Zeno solutions, i.e., solutions that jump indefinitely in a finite continuous time interval. It is typically set to a small value.

• *Activation*. If $a_{ij} = 0$, link (i,j) is not active. Activation is enabled when $\Gamma_{ij}^{\text{on}}(y,a) \geq \varepsilon$ with Γ_{ij}^{on} defined in (3.7). The underlying idea is that link (i,j) should be activated when the difference between opinions i and j, namely $|y_i - y_j|$ is small as compared to the average opinion mismatch of agents i and j with their respective neighbours (individuals with relatively close opinions influence each other).

Parameter η determines how big the mismatch $|y_i - y_j|$ needs to be with respect to the average opinions mismatch of agents i and j with their neighbours to (de)activate the link.

As a result, the overall hybrid model for memoryless interactions as proposed in [58] is given by

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \dot{a} \end{bmatrix} &= \begin{bmatrix} -\Phi y \\ 0 \end{bmatrix}, & (y,a) \in C_{\text{inst}} \\ \begin{bmatrix} y^+ \\ a^+ \end{bmatrix} &\in \begin{bmatrix} y \\ \displaystyle\bigcup_{\substack{(y,a)\in D_{ij,\text{inst}} \\ (i,j)\in\mathcal{E}^+}} g_{ij}(y,a) \end{bmatrix}, & (y,a) \in D_{\text{inst}}, \end{aligned} \tag{3.8a}$$

1. Constant ε is the same for every link of the network in [58], however the results do hold mutatis mutandis when it is link dependent, i.e., when we have different $\varepsilon_{ij} > 0$ for each $(i,j) \in \mathcal{V} \times \mathcal{V}$.

$$
\begin{aligned}
\Gamma_{ij}^{\text{on}}(y,a) &:= \sum_{\ell\neq i,\ \ell\neq j} \Big[(d_j+1)\varphi_{i\ell}(y_i-y_\ell)^2 + (d_i+1)\varphi_{j\ell}(y_j-y_\ell)^2\Big] - \left(1+\frac{\eta^2}{d_i d_j}\right)(y_i-y_j)^2 \\
\Gamma_{ij}^{\text{off}}(y,a) &:= \sum_{\ell\neq i,\ \ell\neq j} \Big[\frac{d_j a_{i\ell}}{(d_i-1)d_\ell}(y_i-y_\ell)^2 + \frac{d_i a_{j\ell}}{(d_j-1)d_\ell}(y_j-y_\ell)^2\Big] - \left(1-\frac{\eta^2}{d_i d_j}\right)(y_i-y_j)^2
\end{aligned}
\tag{3.7}
$$

where we recall that $\Phi = [-\varphi_{ij}]_{(i,j)\in\mathscr{V}\times\mathscr{V}}$ and $\mathscr{E}^+$ is in (3.1), and $\mathbb{X}_{\text{inst}} := \mathbb{R}^n \times \{0,1\}^{\frac{n(n-1)}{2}}$, and

$$
\begin{aligned}
D_{ij,\text{inst}}^{\text{on}} &:= \Big\{(y,a)\in\mathbb{X}_{\text{inst}} \mid a_{ij}=0,\ \Gamma_{ij}^{\text{on}}(y,a)\geq\varepsilon\Big\} \\
D_{ij,\text{inst}}^{\text{off}} &:= \Big\{(y,a)\in\mathbb{X}_{\text{inst}} \mid a_{ij}=1,\ \Gamma_{ij}^{\text{off}}(y,a)\leq-\varepsilon\Big\},
\end{aligned}
\tag{3.8b}
$$

$D_{\text{inst}} := \bigcup_{(i,j)\in\mathscr{E}^+} D_{ij,\text{inst}}^{\text{on}} \cup D_{ij,\text{inst}}^{\text{off}}$, and $C_{\text{inst}} := \overline{\mathbb{X}\setminus D_{\text{inst}}}$. The main stability result of [58] is to prove that all maximal solutions to (3.7), (3.8) are complete and eventually continuous (i.e, they perform a finite number of jumps) and all enjoy a desirable global asymptotic stability property for the following set $\mathscr{A}_{\text{inst}}$, measured by the following function ω_0,

$$
\mathscr{A}_{\text{inst}} := \{(y,a)\in\mathbb{X} \mid a_{ij}(y_i-y_j)^2=0,\ \forall (i,j)\in\mathscr{E}^+\}, \tag{3.9}
$$

$$
\omega_0(y,a) := \min_{(z,a)\in\mathscr{A}_{\text{inst}}} |y-z|. \tag{3.10}
$$

Since ω_0 is not a Euclidean norm in the extended (y,a) space (because a is fixed when defining ω_0 in (3.10)), we deem it more appropriate to use in this chapter the following notion of $\mathscr{KL}$-stability, which combines the approach in [132] with the $\mathscr{KL}$ results in [27, Section 3.5].

Definition 3.1. *Let $\omega : \mathbb{R}^{n_q} \to \mathbb{R}_{\geq 0}$ be continuous. A hybrid system is $\mathscr{KL}$-stable with respect to ω if there exists $\beta \in \mathscr{KL}$ such that all maximal solutions ϕ are complete and satisfy $\omega(\phi(t,j)) \leq \beta(\omega(\phi(0,0)), t+j)$ for all $(t,j)\in \operatorname{dom}\phi$.* □

It is proven in the text beneath [58, Lemma 5] that system (3.8) is $\mathscr{KL}$-stable with respect to ω_0. Due to the structure of $\mathscr{A}_{\text{inst}}$ where $a_{ij}(y_i-y_j)^2 = 0$, this property means that solutions asymptotically form clusters [58, Section 4.3].

A possible criticism of the result of [58] summarized above is that the connectivity variables a_{ij} are only based on the instantaneous opinions mismatch, see (3.8b). If two agents had or had not been in agreement for a long time, their current interaction status is not affected by the past. The main contribution of this chapter is to introduce a novel model with memory-based connectivity features. In the next section, we formalize this intuition via a new hybrid model where the past memory is captured by additional state variables. For this model, we will prove a generalization of the above mentioned $\mathscr{KL}$-stability property.

3.3 Memory-based connectivity

3.3.1 Hybrid model

We define the connectivity between agents i and j, for $(i,j) \in \mathscr{E}^+$, using Γ_{ij}^{on} or Γ_{ij}^{off} in (3.7), but also based on a new *memory state variable* $\theta_{ij} \in \mathbb{R}$ that is a filtered version of the local instantaneous threshold criterion reviewed in Section 3.2. Loosely speaking, θ_{ij} reflects the history of the interaction between agents i and j.

More precisely, for each $(i,j) \in \mathscr{E}^+$, the flow dynamics for θ_{ij} is selected as

$$\begin{aligned}\dot{\theta}_{ij} &= -\beta_{ij}\theta_{ij} + (1-a_{ij})\Gamma_{ij}^{\text{on}}(y,a) + a_{ij}\Gamma_{ij}^{\text{off}}(y,a)\\ &=: f_{\theta,ij}(y,a,\theta_{ij}),\end{aligned} \tag{3.11}$$

where $\beta_{ij} > 0$ are tunable parameters associated to how fast each agent "forgets" the past, and Γ_{ij}^{on} and Γ_{ij}^{off} are given in (3.7). When link (i,j) is active, $a_{ij} = 1$ according to (3.2) and $\dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + \Gamma_{ij}^{\text{off}}(y,a)$ in view of (3.11). Hence, variable θ_{ij} filters $\Gamma_{ij}^{\text{off}}(y,a)$, which is indeed the right term to be monitored for deciding whether link (i,j) should be deactivated, see Section 3.2. Conversely, when link (i,j) is not active, $a_{ij} = 0$ and $\dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + \Gamma_{ij}^{\text{on}}(y,a)$ so that $\Gamma_{ij}^{\text{on}}(y,a)$ is filtered to infer whether or not the link should be activated.

Parameters β_{ij} in (3.11) represent how "nostalgic" each pair of agents are with respect to their common past. When β_{ij} is very large, the past is not given much credit and, as $\beta_{ij} \to \infty$, we recover the criterion of Section 3.2. Conversely, when β_{ij} is small, the past values of Γ_{ij}^{off} or Γ_{ij}^{on} matter more, as compared to the instantaneous ones.

When a jump occurs, i.e., when a link is (de)activated, the memory variable θ_{ij} is unchanged, namely $\theta_{ij}^+ = \theta_{ij}$ for each $(i,j) \in \mathscr{E}^+$. The proposed memory-based (de)activation policy then intuitively generalizes the one of Section 3.2:
- *Activation*. If $a_{ij} = 0$, link (i,j) is not active. Activation is enabled when $\Gamma_{ij}^{\text{on}}(y,a) \geq \varepsilon$ and θ_{ij} is non-negative. Parameter ε plays the same role as in Section 3.2, preventing Zeno solutions, see footnote 1 on page 22.
- *Deactivation*. If $a_{ij} = 1$, link (i,j) is active. Deactivation is then enabled when $\Gamma_{ij}^{\text{off}}(y,a) \leq -\varepsilon$ and θ_{ij} is non-positive. The rationale is similar to the previous case.

The mechanism described above can be written in a compact form extending the memoryless model (3.8). Introducing

$$\begin{aligned}\theta &:= (\theta_{12},\ldots,\theta_{1n},\theta_{23},\ldots,\theta_{n-2,n},\theta_{n-1,n}) \in \mathbb{R}^{\frac{n(n-1)}{2}}\\ x &:= (y,a,\theta) \in \mathbb{X}_{\text{mem}} := \mathbb{R}^n \times \{0,1\}^{\frac{n(n-1)}{2}} \times \mathbb{R}^{\frac{n(n-1)}{2}},\end{aligned}$$

the memory-based hybrid model is given by

$$\begin{aligned}\begin{bmatrix}\dot{y}\\ \dot{a}\\ \dot{\theta}\end{bmatrix} &= f(x) := \begin{bmatrix}-\Phi y\\ 0\\ f_\theta(y,a,\theta)\end{bmatrix}, && x \in C_{\text{mem}}\\ \begin{bmatrix}y^+\\ a^+\\ \theta^+\end{bmatrix} &\in g(x) := \begin{bmatrix} y\\ \bigcup\limits_{\substack{(y,a)\in D_{ij,\text{mem}}\\ (i,j)\in\mathscr{E}^+}} g_{ij}(y,a)\\ \theta\end{bmatrix}, && x \in D_{\text{mem}},\end{aligned} \tag{3.12a}$$

with $x := (y, a, \theta)$, with

$$\begin{aligned} D_{\text{mem}} &:= \bigcup_{(i,j)\in\mathscr{E}^+} D^{\text{on}}_{ij,\text{mem}} \cup D^{\text{off}}_{ij,\text{mem}}, \quad C_{\text{mem}} := \overline{\mathbb{X}_{\text{mem}} \backslash D_{\text{mem}}}\\ D^{\text{on}}_{ij,\text{mem}} &:= \Big\{(y,a,\theta) \in \mathbb{X}_{\text{mem}} \mid a_{ij} = 0,\ \Gamma^{\text{on}}_{ij} \geq \varepsilon,\ \theta_{ij} \geq 0\Big\}\\ D^{\text{off}}_{ij,\text{mem}} &:= \Big\{(y,a,\theta) \in \mathbb{X}_{\text{mem}} \mid a_{ij} = 1,\ \Gamma^{\text{off}}_{ij} \leq -\varepsilon,\ \theta_{ij} \leq 0\Big\}. \end{aligned} \tag{3.12b}$$

System (3.12) satisfies the hybrid basic conditions of [64, As. 6.5], in view of the definition of the flow and jump maps and the flow and jump sets. Then, from [64, Thm 6.30], it is (nominally) well-posed, namely its solutions satisfy a desirable sequential compactness property.

3.3.2 Main stability result

We establish here a $\mathscr{KL}$-stability property for (3.12) generalizing the one established for (3.8) at the end of Section 3.2. To this end, function ω_0 in (3.10) is generalized to

$$\omega(x) := \omega_0((y,a)) + \sum_{(i,j)\in\mathscr{E}^+} (1 - a_{ij}) \max\{0, \theta_{ij}\}, \tag{3.13}$$

for any $x \in \mathbb{X}_{\text{mem}}$ which incorporates the memory variable θ.

Since ω_0 is continuous, then ω is continuous too on $\mathbb{X}_{\text{mem}}$. Our main result below ensures the $\mathscr{KL}$-stability property for model (3.12) with respect to ω, as well as properties of the hybrid time domains of its solutions. The proof of Theorem 3.1 is based on a novel hybrid Lyapunov function characterized in the next section.

Theorem 3.1. *All maximal solutions to system (3.12) are complete and eventually continuous. For each maximal solution x, there exists $x^\star \in \mathbb{X}_{mem}$ such that $x(t,j) \to x^\star$ as $t + j \to \infty$. Moreover, system (3.12) is $\mathscr{KL}$-stable with respect to ω in* (3.13). □

Since the second term in (3.13) is non-negative, the convergence to zero of $\omega(x)$ established in Theorem 3.1 immediately implies that $\omega_0((y,a)) \to 0$. As result, Theorem 3.1 ensures that opinions converge to clusters as time grows. In addition, $(1 - a_{ij}) \max\{0, \theta_{ij}\}$ converges to zero for all $(i,j) \in \mathscr{E}^+$, namely the memory variable θ_{ij} associated to individuals belonging to different clusters is not positive. The asymptotic behavior of solutions is clarified in the following corollary, which is an immediate consequence of eventual continuity of solutions (a eventually settles to a clustering pattern) and convergence of solutions (opinions settle to constant values that coincide within each cluster because $\omega_0((y,a)) \to 0$).

Corollary 3.1. *Each maximal solution of (3.12) converges to a clustering pattern with constant and equal opinions in each cluster.* □

3.3.3 Lyapunov function and proof of Theorem 3.1

Consider the following candidate Lyapunov function,

$$U(x) := V(x) + \gamma \sum_{(i,j)\in\mathcal{E}^+} (1-a_{ij}) \max\{0, \theta_{ij}\}, \tag{3.14}$$

for each $x = (y, a, \theta) \in \mathbb{X}_{\text{mem}}$, $\gamma > 0$ to be selected, and

$$V(x) := \frac{1}{2} y^\top \Phi y = \frac{1}{4} \sum_{(i,j)\in\mathcal{V}\times\mathcal{V}} \varphi_{ij}(y_i - y_j)^2. \tag{3.15}$$

The second equality above arises from the Dirichlet form [52, Prop. 1.9] and the definition of Φ after (3.4). Function V is the Lyapunov function used in [58, eqn. (18)], while the second term in (3.14) accounts for the new dynamics θ. The next proposition states key properties of U.

Proposition 3.1. *Given system (3.12), there exist $\gamma > 0$ in* (3.14) *and $c_1, c_2, c_F, c_J > 0$ such that the following holds.*

(i) U is locally Lipschitz on $\mathbb{X}_{mem}$ and satisfies $c_1\omega(x) \le U(x) \le c_2\omega(x)$ for all $x \in \mathbb{X}_{mem}$.

(ii) For all $x \in C_{mem}$, $U^\circ(x; f(x)) \le -c_F U(x)$.

(iii) For all $x \in D_{mem}$, and $v \in g(x)$, $U(v) - U(x) \le -c_J$.

□

Proof. We prove the three items one by one.

Proof of item (i). Function U is locally Lipschitz on $\mathbb{X}_{\text{mem}}$ in view of its definition in (3.14). According to [58, eqn. (19)], there exist $\tilde{c}_1, \tilde{c}_2 > 0$ such that for any $(y,a) \in \mathbb{X}_{\text{inst}}$, $\tilde{c}_1\omega_0((y,a))^2 \le V(x) \le \tilde{c}_2\omega_0((y,a))^2$. As a result, by the definition of ω in (3.13), we obtain the inequality in (i) with $c_1 := \min\{\tilde{c}_1, \gamma\}$ and $c_2 := \max\{\tilde{c}_2, \gamma\}$.

Proof of item (ii). Given any $x = (y, a, \theta) \in C_{\text{mem}}$, introduce the sets (below, for simplicity, the dependence on x is sometimes omitted)

$$\begin{aligned} \mathcal{E}^c_{>0}(x) &:= \{(i,j) \in \mathcal{E}^+ \mid a_{ij} = 0 \text{ and } \theta_{ij} > 0\}, \\ \mathcal{E}^c_{=0}(x) &:= \{(i,j) \in \mathcal{E}^+ \mid a_{ij} = 0 \text{ and } \theta_{ij} = 0\}, \\ \mathcal{E}^c_{\Gamma}(x) &:= \{(i,j) \in \mathcal{E}^+ \mid \Gamma^{\text{on}}_{ij}(y,a) \ge 0\}. \end{aligned} \tag{3.16}$$

According to [91, Prop. 1.1], in view of (3.11), (3.12), (3.14) and the definition of f_θ, we have that

$$\begin{aligned} U^\circ(x; f(x)) &= \langle \nabla V(x), f(x)\rangle + \gamma \sum_{(i,j)\in\mathscr{E}^c_{>0}} \left(-\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right) + \gamma \sum_{(i,j)\in\mathscr{E}^c_{=0}} \max\left\{0, -\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right\} \\ &= \langle \nabla V(x), f(x)\rangle + \gamma \textstyle\sum_{(i,j)\in\mathscr{E}^c_{>0}} \left(-\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right) + \gamma \textstyle\sum_{(i,j)\in(\mathscr{E}^c_{=0}\cap\mathscr{E}^c_\Gamma)} \Gamma^{\text{on}}_{ij}(y,a) \\ &= \langle \nabla V(x), f(x)\rangle + \gamma \textstyle\sum_{(i,j)\in\widehat{\mathscr{E}}} \left(-\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right), \end{aligned}$$

where $\widehat{\mathscr{E}}(x) := \mathscr{E}^c_{>0}(x) \cup (\mathscr{E}^c_{=0}(x) \cap \mathscr{E}^c_\Gamma)(x))$. Note that [58, eqn. (20)] holds because variables (y,a) obey the same flow dynamics as in (3.8) and (3.12). Then $\langle \nabla V(x), f(x)\rangle \leq -\tilde{c}_F V(x)$ for some $\tilde{c}_F > 0$. Consequently,

$$U^\circ(x; f(x)) \leq -\tilde{c}_F V(x) - \gamma \sum_{(i,j)\in\widehat{\mathscr{E}}} \beta_{ij}\theta_{ij} + \gamma \sum_{(i,j)\in\widehat{\mathscr{E}}} \Gamma^{\text{on}}_{ij}(y,a).$$

The expression of Γ^{on}_{ij} in (3.7), together with (3.15) yields

$$\begin{aligned} \sum_{(i,j)\in\widehat{\mathscr{E}}} \Gamma^{\text{on}}_{ij}(y,a) &= \sum_{(i,j)\in\widehat{\mathscr{E}}} \left(-\left(1 + \frac{\eta^2}{d_i d_j}\right)(y_i - y_j)^2 + \sum_{\ell\neq i,\ \ell\neq j} \left((d_j+1)\varphi_{i\ell}(y_i - y_\ell)^2 + (d_i+1)\varphi_{j\ell}(y_j - y_\ell)^2\right)\right) \\ &\leq n \sum_{(i,j)\in\widehat{\mathscr{E}}} \sum_{\ell\neq i,\ \ell\neq j} \left(\varphi_{i\ell}(y_i - y_\ell)^2 + \varphi_{j\ell}(y_j - y_\ell)^2\right) \\ &\leq n \sum_{(i,\ell)\in\mathscr{V}\times\mathscr{V}} \varphi_{i\ell}(y_i - y_\ell)^2 + n \sum_{(j,\ell)\in\mathscr{V}\times\mathscr{V}} \varphi_{j\ell}(y_j - y_\ell)^2 \\ &= 4nV(x) + 4nV(x) = 8nV(x), \end{aligned} \tag{3.17}$$

which can be substituted in the preceding inequality to get

$$U^\circ(x; f(x)) \leq -(\tilde{c}_F - 8n\gamma)V(x) - \gamma \sum_{(i,j)\in\widehat{\mathscr{E}}} \beta_{ij}\theta_{ij}. \tag{3.18}$$

Now, $\theta_{ij} = (1 - a_{ij})\max\{0, \theta_{ij}\}$ for $(i,j) \in \widehat{\mathscr{E}} \subset \mathscr{E}^c_{>0} \cup \mathscr{E}^c_{=0}$, and $(1 - a_{ij})\max\{0, \theta_{ij}\} = 0$ for $(i,j) \in \mathscr{E}^+ \backslash \widehat{\mathscr{E}}$, because either $a_{ij} = 1$ or $\theta_{ij} \leq 0$ for those edges. Then,

$$U^\circ(x; f(x)) \leq -c_F U(x), \tag{3.19}$$

where $c_F = \min\{\tilde{c}_F - 8n\gamma, \beta\}$ and $\beta := \min_{(i,j)\in\mathscr{E}^+} \beta_{ij} > 0$, which implies item (ii) with any $\gamma \in (0, \frac{\tilde{c}_F}{8n})$.
Proof of item (iii). From (3.12b), for each $x \in D_{\text{mem}}$ and $\upsilon \in g(x)$, there exists $(i,j) \in \mathscr{E}^+$ such that $x \in D^{\text{on}}_{ij,\text{mem}} \cup D^{\text{off}}_{ij,\text{mem}}$, and $a^+ \in g_{ij}(y,a)$. Since $D^{\text{on}}_{ij,\text{mem}}$ and $D^{\text{off}}_{ij,\text{mem}}$ are disjoint, then two cases may occur: case "on" and case "off" below.
Case "on": $x \in D^{\text{on}}_{ij,\text{mem}}$. In this case, $a_{ij} = a_{ji} = 0$, $\Gamma^{\text{on}}_{ij}(y,a) \geq \varepsilon$ and $\theta_{ij} \geq 0$. Since link (i,j) is activated at this jump, $a^+_{ij} = a^+_{ji} = 1$ and from (3.5), (3.6), the other connectivity variables, as well as all the memory variables in view of (3.12), remain constant across the jump. Consequently,

$$U(\upsilon) - U(x) = V(\upsilon) - V(x) - \gamma \max\{0, \theta_{ij}\}. \tag{3.20}$$

Following the proof [58, Lemma 5], we get

$$V(v)-V(x)=\frac{1}{2(d_i+1)(d_j+1)}\left(-\Gamma_{ij}^{\text{on}}(y,a)-\frac{\eta^2}{d_id_j}(y_i-y_j)^2\right).$$

Thus, (3.20) yields

$$U(v)-U(x)\leq\frac{1}{2d_i^+d_j^+}\left(-\varepsilon-\frac{\eta^2}{d_id_j}(y_i-y_j)^2\right),$$

where $d_i^+=d_i+1$ and $d_j^+=d_j+1$. Taking $c_J\in\left(0,\frac{\varepsilon}{2n^2}\right]$, characterizing the least possible decrease with n agents, we prove item (iii) for the case "on".

Case "off": $x\in D_{ij,\text{mem}}^{\text{off}}$. In this case, $a_{ij}=a_{ji}=1$, $\Gamma_{ij}^{\text{off}}\leq-\varepsilon$ and $\theta_{ij}\leq 0$. Link (i,j) is deactivated at this jump and $a_{ij}^+=a_{ji}^+=0$. The other connectivity variables and the memory variables remain constant. Then (3.20) holds again and following again the proof of [58, Lemma 5], we deduce that $V(v)-V(x)=\frac{1}{2d_id_j}\left(\Gamma_{ij}^{\text{off}}(y,a)-\frac{\eta^2}{d_id_j}(y_i-y_j)^2\right)$. Thus, (3.20) yields

$$U(v)-U(x)\leq\frac{1}{2d_id_j}\left(-\varepsilon-\frac{\eta^2}{d_id_j}(y_i-y_j)^2\right).$$

Selecting $c_J\in\left(0,\frac{\varepsilon}{2n^2}\right]$, item (iii) holds in case "off".
Thus, item (iii) holds with $c_J:=\frac{\varepsilon}{2n^2}$. ∎

Proof of Theorem 3.1. To prove that maximal solutions to (3.12) are complete, we invoke [64, Prop. 6.10]. First, the viability condition is satisfied in view of the system definition. Secondly, $g(D_{\text{mem}})\subset C_{\text{mem}}\cup D_{\text{mem}}$. Moreover, using $W(x)=y^\top y$, we have $\langle\nabla W(x),f(x)\rangle=-2y^\top\Phi y\leq 0$, therefore the y components are bounded. Also the memory variables θ are bounded, because they are constant across jumps and the components of its flow map are exponentially stable filters with integrable inputs. Consequently, maximal solutions do not escape in finite time and [64, Prop. 6.10] establishes their completeness. Eventual continuity follows from the fact that the decrease of U across jumps in item (iii) of Proposition 3.1 is constant at each jump and that U does not increase on flows in item (ii) of Proposition 3.1, therefore any solution jumping forever would eventually lead to a negative $U(x)$, contradicting item (i) of Proposition 3.1. About convergence of solutions, state a settles due to eventual continuity, y settles too because $\langle\nabla W(x),f(x)\rangle=-2y^T\Phi y=0$ and symmetry of Φ implies $-2\dot{y}=\Phi y=0$, finally, θ converges too because it is a linear filter with a converging input.

Next we establishes a $\mathcal{KL}$ bound on ω. Let x be a solution to (3.12), from item (ii) of Proposition 3.1 implies that, for each $i\in\{0,...,j\}$ and for almost all $s\in[t_i,t_{i+1}]$,

$$U^\circ(x(s,i);f(x(s,i)))\leq-c_FU(x(s,i)). \tag{3.21}$$

In view of [133, page 99] and (3.21), and since U is locally Lipschitz, it holds that

$$\frac{d}{ds}U(x(s,i)) \leq U^{\circ}(x(s,i); f(x(s,i))), \tag{3.22}$$

and thus, from (3.21) and (3.22), for each $i \in \{0,...,j\}$ and for almost all $s \in [t_i, t_{i+1}]$ we have that

$$\frac{d}{ds}U(x(s,i)) \leq -c_F U(x(s,i)), \tag{3.23}$$

which leads, by applying the comparison principle [73, Lemma 3.4], to

$$U(x(s,i)) \leq e^{-c_F(s-t_i)}U(x(t_i,i)), \quad \forall i \in \{0,...,j\}, \quad \forall s \in [t_i, t_{i+1}]. \tag{3.24}$$

Let $\{(t_{i+1}, i+1), (t_{i+1}, i)\} \in \operatorname{dom} x$, in view of item (iii) of Proposition 3.1, it holds that

$$U(x(t_{i+1}, i+1)) - U(x(t_{i+1}, i)) \leq -c_J = -c_J \frac{\max(U(x(0,0)),1)}{\max(U(x(0,0)),1)} < -c_J \frac{U(x(0,0))}{\max(U(x(0,0)),1)}. \tag{3.25}$$

In view of (3.24) and (3.25), for any $(t,j) \in \operatorname{dom} x$, $U(x(t,j)) \leq U(x(0,0))$, hence

$$U(x(t_{i+1}, i+1)) \leq -c_J \frac{U(x(0,0))}{\max(U(x(0,0)),1)} + U(x(t_{i+1}, i)) = \left(1 - \frac{c_J}{\max(U(x(0,0)),1)}\right) U(x(t_{i+1}, i)), \tag{3.26}$$

where we consider, without loss of generality, $c_J < 1$. Therefore, equations (3.24) and (3.26) yield to

$$U(x(t,j)) \leq e^{-c_F t} \left(1 - \frac{c_J}{\max(U(x(0,0)),1)}\right)^j U(x(0,0)), \quad \forall (t,j) \in \operatorname{dom} x, \tag{3.27}$$

which implies, in view of item (i) of Proposition 3.1, that

$$\omega(x(t,j)) \leq \frac{c_2}{c_1} e^{-c_F t} \left(1 - \frac{c_J}{\max(c_2 \omega(x(0,0)),1)}\right)^j \omega(x(0,0)) =: \beta(\omega(x(0,0)), t, j), \quad \forall (t,j) \in \operatorname{dom} x, \tag{3.28}$$

which establishes a $\mathcal{KLL}$ bound [27] on ω, in view of the fact that we can choose, without loss of generality, $c_J < 1$. Hence, in view of [27, Lemma 6.1] and (3.28), there exists $\overline{\beta} \in \mathcal{KL}$ such that, for all solutions x to (3.12),

$$\omega(x(t,j)) \leq \overline{\beta}(\omega(x(0,0)), t+j), \quad \forall (t,j) \in \operatorname{dom} x.$$

■

Notice the design of the adaptive thresholds 3.7 has also been motivated by the choice of V.

3.4 Simulations

We consider $n = 15$ agents, $\varepsilon = 0.01$ and $\eta = 3$ in (3.7). The initial topology is an Erdős-Rényi random graph, with probability p of having an interconnection between each node pair (i,j), while the initial values of $y_i, i \in \{1, \ldots, n\}$ are selected randomly in the interval $[0, 1]$. The result of simulations [2] done using model (3.8) is reported in Fig. 3.1.

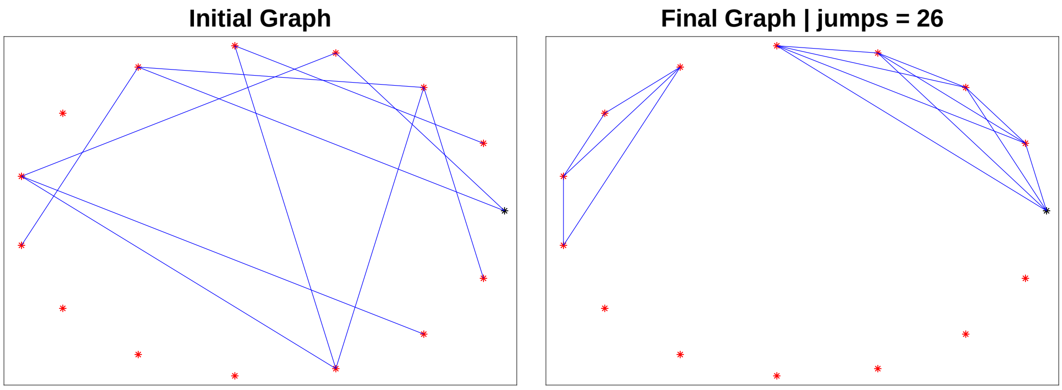

FIGURE 3.1 – Initial and final topologies for given $y(0, 0)$, with $p = 0.1$. Nodes have been sorted counterclockwise to clearly visualize the clusters appearing in the final topology.

