UNIVERSITÉ DE LORRAINE

École doctorale IAEM Lorraine

Analysis on singular space and operators algebras

THÈSE

présentée et soutenue publiquement le 30 septembre 2019

pour l'obtention du

Doctorat de l'Université de Lorraine
(mention Mathématiques)

par

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Remerciements

C’est avec un grand plaisir que je remercie mon directeur de thèse Victor Nistor de m’avoir proposé ce sujet de thèse ainsi que de m’avoir encadré. Je le remercie pour sa disponibilité, sa bonne humeur ainsi que ses grandes connaissances en mathématiques qui ont toujours été rassurantes. En plus de tous son savoir mathématiques, il reste une personne très accessible et m’a aidé à prendre confiance.

Je remercie Serge Richard et Radu Purice d’avoir accepté d’être les rapporteurs de ma thèse. Je remercie également Paulo Carillo-Rouse, Hervé Oyono-Oyono, Angela Pasquale et Lisette Jager d’avoir accepté d’être dans mon jury de thèse.

Je remercie Nicolas Prudhon et Bernd Ammann, cela a été un plaisir de travailler avec eux. J’espère que nous pourrons continuer nos collaborations.

Je tiens à remercier Vladimir Georgescu, Marius Măntoiu et Jean Ludwig pour leur aide dans mon travail. Leurs avis et conseils m’ont été précieux.

C’est avec regret que je vais quitter le laboratoire de l’IECL. Je remercie tous mes anciens enseignants qui sont devenus mes collèges. Je remercie également le personnel administratif de l’IECL en particulier Claude Coppin pour sa bonté et son indulgence.

Je voudrais remercier Hanne Van Den Bosch et Edgardo Stockmeyer pour leur accueil à Santiago. Je remercie Sébastien Breteaux pour son aide à Santiago même si sortir un week-end de jour férié, n’était pas la meilleure idée qu’il ait eue.

La thèse a été beaucoup plus légère grâce à mes collègues doctorants. Je tiens a témoigner mes remerciements à Allan pour nos discussions qui ne pouvaient pas rester sérieuse plus de quelques phrases, à Spyros pour sa simplicité à laisser sortir ses émotions, à Rémi pour la relecture du manuscrit et sa bienveillance, à Souad pour sa patiente à toutes mes interrogations, à Hichem pour les fous rires que nous avons partagé. Je ne remercie pas Benjamin, car justement, c’est absurde de ne pas le faire. Je remercie également P-H, Maxime, Simon, Fabien, Alexandre et Gabriel. À Sarah qui m’a encouragé au début de ce projet et a corrigé bon nombre des mes fautes d’orthographes et de grammaires.

À Kawther qui m’a aidé grâce à son écoute, sa patiente et son sourire.

Enfin je remercie ma famille, ma mère qui a eu une patiente infini pour me faire travailler, mon père qui cherche toujours à me rendre la vie plus simple, mon beau-père qui me considère comme son fils, mon frère avec qui nous n’avons pas besoin de long discours pour se comprendre, ma belle-sœur qui a d’excellents gouts musicaux, et ma sœur, qui par le regard qu’elle porte sur moi, me pousse à être une meilleure personne.

Le monde est encore plus intéressant que triste.
Raphaël Enthoven

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Introduction

1.1 Motivation : the Hamiltonian of N-body problem

The research carried out in this thesis is concerned with the study of operators of Schrodinger type, especially with the operators that model the energy of the N-body problems. These operators are of a crucial importance in Quantum Mechanics. An example of an N-body system is a molecule with N constituents. The operator that models the energy of the system is called a *Hamiltonian.* If we forget the magnetic field, the energy described by the Hamiltonian H will have two different sources. The first one is the free energy and is modeled by the Laplacian Δ and the second one is the potential energy described by an operator of multiplication by a function V. More explicitly we have :

$$H = -\Delta + V \tag{1.1}$$

An early successful approach to study the essential spectrum of the operator H was based on Weyl's essential spectrum theorem. A detailed presentation of this result can be found in [52, Chapter XIII, Section 4]. This theorem tells us, in particular, that, for a self-adjoint operator A, we have

$$\sigma_{ess}(A) = \sigma_{ess}(A + C),$$

where C is a relatively compact perturbation of A. Recall that C is a *relatively compact perturbation* of A if $D(A) \subset D(C)$ and if $C(A + i)^{-1}$ is compact. In the particular case of $A = -\Delta$, there are many explicit sufficient conditions on V to yield a relatively compact perturbation. The following result provides a good example :

Theorem 1.1.1 ([52], Theorem XIII.15). *Let $V \in L^p(\mathbb{R}^n) + L^\infty_\epsilon(\mathbb{R}^n)$ with $p \geq \max\{2, \frac{n}{2}\}$ if $n \neq 4$ and $p > 2$ if $n = 4$. Then V is a relatively compact perturbation of $-\Delta$ and*

$$\sigma_{ess}(-\Delta + V) = \sigma_{ess}(-\Delta) = [0, +\infty). \tag{1.2}$$

Here, $V \in L^p(\mathbb{R}^n) + L^\infty_\epsilon(\mathbb{R}^n)$ means that there exists a sequence $V_k \in L^p(\mathbb{R}^n)$ (the usual Lebesgue space) such that $V - V_k \in L^\infty(\mathbb{R}^n)$ and $\|V - V_k\|_\infty \to 0$.

Let us give now more details on the Hamiltonian H and the potential V of the Hamiltonian of the N-body problem. We model the position of each body of the N-body problem by an element of $\mathbb{R}^3$, hence we consider the real vector space $X = \mathbb{R}^{3N}$. The operator H acts on $L^2(X)$ and is given by

$$H^{(N)} := -\Delta_{\mathbb{R}^{3N}} + \sum_{1 \leq i \leq N} \frac{a_i}{\|x_i\|} + \sum_{1 \leq i < j \leq N} \frac{b_{i,j}}{\|x_i - x_j\|}, \tag{1.3}$$

where $x = (x_1, \ldots, x_N) \in X$ and $\|.\|$ the euclidean norm. The constants a_i and $b_{i,j}$ are positive or negative depending on whether their corresponding forces are attracting or repelling. The function $\frac{b_{i,j}}{\|x_i - x_j\|}$ models the interactions between the i'th body and the j'th body. This function dependents only on the distance between the two objects. The function $\frac{a_i}{\|x_i\|}$ appears because of the assumption that there exits a very heavy body at the origin of X. A typical example is an atom with N electron whose nucleus is very heavy compared to its electrons and was placed at the origin of X. We note that the potential in equation (1.3) fails to go to zero when x goes to ∞, and hence Theorem 1.1.1 cannot by applied to describe the essential spectrum of this Hamiltonian. However, results on the essential spectrum of the Hamiltonian (1.3) were obtained by many people, including Hunziker [29], Van Winter [58], and Žislin [61]. Their main results is the so called HVZ theorem, which is by now a very classical result in the field. We follow [57, Section 11, Theorem 11.2] in recalling the statement of the HVZ theorem for an atom with N electrons :

Theorem 1.1.2 (HVZ). *Let $H^{(N)}$ be the operator defined in* (1.3). *Then $H^{(N)}$ is self-adjoint and is bounded from below. Moreover,*

$$\sigma_{ess}(H^{(N)}) = \inf\left(\sigma(H^{(N-1)})\right) = [\lambda^{N-1}, \infty), \tag{1.4}$$

with λ^{N-1} is negative and is the bottom of the spectrum of $H^{(N-1)}$ and $H^{(N-1)}$ is the Hamiltonian associate to $(N-1)$ body.

A physical interpretation of this theorem is that the energy required to remove an electron of a atom with N electrons is equal to $\lambda^N - \lambda^{N-1}$, where λ^N is the bottom the spectrum of $H^{(N)}$. The classical version of the HVZ theorem required much more notation see [52, Theorem XIII.17] for precise statement.

As mentioned before, the potential in equation (1.3) fails to go to 0 when x goes to ∞. The behavior of the potential is very complicated due to the singularities. These singularities can be found along the following subspaces :

$$\begin{aligned} \mathcal{P}_j &:= \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_j = 0 \in \mathbb{R}^3\}, \quad 1 \le j \le N, \text{ and} \\ \mathcal{P}_{i,j} &:= \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_i = x_j \in \mathbb{R}^3\}, \quad 1 \le i < j \le N. \end{aligned} \tag{1.5}$$

These subspaces are the *collision planes.* For example, if $x \in \mathcal{P}_{i,j}$, then the i'th and the j'th body are at the same positions and hence a collision happens. Several collisions can occurs in the same time, so we will consider $\mathcal{S}$ the family of subspaces of X that contains all the space $\mathcal{P}_i$ and $\mathcal{P}_{i,j}$ and their intersections. This yields a family of subspaces $\mathcal{S}$ of X that is stable by intersections, that is a *semi-lattice* of subspace of X. Boutet de Monvel and Georgescu have associated to every semi-lattice a graded C^*-algebra that they used to recover the HVZ theorem and to obtain Mourre estimates [6, 7, 8]. An exhaustive study of these graded C^*-algebras was carried out by Mageira in her PhD thesis and in the resulting publications [34, 35].

1.2 Summary of contents and results

In my work, I continued the investigation of Hamiltonians of the N-body problem with more general potentials using algebraic and geometric methods, more precisely, using C^*-algebras to determine the essential spectrum of these Hamiltonians and using manifolds with corners and their blow-ups to further the understanding of the spectra of the C^*-algebras appearing in the proofs and the behavior of their eigenfunctions. My work has led to four research papers. Three of these papers have been published, and the last one is in final preparation for submission. The thesis is based, to a very large extent, on these papers. I have, nevertheless, gathered most of the common preliminary material in a separate chapter, to avoid unnecessary repetitions.

In all the papers, we consider *radial compactifications* of real vector spaces, so let us recall this notion now. Let X be a finite dimensional real vector space and $\mathbb{S}_X$ be the set of directions of X, that is, the set of semi-lines $\mathbb{R}_+^* v$, $v \in X \smallsetminus \{0\}$. The *radial compactification* of X is the set $\overline{X} := X \sqcup \mathbb{S}_X$. Its topology is described in Subsection 2.2.2, whereas its smooth structure is a subtle issue discussed in Chapter 6. In particular, a function in $\mathcal{C}(\overline{Z})$ is a continuous function on Z that has uniform radial limits at infinity.

Let us now quickly describe the main contents and results of the chapters of this thesis and, in particular, of the four papers mentioned above. Their results will be described in much more detail in the following sections of the Introduction, with one section being devoted to each paper.

Chapter 2 is devoted to common background material to all the other chapters (and hence to the four papers). In fact, a lot of the background material for these papers is common to most of them, especially to the first three papers. Therefore, instead of repeating preliminary material in every chapter, I have collected most of it in a separate chapter, Chapter 2, which contains background material on C^*-algebras, crossed-products, Fredholm operators, compactifications, and exhaustive families of representations.

Chapter 3 reproduces to a large extent the results of my first paper ("A refined HVZ-theorem for asymptotically homogeneous interactions and finitely many collison planes," joint work with Nistor and Prudhon, published in *Rev. Roumaine Math. Pure Applic.*, 2017, 22 pages). More precisely, we study algebras associated to N-body type Hamiltonians with interactions that are asymptotically homogeneous at infinity on a finite dimensional, vector real space X. More precisely, let $\mathcal{S}$ be a *finite* semilattice of linear subspaces of X and, for each let $Y \in \mathcal{S}$, let $v_Y \in \mathcal{C}(\overline{X/Y})$, that is v_Y is a continuous function on X/Y that has uniform homogeneous radial limits at infinity. (Recall that $\overline{X/Y}$ denotes the radial compactification of X/Y.) We considered in that paper Hamiltonians of the form

$$H := -\Delta + \sum_{Y \in \mathcal{S}} v_Y . \tag{1.6}$$

Georgescu and Nistor have considered the case when $\mathcal{S}$ consists of *all* subspaces $Y \subset X$ [27]. The resulting spectra of C^*-algebras, however, are quite different, and the finite lattice case presents some properties that are not evident in the infinite dimensional case (some of these properties do not even hold true in the infinite case). In fact, in this paper, among other things, we prepare the ground for the more precise identification of these spectra in the fourth paper (see Chapter 6). As in [27], we also consider more general Hamiltonians affiliated to a suitable cross-product algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$.

A first goal of this paper (chapter) is to see which results of [27] carry through to the case when $\mathcal{S}$ (the set of "collision planes") is finite and, for the ones that do not, what is their suitable modification. While the results on the essential spectra of the resulting Hamiltonians and the affiliation criteria carry through, the spectra of the corresponding algebras are quite different, as we have already mentioned above. Identifying these spectra may have implications for the regularity of eigenvalues and numerical methods. Our results also shed some new light on the results of Georgescu and Nistor in the aforementioned paper and, in general, on the theory developed by Georgescu and his collaborators. For instance, we show that, in our case, the closure is not needed in the union of the spectra of the limit operators, more precisely,

$$\sigma_{ess}(T) = \bigcup_{\alpha \in \mathcal{F}} \sigma(T_\alpha) , \tag{1.7}$$

where α ranges through a suitable set of indices. (Usually, this set of indices is the set of orbits for a certain action of a group.) This result will be a paradigm for the following two papers. We also obtain a quotient topology description of the topology on the spectrum of the graded N-body C^*-algebras introduced by Georgescu and studied also by [42].

Chapter 4 reproduces to a large extent the results of my second paper ("Exhaustive families of representations of C^*-algebras associated to the N-body problem Hamiltonians with asymptotically homogeneous interactions," joint work with Prudhon, published in *C.R Math. Acad. Sci. Paris*, 2019, 5 pages), which is an announcement (note). In this note, we continue the analysis of algebras $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ introduced in Chapter 3 in order to study the N-body type Hamiltonians with interactions corresponding to a finite semilattice of vector subspaces. The main new result is that we also consider action of groups and then explain how the properties of the algebras that we consider answer a question of Melrose and Singer [39]. We also consider Dirac type operators and, more generally, pseudodifferential operators. See Subsection 1.4 for more details.

Chapter 5 reproduces to a large extent the results of my third paper, ("Essential spectrum, quasi-orbits and compactifications : Application to the Heisenberg group," singly authored, published *Rev. Roumaine Math. Pure Applic.*, 2019, 19 pages). More precisely, we replace the underlying vector space X with the three dimensional Heisenberg group H and study to what extent the results in the commutative case extend to this new, non-commutative setting. We still consider the radial compactification $\overline{H}$, but now we have several natural options : the first one is to regard $H \simeq \mathbb{R}^3$ using the exponential map and the second one is to use the natural embedding of H in 3×3 matrices to obtain another identification with $\mathbb{R}^3$. We consider both choices, and one of the main results are a decomposition of the essential spectrum of the Schrödinger-type operator $T = -\Delta + V$, where V is a continuous real function on $\overline{H}$, for both types of compactifications. It is remarkable that we obtain *different* essential spectra for the resulting operators in the two compactifications, which shows that there are some significant complications in the case of non-commutative groups. My results extend the classical results of HVZ-type from Euclidean spaces to the Heisenberg group. While many features are preserved in the Heisenberg group case, there are also some notable differences. First, the action of H on itself extends to an action of H on $\overline{H}$ and we compute the quasi-orbits (the closure of the orbits) of the action, whose structure is more complicated in the Heisenberg case than in the Euclidean case. Following [44], we show that the essential spectrum of any operator T contained in (or affiliated to) $\mathcal{C}(\overline{H}) \rtimes H$ is the union of the spectra of a family $(T_\alpha)_{\alpha \in \mathcal{F}}$ of simpler operators indexed by the family $\mathcal{F}$ of quasi-orbits of $\overline{H} \smallsetminus H$ with respect to the action of H, that is, we recover again Equation (1.7). This part of the result holds for any compactification H, but the actual orbits differ depending on the chosen H-equivariant compactification.

Chapter 6 reproduces to a large extent the results of my fourth paper ("A comparison of the Georgescu and Vasy spaces associated to the N-body problems," joint work with Ammann and Nistor, completed, in final preparation for submission). More precisely, we show that the space introduced by Vasy in order to construct a pseudodifferential calculus adapted to the N-body problem can be obtained as the primitive ideal spectrum of one of the N-body algebras considered by Georgescu. In particular, we identify the spectra of the algebras considered in the first two papers (Chapters 3 and 4) as manifolds with corners obtained by successive blow-ups of the radial compactification $\overline{X}$ of the underlying space X along the traces of the collision planes on the sphere at infinity. In the process, we provide an alternative description of the iterated blow-up space of a manifold with corners with respect to a cleanly intersecting semilattice of adapted submanifolds (i.e. p-submanifolds).