We then study model (3.12), the impact of the choices of β_{ij} and the initial values of θ_{ij} on the evolution of the opinions, for the same $y(0, 0)$ as in Fig. 3.1. In particular, we take for $\beta_{ij} = \beta$ with $\beta \in \{0.1, 50\}$, and $\theta_{ij}(0, 0) = \theta^o$ with $\theta^o \in \{0, 0.01, 1\}$ for $a_{ij}(0, 0) = 1$ and $\theta^o = 0$ otherwise, as well as the case where θ^o takes random values in $[-1, 0)$, for all $(i, j) \in \mathcal{E}^+$. The final graphs are depicted in Fig. 3.2 and 3.3.

2. The simulations have been carried out using the Matlab toolbox [114].

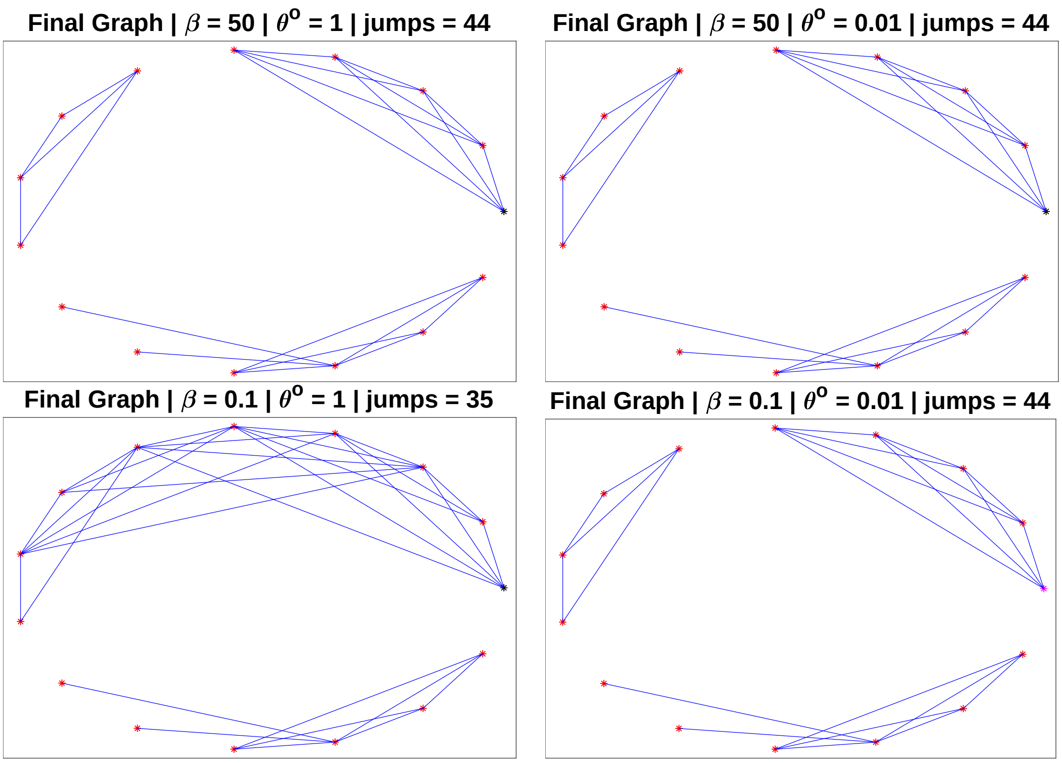

FIGURE 3.2 – Four different final topologies for different couples of $(\beta, \theta(0,0))$.

We can see that the opinions converge to fixed agreement values in each cluster, in agreement with Corollary 3.1. In general these clusters are different from those obtained with model (3.8). There is only one situation where the obtained clusters are the same as in Fig. 3.1: when $\theta^o = 0$, see Fig. 3.3. This can be explained by the fact that the opinions converge quickly to cluster formations and θ has no impact on it.

When comparing Fig. 3.2 and 3.3, we note that positive values of θ tend to preserve the exiting initial interconnections, leading to larger clusters. This suggests that agents remember their mutual relationships with each other. On the other hand, negative values generate clusters made of fewer agents in general, see Fig. 3.2 compared to Fig. 3.3.

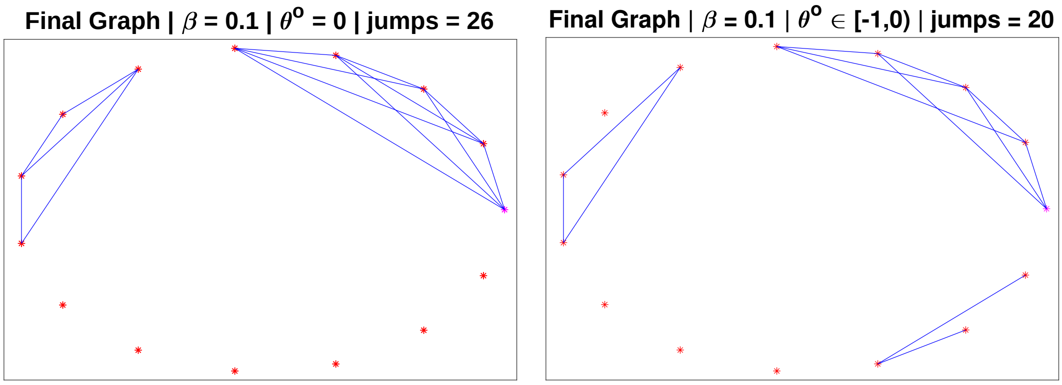

FIGURE 3.3 – Different final topologies for different, non positive $\theta(0,0)$.

The evolutions of y and θ as functions of the continuous-time t are depicted in Fig. 3.4 for $\beta = 0.1$ and $\theta^o = 1$. Two clusters appear as time grows, in agreement with the corresponding plot in Fig. 2. When agents i and j are in the same cluster, θ_{ij} converges to 0 just as $\Gamma^{\text{off}}_{ij}(y,a)$ does in this case. When agents i and j are not in the same cluster, θ_{ij} converges to the same values of $\Gamma^{\text{on}}_{ij}(y,a)/\beta_{ij}$ in view of (3.11). This ratio, $\Gamma^{\text{on}}_{ij}(y,a)/\beta_{ij}$, can take any constant value in $(-\infty, \varepsilon]$ in view of (3.12b). Hence, in some cases we have $\theta_{ij}(t,j) \to 0$ as $t + j \to \infty$ although agents i and j are not in the same cluster. In this context, to distinguish agents from the same clusters, it is more relevant to monitor σ_{ij} defined as:

$$\sigma_{ij} := (2a_{ij} - 1)\frac{|\theta_{ij}|}{|(1-a_{ij})\Gamma^{\text{on}}_{ij}(y,a) + a_{ij}\Gamma^{\text{off}}_{ij}(y,a)|}. \tag{3.29}$$

At steady-state, σ_{ij} either converges to $1/\beta_{ij}$ if agents i and j belong to the same cluster, or to $-1/\beta_{ij}$ otherwise. The influence of β_{ij} is clear here: the smaller β_{ij}, the more important the past is and the bigger σ_{ij}, and vice versa.

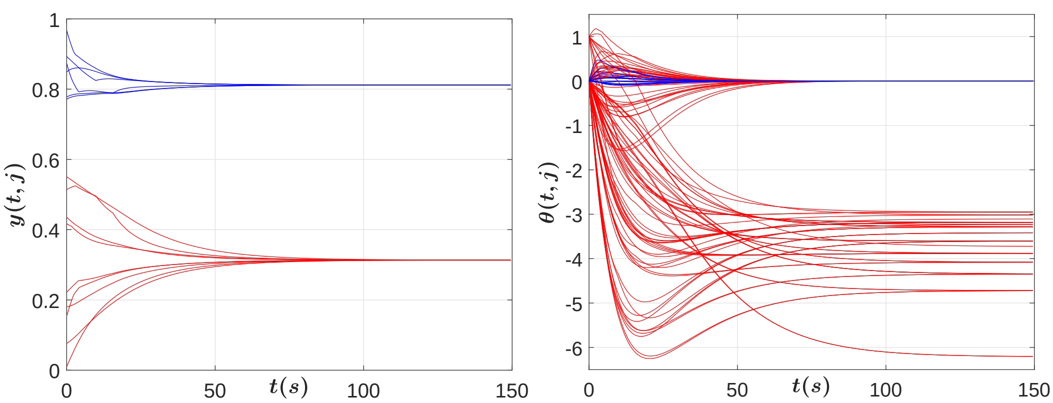

FIGURE 3.4 – Values of y (left) and θ (right) for $\beta = 0.1$ and $\theta^o = 1$. Different colors have been used for different clusters in the final topology.

3.5 Conclusions

We have presented a hybrid model of opinion dynamics where the connectivity among individuals takes into account both the present and the past values of the opinions of the respective individuals. Filtered versions of the adaptive thresholds advocated in [58] have been introduced for this purpose. The overall system has been modeled using the hybrid formalism of [64]. A global asymptotic stability property of a given closed set has been proven using a new hybrid Lyapunov function. This stability property shows that individuals form clusters as time grows. The model depends on two types of parameters: those related to the connectivity of each agent, and those related to the filter dynamics and thus how much the past needs to be considered to define connectivity. We believe that the idea of taking into account the past when defining connectivity in opinion dynamics is appealing and relevant, and that it has been largely unexplored so far. Fruitful directions to investigate could be to extend this framework to multi-topic models of opinion dynamics and to the scenarios where the interactions among the agents are asymmetric. Further discussions on possible scientific directions are presented in Chapter 6.

Chapter 4

Hybrid coupling rules for heterogeneous oscillators

Contents

Abstract - We investigate in this chapter the engineering scenario where the objective is to synchronize a network of heterogeneous oscillators in a distributed fashion. The internal dynamics of each oscillator are general enough to capture their time-varying natural frequency as well as physical couplings and unknown bounded terms. A communication layer is set in place to allow the oscillators to exchange synchronizing coupling actions through a tree-like leaderless network. In particular, we present a class of hybrid coupling rules depending only on local information that can recover locally

the behaviour of Kuramoto oscillators, while ensuring a uniform global practical or asymptotic stability of the synchronization set, which is impossible with Kuramoto models. We further show that the synchronization set can be made uniformly globally finite-time stable by selecting the coupling function to be discontinuous at the origin. Novel mathematical tools on non-pathological functions and set-valued Lie derivatives are developed to carry out the stability analysis. Numerical simulation results illustrate the advantages of the proposed hybrid coupling rules with respect to the non-uniform behavior typically found with Kuramoto models. The effectiveness of the approach is illustrated in simulations where we apply our synchronizing hybrid coupling rules to models of power grids previously used in the literature.

4.1 Introduction

The Kuramoto model [75] is used in various research fields to describe and analyze the dynamics of a broad family of systems with oscillatory behavior [2] including neuroscience [9, 37, 130], chemistry [54], power networks [47] and natural sciences [76], to cite a few (see also [126]). The many application areas where Kuramoto dynamics emerged from physical considerations motivated a detailed analysis of the synchronization properties of the model, first for the all-to-all connection case [3], as originally described by Kuramoto, then for a general interconnection layout [70], with a focus on the derivation of the least conservative lower bound for a stabilizing coupling gain [31, 51, 71].

Given its simple and accurate description of natural synchronization phenomena, the Kuramoto model has also inspired the design of distributed communication protocols in engineering applications where the coupling function among different agents can be arbitrarily assigned to achieve synchronization, as in the bio-inspired synchronization of moving particles in [120], the synchronized acquisition of oceanographic data from Autonomous Underwater Vehicles [15], in clock synchronization [74], in mobile sensors networks modeled as particles with coupled oscillator dynamics [97], in monotone coupled oscillators [84] or in other engineering applications surveyed in [49].

While the sinusoidal coupling of Kuramoto models provides a powerful tool to obtain synchronization in coupled networks of oscillators, it also introduces some undesirable properties for engineering applications. For example, when the network comprises oscillators with the same natural frequency, it is now well-known that a system of Kuramoto oscillators admits, in addition to stable equilibria coinciding with the synchronization set, equilibria that are unstable (see, e.g., [120, 125]). The downside of this result is that the closer a solution is initialized to an unstable equilibrium, the longer it will take for phase synchronization to arise: we talk of *non-uniform* convergence [120]. Although non-uniform synchronization may naturally characterize certain physical [96] and biological systems, in general it is not a desirable property for engineering applications. Indeed, the lack of uniformity may induce arbitrarily slow convergence to the attractor set and poor robustness properties [88]. Secondly, it may occur in the Kuramoto model that the angular phase mismatch between adjacent oscillators remains constant and different from zero indefinitely: in this case we talk of *phase locking* [3], which hampers the capability to reach asymptotic collective synchronization. Thirdly, in critical applications, finite-time stability, instead of only asymptotic synchronization, may be a mandatory requirement [104].

In this work, we investigate the engineering scenario where the goal is to synthesize local coupling rules to synchronize a set of heterogeneous oscillators. We assume the model of the oscillators to be general enough to capture not only their (time-varying) natural frequency but also physical coupling actions and other unknown bounded terms, thus being able to represent, among many possibilities, networks of Kuramoto oscillators with heterogeneous time-varying natural frequencies. Furthermore, without loss of generality, we introduce suitable resets of the oscillators' phase coordinates, so that they are unwrapped to evolve in a compact set, which includes $[-\pi, \pi]$ consistently with their angular nature. Consequently, we define hybrid 2π-unwinding mechanisms to ensure the forward completeness of the oscillating solutions.

To achieve uniform global phase synchronization, thereby overcoming the limitations of Kuramoto models, we equip the oscillators with a leaderless tree-like communication network to locally exchange coupling actions based on local information. This approach has been already exploited in the context of DC microgrids as in, e.g., ([36]), or ([61]), for a network of Kuramoto oscillators equipped with a leader. The selection of a tree-like graph, which can always be derived in a distributed way by using the algorithms surveyed in ([98]), is also not new while addressing a problem of distributed cooperative control: see ([85]) in the context of hybrid dynamical systems, or ([14]) and ([4]) for continuous-time networked systems and power grids, respectively. To define the coupling actions, we present novel hybrid coupling rules for which a Lyapunov-based analysis ensures uniform global (practical or asymptotic) phase synchronization. This result overcomes both the lack of uniform convergence and the phase-locking issues characterizing the Kuramoto model ([120]). Interestingly, we can design the coupling rules in such a way that the network of oscillators behaves like the original Kuramoto models when the oscillators are near phase synchronization. Furthermore, due to the mild properties that we require for our hybrid coupling function, discontinuous selections are allowed, like in ([35]). When the discontinuity is at the origin, we prove finite-time stability properties. In particular, exact synchronization can be reached in a prescribed finite-time ([121]), and convergence is thus independent of the initial conditions. Compared to the related works in ([84]) and ([141]), the finite-time stability property we ensure is global and the convergence time can be arbitrarily prescribed, respectively. We resort for this purpose to non-smooth Lyapunov theory, in particular non-pathological Lyapunov functions and set-valued Lie derivatives ([13]), for which we provide new results and novel proof techniques that are of independent interest. Due to the possible presence of discontinuities in the coupling function, the stability analysis is carried out by focusing on the regularization of the dynamics, as typically done in the hybrid formalism of [64, Ch. 4]. Finally, simulations are provided to illustrate the theoretical guarantees and demonstrate the potential strength of our hybrid theoretical tools to address both first and second-order oscillators modeling generators in power grids considered in ([48]).

The recent submission ([19]) (see also ([20])) also uses hybrid tools to obtain uniform global synchronization guarantees in a Kuramoto setting but in a different context, namely for second-order oscillators (where the ω_i's are states rather than external inputs) and, most importantly, for a network with a leader, which significantly changes the setting compared to the leaderless scenario investigated in this work, where no oscillator is insensible to the coupling actions from its neighbours. With respect to the preliminary version of this work in ([18]), we include the next novel elements: relaxed requirements on the coupling function, time-varying, phase-dependent, (possibly) non-identical natural frequencies, generalizing the two-agents theorems of ([18]) to the case of n oscillators in addition to establishing a set of new stability results missing in ([18]) (finite-time, practical properties and other ancillary results).

The rest of the chapter is organized as follows. The local hybrid coupling rules and oscillators network model are derived in Section 4.2. In Section 4.3, we introduce the regularized version of the dynamics presented in Section 4.2. In Section 4.4, we present Lyapunov-based analysis tools establishing the asymptotic properties of our model, while prescribed finite-time results are given in Section 4.5. Numerical illustrations are provided in Section 4.6, while most of the technical aspects of our proofs requiring non-smooth analysis concepts are gathered in Section 4.7. A few proofs of minor importance are relegated to Annex A.

4.2 Oscillators with hybrid coupling

4.2.1 Flow dynamics

Consider a networked system of n heterogeneous oscillators. To achieve synchronization, the oscillators locally exchange coupling actions through the *unweighted undirected tree* [3] $\mathscr{G}_u := (\mathscr{V}, \mathscr{E}_u)$ made of n nodes and thus $m = n-1$ edges, $n \in \mathbb{Z}_{>1}$. We assign an arbitrary orientation to $\mathscr{G}_u$, which leads to the oriented tree $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. In this scenario, the oscillator phase corresponding to node i, with $i \in \mathscr{V}$, is denoted θ_i and has the next flow dynamics

$$\begin{aligned}\dot{\theta}_i = \omega_i(\theta, \mathfrak{t}) + \kappa \sum_{j \in \mathscr{O}_i} \sigma(\theta_j - \theta_i + 2q_{ij}\pi) \\ - \kappa \sum_{j \in \mathscr{I}_i} \sigma(\theta_i - \theta_j + 2q_{ji}\pi), \quad (\theta, q) \in C \end{aligned} \tag{4.1}$$

where $\omega_i(\theta, \mathfrak{t})$ is a possibly an unknown term modeling the dynamics of the i-th oscillator, which can capture physical coupling actions, its time-varying natural frequency, and any other unknown bounded dynamics affecting the oscillator; see Section 4.6 for a numerical example. We assume that ω_i is locally bounded, measurable in $\mathfrak{t}$, piecewise continuous in θ and such that $\omega_i(\theta, \mathfrak{t}) \in \Omega := [\omega_{\mathrm{m}}, \omega_{\mathrm{M}}]$ for any time $\mathfrak{t} \geq 0$ and $(\theta, q) \in C$, with $\omega_{\mathrm{m}} \leq \omega_{\mathrm{M}} \in \mathbb{R}$, namely Ω is a compact interval of values [4]. Since (4.1) possibly has a discontinuous right-hand side, the notion of solution should be carefully defined, and we postpone this discussion to Section 4.3 (where we also prove the existence of solutions) to avoid overloading the exposition. For now it suffices to say that a function θ is a

3. As mentioned in the introduction, we can obtain a spanning tree using any of the distributed, finite-time algorithms described in ([98]).

4. The assumption that ω_i, for any $i \in \mathscr{V}$, takes values in the compact set Ω could be relaxed by only assuming boundedness of the mismatch $\sup_{(\mathfrak{t},\theta)\in\mathbb{R}_{\geq 0}\times[-\pi-\delta,\pi+\delta]} |\omega_i(\mathfrak{t}, \theta) - \omega_j(\mathfrak{t}, \theta)|$ for any pair $(i, j) \in \mathscr{E}$, and adapting the proofs accordingly.

solution of (4.1) if it is absolutely continuous (i.e., it coincides with the integral of its derivative) and satisfies (4.1) almost everywhere.

Phase θ_i in (4.1) evolves in the set $[-\pi-\delta,\pi+\delta]$, with $\delta\in(0,\pi)$, which thus covers the unit circle corresponding to phases taking values in $[-\pi,\pi]$. Parameter $\delta>0$ inflates the set of angles $[-\pi,\pi]$ to rule out Zeno solutions as explained in the following, see Section 4.2.3. Thus, δ is a regularization parameter chosen to be the same for each oscillator. Variable q_{ij}, with $(i,j)\in\mathcal{E}$, is a logic state taking values in $\{-1,0,1\}$, which is constant during flows. Its role is to unwind the difference between the two phases θ_j and θ_i through jumps. Indeed, since θ_j and θ_i are angles, to evaluate their mismatch, loosely speaking, we have to consider their minimum mismatch modulo 2π: q_{ij} is introduced for this purpose as clarified in Section 4.2.2. The vectors θ and q collect all the states θ_i, $i\in\mathcal{V}$, and q_{ij}, $(i,j)\in\mathcal{E}$, respectively, as formalized in the following, together with the formal definition of the flow set C, namely a compact subset of the state-space where the solutions are allowed to evolve continuously. The gain $\kappa\in\mathbb{R}_{>0}$ is associated with the intensity of each coupling action and it is the same for each interconnection. Finally, the coupling action between each pair of nodes $(i,j)\in\mathcal{E}$ is defined as $\sigma(\theta_j-\theta_i+2q_{ij}\pi)$, where σ is the function used to penalize the phase mismatch $\theta_j-\theta_i+2q_{ij}\pi$ between phases θ_j and θ_i, and it satisfies the next property.

Property 4.1. *Function σ is piecewise continuous on* $\operatorname{dom}\sigma:=[-\pi-\delta,\pi+\delta]$ *and satisfies*

a) $\sigma(s)=-\sigma(-s)$ *for any* $s\in\operatorname{dom}\sigma$,

b) there exists $\alpha\in\mathcal{K}$ *such that* $\operatorname{sign}(s)\sigma(s)\geq\alpha(|s|)$ *for any* $s\in\operatorname{dom}\sigma\setminus\{0\}$. □

Item a) of Property 4.1 ensures that σ is an odd function and thus implies $\sigma(0)=0$, while item b) of Property 4.1 guarantees that $\sigma(s)$ can only be zero at $s=0$. Notice that the sine function, customarily used in the classical Kuramoto model, satisfies item a) but not item b) of Property 4.1, which is fundamental to establish the global uniform stability result of this work. Examples of functions σ satisfying Property 4.1 are depicted in Fig. 4.1, together with the sine function for the sake of comparison. We emphasize that the mild assumptions of Property 4.1 allow considering, among others, intuitive discontinuous selections such as the sign function of Fig. 4.1, which leads to an interesting parallel between (4.1) and the ternary controllers considered in [39]. Another possible example of σ enjoying Property 4.1 is $\sigma(s)=\sin(s)+u(s)$, where u is such that items a) and b) of Property 4.1 hold. Note that when u is negligible compared to sin in a neighborhood of the origin, the model behaves locally like the classical Kuramoto network, as illustrated in Section 4.6. Also, Property 1 comes with no loss of generality as we consider the scenario where we have the freedom to design the coupling rules among the oscillators and thus σ.

Function σ is only defined on $\operatorname{dom}\sigma=[-\pi-\delta,\pi+\delta]$ according to Property 4.1. We ensure in the sequel that the argument of σ in (4.1), namely $\theta_j-\theta_i+2q_{ij}\pi$, belongs to $\operatorname{dom}\sigma$ for all $(i,j)\in\mathcal{E}$, whenever $x\in C$, so that (4.1) is well-defined, see Section 4.2.2.

Collecting in the vector $\boldsymbol{\sigma}(x)\in\mathbb{R}^m$ all the coupling actions $\sigma(\theta_j-\theta_i+2q_{ij}\pi)$, with $(i,j)\in\mathcal{E}$,

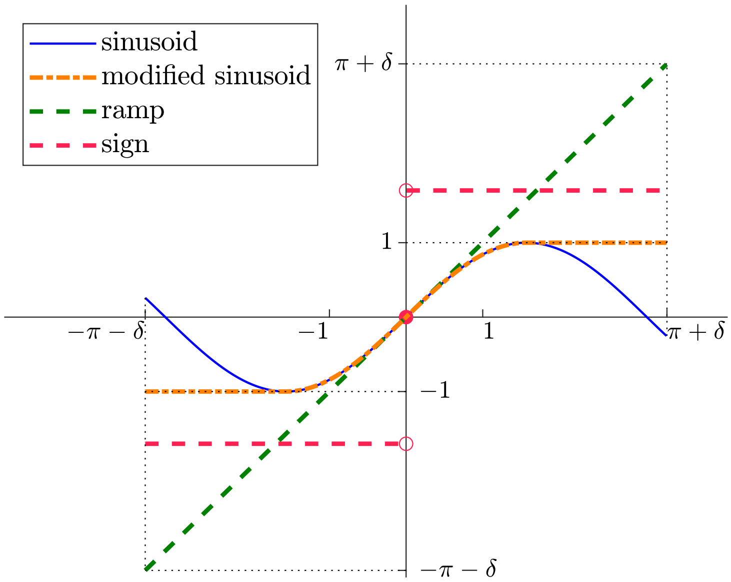

FIGURE 4.1 – Examples of functions σ satisfying Property 4.1, together with the sine function (which does not satisfy Property 4.1).

using the same order as the columns of B, the flow dynamics in (4.1) is written as

$$\dot{x}=\begin{bmatrix}\dot{\theta}\\ \dot{q}\end{bmatrix}=f(x,\boldsymbol{\omega}(\theta,\mathrm{t})):=\begin{bmatrix}\boldsymbol{\omega}(\theta,\mathrm{t})-B\kappa\boldsymbol{\sigma}(x)\\ \mathbf{0}_m\end{bmatrix},\qquad x\in C, \tag{4.2}$$

with $\theta := (\theta_1,\dots,\theta_n)\in[-\pi-\delta,\pi+\delta]^n$, $\boldsymbol{\omega}(\mathrm{t}) := (\omega_1(\theta,\mathrm{t}),\dots,\omega_n(\theta,\mathrm{t}))\in\Omega^n$, and where $q\in\{-1,0,1\}^m$ is the vector stacking all the q_{ij}'s for $(i,j)\in\mathscr{E}$, ordered as in $\boldsymbol{\sigma}(x)$. Thus, the overall state $x:=(\theta,q)$ evolves in the compact state space defined as

$$X:=[-\pi-\delta,\pi+\delta]^n\times\{-1,0,1\}^m. \tag{4.3}$$

The flow set C in (4.2) will be selected as the closed complement of the jump set D introduced next.

4.2.2 Jump dynamics

We introduce jump rules to constrain each phase θ_i to take values in $[-\pi-\delta,\pi+\delta]$ as well as to guarantee that the argument $\theta_j-\theta_i+2q_{ij}\pi$ of σ in (4.1) belongs to $\operatorname{dom}\sigma=[-\pi-\delta,\pi+\delta]$ when flowing. To guarantee the latter property, define, for any $(i,j)\in\mathscr{E}$, the jump set

$$D_{ij}:=\left\{x\in X: |\theta_j-\theta_i+2q_{ij}\pi|\geq\pi+\delta\right\}, \tag{4.4a}$$

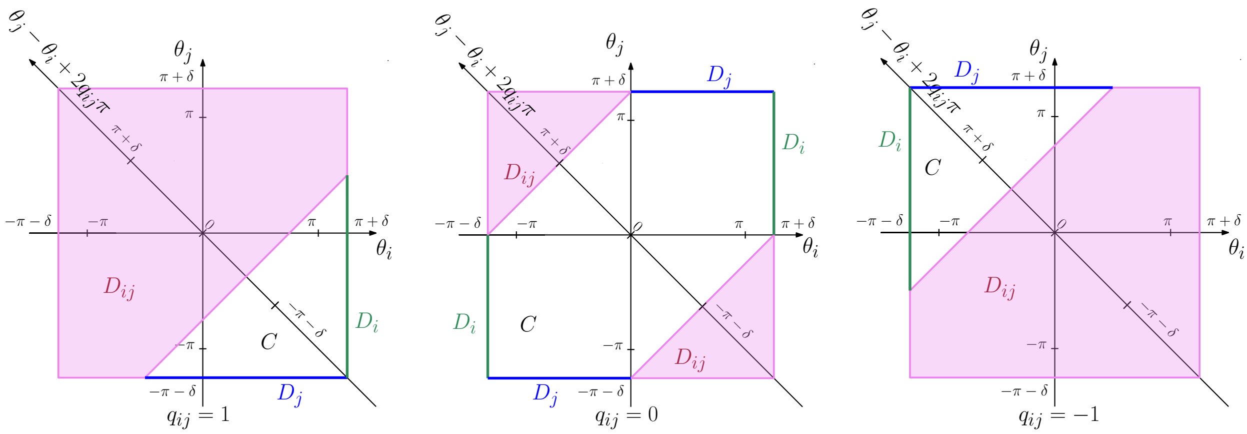

FIGURE 4.2 – Projection of the flow and jump sets on (θ_i, θ_j) for each value of q_{ij}.

and the associated difference inclusion

$$x^+ = \begin{bmatrix} \theta^+ \\ q^+ \end{bmatrix} \in G_{ij}^{\text{ext}}(x) := \begin{bmatrix} \theta \\ G_{ij}(x) \end{bmatrix}, \qquad x \in D_{ij}, \tag{4.4b}$$

where the entries of $G_{ij} : X \rightrightarrows \{-1,0,1\}^m$ are given by

$$(G_{ij})_{(u,v)} := \begin{cases} \underset{h\in\{-1,0,1\}}{\operatorname{argmin}} \, |\theta_j - \theta_i + 2h\pi|, & \text{if } (u,v) = (i,j), \\ \{q_{uv}\}, & \text{otherwise}, \end{cases} \tag{4.4c}$$

with $(u,v),(i,j) \in \mathcal{E}$. Set D_{ij} in (4.4a) enforces a jump when $\theta_j - \theta_i + 2q_{ij}\pi$ is not in $\operatorname{dom}\sigma$ for $(i,j) \in \mathcal{E}$. Across a jump, according to (4.4b), only q_{ij} changes in such a way that $|\theta_j - \theta_i + 2q_{ij}\pi| < \pi + \delta$ after a jump as formalized in the next lemma whose proof is given in Appendix A.1 to avoid breaking the flow of the exposition.

Lemma 4.1. *For any* $(i,j) \in \mathcal{E}$ *and* $x \in D_{ij}$*, any* $x^+ \in G_{ij}^{\text{ext}}(x)$ *as per* (4.4b) *satisfies* $x^+ \in X$ *and* $|\theta_j^+ - \theta_i^+ + 2q_{ij}^+\pi| < \pi + \delta$. □

A second jump rule is introduced for when one of the oscillators $i \in \mathcal{V}$ reaches $|\theta_i| = \pi + \delta$. In this case, a jump of 2π is enforced so that the phase then belongs to $(-\pi-\delta, \pi+\delta)$ while remaining the same modulo 2π. We define for this purpose

$$x^+ = \begin{bmatrix} \theta^+ \\ q^+ \end{bmatrix} = g_i(x) := \begin{bmatrix} g_{i,\theta}(x) \\ g_{i,q}(x) \end{bmatrix}, \qquad x \in D_i, \tag{4.5a}$$

where the entries of $g_{i,\theta} : X \to [-\pi-\delta, \pi+\delta]^n$ and $g_{i,q} : X \to \{-1,0,1\}^m$ are defined as

$$(g_{i,\theta})_j := \begin{cases} \theta_i - \operatorname{sign}(\theta_i)2\pi, & \text{if } j = i, \\ \theta_j, & \text{otherwise}, \end{cases} \tag{4.5b}$$

$$(g_{i,q})_{(u,v)} := \begin{cases} q_{uv} + \operatorname{sign}(\theta_i), & \text{if } v = i, \\ q_{uv} - \operatorname{sign}(\theta_i), & \text{if } u = i, \\ q_{uv}, & \text{otherwise,} \end{cases} \tag{4.5c}$$

with $j \in \mathcal{V}$ and $(u,v) \in \mathcal{E}$. The set D_i, $i \in \mathcal{V}$, is defined as

$$D_i := \operatorname{cl}\big(\{x \in X : x \notin D_{uv} \text{ for any } (u,v) \in \mathcal{E}, \text{and } |\theta_i| = \pi + \delta\}\big). \tag{4.5d}$$

In view of (4.5d), the jump rule (4.5a) is allowed when both $|\theta_i| = \pi + \delta$ and x is not in the interior of D_{uv} for any $(u,v) \in \mathcal{E}$, where a jump may occur according to (4.4).

Note that each function g_i is continuous on its (not connected) domain, because D_i does not contain points with $\theta_i = 0$ for any $i \in \mathcal{V}$.

Finally, switching/jumping ruled by (4.5) unwinds the phase θ_i without changing the phase mismatches between neighbours, defined as $(\theta_j - \theta_i + 2q_{ij})$, as shown in the next lemma, whose proof is given in Appendix A.2.

Lemma 4.2. *For each $i \in \mathcal{V}$ and $x \in D_i$, $x^+ = g_i(x)$ implies $x^+ \in X$ and, for all $(u,v) \in \mathcal{E}$,*

$$\begin{cases} \theta_v^+ - \theta_u^+ + 2q_{uv}^+\pi = \theta_v - \theta_u + 2q_{uv}\pi, \\ |\theta_i^+| = \pi - \delta < \pi + \delta. \end{cases} \tag{4.6}$$

□

Remark 4.1. *Since we envision engineering applications, each phase θ_i with $i \in \mathcal{V}$ may be reconstructed from the angular measurements provided by sensors. Due to the wide variety of outputs provided by commercial sensors, a relevant task is to extrapolate a continuous measurement from a sensor that may return values whose wrapping around 2π is unknown; see, for example, ([109]) and ([6]). In this scenario, we can implement an algorithm to extract a continuous measurement of the phase satisfying* (4.2). *In particular, following a rationale similar to that proposed in [86, Figure 1] for a setting with sampled measurements, we may continuously update an estimate $\theta_{i,e}$ of θ_i. Indeed, for each sensor output $\theta_{i,so}$, we may extract the lifted measurement as the closest one to $\theta_{i,e}$ when performing 2π-wraps $\theta_{i,e}^+ = \theta_{i,so} + 2\pi \operatorname{argmin}_{h\in\{-1,0,1\}} |\theta_{i,so} - \theta_{i,e} + 2h\pi|$. This rule parallels the selection of (15) and (27b) of ([86]) for the simpler case of $\mathbb{S}^1$ and scalar angular measurements.* □

4.2.3 Overall model

In view of Sections 4.2.1-B, the overall hybrid model is given by

$$\begin{cases} \dot{x} \ = f(x, \boldsymbol{\omega}(\mathrm{t})), & x \in C, \\ x^+ \in G(x), & x \in D, \end{cases} \tag{4.7a}$$

where f is defined in (4.2), and using (4.4a) and (4.5d),

$$D := \Big(\bigcup_{i=1}^{n} D_i\Big) \cup \Big(\bigcup_{(i,j)\in\mathscr{E}} D_{ij}\Big), \tag{4.7b}$$

$$C = \mathrm{cl}(X \setminus D), \tag{4.7c}$$

with X defined in (4.3). The set-valued jump map G is defined in terms of its graph, which is given by

$$\mathrm{gph}\ G := \Big(\bigcup_{i=1}^{n} \mathrm{gph}\ g_i\Big) \cup \Big(\bigcup_{(i,j)\in\mathscr{E}} \mathrm{gph}\ G_{ij}^{\mathrm{ext}}\Big), \tag{4.7d}$$

with g_i and G_{ij}^{ext} as per (4.4b), (4.5a)-(4.5c). Fig. 4.2 shows three projections of the state space X on the plane (θ_i, θ_j) for some $(i, j) \in \mathscr{E}$, which corresponds to a union of three squares, one for each value of q_{ij}.

4.3 Regularized hybrid dynamics

Model (4.7) is a time-varying hybrid system with a possibly discontinuous right-hand side, due to the mild properties of σ, see Property 4.1. Hence, solutions may be understood in the generalized sense of [64]. We consider for this purpose the Krasovskii regularization of (4.7), so that stability properties for the regularized system carry over to the nominal and generalized solutions of (4.7). In particular, following [64, Page 79] we consider

$$\begin{cases} \dot{x} \ \in F(x), & x \in C, \\ x^+ \in G(x), & x \in D, \end{cases} \tag{4.8a}$$

where C, D, and G coincide with those in (4.7), and the set-valued map F regularizes f in (4.2) as

$$F(x) := \begin{bmatrix} \widehat{\Omega} - B\kappa\widehat{\Sigma}(x) \\ \mathbf{0}_m \end{bmatrix}, \quad \forall x \in X, \tag{4.8b}$$

with the sets $\widehat{\Omega} := \Omega \times \cdots \times \Omega = [\omega_{\mathrm{m}}, \omega_{\mathrm{M}}]^n$ and $\widehat{\Sigma}$ being the Krasovskii regularization of the function $\boldsymbol{\sigma}$ in (4.2), see for more details Chapter 2. More specifically, following [64, Def. 4.13], $\widehat{\Sigma}(x) := \bigcap_{s>0} \overline{\mathrm{co}}\, \boldsymbol{\sigma}((x + s\mathbb{B}) \cap C)$. It is readily verified that, denoting by $\widehat{\sigma}$ the Krasovskii regularization of the scalar function σ, namely

$$\widehat{\sigma}(\tilde{\theta}) := \bigcap_{s>0} \overline{\mathrm{co}}\, \sigma([\tilde{\theta} - s, \tilde{\theta} + s] \cap [-\pi - \delta, \pi + \delta]), \tag{4.9}$$

for any $\tilde{\theta} \in [-\pi - \delta, \pi + \delta]$, then the set-valued map $\widehat{\Sigma}(x)$ is the stacking (with the same ordering as in $\boldsymbol{\sigma}$) of the set-valued maps $\widehat{\sigma}_{ij}$ defined as

$$\widehat{\sigma}_{ij} := \widehat{\sigma}(\tilde{\theta}_{ij}), \qquad \tilde{\theta}_{ij} := \theta_j - \theta_i + 2q_{ij}\pi, \tag{4.10}$$

for all $(i, j) \in \mathscr{E}$.

Since the jump set, flow set, and jump map of hybrid system (4.8) coincide with those of (4.7), and for any $x \in X$ and $\boldsymbol{\omega} \in \widehat{\Omega}$, $f(x, \boldsymbol{\omega}) \in F(x)$, we study the stability properties of solutions of (4.7) by concentrating on the regularized dynamics (4.8). In addition to clarifying the nature of solutions of (4.7), which may, among other things, present sliding behavior (see Section 4.5), the advantage of using (4.8) instead of (4.7) is that (4.8) satisfies the so-called hybrid basic conditions [64, Assumption 6.5], which ensure its well-posedness [64, Thm. 6.30].

Lemma 4.3. *System* (4.8) *satisfies the hybrid basic conditions (Assumption 2.1).* □

Proof: Sets C and D, as defined in (4.4a), (4.5d), (4.7b), (4.7c) are closed, as required by [64, Assumption 6.5 (A1)]. On the other hand, F is the Krasovskii regularization of a function, which satisfies the HBC in view of its locally boundedness on C and [64, Lemma 5.16] as shown in [64, Ex. 6.6], thus [64, Assumption 6.5 (A2)] is satisfied. Lastly, each g_i and G_{ij}^{ext} has a closed graph, and so due to (4.7d) the graph of G is closed as well. As consequence, according to [64, Lemma 5.10], G is outer semicontinuous and it is also locally bounded relative to D, thereby satisfying [64, Assumption 6.5 (A3)]. ■

Among other useful properties, Lemma 4.3 guarantees intrinsic robustness of the stability property established later in Sections 4.4 and 4.5, see [64, Ch. 7]. To conclude this section, we note that all maximal solutions to (4.8) are complete [5] and exhibit a (uniform) average dwell-time property, thereby excluding Zeno phenomena. We emphasize that, through Lemmas 4.1 and 4.2, the parameter δ plays a key role in establishing that no complete discrete solution exists. In particular, the fact that $\delta \neq 0$ and $\delta \neq \pi$ is key for being able to exclude Zeno solutions.

Proposition 4.1. *All solutions to* (4.8) *enjoy a uniform average dwell-time property. Namely, there exist* $\tau_D \in \mathbb{R}_{>0}$ *and* $J_0 \in \mathbb{Z}_{\geq 0}$ *such that, for any solution* x *to* (4.8) *and for any pair of hybrid times such that* $\mathfrak{t} + \mathfrak{j} \geq \mathfrak{s} + \mathfrak{r}$ *with* $(\mathfrak{s}, \mathfrak{r}), (\mathfrak{t}, \mathfrak{j}) \in \operatorname{dom} x$, $\frac{1}{\tau_D}(\mathfrak{t} - \mathfrak{s}) + J_0 \geq (\mathfrak{j} - \mathfrak{r})$. *Moreover, if* x *is maximal, then it is* $\mathfrak{t}$*-complete, i.e.,* $\sup_{\mathfrak{t}} \operatorname{dom} x = \sup\{\mathfrak{t} \in \mathbb{R}_{\geq 0} : \exists \mathfrak{j} \in \mathbb{Z}_{\geq 0}, (\mathfrak{t}, \mathfrak{j}) \in \operatorname{dom} x\} = +\infty$. □

Proof: We first recall that, in view of Lemma 4.2, for any $i \in \mathscr{V}$, $x \in D_i$ and $x^+ = g_i(x)$

$$|\theta_i^+| = \pi - \delta < \pi + \delta, \tag{4.11}$$

while the other θ_j, $j \neq i \in \mathscr{V}$ remain unchanged across such a jump and so does $\theta_j - \theta_i + 2q_{ij}\pi$ for

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all $(i,j)\in\mathscr{E}$. We also recall that, from Lemma 4.1, for any $(i,j)\in\mathscr{E}$, $x\in D_{ij}$ and $x^+\in G_{ij}^{\text{ext}}(x)$

$$|\theta_j^+ - \theta_i^+ + 2q_{ij}^+\pi| \le \max(2\delta,\pi) < \pi+\delta, \tag{4.12}$$

while the θ_u and the other $\theta_h-\theta_k+2q_{hk}\pi$ remain unchanged for any $u\in\mathscr{V}$ and $(h,k)\neq(i,j)\in\mathscr{E}$. From uniform global boundedness of the right hand-side $F(x)$ of the flow dynamics (a consequence of the local boundedness of F and of the boundedness of X), all solutions satisfy a global Lipschitz property with respect to the flowing time and (4.11) and (4.12) imply a uniform average dwell time on the jumps from $\bigcup_{i=1}^{n} D_i$ and $\bigcup_{(i,j)\in\mathscr{E}} D_{ij}$, respectively. Finally, the uniform average dwell time property of solutions jumping from D derives directly from Lemma 4.2 and (4.4b)-(4.4c). Indeed, (4.4b)-(4.4c) imply that jumping from D_{ij} does not affect the triggering condition in D_u or D_{hk}, for any $u\neq i\in\mathscr{V}$ and $(i,j)\neq(h,k)\in\mathscr{E}$. In a similar way, Lemma 4.2 implies that jumping from D_i does not affect the triggering condition in D_j or D_{ij} or D_{ji}, for any $j\neq i\in\mathscr{V}$ and $(i,j)\in\mathscr{E}$ or $(j,i)\in\mathscr{E}$. Thus, a uniform average dwell time on jumps from D stems from the global Lipschitz property of the solutions with respect to the flowing time, along with (4.11) and (4.12).

We now check that maximal solutions of (4.8) are complete by proving that the conditions of [64, Prop. 6.10] hold. First, consider $\xi\in C\setminus D$. Because $\partial C\subset D$, then $\xi\in\text{int}(C)$. Therefore, there exists a neighbourhood U of ξ such that $U\subset C\setminus D$. Thus, for any $x\in U\subset C\setminus D$, the tangent cone [6] to C at x is $T_C(x)=\mathbb{R}^n\times\{0\}^m$. Hence, any $\psi\in F(x)$ in (4.8b) satisfies $\{\psi\}\cap T_C(x)=\{\psi\}\neq\emptyset$, so that [64, Prop. 6.10 (VC)] holds for any $\xi\in C\setminus D$. On the other hand, the state space X in (4.3) is bounded, thus item (b) in [64, Prop. 6.10] is excluded. To rule out item (c) in [64, Prop. 6.10], from Lemmas 4.1 and 4.2, we have $G(D)\subset X=C\cup D$. Hence, we can apply [64, Prop. 6.10] to conclude that all maximal solutions are complete, thus obtaining $\mathfrak{t}$-completeness of solutions in view of their uniform average dwell time property established above. ■

4.4 Asymptotic stability properties

4.4.1 Synchronization set and its stability property

To analyze the synchronization properties of system (4.8), consider the set

$$\mathscr{A} := \{x\in X : \theta_i=\theta_j+2q_{ij}\pi,\ \forall(i,j)\in\mathscr{E}\}. \tag{4.13}$$

Because the network is a tree, for any $x\in\mathscr{A}$, the phases θ_i and θ_j coincide modulo 2π not only for any $(i,j)\in\mathscr{E}$ but also for any $i\in\mathscr{V}$ and $j\in\mathscr{V}\setminus\{i\}$. In other words, when $x\in\mathscr{A}$, all the oscillators are synchronized even if they do not share a direct link. We therefore call $\mathscr{A}$ the *synchronization* set. Our main result below establishes a practical asymptotic stability result for $\mathscr{A}$, as a function of the coupling gain κ appearing in the flow map (4.1). The "practical" tuning of κ depends on the

6. See Chapter 2.

following two parameters:

$$\underline{\lambda} := \lambda_{\min}(B^\top B),\ \overline{\omega} := (n-1)|\omega_{\rm M} - \omega_{\rm m}| \geq \max_{\widehat{\boldsymbol{\omega}}\in\widehat{\Omega}} |B^\top \widehat{\boldsymbol{\omega}}|_1. \tag{4.14}$$

Parameter $\underline{\lambda}$ ensures a detectability property of the distance $|x|_{\mathcal{A}}$ in (4.13) from the norm $|\widehat{\boldsymbol{\sigma}}|$, for any $\widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}(x)$. In particular, we have from the results in [63].

Lemma 4.4. *Since $\mathcal{G}$ is a tree, $\underline{\lambda} := \lambda_{\min}(B^\top B) > 0$.* □

Proof: Consider a tree graph $\mathcal{G}$ with incidence matrix $B \in \mathbb{R}^{n\times n-1}$. By definition $\mathcal{G}$ has only one bipartite connected component. From [63, Thm. 8.2.1], $\mathrm{rank}(B) = n-1$ and thus $\dim(\ker(B)) = 0$ by way of the fundamental theorem of linear algebra. Therefore, $By \neq 0$ for any $y \in \mathbb{R}^{n-1} \setminus \{\mathbf{0}_{n-1}\}$ which implies $y^\top B^\top By = |By|^2 > 0$ for any $y \in \mathbb{R}^{n-1} \setminus \{\mathbf{0}_{n-1}\}$. Consequently all the eigenvalues of $B^\top B$ are strictly positive, thus completing the proof. ■

Remark 4.2. *The smallest eigenvalue $\underline{\lambda}$ of $B^\top B$ and its positivity established in Lemma 4.4 play a fundamental role on the speed of convergence of the closed-loop solutions to the synchronization set. As $\mathcal{G}$ is a tree, positivity of $\underline{\lambda}$ is ensured by Lemma 4.4. In more general cases with $\mathcal{G}$ not being a tree, the leaderless context considered in this chapter, where the synchronized motion emerges from the network, poses significant obstructions to achieving global results. A simple insightful example of a cyclic graph is discussed in Section 4.4.2, which provides a clear illustration of the motivation behind requiring that $\mathcal{G}$ is a tree. We emphasize that a similar obstruction is experienced in prior work [85] where, in a different context, a similar assumption on the network is required.* □

We are now ready to state the main result of this chapter, corresponding to a practical $\mathcal{KL}$ bound on the distance of x from $\mathcal{A}$ that is uniform in κ. We state the bound in our main theorem below, whose proof is given in Section 4.7.2, and then illustrate its relevance on a number of corollaries given next.

Theorem 4.1. *Given set $\mathcal{A}$ in* (4.13)*, there exists a class $\mathcal{KL}$ function $\beta_\circ$ and a class $\mathcal{K}$ gain $\gamma_\circ$, both of them independent of κ, such that, for any $\kappa > 0$, all solutions x of* (4.8) *satisfy*

$$|x(\mathrm{t},\mathrm{j})|_{\mathcal{A}} \leq \beta_\circ(|x(0,0)|_{\mathcal{A}}, \kappa\mathrm{t}) + \gamma_\circ((\kappa\underline{\lambda})^{-1}c\overline{\omega}), \tag{4.15}$$

for all $(\mathrm{t},\mathrm{j}) \in$ *dom* x *and with* $c := \max\limits_{s\in dom\,\sigma} \widehat{\sigma}(s)$. □

The bound (4.15) in Theorem 4.1 is the sum of two terms: $\beta_\circ$ captures the phases tendency to synchronize, while function $\gamma_\circ$ depends on the mismatch among the (possibly) non-identical, time-varying natural frequencies of the oscillators, which hampers asymptotic phase synchronization in general. Therefore, Theorem 4.1 provides an insightful bound (4.15) illustrating the trend of the continuous-time evolution of the hybrid solutions to (4.8). Notice that $\beta_\circ$ and $\gamma_\circ$ can be constructed by following similar steps as the ones in [122, Lemma 2.14], noting that the resulting bound is often subject to some conservatism. On the other hand, because $\beta_\circ$ and $\gamma_\circ$ are independent of κ and $\underline{\lambda}$, (4.15) still provides valuable quantitative information. Indeed, in view of (4.15), increasing κ speeds up the transient and reduces the asymptotic phase disagreement caused by the non-identical time-varying natural frequencies. Equation (4.15) also highlights the impact of the algebraic connectivity $\underline{\lambda}$ of $\mathscr{G}$ [87, pages 23-24] on the phase synchronization, by giving information on the scalability of our algorithm. Recall that $\underline{\lambda}$ is influenced by several parameters of the undirected graph, such as the maximum degree and the number of nodes [108]. This continuous-time focus in (4.15) in Theorem 4.1 is motivated by Proposition 4.1. It is also of interest to establish a bound similar to (4.15) while measuring the elapsed time in terms of $t+j$ and not only in terms of t, as usually done when defining bounds for solutions to hybrid systems, which allows us to ensure stronger stability properties, in particular, uniformity and robustness [64, Chp.7]. Hence, combining Theorem 4.1 with Proposition 4.1, we obtain the following second main result.

Theorem 4.2. *For each value $\kappa > 0$, there exists a class $\mathscr{KL}$ function β such that all solutions to* (4.8) *satisfy*

$$|x(t,j)|_{\mathscr{A}} \leq \beta(|x(0,0)|_{\mathscr{A}}, t+j) + \gamma_\circ((\kappa\underline{\lambda})^{-1} c\overline{\omega}), \tag{4.16}$$

for all $(t,j) \in \mathrm{dom}\, x$*, with $\gamma_\circ$ as in Theorem 4.1.* □

Proof: Let $\kappa > 0$ and x be a solution to (4.8). In view of Proposition 4.1, $\frac{1}{\tau_D}t + J_0 \geq j$ for any $(t,j) \in \mathrm{dom}\, x$, which is equivalent to $\frac{1}{2}t \geq \frac{\tau_D}{2}(j - J_0)$. Hence, we derive from (4.15), for any $(t,j) \in \mathrm{dom}\, x$,

$$\begin{aligned}\beta_\circ(|x(0,0)|_{\mathscr{A}}, \kappa t) &= \beta_\circ\left(|x(0,0)|_{\mathscr{A}}, \kappa(\frac{1}{2}t + \frac{1}{2}t)\right)\\ &\leq \beta_\circ\left(|x(0,0)|_{\mathscr{A}}, \kappa \max\left\{0, \frac{1}{2}t + \frac{\tau_D}{2}j - \frac{\tau_D}{2}J_0\right\}\right)\\ &\leq \beta_\circ\left(|x(0,0)|_{\mathscr{A}}, \frac{\kappa}{2}\max\{0, \min(1,\tau_D)(t+j) - \tau_D J_0\}\right) =: \beta(|x(0,0)|_{\mathscr{A}}, t+j).\end{aligned} \tag{4.17}$$

Function β is of class $\mathscr{KL}$. Hence, (4.15) and (4.17) yield (4.16), thus completing the proof. ■

Theorem 4.2 implies that the oscillator phases uniformly converge to any desired neighborhood of $\mathscr{A}$ by taking κ sufficiently large, thus the practical nature of the result. We also immediately conclude from Theorem 4.2 and Lemma 4.3 that the stability property in (4.16) is robust in the sense of item (a) of [64, Def. 7.18], according to [64, Thm. 7.21].

We may draw an important additional conclusion from Theorem 4.2 corresponding to a global practical $\mathscr{KL}$ bound stemming from the fact that the function $\gamma_\circ$ in (4.16) is independent of κ.

Corollary 4.1. *Set $\mathscr{A}$ is uniformly globally practically $\mathscr{KL}$ asymptotically stable for system* (4.8)*, i.e., for each $\varepsilon > 0$, there exists $\kappa^\star > 0$ such that, for all $\kappa \geq \kappa^\star$, there exists $\beta \in \mathscr{KL}$ such that any solution x verifies $|x(t,j)|_{\mathscr{A}} \leq \beta(|x(0,0)|_{\mathscr{A}}, t+j) + \varepsilon$, for all $(t,j) \in \mathrm{dom}\, x$.* □

Lastly, in the case of uniform frequencies $\boldsymbol{\omega}(t) = \mathbf{1}_n \omega(t)$, for all $t \geq 0$, with $\omega(t) \in \Omega$, we have

$B^\top \boldsymbol{\omega}(t) = 0$, for all $t \geq 0$. Then, we can exploit the fact that the term $|\overline{\omega}|$ at the right-hand side of (4.16) stems from upper bounding $|B^\top \widehat{\boldsymbol{\omega}}|_1$ as in (4.14), which allows obtaining the following asymptotic property of $\mathscr{A}$.

Corollary 4.2. *If $\boldsymbol{\omega}(t) = \mathbf{1}_n \omega(t)$, for all $t \geq 0$, with $\omega(t) \in \Omega$, then set $\mathscr{A}$ is uniformly globally $\mathscr{KL}$ asymptotically stable for system* (4.8)*, i.e., for each $\kappa > 0$, there exists $\beta \in \mathscr{KL}$ such that any solution x verifies*

$$|x(t,j)|_{\mathscr{A}} \leq \beta(|x(0,0)|_{\mathscr{A}}, t+j), \quad \forall (t,j) \in dom\, x. \tag{4.18}$$

□

4.4.2 Cyclic graphs and their potential issues

Before proceeding with the technical derivations needed to prove Theorem 4.1, we devote some attention to the issues pointed out in Remark 4.2 about the need for the graph $\mathscr{G}$ to be a tree, similar to [85].

Consider system (4.7) with $n = 3$ and with $\mathscr{G}_u$ an all-to-all undirected graph, thus not a tree. Let $\mathscr{G}$ be the orientation of $\mathscr{G}_u$ with the incidence matrix $B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. We take $\delta = \frac{1}{180}\pi$, any $\kappa > 0$, any σ satisfying Property 4.1, and we select for convenience $\boldsymbol{\omega}(t) = \mathbf{0}_3$ for any time $t \geq 0$. Let $x = (\theta, q)$ be a solution to the corresponding system (4.7) initialized at $\left(-\frac{2}{3}\pi, 0, \frac{2}{3}\pi, 0, 0, 1\right)$. We have that $x(0,0) \in \text{int}(C) \setminus \mathscr{A}$ and $B^\top \theta(0,0) + 2\pi q(0,0) = \frac{2}{3}\pi \mathbf{1}_3$, thus implying $\boldsymbol{\sigma}(x(0,0)) = \sigma(\frac{2}{3}\pi)\mathbf{1}_3$. Hence, because $B\mathbf{1}_3 = 0$, from (4.2) it holds that $\dot{\theta}(0,0) = \mathbf{0}_3$, and consequently $\text{dom}\, x \subset [0, \infty) \times \{0\}$, and $x(t,0) = x(0,0)$ and $x(t,0) \in \text{int}(C) \setminus \mathscr{A}$ for all $(t,0) \in \text{dom}\, x$. As a result, solution x does not converge to the synchronization set.

More generally, when the graph is not a tree, the kernel of matrix B, also known as the *flow space* of B [63, Ch. 14], contains additional elements besides the zero vector. Consequently, we can have $B\boldsymbol{\sigma}(x) = \mathbf{0}_n$ even when $x \notin \mathscr{A}$, and $\underline{\lambda} = 0$ in (4.14). As a result, $\mathscr{A}$ is not globally attractive. The additional undesired equilibria one may find when studying the synchronization of continuous-time oscillators revolving on the unit circle may correspond to symmetric configurations for all-to-all networks [120], or to asymmetric ones, as for networks with Harary graphs [29].

4.4.3 A Lyapunov-like function and its properties

To prove Theorem 4.1, we rely on the Lyapunov function V, defined as

$$V(x) := \sum_{(i,j)\in\mathscr{E}} V_{ij}(x), \qquad \forall x \in X, \tag{4.19}$$

$$V_{ij}(x) := \int_0^{\theta_j - \theta_i + 2q_{ij}\pi} \sigma(\text{sat}_{\pi+\delta}(s))ds, \tag{4.20}$$

with $\text{sat}_{\pi+\delta}(s)$ given by

$$\text{sat}_{\pi+\delta}(s) := \max\left\{\min\{s, \pi+\delta\}, -\pi-\delta\right\}, \quad \forall s \in \mathbb{R}.$$

Function V enjoys useful relations with the distance of x from the synchronization set $\mathscr{A}$, as formalized next.

Lemma 4.5. *Given function V in* (4.19)-(4.20)*, there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ independent of $\overline{\omega}$ in* (4.14) *and of κ, such that*

$$\alpha_1(|x|_{\mathscr{A}}) \le V(x) \le \alpha_2(|x|_{\mathscr{A}}), \quad \forall x \in X. \tag{4.21}$$

□

Proof: For each $x \in \mathscr{A}$, $V(x) = 0$ in view of (4.13), (4.19), (4.20), and for each $x \in (C \cup D) \setminus \mathscr{A}$, $V(x) > 0$ in view of item b) of Property 4.1. In addition, V is (vacuously) radially unbounded as X is compact. Hence, (4.21) holds from [64, Page 54]. ■

To prove Theorem 4.1, it is also fundamental to formalize the relation between the distance of x from the set $\mathscr{A}$ and $\widehat{\Sigma}(x)$ in (4.8), as done in the next lemma.

Lemma 4.6. *There exists a class $\mathscr{K}_\infty$ function η such that, for each $x \in X$, $\eta(|x|_{\mathscr{A}}) \le |\widehat{\boldsymbol{\sigma}}|^2$, $\forall \widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}(x)$.* □

Proof: From items a) and b) of Property 4.1, $|\sigma(s)| \ge \alpha(|s|)$ for any $s \in \operatorname{dom}\sigma$. Thus, in view of (4.9), $|\varsigma| \ge \alpha(|s|)$, for any $s \in \operatorname{dom}\sigma$ and $\varsigma \in \widehat{\sigma}(s)$. We recall that, for any $x \in X$, $\widehat{\Sigma}(x)$ is the stacking of all the set-valued maps $\widehat{\sigma}_{ij}$ defined in (4.10), $(i,j) \in \mathscr{E}$. Hence, by definition of $\mathscr{A}$, for any $x \in X \setminus \mathscr{A}$, there exists at least one element $\tilde{\theta}_{ij} \ne 0$, with $(i,j) \in \mathscr{E}$, thus $\widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}(x)$ implies that $|\widehat{\boldsymbol{\sigma}}| \ge |\widehat{\sigma}_{ij}(x)| \ge \alpha(|\tilde{\theta}_{ij}|)$. Similarly, for any $x \in \mathscr{A}$, $|\widehat{\boldsymbol{\sigma}}| \ge |\widehat{\sigma}_{ij}(x)| \ge \alpha(|\tilde{\theta}_{ij}|) = 0$. Therefore, $\max_{(i,j)\in\mathscr{E}} \alpha(|\tilde{\theta}_{ij}|)$ is a suitable lower bound for $|\widehat{\boldsymbol{\sigma}}|$, for any $x \in X$. Since $\tilde{\theta}_{ij}$ is a function of the states and $\max_{(i,j)\in\mathscr{E}} \alpha(\cdot)$ is positive definite and radially unbounded, as X is compact, then [64, Page 54] implies that there exists $\eta \in \mathscr{K}_\infty$ such that $\eta(|x|_{\mathscr{A}}) \le |\widehat{\boldsymbol{\sigma}}|^2$ holds for each $x \in X$ and for all $\widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}(x)$, thus concluding the proof. ■

Function V is locally Lipschitz due to the properties of σ and characterizing its variation when evaluated along the solutions of the hybrid inclusion (4.8) requires using tools from non-smooth analysis. To avoid breaking the flow of the exposition, we postpone to Section 4.7.2 those technical derivations and summarize the corresponding conclusions in the next proposition, a key result for proving Theorem 4.1.