Let me now describe in detail the results of my thesis by describing the results of each of these papers (all of which are included in this thesis, after some mathematically minor modifications to exclude redundancies).

1.3 Introduction to : “A refined HVZ-theorem for asymptotically homogeneous interactions and finitely many collison planes”

Let me now describe the results of my first paper, “A refined HVZ-theorem for asymptotically homogeneous interactions and finitely many collison planes,” joint work with Nistor and Prudhon. We continue the study begun by Georgescu and Nistor [27] of Hamiltonians of N-body type with interactions that are asymptotically homogeneous at infinity on a finite dimensional Euclidean space X. The Hamiltonians considered in that paper were obtained by a procedure (described below) that was employing *all* subspaces $Y \subset X$, whereas in this manuscript, we only consider those subspaces Y that belong to a suitable semilattice $\mathcal{S}$ of subspaces of X satisfying $X \in \mathcal{S}$. (Thus $Z_1 \cap Z_2 \in \mathcal{S}$ if $Z_1, Z_2 \in \mathcal{S}$.) Whenever possible, we follow the broad lines of [27]. Eventually, we shall assume that $\mathcal{S}$ is finite, but we begin with the general case. To fix ideas, let us mention right away an important example of a semilattice that arises in the study of quantum N-body problems. Namely, it is the semi-lattice $\mathcal{S}_N$ of subspaces of $X := \mathbb{R}^{3N}$ generated by the subspaces X and $\mathcal{P}_j$ and $\mathcal{P}_{ij}$ of Equation (1.5).

Let us fix a semilattice $\mathcal{S}$ with $X \in \mathcal{S}$. It turns out that the results in [27] on essential spectra and on the affiliation of operators carry through to this arbitrary semilattice $\mathcal{S}$. This is easy to see and is explained in this introduction. However, some important intermediate results on the representations of the cross-product algebras $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ that control the Fredholm property, are different in the general case. (See Equation (1.8) for the definition of the algebra $\mathcal{E}_{\mathcal{S}}(X)$.) A careful study of the representations of these algebras also allows us to sharpen the results on the essential spectra by removing the closure in the union of the spectra of the limit operators when $\mathcal{S}$ is finite. (See Theorem 1.3.1.)

Possible applications of the extensions presented in this paper (or chapter) are to regularity results and hence to numerical methods for the resulting Hamiltonians and the study of the fine structure of their spectrum. We include complete details of the proof that we can remove the closure in the union of the spectra in Theorem 1.3.1. The proof of this result may also be useful for other applications.

Let us now discuss the settings of the paper and state our first result on essential spectra, Theorem 1.3.1. For any real vector space Z, we continue to denote by $\overline{Z}$ its spherical compactification, already defined. For any subspace $Y \subset X$, $\pi_Y : X \to X/Y$ denotes the canonical projection. As in Equation (1.6), let $H := -\Delta + \sum_{Y \in \mathcal{S}} v_Y$, where $v_Y \in \mathcal{C}(\overline{X/Y})$ is seen as a function on X via the projection $\pi_Y : X \to X/Y$, the sum is over *all* subspaces $Y \subset X$, $Y \in \mathcal{S}$(we assume the sum to be convergent). One of the main result of [27] describes, in particular, the essential spectrum of H on $L^2(X)$ when $\mathcal{S}$ consists of *all* subspaces of X as $\sigma_{ess}(H) = \overline{\cup}_{\alpha \in \mathbb{S}_X} \sigma(\tau_\alpha(H))$, with the notation being the one used in Theorem 1.3.1. This result directly extends the celebrated HVZ theorem [13, 52, 57]. A first goal of this paper is to explain how the results and methods of [27] are affected by assuming that $\mathcal{S}$ is finite. We include also some extensions of the results in [27].

On a technical level, we obtain smaller algebras than the ones in [27], so the results on the affiliation of operators and on their essential spectra do not change. We will thus review just a small sample of results of this kind. On the other hand, for a possible further study of Hamiltonians of the form (1.6), it may be useful to have an explicit description of the spectra of the intermediate algebras involved (the algebras $\mathcal{E}_{\mathcal{S}}(X)$ and $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ introduced next). These spectra change dramatically in the case $\mathcal{S}$ finite. Concretely, let

$$\mathcal{E}_{\mathcal{S}}(X) := \langle \mathcal{C}(\overline{X/Y}) , \quad Y \in \mathcal{S} \rangle. \tag{1.8}$$

In other words, $\mathcal{E}_{\mathcal{S}}(X)$ is the closure in norm of the algebra of functions on X generated by all functions of the form $u \circ \pi_Y$, where $Y \in \mathcal{S}$ and $u \in \mathcal{C}(\overline{X/Y})$. Since X acts continuously by translations on $\mathcal{E}_{\mathcal{S}}(X)$, we can define the crossed product C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$, which can be regarded as an algebra of operators on $L^2(X)$. It is the algebra generated by operators of multiplication by functions in $\mathcal{E}_{\mathcal{S}}(X)$ and operators of convolution, that is, by operators of the form $m_f C_\phi$, where m_f is the operator of multiplication by $f \in \mathcal{E}_{\mathcal{S}}(X)$ and $C_\phi u(x) := \int_X \phi(y)u(x-y)dy$ is the operator of convolution by $\phi \in \mathcal{C}_c^\infty(X)$. Let $V \in \mathcal{E}_{\mathcal{S}}(X)$ (for instance, we could take $V := \sum_{Y\in\mathcal{S}} v_Y$, as in Equation (1.6)). We then obtain

$$(H+\mathrm{i})^{-1} \;=\; \big[(-\Delta+\mathrm{i})+V\big]^{-1} \;=\; (-\Delta+\mathrm{i})^{-1}[1+V(-\Delta+\mathrm{i})^{-1}\big]^{-1} \,\in \mathcal{E}_{\mathcal{S}}(X)\rtimes X\,. \quad (1.9)$$

This means that the operator H of Equation (1.6) is affiliated to $\mathcal{E}_{\mathcal{S}}(X)\rtimes X$. (see Definition 2.1.27). This is, in fact, one of the starting points of the theory developed by Georgescu and his collaborators [14, 15, 24, 26].

For each $x \in X$, we let $(T_x f)(y) := f(y-x)$ denote the translation on $L^2(X)$. Let $\mathbb{S}_X$ be the set of half-lines in X, that is

$$\mathbb{S}_X \;:=\; \{\,\hat{a},\; a\in X, a\neq 0\,\}\,, \quad (1.10)$$

where $\hat{a} := \{ra|\, r>0\}$. For any operator P on $L^2(X)$, we let

$$\tau_\alpha(P) := s-\lim_{r\to+\infty} T^*_{ra}PT_{ra}\,,\quad \text{if}\;\; \alpha=\hat{a}\in\mathbb{S}_X\,, \quad (1.11)$$

whenever the *strong* limit exists. We identify $\mathbb{S}_Z = \overline{Z}\setminus Z$ for any real vector space Z.

Theorem 1.3.1. *The operator H of Equation* (1.6) *is self-adjoint and affiliated to $\mathcal{E}_{\mathcal{S}}(X)\rtimes X$. Let H be any self-adjoint operator affiliated to $\mathcal{E}_{\mathcal{S}}(X)\rtimes X$ and $\alpha=\hat{a}\in\mathbb{S}_X$. Then the limit $\tau_\alpha(H) := s-\lim_{r\to+\infty} T^*_{ra}HT_{ra}$ exists and, if $\mathcal{S}$ is finite, and $0\in\mathcal{S}$, then*

$$\sigma_{\mathrm{ess}}(H) \;=\; \cup_{\alpha\in\mathbb{S}_X}\sigma(\tau_\alpha(H))\,.$$

Most of this theorem is (essentially) contained in [27], however, in that paper, only the relation $\sigma_{\mathrm{ess}}(H) \;=\; \overline{\cup}_{\alpha\in\mathbb{S}_X}\sigma(\tau_\alpha(H))$ was proven, but without restrictions on $\mathcal{S}$. This amounts to the fact that the family $\{\tau_\alpha\}$ is a *faithful* family of representations of $\mathcal{E}_{\mathcal{S}}(X)\rtimes X$. Our stronger result is obtained by showing that the family $\{\tau_\alpha\}$ is actually an *exhausting* family of representations of $\mathcal{E}_{\mathcal{S}}(X)\rtimes X$ (Theorem 3.4.5). We notice that if $0\notin\mathcal{S}$, then the part of the above theorem on the essential spectrum simply states that $\sigma_{\mathrm{ess}}(H)=\sigma(H)$, as H is among the operators $\tau_\alpha(H)$, since H is invariant with respect to a non-trivial subgroup of X.

One of the main points of Theorem 1.3.1 is that the operators $\tau_\alpha(H)$ are generally simpler than H (if $0\in\mathcal{S}$) and (often) easy to identify. The operators $\tau_\alpha(H)$ are sometimes called *limit operators*.

Here is a typical example. If $u : X\to\mathbb{C}$, we shall write $\text{av-lim}_\alpha\, u = c\in\mathbb{C}$ if $\lim_{a\to\alpha}\int_{a+\Lambda}|u(x)-c|\mathrm{d}x = 0$ for some (hence any !) bounded neighborhood of Λ of $0\in X$. Here $a\in X\subset\overline{X} := X\cup\mathbb{S}_X$, $\alpha\in\mathbb{S}_X$, and the convergence is in the natural topology of the spherical compactification $\overline{X}$ of X. For instance, let us assume that we are given real valued functions v_Y, $Y\in\mathcal{S}$, such that $\text{av-lim}_\alpha\, v_Y$ exists for all $\alpha\in\mathbb{S}_{X/Y}$ and $v_Y=0$ except for finitely many subspaces Y. Let $V := \sum_Y v_Y$. If $\alpha\not\subset Y$ then $\pi_Y(\alpha)\in\mathbb{S}_{X/Y}$ is a well defined half-line in the quotient and we may define $v_Y(\alpha) \;:=\; \text{av-lim}_{\pi_Y(\alpha)}\, v_Y$. Then Proposition 1.3 of [27] gives that

$$\tau_\alpha(H) \;=\; -\Delta + \sum_{Y\supset\alpha} v_Y + \sum_{Y\not\supset\alpha} v_Y(\alpha)\,. \quad (1.12)$$

For the usual N-body type Hamiltonians, we have that $v_Y : X/Y\to\mathbb{R}$ vanish at infinity. In that case $\tau_\alpha(H) \;=\; -\Delta+\sum_{Y\supset\alpha} v_Y$, which is the usual version of the HVZ theorem. This calculation remains valid for operators of the form (1.13).

For the result of Theorem 1.3.1 to be effective, we need some concrete examples of self-adjoint operators on $L^2(X)$ affiliated to $\mathcal{E}_{\mathcal{S}}(X)\rtimes X$. Let us briefly recall the affiliation criteria of [27] and see that they work in our setting as well.

Let $\mathcal{B}(\overline{X})$ be the set of functions $u \in L^\infty(X)$ such that the “averaged limits” av-$\lim_\alpha u$ (defined earlier) exist for any $\alpha \in \mathbb{S}_X$ and let $\mathcal{E}^\sharp_{\mathcal{S}}(X) \subset L^\infty(X)$ be the norm closed subalgebra of $L^\infty(X)$ generated by the algebras $\mathcal{B}(\overline{X/Y})$, when $Y \in \mathcal{S}$. Let h be a proper real function $h : X^* \to [0, \infty)$ (i.e. $|h(k)| \to \infty$ for $k \to \infty$). Also, let $\mathcal{F} : L^2(X) \to L^2(X^*)$ be the Fourier transform and $h(p) := \mathcal{F}^{-1} m_h \mathcal{F}$ be the associated convolution operator. We consider then $v \in L^1_{loc}(X)$ a real valued function such that there exists a sequence $v_n \in \mathcal{E}^\sharp_{\mathcal{S}}(X)$ of real valued functions with the property that $(1 + h(p))^{-1} v_n$ is convergent in norm to $(1 + h(p))^{-1} v$. Then

$$H \; := \; h(p) + v \tag{1.13}$$

is affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. This allows us to consider potentials v with Coulomb type singularities (in particular, unbounded). We also note the similar approach to magnetic Schrödinger operators [36, 44].

We now briefly describe the contents of this paper (Chapter 3). In Section 3.1, we introduce the basic algebras $\mathcal{E}_{\mathcal{S}}(X)$ and study radial limits at infinity for functions in these algebras. We also characterize the irreducible representations of the algebras $\mathcal{E}_{\mathcal{S}}(X)$ and $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$, which lead to our results on the essential spectrum, that is, to the Equation (1.7). In order to describe the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$, we introduce the concept of an $\mathcal{S}$-chain. Section 3.2 contains some results on the topology on the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$. In Section 3.3, we use the result of Section 3.2 to give a description of the spectrum of Georgescu’s algebra introduced in his study of the N-body problem. The final section 3.4 studies the crossed product algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. We use result of [63] in order to describe its primitive ideals space and to show that localizations at infinity provide an exhausting family of representations, which leads to the more precise result on the essential spectrum in Theorem 1.3.1.

1.4 Introduction to : “Exhaustive families of representations of C^*-algebras associated to the N-body Hamiltonians with asymptotically homogeneous interactions.”

This note (or chapter) extends the results of the previous chapter by relaxing the conditions on the family $\mathcal{S}$. We thus continue the analysis of algebras $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ introduced in Chapter 3 in order to study the N-body type Hamiltonians with interactions corresponding to a finite semilattice of vector subspaces. We consider, in this note, Hamiltonians of the form (1.6), where the subspaces $Y \subset X$ belong to some given family $\mathcal{S}$ of subspaces (not necessarily a semilattice). We develop new techniques to prove the result on the essential spectrum of the Hamiltonian in the case where $\mathcal{S}$ is a general family of subspaces that contains $\{0\}$ (so not necessarily a semilattice), and extend those results to other operators affiliated to a larger algebra of pseudodifferential operators associated to the action of X. In addition, we exhibit Fredholm conditions for such elliptic operators, and hence we obtain determinations of their essential spectrum of the type (1.7). Our results are valid for other interesting operators, such as the Dirac operators. We also consider the action of groups and then explain how the properties of the algebras that we consider answer a question of Melrose and Singer [39]. The question was how to build a suitable compactification of $\mathbb{R}^{3N}$ stable under the action of the permutation group $\mathfrak{S}_N$.

1.5 Introduction to : “Essential spectrum, quasi-orbits and compactifications : applications to the Heisenberg group”

We study the essential spectrum and Fredholm conditions for Schrödinger-type operators with homogeneous potentials at the infinity acting on $L^2(H)$, where H is the Heisenberg group. The approach of this paper is based on the articles [26, 27, 41, 44]. The techniques presented here also work for more general locally compact groups.

We consider $\overline{H}$ a compactification of H such that the algebra of continuous function on the compactification $\mathcal{C}(\overline{H})$ is separable. The compactification induces a natural family (R_α) of translations at the "infinity" . We also show that the action of H on itself extends to the compactification and we compute the quasi-orbits. Recall that a quasi-orbit is the closure of an orbit. We show that the family R_α is, in fact, an *exhaustive family* of morphisms for a suitable operator algebra contained in the Calkin algebra. In particular, for the operator $T = -\Delta + V$ with potential V, a continuous function on the compactification $\overline{H}$, we obtain :

$$\sigma_{ess}(T) = \bigcup_{\alpha \in \mathcal{F}} \sigma(T_\alpha).$$

Here $\mathcal{F}$ is a covering by quasi-orbits of the part at the infinity of the compactification and $T_\alpha = -\Delta + V_\alpha$, with V_α a simpler potential associated to a quasi-orbit corresponding of α. The operator algebra that we consider contains the resolvent of the Schrödinger-type operator and is generated by some integral operators. For more details on the concept of exhaustive family of morphisms, see [46] and [55]. The idea of considering the algebra of the resolvent operators is not new and has been well developed by Georgescu and others in [3, 26]. The method of spectral decomposition used here involved crossed product of C^*-algebras and has been applied to magnetic fields [42] and to the N-body problem [27, 40]. However, every crossed product of a commutative C^*-algebra can be viewed as a C^*-algebra of a groupoid [53]. Examples of C^*-algebras associated to groupoids, and more generally algebras of pseudo-differential operators on groupoids, can be found in [11, 16, 17] and the references therein. The advantage of the crossed product C^*-algebras is that most of them have the *quasi-regularity* property. The quasi-regularity allows us to express the spectrum of the crossed product C^*-algebra in term of quasi-orbits of the action of the group on the spectrum of the initial C^*-algebra. More details on quasi-regularity will be given in Section 5.3 and can be found in [63]. For results on Fredholm conditions and decompositions of the essential spectrum, see [10].

Let us briefly describe the contents of the Chapter 5. In Section 5.1, we discuss the quasi-regularity property. Thanks to Williams [63], we have convenient conditions that imply the quasi-regularity. In Section 5.2, we compute explicitly the quasi-orbits for two differents (but similar) compactification of H. Section 5.3 is devoted to the study of translations at the infinity and to the proof of the main theorem. The last section, Section 5.4, describes other kinds of algebras of potentials associated to repeated compactifications. We also discuss the relation between our results and a result of Power in [48] characterizing the spectrum of mixed algebras.

1.6 Introduction to : " A comparison of the Georgescu and Vasy spaces associated to the N-body problems"

We show that the space introduced by Vasy in [59, 60] in relation to the N-body problem coincides with the primitive ideal spectrum of the algebras considered by Georgescu and others in [15, 24, 27, 40]. Here we use the variant introduced in [40](see Chapter 3). The definitions of all these spaces is recalled in the main body of the paper.

The space considered by Vasy was defined using blow-ups of manifolds with corners. Let throughout this paper M be a manifold with corners. A submanifold will be called *closed* if it is a closed subset of the larger manifold in the sense of a topological spaces. Recall that a *p-submanifold* $P \subset M$ is a closed submanifold of M that has a suitable tubular neighborhood in M (Definition 6.1.7), and hence one that has a tubular neighborhood $P \subset U_P \subset M$ in M.

For any p-submanifold $P \subset M$, recall that the *blow-up* $[M : P]$ *of* M *with respect to* P, is defined by replacing P with the set $N_+^M P$ of interior directions in the normal bundle $N^M P$ of P in M (see [30, 37], or Definiton 6.2.1). This construction has some special properties if $P = \emptyset$ or if P contains an open subset of M. For this reason, a manifold with one of these two properties will be called a *trivial submanifold* of M and often will be excluded from our consideration.

Since Vasy's construction uses blow-ups, we begin this paper with their study. More precisely, we shall study and use the blow-up of a manifold with corners with respect to a *family* of p-submanifolds. If this family has clean intersections, the blow-up can be defined iteratively as in [1, 33, 37] and in other papers. Our method is based on an alternative definition of the blow-up with respect to a family of p-submanifolds. More precisely, let us consider a locally finite family $\mathcal{F}$ of *non-trivial* p-submanifolds of M and let $M_{\mathcal{F}} := M \smallsetminus \bigcup_{P\in\mathcal{F}} Y$ be the complement of all the submanifolds in $\mathcal{F}$. Then $M_{\mathcal{F}}$ is open and dense in M and is contained in each of the blow-up manifolds $[M : P]$, $P \in \mathcal{F}$. Then we define the *graph blow-up* $\{M : \mathcal{F}\}$ of M with respect to the family $\mathcal{F}$ as the closure

$$\{M : \mathcal{F}\} \;:=\; \overline{\delta(M_{\mathcal{F}})} \;\subset\; \prod_{P\in\mathcal{F}} [M : P]\,, \tag{1.14}$$

where δ is the diagonal embedding (see Definition 6.3.1 and the discussion following it for more details). For simplicity, in this paper, we shall consider the graph blow-up only with respect to a *locally finite* family of non-trivial p-submanifolds that are closed subsets of the ambient manifold.

In order to have a well-behaved graph blow-up, we shall impose some additional assumptions on $\mathcal{F}$. Recall that a family $\mathcal{S}$ of subsets of M is a *semilattice* (with respect to the inclusion) if, for all $P_1, P_2 \in \mathcal{S}$, we have $P_1 \cap P_2 \in \mathcal{S}$. Let then $\mathcal{S}$ be a semilattice of p-submanifolds of M and arrange its elements in an order $(\emptyset = P_0, P_1, P_2, \ldots, P_n)$ that is assumed to be *compatible with the inclusion*, in the sense that

$$P_i \subset P_j \;\Rightarrow i \le j\,. \tag{1.15}$$

We assume also that all non-empty members of $\mathcal{S}$ are non-trivial p-submanifold of M, that they are closed subsets of M, and that any finite subset of manifold in $\mathcal{S}$ intersect cleanly (in other words, we assume that $\mathcal{S}$ is a *cleanly intersecting semilattice*, see Definiton 6.4.3). Then we can successively blow-up M with respect to $(\emptyset = P_0, P_1, P_2, \ldots, P_n)$ by first doing so with respect to P_1, then doing so with respect to the lift of P_2, and so on, to obtain in the end the *iterated blow-up* $[M : \mathcal{S}]$ [1, 33, 37]. One of our main results is to prove that, if $\mathcal{S}$ is a finite cleanly intersecting semilattice, then there exists a natural diffeomorphism

$$[M : \mathcal{S}] \;\simeq\; \{M : \mathcal{S}\} \tag{1.16}$$

that is the identity on the common open subset $M_{\mathcal{S}} := M \smallsetminus \bigcup_{k=1}^{n} P_k$ (see Corollary 6.4.11). In particular, $[M : \mathcal{S}]$ is independent on the initially chosen order on $\mathcal{S}$, as long as it is compatible with the inclusion.

We apply these results to the study of the N-body problem in the following way. Let $\overline{X}$ denote the spherical compactification of a finite-dimensional vector space X, with boundary at infinity $\mathbb{S}_X := \overline{X} \smallsetminus X$, with smooth structure defined by the central projection of Lemma 6.5.1. Let $\mathcal{F}$ be a finite semilattice of linear subspaces of X. To $\mathcal{F}$ we associate the semilattice $\mathcal{S} := \{\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F}\} \cup \{\emptyset\}$. In this application to the N-body problem, the role of M will be played by $\overline{X}$. Vasy has considered the space $[\overline{X} : \mathcal{S}]$ in order to define a pseudodifferential calculus adapted to the N-body problem, see [59, 60] and the references therein. Inspired by Georgescu [15, 24, 27, 40], let us consider the norm closed algebra generated by all the spaces $\mathcal{C}(\overline{X/Y})$ in $L^\infty(X)$, with $Y \in \mathcal{F}$. This algebra will be denoted $\mathcal{E}_{\mathcal{S}}(X)$ (and not $\mathcal{E}_{\mathcal{F}}(X)$!) in what follows, in order to keep with the notation of [40]. It was proved in [40] that the spectrum $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ of $\mathcal{E}_{\mathcal{S}}(X)$ is the closure of the image of X in the product $\prod_{Y\in\mathcal{F}} \overline{X/Y}$. In this paper, we show that this closure coincides with $\{\overline{X} : \mathcal{S}\}$. Hence the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ introduced by Georgescu is canonically homeomorphic to the space $[\overline{X} : \mathcal{S}]$ introduced by Vasy, which will be henceforth called the "Georgescu-Vasy space."

Here are the contents of Chapter 6. In Section 6.1 we recall the concept of a manifold with corners using atlases. We also recall the concept of a p-submanifold. Then, in Section 6.2, we recall the definition of the blow-up of a manifold with corners by a p-submanifold, in particular, we describe its smooth structure. In Section 6.3, we present the notion of *graph blow-up* of a manifold with corners with respect to a rather general subfamily of manifolds. This notion is crucial to relate the results of Chapter 3 and Chapter 5. In Section 6.4, we show that for a good family of p-submanifolds, the graph blow-up and the iterated blow-up are diffeomorphic. As consequence, we recover results of Melrose, Vasy, and Kottke and the (almost) commutativity of the iterated blow-up. The last Section makes explicit the relation between the graph blow-up, the iterated blow-up, the spectrum of the algebra $\mathcal{E}_{\mathcal{S}}(X)$, and the N-body problem. In particular, we show that the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ introduced (essentially) by Georgescu and the iterated blow-up space $[\overline{X} : \mathcal{S} \cap \mathbb{S}_X]$ introduced by Vasy are canonically homeomorphic. This, on the one hand, defines a smooth structure on the Georgescu space and makes them more concrete and, on the other hand, provides natural group actions on the Vasy space, since the two spaces are homeomorphic (for the same semilattice of subspaces).

Première partie

Operator algebra and N-body problem

The first part of this manuscript contains results using mostly C^*-algebra techniques, with the goal, however, of determining the essential spectrum of operators appearing in Quantum Mechanics. Except for the chapter devoted to preliminary material ("Background material," next), the other chapters have the same titles as the papers on which they are based.

2

Background material

In this chapter, we have collected background material common to most of the four papers included in this thesis. More information can be found in [19, 47, 62, 63].

2.1 C^*-algebras

By $\mathcal{H}$ we will always denote a Hilbert space and by $\mathcal{B}(\mathcal{H})$ we will always denote the space of bounded operators acting on $\mathcal{H}$.

2.1.1 Definition and notations

Let us now recall some basics definitions on C^*-algebra and our main examples.

Definition 2.1.1. *A* C^*-algebra A *is an algebra over the field of complex number* $\mathbb{C}$ *with a norm* $\|.\|$ *and with a map* $* : A \to A$ *such that* A *is a Banach algebra and for every* $\lambda, \mu \in \mathbb{C}$ *and* $a, b \in A$*, we have*

(i) $(a^*)^* = a$,

(ii) $(ab)^* = b^*a^*$,

(iii) $(\lambda a + \mu b)^* = \overline{\lambda}a^* + \overline{\mu}b^*$,

(iv) $\|aa^*\| = \|a\|^2$.

The C^**-algebra is* commutative *if* $ab = ba$ *for all* $a, b \in A$*.*

Examples 2.1.2. Examples of C^*-algebras

(i) We endow the complex number $\mathbb{C}$ with $z^* = \overline{z}$, where $\overline{z}$ is the complex conjugate of z, and $\|z\| = |z|$, the module of z. With this choice, the complex numbers define a commutative C^*-algebra.

(ii) Let X be a locally compact space and $\mathcal{C}_0(X)$ be the space of continuous function $f : X \to \mathbb{C}$ such that f vanishes at the infinity. We define $f(x)^* = \overline{f(x)}$ and $\|f\| = \|f\|_\infty = \sup_{x \in X} |f(x)|$. With this involution and norm, $\mathcal{C}_0(X)$ is a commutative C^*-algebra.

(iii) Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the bounded operator acting on $\mathcal{H}$. The space $\mathcal{B}(\mathcal{H})$ is a C^*-algebra for the adjoint defined with respect to the inner product of $\mathcal{H}$ and the norm operator.

(iv) Let A be C^*-algebra and $B \subset A$ be subalgebra of A and closed for the norm and stable for the involution. The space B is a C^*-algebra and a C^*- subalgebra of A.

(v) Let A and B be two C^*-algebra. The cartesian product $A \times B$ is C^*-algebra for the canonical structure $(a,b)^* = (a^*, b^*)$ and $\|(a,b)\| = \sup\{\|a\|, \|b\|\}$.

For a C^*-algebra A and an element $a \in A$, the element a^* is the *adjoint* of a. If $a = a^*$, we say that a is *self-adjoint*. Let $B \subset A$, we define $B^* = \{b^*, b \in B\}$ the adjoint of B. If $B^* = B$, we say that B is self-adjoint or stable by involution.

If A and A' are two C^*-algebras, a $*$-*morphism* is a morphism for the algebra structure that preserve the involution : $\forall a \in A,\ \phi(a^*) = (\phi(a))^*$.

Lemma 2.1.3. *Let A be C^*-algebra, then its involution is isometric, namely,* $\forall a \in A, \|a\| = \|a^*\|$.

A Banach algebra with an isometric involution is an *involutive Banach algebra*. In view of Lemma 2.1.3, every C^*-algebra is a involutive Banach algebra. Hence the equality $\|aa^*\| = \|a\|^2$ is a more restrictive conditions. This leads to the definition of the C^*-norm.

Definition 2.1.4. *Let A be an involutive Banach algebra for the norm $\|.\|$. We say that $\|.\|$ is a C^*-norm if for every $a \in A$, $\|aa^*\| = \|a\|^2$.*

Given an involutive Banach algebra $(A, \|.\|)$, we can always define a new norm $\|.\|_{full}$ on A such that the closure of A for $\|.\|_{full}$ is a C^*-algebra. An example of the construction of $\|.\|_{full}$ in the particular case of the crossed-product is discussed in Section 2.3.

2.1.2 Spectrum of an element and unitarization

Definition 2.1.5. *Let A be a unital C^*-algebra. For each $a \in A$, we define* the spectrum of a in A *by*

$$\sigma_A(a) := \{\lambda \in \mathbb{C} \mid a - \lambda \textit{ is not inversible}\}. \tag{2.1}$$

If A is non-unital, we need to adapt Definition 2.1.5. The spectrum of an element will be defined in *a unitarization of A*. We shall need the definition of an *essential ideal* to define the unitarization. Recall that I is a two-sided ideal of a C^*-algebra A if $\forall a \in A,\ i \in I$ we have $ai \in I$ and $ia \in I$. Unless it is otherwise stated, all ideals we consider will be two-sided (see Subsection 2.1.4 for more details on ideals of a C^*-algebra).

Definition 2.1.6. *Let I be an ideal of a C^*-algebra A. The ideal is an* essential ideal *if for every J, non-trivial ideal of A, we have $J \cap I \neq \{0\}$.*

An equivalent definition of an essential ideal is if $a \in A$ such that $aI = \{0\}$ then $a = 0$.

Definition 2.1.7. *Let A be a non-unital C^*-algebra. We say that B is a* unitarization *of A if B is a unital C^*-algebra such that there exists $\phi : A \to B$ an injective morphism such that $\phi(A)$ is an essential ideal of B.*

Among all unitarizations, there exists two distinguished ones, the smallest one and the biggest one (for the inclusion). Details on the maximal unitarization require more definitions, which are discussed in Subsection 2.1.5. We now describe the smallest unitarization.

Example 2.1.8. Let A be a non-unital C^*-algebra. We consider $A^+ = A \times \mathbb{C}$ and we endow A^+ with the following C^*-algebra structure. For every $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$, we define

(i) $(a,\lambda) + (b,\mu) = (a+b, \lambda+\mu)$,
(ii) $(a,\lambda)(b,\mu) = (ab + \lambda b + \mu a, \lambda\mu)$,
(iii) $(a,\lambda)^* = (a^*, \overline{\lambda})$,
(iv) $L_{(a,\lambda)}(c) = ac + \lambda c$, for each $c \in A$,

(v) $\|(a,\lambda)\| = \sup\{L_{(a,\lambda)}(c)\|, c \in A, \|c\| \leq 1\}$.
In this case $(0,1)$ is the unity of A^+. The identification of A with $A \times \{0\}$ is injective morphism hence A^+ is unitarization of A and will use the notation $a + \lambda$ for the couple (a, λ).

Definition 2.1.9. *Let A be a non-unital C^*-algebra. For each $a \in A$, we define* the spectrum of a in A *by*

$$\sigma_A(a) := \sigma_{A^+}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \textit{ is not inversible in } A^+\}. \tag{2.2}$$

If A is a unital C^*-algebra, the C^*-algebra A^+ is isomorphic to $A \times \mathbb{C}$ (see (v) of Example 2.1.2). The isomorphism is the map $(a + \lambda e, \lambda) \mapsto (a, \lambda)$, where e is the unit of A. The isomorphism implies that

$$\sigma_{A^+}(a) = \sigma_A(a) \cup \{0\}.$$

2.1.3 Spectrum of a C^*-algebra

Spectrum of a commutative C^*-algebra

Every commutative C^*-algebra can identified with $\mathcal{C}_0(X)$ for the structure defined in (ii) Example 2.1.2. We give some details on this identification.