Proposition 4.2. *Consider system* (4.8) *and function V in* (4.19)-(4.20)*. There exist $\alpha_3 \in \mathscr{K}_\infty$ independent of $\overline{\omega}$ in* (4.14) *and of κ, such that for any $\kappa > 0$, any solution x of* (4.8) *satisfies (denoting $\operatorname{dom} x = \bigcup_{\mathrm{j}=0}^{J} [\mathrm{t_j}, \mathrm{t_{j+1}}] \times \{\mathrm{j}\}$, possibly with $J = +\infty$)*
(i) for all $\mathrm{j} \in \{0, \dots, J\}$ and almost all $\mathrm{t} \in [\mathrm{t_j}, \mathrm{t_{j+1}}]$,

$$\frac{d}{d\mathrm{t}} V(x(\mathrm{t},\mathrm{j})) \le -\kappa \underline{\lambda} \alpha_3(V(x(\mathrm{t},\mathrm{j}))) + c\overline{\omega}, \tag{4.22a}$$

with c defined in Theorem 4.1;
(ii) for all $j \in \{0, \dots, J-1\}$,

$$V(x(t, j+1)) \leq V(x(t, j)). \tag{4.22b}$$

□

We are now ready to present the proof of Theorem 4.1.

Proof of Theorem 4.1: Let $\kappa > 0$ and x be a solution (4.8) and denote, with a slight abuse of notation, $\operatorname{dom} x = \bigcup_{j=0}^{J} [t_j, t_{j+1}] \times \{j\}$ with $J \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$. We scale the continuous time as $\tau := (\kappa\underline{\lambda})t \in \mathbb{R}_{\geq 0}$ and we denote $\tau_j := (\kappa\underline{\lambda})t_j$ and $V'(x(\cdot,\cdot))$ the time-derivative of V with respect to τ. From item (i) of Proposition 4.2, for all $j \in \{0, \dots, J\}$ and almost all $\tau \in [\tau_j, \tau_{j+1}]$,

$$\begin{aligned} V'(x(\tau, j)) &= (\kappa\underline{\lambda})^{-1} \dot{V}(x(t, j)) \\ &\leq -\alpha_3(V(x(t, j))) + (\kappa\underline{\lambda})^{-1} c\overline{\omega}. \end{aligned} \tag{4.23}$$

Combining (4.23) with the non-increase condition in (4.22b), we follow the steps of the proof of [122, Lemma 2.14] to obtain an input-to-state stability bound on $V(\tilde{x}(\cdot,\cdot))$ where $\tilde{x}(\cdot,\cdot) := x((\kappa\underline{\lambda})^{-1}(\cdot), (\cdot)) = x(\cdot,\cdot)$, which can then be converted to a bound on $|x(\cdot,\cdot)|_{\mathscr{A}}$ using (4.21), thus leading to (4.15), where $\beta_\circ$ and $\gamma_\circ$ only depend on α_1, α_2 and α_3 and are therefore independent of $\overline{\omega}$ and κ. Note that the dependence on $\underline{\lambda}$ is left implicit in $\beta_\circ$ and $\gamma_\circ$. ■

4.5 Prescribed finite-time stability properties

A useful outcome of the mild regularity conditions that we require from σ in Property 4.1 is that defining σ to be discontinuous at the origin, as in the sign function represented in Fig. 4.1, leads to desirable sliding-like behavior of the solutions in the attractor $\mathscr{A}$. This sliding property induces interesting advantages of the behavior of solutions, as compared to the general asymptotic and practical properties characterized in Section 4.4.

A first advantage is that, even with non-uniform natural frequencies, we prove uniform global $\mathscr{K}\mathscr{L}$ asymptotic stability of $\mathscr{A}$ for a large enough coupling gain κ, due to the well-known ability of sliding-mode mechanisms to dominate unknown additive bounded disturbances acting on the dynamics. A second advantage is that the Lyapunov decrease characterized in Proposition 4.2 can be associated with a guaranteed constant negative upper bound outside $\mathscr{A}$, which implies finite-time convergence. Finally, since this constant upper bound can be made arbitrarily negative by taking κ sufficiently large, we actually prove prescribed finite-time convergence (see [121]) when using these special discontinuous functions σ, whose peculiar features are characterized in the next lemma.

Lemma 4.7. *Given a function σ satisfying Property 4.1, if σ is discontinuous at the origin, then there exists $\mu > 0$ such that, for any $x \in X \setminus \mathscr{A}$, $|\widehat{\boldsymbol{\sigma}}| \geq \mu$, for all $\widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}(x)$.* □

Proof: Since σ is discontinuous at 0 and it is piecewise continuous, there exists $\varepsilon > 0$ such that σ is continuous in $[-\varepsilon, 0)$ and $(0, \varepsilon]$. By item a) of Property 4.1, $\lim_{s\to 0^+} \sigma(s) = -\lim_{s\to 0^-} \sigma(s) =: \sigma_\circ \neq 0$ as σ is discontinuous at 0 and $\sigma(0) = 0$. Then there exists $\varepsilon_\circ \in (0, \varepsilon]$ such that $\sigma(s) \geq \frac{\sigma_\circ}{2}$ for all $s \in (0, \varepsilon_\circ]$. From item b) of Property 4.1, for any $s \in [\varepsilon_\circ, \pi + \delta]$, $\sigma(s) \geq \alpha(\varepsilon_\circ) > 0$. Hence, due to item a) of Property 4.1, $|\sigma(s)| \geq \mu := \min(\frac{\sigma_\circ}{2}, \alpha(\varepsilon_\circ))$ for all $s \in \operatorname{dom}\sigma \setminus \{0\}$. Moreover, in view of (4.9), for any $s \in \operatorname{dom}\sigma \setminus \{0\}$ and any $\varsigma \in \widehat{\sigma}(s)$, $|\varsigma| \geq \mu$. Since, for any $x \in X$, $\widehat{\Sigma}(x)$ is the stacking of all the set-valued maps $\widehat{\sigma}_{ij} = \widehat{\sigma}$, $(i, j) \in \mathcal{E}$, and by definition of $\mathcal{A}$, for any $x \in X \setminus \mathcal{A}$, there exists at least one nonzero element $\tilde{\theta}_{ij} \neq 0$ for some $(i, j) \in \mathcal{E}$. Then $\widehat{\boldsymbol{\sigma}} \in \widehat{\Sigma}(x)$ implies $|\widehat{\boldsymbol{\sigma}}| \geq |\widehat{\sigma}_{ij}(x)| \geq \mu$, thus concluding the proof. ■

Paralleling the structure of Proposition 4.2, the next proposition, whose proof is postponed to Section 4.7.3, is a key result for proving Theorem 4.3.

Proposition 4.3. *Consider system* (4.8) *and function V in* (4.19)-(4.20)*. If σ is discontinuous at the origin, then there exist $\mu \in \mathbb{R}_{>0}$ independent of $\overline{\omega}$ in* (4.14) *and $\kappa^\star > 0$ such that for each $\kappa \geq \kappa^\star$ any solution x of* (4.8) *satisfies (denoting $\operatorname{dom} x = \bigcup_{\mathrm{j}=0}^{J} [\mathrm{t_j}, \mathrm{t_{j+1}}] \times \{\mathrm{j}\}$, possibly with $J = +\infty$)*
(i) for all $\mathrm{j} \in \{0, \ldots, J\}$ and almost all $\mathrm{t} \in [\mathrm{t_j}, \mathrm{t_{j+1}}]$ such that $x(\mathrm{t}, \mathrm{j}) \notin \mathcal{A}$,

$$\frac{d}{d\mathrm{t}} V(x(\mathrm{t}, \mathrm{j})) \leq -\frac{1}{2}\kappa\underline{\lambda}\mu^2; \tag{4.24a}$$

(ii) for all $\mathrm{j} \in \{0, \ldots, J-1\}$,

$$V(x(\mathrm{t}, \mathrm{j}+1)) \leq V(x(\mathrm{t}, \mathrm{j})). \tag{4.24b}$$

□

Exploiting Lemma 4.7 and Proposition 4.3, we can follow similar steps to those in the proof of Theorem 4.1 to show the following main result on uniform global $\mathcal{KL}$ asymptotic stability and prescribed finite-time stability of $\mathcal{A}$ for (4.8).

Theorem 4.3. *If σ is discontinuous at the origin, then set $\mathcal{A}$ in* (4.13) *is prescribed finite-flowing-time stable for* (4.8)*, i.e., for each $T > 0$ there exists $\kappa^\star > 0$ such that for each $\kappa \geq \kappa^\star$:*
(i) there exists $\beta \in \mathcal{KL}$ such that all solutions x satisfy (4.18)*;*
(ii) all solutions x satisfy, $x(\mathrm{t}, \mathrm{j}) \in \mathcal{A}$ for all $(\mathrm{t}, \mathrm{j}) \in \operatorname{dom} x$ with $\mathrm{t} \geq T$. □

Proof: We start showing that $G(D \cap \mathcal{A}) \subset \mathcal{A}$. Indeed, we notice that $D_{ij} \cap \mathcal{A} = \emptyset$ for any $(i, j) \in \mathcal{E}$, and thus $G(D_{ij} \cap \mathcal{A}) \subset \mathcal{A}$ trivially holds. Moreover, from Lemma 4.2, it holds that $G(D_i \cap \mathcal{A}) \subset \mathcal{A}$ for any $i \in \mathcal{V}$. Hence, from (4.7b), we conclude that $G(D \cap \mathcal{A}) \subset \mathcal{A}$. To establish that $\mathcal{A}$ is (strongly) forward invariant for (4.8), it is left to prove that solutions cannot leave $\mathcal{A}$ while flowing. We proceed by contradiction and for this purpose suppose there exists a solution x_{bad} to (4.8) such that $x_{\text{bad}}(0, 0) \in \mathcal{A}$ and $x_{\text{bad}}(\mathrm{t}^*, 0) \notin \mathcal{A}$ for some $\mathrm{t}^* > 0$ with $(\mathrm{t}^*, 0) \in \operatorname{dom} x_{\text{bad}}$. From continuity of flowing solutions between any two successive jumps and closedness of $\mathscr{A}$, there exists $x_{\text{bad}}(t,0) \in \mathscr{A}$ for all $t \in [0,t^*)$ and $x_{\text{bad}}(t^*,0) \notin \mathscr{A}$. Hence, from (4.19) and (4.20) and positive definiteness of V, we have $0 = V(x_{\text{bad}}(0,0)) < V(x_{\text{bad}}(t^*,0))$. However, since the solution is flowing, integrating (4.24a) over the continuous time interval $[0,t^*]$ we obtain $V(x_{\text{bad}}(t^*,0)) < V(x_{\text{bad}}(0,0))$, which establishes a contradiction. Consequently, a solution cannot leave $\mathscr{A}$ while flowing. We have proven that the set $\mathscr{A}$ is (strongly) forward invariant, implying that if $x(t,j) \in X \setminus \mathscr{A}$ then $x(t',j') \in X \setminus \mathscr{A}$, for any $t'+j' \leq t+j$, with $(t',j'),(t,j) \in \operatorname{dom} x$. Let $\kappa \geq \kappa^\star$ with $\kappa^\star$ defined in Proposition 4.3 and x be a solution to (4.8). Combining (4.24a) with the non-increase condition (4.24b) and the forward invariance of $\mathscr{A}$, we obtain by integration for any $(t,j) \in \operatorname{dom} x$

$$V(x(t,j)) \leq -\frac{1}{2}\kappa\underline{\lambda}\mu^2 t + V(x(0,0)), \tag{4.25}$$

whenever $x(t,j) \in X \setminus \mathscr{A}$, and thus

$$V(x(t,j)) \leq \max(-\frac{1}{2}\kappa\underline{\lambda}\mu^2 t + V(x(0,0)), 0), \tag{4.26}$$

for any $x(t,j) \in X$. Equation (4.26) can then be converted to a bound on $|x(t,j)|_{\mathscr{A}}$ using (4.21). Hence, we follow the same steps used in the proofs of Theorem 4.1 and 4.2 to obtain (4.18), thus concluding the proof of item (i) in Theorem 4.3. In view of (4.25) and from the positive definiteness of V with respect to $\mathscr{A}$, we conclude that solutions to (4.8) reach the synchronization set $\mathscr{A}$ flowing at most for $T := \frac{1}{\kappa}\frac{2\overline{v}}{\underline{\lambda}\mu^2}$, where $\overline{v} := \max_{x \in X} V(x)$. Hence, in view of the forward invariance of $\mathscr{A}$, item (ii) in Theorem 4.3 holds thus completing the proof. ■

Notice that phases synchronize at most at continuous time $T = \frac{1}{\kappa}\frac{2\overline{v}}{\underline{\lambda}\mu^2}$, in view of (4.26). Therefore, we may decrease T at will by selecting a larger μ as in Lemma 4.7 and/or by increasing the coupling gain κ.

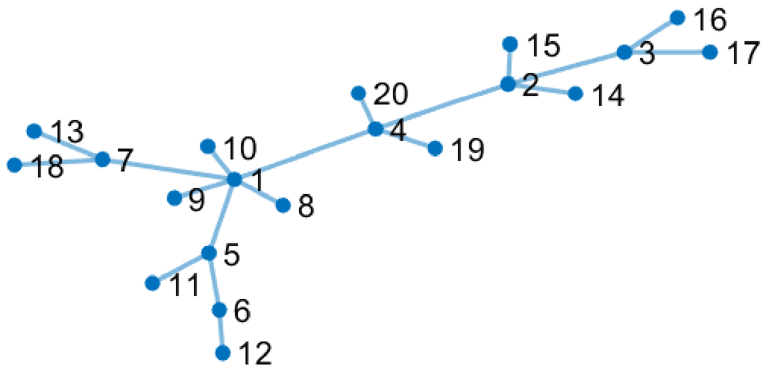

FIGURE 4.3 – The undirected graph $\mathscr{G}$ with $n = 20$ used in Section 4.6.

4.6 Numerical illustration

WIn this section, we first apply the proposed methodology to a generalization of the all-to-all network of Kuramoto oscillators introduced in ([48]) to model a network of generators. The dynamics of a strongly damped generator can be approximated by nonuniform first-order Kuramoto oscillators, see equation (2.8) in ([48]). We thus consider $n = 10$ first-order heterogeneous Kuramoto oscillators interconnected via the *all-to-all* graph $\mathcal{G}_p$ in Figure 4.3 (in blue). In the absence of a cyber-physical coupling, the right-hand side of the differential equation describing the dynamics of each oscillator is defined by

$$\begin{aligned}\omega_i(\theta, \mathrm{t}) := &\frac{1}{\zeta_i}\bigg(\widetilde{\omega}_i\Big(1 + \frac{3}{10}\sin(\widetilde{\omega}_{\mathrm{t},i}\mathrm{t} + \phi_{\mathrm{t},i})\Big) + d_{t,i}(\mathrm{t}) \\ &- \widetilde{\kappa}_{ij}\sum_{j\in\mathcal{V}\setminus\{i\}}\sin(\theta_j - \theta_i + \phi_{ij})\bigg),\end{aligned} \tag{4.27}$$

with $i \neq j \in \mathcal{V}$ and where $\widetilde{\omega}_{\mathrm{t},i} \in \mathrm{uni}([-1,1])$, $\widetilde{\omega}_i \in \mathrm{uni}([-5,5])$, $\phi_{\mathrm{t},i}, \phi_{ij} \in \mathrm{uni}([0, \mathrm{atan}(0.25)])$, $\widetilde{\kappa}_{ij} = \widetilde{\kappa}_{ji} \in \mathrm{uni}([0.7, 1.2])$ and $\zeta_i \in \mathrm{uni}([20, 30]\frac{1}{120\pi})$. Each high-frequency disturbance $d_{t,i} : \mathbb{R}_{\geq 0} \to [0,5]$ is defined as $d_{t,i}(s) = 0$ if $s \in [0, 5.2] \cup [6.0, 11]$ and $d_{t,i}(s) = 5\sin(50\omega_{\mathrm{t},i}s + \phi_{\mathrm{t},i})$ if $s \in (5.2, 6)$. The parameters are selected as in ([48]) to model realistic, strongly damped, generators. We equip the oscillators with the communication graph $\mathcal{G}_u$ depicted in red in Figure 4.3. We initialize the oscillators with $q(0,0) = \mathbf{0}_9$ and the initial phases are chosen in such a way that the oscillators are equally spaced on the unit circle. Finally, we select $\delta = \frac{\pi}{4}$.

The evolution of the phases, $\theta_i's$, and the angular errors between any two neighbours in $\mathcal{G}$, namely $\min_{h\in\{-1,0,1\}}(|\theta_j - \theta_i + 2h\pi|)$, are reported [7] in the top two rows of Figure 4.4 and in Figure 4.5, for different selections of σ, and $\kappa = \frac{576\pi}{10}$, which ensure finite-time synchronization due to the bound reported in Lemma 4.10 in Section 4.7.3. When no communication layer is considered (left plots), the oscillators do not synchronize. When the communication layer is implemented and σ is given as the ramp function, practical synchronization is achieved as established in Theorem 4.1 and shown in Figures 4.4 and 4.5. (Non-uniform) practical synchronization can also be achieved in absence of a communication layer by selecting larger values for the $\widetilde{\kappa}_{ij}$'s. On the other hand, the sign function, which is discontinuous at 0, also leads to a finite-time synchronization property in an agreement with Theorem 4.3, see Figures 4.4 and 4.5. Furthermore, the set of plots in the bottom row of Figure 4.4 shows that each phase θ_i maps continuously the angular values identifying oscillator i on the unit circle, in agreement with Section 4.2.1 and Lemma 4.2.

Motivated by ([48]) and by the generality of the considered ω_i in (4.1), we conclude by showing that our control scheme is also effective in synchronizing the phase of a network of non-strongly damped generators modeled as second-order heterogeneous oscillators in equation (2.3) of ([48]). Again, we consider the generators to be physically interconnected by an all-to-all network. The dynamics of the natural frequency of each oscillator is defined by the same parameters that we used for the

7. The simulations have been carried out using the Matlab toolbox HyEQ ([114]).

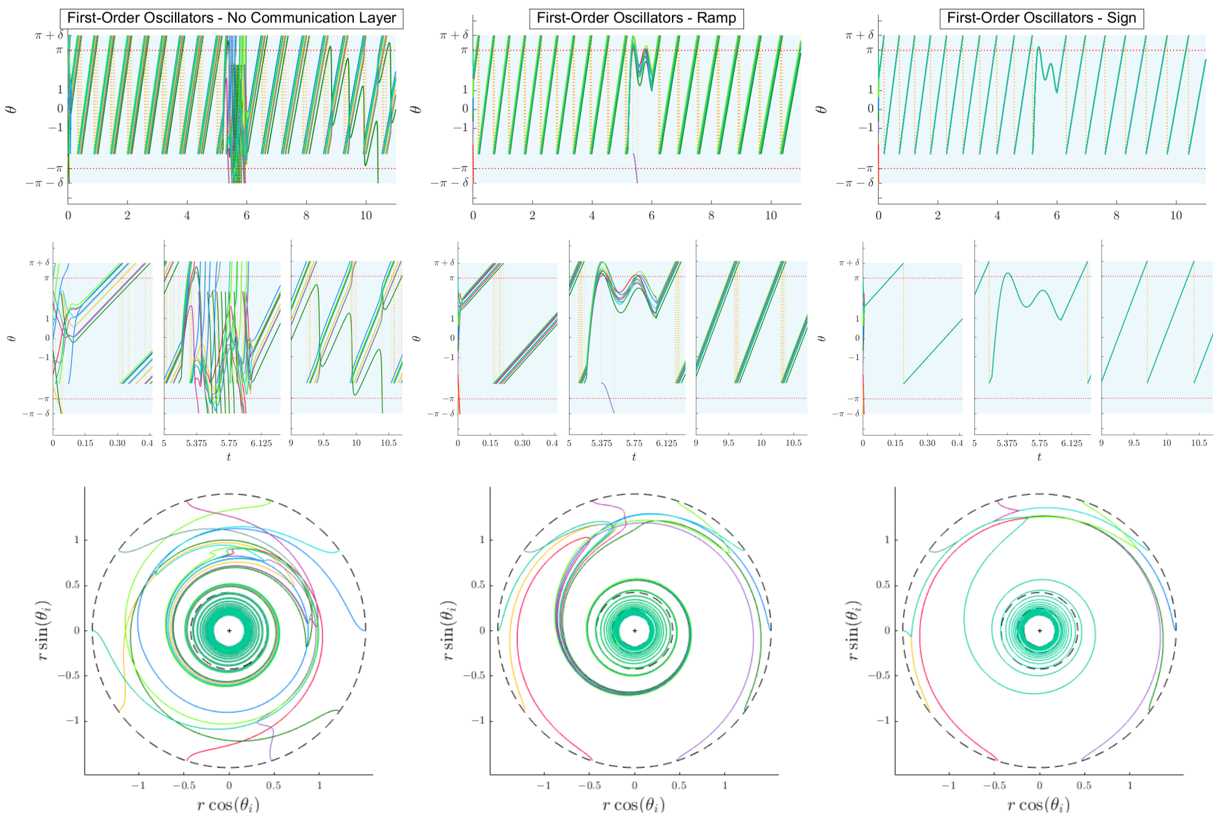

FIGURE 4.4 – (Top) Phase evolution for $\kappa = \frac{576\pi}{10}$, $\delta = \frac{\pi}{4}$ and different selections of σ and communication configurations. (Middle) Phase evolution in the time intervals $[0, 0.41]$, $[5, 5.2]$ and $[9, 10.75]$. (Bottom) Evolution of the pair $(r\cos(\theta_i), r\sin(\theta_i))$, with $r(t) = (1.55\sqrt{t} + 0.66)^{-1}$, showing radially the continuous-time evolution for the phases generated by our hybrid modification (4.7). The black dashed lines are isotime (0 (outer), 3.66, 7.33 and 11 (inner) time units).

previous set of simulations, with the addition of the mass parameters $m_i \in \text{uni}([2, 12]\frac{1}{120\pi})$ for each $i \in \mathcal{V}$. We equip the oscillators with the same network $\mathcal{G}_u$ as in Figure 4.3, we initialize q and θ as in the previous simulations and $\dot{\theta}_i(0, 0) \in \text{uni}([-0.1, 0.1])$ for each $i \in \mathcal{V}$. Finally, we still select $\delta = \frac{\pi}{4}$ in (4.3). The evolution of the phases is reported in Figure 4.6, for different selections of σ, and $\kappa = \frac{576\pi}{10}$. Similarly to what happens for the first-order oscillators, when no communication layer is considered, the second-order oscillators do not synchronize. When the generators are equipped with the communication network and σ is instead defined as the ramp function, practical synchronization is achieved, as predicted by Theorem 4.1 and as shown in Figure 4.6. On the other hand, considering the (discontinuous at 0) sign function to generate the synchronizing hybrid coupling actions again leads to a finite-time synchronization property, thus confirming Theorem 4.3.

To conclude, we provide a numerical illustration showing that, when σ is the modified sinusoid of Fig.4.1, the phases generated by our hybrid system (4.7) evolve like those given by the Kuramoto model

$$\dot{\theta}_i = \omega_i + \kappa \sum_{j \in \{1,\ldots,n\}} \sin(\theta_j - \theta_i), \quad \theta_i \in \mathbb{R}, \quad i \in \{1, \ldots, n\} \tag{4.28}$$

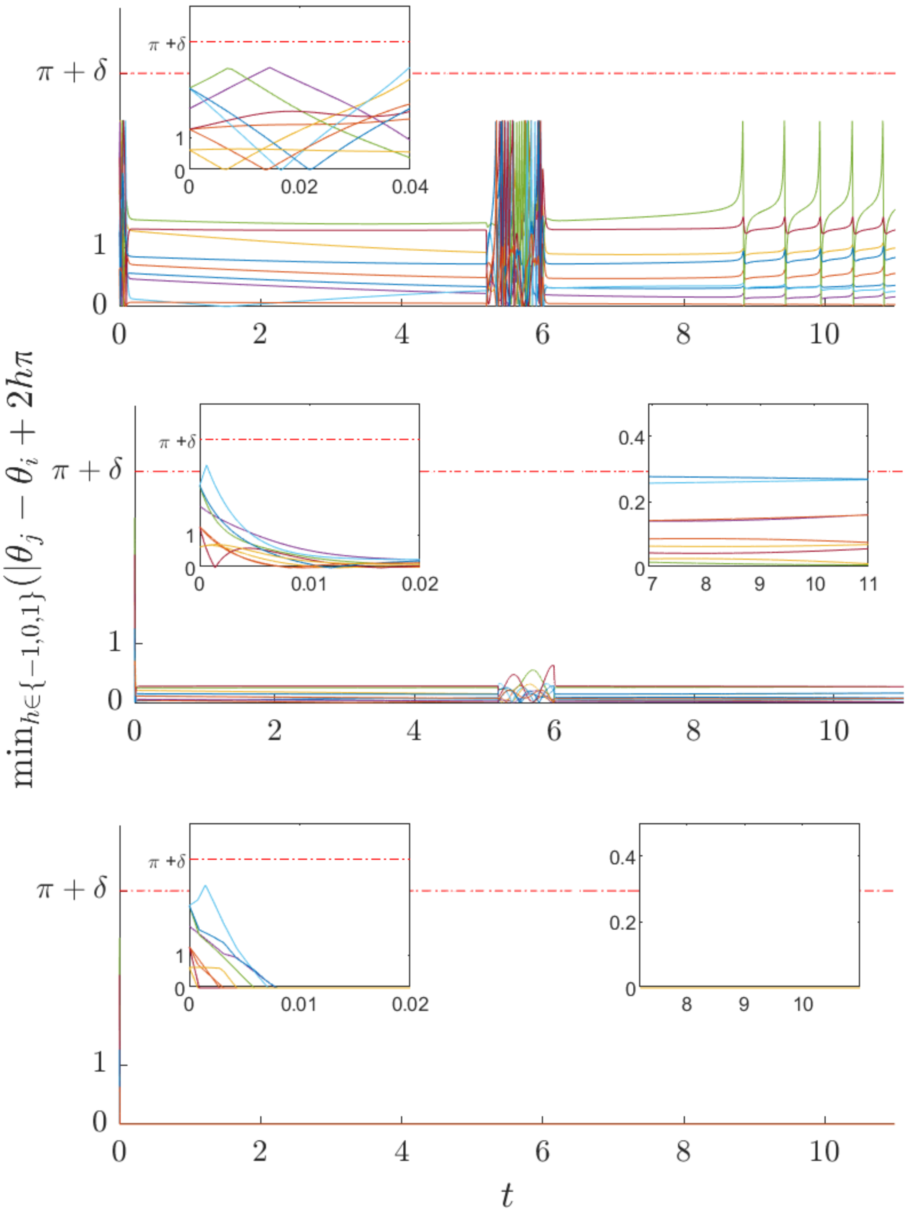

FIGURE 4.5 – Evolution of the phase errors for $\kappa = \frac{576\pi}{10}$ and different selections of σ and communication configurations (from top to bottom: no communication layer, ramp and sign functions).

up to relatively large values of the initial errors. For this purpose, we consider $n = 5$ oscillators, $\kappa = 1$ both in (4.7) and (4.28), so that the coupling gains are the same, $\omega_i = 1$, $i \in \{1, \ldots, 5\}$ and δ as above. Fig.4.6 shows the corresponding simulation results. Note that the phases generated by (4.28) and (4.7) coincide up to a relatively large initial phase mismatch (first two rows). For larger initial errors, our modified sinusoid induces uniform synchronisation, whereas slow transients are generated by (4.28) (bottom row). Similar results have been obtained in the case where the interconnection graph is a tree. Even though our main results in Theorems 4.2 and 4.3 require tree-

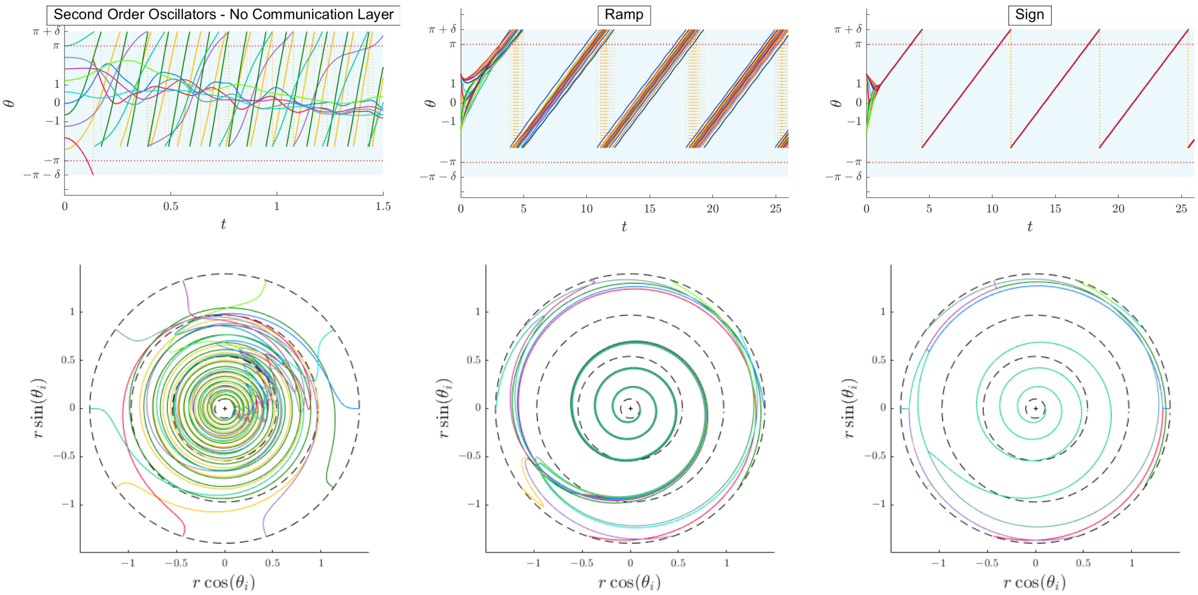

FIGURE 4.6 – (Top) Phase evolution associated of second-order oscillators for $\kappa = \frac{576\pi}{10}$, $\delta = \frac{\pi}{4}$ and different selections of σ and communication configurations. (Bottom) Evolution of the pair $(r\cos(\theta_i), r\sin(\theta_i))$, with $r(t) = -0.255\sqrt{t} + 1.4$, showing radially the continuous-time evolution for the phases generated by our hybrid modification of second-order oscillators. The black dashed lines are isotime (0 (outer), 0.5, 1 and 1.5 (inner) time units).

like networks and the counterexample discussed in Section 4.4.2 remains an important obstruction to removing that assumption, Fig.4.6 shows that our solution (4.7) may provide desirable uniform synchronization also with an “all-to-all" network.