Let A be a C^*-algebra, a *character* of A is non-zero $*$-morphism $\chi : A \to \mathbb{C}$. If A is a commutative C^*-algebra, the set of all characters of A denoted by $\widehat{A}$ and is called the *spectrum of* A. We endow $\widehat{A}$ with the weak topology. The space $\widehat{A}$ becomes a locally compact space. The space $\widehat{A}$ is compact if and only if A is unital. The *Gelfand transformation* is defined as follow :

$$G : A \to \mathcal{C}_0(\widehat{A}),\ a \mapsto ev_a \text{ with } \forall \chi \in \widehat{A}, ev_a(\chi) = \chi(a). \tag{2.3}$$

Using the Gelfand transformation, we have the following results of characterisation of commutative C^*-algebra.

Theorem 2.1.10. *Let A be a commutative C^*-algebra. The Gelfand transformation of Equation* (2.3) *is an isometric isomorphism . This implies that*

1. *If A is unital, $A \simeq \mathcal{C}(\widehat{A})$,*
2. *If A is not unital, $A \simeq \mathcal{C}_0(\widehat{A})$.*

Moreover, for all $a \in A$,

$$\sigma(a) = \{\chi(a),\ \chi \in \widehat{A}\}. \tag{2.4}$$

Spectrum of a non-commutative C^*-algebra

We want to generalize the notion of the spectrum of a C^*-algebra to the non-commutative case. We will define two different candidates for this generalization and will compare them. To do this, we represent the C^*-algebra in the space of bounded operator of a Hilbert space : $\mathcal{B}(\mathcal{H})$. Another aim of a representation is to obtained a concrete mathematical object compare to the one of Definition 2.1.1.

Definition 2.1.11. *Let A be a C^*-algebra. A* representation of A *is a pair $(\pi, \mathcal{H})$ such that $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-morphism. The representation is* non-degenerate *if the set $\{\pi(a)\xi, a \in A, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.*

In the rest of this manuscript, we will only consider non-degenerate representations. If there is no ambiguity, we will only write π for a representation instead of the couple $(\pi, \mathcal{H})$. A main results of the theory of the C^*-algebra is the existence of a particular representation called the GNS representation in honor of Gelfand, Naimark and Segal.

Theorem 2.1.12 (GNS)**.** *Let A be a C^*-algebra. There exists $\pi : A \to \mathcal{B}(\mathcal{H})$ a representation of A such that π is injective.*

As immediate consequence of the Theorem GNS, we obtain

Corollary 2.1.13. *Every C^*-algebra is isomorphic and isometric to a closed subset stable by involution of $\mathcal{B}(\mathcal{H})$*

We want to study the representations of A, and hence we consider all representations of a C^*-algebra. However this is not a set, because there are "too many representations." We introduce thus the *set of equivalence classes of representations.*

Definition 2.1.14. *Let A be a C^*-algebra. We consider two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ of A. We say that two representation are equivalent if there exists an isomorphism $\phi : \mathcal{H} \to \mathcal{H}'$ such that*

$$\forall a \in A, \quad \phi \circ \pi(a) = \pi'(a) \circ \phi.$$

The set of class of equivalence classes representations of A *will be denoted by* $Rep(A)$.

We want to decompose a given representation in "elementary blocks."

Definition 2.1.15. *Let A be a C^*-algebra and $\pi \in Rep(A)$. A subspace V of $\mathcal{H}$ is* invariant *if*

$$\forall a \in A, v \in V, \quad \pi(a)v \in V.$$

The representation π is said irreducible *if its only closed invariant subspaces are $\{0\}$ and $\mathcal{H}$.*

We can now define the spectrum of C^*-algebra.

Definition 2.1.16. *Let A be a C^*-algebra. The set of equivalence classes of irreducible representation of A is called the* spectrum *of A and is denoted by $\widehat{A}$.*

There is a natural topology on $\widehat{A}$ see [19, Section 3.5] for more details.

Recall that for a commutative C^*-algebra, we define the spectrum $\widehat{A}$ as the set of characters of A. This definition is not in contradiction with Definition 2.1.16 because all irreducible representations of a commutative C^*-algebra are of dimension one (by the Schur lemma). Hence, every irreducible representation of a commutative C^*-algebra is a $*$-morphism with value in $\mathbb{C}$, so it is a character.

2.1.4 The space of primitive ideals : $\mathrm{Prim}(A)$

We recall basics fact about the closed ideals.

Proposition 2.1.17. *Let I be a closed two-sided ideal of a C^*-algebra A then*

1. *I is self-adjoint,*
2. *I is a C^*-algebra,*
3. *The quotient A/I is a C^*-algebra with the canonical norm $\|a + I\| = \inf_{i \in I} \|a + i\|$.*

Unless otherwise specified, all ideal will be closed. If $\phi : A \to B$ is a $*$-morphism between two C^*-algebra, then $\ker(\phi)$ is an ideal of A.

Definition 2.1.18. *Let A be a C^*-algebra and recall that $\widehat{A}$ is the set (of class of equivalence) of irreducible representation of A. We define*

$$Prim(A) \;:=\; \{\ker(\pi),\ \pi \in \widehat{A}\}. \tag{2.5}$$

Two different morphisms can have the same kernel hence we have a onto (but not necessary into) map $\widehat{A} \twoheadrightarrow \mathrm{Prim}(A)$. When A is a commutative C^*-algebra, $\widehat{A}$ and $\mathrm{Prim}(A)$ are in bijection. We endow $Prim(A)$ with a natural topology. This topology is defined using a Kuratowski closure operator.

Definition 2.1.19. *Let X be a non empty set and $\mathcal{P}(X)$ the power set of X. Let $cl : \mathcal{P}(X) \to \mathcal{P}(X)$, the application cl is a Kuratowski closure operator if :*

1. $cl(\emptyset) = \emptyset$,
2. *For every $A \in \mathcal{P}(X), A \subseteq cl(A)$,*
3. *For every $A, B \in \mathcal{P}(X), cl(A \cup B) = cl(A) \cup cl(B)$,*
4. *For every $A \in \mathcal{P}(X), cl(cl(A)) = cl(A)$.*

Lemma 2.1.20. *Let X be a non empty set and cl a Kuratowksi closure operator. We consider*

$$\mathcal{F} := \{A \in \mathcal{P}(X) | cl(A) = A\}. \tag{2.6}$$

Then $(X, \mathcal{F})$ is topological space where the closed sets are the elements of $\mathcal{F}$.

Definition 2.1.21. *Let A be a C^*-algebra and we consider $T \subset Prim(A)$. Let $I(T) := \cap_{J \in T} J$. As intersection of ideals of A, $I(T)$ is an ideal of A. We define*

$$cl(T) = \{J \in Prim(A) | I(T) \subset J\}. \tag{2.7}$$

The Jacoboson topology on $Prim(A)$ is the topology induces by the cl in (2.7)*, which is a Kuratowski operator.*

To show that cl fulfilled the axiom of Definition 2.1.19 see [19, Chap 3]. The Jacobson topology can be defined on any algebra A over a commutative field.

For I an ideal of a C^*-algebra A, we want to decompose $Prim(A)$ using $Prim(A/I)$ and $Prim(I)$. We shall need the notation :

$$Prim_I(A) = \{J \in Prim(A) \mid I \subset J\}, \quad Prim^I(A) = Prim(A) \setminus Prim_I(A). \tag{2.8}$$

Lemma 2.1.22. *Let I be an ideal of a C^*-algebra A.*

1. *The map $J \mapsto J/I$ is a homeomorphism between $Prim_I(A)$ and $Prim(A/I)$.*
2. *The map $J \mapsto J \cap I$ is a homeomophism between $Prim^I(A)$ and $Prim(I)$.*

Moreover, we have the exact sequence :

$$\{0\} \longrightarrow Prim(I) \longrightarrow Prim(A) \longrightarrow Prim(A/I) \longrightarrow \{0\} \tag{2.9}$$

Note that for an ideal I, $cl(I) = \mathrm{Prim}_I(A)$ hence $\mathrm{Prim}_I(A)$ is closed and $\mathrm{Prim}^I(A)$ is open. The topology on $\mathrm{Prim}(A)$ can be very singular. For example, if we consider the closure of $\mathrm{Prim}^I(A)$ when I is an essential ideal.

Lemma 2.1.23. *Let I be an ideal of a C^*-algebra A. We have the equivalence between*

1. *I is an essential ideal of A,*
2. $\mathrm{Prim}^I(A)$ *is dense in* $\mathrm{Prim}(A)$.

An immediate consequence, we have that if I is a primitive ideal and an essential then the singleton $\{I\}$ is dense in $\mathrm{Prim}(A)$. For example, if $\{0\}$ is primitive as ideal, then $\{0\}$ is dense in $\mathrm{Prim}(A)$.

2.1.5 Multiplier algebra

There exist several definitions of the multiplier algebra. We give two different definitions and then we focus on the case of commutative C^*-algebras. See [62, Chapter 2] for details on the multiplier algebra.

Definition 2.1.24. *Let A be a non-unital C^*-algebra. The* multiplier algebra *of A, denoted $\mathcal{M}(A)$, is the largest unitarization of A defined by this universal property. For every unitarization B of A with $\phi : A \to B$ an embedding of A in B as an essential ideal of B, there exists $\psi : B \to \mathcal{M}(A)$ such that $\iota = \psi \circ \phi$, where ι is the embedding of A in $\mathcal{M}(A)$. We can summarize this property by the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow^{\iota} & \downarrow \psi \\ & & \mathcal{M}(A) \end{array} \quad . \tag{2.10}$$

Proposition 2.1.25. *Let A be a non-unital $\mathcal{M}(A)$ and $\pi : A \to \mathcal{B}(\mathcal{H})$ the GNS representation of A. We identify A and $\pi(A)$.*

$$B := \{T \in \mathcal{B}(\mathcal{H}) | TA \subset A \text{ and } AT \subset A\} \tag{2.11}$$

The multiplier algebra $\mathcal{M}(A)$ and B are isomorphic as C^*-algebras.

Proposition 2.1.26. *We consider a commutative non-unital C^*-algebra $A = \mathcal{C}_0(\Omega)$. The multiplier algebra of A is isomorphic to $\mathcal{C}_b(\Omega)$, the continuous and bounded functions on Ω.*

2.1.6 Affiliated operators

In view of Theorem 2.1.12, every C^*-algebra is a subalgebra of $\mathcal{B}(\mathcal{H})$. Hence the elements of a C^*-algebra are only bounded operator. To study the unbounded operator, we shall the notion of affiliated operators.

Definition 2.1.27. *Let P be a self-adjoint, not necessarily bounded operator acting on $\mathcal{H}$. We consider a C^*-algebra A represented in $\mathcal{B}(\mathcal{H})$. We say that P is affiliated to A, writing $P \in' A$, if for all $\varphi \in \mathcal{C}_0(\mathbb{R})$ the operator $\varphi(P) \in A$.*

We have the convenient criteria to show that an operator if affiliated to a C^*-algebra.

Lemma 2.1.28. *The self adjoint operator P is affiliated to A if and only if $(P - z)^{-1} \in A$ for some z in the resolvent set of P.*

Of course every self-adjoint operator is affiliated to $\mathcal{B}(\mathcal{H})$, however, we are interested in smaller C^*-algebras. If $P \in' A$ and $\phi : A \to B$ is a morphism of C^*-algebras, then $\phi(P)$ is defined. More details on the notion of affiliated operator can be found in [22] and reference therein, this article contains a generalization of affiliated operators for densely defined and non self-adjoints operators. Moreover affiliated operators have been also studied in [5] and [64].

2.2 Compactification

In this section, we detail the concept of compactification of a locally compact space. We stress two advantages of the process of the compactification. The first advantage, using the Gelfand transform, we can make the parallel between the process of compactified of a locally compact Ω space and the unitarization of the commutative C^*-algebra $\mathcal{C}_0(\Omega)$. The second advantage is that a continuous function V on a compactification can be view as a continuous function on Ω with a controlled behavior at the infinity.

Definition 2.2.1. *Let Ω be a locally compact space. A* compactification of Ω *is a space $\overline{\Omega}$ such that :*

1. *The space $\overline{\Omega}$ is compact,*
2. *Ω is contained in $\overline{\Omega}$ as open dense subspace.*

We will denote by $\Omega_\infty = \overline{\Omega} \setminus \Omega$, the points added point by the compactification. The elements of Ω_∞ we be called the points at infinity *of the compactification.*

By Condition 2. of Definition 2.2.1, the space Ω_∞ is a closed subspace of $\overline{\Omega}$ hence a compact space.

2.2.1 Compactifications and commutative C^*-algebras

Recall by Theorem 2.1.10 that every commutative C^*-algebra is of the form $\mathcal{C}_0(\Omega)$ for Ω a locally compact space.

Lemma 2.2.2. *We keep the notation of Definition 2.2.1. Let $i : \mathcal{C}_0(\Omega) \to \mathcal{C}(\overline{\Omega})$ be the inclusion map where $i(f)$ is the extension by 0 of the function f and let $j : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\Omega_\infty)$ be the restriction map. We have the following exact sequence :*

$$\{0\} \longrightarrow \mathcal{C}_0(\Omega) \xrightarrow{\ i\ } \mathcal{C}(\overline{\Omega}) \xrightarrow{\ j\ } \mathcal{C}(\Omega_\infty) \longrightarrow \{0\}\ . \tag{2.12}$$

We make the parallel between the compactification and the unitarization of a C^*-algebra.

Proposition 2.2.3. *Let $A = \mathcal{C}_0(\Omega)$ and $A' = \mathcal{C}(\Omega')$ be two commutative C^*-algebras such that A is non-unital and A' is unital. We have the following equivalence :*

1) *A' is a unitarization of A.*
2) *Ω' is a compactification of Ω.*

Proof. By lemma 2.1.23, we know that an essential ideal is dense in $\mathrm{Prim}(A)$. In the commutative case, the spectrum and $\mathrm{Prim}(A)$ coincide. □

2.2.2 Examples of compactification

We give some typical examples of compactifications and describe the associated topologies and the sets of continuous functions.

The one-point compactification

Let Ω be a locally compact space. The one-point compactification or the Alexandrov compactification is $\Omega^+ = \Omega \sqcup \{\infty\}$. The open sets of Ω^+ are the open sets of Ω and $K^c \sqcup \{\infty\}$, where K is a compact subset of Ω. This compacitification is the smallest one for the inclusion. A function $f \in \mathcal{C}(\Omega^+)$ is a continuous function on Ω that have a finite limit at infinity. Moreover, we can show that $\mathcal{C}(\Omega^+) \simeq (\mathcal{C}_0(\Omega))^+$ as C^*-algebras.

The spherical compactification

Let X be a finite dimensional real vector space, its *radial compactification*, noted $\overline{X}$ is the set $\overline{X} := X \sqcup \mathbb{S}_X$, where $\mathbb{S}_X$ is the set of directions of X, that is, the set of semi-lines $\mathbb{R}_+ * v$, $v \in X \smallsetminus \{0\}$. Thus, if $X \simeq \mathbb{R}^n$, then $\mathbb{S}_X \simeq \mathbb{S}^{n-1}$. A positive cone of X is a subset C of X stable by $\mathbb{R}_+$. An open truncated cone $C^\dagger$ is $C^\dagger = C \cap (B(0,r))^c$, where C is a positive cone and $(B(0,r))^c$ is the complementary of an open ball of radius $r > 0$. The open subsets of $\overline{X}$ are the open subsets of X and the truncated open cones. A function $f \in \mathcal{C}(\overline{X})$ is a continuous function on X that have radial limit. More precisely, for each $x \in X$, $x \neq 0$, the limit $\lim\limits_{r \to \infty} f(rx)$ is finite.

The Stone-Čech compacitfication

Let Ω be a locally compact space, the Stone-Čech compactification of Ω, noted $\beta\Omega$ is the maximal compactification of Ω. It is defined by the following universal propriety : Let $\iota : \Omega \to \beta\Omega$ be the canonical inclusion. For any K compact set and any continuous map $\phi : \Omega \to K$ such that ϕ is an homeomorphism between Ω and $\phi(\Omega)$, there exits a continuous map $\psi : \beta\Omega \to K$ such that the following diagram is commutative.

$$\begin{array}{ccc} \Omega & \xrightarrow{\ \phi\ } & K \\ & \searrow^{\iota} & \uparrow{\scriptstyle \psi} \\ & & \beta\Omega \end{array} \tag{2.13}$$

The set $\beta\Omega$ is the spectrum of the multiplier algebra of $\mathcal{M}(\mathcal{C}_0(\Omega))$. By Proposition 2.1.26, we have $\mathcal{M}(\mathcal{C}_0(\Omega)) \simeq \mathcal{C}_b(\Omega)$. Using the Gelfand isomorphism, we obtain $\mathcal{C}(\beta\Omega) \simeq \mathcal{C}_b(\Omega)$. In other words, any continuous function on the Stone-Čech compactification can be view as a continuous and bounded function on Ω.

2.3 C^*-dynamical systems

We recall now some basics facts concerning C^*-dynamical systems and their associated crossed products, see [47], [63] for more details.
A *C^*-dynamical system* consists of a C^*-algebra A and a locally compact group G with a strongly continuous action $\theta : G \to Aut(A)$. Let $\mathcal{LUC}(G)$ be the C^*-algebra of left uniformly continuous functions on G. We focus our study on the case when A is an unital C^*-subalgebra of $\mathcal{LUC}(G)$ and $\theta_y(f)(x) = f(y^{-1}x)$ for $x, y \in G$. Note that, with this choice of A and θ, the C^*-algebra A is commutative and we have to assume that A is invariant by translation. We consider $L^1(G, A)$, the Bochner space of integrable functions. The norm on $L^1(G, A)$ is defined using the norm of A. We endow $L^1(G, A)$ with a structure of $*$-algebra with the product and the involution define by :

$$\phi * \psi(x) = \int_G \phi(y)\theta_y[\psi(y^{-1}x)]d\mu(y), \quad \phi^*(x) = m(x)^{-1}\theta_x[\phi(x^{-1})^*], \tag{2.14}$$

where $\phi, \psi \in L^1(G, A), x \in G$ and μ is the Haar measure on G, and m is the modular function of G. The *(full) crossed-product,* $A \rtimes_\theta G$ is defined as the completion of $L^1(G, A)$ for the norm $||\phi|| := sup_\Pi ||\Pi(\phi)||$, where the supremum is taken over all non-denegerate $*$-representation $\Pi : L^1(G, A) \to \mathcal{B}(\mathcal{H})$. There exists another natural completion of $L^1(G, A)$ called the *reduced crossed product* but we only need the full crossed-product in this manuscript. Indeed, we consider amenable group, for which the reduced and full crossed-product are isomorphic.
To characterize the representations of $A \rtimes_\theta G$, we need the notion of covariant pair.

Definition 2.3.1. *For a C^*-dynamical system (A, θ, G), a triplet $(\pi, U, \mathcal{H})$ is* covariant pair *of (A, θ, G) if :*

- *$\mathcal{H}$ is a Hilbert space,*
- *$\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*-$representation of A,*
- *$U : G \to \mathcal{B}(\mathcal{H})$ is a strongly continuous unitary morphisms,*
- *for all $a \in A, g \in G$, we have $U(g)\pi(a)U(g^{-1}) = \pi(\theta_g(a))$.*

If there is no ambiguity, we will drop the Hilbert space $\mathcal{H}$ and simply write (π, U) for a covariant pair.

Definition 2.3.2. *Let (A, θ, G) be a C^*-dynamical system and (π, U) be a covariant pair. To (A, θ, G), we associate a representation $\pi \rtimes U : A \rtimes_\theta G \to \mathcal{B}(\mathcal{H})$ called the* integrated form *of (π, U). This* *integrated form is defined by :*

$$\pi \rtimes U(f) = \int_G \pi(f(x)) U_x d\mu(x).$$

The following proposition corresponds to Proposition 2.40 in [63].

Proposition 2.3.3 (Williams)**.** *Let (A, θ, G) be a dynamical system with A not necessarily commutative. Every non-degenerate representation Π of $A \rtimes_\theta G$ comes from a unique integrated form of a covariant pair of the dynamical system (A, θ, G). That is $\Pi = \pi \rtimes U$, where the pair (π, U) is a covariant representation of the (A, θ, G) and the pair (π, U) is unique (up to equivalence).*

If A is a unital C^*-subalgebra of $\mathcal{LUC}(G)$, any function $\phi : G \to A$ is identified with a function $G^2 \to \mathbb{C}$. Let $x, y \in G$, we will use the notation $\phi(x; y) = [\phi(x)](y)$ to keep in mind the dependence on the two variables. The following proposition gives a convenient representation of the crossed-product $A \rtimes G$.

Proposition 2.3.4. *Let $\phi \in L^1(G, A)$, we define $Sch(\phi)$, an operator on $L^2(G)$, by*

$$[Sch(\phi) f](x) = \int_G \phi(x; y) f(y^{-1}x) d\mu(y).$$

where $f \in L^2(G)$ and $x \in G$. The application Sch can be extended from $L^1(G, A)$ to $A \rtimes_\theta G$. The extension of Sch is a faithful representation of $A \rtimes_\theta G$ on $\mathcal{B}(L^2(G))$.

The preceding proposition corresponds to the Proposition 7.9 in [43].

2.3.1 The C^*-algebra of a group

We consider the particular C^*-dynamical system $(\mathbb{C}, \theta, G)$ with $\theta_y(f)(x) = f(y^{-1}x)$ for $f \in L^1(G, \mathbb{C}) = L^1(G)$. In this case, the crossed product $\mathbb{C} \rtimes G$ is called *the full C^*-algebra of the group G* and is noted $C^*_{full}(G)$ or $C^*(G)$. By definition, $C^*(G)$ is just the completion of $L^1(G)$ for the universal norm.

2.3.2 The case $G = \mathbb{R}^n$

Assume $G = \mathbb{R}^n$, regarded as commutative group. We denote by $\mathcal{B}(L^2(G))$ the algebra of bounded linear operator on $L^2(G)$ and by $\mathcal{K}(G)$ the subalgebra of compact operators of $\mathcal{B}(L^2(G))$.

Recall that $\mathcal{C}_b(G)$ be the algebra of bounded continuous functions on G with complex values, $\mathcal{C}_c(G)$ the subalgebra of functions with compact support and $\mathcal{C}_0(G)$ the subalgebra of functions that go to zero at infinity.

Any measurable function $\phi : G \to \mathbb{C}$ induces an operator of multiplication by ϕ acting on $L^2(G)$. To avoid the ambiguity, the operator of multiplication will be denoted by $\phi(q)$. We also introduces the operator $\phi(p)$ defined using the Fourier transform : $\phi(p) := \mathcal{F}^{-1}\phi(q)\mathcal{F}$. When $\phi \in \mathcal{C}_0(G)$, the operator $\phi(p)$ is an element of $\mathcal{B}(L^2(G))$. There is a natural action by translations of G on $\mathcal{C}_b(G)$, we denote by τ_x this translation for each $x \in G$. With this notation, we can describe the crossed product $\mathcal{A} = A \rtimes G$, where A is a C^*-algebra of functions on G such that $1 \in A$ and $\tau_x(A) \subset A$ for every $x \in X$. Moreover, we suppose that $\mathcal{C}_0(G) \subset A \subset \mathcal{C}_b(G)$. We shall denote by $\langle S \rangle$ the norm closed algebra generated by a subset S of a Banach algebra. With the last assumption, $A \rtimes G$ can be represented into $\mathcal{B}(L^2(G))$ by the following norm closure of linear subspace stable by involution :

$$\langle \phi(q)\psi(p),\ \phi \in A,\ \psi \in \mathcal{C}_0(G^*) \rangle. \tag{2.15}$$

See the remark 2.3.6 to make the connection between the representation $\phi(q)\psi(p)$ and Proposition 2.3.4.We recall that $\widehat{A}$ is compact if and only if A has an unit and there is an isomorphism between A and $\mathcal{C}(\widehat{A})$. Each element $x \in G$ can be identified with a character χ_x of A defined by $\chi_x(\phi) = \phi(x)$. The C^*-algebra is a unitarization of $\mathcal{C}_0(G)$ hence $\widehat{A}$ is a compactification of G and let $\mathcal{A}^\dagger$ be the part at infinity of this compactification space, that is

$$A^\dagger = \widehat{A} \setminus G = \{\varkappa \in \widehat{A} \mid \varkappa(\phi) = 0, \forall \phi \in \mathcal{C}_0(G)\}$$

Let $\varkappa \in A^\dagger$ and let $\phi \in A$, then there exists a net $(x_i)_{i\in I}$ of elements of G such that $\lim_{i\in I} x_i = \varkappa$. Using this net, we can define $\tau_\varkappa$, the translation at the infinity by $\varkappa$, that is :

$$\tau_\varkappa[\phi](x) = \lim_{i\in I} \tau_{x_i}[\phi](x) = \lim_{i\in I} \phi(x + x_i) = \phi(x + \varkappa).$$

We can extend the translations τ_x and $\tau_\varkappa$ from A to $A \rtimes G$ such these extensions leave invariant the operator $\psi(p)$ and acts on $\phi(q)$ as on ϕ viewed as a element of A. We keep the same notation for the extension. For $P \in A \rtimes G$, we have $\tau_x(P) = e^{ixp}Pe^{-ixp}$, where e^{ixp} is the translation operator by x and

$$\tau_\varkappa(P) = \lim_{j\in I} e^{ix_jp}Pe^{-ix_jp} := P_\varkappa.$$

The convergence holds for the topology induces by the family of semi-norms $||P||_\theta = ||P\theta(q)||$, with $\theta \in \mathcal{C}_0(G)$ on $\mathcal{B}(L^2(G))$. We have the following theorem in [22].

Theorem 2.3.5 (Georgescu)**.** *For any operator $P \in A \rtimes G$, we have*

$$\sigma_{ess}(P) = \bigcup_{\varkappa\in A^\dagger} \sigma(P_\varkappa).$$

Remark 2.3.6. There exists a natural identification between $A \rtimes_{full} G$ and the projective tensor product $A\overline{\otimes}L^1(G)$. This identification is given by the extension of $(Id_A \otimes \mathcal{F})$ to the crossed product. The assumption $1 \in A$ allow us to consider $Sch(1 \otimes \psi)$ for $\psi \in L^1(G)$. In this case $Sch(1 \otimes \psi)$ is an operator of convolution acting on $L^2(G)$. The discussion in Section 4 of [26] makes explicit the bijection between the convolution operator by functions of $L^1(G)$ and the operator $\psi(p)$ for $\psi \in \mathcal{C}_0(G)$. The equality $Sch(\phi \otimes \psi) = \phi(q)\psi(p)$ follows then easily.

Remark 2.3.7. The category of commutative C^*-algebras with $*$-morphisms and the category of locally compact spaces with proper functions are equivalent, via the isomorphism $A \simeq \mathcal{C}_0(\widehat{A})$. A C^*-dynamical system (A, θ, G), where A is a commutative C^*-algebra, can be defined by a triplet (Ω, θ, G), where θ is a continuous action of the group G on the locally compact space $\Omega = \widehat{A}$. In the case when A is a commutative and unital C^*-algebra, the corresponding space Ω is compact and we will call the triplet (Ω, θ, G) a compact dynamical system. We will often navigate between the C^*-algebra formalism and the locally compact space formalism for C^*-dynamical system.

For a compact dynamical system (Ω, θ, G), we will denote by O^ω, the orbit of ω. The closure of O^ω is called the *quasi-orbit* of ω and we will denote it by Q^ω.

2.4 Essential spectrum

The HVZ-theorem and a result of this manuscript describe the essential spectrum of an Hamiltonian associated to the N-body problem. We give details on the definition of the essential spectrum.

2.4.1 Compact operator and Calkin algebra

We recall the definition of compact operators and of the Calkin algebra. We establish the connection between the essential spectrum of an operator, the Calkin algebra and the Fredholm operators. We also realize the algebra compact operators on a Lie group as a crossed product.

Definition 2.4.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is* compact *if for any bounded sequence $(\xi_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$, we can extract a convergent subsequence of $(T\xi_n)_{n\in\mathbb{N}}$. The set of compact operator is denoted by $\mathcal{K}(\mathcal{H})$.*

We have the following equivalent characterization.

Lemma 2.4.2. *We have $T \in \mathcal{K}(\mathcal{H})$ if, and only if, there exists a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ of operator of finite rank such $\lim_{n\to+\infty} T_n = T$. The convergence is in the operator norm.*

Recall that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. The set $\mathcal{K}(\mathcal{H})$ has a particular structure.

Proposition 2.4.3. *The set $\mathcal{K}(\mathcal{H})$ is an essential ideal of $\mathcal{B}(\mathcal{H})$.*

In particular, $\mathcal{K}(\mathcal{H})$ is a non-unital C^*-algebra and using Proposition 2.1.25 the multiplier of $\mathcal{K}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$. See [19, Corollary 4.1.7] for the following standard result.

Lemma 2.4.4. *The algebra of compact operators has (up to equivalence) only one irreducible representation, which is hence injective. In particular, $\mathrm{Prim}(\mathcal{K}(\mathcal{H}))$ consists of a single element, the zero ideal.*

As an ideal, we can consider the quotient by the compact operator.

Definition 2.4.5. *The* Calkin algebra *is the quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.*

We can now define the essential spectrum.

Definition 2.4.6. *Let $T \in \mathcal{B}(\mathcal{H})$. The* essential spectrum*, of T, noted $\sigma_{ess}(T)$, is the spectrum of the image of T in the Calkin algebra. That is*

$$\sigma_{ess}(T) := \sigma_{\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not inversible modulo a compact operator } \}.$$

The null operator is compact hence for any operator we have $\sigma_{ess}(T) \subset \sigma(T)$.

Let G be a locally compact amenable group. We denote by $\mathcal{K}(G)$ the algebra of compact operators of the Hilbert space $L^2(G)$. We consider the C^*- dynamical system $(\mathcal{C}_0(G), \tau, G)$ and where τ is the action by translation of G on $\mathcal{C}_0(G)$. By [63, Theorem 4.24], we have

Theorem 2.4.7. *Let G be an amenable locally compact group. We have*

$$\mathcal{C}_0(G) \rtimes G \simeq \mathcal{K}(G). \tag{2.16}$$

2.4.2 Fredholm operator

The essential spectrum can be also defined using the notion of Fredholm operators.

Definition 2.4.8. *Let T an operator acting on $\mathcal{H}$. The operator T is a* Fredholm operator *if these two conditions are fulfilled :*

1. $\dim(ker\ T)$ *is finite,*
2. $\dim(\mathrm{coker}\ T) = \dim(\mathcal{H}/\ker\ T)$ *is finite.*

An important result for Fredholm operators is Atkinson's theorem

Theorem 2.4.9. *Let T be an operator acting on a Hilbert space. We have the equivalence between*

1. *T is a Fredholm operator,*
2. *T is inversible in the Calkin algebra.*

As an immediate consequence, we have another characterization of the essential spectrum.

Corollary 2.4.10. *Let T be an operator acting on an Hilbert space. We have*

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not a Fredholm operator}\}. \tag{2.17}$$

2.5 Exhaustive families of morphisms of a C^*-algebra

In [46, 55], the authors introduce the concept of *exhaustive families* of C^*-algebra. This concept is very useful to obtain a decomposition of spectrum of an affiliated operator at C^*-algebra. We used the exhaustive families each chapter except the last one.

Let $\phi : A \to B$ be a $*$-morphism between two C^*-algebra, we define *the support of ϕ* by $\mathrm{supp}(\phi) := \mathrm{Prim}^{ker(\phi)}(A)$.

Definition 2.5.1. *A family $(\phi_i)_{i\in I}$ of morphisms of a C^*-algebra A is said to be* exhaustive *if*

$$\mathrm{Prim}(A) = \cup_{I\in I}\mathrm{supp}(\phi_i).$$

Or equivalently, $(\phi_i)_{i\in I}$ is an exhaustive families if any primitive ideal contains at least one $\ker \phi_i$ *for some $i \in I$.*

To obtain a decomposition of the spectrum of an element, we shall need the notion of *spectral families*.

Definition 2.5.2. *A family $(\phi_i)_{i\in I}$ of morphisms of a unital C^*-algebra A is said to be* strictly spectral *if*

$$\forall a \in A, \quad \sigma(a) = \cup_{i\in I}\sigma(\phi_i(a))$$

We give results of [46, 55].