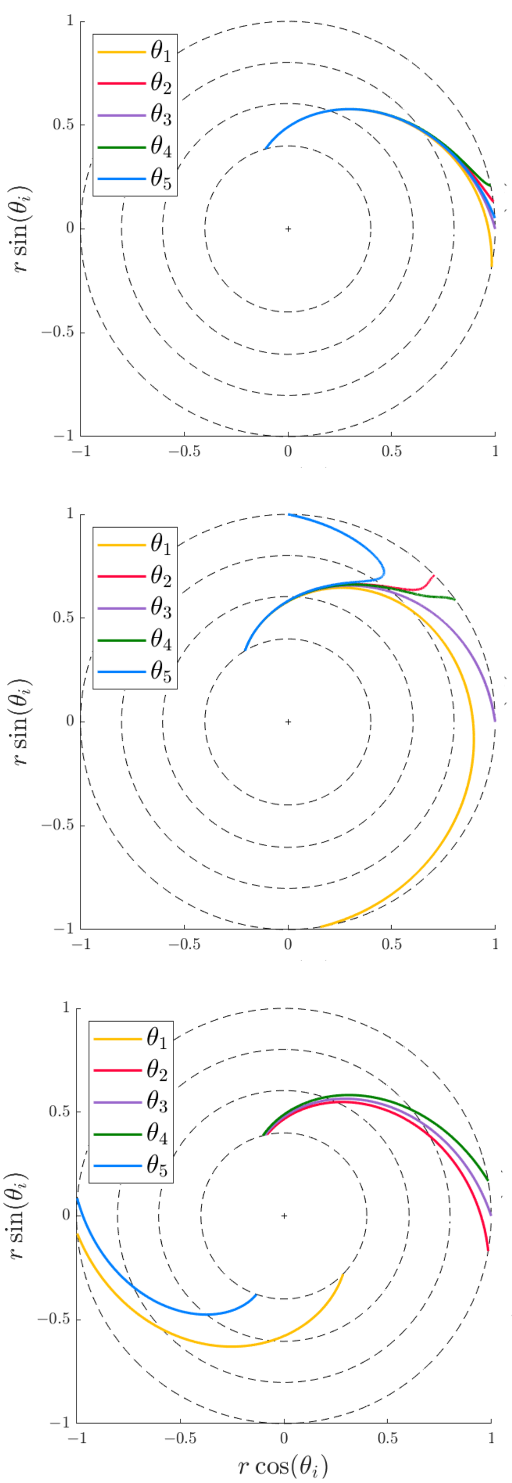

FIGURE 4.7 – Evolution of the pair $(r\cos(\theta_i), r\sin(\theta_i))$, with $r(t) = -0.2t + 1$, showing radially the continuous-time evolution for the phases generated by the classical Kuramoto model (4.28) (left) and our hybrid modification (4.7) with $q(0,0) = \mathbf{0}_{10}$ (right).The initial phase mismatch is increasingly large from top to bottom. The black dashed lines are isotime (0 (outer), 0.6, 1.2 and 1.8 (inner) time units).

4.7 Proof of Propositions 4.2 and 4.3

4.7.1 Results on scalar non-pathological functions

The proofs of Propositions 4.2 and 4.3, which are instrumental for proving our main results of Sections 4.4 and 4.5, require exploiting results from non-smooth analysis, due to the fact that V in (4.19) is not differentiable everywhere. A further complication emerges from the fact that, since σ may be discontinuous, the flow map in dynamics (4.8) is outer semicontinuous, but not inner semi-continuous. The lack of inner semicontinuity prevents us from exploiting the "almost everywhere" conditions of [45] and references therein. Instead, one could resort to conditions involving Clarke's generalized directional derivative and Clarke's generalized gradient, which can be defined as (see [33, page 11])

$$\partial V(x) := \text{co}\{\lim_{i\to\infty} \nabla V(x_i) : x_i \to x,\, x_i \notin \mathscr{X},\, x_i \notin \Omega_u\}, \tag{4.29}$$

where Ω_u is the set (of Lebesgue measure zero) where U is not differentiable, and $\mathscr{X}$ is any other set of Lebesgue measure zero. However, due to the peculiar dynamics considered here (resembling, for example, the undesirable conservativeness highlighted in [43, Ex. 2.2]), Lyapunov decrease conditions based on Clarke's generalized gradient would be too conservative and impossible to prove. Due to the above motivation, in this section we prove Propositions 4.2 and 4.3 by exploiting the results of [13, 136], whose proof is also reported in [43, Lemma 2.23], establishing a link between the time derivative $\frac{d}{dt}V(\phi(t))$ of a Lyapunov-like function V evaluated along a generic solution ϕ of a continuous-time system, and the so-called set-valued Lie derivative [12]

$$\dot{\overline{V}}_F(x) := \{a \in \mathbb{R} |\, \exists f \in F(x) : \langle v, f\rangle = a,\ \forall v \in \partial V(x)\}, \tag{4.30}$$

with ∂V defined in (4.29). In the following we characterize some features of the set-valued Lie derivative, useful for the next technical derivations.

Lemma 4.8. *Consider a set $S \subset \mathbb{R}^n$, $F : \text{dom}\, F \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $S \subset \text{dom}(F)$ and a locally Lipschitz $V : \text{dom}\, V \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $S \subset \text{dom}(V)$. Given any function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\varphi(x, f) \in \partial V(x)$, $\forall x \in S$, $\forall f \in F(x)$, it holds that*

$$\sup \dot{\overline{V}}_F(x) \leq \sup_{f\in F(x)} \langle \varphi(x, f), f\rangle. \tag{4.31}$$

□

Proof: Consider any $x \in S$. In view of (4.30), and by the fact that $\varphi(x, f) \in \partial V(x)$ for all $f \in F(x)$, we have that $a = \langle v, f(x)\rangle \in \dot{\overline{V}}_F(x)$ implies $a = \langle \varphi(x, f), f\rangle$. Hence, we derive that $\dot{\overline{V}}_F(x) \subseteq \bigcup_{f\in F(x)} \{\langle \varphi(x, f), f\rangle\}$ and thus $\sup \dot{\overline{V}}_F(x) \leq \sup_{f\in F(x)} \langle \varphi(x, f), f\rangle$ as to be proven. ■

Whenever function V is non-pathological (according to the definition given next), [43, Lemma 2.23] ensures that $\frac{d}{dt}V(\phi(t)) \in \dot{\overline{V}}_F(\phi(t))$ for almost all t in the domain of ϕ. We provide below the definition of non-pathological functions and we prove that function V in (4.19) enjoys that property. Our result below can be seen as a corollary of the fact that piecewise C^1 functions (in the sense of [118]) are non-pathological. This result has been recently published in [46, Lemma 4]. The scalar case is a consequence of Proposition 5 in ([136]). An alternative proof is reported here.

Definition 4.1. *[136] A locally Lipschitz function $W : \mathrm{dom}\, W \subseteq \mathbb{R} \to \mathbb{R}$ is* non-pathological *if, given any absolutely continuous function $\phi : \mathbb{R}_{\geq 0} \to \mathrm{dom}\, W$, we have that for almost every $\mathfrak{t} \in \mathbb{R}_{\geq 0}$ there exists $a_{\mathfrak{t}} \in \mathbb{R}$ satisfying*

$$\langle w, \dot{\phi}(\mathfrak{t})\rangle = a_{\mathfrak{t}}, \quad \forall w \in \partial W(\phi(\mathfrak{t})). \tag{4.32}$$

In other words, for almost every $\mathfrak{t} \in \mathbb{R}_{\geq 0}$, $\partial W(\phi(\mathfrak{t}))$ is a subset of an affine subspace orthogonal to $\dot{\phi}(\mathfrak{t})$. □

Proposition 4.4. *Any function $W : \mathrm{dom}\, W \subseteq \mathbb{R} \to \mathbb{R}$ that is piecewise continuously differentiable is non-pathological.* □

For this chapter, the interest of Proposition 4.4 stands in the fact that it implies that function V in (4.19) is non-pathological [12, 136], being the sum of piecewise continuously differentiable scalar functions.

Remark 4.3. *An alternative proof of Proposition 4.4, might be to first establish that piecewise C^1 functions from $\mathbb{R} \to \mathbb{R}$ can be represented as a* max-min *of C^1 functions from $\mathbb{R} \to \mathbb{R}$ (a similar result has been proven, for example, in [142, Thm. 1 and Prop. 1] with reference to piecewise affine functions), and then obtain Proposition 4.4 as a corollary of [12, 43, 136, Lemma 2.20], which establish non-pathological properties of* max-min *functions.* □

Proof of Proposition 4.4: Let $W : \mathrm{dom}\, W \subseteq \mathbb{R} \to \mathbb{R}$ be piecewise continuously differentiable. Let $\phi : \mathbb{R}_{\geq 0} \to \mathrm{dom}\, W$ be an absolutely continuous scalar function and suppose it is differentiable at $\bar{\mathfrak{t}} \in \mathbb{R}_{\geq 0}$. We split the analysis in three cases.

a) $\dot{\phi}(\bar{\mathfrak{t}}) = 0$. Then, for all $w \in \partial W(\phi(\bar{\mathfrak{t}}))$, $\langle w, \dot{\phi}(\bar{\mathfrak{t}})\rangle = \langle w, 0\rangle = 0$. Thus, (4.32) is satisfied with $a_{\mathfrak{t}} = 0$.

b) $\dot{\phi}(\bar{\mathfrak{t}}) > 0$. If W is continuously differentiable at $\phi(\bar{\mathfrak{t}})$ then, $\partial W(\phi(\bar{\mathfrak{t}})) = \nabla W(\phi(\bar{\mathfrak{t}}))$ and (4.32) holds. Consider now the case where W is not differentiable at $\bar{\mathfrak{t}}$. We recall that, in view of the fact that W is piecewise continuously differentiable, $\partial W(\phi(\mathfrak{t})) = \nabla W(\phi(\mathfrak{t}))$ for any $\mathfrak{t}$ in a sufficiently small neighbourhood of $\bar{\mathfrak{t}}$. From the absolute continuity of ϕ, there exists $\varepsilon > 0$ such that $\lim\limits_{h\to 0} \frac{\phi(\bar{\mathfrak{t}}+\mathfrak{h})-\phi(\bar{\mathfrak{t}})}{\mathfrak{h}} \geq \varepsilon > 0$. Hence, there exists $\rho_1 \in \mathbb{R}_{>0}$ such that for any $\rho \in (0, \rho_1]$, we have $\frac{\phi(\bar{\mathfrak{t}}+\rho)-\phi(\bar{\mathfrak{t}})}{\rho} \geq \frac{\varepsilon}{2}$ and thus $\phi(\bar{\mathfrak{t}}+\rho) \geq \frac{\rho\varepsilon}{2} + \phi(\bar{\mathfrak{t}}) > \phi(\bar{\mathfrak{t}})$. With a similar reasoning, there exist $\rho_2 \in \mathbb{R}_{\geq 0}$ such that $\rho_2 > 0$ implies $\phi(\bar{\mathfrak{t}} - \rho) < \phi(\bar{\mathfrak{t}})$ for any $\rho \in (0, \rho_2]$. Hence, there exists a neighbourhood of $\bar{\mathfrak{t}}$ contained in $[\bar{\mathfrak{t}} - \rho_2, \bar{\mathfrak{t}} + \rho_1]$, for which $\partial W(\phi(\cdot))$ is defined and coincides with $\nabla W(\phi(\mathfrak{t}))$. Therefore, for almost every $\tilde{\mathfrak{t}} \in \mathscr{T} \subseteq [\bar{\mathfrak{t}} - \rho_2, \bar{\mathfrak{t}} + \rho_1]$ there exists $a_{\mathfrak{t}} \in \mathbb{R}$ such that (4.32) is satisfied.

c) $\dot{\phi}(\bar{t}) < 0$. This case is identical to the previous one by changing all the signs, therefore implying that for almost every t there exists $a_t \in \mathbb{R}$ in a compact neighbourhood of $\bar{t}$ such that (4.32) is satisfied.

Since ϕ is absolutely continuous, then the set Φ where it is not differentiable is of Lebesgue measure zero. We conclude that (4.32) is satisfied for almost all $t \in \mathbb{R}_{\geq 0}$ because we have arbitrarily selected $\bar{t} \in \mathbb{R}_{\geq 0} \setminus \Phi$, as to be proven. ■

4.7.2 Proof of Proposition 4.2

The following lemma establishes geometric properties of V that are used, together with Lemma 4.5 and Proposition 4.4, to show the trajectory-based results of Proposition 4.2.

Lemma 4.9. *Consider system* (4.8)*, function V in* (4.19)-(4.20) *and c as in Theorem 4.1. There exists $\alpha_3 \in \mathcal{K}_\infty$, independent of $\overline{\omega}$ in* (4.14) *and of κ, such that*

$$\sup \dot{\overline{V}}_F(x) \leq -\kappa \underline{\lambda} \alpha_3(V(x)) + c\overline{\omega}, \qquad \forall x \in C, \tag{4.33a}$$

$$\Delta V(x) := \sup_{g \in G(x)} V(g) - V(x) \leq 0, \qquad \forall x \in D. \tag{4.33b}$$

□

Proof: We prove the two equations one by one.

Proof of (4.33a). For each $x \in C$ and each $f \in F(x)$, there exist $\widehat{\boldsymbol{\omega}} \in \widehat{\Omega}$ and $\boldsymbol{\sigma}_f \in \widehat{\Sigma}(x)$ such that $f = (\widehat{\boldsymbol{\omega}} - B\kappa\boldsymbol{\sigma}_f, 0) \in F(x)$. Define $\varphi(x, f)$ in Lemma 4.8 as $\varphi(x, f) := (B\boldsymbol{\sigma}_f, 2\pi\boldsymbol{\sigma}_f) \in \partial V(x)$. From (4.30) in Lemma 4.8, we have

$$\sup \dot{\overline{V}}_F(x) \leq \sup_{\substack{\widehat{\boldsymbol{\omega}} \in \widehat{\Omega}, \\ \boldsymbol{\sigma}_f \in \widehat{\Sigma}(x)}} (-\kappa \boldsymbol{\sigma}_f^\top B^\top B \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_f^\top B^\top \widehat{\boldsymbol{\omega}}). \tag{4.34}$$

Thus, in view of (4.14) and Lemma 4.4, (4.34) yields

$$\begin{aligned} \sup \dot{\overline{V}}_F(x) &\leq \sup_{\boldsymbol{\sigma}_f \in \widehat{\Sigma}(x)} (-\kappa \boldsymbol{\sigma}_f^\top B^\top B \boldsymbol{\sigma}_f) + c\overline{\omega} \\ &\leq \sup_{\boldsymbol{\sigma}_f \in \widehat{\Sigma}(x)} -\kappa \underline{\lambda} |\boldsymbol{\sigma}_f|^2 + c\overline{\omega}. \end{aligned} \tag{4.35}$$

Moreover, in view of (4.21) and Lemmas 4.5 and 4.6, we have that, for any $\boldsymbol{\sigma}_f \in \widehat{\Sigma}(x)$,

$$\eta \circ \alpha_2^{-1}(V(x)) \leq \eta(|x|_{\mathcal{A}}) \leq |\boldsymbol{\sigma}_f|^2. \tag{4.36}$$

By defining $\alpha_3 := \eta \circ \alpha_2^{-1} \in \mathcal{K}_\infty$, (4.33a) stems from (4.35) and (4.36).

Proof of (4.33b). We split the analysis in two cases.

First, let $i \in \mathcal{V}$, $x \in D_i$ and $x^+ = g_i(x)$ as in Lemma 4.2. In view of the equality in (4.6) and the definition of V_{ij} in (4.20), we have

$$\begin{aligned} V_{ij}(x^+) &= \int_0^{\theta_j^+ - \theta_i^+ + 2q_{ij}^+\pi} \sigma(\text{sat}_{\pi+\delta}(s))ds \\ &= \int_0^{\theta_j - \theta_i + 2q_{ij}\pi} \sigma(\text{sat}_{\pi+\delta}(s))ds = V_{ij}(x), \end{aligned}$$

and thus $V(x^+) = \sum_{(i,j)\in\mathcal{E}} V_{ij}(x^+) = \sum_{(i,j)\in\mathcal{E}} V_{ij}(x) = V(x)$.

Second, let $(i,j) \in \mathcal{E}$, $x \in D_{ij}$ and $x^+ \in G_{ij}(x)$ as in Lemma 4.1. In view of item a) of Property 4.1,

$$V_{ij}(x) = \int_0^{|\theta_j - \theta_i + 2q_{ij}\pi|} \sigma(\text{sat}_{\pi+\delta}(s))ds. \tag{4.37}$$

On the other hand, in view of (4.4a), (4.4b) and Lemma 4.1,

$$\begin{aligned} |\theta_j^+ - \theta_i^+ + 2q_{ij}^+\pi| &= |\theta_j - \theta_i + 2q_{ij}^+\pi| \\ &< \pi + \delta \le |\theta_j - \theta_i + 2q_{ij}\pi|, \end{aligned}$$

because $x \in D_{ij}$. Consequently, in view of (4.4a) and item b) of Property 4.1

$$\begin{aligned} V_{ij}(x^+) &= \int_0^{|\theta_j^+ - \theta_i^+ + 2q_{ij}^+\pi|} \sigma(\text{sat}_{\pi+\delta}(s))ds \\ &< \int_0^{|\theta_j - \theta_i + 2q_{ij}\pi|} \sigma(\text{sat}_{\pi+\delta}(s))ds = V_{ij}(x). \end{aligned}$$

On the other hand, from the definition of G_{ij} in (4.4b), $V_{uv}(x^+) = V_{uv}(x)$ for any $(u,v) \neq (i,j) \in \mathcal{E}$. Therefore $V(x^+) - V(x) = V_{ij}(x^+) - V_{ij}(x) < 0$, since the arguments of all the other elements of the summation in (4.19) do not change. ■

Based on Lemma 4.9, we can now prove Proposition 4.2.

Proof of Proposition 4.2: Item (ii) of Proposition 4.2 is a direct consequence of (4.33b) in Lemma 4.9. To prove item (i) of Proposition 4.2 we exploit the fact that, in view of [43, Lemma 2.23] and V being non pathological, for each solution x to (4.8), for all $\mathrm{j} \in \{0, \dots, J\}$ and almost all $\mathrm{t} \in [\mathrm{t_j}, \mathrm{t_{j+1}}]$ in $\operatorname{dom} x$, $\frac{dV(x(\mathrm{t},\mathrm{j}))}{d\mathrm{t}} \in \dot{\overline{V}}_F(x(\mathrm{t},\mathrm{j}))$. Hence, as a consequence, $\frac{d}{d\mathrm{t}} V(x(\mathrm{t},\mathrm{j})) \le -\kappa \underline{\lambda} \alpha_3(V(x(\mathrm{t},\mathrm{j}))) + c\overline{\omega}$, thus concluding the proof. ■

4.7.3 Proof of Proposition 4.3

Paralleling Section 4.7.2, we establish via the next lemma geometric properties of V that can be used, together with Lemma 4.5 and Proposition 4.4, to show the trajectory-based result of Proposition 4.3.

Lemma 4.10. *If σ is discontinuous at the origin, then there exist $\mu \in \mathbb{R}_{>0}$ independent of $\overline{\omega}$ in (4.14) and $\kappa^\star > 0$ such that for each $\kappa > \kappa^\star$*

$$\sup \dot{\overline{V}}_F(x) \leq -\frac{1}{2}\kappa\underline{\lambda}\mu^2, \qquad \forall x \in C \setminus \mathscr{A}, \tag{4.38a}$$

$$\Delta V(x) := \sup_{g\in G(x)} V(g) - V(x) \leq 0, \qquad \forall x \in D. \tag{4.38b}$$

□

Proof: We only prove (4.38a), as (4.38b) follows from the same arguments as those proving (4.33b) in Proposition 4.2. For each $x \in C \setminus \mathscr{A}$ and each $f \in F(x)$, we may proceed as in (4.34) and then exploit from Lemma 4.7 that,

$$\sup \dot{\overline{V}}_F(x) \leq -\kappa\underline{\lambda}\mu^2 + c\overline{\omega}. \tag{4.39}$$

By selecting $\kappa \geq \kappa^\star = \frac{2c\overline{\omega}}{\underline{\lambda}\mu^2}$, (4.39) yields $\sup \dot{\overline{V}}_F(x) \leq -\frac{1}{2}\kappa\underline{\lambda}\mu^2$, which proves (4.38a). ■

Based on Lemma 4.10, we are now ready to prove Proposition 4.3.

Proof of Proposition 4.3: Item (ii) of Proposition 4.3 is a direct consequence of (4.38b) in Lemma 4.10. In view of [43, Lemma 2.23] and V being non pathological, for each solution x to (4.8), for all $\mathrm{j} \in \{0, \ldots, J\}$ and almost all $\mathrm{t} \in [\mathrm{t_j}, \mathrm{t_{j+1}}]$ in $\operatorname{dom} x$, $\frac{dV(x(\mathrm{t},\mathrm{j}))}{dt} \in \dot{\overline{V}}_F(x(\mathrm{t},\mathrm{j}))$. Hence, as a consequence, $\frac{d}{dt}V(x(\mathrm{t},\mathrm{j})) \leq -\frac{1}{2}\kappa\underline{\lambda}\mu^2$, whenever $x(\mathrm{t},\mathrm{j}) \notin \mathscr{A}$, thus proving item (i) of Proposition 4.3 which concludes the proof. ■

4.8 Conclusions

We presented a cyber-physical hybrid model of leaderless heterogeneous first-order oscillators, where global uniform asymptotic and/or finite-time synchronization is obtained in a distributed way via hybrid coupling. More specifically we establish that the synchronization set for the proposed model enjoys uniform asymptotic practical stability property. Thanks to the mild requirements on the coupling function, the stability result was strengthened to a prescribed finite-time property when the coupling function is discontinuous at the origin. Finally, we proved a useful statement on scalar non-pathological functions, exploited here for the non-smooth Lyapunov analysis in our main theorems. We believe that this work demonstrates the potential of hybrid systems theoretical tools to overcome the fundamental limitations of continuous-time systems.

Future extensions of this work includes addressing graphs with cycles (not trees) and investigating the case with leaders as done for a second-order Kuramoto model in [19]. Additional challenges may include studying the converse problem of globally de-synchronizing the network [56] via hybrid approaches. For detailed discussions on possible additional scientific directions, refer to Chapter 6.

In the next chapter, we are going to further exploit the results on nonpathological functions and set-valued Lie derivatives given in Section 4.7, to analyze the stability properties of continuous-time Lur'e systems with piecewise continuous nonlinearities, thanks to the similarities between the Lyapunov functions and continuous-time dynamics appearing in Chapters 4 and 5.

Chapter 5

Lur'e systems with piecewise continuous nonlinearities:finite-time stability

Contents

Abstract - In this chapter, we analyze the stability properties of Lur'e systems with piecewise continuous nonlinearities by exploiting the notion of set-valued Lie derivative for Lur'e-Postnikov Lyapunov functions by exploiting the results on nonpathological functions and set-valued Lie derivative provided in Chapter 4. We first extend an existing result of the literature to establish the global asymptotic stability of the origin under a more general sector condition. We then present the main results of this chapter, namely additional conditions under which output and state finite-time stability properties also hold for the considered class of systems. We highlight the relevance of these results by certifying the stability properties of two engineering systems of known interest: mechanical systems affected by friction and cellular neural networks.

5.1 Introduction

Defining conditions to ensure stability properties of continuous-time linear systems subject to a cone-bounded nonlinear output feedback, namely, the so-called the Lur'e problem, has been widely investigated in the literature see, e.g., [73, 80, 106, 143]. This class of systems is ubiquitously used in various engineering domains, such as mechanical engineering to describe dynamical systems affected by friction and/or unilateral constraints [26], electrical and electronic engineering to capture the behavior of electrical circuits with switches or electronic devices [1, 138], or neural networks [124]; see [25] for additional examples. However, to the authors' best knowledge, very few results are available on the finite-time stability properties of Lur'e systems, see [129], which concentrates on cluster synchronization of networks of Lur'e systems. Finite-time stability properties are gaining increasing attention due to their relevance in many applications such as high-order sliding mode algorithms [105], controllers for mechanical systems [16], spacecraft stabilization [140], observer design problems [7]; see [139] for additional examples. There is therefore a need for analytical tools to establish finite-time stability properties for this class of systems. In this context, we investigate the output and state finite-time stability properties of Lur'e system with piecewise continuous nonlinearities.

Historically, two different types of Lyapunov functions have been used to analyze the (absolute) stability of continuous-time Lur'e systems: quadratic functions of the state and the so-called Lur'e-Postnikov Lyapunov functions, which are the sum of a quadratic function of the state and a weighted sum of the integrals of the feedback nonlinearities [73]. Lur'e-Postnikov Lyapunov functions are generally used to draw less conservative sufficient stability conditions [143]. However, when the nonlinearities are piecewise continuous, as in, e.g., mechanical systems [26], neural networks [55], see also [25], the challenge is that Lur'e-Postnikov Lyapunov functions become only differentiable almost everywhere (being locally Lipschitz continuous) due to the discontinuity points of the nonlinearities. Indeed, when the system nonlinearities are piecewise continuous and a Lur'e-Postnikov Lyapunov function is considered, the standard tools used in the nonsmooth analysis, like Clarke's generalized directional derivatives, may lead to conservative algebraic Lyapunov conditions as we show in this chapter; see also [43]. This limitation is overcome in [26], where trajectory-based arguments are used to prove an input-to-state (ISS) stability property, but no finite-time stability property is provided.

In this chapter, we first extend one of the results in [26] to establish the global asymptotic stability of the origin for Lur'e systems with piecewise continuous nonlinearities under a more general sector condition. We resort for this purpose to a nonsmooth Lur'e-Postnikov Lyapunov function. We present algebraic Lyapunov decrease conditions by using the notion of set-valued Lie derivative [12, 136]. The set-valued Lie derivative is the key to overcoming the conservatism which the customarily used Clarke's generalized directional derivative may give, as we illustrate in a dedicated example. It has to be noted that in [129] set-valued Lie derivatives are also used in the analysis of these interconnections, however, the Lyapunov function is quadratic (thus continuously differentiable), which, as mentioned above, leads to more conservative conditions and, more importantly, the problem setting is different. Our main results establish output and state finite-time stability properties for the considered Lur'e systems. To illustrate the usefulness of our results we focus on two engineering applications, considered respectively in [26, 55] and that can be modeled as Lur'e systems. Indeed, we establish output finite-time and state-independent local asymptotic stability properties for mechanical systems subject to friction, which is a novelty compared to [26]. Furthermore, we certify that the cellular neural networks modeled as in [55] are state finite-time stable, thus retrieving the results in [55, Thm. 4] while coping with a more general class of Lur'e systems.

The rest of the chapter is organized as follows. Notation and background material are given in Section. The class of Lur'e systems under consideration is introduced in Section 5.2. In Section 5.3, we address asymptotic stability characterizations with a novel algebraic Lyapunov proof. Finite-time stability results are given in Section 5.4, while we discuss applications of these results in Section 5.5. In Section 5.6 we give conclusions and some perspectives.

5.2 Problem statement

Consider the system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\boldsymbol{\psi}(y), \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state, $u, y \in \mathbb{R}^p$ are respectively the input and the output and A, B and C are real matrices of appropriate dimensions. The function $\boldsymbol{\psi} : \mathbb{R}^p \to \mathbb{R}^p$ is decentralized, namely for any $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$, $\boldsymbol{\psi}(y) = (\psi_1(y_1), \ldots, \psi_p(y_p))$. We suppose that ψ satisfies the next sector condition.

Assumption 5.1. *For any $i \in \{1, \ldots, p\}$, ψ_i is piecewise continuous and there exists $\zeta_i \in (0, +\infty]$ such that*

$$\psi_i(y_i)(\psi_i(y_i) - \zeta_i y_i) \leq 0, \qquad \forall y_i \in \mathbb{R}. \tag{5.2}$$

□

The sector condition (5.2) is more general than the one considered in [26], that is recovered when $\zeta_i = +\infty$ for all $i \in \{1, \ldots, p\}$, in which case (5.2) reads

$$-\psi_i(y_i) y_i \leq 0, \qquad \forall y_i \in \mathbb{R} \qquad \forall i \in \{1, \ldots, p\}. \tag{5.3}$$

This generalization allows to derive less conservative stability conditions when the nonlinearities satisfy (5.2) with some finite ζ_i. Assumption 5.1 characterizes a so-called Lur'e system [73, Ch. 7], [143].

In view of Assumption 5.1, system (5.1) may have a discontinuous right-hand side. Therefore, when we refer to the solutions to system (5.1), we consider its so-called (generalized) Krasovskii solutions, which coincide with the solutions obtained by the Krasovskii regularization [66] of (5.1), that is

$$\dot{x} \in F(x) := Ax - B\mathbf{\Psi}(Cx), \qquad y = Cx, \tag{5.4}$$

where $\mathbf{\Psi}(y) = (\mathbf{\Psi}_1(y_1), \ldots, \mathbf{\Psi}_p(y_p))$ is the Krasovskii regularization of $\boldsymbol{\psi}$ in (5.1), whose components are defined as $\mathbf{\Psi}_i(y_i) := \bigcap_{s>0} \overline{\text{co}}\, \boldsymbol{\psi}_i(y_i + s\mathbb{B})$, $i \in \{1, \ldots, p\}$, for any $y_i \in \mathbb{R}$. Observe that, by Assumption 5.1, F is outer semicontinuous and locally bounded on $\mathbb{R}^n$ and $F(x)$ is convex for any $x \in \mathbb{R}^n$, thus local existence of solutions to (5.4) is guaranteed by Theorem 3 in [11, Ch. 2.1]. Moreover, by definition, each $\mathbf{\Psi}_i : \mathbb{R} \rightrightarrows \mathbb{R}$ is set-valued only on a set of isolated points, therefore it is locally integrable: a property that will be exploited in the following.