Proposition 2.5.3. *If $(\phi_i)_{i\in I}$ is an exhaustive family of a C^*-algebra A, then $(\phi_i)_{i\in I}$ is a spectral family of A.*

In this manuscript, we show that suitable families of morphisms are exhaustive families of morphisms because it is an easier condition to check rather than the condition of being a spectral families. Then, we use Proposition 2.5.3 to obtain the decomposition of the spectrum.

Recall that for a closed ideal I of a C^*-algebra A, every $*$-representation $\phi : I \to \mathcal{B}(\mathcal{H})$ extends to a morphism $A \to \mathcal{B}(\mathcal{H})$.

Proposition 2.5.4. *Let I be closed ideal of a C^*-algebra A. We consider*

1. *$(\phi_j)_{j\in J}$ an exhaustive family of I.*
2. *$(\widetilde{\phi_j})_{j\in J}$ the induced family of $*$-morphism defined on A.*
3. *$(\psi_k)_{k\in K}$ an exhaustive family of A/I.*
4. *$\pi_I : A \to A/I$ the canonical projection.*

Then the union of the two families $(\widetilde{\phi_j})_{j\in J}$ and $(\psi_k(\pi_I))_{k\in K})$ is an exhaustive family of A.

As immediate consequence, we obtain

Corollary 2.5.5. *We keep the notation of Proposition2.5.4. Let P be an operator affiliated to A, then*

$$\sigma(P) \;=\; \cup_{j\in J}\sigma(\widetilde{\phi_j(P)})\bigcup\cup_{k\in K}\sigma(\psi_k(\pi(P)). \tag{2.18}$$

2.6 Operators on a Lie group

2.6.1 The Laplacian on a Lie group

This section is dedicated to a convenient presentation of the Laplacian on the Lie group and to establish connection with the notion of affiliated operators to a C^*-algebra. Let G be a Lie group. We denote by $Lie(G)$ the Lie algebra of G. It consists of first-order differential operators without constant term acting on $\mathcal{C}_c^\infty(G)$ and commuting with the right translation. We take $X_1, \ldots, X_n$ an orthonormal basis of $Lie(G)$ with respect to an right invariant metric and we consider

$$L = \sum_{i=1}^{n} X_i^2$$

The operator L is an unbounded operator acting on $L^2(G)$ with core $\mathcal{C}_c^\infty(G)$, which is dense in $L^2(G)$. The Laplacian is defined as

$$\Delta = L^*. \tag{2.19}$$

The following proposition is from [28].

Proposition 2.6.1. *For $\lambda > 0$, the resolvent $\rho_\lambda = (\lambda - \Delta)^{-1}$ is a bounded operator acting on $L^2(G)$. Moreover there exists ν_λ, an absolutely continuous measure with respect to the Haar measure of G, such that, for every $f \in L^2(G)$, we have :*

$$\rho_\lambda f = \nu_\lambda * f.$$

The Radon-Nikodym derivative of ν_λ, denoted by k_λ, is an element of $L^1(G)$ and is given by

$$k_\lambda = \int_0^\infty e^{-\lambda t} p_t dt,$$

where, for every $t > 0$, p_t is a non-negative function that fulfills the following conditions :

$$p_t \in L^1(G) \cap L^2(G), \quad ||p_t||_1 = 1.$$

Corollary 2.6.2. *We have the following consequence of Remark 2.3.6, the crossed-product contains operator $Sch(1 \otimes \psi) = \psi(p)$ for $\psi \in L^1(G)$. In particular, ρ_λ is an operator of convolution by a function in $L^1(G)$. That means that Δ is affiliated to $\mathcal{C}(G^+) \rtimes G$, where G^+ is the one-point compactification of G (see Subsection 2.2.2).*

Remark 2.6.3. Let G a Lie group (not necessarily connected) and G_0 the connected component of G of the identity of G. The inclusion $\mathcal{C}(G_0^+) \rtimes G \subset \mathcal{C}(G^+) \rtimes G$ and Corollary 2.6.2 implies that Δ is affiliated to $\mathcal{C}(G^+) \rtimes G$ even if G is not connected.

2.6.2 The Heseinberg group

Definition 2.6.4. *We recall that the Heisenberg group H is the sub-group of $GL(3, \mathbb{R})$ of upper triangular matrices and that we have a natural bijection between H and $\mathbb{R}^3$:*

$$H \ni \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow (a, b, c) \in \mathbb{R}^3 \tag{2.20}$$

Let $\mathfrak{h}$ be the Lie algebra of H, it's well known that $\mathfrak{h}$ is the space of strictly upper triangular matrix. Again, there is a natural bijection between $\mathfrak{h}$ and $\mathbb{R}^3$:

$$\mathfrak{h} \ni \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow (a, b, c)_0 \in \mathbb{R}^3 \tag{2.21}$$

The Heisenberg group is nipoltent, hence the exponential map is a homeomorphism from the Lie algebra $\mathfrak{h}$ to H. Recall that the exponential map is given by $\exp((a, b, c)_0) = (a, b, c + \frac{ab}{2})$ with inverse $\exp^{-1}((a, b, c)) = (a, b, c - \frac{ab}{2})_0$. For each $X \in H$, let L_X (resp R_X): $H \to H$, the left (resp right) multiplication by X. In the next sections, we will consider potentials that are equivariant for various compactifications.

See also [3, 13, 18, 52] for a general introduction to the basics of the problems studied here and [27] for some more specific references. In addition to the works of Georgescu and his collaborators mentioned above, essential spectra have been studied using algebraic methods by many people, including [49, 50, 51, 54, 56].

3

A refined HVZ-theorem for asymptotically homogeneous interactions and finitely many collision planes

We refer the reader to Section 1.3 for a detailed introduction of this Chapter.

Recall that we denote by $\langle S\rangle$ the norm closed algebra generated by a subset S of a Banach algebra. Recall that for $X = \mathbb{R}^n$ and $\mathcal{S}$ a family of subspace of X, $\mathcal{E}_{\mathcal{S}}(X)$ was defined in the Introduction, in Equation (1.8) as the norm closed algebra generated by all the algebras $\mathcal{C}_0(\overline{X/Y})$ and the crossed product for the natural action of X on $\mathcal{E}_{\mathcal{S}}(X)$ is given by

$$\mathcal{E}_{\mathcal{S}}(X) \rtimes X = \langle \phi(q)\psi(p), \phi \in \mathcal{E}_{\mathcal{S}}(X),\ \psi \in \mathcal{C}_0(X)\rangle \subset \mathcal{B}(L^2(X)). \tag{3.1}$$

3.1 Character spectrum and chains of subspaces

In this section, we determine the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$ *as a set*. The topology of the spectrum will be discussed in the next section.

Recall that in this paper $\mathcal{S}$ denotes a (non empty) semilattice of subspaces of X, that is, $Z_1 \cap Z_2 \in \mathcal{S}$ if $Z_1, Z_2 \in \mathcal{S}$. If $X \notin \mathcal{S}$, then $\mathcal{S}' = \mathcal{S} \cup \{X\}$ is a lattice of subspaces of X with $\mathcal{E}_{\mathcal{S}'}(X) = \mathcal{E}_{\mathcal{S}}(X)$. There is thus no loss of generality to assume that $X \in \mathcal{S}$, which we shall do from now on.

Remark 3.1.1. The algebras $\mathcal{E}_{\mathcal{S}}(X)$ make sense for any non empty family $\mathcal{S}$ of subspaces of X. It is convenient however for us to assume that $\mathcal{S}$ is a semilattice since then $\mathcal{S}$ has a least element Y_0 and then $\mathcal{E}_{\mathcal{S}}(X)$ is isomorphic to $\mathcal{E}_{\mathcal{S}'}(X/Y_0)$, where $\mathcal{S}'$ is the induced semilattice on X/Y_0. We have $0 \in \mathcal{S}'$, which may not be the case for $\mathcal{S}$. Also note that $\mathcal{C}_0(X) \subset \mathcal{E}_{\mathcal{S}}(X)$ if, and only if, $0 \in \mathcal{S}$. Hence, we thus need to assume that $0 \in \mathcal{S}$. In the important example of the lattice $\mathcal{S}_N$ mentioned in the Introduction, we do have that $0 \in \mathcal{S}_N$, but that is not true for the semilattice generated just by the subspaces $\mathcal{P}_{ij}$. If $0 \notin \mathcal{S}$, then H is among the operators $\tau_\alpha(H)$, so Theorem 1.3.1 simply asserts that $\sigma_{\text{ess}}(H) = \sigma(H)$, which is clear anyway, since H is invariant with respect to the minimal element of the lattice $\mathcal{S}$, which is non-zero if $0 \notin \mathcal{S}$.

3.1.1 Translation to infinity

The natural projection $\pi_Y : X \to X/Y$ extends by continuity to a map $\tilde{\pi}_Y : \overline{X} \smallsetminus \mathbb{S}_Y \to \overline{X/Y}$ satisfying $\tilde{\pi}_Y(\mathbb{S}_X \smallsetminus \mathbb{S}_Y) \subset \mathbb{S}_{X/Y}$. More precisely, if $\alpha \in \mathbb{S}_X \smallsetminus \mathbb{S}_Y$, then it is a half-line $\mathbb{R}_+ a$ in X, with $a \in X \smallsetminus Y$. Then $\tilde{\pi}_Y(\alpha)$ correspond at the half-line $\mathbb{R}_+\pi_Y(a)$ in X/Y. We note, however, that π_Y will not have a limit at $\alpha \in \mathbb{S}_Y$. Indeed, for each vector in $y \in \overline{X/Y}$, we can find a sequence $(x_n) \in X$ such that $\lim_{n\to+\infty} x_n = \alpha$ and $\lim_{n\to+\infty} \pi_Y(x_n) = y$.

Let $\alpha = \hat{a} \in \mathbb{S}_X$ (so $a \neq 0$). As in [27], if $u \in \mathcal{C}(\overline{X/Y}), x \in X$, then

$$\tau_\alpha(u)(x) \;:=\; \lim_{r\to+\infty} u(ra+x) \;=\; \begin{cases} u(x) & \text{if } \alpha \subset Y \text{ (i.e., } a \in Y) \\ u(\tilde{\pi}_Y(\alpha)) \in \mathbb{C} & \text{otherwise} \end{cases} \tag{3.2}$$

exists, and hence the limit $\tau_\alpha(u)$ exists for all $u \in \mathcal{E}_{all}(X)$ (the algebra obtained by considering the case of *all* subspaces of X, as in [27]). In particular, we have that $\tau_\alpha(u) \in \mathcal{E}_\mathcal{S}(X)$, if $u \in \mathcal{E}_\mathcal{S}(X)$, and hence τ_α defines an endomorphism of the algebra $\mathcal{E}_\mathcal{S}(X)$. Note that the limit defining τ_α is both in pointwise sense for functions and in strong sense for operators on $L^2(X)$.

For $\alpha \in \mathbb{S}_X$, we shall denote by $\chi_\alpha(f) := f(\alpha)$, the evaluation character at α for $f \in \mathcal{C}(\overline{X})$. We have the following lemma [27]

Lemma 3.1.2. *Let $Y \subset X$ be a subspace, let B be the C^*–algebra generated by $\mathcal{C}(\overline{X})$ and $\mathcal{C}(\overline{X/Y})$ in $\mathcal{C}_b(X)$, and let $\alpha \in \mathbb{S}_X \smallsetminus \mathbb{S}_Y$. Then the character χ_α of $\mathcal{C}(\overline{X})$ extends to a unique character of B. This extension is the restriction of τ_α to B.*

We shall need the following notation. Let $\alpha \in \mathbb{S}_X$ and

$$\mathcal{S}_\alpha \;:=\; \{Y \in \mathcal{S} |\, \alpha \subset Y\}\,, \quad Z(\alpha) \;:=\; \bigcap_{Y\in\mathcal{S}_\alpha} Y\,, \quad \mathcal{S}/\alpha := \{Y/Z(\alpha) |\, Y \in \mathcal{S}_\alpha\}\,. \tag{3.3}$$

Then $\mathcal{S}_\alpha$ is again a semilattice. Therefore $Z(\alpha) \in \mathcal{S}_\alpha$ since $\dim(X) < \infty$, and hence it is the smallest element of $\mathcal{S}_\alpha$. Similarly, $\mathcal{S}/\alpha$ is the induced lattice of subspaces of $X/Z(\alpha)$.

The semilattices $\mathcal{S}_\alpha$ and $\mathcal{S}/\alpha$ will play a fundamental role in what follows. For instance

$$\tau_\alpha(\mathcal{E}_\mathcal{S}(X)) \;=\; \mathcal{E}_{\mathcal{S}_\alpha}(X) \tag{3.4}$$

and $\mathcal{E}_{\mathcal{S}_\alpha}(X)$ is naturally isomorphic to $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$ via $\pi_{Z(\alpha)} : X \to X/Z(\alpha)$. We note that, unlike in the case of all subspaces of X, the semilattices $\mathcal{S}_\alpha$ and $\mathcal{S}/\alpha$ depend on $\mathcal{S}$, and not just on $\alpha \in \mathbb{S}_X$. We identify $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$ with the subalgebra $\mathcal{E}_{\mathcal{S}_\alpha}(X)$ of $\mathcal{E}_\mathcal{S}(X)$ using the projection $\pi_Y : X \to X/Y$.

Lemma 3.1.3. *The morphism τ_α descends to a surjective morphism*

$$\tilde{\tau}_\alpha : \mathcal{E}_\mathcal{S}(X) \to \mathcal{E}_{\mathcal{S}/\alpha}\,(X/Z(\alpha))\;.$$

Let $\alpha \in \mathbb{S}_X$, regarded as a half line in X. Let $Z(\alpha)$ be the smallest subspace in $\mathcal{S}$ containing α, as before. Also, let $X' := X/Z(\alpha)$ and $\mathcal{S}' := \mathcal{S}/\alpha$. (Recall that $\mathcal{S}/\alpha := \{Y/Z(\alpha) \subset X' |\, Z(\alpha) \subset Y \in \mathcal{S}\}$.) Then we consider

$$\tau^*_\alpha \;:\; \widehat{\mathcal{E}_{\mathcal{S}'}(X')} \;\cong\; \mathrm{Prim}(\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))) \;\to\; \widehat{\mathcal{E}_\mathcal{S}(X)}\,, \tag{3.5}$$

the map dual to $\tilde{\tau}_\alpha$, that is, $\tau^*_\alpha(\chi) \;:=\; \chi \circ \tilde{\tau}_\alpha$. The above lemma gives that τ^*_α is continuous and a homeomorphism onto its image, which is a closed, compact subset of $\widehat{\mathcal{E}_\mathcal{S}(X)}$. The following lemma identifies the image of τ^*_α with the set of characters of $\mathcal{E}_\mathcal{S}(X)$ that restrict to χ_α on $\mathcal{C}(\overline{X})$ when $\mathcal{C}(\overline{X}) \subset \mathcal{E}_\mathcal{S}(X)$, that is when $0 \in \mathcal{S}$. In view of Remark 3.1.1, *we assume from now that* $0 \in \mathcal{S}$.

Lemma 3.1.4. *Let $\alpha \in \mathbb{S}_X$ and $\Omega_\alpha := \{\chi \in \widehat{\mathcal{E}_\mathcal{S}(X)} |\, \chi|_{\mathcal{C}(\overline{X})} = \chi_\alpha\}$ (recall that $0 \in \mathcal{S}$). Then*

$$\Omega_\alpha \;=\; Im(\tau^*_\alpha) \;\cong\; \mathrm{Prim}(\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha)))\;.$$

In other words, we have that a character $\chi \in \widehat{\mathcal{E}_{\mathcal{S}}(X)}$ restricts to the character χ_α on $\mathcal{C}(\overline{X})$ if, and only if, it is of the form $\chi = \chi' \circ \tilde{\tau}_\alpha$, for some character χ' of $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$.

Conversely, given a character χ of $\mathcal{E}_{\mathcal{S}}(X)$, let us consider its restriction to a character of $\mathcal{C}(\overline{X}) \subset \mathcal{E}_{\mathcal{S}}(X)$. Hence there exists $\alpha \in \overline{X}$ such that $\chi = \chi_\alpha$ on $\mathcal{C}(\overline{X})$. If $\alpha \in X \subset \overline{X}$, then, in fact, χ is uniquely determined by α, since $X \cong \widehat{\mathcal{C}_0(X)}$ and every character of an ideal extends *uniquely* to the algebra. In particular, we obtain that X identifies with an open subset of $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. We shall write $X \subset \widehat{\mathcal{E}_{\mathcal{S}}(X)}$, by abuse of notation. If $\alpha \notin X$, we have that $\alpha \in \mathbb{S}_X$, and hence $\chi \in \Omega_\alpha \cong \mathrm{Prim}(\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha)))$.