We analyze the stability properties of system (5.4) in the sequel, thereby ensuring the same stability properties for the Krasovskii solutions of (5.1). As customary in the Lur'e systems literature and as shown in Fig. 5.1, we perform a loop transformation to interpret system (5.4) as the feedback interconnection of two passive systems.

By following the steps in [73, Ch. 7.1.2] and [26] and by adopting the same mathematical notation found in [73, Ch. 7], we define the dynamic multiplier with transfer function

$$\mathscr{M}(s) := I + \Gamma s \quad \forall s \in \mathbb{C}, \tag{5.5}$$

where $\Gamma := \text{diag}(\gamma_1, \ldots, \gamma_p)$ and $\gamma_1, \ldots, \gamma_p > 0$ are suitable parameters, as detailed in the sequel. We thus interpret system (5.4) as the feedback interconnection of the linear system Σ_1

$$\Sigma_1 : \begin{cases} \dot{x} &= Ax + Bu \\ \overline{y} &= (C + \Gamma CA)x + (\Gamma CB + Z)u, \end{cases} \tag{5.6}$$

where $Z := \text{diag}(\zeta_1^{-1}, \ldots, \zeta_p^{-1})$ with [8] $\zeta_i \in (0, +\infty]$ in Assumption 5.1, with the nonlinear system Σ_2

$$\Sigma_2 : \begin{cases} \dot{y} &= g(y, \overline{y}) := -\Gamma^{-1}y + \Gamma^{-1}(\overline{y} - Zu) \\ u &\in -\mathbf{\Psi}(y). \end{cases} \tag{5.7}$$

We assume that system Σ_1 is strictly passive from u to $\overline{y}$ with quadratic storage function U_1 defined as

$$U_1(x) := \frac{1}{2}x^\top Px, \quad \forall x \in \mathbb{R}^n, \tag{5.8}$$

with $P \in \mathbb{R}^{n\times n}$ symmetric and positive definite [9], as formalized next.

Assumption 5.2. *System Σ_1 in (5.6) is strictly passive from u to $\overline{y}$ with storage function U_1 in (5.8)*

8. When $\zeta_i = +\infty$ we use the convention $\zeta_i^{-1} = 0$.
9. We define a symmetric matrix $P \in \mathbb{R}^{n\times n}$ with $n \in \mathbb{Z}_{>0}$ to be positive (negative) definite, i.e., $P > 0\,(P < 0)$, if all its eigenvalues are real and positive (negative).

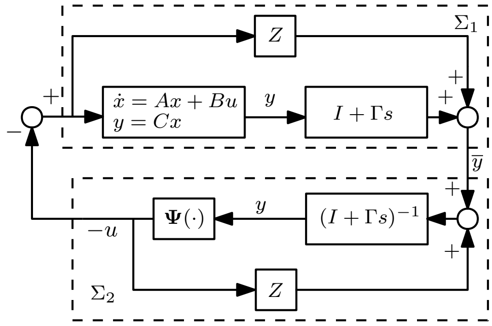

FIGURE 5.1 – System (5.4) as the feedback interconnection of systems Σ_1 and Σ_2.

[73, Def. 6.3], i.e., there exist matrices $\Gamma > 0$ *diagonal,* $P = P^\top > 0$ *and a scalar* $\eta > 0$ *such that* [10]

$$M := \begin{bmatrix} PA + A^\top P + \eta I_n & PB - (C + \Gamma CA)^\top \\ B^\top P - (C + \Gamma CA) & -2Z - \Gamma CB - (\Gamma CB)^\top \end{bmatrix} \leq 0. \tag{5.9}$$

□

The linear matrix inequality (5.9) in Assumption 5.2 can be efficiently tested numerically. Several tools are also available in the literature to certify (5.9): the Kalman-Yakubovich-Popov lemma [73, Lemma 6.3] and the equivalent conditions given in [24, Ch. 3.1] for minimal realizations or the results surveyed in [24, Ch. 3.3] for nonminimal ones, to cite a few.

On the other hand, it can be proven, as clarified later in Remark 5.1 in Section 5.3.2, that Σ_2 in (5.7) is passive from input $\overline{y}$ to output $-u$, by considering the piecewise continuously differentiable storage function U_2 defined as

$$U_2(y) := \sum_{i=1}^{p} \gamma_i \int_0^{y_i} \psi_i(\sigma) d\sigma, \quad \forall y \in \mathbb{R}^p, \tag{5.10}$$

where $\gamma_i > 0$ for $i \in \{1, \ldots, p\}$ are the diagonal elements of Γ as defined after (5.5).

We are now ready to proceed with the stability analysis of (5.4). First, we provide sufficient conditions to ensure global asymptotic stability of the origin for system (5.4) in Section 5.3. We then analyze its finite-time stability properties in Section 5.4.

5.3 Asymptotic stability

5.3.1 Nonsmooth Lur'e-Postnikov Lyapunov functions

Inspired by [26, 73] where Lur'e systems with continuous nonlinearities are considered, we characterize the stability of the origin for system (5.4) with a Lur'e-Postnikov Lyapunov function V given

10. We say that P is positive (negative) semidefinite, i.e., $P \geq 0\,(P \leq 0)$, if all its eigenvalues are real and non-negative (non-positive).

by

$$V(x) := U_1(x) + U_2(Cx)$$
$$= \frac{1}{2}x^\top P x + \sum_{i=1}^{p} \gamma_i \int_0^{C_i x} \psi_i(\sigma) d\sigma, \quad \forall x \in \mathbb{R}^n, \tag{5.11}$$

where P comes from Assumption 5.2. Function V is piecewise continuously differentiable, and thus locally Lipschitz, therefore there are points where its gradient is not defined. A standard tool to circumvent this is Clarke's generalized directional derivative, defined for each direction $f \in \mathbb{R}^n$ at each $x \in \mathbb{R}^n$ as [33, page 11]

$$V^\circ(x; f) := \max\{\langle v, f\rangle : v \in \partial V(x)\},$$

where $\partial V(x)$ denotes Clarke's generalized gradient of V at x given by

$$\partial V(x) := \{v \in \mathbb{R}^n | v \in x^\top P + (\mathbf{\Psi}(Cx))^\top \Gamma C\}. \tag{5.12}$$

However, the Lyapunov analysis of system (5.4) using Clarke's generalized directional derivative of V is often too conservative to establish asymptotic stability of the origin. Roughly speaking, for some $x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$ there may exist a selection $f_{\text{bad}} \in F(x)$ that is never viable for any solution to (5.4) and such that $V^\circ(x, f_{\text{bad}}) > 0$, thereby preventing to prove that the origin of the system is globally asymptotically stable, as illustrated in the next example.

Example 5.1. *Consider system* (5.4) *with* $n = 2$*,* $p = 1$ *(SISO case),*

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and where $\mathbf{\Psi}$ *is the Krasovskii regularization of* $\psi : \mathbb{R} \to [-\frac{1}{4}, 1]$*, defined as* $\psi(s) = 1$ *if* $s > 0$*,* $\psi(s) = -\frac{1}{4}$ *if* $s < 0$*, and* $\psi(s) = 0$ *if* $s = 0$*; hence* $\mathbf{\Psi}(0) = [-1/4, 1]$*. This function* ψ *satisfies Assumption 5.1 with* $\zeta_1 = +\infty$ *(namely* (5.3)*). Consider* V *as in* (5.11) *with* $P = I_2$ *and* $\Gamma = \gamma_1 = 1$*. The proposed selection of matrices* A*,* B*,* C*,* P *and* Γ *and the set-valued map* $\mathbf{\Psi}$ *are such that Assumptions 5.1 and 5.2 are satisfied. Function* (5.11) *in this case is given by* $V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \int_0^{x_1} \psi(\sigma) d\sigma$ *for any* $x \in \mathbb{R}^2$*. We have that* V *is positive definite and radially unbounded. Furthermore,* V *is not differentiable at* $\{0\} \times \mathbb{R}$*. By following [34, Ch. 4] as summarized in [43, Ch. 2.4.2], to analyze the stability of the origin for the considered system, we study at any* $x \in \mathbb{R}^2$ *the maximum of* $V^\circ(x; f)$ *over all allowable directions* $f \in F(x)$ *with* F *as in* (5.4)*. In this regard, consider* $x = (0, \frac{1}{2})$*,*

$$\max_{f \in F(0, \frac{1}{2})} V^\circ((0, \tfrac{1}{2}); f) = \max\left\{\langle v, f\rangle | v \in [-\tfrac{1}{4}, 1] \times \left\{\tfrac{1}{2}\right\}, f \in \left[-\tfrac{3}{2}, -\tfrac{1}{4}\right] \times \left\{-\tfrac{1}{2}\right\}\right\} = \tfrac{1}{8}. \tag{5.13}$$

With this positive upper bound, in view of [43, Def. 2.16], we cannot establish asymptotic stability of *the origin* [11] *[43, Thm. 2.18]. Nevertheless a direct inspection shows that V strictly decreases along all solutions outside the origin. The issue is overcome in the following by exploiting the notion of set-valued Lie derivative of V [136].* □

In [26], the authors overcame the limitations discussed in Example 5.1 by using *trajectory-based* Lyapunov arguments when Assumption 5.1 holds with $\zeta_i = +\infty$ for any $i \in \{1,\ldots,p\}$. In the next theorem, we establish global asymptotic stability of the origin for system (5.4). Compared to [26], the result relies on the more general sector condition in (5.2), and, importantly for the sequel, its proof uses *algebraic* Lyapunov arguments.

Theorem 5.1. *Consider system* (5.4) *and suppose that Assumptions 5.1 and 5.2 hold. Then the origin is GAS, i.e., there exists $\beta \in \mathcal{KL}$ such that all solutions x satisfy*

$$|x(t)| \le \beta(|x(0)|, t), \quad \forall t \in \mathbb{R}_{\ge 0}. \tag{5.14}$$

□

The proof of Theorem 5.1 is given in Section 5.3.2, where we use the concept of set-valued Lie derivative that we now recall.

5.3.2 Set-valued Lie derivative and its properties

The set-valued Lie derivative of V with respect to F in (5.4) at $x \in \mathbb{R}^n$ is defined as [12]

$$\dot{\overline{V}}_F(x) := \{a \in \mathbb{R} |\, \exists f \in F(x) : \langle v, f\rangle = a,\ \forall v \in \partial V(x)\}, \tag{5.15}$$

with $\partial V(x)$ given in (5.12). Note that $\dot{\overline{V}}_F(x)$ is a subset of $\{\langle v, f\rangle | v \in \partial V(x), f \in F(x)\}$ and that, by definition, at any x where V is continuously differentiable, so that $\partial V(x)$ is a singleton, this reduces to the set of all standard directional derivatives of V in any direction of $f \in F(x)$. Notice that $\dot{\overline{V}}_F(x)$ may be the empty set as illustrated later in Example 5.2. In the next lemma, a useful and intuitive upper bound of the set-valued Lie derivative of V in (5.11) along dynamics (5.4) is provided.

Lemma 5.1. *Given function V in* (5.11) *and F in* (5.4)*,*

$$\sup \dot{\overline{V}}_F(x) \le \sup_{u \in -\Psi(Cx)} \big((x^\top P - u^\top \Gamma C)(Ax + Bu)\big), \quad \forall x \in \mathbb{R}^n, \tag{5.16}$$

where we use the convention $\sup \emptyset = -\infty$ *fot the left-hand side when* $\sup \dot{\overline{V}}_F(x) = \emptyset$. □

Proof: For each $x \in \mathbb{R}^n$ and each element $f = Ax + Bu \in F(x)$ with $u \in -\Psi(Cx)$, as in (5.4), denote $\rho(x, f) := Px - C^\top \Gamma u$ and note that $\rho(x, f) \in \partial V(x)$. Notice that ρ and f are defined by selecting the same $u \in -\Psi(Cx)$. In view of Lemma 4.8, exploiting this selection we have that $\sup \dot{\overline{V}}_F(x) \le \sup_{u \in -\Psi(Cx)} (\rho(x, f)^\top f)$, thus concluding the proof. ■

11. The Ryan's invariance principle [111] is also not applicable to guarantee asymptotic stability of the origin for this example.

Exploiting (5.15) and Lemma 5.1, we can establish the next algebraic Lyapunov conditions for system (5.4).

Proposition 5.1. *Consider F in* (5.4) *and suppose that Assumptions 5.1 and 5.2 hold. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that function V in* (5.11) *satisfies*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n, \tag{5.17}$$

$$\sup \overline{\dot{V}}_F(x) \leq -\alpha_3(V(x)), \quad \forall x \in \mathbb{R}^n. \tag{5.18}$$

□

Proof: From (5.2) and (5.11), V is positive definite, continuous on $\mathbb{R}^n$ and radially unbounded. Therefore, (5.17) holds by [73, Lemma 4.3]. Let $x \in \mathbb{R}^n$, we have from Lemma 5.1 that

$$\begin{aligned}\sup \overline{\dot{V}}_F(x) \leq \sup_{u \in -\Psi(Cx)} \Big[& (x^\top P - u^\top \Gamma C)(Ax + Bu) \\ & - u^\top(-Zu - Cx) - \frac{\eta}{2}|x|^2 \\ & + u^\top(-Zu - Cx) + \frac{\eta}{2}|x|^2 \Big],\end{aligned} \tag{5.19}$$

with Z as in (5.6). Therefore,

$$\sup \overline{\dot{V}}_F(x) \leq \sup_{u \in -\Psi(Cx)} \left(\frac{1}{2} \begin{bmatrix} x \\ u \end{bmatrix}^\top M \begin{bmatrix} x \\ u \end{bmatrix} + u^\top(Zu + Cx) \right) - \frac{\eta}{2}|x|^2, \tag{5.20}$$

with M as in (5.9). In view of Assumption 5.1, it holds that $u^\top(Zu + Cx) \leq 0$ for all $u \in -\Psi(Cx)$ and any $x \in \mathbb{R}^n$, as $\Psi(Cx)$ is convex. Moreover, $M \leq 0$ in view of Assumption 5.2. Therefore, we have from (5.20)

$$\begin{aligned}\sup \overline{\dot{V}}_F(x) &\leq \sup_{u \in -\Psi(Cx)} (u^\top(Zu + Cx)) - \frac{\eta}{2}|x|^2 \\ &\leq -\frac{\eta}{2}|x|^2 \leq -\frac{\eta}{2}(\alpha_2^{-1}(V(x)))^2 =: -\alpha_3(V(x)),\end{aligned} \tag{5.21}$$

with $\alpha_3 \in \mathcal{K}_\infty$, which shows (5.18) and the proof is complete. ■

Based on Proposition 5.1 we provide an algebraic proof of Theorem 5.1.

Proof of Theorem 5.1. Let x be a solution to (5.4). In view of Proposition 4.4 and [43, Lemma 2.20], V is non-pathological, and thus [43, Lemma 2.23] ensures that $\frac{d}{dt}V(x(t)) \in \overline{\dot{V}}_F(x(t))$ for almost all $t \in \operatorname{dom} x$. Hence, in view of (5.18) in Proposition 5.1, we have that

$$\dot{V}(x(t)) \leq -\alpha_3(V(x(t))), \quad \text{for almost all } t \in \operatorname{dom} x. \tag{5.22}$$

By following the steps of the proof of [122, Lemma A.4], we have that $\operatorname{dom} x = \mathbb{R}_{\geq 0}$ and there exists

$\overline{\beta} \in \mathcal{KL}$ (independent of x) such that

$$V(x(t)) \leq \overline{\beta}(V(x(0)), t), \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{5.23}$$

Equations (5.17) and (5.23) imply $|x(t)| \leq \alpha_1^{-1}(V(x(t))) \leq \alpha_1^{-1}(\overline{\beta}(\alpha_2(|x(0)|), t)) =: \beta(|x(0)|, t)$ for any $t \in \mathbb{R}_{\geq 0}$, with $\beta \in \mathcal{KL}$, thus concluding the proof. ∎

With the help of Theorem 5.1, we can now establish that the origin of the system in Example 5.1 is GAS.

Example 5.2. *The system in Example 5.1 satisfies both Assumptions 5.1 and 5.2 with the given selections of Z, Γ and P. As a result $x = \mathbf{0}_2$ is GAS in view of Theorem 5.1. It is instructive to see how the notion of set-valued Lie derivative helps overcoming the issue highlighted in Example 5.1. In particular, the set-valued Lie derivative of V with respect to F at $x = (0, \frac{1}{2})$ is the empty set. Indeed, for each $f \in F(0, \frac{1}{2})$ and any two different directions $v_1, v_2 \in \partial V(0, \frac{1}{2})$ with $v_1 \neq v_2$, we have $\langle f, v_1 \rangle \neq \langle f, v_2 \rangle$, thus there exists no $a \in \mathbb{R}$ satisfying the condition in* (5.15). *More specifically, given $F(0, \frac{1}{2}) = \big[-\frac{3}{2}, -\frac{1}{4}\big] \times \big\{-\frac{1}{2}\big\}$ and $\partial V(0, \frac{1}{2}) = [-\frac{1}{4}, 1] \times \big\{\frac{1}{2}\big\}$, by selecting $v_1, v_2 \in \partial V(0, \frac{1}{2})$ with $v_1 \neq v_2$, and $f \in F(0, \frac{1}{2})$ we have that $\langle f, v_1 \rangle = f_1 v_{1,1} - \frac{1}{4}$ and $\langle f, v_2 \rangle = f_1 v_{2,1} - \frac{1}{4}$ with $f_1 \in \big[-\frac{3}{2}, -\frac{1}{4}\big]$ and $v_{1,1} \neq v_{2,1} \in [-\frac{1}{4}, 1]$. Therefore, $\langle f, v_1 \rangle = \langle f, v_2 \rangle$ if and only if $f_1(v_{1,1} - v_{2,1}) = 0$, which is impossible for the specified selection of f, v_1 and v_2. Hence, there exists no $a \in \mathbb{R}$ and $f \in F(0, \frac{1}{2})$ such that $\langle f, v \rangle = a$ for all $v \in \partial V(0, \frac{1}{2})$, thus implying that $\dot{\overline{V}}_F(0, \frac{1}{2}) = \emptyset$. Besides this specific illustrative analysis, by exploiting Lemma 5.1 we may actually show that $\sup \dot{\overline{V}}_F(x) < 0$ for all $x \in \mathbb{R}^2 \setminus \{\mathbf{0}_2\}$. Indeed, we have that $\sup \dot{\overline{V}}_F(x) \leq \sup_{u \in -\Psi(x_1)} (-x_1^2 + 2ux_1 - x_2^2 + ux_2 - u^2) = \sup_{u \in -\Psi(x_1)} \Big(-x_1^2 + 2ux_1 - \big(\frac{1}{2}x_2 - u\big)^2 - \frac{3}{4}x_2^2\Big) < 0$, because Assumption 5.1 implies $ux_1 < 0$. We, therefore, obtain that the supremum of the set-valued Lie derivative of V with respect to $F(x)$ is strictly negative outside the origin, which was not possible to prove using the conservative upper bound* (5.13). □

We explain in the next remark why Σ_2 in (5.7) is passive.

Remark 5.1. *System Σ_2 is passive from $\overline{y}$ to $-u$ as discussed at the end of Section 5.2. Let g as in* (5.7), *we have that, by Lemma 4.8*

$$\sup \dot{\overline{U}}_{2,g}(y) \leq \sup_{u \in -\Psi(y)} (u^\top(y + Zu) - u^\top \overline{y}), \quad \forall y \in \mathbb{R}^p. \tag{5.24}$$

Let y be a solution of (5.6) *with input $\overline{y}$ and output u. From [43, Lemma 2.23], $\frac{d}{dt}U_2(y(t)) \leq \sup \dot{\overline{U}}_{2,g}(y)$. Then, since $u^\top(Zu + y) \geq 0$ by Assumption 5.1 and* (5.24),

$$\frac{d}{dt}U_2(y(t)) \leq \sup_{u \in -\Psi(y(t))} -u^\top \overline{y}(t), \quad \textit{for almost all } t \in \operatorname{dom} y. \tag{5.25}$$

Hence, we obtain

$$U_2(y(t)) \leq U_2(y(0)) + \int_0^t -u(s)^\top y(s)\, ds, \qquad \forall t \in \operatorname{dom} y \subseteq \mathbb{R}_{\geq 0},$$

by integrating (5.25). *Thus* Σ_2 *is passive from* $\overline{y}$ *to* $-u$ *as per Definitions 2.1 and 2.2 in [119]. Notice that, even if Definitions 2.1 and 2.2 are stated for single-valued outputs, we can apply these same definitions without loss of generality to our case where* $u \in -\boldsymbol{\Psi}(y)$. □

We conclude this section with a discussion about how the conditions of Theorem 5.1 can be extended to.

5.3.3 Extension under special properties of plant (5.1)

Since the conditions in Theorem 5.1 are only sufficient, we may prove that the origin is GAS for system (5.4) via Lyapunov analysis by exploiting additional structural properties of matrices A, B and C in (5.4). A set of alternative exploitable properties for system (5.4) is given next.

Property 5.1. *The following holds for system* (5.4).

(i) *Assumption 5.1 is satisfied.*

(ii) *There exist matrices* $\Gamma > 0$ *diagonal,* $P = P^\top > 0$ *and a scalar* $\eta > 0$ *such that*

$$\overline{M} := \begin{bmatrix} PA + A^\top P + \eta I_n & PB \\ B^\top P & -2Z - \Gamma CB - (\Gamma CB)^\top \end{bmatrix} \leq 0. \tag{5.26}$$

(iii) *There exist* $H := \operatorname{diag}(h_1, \ldots, h_p)$ *such that* $\Gamma CA = HC$ *and, for all* $i \in \{1, \ldots, p\}$, *either* $h_i \leq -1$ *holds, or* $h_i \leq 0$ *and* $Z = O_p$ *holds, with* Z *in* (5.6). □

The conditions in items (ii) and (iii) in Property 5.1 impose extra properties of the matrices C and A (item (ii)) and a different matrix inequality compared to (5.9) (item (iii)), indeed as the off-diagonal terms of $\overline{M}$ differ from those in M in (5.9). We show in the next lemma that Property 5.1 implies GAS of the origin for system (5.4). We will invoke this extension in Section 5.5 to analyze the stability properties of the neural networks studied in [55].

Lemma 5.2. *Suppose that system* (5.4) *satisfies items (i)-(iii) of Property 5.1. Then the origin is GAS for system* (5.4). □

Proof: Let $x \in \mathbb{R}^n$ and consider V in (5.11). We have from Lemma 5.1 that, for any $x \in \mathbb{R}^n$,

$$\sup \dot{\overline{V}}_F(x) \leq \sup_{u \in -\boldsymbol{\Psi}(Cx)} \Big[(x^\top P - u^\top \Gamma C)(Ax + Bu) - u^\top Z u - \frac{\eta}{2}|x|^2 + u^\top Z u + \frac{\eta}{2}|x|^2 \Big]. \tag{5.27}$$

with Z as in (5.6). Therefore,

$$\sup \dot{\overline{V}}_F(x) \leq \sup_{u \in -\boldsymbol{\Psi}(Cx)} \left(\frac{1}{2} \begin{bmatrix} x \\ u \end{bmatrix}^\top \overline{M} \begin{bmatrix} x \\ u \end{bmatrix} + u^\top (Zu - HCx) \right) - \frac{\eta}{2}|x|^2. \tag{5.28}$$

We note that, in view of Assumption 5.1, it holds that $u^\top (Zu - HCx) \leq 0$ for all $u \in -\boldsymbol{\Psi}(Cx)$ and any $x \in \mathbb{R}$. Indeed, because each entry of $\boldsymbol{\Psi}(y) = \boldsymbol{\Psi}(Cx)$ is convex for any $x \in \mathbb{R}^n$, by (5.2) it holds that $\alpha_i u_i y_i \leq -\frac{\alpha_i}{\zeta_i} u_i^2$, and $u_i y_i + \frac{1}{\zeta_i} u_i^2 + \alpha_i u_i y_i - \alpha_i u_i y_i \leq 0$, for any $\alpha_i \geq 0$, $u \in -\boldsymbol{\Psi}(Cx)$ and

$i \in \{1,\dots,p\}$. Therefore, we have $\frac{1}{\zeta_i}u_i^2 + (1+\alpha_i)u_i y_i \le \alpha_i u_i y_i \le -\frac{\alpha_i}{\zeta_i}u_i^2 \le 0$. When $h_i \le -1$, taking $\alpha_i = -1 - h_i \ge 0$, we deduce that, for any $u \in -\boldsymbol{\Psi}(Cx)$ and $i \in \{1,\dots,p\}$, $\frac{1}{\zeta_i}u_i^2 - h_i u_i y_i \le 0$ and thus $u^\top(Zu - HCx) \le 0$ for all $u \in -\boldsymbol{\Psi}(Cx)$. In the particular case where $Z = O_p$, $-u^\top HCx \le 0$ is true for any negative semidefinite matrix diagonal H by (5.3), for all $u \in -\boldsymbol{\Psi}(Cx)$ and $i \in \{1,\dots,p\}$. Moreover, we assumed $\overline{M} \le 0$ in item (ii) of Lemma 5.2. Therefore, similar to (5.21), from (5.28) we have

$$\sup \dot{\overline{V}}_F(x) \le -\frac{\eta}{2}|x|^2 \le -\frac{\eta}{2}(\alpha_2^{-1}(V(x)))^2 =: -\overline{\alpha}_3(V(x)), \tag{5.29}$$

with $\overline{\alpha}_3 \in \mathcal{K}_\infty$. Then, as anticipated, by exploiting (5.17) and (5.29), and following similar steps of those in the proof of Theorem 5.1, we conclude that the origin is GAS for system (5.4). ■

5.4 Finite-time stability

5.4.1 Definitions and assumptions

In this section, we provide conditions to guarantee output and state finite-time stability properties for system (5.4). In particular, we consider the next stability notions, see [123, 145].

Definition 5.1. *Consider system* (5.4). *If its solutions are all forward complete* [12], *then we say that the system is:*

(i) output globally asymptotically stable *(oGAS) if there exists $\beta \in \mathcal{KL}$ such that for any solution x*

$$|y(t)| \le \beta(|x(0)|, t), \qquad \forall t \in \mathbb{R}_{\ge 0};$$

(ii) state-independent output locally asymptotically stable *(SIoLAS) if there exist $r > 0$ and $\beta \in \mathcal{KL}$ such that for all solution x,*

$$|x(0)| < r \Rightarrow |y(t)| \le \beta(|y(0)|, t), \qquad \forall t \in \mathbb{R}_{\ge 0};$$

(iii) output finite-time stable *(OFTS) if it is oGAS and for each solution x there exists $0 \le T < +\infty$ such that $y(t) = \mathbf{0}_p$ for all $t \ge T$;*

(iv) state finite-time stable *(SFTS) if the origin is GAS and for each solution x there exists $0 \le T < +\infty$ such that $x(t) = \mathbf{0}_n$ for all $t \ge T$.* □

To be able to prove the output stability properties in Definition 5.1, we make the next assumption.

Assumption 5.3. *The following holds.*

(i) Matrix CB is Lyapunov diagonally stable *(LDS) [68, Def. 5.3], i.e., there exists a diagonal matrix $\overline{\Gamma} > 0$ of appropriate dimensions such that $\overline{\Gamma}CB + (CB)^\top\overline{\Gamma} > 0$.*

(ii) The origin is GAS for system (5.4).

12. A solution is forward complete if its domain is unbounded [8].

(iii) Each ψ_i, with $i \in \{1,\dots,p\}$, is discontinuous at the origin and both its left and right limits are non-zero, i.e., for any $i \in \{1,\dots,p\}$ $\lim_{s\to 0^+} \psi_i(s) > 0$ and $\lim_{s\to 0^-} \psi_i(s) < 0$. □

Item (i) of Assumption 5.3 imposes extra conditions on the matrices C and B of system (5.1). Sufficient conditions to ensure item (ii) of Assumption 5.3 are provided in Theorem 5.1 and Lemma 5.2. Finally, item (iii) of Assumption 5.3 requires each $\psi_i, i \in \{1,\dots,p\}$, to be non-zero at the origin and to have non-zero left and right limit at zero as well. Examples of engineering systems satisfying Assumption 5.3 (as well as Assumptions 5.1 and 5.2) are provided in Section 5.5.

5.4.2 Output and state finite-time stability

We are now ready to present the main result of this section, whose proof is given in Section 5.4.3.

Theorem 5.2. *Consider system* (5.4) *and suppose that Assumptions 5.1 and 5.3 hold, then system* (5.4) *is OFTS and SIoLAS.* □

Theorem 5.2 establishes output finite-time stability properties for system (5.4). A natural question is then whether *state* finite-time stability properties can also be guaranteed. An answer to this question is given in the next theorem which establishes that, whenever Assumptions 5.1 and 5.3 are satisfied, system (5.4) is SFTS if and only if C is invertible.

Theorem 5.3. *Consider system* (5.4) *and suppose that Assumptions 5.1 and 5.3 are verified. Then the system is SFTS if and only if matrix C is inveritble.* □

Proof: We start by proving that there exists $\varepsilon > 0$ such that, for any $\xi \in \ker(C) \cap \varepsilon\mathbb{B}_n$, $u = -(CB)^{-1}CA\xi$ belongs to $\mathbf{\Psi}(\mathbf{0}_p)$ and $CA\xi + CBu = \mathbf{0}_p$. First, note that CB is invertible as it is LDS by item (i) of Assumption 5.3. Hence, for any $\xi \in \ker(C) \cap \varepsilon\mathbb{B}_n$, $u = -(CB)^{-1}CA\xi$ is well-defined and $CA\xi + CBu = \mathbf{0}_p$. Secondly, in view of item (iii) of Assumption 5.3 there exists $\psi_\circ \in \mathbb{R}_{>0}$ such that $[-\psi_\circ, \psi_\circ]^p \subseteq \mathbf{\Psi}(\mathbf{0}_p)$. Therefore, there exists $\varepsilon > 0$ such that, for any $\xi \in \ker(C) \cap \varepsilon\mathbb{B}_n$ and any $i \in \{1,\dots,p\}$, $|((CB)^{-1}CA)_i\xi| \le \psi_\circ$, thus implying $u = -(CB)^{-1}CA\xi \in [-\psi_o, \psi_o]^p \subseteq \mathbf{\Psi}(\mathbf{0}_p)$, as to be proven.

Now we are ready to prove the necessary and sufficient conditions of Theorem 5.3. The sufficient condition in Theorem 5.3 is a direct consequence of Theorem 5.2. We proceed by contradiction to prove the necessary condition in Theorem 5.3. We thus assume that C is not invertible and consider $\varepsilon > 0$ as at the beginning of this proof. Since for any $x \in \ker(C) \cap \varepsilon\mathbb{B}_n$ we can select $u = -(CB)^{-1}CAx$ that belongs to $\mathbf{\Psi}(\mathbf{0}_p)$, we consider below solutions to (5.4) satisfying

$$\dot{x} = Ax - B(CB)^{-1}CAx, \quad x \in \ker(C) \cap \varepsilon\mathbb{B}_n, \tag{5.30}$$

which implies

$$\dot{y} = C\dot{x} = (CA - CB(CB)^{-1}CA)x = \mathbf{0}_p, \quad x \in \ker(C) \cap \varepsilon\mathbb{B}_n. \tag{5.31}$$

We now exploit (5.31) to attain a contradiction. By item (ii) of Assumption 5.3, there exists $\delta > 0$ such that any solution starting in $\delta\mathbb{B}_n$ does not leave $\varepsilon\mathbb{B}_n$ for all times. Let x_p be a nonzero solution starting in $\ker(C) \cap \delta\mathbb{B}_n$, with output $y_p = Cx_p$, which evolves according to (5.30) and (5.31). Then $y_p(0) = Cx_p(0) = \mathbf{0}_p$ and equation (5.31) imply $y_p(t) = Cx_p(t) = \mathbf{0}_p$ and $\dot{x}_p(t) = (A - B(CB)^{-1}CA)x_p(t) \neq 0$ for all $t \geq 0$. As a consequence, x_p exponentially converges to the origin but does not converge in finite-time. Such a solution establishes a contradiction, thus completing the proof. ■

We can now analyze the finite-time stability property of the system in Example 5.1 in light of Theorems 5.2 and 5.3.

Example 5.3. *Consider the system in Example 5.1. Assumption 5.3 holds with* $\overline{\Gamma} = 1$. *As a result, the system is OFTS and SIoLAS. We also know from Theorem 5.2 that the system is not SFTS as C is not invertible. Another way to see it is to consider* $x(0) \in X := \{0\} \times [-\frac{1}{4}, \frac{1}{4}]$. *A possible solution to* (5.4) *is* $x_p(t) = (0, x_2(0)e^{-t})$*, which belongs to the set X for all* $t \geq 0$. *Moreover, we have that* $y_p(t) = 0$ *and* $\dot{y}_p(t) = 0$ *for all* $t \geq 0$. *Clearly,* x_p *converges exponentially to the origin, but not in finite-time.* □

5.4.3 Proof of Theorem 5.2

The proof of Theorem 5.2 relies on the next lemma and proposition.

Lemma 5.3. *Under Assumption 5.1 and item (iii) of Assumption 5.3, there exist* $\nu > 0$ *and* $c > 0$*, such that*

$$|u| \geq c, \quad \forall u \in -\mathbf{\Psi}(y), \quad \forall y \in \nu\mathbb{B}_p \setminus \{\mathbf{0}_p\}. \tag{5.32}$$

□

Proof: In view of item (iii) of Assumption 5.3, there exist positive parameters $\nu_\circ$ and c such that, for each $i \in \{1, \dots, p\}$, ψ_i is continuous in the intervals $[-\nu_\circ, 0)$ and $(0, \nu_\circ]$, and $\min(|\lim_{s\to 0^+} \psi_i(s)|, |\lim_{s\to 0^-} \psi_i(s)|) \geq 2c$. Hence, there exists $\nu \in (0, \nu_\circ]$ such that, for any $i \in \{1, \dots, p\}$ and $s \in [-\nu, 0) \cup (0, \nu]$, $|\psi_i(s)| \geq c$. Therefore, we have that for any $y \in \nu\mathbb{B}_p \setminus \{\mathbf{0}_p\}$ there exists $i \in \{1, \dots, p\}$ such that $|u| \geq |u_i| \geq c$ for all $u \in -\mathbf{\Psi}(y)$ thus concluding the proof. ■

We also invoke the next proposition, which states algebraic properties of a piecewise continuously differentiable function, which is similar to the one in (5.10)

$$W(Cx) := 2\sum_{i=1}^{p} \overline{\gamma}_i \int_0^{C_i x} \psi_i(\sigma) d\sigma, \quad \forall x \in \mathbb{R}^n, \tag{5.33}$$

where $\overline{\gamma}_1, \dots, \overline{\gamma}_p > 0$ are positve parameters selected such that $\overline{\Gamma}CB + (CB)^\top \overline{\Gamma} > 0$, with $\overline{\Gamma} = \mathrm{diag}(\overline{\gamma}_1, \dots, \overline{\gamma}_p)$, which exist by item (i) of Assumption 5.3. Function W enjoys the following properties.

Proposition 5.2. *Suppose that Assumption 5.1 and items (i) and (iii) of Assumption 5.3 hold. Given* *function W in* (5.33)*, there exist $\mu \in (0, \nu]$, with ν as in Lemma 5.3, and $\alpha_4, \alpha_5 \in \mathcal{K}_\infty$ such that*

$$\alpha_4(|Cx|) \leq W(Cx) \leq \alpha_5(|Cx|), \qquad \forall x \in \mu\mathbb{B}_n, \tag{5.34}$$

$$\sup \dot{\overline{W}}_F(Cx) \leq -c\omega, \qquad \forall x \in \mu\mathbb{B}_n \setminus \ker(C), \tag{5.35}$$

with c as in Lemma 5.3, $\omega := \lambda_1(c - 2\mu\frac{\lambda_2}{\lambda_1}) > 0$, λ_1 is the smallest eigenvalue of $\overline{\Gamma}CB + (CB)^\top\overline{\Gamma}$, and $\lambda_2 := |\overline{\Gamma}CA|$. □

Proof: From (5.33) and Lemma 5.3, for any $x \in \mu\mathbb{B}_n \setminus \ker(C)$, $W(Cx) > 0$ while $W(Cx) = 0$ for any $x \in \ker(C) \cap \mu\mathbb{B}_n$. Moreover, we have that W is continuous on $\mu\mathbb{B}_n$. Therefore, (5.34) holds in view of [73, Lemma 4.3]. Let $x \in \mu\mathbb{B}_n \setminus \ker(C)$, from Lemma 5.1, by imposing $P = 0$ and $\Gamma = \overline{\Gamma}$ in (5.16), we have

$$\sup \dot{\overline{W}}_F(Cx) \leq \sup_{u \in -\Psi(Cx)} (-2u^\top\overline{\Gamma}C(Ax + Bu)). \tag{5.36}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\sup \dot{\overline{W}}_F(Cx) \leq \sup_{u \in -\Psi(Cx)} (&-u^\top(\overline{\Gamma}CB + (CB)^\top\overline{\Gamma})u \\ &+ 2|\overline{\Gamma}CA||x||u|).\end{aligned} \tag{5.37}$$

Thus, in view of item (i) of Assumption 5.3, we have that

$$\begin{aligned}\sup \dot{\overline{W}}_F(x) &\leq \sup_{u \in -\Psi(Cx)} (-\lambda_1|u|^2 + 2|\overline{\Gamma}CA||x||u|), \\ &= \sup_{u \in -\Psi(Cx)} (-(\lambda_1|u| - 2|\overline{\Gamma}CA||x|)|u|), \\ &\leq \sup_{u \in -\Psi(Cx)} \Big(-\lambda_1\big(|u| - 2\frac{\lambda_2}{\lambda_1}|x|\big)|u|\Big).\end{aligned} \tag{5.38}$$

Hence, in view of Lemma 5.3, by selecting $\mu \in (0, \nu]$ we have that $\sup \dot{\overline{W}}_F(Cx) \leq \sup_{u \in -\Psi(Cx)} (-\omega|u|) \leq -c\omega$, where $\omega = \lambda_1\big(c - 2\frac{\lambda_2\mu}{\lambda_1}\big) > 0$, thus concluding the proof. ■

We are now ready to prove Theorem 5.2. To prove the OFTS property of system (5.4), we proceed by steps. We first show that, for solutions to (5.4) initialized in a neighborhood of the origin, the corresponding output converges to the origin in finite-time and then, leveraging the GAS property of the origin for (5.4), we prove OFTS of (5.4).

Proof of Theorem 5.2. We start by proving that solutions initialized sufficiently close to the origin converge to $\ker(C)$ in finite time by integrating (5.35). To do so, we recall that, by the GAS property of the origin, there exists $\kappa > 0$ such that solutions starting in $\kappa\mathbb{B}$ will not leave $\mu\mathbb{B}$, with μ as in Proposition 5.2 and we note that the set $\mu\mathbb{B}_n \cap \ker(C)$ is forward invariant for any solution starting

$\kappa\mathbb{B}_n \cap \ker(C)$. Indeed, suppose that there exists a solution x_{bad} to (5.4) such that $x_{\text{bad}}(0) \in \kappa\mathbb{B}_n \cap \ker(C)$ and $x_{\text{bad}}(t^*) \notin \mu\mathbb{B}_n \cap \ker(C)$ for some $t^* > 0$ with $t^* \in \operatorname{dom} x_{\text{bad}}$. Since x_{bad} is continuous with respect to the time, we can choose $t^* > 0$ such that $x_{\text{bad}}(t) \in \kappa\mathbb{B}_n \cap \ker(C)$ for all $t \in [0, t^*)$ and $x_{\text{bad}}(t^*) \in \mu\mathbb{B}_n \setminus \ker(C)$. Hence, from (5.33) and (5.35), and from the fact that W is positive definite on $\mu\mathbb{B}_n$ and non-pathological, we have $0 = W(Cx_{\text{bad}}(t)) < W(Cx_{\text{bad}}(t^*))$, for all $t \in [0, t^*)$, which establishes a contradiction by the continuity property of W. Consequently, solutions cannot leave $\mu\mathbb{B}_n \cap \ker(C)$ after reaching the set $\kappa\mathbb{B}_n \cap \ker(C)$. Therefore, by combining (5.35) with the fact that W is non-pathological, and the forward invariance of $\mu\mathbb{B} \cap \ker(C)$ for solutions starting in $\kappa\mathbb{B} \cap \ker(C)$, for any solution x initialized so that $x(0) \in \kappa\mathbb{B}_n \setminus \ker(C)$, we obtain by integration for any $t \in \operatorname{dom} x$ such that $x(t) \in \mu\mathbb{B}_n \setminus \ker(C)$

$$W(Cx(t)) \leq -c\omega t + W(Cx(0)), \tag{5.39}$$

and thus

$$W(Cx(t)) \leq \max(-c\omega t + W(Cx(0)), 0), \forall x(0) \in \kappa\mathbb{B}_n, \forall t \in \mathbb{R}_{\geq 0}. \tag{5.40}$$

Thus, in view of (5.40) and by the GAS property of the origin, we conclude that, for any solutions starting in $\kappa\mathbb{B}$, there exists a T_y, depending on κ, such that $x(t) \in \kappa\mathbb{B} \cap \ker(C)$ for any $t \geq T_y$. We now leverage the GAS property of the origin to prove that (5.4) is OFTS. We recall that, for any solution x to (5.4), by the GAS property of the origin there exists a time $T_\kappa \geq 0$ such that $x(t) \in \kappa\mathbb{B}$ for all $t \geq T_\kappa$. Therefore, we conclude that $y(t) = \mathbf{0}_p$ for all $t \geq T := T_\kappa + T_y$. We have proved that, for any solution x, there exists $T \geq 0$ such that $y(t) = \mathbf{0}_p$, for all $t \geq T$. Moreover, system (5.4) is oGAS because it is GAS from item (i) of Assumption 5.3 and because $|y| \leq |C||x|$. Therefore, system (5.4) is OFTS.

Finally, we prove that system (5.4) is also SIoLAS. Indeed, combining (5.34) and (5.40) yields, for any solution x with $x(0) \in \kappa\mathbb{B}$,

$$\begin{aligned} |y(t)| &\leq \alpha_4^{-1}(\max(-c\omega t + \alpha_5(Cx(0)), 0)) \\ &=: \beta_\circ(|y(0)|, t), \quad \forall x(0) \in \kappa\mathbb{B}_n, \quad \forall t \in \mathbb{R}_{\geq 0}, \end{aligned} \tag{5.41}$$

with $\beta_\circ \in \mathscr{KL}$, thus ending the proof. ■

5.5 Applications

In this section, we present two applications of the results of Sections 5.3 and 5.4.

5.5.1 Mechanical system affected by friction [26]

Consider the rotor dynamic system with friction system given in [26, Sec. 5], i.e.,

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\omega}_u \\ \dot{\omega}_\ell \end{bmatrix} \in \begin{bmatrix} \omega_u - \omega_\ell \\ -\frac{k_\theta}{J_u}\alpha - \frac{b}{J_u}(\omega_u - \omega_\ell) - \frac{1}{J_u}T_{fu}(\omega_u) + \frac{k_u}{J_u}v \\ \frac{k_\theta}{J_\ell}\alpha + \frac{b}{J_\ell}(\omega_u - \omega_\ell) - \frac{1}{J_\ell}T_{f\ell}(\omega_\ell) \end{bmatrix}, \tag{5.42}$$

with $x = (\alpha, \omega_u, \omega_\ell) \in \mathbb{R}^3$, where α is the angular mismatch between two rotating discs connected by an angular spring and an angular dumper, and ω_u and ω_ℓ are the angular velocities of these two discs. Scalars J_u, J_ℓ, k_u, k_θ and b are positive system parameters whose values are reported in Table 5.5.1. The control input $v \in \mathbb{R}$ is used for state-feedback stabilization, while the set-valued maps T_{fu} and $T_{f\ell}$ in (5.42) are defined as

$$\begin{aligned} T_{fu}(s) &:= \begin{cases} f_u(s)\mathrm{sign}(s), & \forall s \in \mathbb{R} \setminus \{0\} \\ [-f_{u,\circ} + \Delta f_u, f_{u,\circ} + \Delta f_u], & \text{otherwise,} \end{cases} \\ f_u(s) &:= f_{u,\circ} + \Delta f_u \mathrm{sign}(s) + q_1|s| + q_2 s, \quad \forall s \in \mathbb{R}, \\ T_{f\ell}(s) &:= \begin{cases} f_\ell(s)\mathrm{sign}(s), & \forall s \in \mathbb{R} \setminus \{0\} \\ [-f_{\ell,\circ}, f_{\ell,\circ}], & \text{otherwise,} \end{cases} \\ f_\ell(s) &:= f_{\ell,\circ} + (\Delta f_\ell - f_{\ell,\circ})e^{-q_3|s|} + q_4|s|, \quad \forall s \in \mathbb{R}, \end{aligned}$$

for suitable positive scalars $f_{u,\circ}$, $f_{\ell,\circ}$, Δf_u, Δf_ℓ, q_1, q_2, q_3 and q_4 we give in Table 5.5.1 and with function sign : $\mathbb{R} \to [-1, 1]$ defined as $\mathrm{sign}(s) = 1$ if $s > 0$, $\mathrm{sign}(s) = -1$ if $s < 0$, and $\mathrm{sign}(s) = 0$ if $s = 0$, and for which we have that (5.2) is satisfied with $\zeta_1 = \zeta_2 = \infty$.

Like in [50, Ch. 6], by considering the selection $v = v_p + v_{\text{lin}}$ in (5.42), where $v_{p_1} := Kx$, $K = [k_1, k_2, k_3] \in \mathbb{R}^{3\times 1}$ and $v_{\text{lin}} := \frac{1}{k_u}T_{fu}(\omega_u)$, we obtain

$$\begin{bmatrix} \alpha \\ \omega_u \\ \omega_\ell \end{bmatrix} \in \begin{bmatrix} \omega_u - \omega_\ell \\ -\frac{k_\theta}{J_u}\alpha - \frac{b}{J_u}(\omega_u - \omega_\ell) + \frac{k_u}{J_u}(k_1\alpha + k_2\omega_u + k_3\omega_\ell) \\ \frac{k_\theta}{J_\ell}\alpha + \frac{b}{J_\ell}(\omega_u - \omega_\ell) - \frac{1}{J_\ell}T_{f\ell}(\omega_l) \end{bmatrix}, \tag{5.43}$$

which can be written in the Lur'e form (5.4), with $n = 3$ and $p = 1$, and $A = A_{\text{free}} + H_1K + H_2$,

$$A_{\text{free}} = \begin{bmatrix} 0 & 1 & -1 \\ -\frac{k_\theta}{J_u} & -\frac{b}{J_u} & \frac{b}{J_u} \\ \frac{k_\theta}{J_\ell} & \frac{b}{J_\ell} & -\frac{b}{J_\ell} \end{bmatrix}, \; H_1K = \begin{bmatrix} 0 & 0 & 0 \\ \frac{k_u k_1}{J_u} & \frac{k_u k_2}{J_u} & \frac{k_u k_3}{J_u} \\ 0 & 0 & 0 \end{bmatrix},$$

$H_2 = \mathrm{diag}(0, 0, \frac{m}{J_u})$ and $m \in \mathbb{R}$, $B = (0, 0, \frac{1}{J_\ell})$, $C = [0, 0, 1]$ and $\mathbf{\Psi}(Cx) = \mathbf{\Psi}(\omega_\ell) = T_{f\ell}(\omega_\ell) + m\omega_\ell$.

Assumption 5.2 is satisfied with the selection $m = 0.052$, $\Gamma = \gamma_1 = 10$, $\eta = 8.492$, $K =$

b	$[Nm^2/\text{rad } s]$	0
$f_{u,\circ}$	[N m]	0.38
Δf_u	[N m]	-0.006
$f_{\ell,\circ}$	[N m]	0.0009
Δf_ℓ	[N m]	0.68
J_u	[kg m^2]	0.4765
J_ℓ	[kg m^2]	0.035
k_u	[N m/V]	4.3228
k_θ	[N m/rad]	0.075
q_1	[kg m^2/rad s]	2.4245
q_2	[kg m^2/rad s]	-0.0084
q_3	[s/rad]	0.05
q_4	[kg m^2/rad s]	0.26

TABLE 5.1 – Parameters identifying the system given in [26, Sec. 5].

$[-12.8282, 3.7216, -8.4816]$ and

$$P = \begin{bmatrix} 0.5636 & 0.0340 & 0.3793 \\ 0.0340 & 0.0062 & 0.0186 \\ 0.3793 & 0.0186 & 0.2642 \end{bmatrix}.$$

Since Assumption 5.1 also satisfied, Theorem 5.1 implies that the origin is GAS for (5.4), thus retrieving the result originally presented in [26]. In addition, because Assumption 5.3 holds for the considered system, we establish here, from Theorem 5.2, that system (5.4) is OFTS and SIoLAS, which is a novelty compared to [26].

5.5.2 Cellular neural networks from [55]

In [55], cellular neural networks are modeled by system (5.1) (see [55, eq. (N1)-(N2)]), where the system data satisfies the next property according to [55, Prop. 3 and 4].

Property 5.2. *The following holds for system* (5.1).

(i) *A is a diagonal, negative definite matrix.*

(ii) *B is LDS (as per Assumption 5.3).*

(iii) $C = I_n$.

(iv) *For any $i \in \{1, \dots, n\}$, function ψ_i is nondecreasing, i.e., for any $a > b \in \text{dom}\,\psi_i$ it holds that $\psi_i(a) \geq \psi_i(b)$, is piecewise continuous and satisfies Assumption 5.1 with $\zeta_i = +\infty$ and item (iii) of Assumption 5.3.* □

Property 5.2 trivially implies Assumption 5.1 and items (i) and (iii) of Assumption 5.3. We show below that it also implies item (ii) of Assumption 5.3 so that we can invoke Theorems 5.1 and 5.2 to prove GAS of the origin for system (5.4) and that system (5.4) is SFTS, thus providing alternative proofs of the stability results given in [55, Thm. 3 and 4]. Indeed, we recall that, by proving stability properties for system (5.4), we ensure the same stability properties for the Krasovskii solutions of (5.1).

Lemma 5.4. *Suppose that system* (5.1) *satisfies Property 5.2. Then the origin is GAS for system* (5.4)*, and system* (5.4) *is SFTS.* □

Proof: We prove below that there exist matrices $\Gamma > 0$ diagonal, $P = P^\top > 0$ and a scalar $\eta > 0$ satisfying (5.26). Since B is LDS, there exists a $\Gamma > 0$ diagonal such that $\Gamma B + (\Gamma B)^\top =: \Sigma > 0$ and such that $\Gamma A \leq -I_n$. With this selection, we can rewrite matrix $\overline{M}$ in (5.26) as,

$$\overline{M} = \begin{bmatrix} PA + A^\top P & PB \\ B^\top P & -\Sigma \end{bmatrix} + \mathrm{diag}(\eta I_n, \mathbf{0}_n), \tag{5.44}$$

noting that Z is the null matrix due to item (iv) of Property 5.2. Define

$$\widetilde{M} := \begin{bmatrix} PA + A^\top P & PB \\ B^\top P & -\Sigma \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I_n \end{bmatrix} \underbrace{\begin{bmatrix} SA^\top + AS & B \\ B^\top & -\Sigma \end{bmatrix}}_{=:N} \begin{bmatrix} P & 0 \\ 0 & I_n \end{bmatrix},$$

where $S = P^{-1}$. Since A is Hurwitz by item (i) of Property 5.2, there exists $S_\circ = S_\circ^\top > 0$ such that $S_\circ A^\top + AS_\circ = \Pi < 0$. Therefore, by selecting $S = \alpha S_\circ$ with $\alpha > 0$, to be chosen, we have that

$$N = \begin{bmatrix} \alpha\Pi & B \\ B^\top & -\Sigma \end{bmatrix} < 0 \quad \forall \alpha > \alpha^\star, \tag{5.45}$$

where $\alpha^\star > 0$ satisfies $-\alpha^\star \lambda_\Pi > |B\Sigma B^\top|$, with $\lambda_\Pi > 0$ denoting the smallest eigenvalue of Π. Hence, with the given selection of α and P, matrix N and thus $\widetilde{M}$ are negative definite. Therefore, by selecting $0 < \eta < -|\widetilde{M}|$ we have that $\overline{M} \leq 0$ thus proving (5.26) and item (ii) of Property 5.1. Consider now item (iii) of Property 5.1 and note that matrix $H = \Gamma A < 0$ is diagonal negative definite and satisfies $\Gamma CA = \Gamma A = H = HC$. Since $Z = O_p$ due to item (iv) of Property 5.2, then item (iii) of Property 5.1 holds and we can invoke Lemma 5.2 to certify that the origin is GAS for system (5.4) and render Property 5.2. Furthermore, since Assumptions 5.1 and 5.3 hold, then, by Theorem 5.3, system (5.4) is also SFTS because C is invertible. ■

We envision applying our results to a broader class of neural networks with piecewise continuous activation functions, however, we do not pursue such generalizations here and we regard them as future work.

5.6 Conclusion

We have analyzed the stability of the origin for Lur'e systems with piecewise continuous nonlinearities. We have first established the global asymptotic stability of the origin under a milder sector condition compared to [26] and by relying on a different, algebraic Lyapunov proof based on the concept of set-valued Lie derivative. We have then presented conditions under which finite-time stability properties can or cannot be established for the considered class of systems. These results have been applied to two engineering systems of interest: mechanical systems with friction and cellular neural networks.

Future research directions may include: systems affected by exogenous disturbances; weak stability analysis for the considered class of systems in the sense that only some solutions exhibit the desired stability properties; as well as the synchronization of interconnected Lur'e systems with piecewise continuous nonlinearities following the path of paved by [23, 129]. For detailed discussions on possible additional scientific directions refer to Chapter 6.

Chapter 6

Conclusions

6.1 Summary

After providing a general introduction and the mathematical preliminaries in Chapters 1 and 2, we investigated two case studies of networked hybrid systems. In Chapter 3 we presented a hybrid dynamical model of opinion dynamics with memory-based connectivity, while Chapter 4 was devoted to studying the synchronization property of a network of heterogeneous 1-DoF oscillators with time-varying natural frequencies interconnected through a tree-like graph via hybrid couplings. Furthermore, we exploited some of the tools developed in Chapter 4 to prove stability to analyze the finite-time stability properties of Lur'e systems with piecewise continuous nonlinearities in Chapter 5. Below is a more detailed description of the contributions of Chapters 3-5.

— In **Chapter 3**, we investigated social networks where each individual locally interacts with other individuals depending on their respective opinions mismatchs. We proposed for this purpose a novel hybrid model of opinion dynamics with memory-based connectivity. The underlying rationale is that, when the individuals know their respective identity, interactions among them not only depend on their current opinion but also on their past ones. For this purpose, we designed the (de)activation of each link in the network to not only depend on memoryless adaptive thresholds as proposed in [58] but it is also ruled according to the values of memory variables. We have then been able to guarantee the asymptotic convergence of the agents' opinion to clusters arising from the continuous consensus dynamics (Corollary 3.1) by proving a $\mathscr{KL}$ stability property with respect to the indicator function ω in (3.13) (Theorem 3.1). Furthermore, we illustrate these results numerically, while also showing the influence of the parameters defined in the model over the characteristics of the final clusters.

— In **Chapter 4**, we studied the synchronization properties of a network of 1-DoF (possibly) heterogeneous oscillators with time-varying natural frequencies when the freedom of designing the interconnections is given. In particular, we assumed the model of the oscillators to be general enough to capture not only their (time-varying) natural frequency but also physical coupling actions and other unknown bounded terms, thus being able to represent, among many possibilities, networks of Kuramoto oscillators with heterogeneous time-varying natural frequencies. Furthermore, a communication layer is set in place to allow the oscillators to exchange synchronizing coupling actions through a tree-like leaderless network. We present a class of hybrid coupling rules depending only on local information to achieve (practical) uniform global asymptotic synchronization when the (heterogenous) oscillators are interconnected via a tree-like network (Theorems 4.1 and 4.2 and Corollaries 4.1 and 4.2). Furthermore, when suitable piecewise continuous coupling functions are considered, we achieved prescribed finite-flowing-time synchronization (Theorem 4.3). Notice that the proven $\mathcal{KL}$ bounds give quantitative information on the influence of the design parameters (such as the coupling gain κ in (4.1) and the impact of μ in Lemma 4.7) on the synchronization rate and the scalability of our control algorithm. In Proposition 4.1, we also proved that solutions to (4.8) enjoy an average dwell-time property and that the maximal ones are t-complete, thus certifying the absence of Zeno solutions. Furthermore, we showed the subtleties of considering a network with a cycle in place of a tree-like graph in Section 4.4.2, meanwhile, in Section 4.6 we provided numerical illustrations of the theoretical results while also showing how the selection of the coupling functions and the tuning of the coupling gain influence the synchronization behavior of the oscillators. Finally, ancillary results on set-valued Lie derivatives and non-pathological functions were given in Section 4.7 (Lemma 4.8 and Proposition 4.4).

— In **Chapter 5** we analyzed the stability properties of Lur'e systems with piecewise continuous nonlinearities by exploiting the novel mathematical tools presented in Chapter 4, in view of the underlying similarities between the dynamics of these systems and the continuous-time dynamics of the synchronizing oscillators in Chapter 4. In particular, we provided conditions to ensure state and output finite-time stability properties for Lur'e systems with suitable piecewise continues nonlinearities by exploiting the set-valued Lie derivatives in Theorems 5.2 and 5.3. Furthermore, we extended the GAS results in [26] to Lur'e systems with piecewise continuous nonlinearities satisfying general sector conditions (Theorem 5.1), while also providing a novel algebraic Lyapunov-based proof based on a Lur'e-Postnikov Lyapunov function and set-valued Lie derivatives. Moreover, in Examples 5.1 and 5.2 we illustrated, by exploiting this last notion of derivative, the possible shortcomings of adopting the customarily used tools in nonsmooth analysis (Clarke's generalized directional derivative) to study the stability properties of dynamics are defined by differential inclusions (or differential equations with discontinuous right-hand side) which are only outer semicontinuous, while using nonsmooth Lyapunov functions. Finally, we provided two examples of engineering systems whose state and output stability properties can be studied using the tools we propose in Section 5.5.

6.2 Perspectives

The results presented in this thesis allow envisioning various promising research directions, some of which are listed next.

6.2.1 Opinion dynamics

In Chapter 3, we assume that each agent has perfect knowledge of all the other agents' identities and opinions in the social network. Furthermore, we consider that each agent's opinion is possibly equally influenced by the opinion of any other agent in the network and that the opinions are related to a single topic. The same perfect knowledge and symmetrical influence are assumed while defining the dynamics of the memory variables. Therefore, the first two natural directions to investigate are to consider a model of opinion dynamics with memory-based connectivity depending on adaptive thresholds:

1. that is multi-topics [38]. In this scenario, the agents will interact and adjust their interactions based on their opinion (and memory variables) and their neighbors' opinions on a set of topics. Topics that may carry possibly different weights in the decision process of (de)activating links;
2. whose network is a (weighted) directed graph [107]. In this way, different sociality and impressibility for each agent in the network can be represented, thus allowing to model leaders or stubborn agents in the social network.

Similarly, we can define the memory dynamics to take into account differences between agents, e.g., by possibly imposing $\beta_{ij} \neq \beta_{ji}$ in (3.11). Finally, we could consider external inputs in the dynamics to simulate instantaneous and/or extremely fast changes in the memory and/or opinion variables after unexpected meaningful events. In this scenario, a redesign of the dynamics of the memory variables and/or the triggering rules managing (de)activation of the links could be needed, to avoid sudden unwanted changes in the topology. Indeed, we recall that, due to how dynamics in (3.11) is defined, the memory variables associated to links between agents in the same final cluster converge to zero.