Lemma 3.1.5. *Assume $0 \in \mathcal{S}$, as before. The restriction map $R : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \overline{X}$ associated to the inclusion $\mathcal{C}(\overline{X}) \subset \mathcal{E}_{\mathcal{S}}(X)$ gives rise to a disjoint union decomposition*

$$\widehat{\mathcal{E}_{\mathcal{S}}(X)} \;=\; R^{-1}(X) \cup_{\alpha \in \mathbb{S}_X} R^{-1}(\{\alpha\}) \;=:\; X \cup_{\alpha\in\mathbb{S}_X} \Omega_\alpha\,.$$

This allows us an inductive determination of the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$ since Ω_α identifies with the spectrum of $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$. This inductive determination is conveniently formulated in terms of "chains," which we introduce next. We note that the subsets X and $\{\alpha\}$ in the above lemma are exactly the orbits of X acting on $\overline{X}$. A similar chain structure has appeared also in [9, 27].

3.1.2 $\mathcal{S}$-chains

The spectrum of the algebra $\mathcal{E}_{\mathcal{S}}(X)$ is conveniently described in terms of $\mathcal{S}$*-chains* $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$, in a manner similar, but different to the one in [27]. To introduce the concept of $\mathcal{S}$-chains, we shall use the notation introduced in 3.3. An $\mathcal{S}$-chain $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$, $0 \le k \le \dim(X)$, is required to satisfy the following recursive conditions, which involves also a sequence Z_j that is defined recursively as follows :

1. $Z_0 = 0$;
2. $\alpha_j \in \mathbb{S}_{X/Z_{j-1}}$, (a half-line in X/Z_{j-1}), $j = 1, 2, \dots, k$;
3. $Z_j \in \mathcal{S}$ is the least subspace containing Z_{j-1} and α_j, for $j \le k$.

In (3) above, we have used that $\alpha_j \in \overline{X/Z_{j-1}}$ is a point in X/Z_{j-1} or a half line in X/Z_{j-1} and hence, in turn, a subset of X. In particular, we obtain $\alpha_1 \in \mathbb{S}_X$ and $Z_1 = Z(\alpha_1)$, the least subspace of $\mathcal{S}$ containing α_1. We say that the $\mathcal{S}$-chain $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$ has *length* k. There is only one $\mathcal{S}$-chain of length zero : the empty set $\emptyset$.

The $\mathcal{S}$-chain $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$ determines the spaces Z_j, $0 \le j \le k$ as follows. Let α'_j be a representative in X of $\alpha_j \in S_{X/Z_{j-1}}$. That is, $\alpha_j = \mathbb{R}_+^* \alpha'_j + Z_{j-1} \subset Z_j \in \mathcal{S}$. The subspace $[\alpha'_1, \alpha'_2, \dots, \alpha'_j] \subset X$ linearly generated by the $\alpha'_1, \alpha'_2, \dots, \alpha'_j$ may depend on the choices of the α'_j, but the *least* subspace $Z \subset \mathcal{S}$ containing it will *not* depend on the choices of the representatives and $Z_j = Z$. We shall occasionally also use the more complete notation

$$Z(\alpha_1, \alpha_2, \dots, \alpha_j) := Z_j \tag{3.6}$$

and $Z(\overrightarrow{\alpha}) := Z(\alpha_1, \alpha_2, \dots, \alpha_k)$ if $\overrightarrow{\alpha}$ has length k. If $\overrightarrow{\alpha} = \emptyset$ (that is, if $k = 0$), we let $Z(\overrightarrow{\alpha}) = 0$. The symbol $\tilde{\Xi}_X^{(k)}$ will denote the set of $\mathcal{S}$-chains of length k.

A sequence $0 \neq Z_1 \subsetneq Z_2 \subsetneq \dots \subsetneq Z_k$ of subspaces in $\mathcal{S}$ will be called an $\mathcal{S}$*-flag* (of length k). Each $\mathcal{S}$-flags of length k corresponds to at least one $\mathcal{S}$-chains of length k.

An *augmented $\mathcal{S}$-chain* is a pair $(a, \overrightarrow{\alpha})$, where $\overrightarrow{\alpha}$ is an $\mathcal{S}$-chain and $a \in X/Z(\overrightarrow{\alpha})$. By $\Xi_X^{(k)}$ we shall denote the set of augmented $\mathcal{S}$-chains of length k :

$$\Xi_X^{(k)} \;:=\; \{(a, \overrightarrow{\alpha}) |\; \overrightarrow{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \tilde{\Xi}_X^{(k)},\; a \in X/Z(\overrightarrow{\alpha})\}\,. \tag{3.7}$$

We let $\Xi_X := \cup_k \Xi_X^{(k)}$ denote the set of all augmented $\mathcal{S}$-chains.

Assume $(a, \overrightarrow{\alpha}) = (a, \alpha_1, \alpha_2, \dots, \alpha_k) \in \Xi_X$ and let $\mathcal{S}_j := \{Y/Z_j | \ Z_j \subset Y, Y \in \mathcal{S}\}$ be the induced semilattice of subspaces of X/Z_j, as before, see Equation (3.6). We obtain for each j (so $\alpha_j \in X/Z_{j-1}$) a morphism

$$\tilde{\tau}_{\alpha_j} : \mathcal{E}_{\mathcal{S}_{j-1}}(X/Z_{j-1}) \ \to \ \mathcal{E}_{\mathcal{S}_j}(X/Z_j) \,. \tag{3.8}$$

Recall that if $a \in X$, the character $\chi_a : \mathcal{E}_{\mathcal{S}}(X) \to \mathbb{C}$ is the evaluation at a.

Definition 3.1.6. *For each augmented $\mathcal{S}$-chain $(a, \alpha_1, \alpha_2, \dots, \alpha_k) \in \Xi_X^{(k)}$, we define*

$$\tau_{\overrightarrow{\alpha}} \ := \ \tilde{\tau}_{\alpha_k}\tilde{\tau}_{\alpha_{k-1}} \dots \tilde{\tau}_{\alpha_1} : \mathcal{E}_{\mathcal{S}}(X) \to \mathcal{E}_{\mathcal{S}_k}(X/Z_k) \quad \textit{and} \quad \chi_{a,\overrightarrow{\alpha}} \ := \ \chi_a \tau_{\overrightarrow{\alpha}} : \mathcal{E}_{\mathcal{S}}(X) \to \mathbb{C} \,.$$

Lemma 3.1.7. *The map $\tau_{\overrightarrow{\alpha}}$ of Definition 3.1.6 is a surjective morphism $\mathcal{E}_{\mathcal{S}}(X) \to \mathcal{E}_{\mathcal{S}_k}(X/Z_k) = \mathcal{E}_{S/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$. Consequently, $\chi_{a,\overrightarrow{\alpha}}$ defines a character of $\mathcal{E}_{\mathcal{S}}(X)$.*

It will be convenient to use also the more complete notation

$$\mathcal{S}/(\alpha_1, \alpha_2, \dots, \alpha_j) := \{Y/Z(\alpha_1, \alpha_2, \dots \alpha_j) | \ [\alpha_1', \alpha_2' \dots \alpha_j'] \subset Y \in \mathcal{S}\} = \mathcal{S}_j$$

.

Proof. The first assertion is a successive application of lemma 3.1.3. More precisely, we have the following sequence of surjective maps :

$$\mathcal{E}_{\mathcal{S}}(X) \xrightarrow{\tau_{\alpha_1}} \mathcal{E}_{\mathcal{S}/\alpha_1}(X/Z(\alpha_1)) \xrightarrow{\tau_{\alpha_2}} \mathcal{E}_{\mathcal{S}/(\alpha_1,\alpha_2)}(X/Z(\alpha_1,\alpha_2)) \xrightarrow{\tau_{\alpha_3}} \dots \xrightarrow{\tau_{\alpha_k}} \mathcal{E}_{S/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$$

Then the second assertion is direct consequence of the first one because, if $a \in X/Z(\overrightarrow{\alpha})$, then χ_a is a character of $\mathcal{E}_{S/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$. □

Remark 3.1.8. We distinguish two special cases :
— If $\overrightarrow{\alpha} = \emptyset$, we have $\tau_{\overrightarrow{\alpha}} = Id$, and hence $\chi_{a,\emptyset} := \chi_a$ $(a \in X)$.
— If $Z(\overrightarrow{\alpha}) = X$, we have $\chi_{a,\overrightarrow{\alpha}} := \tau_{\overrightarrow{\alpha}}$, since there is only one $a \in X/X = 0$.

We obtain that the spectrum of our algebra $\mathcal{E}_{\mathcal{S}}(X)$ identifies naturally with the set Ξ_X of augmented $\mathcal{S}$-chains.

Theorem 3.1.9. *Assume that $0 \in \mathcal{S}$, then we have a bijective map $\Theta : \Xi_X \to \widehat{\mathcal{E}_{\mathcal{S}}(X)}$,*

$$\Theta(a, \overrightarrow{\alpha}) \ := \ \chi_{a,\overrightarrow{\alpha}} \ := \ \chi_a \tau_{\overrightarrow{\alpha}} \,.$$

Proof. This is obtained by induction on $\dim(X)$, using Lemmas 3.1.3 and 3.1.4. □

Let us explain now how the characters $\chi_{a,\overrightarrow{\alpha}}$ act on $\mathcal{E}_{\mathcal{S}}(X)$. If $Z \subset Y \subset X$, we shall use the similar notation $\pi_{Y/Z} : X/Z \to X/Y$ for the linear projection, which we extend by continuity to $\tilde{\pi}_{Y/Z} : \overline{X/Z} \setminus \mathbb{S}_{Y/Z} \to \overline{X/Y}$.

Remark 3.1.10. Let $(a, \overrightarrow{\alpha}) \in \Xi_X$.

1. If $\overrightarrow{\alpha} = \emptyset$, then $(a, \overrightarrow{\alpha}) = a$ and

$$\chi_{(a,\emptyset)}(f) \ = \ \chi_a(f) \ = \ f(a) \,.$$

2. If $\overrightarrow{\alpha} \neq \emptyset$ has length $k \geq 1$ and $f \in \mathcal{C}(\overline{X/Y})$, with $Y \in \mathcal{S}$, we have

$$\chi_{(a,\overrightarrow{\alpha})}(f) \;=\; \begin{cases} f(\pi_{Y/Z(\overrightarrow{\alpha})}(a)) & \text{if } Z(\overrightarrow{\alpha}) \subset Y \\ f(\tilde{\pi}_{Y/Z_{p-1}}(\alpha_p)) & \text{if } Z_{p-1} \subset Y, \text{ but } Z_p \not\subset Y \,. \end{cases} \tag{3.9}$$

In the first case of Equation (3.9), $\pi_{Y/Z(\overrightarrow{\alpha})}(a) \in X/Y$ is well defined since $a \in X/Z(\overrightarrow{\alpha})$ and $Z(\overrightarrow{\alpha}) \subset Y$. In the second case, the index $0 < p \leq k$ is determined to be the largest satisfying $Z_{p-1} := Z(\alpha_1, \ldots, \alpha_{p-1}) \subset Y$, (so $Z_p := Z(\alpha_1, \ldots, \alpha_p) \not\subset Y$). This follows by repeatedly using Equation (3.2). We also notice that the relation $Z_p \not\subset Y$ is equivalent to $\alpha_p \notin Y/Z_{p-1}$. Again, $\tilde{\pi}_{Y/Z(\overrightarrow{\alpha})}(\alpha_p) \in \mathbb{S}_{X/Y}$ is defined since $\alpha_p \in \mathbb{S}_{X/Z_{p-1}}$, $Z_{p-1} \subset Y$, and $\alpha_p \notin \mathbb{S}_{Y/Z_{p-1}}$. See the definition of the extensions $\tilde{\pi}_Y$ at the beginning of this section.

From this remark it follows that the induced action of X on the set of augmented $\mathcal{S}$-chains Ξ_X is by translation on the first component :

$$x \cdot (a, \overrightarrow{\alpha}) \;=\; (\pi_{Z(\overrightarrow{\alpha})}(x) + a, \overrightarrow{\alpha}) \,, \quad x \in X, \text{ and } (a, \overrightarrow{\alpha}) \in \Xi_X \,. \tag{3.10}$$

In particular, if $Z(\overrightarrow{\alpha}) = X$, then $\overrightarrow{\alpha}$ is invariant for the action of X.

We would like next to study the topology on the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ of characters of $\mathcal{E}_{\mathcal{S}}(X)$ and the topology that it induces on $\Xi_X := \cup_{0 \leq k \leq \dim(X)} \Xi_X^{(k)}$, since this will be useful in proving that the family of morphisms $\{\tau_\alpha | \ \alpha \in \mathbb{S}_X\}$ is *exhaustive* (the notion of exhausting families was introduced in [46] and will be recalled in the last section (see Definition 2.5.1).

3.2 The topology on the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$

We now give a first description of the *topology* on the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$ by identifying it with a closed subset of the product $\prod_{Y \in \mathcal{S}} \overline{X/Y}$. We continue to assume in this section and thereafter, for simplicity, that $0, X \in \mathcal{S}$, even if some results hold in greater generality.

We refer the reader to Subsection 2.1.4 for details on $\mathrm{Prim}(A)$ and its topology. In particular, we recall Proposition 2.1.23 :

Proposition 3.2.1. *If J is an essential ideal of A then $Prim^J(A)$ is dense in $Prim(A)$.*

The converse is obviously true.

Remark 3.2.2. We shall use this result for $\mathcal{E}_{\mathcal{S}}(X)$ and $\mathcal{C}_0(X)$ and for their cross-products by X. In the first case, that is, for $A = \mathcal{E}_{\mathcal{S}}(X)$ and $J = \mathcal{C}_0(X)$, it follows from the definition that $\mathcal{C}_0(X)$ is essential in $\mathcal{E}_{\mathcal{S}}(X)$ (since it is essential in $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$), and hence that $X \cong \widehat{\mathcal{C}_0(X)}$ (or rather that its image) is dense in $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. In the second case, that is for $J := \mathcal{K}(X) \cong \mathcal{C}_0(X) \rtimes X \subset \mathcal{E}_{\mathcal{S}}(X) \rtimes X =: A$, we have already seen in Lemma 2.4.4 that $\mathrm{Prim}^J = \{0\}$ (it contains only the zero ideal). Indeed, this follows by taking $\mathcal{A} := \mathcal{E}_{\mathcal{S}}(X)$ in that remark. It thus follows that 0 is a dense point in the primitive ideal spectrum of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$: $\overline{\{0\}} = \mathrm{Prim}(\mathcal{A} \rtimes X)$.

$$G^{\mathcal{S}} \;:=\; \prod_{Y \in \mathcal{S}} \pi_Y \;: X \;\to\; \prod_{Y \in \mathcal{S}} \overline{X/Y} \,. \tag{3.11}$$

Let us similarly consider all the restrictions $\widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \widehat{\mathcal{C}(\overline{X/Y})} \cong \overline{X/Y}$. Combining all these restrictions, we obtain the map $\Phi : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \prod_{Y \in \mathcal{S}} \overline{X/Y}$

$$\Phi(\chi) \;=\; (x_Y) \;\in\; \prod_{Y \in \mathcal{S}} \overline{X/Y} \,, \quad \text{where } \chi(f) = f(x_Y) \,, \;\; f \in \mathcal{C}(\overline{X/Y}), \; Y \in \mathcal{S} \,. \tag{3.12}$$

Lemma 3.2.3. *The map* Φ *of Equation* (3.12) *is continuous and a homeomorphism onto its image.*

Proof. The continuity of Φ is due to the fact that the dual map defined by restriction for characters is continuous. The injectivity comes from the fact that the algebras $\mathcal{C}(\overline{X/Y})$ generate $\mathcal{E}_{\mathcal{S}}(X)$. The proof is completed by recalling that a continuous bijection of compact spaces is a homeomorphism. □

Let $j : X \to \widehat{\mathcal{E}_{\mathcal{S}}(X)}$ be the inclusion defined by $\mathcal{C}_0(X) \subset \mathcal{E}_{\mathcal{S}}(X)$. Also, recall the map Φ defined in Equation (3.12) and $G^{\mathcal{S}}$ defined in Equation (3.11). The following theorem describes the topology on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$.