6.2.2 Synchronizing 1-DoF oscillators

While assuming a tree-like network is not new in the scenario of controlling networked systems via distributed control laws [86], it could be a restrictive assumption. Ad-hoc solutions to solve this possible shortcoming can be thought of when dealing with cycles, as, for example, the one given in the next conjecture, however little can be said for the general case yet.

Conjecture 6.1. *Consider*

$$\begin{cases} \dot{x} \ \in F_c(x), & x \in C, \\ x^+ \in G(x), & x \in D, \end{cases} \tag{6.1}$$

where C, D, and G as in (4.7)*, and the set-valued map F_c defined as*

$$F_c(x) := \begin{bmatrix} \widehat{\Omega} - \kappa B K \widehat{\Sigma}(x) \\ \mathbf{0}_m \end{bmatrix}, \quad \forall x \in X, \tag{6.2}$$

with the set $\widehat{\Omega}$ and set-valued map $\widehat{\Sigma}$ as in (4.8b) *and where $K := \operatorname{diag}(\overline{\kappa})$, with $\overline{\kappa}$ being a vector of m strictly positive coupling gains $\overline{\kappa}_{ij} \neq \overline{\kappa}_{hs} \in \mathbb{R}_{>0}$ with $(i,j) \neq (h,s) \in \mathcal{E}$, using the same order as the columns of B. If the coupling function σ is selected as the sign function, and the considered graph $\mathcal{G}$ is* *an interconnection of cycles via trees, then set* $\mathscr{A}$ *in* (4.13) *is prescribed finite-flowing-time stable for* (6.1). □

Roughly speaking, due to the shape of the flow space of $\mathscr{G}$, in Conjecture 6.1 the coupling actions are designed so that the phase locking occurs only if the oscillators are synchronized. Furthermore, to generate the unique gains required by the control algorithm in Conjecture 6.1, one may adopt pseudo-unique IDs generated in a distributed way [101].

So, the first line of research to explore would be to generalize the work in Chapter 4, and thus define coupling actions to synchronize 1-DoF oscillators interconnected via general topologies. We think that adding new dynamics to the control scheme and/or modifying the existing system dynamics will be required, e.g., by including a (de)activation mechanism for the links in the network ruled by a newly defined control variable or by using stochastic processes [134] to reach synchronization as done in [117]. This reasoning is motivated by the next conjecture.

Conjecture 6.2. *Consider system* (4.8) *where the oscillators are interconnected via a connected graph. Furthermore, suppose that* $x(0,0) \in X \setminus \bigcup_{(i,j)\in\mathscr{E}} D_{ij}$ *and* $q(0,0) \in \{z \in \{-1,0,1\} | z = B^\top h, h \in \mathbb{Z}^m\}$. *Furthermore, assume that all solutions to* (4.8) *will remain in the set* $X \setminus \bigcup_{(i,j)\in\mathscr{E}} D_{ij}$ *for all* $(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. *Then set* $\mathscr{A}$ *is uniformly semi-globally practically* $\mathscr{KL}$ *asymptotically stable for system* (4.8). □

This research directions, aim to increase the applicability of our results to engineering applications such as clock synchronization, control of smart grids, distributed control of attitudes in flocks/swarms of unmanned drones and more [49]. Having these potential applications in mind, considering an unknown delay/packet loss in the system control input could be a further interesting direction to explore, these being some of the known issues affecting wireless communications [127]. For example, one could consider the phases in the coupling functions as instantaneous samples of the real-time analog signal where the update of these variables would happen periodically or could be event-triggered. Update mechanisms that could be affected by delays and/or faults [79]. Finally, we think that the developed model could be adapted to solve the problem of desynchronizing oscillators, see, for example, [56] to get further insight into the problem.

6.2.3 Lur'e systems

In Chapter 5 we have discussed the state and output stability properties of Lur'e systems with piecewise continuous nonlinearities. The assumptions we made are reasonable enough to be applied to the engineering systems given in [26, 55], however, we can think of relevant applications where a selection of the items in Assumptions 5.2 and/or 5.3 is not satisfied (for example, when A in (5.4) is not invertible), see [24, Chapter 3.3] for additional insights on this scenario. Hence, an interesting research direction is to define less stringent or alternative conditions to guarantee output and state asymptotic and finite-time stability properties for system (5.4), as in the next conjecture.

Conjecture 6.3. *Consider system* (5.4) *and suppose that Assumption 5.1 holds. Furthermore, suppose that:*

— *There exist matrices* Γ *positive definite and diagonal,* $P = P^\top \geq 0$ *and a scalar* $\eta > 0$ *such that*

$$\begin{bmatrix} PA + A^\top P & PB - (C + \Gamma CA)^\top \\ B^\top P - (C + \Gamma CA) & -2Z - \Gamma CB - (\Gamma CB)^\top + \eta I_p \end{bmatrix} \leq 0. \tag{6.3}$$

— *There exists* $\alpha \in \mathcal{K}$ *such that* $\mathrm{sign}(y_i)\psi_i(y_i) \geq \alpha(|y_i|)$ *for any* $y_i \in \mathbb{R}$ *and* $i \in \{1, \ldots, p\}$.

Then system (5.4) *is oGAS. Moreover, if C in* (5.4) *is invertible, the origin is GAS for* (5.4).

Furthermore, if item (iii) of Assumption 5.3 holds, then the system is OFTS. Finally, (5.4) *is also SFTS if and only if C in* (5.4) *is invertible.* □

We could also investigate conditions to ensure state and output weak stability properties, which would guarantee, for example, that there exists at least one solution whose states and/or outputs satisfy a $\mathcal{KL}$-stability property as in (5.14) [21]. These results often allow one to assume the least stringent conditions on the systems and may find usage in non-critical applications and/or when discriminating between solutions is a possibility. Furthermore, following the path paved by [129], studying the properties of interconnected Lur'e systems could bring relevant contributions to the field, aiming, for example, to defining tools for designing observers, as, for example, in [23]. Finally, similarly to what is done in [26], studying the state and output (finite-time) stability of Lur'e system with piecewise continuous nonlinearity in presence of exogenous disturbances could bring interesting and exploitable results.

6.2.4 Networked hybrid systems

Different approaches can be thought of when dealing with the analysis of networked hybrid dynamical systems. The approach envisioned in [64] to study any hybrid dynamical system (and thus also, the networked ones), and consequently, many of the existing results and developed analytical tools require that a single global model is specified. While, in this thesis, we have chosen to follow this route, as it has been done for the opinion dynamics model in [58] or the synchronizing hybrid dynamical systems in [103], we see large potential of studying networked hybrid systems by lifting the local properties of the subsystems defining the network, to the global model of the network, as envisioned in [40]. Some visionary works in this direction are [131], where the author proposes to decompose global systems in subparts and exploit their dissipativity properties to derive stability results, or [78], in which the authors use some Lyapunov stability assumptions and a small-gain condition to certify the (ISS) stability of the interconnections of hybrid dynamical systems. Following the path paved by the aforementioned works, an interesting research effort could be devoted to defining criteria to asses the stability properties of networked hybrid systems starting from the structural properties of each subsystem (ISS and ISS-like stability properties, dissipativity properties, etc.). Furthermore, defining criteria to obtain information on the properties of the (maximal) solutions (average-dwell time, t—completeness, etc.) of the whole network dynamics by lifting the properties of each subsystem in the network would be also of major interest. A fundamental follow-up would be to define constructive methodologies to build distributed control laws for networked hybrid systems and develop control strategies that can be used to optimize the performances of the networked hybrid dynamical system, such as by minimizing energy consumption or communications or maximizing the speed of convergence towards the desired collective behavior. Results that should be thought to be easily implementable in the software environments typically used by the engineering community.

Chapter 7

Résumé détaillé

7.1 Introduction

Lors de l'étape de modélisation, nous classons souvent un système dynamique selon qu'il dépende du temps de façon continue, comme, par exemple, les systèmes mécaniques ou électriques, ou de façon discrète à l'instar des processus numériques. Cependant, les systèmes qui échappent à cette dichotomie sont omniprésents : tout système physique dont la dynamique est affectée par des variables logiques, comme les systèmes mécaniques contrôlés par des ordinateurs numériques, les systèmes impulsifs où les changements quasi-instantanés affectant la dynamique continue peuvent être modélisés par des événements discrets (voir le célèbre exemple de la balle rebondissante dans [64]), ou tout système biologique où la dynamique en temps continu est affectée par un mécanisme de réinitialisation, comme pour la dynamique décrivant la synchronisation des lucioles clignotantes [64, Ch. 2] ou des neurones [69, Ch. 8]. Les systèmes de commutation constituent une autre classe importante de systèmes échappant à la dichotomie susmentionnée. Il s'agit de systèmes à temps continu dont les équations différentielles décrivant leur dynamique sont choisies parmi un ensemble fini ou dénombrable d'équations différentielles possibles et sont modifiées conformément à une règle de commutation. Il est intéressant de noter que la richesse du comportement de tels systèmes ne peut pas être capturée en utilisant uniquement des équations différentielles (ou des inclusions différentielles) ou des équations de différences (ou des inclusions de différences), et qu'ils ne peuvent donc pas être décrits uniquement comme des systèmes dynamiques à temps continu ou à temps discret. Seule une combinaison bien pensée de ces deux dynamiques permet d'obtenir des représentations complètes et exhaustives de ces systèmes, ce que l'on appelle les *systèmes dynamiques hybrides*.

Dans la littérature, différents cadres ont été proposés pour modéliser les dynamiques hybrides, notamment pour les systèmes de commutation dans [77], pour les systèmes dynamiques impulsifs dans [65], pour les automates hybrides dans [81] ou pour les systèmes de complémentarité dans [22]. Dans cette thèse, nous choisissons d'adopter un autre formalisme pour modéliser les systèmes dynamiques hybrides : celui développé dans [64] par Andrew Teel, Rafał Goebel, et Ricardo Sanfelice. Intuitivement, les systèmes définis par ce formalisme peuvent évoluer selon des dynamiques à temps continu et à temps discret, en alternant éventuellement ces comportements. Le choix entre ces évolutions est régi par des conditions définissant deux régions de l'espace d'état (qui peuvent se chevaucher partiellement ou totalement). Dans le cadre le plus général, les dynamiques continues et discrètes sont représentées par des inclusions différentielles et de différences respectivement. De plus amples explications sur le modèle sont fournies au Chapitre 2.3 où la terminologie et la notation utilisées sont définies. Nous avons donc choisi d'adopter le formalisme de [64] pour :

(i) la généralité offert par ce cadre de modélisation, qui permet de capturer une riche classe de systèmes dynamiques hybrides, y compris les systèmes à commutation, les systèmes impulsifs et les automates hybrides ;

(ii) la notion de solutions associée, qui généralise les notions existantes pour les inclusions différentielles/de différences ;

(iii) la solide théorie de la stabilité de Lyapunov associée à ce cadre, qui couvre de manière cohérente les résultats existants pour les systèmes à temps continu et à temps discret en tant que cas particuliers ;

(iv) la façon dont la question de la robustesse est traitée, ce qui est extrêmement difficile lorsque l'on traite de tels systèmes discontinus.

En ce qui concerne le dernier point, les auteurs de [64] ont ainsi identifié un ensemble de conditions générales, sous lesquelles un système dynamique hybride est dit bien posé, les conditions de base hybrides rappelées dans le Chapitre 2.3, impliquant ainsi que la limite de toute suite de solutions (graphiquement) convergentes est solution du système; on parle de compacité séquentielle. Cette dernière propriété semble être essentielle pour établir les propriétés de robustesse.

De nombreux résultats concernant les systèmes à temps continu et à temps discret ont été étendus aux systèmes dynamiques hybrides dans le formalisme de [64]. Dans [116] et [115], les principes d'invariance de LaSalle et un théorème de Matrosov pour les systèmes dynamiques hybrides sont fournis. La caractérisation des propriétés de stabilité d'entrée à l'état pour les systèmes dynamiques hybrides est donnée dans [28], tandis que dans [92], les auteurs décrivent les propriétés de stabilité à gain fini $\mathscr{L}_2$ pour les systèmes dynamiques hybrides tout en fournissant également des résultats supplémentaires sur les systèmes dynamiques hybrides homogènes. Les résultats dans [78] fournissent des théorèmes de petit gain basés sur Lyapunov pour les systèmes dynamiques hybrides afin de certifier la stabilité d'entrée à l'état (ISS) des systèmes hybrides avec entrées et la stabilité asymptotique globale (GAS) des systèmes dynamiques hybrides sans entrées. Dans [94], les auteurs étendent l'inégalité de Gronwall, un résultat bien connu et utile à la fois pour les signaux à temps continu et à temps discret, aux signaux dynamiques hybrides. La récente contribution au domaine faite dans [113] vise plutôt à généraliser les résultats sur la conception de contrôleurs pour les systèmes dynamiques à temps continu et à temps discret aux systèmes hybrides.

Dans cette thèse, nous visons à démontrer la puissance des outils théoriques hybrides pour modéliser et contrôler des classes importantes de systèmes *en réseau* conduisant à des *systèmes dynamiques hybrides en réseau*. En effet, les réseaux sociaux, l'industrie 4.0, les réseaux intelligents et de nombreux autres systèmes dynamiques en réseau sont omniprésents dans le monde d'aujourd'hui. Ces systèmes peuvent être représentés de comme des réseaux d'agents évoluant vers un objectif global commun tout en ayant connaissance que d'informations locales. Le comportement de ces systèmes dans leur ensemble est souvent décrit par une dynamique à temps continu affectée par d'éventuels changements instantanés, qui peuvent se produire naturellement ou être imposés pour obtenir un comportement souhaité du réseau. La superposition de ces deux dynamiques génère donc ce que l'on appelle un *système dynamique hybride en réseau*.

Dans ce contexte, nous pouvons identifier trois phénomènes principaux provoquant des changements instantanés dans les systèmes (qui, en général, ne s'excluent pas mutuellement) :

(i) la dynamique intrinsèque de chaque nœud, comme pour les lucioles clignotantes dans [64, Ch. 2] ou la dynamique des neurones dans [102] ;

(ii) les degrés de liberté intrinsèques des actions de contrôle, comme dans les convertisseurs de puissance au sein des réseaux intelligents [17, 135], ou dans les systèmes en réseau soumis à des contraintes de calcul et de communication, comme décrit dans [95] pour le contrôle déclenché par un événement pour les systèmes multi-agents et [40] où la coordination d'un réseau d'agents dans un environnement cyber-physique est abordée via le remaniement de Lyapunov ;

(iii) la création/perte de liens ou l'ajout/la suppression de liens ou l'ajout/la suppression de nœuds comme dans les modèles de dynamique d'opinion [58].

Dans cette thèse, nous nous concentrons sur le point (iii), et notre objectif est de démontrer les forces des outils hybrides pour modéliser, analyser et contrôler les systèmes en réseau via deux études de cas. Le formalisme hybride a déjà démontré son efficacité pour généraliser et relâcher des limites fondamentales les travaux existants dans le cadre purement temps continu ou temps discret cf. [93], [64], [100]. Nous sommes convaincus que les outils hybrides joueront également un rôle important pour les systèmes en réseau comme les travaux de cette thèse tendent à le démontrer.

Dans la littérature, il existe déjà des résultats qui étudient les systèmes en réseau via le formalisme hybride dans [64]. Dans son travail visionnaire [131], l'auteur propose d'exploiter les notions de dissipativité pour établir la stabilité asymptotique des systèmes hybrides à grande échelle, généralisant ainsi les résultats pour les systèmes à temps continu dans [90], et les outils de dissipativité sont connus pour être des outils puissants pour analyser le comportement collectif dans les systèmes en réseau [10]. Dans [112] et [78], qui fournissent des théorèmes de petit gain pour analyser les interconnexions des systèmes dynamiques hybrides, des outils très utiles pour établir des garanties de stabilité pour les systèmes interconnectés sont fournis. Il reste encore beaucoup à faire pour exploiter le potentiel des techniques hybrides afin de modéliser, d'analyser et de contrôler les systèmes en réseau.

7.2 Contributions

Dans cette thèse, nous visons à montrer l'efficacité des outils théoriques hybrides pour modéliser et contrôler de manière distribuée d'importantes classes de systèmes en réseau. Nous démontrons d'abord comment le formalisme hybride dans [64] peut être exploité pour modéliser l'évolution des opinions dans un réseau social, dans un scénario où les interactions entre les individus dépendent à la fois de leurs opinions passées et actuelles. Il s'agit d'une hypothèse raisonnable lorsque chaque individu connaît l'identité des autres membres du réseau. Pour ce faire, nous fusionnons les caractéristiques du modèle de Friedkin-Johnsen (FJ) [59] (où l'importance est donnée au passé, aux conditions initiales) et celles du modèle de Hegselmann-Krause (HK) [67] où le mécanisme de confiance limitée repose sur les opinions actuelles). Nous tenons compte du passé en filtrant la non-concordance des opinions des voisins. Nous exploitons ensuite ces données filtrées en concevant la (dé)activation de chaque lien dans le réseau de manière à ce qu'elle ne dépende pas seulement de seuils adaptatifs sans mémoire, comme proposé dans [58], mais qu'elle soit également réglée en fonction des valeurs des variables de mémoire. Un nouveau modèle est alors développé dans le formalisme de [64]. Nous analysons ensuite les propriétés de stabilité du modèle à l'aide d'une nouvelle fonction de Lyapunov, qui garantit une propriété de stabilité $\mathcal{KL}$ appropriée assurant la convergence asymptotique des opinions vers les clusters/foyers (Théorème 3.1 et Corollaire 3.1 du Chapitre 3). En outre, il est prouvé que les solutions ne génèrent pas de phénomène de Zénon et qu'elles sautent un nombre fini de fois. Nous illustrons numériquement l'impact de la dynamique de la mémoire et des paramètres de conception sur le consensus atteint par les agents. Ce travail illustre donc comment des outils hybrides peuvent être utilisés pour modéliser et analyser la dynamique des opinions.

Dans la deuxième étude de cas, nous surmontons les limites fondamentales du couplage en temps continu pour synchroniser un réseau d'oscillateurs en exploitant des techniques hybrides. En effet, ce problème est historiquement abordé dans la littérature en recourant au célèbre modèle de Kuramoto dont l'inspiration originale provient des réseaux biologiques et physiques. Cependant, le modèle de Kuramoto présente des lacunes majeures pour les applications techniques, à savoir l'absence de synchronisation uniforme et la possibilité de verrouillage de phase en dehors de l'ensemble de synchronisation [120]. En particulier, nous envisageons un scénario d'ingénierie dans lequel l'objectif est de concevoir des règles de couplage pour synchroniser globalement et uniformément les phases d'oscillateurs hétérogènes. Chaque oscillateur a une fréquence naturelle variable dans le temps qui prend des valeurs dans un ensemble compact. Par conséquent, pour obtenir une synchronisation uniforme et globale, nous contraignons les graphes d'interconnexion à être un arbre non orienté, qui peut toujours être dérivé de graphes connectés génériques de manière distribuée en utilisant les algorithmes étudiés dans [98]. Nous introduisons des réinitialisations appropriées (un mécanisme de désenroulement de 2π) des coordonnées des phases des oscillateurs, de sorte qu'elles sont déroulées pour évoluer dans un ensemble compact, qui comprend $[-\pi, \pi]$ de manière cohérente avec leur nature angulaire. Dans ce cadre, nous présentons de nouvelles règles de couplage hybrides pour lesquelles une analyse basée sur Lyapunov garantit la synchronisation asymptotique globale uniforme (pratique) lorsque les oscillateurs (hétérogènes) sont interconnectés via un réseau arborescent (Théorèmes 4.1 et 4.2 et Corollaires 4.1 et 4.2 du Chapitre 4). Curieusement, étant donné que l'ensemble des fonctions de couplage admissibles est assez général, il existe des sélections de règles de couplage qui permettent de retrouver le comportement du modèle de Kuramoto original lorsque les oscillateurs sont proches de la synchronisation de phase. En outre, en raison des propriétés légères que nous exigeons pour notre fonction de couplage hybride, les sélections discontinues sont autorisées, comme dans [35]. Lorsque la discontinuité est à l'origine, nous prouvons la synchronisation prescrite en temps fini (Théorème 4.3 du Chapitre 4). En particulier, la synchronisation exacte peut être atteinte dans un temps fini prescrit [121], et la convergence est donc indépendante des conditions initiales. Par rapport aux travaux connexes dans [84] et [141], la propriété de stabilité en temps fini que nous assurons est globale et le temps de convergence peut être prescrit arbitrairement, respectivement. Remarquons que les bornes $\mathscr{K}\mathscr{L}$ prouvées donnent des informations quantitatives sur l'influence des paramètres de conception (tels que le gain de couplage κ dans (4.1) et l'impact de μ dans le Lemme 4.7) sur le taux de synchronisation et l'extensibilité de notre algorithme de contrôle. En raison de la présence possible de discontinuités dans la fonction de couplage, l'analyse de stabilité est effectuée en se concentrant sur la régularisation de Krasovskii de la dynamique à temps continu. Nous avons pu prouver que les solutions à la dynamique régularisée jouissent d'une propriété de temps de latence moyen et que les solutions maximales sont complètes. Par conséquent, comme ces deux propriétés impliquent que les solutions maximales sont t—complètes, nous certifions l'absence du phénomène de Zénon. Par ailleurs, nous avons montré les subtilités de la prise en compte d'un réseau avec un cycle au lieu d'un graphe arborescent dans la section 4.4.2, tandis que dans la section 4.6 nous avons fourni des illustrations numériques des résultats théoriques tout en montrant également comment la sélection des fonctions de couplage et l'accord du gain de couplage influencent le comportement de synchronisation des oscillateurs. Enfin, nous fournissons des résultats auxiliaires sur les fonctions non pathologiques et les dérivées de Lie à valeur d'ensemble (Lemma 4.8 et Proposition 4.4), motivés par la nécessité de déployer des outils nouveaux et efficaces pour étudier les propriétés de stabilité des systèmes dynamiques hybrides avec des dynamiques continues à valeurs définies qui sont localement limitées et semi-continues externes mais pas semi-continues internes (comme c'est typiquement le cas pour la régularisation de Krasovskii des fonctions discontinues à valeur unique par morceaux), tout en utilisant des fonctions de Lyapunov non lisses.

En raison de la similitude entre les fonctions de Lyapunov de Lur'e-Postinkov et les fonctions de Lyapunov développées au Chapitre 4, et de par la nature de la nature similaire des non-linéarités étudiées, nous exploitons les outils mathématiques susmentionnés dans le Chapitre 5, pour analyser la stabilité des systèmes de Lur'e en temps continu avec des non-linéarités continues par morceaux. Les systèmes de Lur'e étant omniprésents dans de nombreux domaines de l'ingénierie, on trouve dans la littérature un nombre considérable de publications étudiant leurs propriétés de stabilité, à la fois dans un cadre général, comme [143], puis pour des applications spécifiques, à l'instar de [26]. Cependant, la littérature est encore insuffisante pour caractériser l'état et les propriétés de stabilité en temps fini des systèmes de Lur'e avec des non-linéarités continues par morceaux, dont la pertinence est évidente dans [144] et [105]. Par conséquent, en exploitant la notion de dérivée de Lie à valeur d'ensemble, nous avons fourni des conditions pour garantir les propriétés de stabilité à temps fini de l'état et de la sortie pour les systèmes de Lur'e avec des non-linéarités continues par morceaux appropriées dans les théorèmes 5.2 et 5.3. En outre, nous généralisons un résultat existant sur la stabilité asymptotique de l'origine pour cette classe de systèmes en utilisant les mêmes outils. De plus, nous étendons les résultats GAS dans [26] aux systèmes de Lur'e avec des non-linéarités continues par morceaux satisfaisant des conditions générales de secteur (Théorème 5.1), tout en fournissant également une nouvelle preuve algébrique de Lyapunov basée sur une fonction de Lyapunov de Lur'e-Postnikov et des dérivées de Lie à valeur définie. Finalement, dans les exemples 5.1 et 5.2, nous illustrons, en exploitant cette dernière notion de dérivée, les lacunes possibles de l'adoption des outils habituellement utilisés en analyse non lisse (dérivée directionnelle généralisée de Clarke) pour étudier les propriétés de stabilité des dynamiques définies par des inclusions différentielles (ou des équations différentielles avec un côté droit discontinu) tout en utilisant des fonctions de Lyapunov non lisses. Nous illustrons l'utilité de nos résultats pour certifier les propriétés de stabilité de sortie et d'état dans deux applications d'ingénierie, qui sont respectivement considérées dans [26, 55]. En effet, nous établissons des propriétés de stabilité asymptotique locale en temps fini et indépendante de l'état pour les systèmes mécaniques soumis au frottement, ce qui est une nouveauté par rapport à [26]. De plus, nous certifions que les réseaux de neurones cellulaires modélisés comme dans [55] sont stables en temps fini, retrouvant ainsi les résultats de [55, Thm. 4] tout en faisant face à une classe plus générale de systèmes de Lur'e.

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Annex A

Proofs of Chapter 4

A.1 Proof of Lemma 4.1

Let $(i, j) \in \mathscr{E}$, $x \in D_{ij}$ and x^+ satisfies (4.4b). Then, $\theta^+ = \theta$ while $q^+ \in \{-1, 0, 1\}^m$ in view of (4.4). Thus, $x^+ \in X$ and the first part of the statement is proved. Let $\Delta\theta_{ij} := \theta_j - \theta_i$, so that $\theta_j - \theta_i + 2q_{ij}\pi = \Delta\theta_{ij} + 2q_{ij}\pi$. Since $x^+ \in X$, by definition of X in (4.3), $|\Delta\theta_{ij}| < 2\pi + 2\delta$.

We now prove that $|\Delta\theta_{ij}^+ + 2q_{ij}^+\pi| = |\Delta\theta_{ij} + 2q_{ij}^+\pi| < \pi + \delta$ by exploiting the fact that $q_{ij}^+ = h^* \in \underset{h\in\{-1,0,1\}}{\text{argmin}}\ |\Delta\theta_{ij} + 2h\pi|$ according to (4.4c), and splitting the analysis in five cases.

a) $\Delta\theta_{ij} \in (-\pi, \pi)$. Then, the minimizer is $h^* = 0$ and $|\Delta\theta_{ij} + 2h^*\pi| < \pi < \pi + \delta$.

b) $\Delta\theta_{ij} = \pi$. Then, the minimizer is $h^* \in \{-1, 0\}$ and $|\Delta\theta_{ij} + 2h^*\pi| \leq \pi < \pi + \delta$.

c) $\Delta\theta_{ij} = -\pi$. This case is identical to the previous one by changing all the signs, therefore $h^* \in \{0, 1\}$ and $|\Delta\theta_{ij} + 2h^*\pi| \leq \pi < \pi + \delta$.

d) $\Delta\theta_{ij} \in (\pi, 2\pi + 2\delta]$. Then, the minimizer is $h^* = -1$ and $\Delta\theta_{ij} + 2h^*\pi \in (-\pi, 2\delta]$, which implies $|\Delta\theta_{ij} + 2h^*\pi| < \pi + \delta$, since $\max(2\delta, \pi) < \pi + \delta$.

e) $\Delta\theta_{ij} \in [-2\pi - 2\delta, -\pi)$. In this case, the minimizer is $h^* = 1$ and $\Delta\theta_{ij} + 2h^*\pi \in [-2\delta, -\pi)$, which implies $|\Delta\theta_{ij} + 2h^*\pi| < \pi + \delta$, since $2\delta < \pi + \delta$.

Hence, we obtain, in view of all the previous cases, $|\theta_j^+ - \theta_i^+ + 2q_{ij}^+\pi| \leq \max(2\delta, \pi) < \pi + \delta$ thus concluding the proof since we have arbitrarily selected $(i, j) \in \mathscr{E}$.

A.2 Proof of Lemma 4.2

Let $i \in \mathscr{V}$, $x \in D_i$ and $x^+ = g_i(x)$. Let $(u, v) \in \mathscr{E}$, in view of (4.5), if $u \neq i$ and $v \neq i$ the first equality in (4.6) trivially holds. If $u = i$, then we have

$$\begin{aligned}
&\theta_v^+ - \theta_u^+ + 2q_{uv}^+\pi \\
&\qquad = \theta_v - \theta_u + \text{sign}(\theta_u)2\pi + 2(q_{uv} - \text{sign}(\theta_u))\pi \\
&\qquad = \theta_v - \theta_u + 2q_{uv}\pi.
\end{aligned}$$

Similarly, we obtain for $v = i$ that $\theta_v^+ - \theta_u^+ + 2q_{uv}^+\pi = \theta_v - \theta_u + 2q_{uv}\pi$, thus proving the first equality in (4.6). On the other hand, in view of (4.5b), $|\theta_i^+| = \pi - \delta < \pi + \delta$. Thus, all the elements of (4.6) are proved.

Let us now prove that $x^+ \in X$. In particular, we need to make sure that $q^+ \in \{-1, 0, 1\}^m$. For any $j \neq i \in \mathscr{V}$ we have that $\theta_j^+ = \theta_j$ and $|\theta_i^+| = \pi - \delta$ in view of (4.5b). Moreover, if j is such that $(i, j) \in \mathscr{E}$, then from (4.4a), (4.5d) we prove next that $q_{ij} \neq -\text{sign}(\theta_i)$. Indeed, if $q_{ij} = -\text{sign}(\theta_i)$ then we would have

$$\begin{aligned}|\theta_j - \theta_i + 2q_{ij}\pi| &= \left|\theta_j - \text{sign}(\theta_i)|\theta_i| - \text{sign}(\theta_i)2\pi\right| \\ &= \left|\theta_j - \text{sign}(\theta_i)(3\pi + \delta)\right| \geq 2\pi > \pi + \delta,\end{aligned}$$

meaning that $x \in \text{int}(D_{ij})$ and consequently $x \notin D_i$. Thus, $q_{ij} \neq -\text{sign}(\theta_i)$ and, in view of (4.5c), we obtain $q_{ij}^+ \in \{0, -\text{sign}(\theta_i)\}$. With a similar reasoning, we conclude that if j is such that $(j, i) \in \mathscr{E}$ then we must have $q_{ji} \neq \text{sign}(\theta_i)$, implying that $q_{ji}^+ \in \{0, \text{sign}(\theta_i)\}$ in view of (4.5c). Hence, $x^+ \in X$.