Theorem 3.2.4. *The following diagram is commutative*

$$\begin{array}{ccc} \widehat{\mathcal{E}_{\mathcal{S}}(X)} & \xrightarrow{\quad\Phi\quad} & \prod_{Y\in\mathcal{S}} \overline{X/Y} \\ & {\scriptstyle j}\nwarrow \quad \nearrow{\scriptstyle G^{\mathcal{S}}} & \\ & X & \end{array} \tag{3.13}$$

In particular, Φ *induces a homeomorphism of* $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ *onto* $\overline{G_{\mathcal{S}}(X)}$ *that is functorial in* $\mathcal{S}$.

Proof. Each component of the composition $\Phi \circ j$ is obtained by extending a character χ_x of $\mathcal{C}_0(X/Y)$ to $\mathcal{E}_{\mathcal{S}}(X)$ and then restricting to $\mathcal{C}(\overline{X/Y})$. This extension is unique and correspoinds to the evaluation at x, that is, to χ_x. Since $\mathcal{C}_0(X)$ is an essential ideal in $\mathcal{E}_{\mathcal{S}}(X)$, X is dense in $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. By continuity

$$\Phi(\widehat{\mathcal{E}_{\mathcal{S}}(X)}) \subset \overline{\Phi(j(X))} = \overline{G^{\mathcal{S}}(X)}\,.$$

Moreover, the image contains X and is closed, since it is compact. Hence we have equality. The result then follows from Lemma 3.2.3. □

The functoriality in $\mathcal{S}$ refers to the inclusion $\mathcal{E}_{\mathcal{S}}(X) \subset \mathcal{E}_{\mathcal{S}'}(X)$ if $\mathcal{S} \subset \mathcal{S}'$.

The meaning of Theorem 3.2.4 is that it provides also an elementary geometric construction of the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$, which, as we have already mentioned, may be useful for numerical methods. The description of the topology on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ is, however, not completely satisfactory at this point, since we do not have a good understanding of $\overline{G_{\mathcal{S}}(X)}$ yet. We have good reasons to believe, however, that it is a manifold with corners obtained by successively blowing-up the singular strata and that it coincides with a space introduced by Vasy [59]. We develop this idea in Chapter 6.

A natural question then is to identify the composite map $\Phi \circ \Theta : \Xi_X \to \overline{G_{\mathcal{S}}(X)}$. Recall that $\pi_{Y/Z} : X/Z \to X/Y$ is, as usual, the projection, and that it extends to a continuous map $\tilde{\pi}_{Y/Z} : \overline{X/Z} \smallsetminus \mathbb{S}_{Y/Z} \to \overline{X/Y}$. Given that $\Phi : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \prod_{Y\in\mathcal{S}} \overline{X/Y}$ is defined by restrictions to the generating subalgebras $\mathcal{C}(\overline{X/Y})$, see (3.12), Remark 3.1.10 tells us that the Y component $\big(\Phi(\chi_{(a,\overrightarrow{\alpha})})\big)_Y \in \overline{X/Y}$ of $\Phi(\chi_{(a,\overrightarrow{\alpha})}) \in \prod_{Y\in\mathcal{S}} \overline{X/Y}$ is

$$\big(\Phi(\chi_{(a,\overrightarrow{\alpha})})\big)_Y = \begin{cases} \pi_{Y/Z(\overrightarrow{\alpha})}(a) & \text{if } Z(\overrightarrow{\alpha}) \subset Y \\ \tilde{\pi}_{Y/Z_{p-1}}(\alpha_p) & \text{if } Z_{p-1} \subset Y, \text{ but } Z_p \not\subset Y\,, \end{cases} \tag{3.14}$$

where we have used the notation of that remark. Let $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_k)$. We note that the component of $\Phi(\chi_{(a,\overrightarrow{\alpha})})$ corresponding to $Y = Z_j$, $j = 0, \ldots, k-1$, is α_{j+1}, whereas the component of $\Phi(\chi_{(a,\overrightarrow{\alpha})})$ corresponding to $Y = Z_k$ is a. Thus all other components of $\Phi(\chi_{(a,\overrightarrow{\alpha})}) = \Phi(\Theta(a, \overrightarrow{\alpha}))$ are determined by these components (a and α_j), as explained. More precisely, to determine the $Y \in \mathcal{S}$ component of

$\Phi(\chi_{(a,\overrightarrow{\alpha})})$, we need to find the largest p such that $Z_{p-1} \subset Y$, and then the component corresponding to Y will be the projection onto $\overline{X/Y}$ of α_p, if $p < k$, or of a, if $p = k$.

Let us consider the augmented $\mathcal{S}$-chains $(a, \overrightarrow{\alpha}) \in \Xi_X^{(k)}$ that have the same fixed $\mathcal{S}$-flag $\mathcal{Z} := (Z_1, Z_2, ..., Z_k)$, where $Z_j := Z(\alpha_1, \alpha_2, \ldots, \alpha_j) \in \mathcal{S}$, as before, and hence $Z_1 \subset Z_2 \subset ... \subset Z_k$. If $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, then $\alpha_1 \in \mathcal{Y}_1 := \mathbb{S}_{Z_1} \smallsetminus \cup_{Y \in \mathcal{S}, Y \subsetneq Z_1} \mathbb{S}_Y$, and this set has a natural smooth structure and hence a natural topology. Similarly,

$$\alpha_j \in \mathcal{Y}_j := \mathbb{S}_{Z_j/Z_{j-1}} \smallsetminus \cup_{Y \in \mathcal{S}, Z_{j-1} \subset Y \subsetneq Z_j} \mathbb{S}_{Y/Z_{j-1}} ,$$

and hence we can endow the set of $\mathcal{S}$-chains with the given flag $\mathcal{Z}$ with the induced topology of the product manifold $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_k$ and the set of augmented $\mathcal{S}$-chains with the given flag $\mathcal{Z}$ with the induced topology of the product manifold

$$\mathcal{X}_{\mathcal{Z}} := X/Z(\overrightarrow{\alpha}) \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_k . \tag{3.15}$$

We then see that $\Phi \circ \Theta$ restricts to a diffeomorphism from $\mathcal{X}_{\mathcal{Z}}$ onto its image in $\prod_{Y \in \mathcal{S}} \overline{X/Y}$. Indeed, it is enough to consider the components of $\Phi \circ \Theta(a, \overrightarrow{\alpha})$ corresponding to all Z_j, $j = 0, \ldots, k$ (with $\mathcal{Y}_j$ projecting onto $\overline{X/Z_{j-1}}$). Clearly, all the sets $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$ are disjoint and $\overline{G_{\mathcal{S}}(X)} = \cup_{\mathcal{Z}} \Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$, since to each augmented $\mathcal{S}$-chain there corresponds exactly one $\mathcal{S}$-flag.

We endow the set of $\mathcal{S}$-flags with the lexicographic order. Namely, let $\mathcal{Z} := (Z_1, Z_2, \ldots, Z_k)$ and $\mathcal{Z}' := (Z_1', Z_2', \ldots, Z_n')$. Then

$$\mathcal{Z} < \mathcal{Z}' \text{ if } Z_1 = Z_1', Z_2 = Z_2', \ldots, Z_{j-1} = Z_{j-1}', \text{ but } Z_j \supsetneq Z_j', \tag{3.16}$$

for some $j \leq \min\{k, n\}$. Clearly, if $\mathcal{Z}' < \mathcal{Z}''$ and $\mathcal{Z} < \mathcal{Z}'$, then $\mathcal{Z} < \mathcal{Z}''$. We can now look at the relation between the sets $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$.

Lemma 3.2.5. *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two $\mathcal{S}$-flags such that $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}'})$ intersects the closure of $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$. Then $\mathcal{Z} < \mathcal{Z}'$.*

Proof. Let $j \geq 1$ be the smallest integer such that $Z_0 = Z_0'$, $Z_1 = Z_1'$, $Z_2 = Z_2'$, $\ldots$, $Z_{j-1} = Z_{j-1}'$, but $Z_j' \neq Z_j$. Let $(a', \overrightarrow{\alpha}') = (a', \alpha_1', \alpha_2', \ldots, \alpha_n')$ be an augmented $\mathcal{S}$-chain with flag $\mathcal{Z}'$ that maps to the closure of $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$. Then each Y-component of $\Phi(\Theta(a', \overrightarrow{\alpha}'))$ is the limit of Y-components of points in $\mathcal{X}_{\mathcal{Z}}$, $Y \in \mathcal{S}$. This is true, in particular, for the Z_{j-1} component, which is in $\overline{X/Z_{j-1}}$. Then we see that $\alpha_j' \in \mathbb{S}_{Z_j/Z_{j-1}} \subset \overline{Z_j/Z_{j-1}}$, which gives $Z_j' \subset Z_j$, since Z_j' is the least subspace of $\mathcal{S}$ containing $Z_{j-1}' = Z_{j-1} \subset Z_j$ and α_j'. Hence $\mathcal{Z} < \mathcal{Z}'$, by definition. □

Recall that a set in a topological space is *locally closed* if it is open in its closure, or, which is the same thing, if it is the intersection of an open subset and of a closed subset. We shall need the following corollary.

Corollary 3.2.6. *If $\mathcal{S}$ is finite, then for each $\mathcal{S}$-flag $\mathcal{Z}$, the set $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$ is locally closed in $\overline{G_{\mathcal{S}}(X)}$.*

Proof. Let F be the union of all the sets $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}'})$ with $\mathcal{Z} < \mathcal{Z}'$. Lemma 3.2.5 shows that the sets F and $F \cup \Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$ are closed, since "$<$" is transitive. □

3.3 Georgescu's algebra

We now use the results of the previous subsection to identify the topology on the spectrum of Georgescu's graded algebras [14].

Let us start from the same data : that is, we continue to assume that X is a finite dimension vector space and that $\mathcal{S}$ is a family of sub vector space of X with the condition $0, X \in \mathcal{S}$. In the framework of the *true* N-body problems, the interactions vanish at infinity, so it is more natural to consider the following algebra of interactions (potentials)

$$\mathcal{G}_{\mathcal{S}}(X) := \langle \mathcal{C}_0(X/Y) \rangle , \quad Y \in \mathcal{S} , \tag{3.17}$$

see [14] and the references therein. Notice that $X \in \mathcal{S}$ implies that $1 \in \mathcal{G}_{\mathcal{S}}(X)$.

As in the case of the algebra $\mathcal{E}_{\mathcal{S}}(X)$, we want to describe the spectrum of the C^*-algebra $\mathcal{G}_{\mathcal{S}}(X)$. Since the natural map $\iota : \mathcal{G}_{\mathcal{S}}(X) \subset \mathcal{E}_{\mathcal{S}}(X)$ is an inclusion (injective), we have that the resulting dual map

$$\iota^* : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \widehat{\mathcal{G}_{\mathcal{S}}(X)} , \quad \iota^*(\chi) := \chi_{|\mathcal{G}_{\mathcal{S}}(X)} , \tag{3.18}$$

is continuous and onto. As we already know explicitly $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$, it remains to determine the equivalence relation induced by ι^*.

Let $Y \in \mathcal{S}$ and $f \in \mathcal{C}_0(X/Y)$. Using Equation (3.9), we obtain

$$\chi_{(a,\overrightarrow{\alpha})}(\iota(f)) = \begin{cases} f(\pi_{Y/Z(\overrightarrow{\alpha})}(a)) & \text{if } Z(\overrightarrow{\alpha}) \subseteq Y \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

In summary the spectrum of the Georgescu algebra is then given by the following theorem.

Theorem 3.3.1. *Let $0, X \in \mathcal{S}$. The space $\widehat{\mathcal{G}_{\mathcal{S}}(X)}$ has the quotient topology for the map*

$$\iota^* : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \widehat{\mathcal{G}_{\mathcal{S}}(X)} , \quad \iota^*(\chi) := \chi_{|\mathcal{G}_{\mathcal{S}}(X)} , \tag{3.20}$$

Moreover two characters $(a, \overrightarrow{\alpha})$ and $(b, \overrightarrow{\beta})$ in Ξ_X are equal on $\mathcal{G}_{\mathcal{S}}(X)$ if and only if $Z(\overrightarrow{\alpha}) = Z(\overrightarrow{\beta})$ and $a = b \in X/Z(\overrightarrow{\alpha}) = X/Z(\overrightarrow{\beta})$.

It is known that $\widehat{\mathcal{G}_{\mathcal{S}}(X)}$ is in a natural bijection with the disjoint union of the spaces X/Y, $Y \in \mathcal{S}$ [35]. The restriction map then becomes $\iota^*(a, \overrightarrow{\alpha}) = \pi_{Y/Z(\overrightarrow{\alpha})}(a) \in X/Y$, where $Z(\overrightarrow{\alpha}) = Y$.

With our description of the spectrum and the notion of augmented $\mathcal{S}$-chain, we can then specify the topology of the of spectrum of $\mathcal{G}_{\mathcal{S}}(X)$. Indeed, with the preceding result, we see that for $Y \in \mathcal{S}$, if we denote by Ξ_Y the set of augmented $\mathcal{S}$-chains such that the associated $\mathcal{S}$-flag ends with Y, then the restriction $\iota^*|_{\Xi_X(Y)}$ can be viewed as the projection from $\mathcal{X}_{\mathcal{Z}}$ to X/Y, (see Equation 3.15), where $\mathcal{Z}$ is a flag that ends with $Z_k = Y$.

3.4 Exhausting families and a precise result on the essential spectrum

In order to apply our results to Hamiltonians such as the one given in Equation (1.6), we need to study the cross-product C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. This C^*algebra is noncommutative, so we will consider exclusively its primitive ideal spectrum. We assume in this section that $\mathcal{S}$ is finite in order to use Corollary 3.2.6 and hence to be able to use the results in [63]. In particular, its spectrum $\widehat{\mathcal{E}_{\mathcal{S}}(X)} \cong \mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X))$ is second countable (i.e. it will have a countable basis of open subsets). Moreover, as we will see below, the action of X on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ has locally closed orbits. This means that the primitive ideal spectrum of the cross-product C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ can be completely understood using, for instance, the theory explained in [20, 63].

More precisely, let us consider an arbitrary *locally compact, second countable space* Ω and assume that a *locally compact, second countable, abelian group* G, acts continuously on Ω. For simplicity, we shall assume that the orbits of G are locally closed in Ω, that is, that each orbit is open in its closure in Ω. The primitive ideal spectrum of $\mathcal{C}_0(\Omega) \rtimes G$ then consists of the set of pairs $(\mathcal{O}, \xi)$, where $\mathcal{O}$ is an orbit of G in Ω and ξ is a character of the stabilizer $G_{\mathcal{O}}$ of $\mathcal{O}$. (Recall that the stabilizer of the orbit $\mathcal{O} := G\omega$ is given by the set $G_\omega := \{g \in G | \ g\omega = \omega\}$, and this is independent of ω in the orbit $\mathcal{O}$, since G is commutative.) Moreover, the topology is the quotient topology of $\Omega \times \hat{G}$ with respect to the quotient map

$$\Phi_{\Omega,G} : \Omega \times \hat{G} \ \to \ \mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G) \ ,$$

given by $\Phi_{\Omega,G}(\omega, \chi) := (G\omega, \chi|_{G_\omega})$, see Theorem 8.39 in [63] for details. This map is also natural with respect to restriction morphisms, in the following sense :

Proposition 3.4.1. *Assume that $\Omega' \subset \Omega$ is a closed, G-invariant subset, with Ω and G locally compact, second countable, as above. Then Ω' also has locally closed orbits and the inclusion $j : \Omega' \to \Omega$ induces a surjective morphism $j \rtimes G : \mathcal{C}_0(\Omega) \rtimes G \to \mathcal{C}_0(\Omega') \rtimes G$ and hence an injective map $(j \rtimes G)^* : \mathrm{Prim}(\mathcal{C}_0(\Omega') \rtimes G) \to \mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G)$ such that*

$$(j \rtimes G)^* \circ \Phi_{\Omega',G} = \Phi_{\Omega,G} \circ (j \times id) \ : \ \Omega' \times \hat{G} \ \to \ \mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G) \ .$$

A similar statement holds for open inclusions (but with the arrows reversed).

Proof. This follows from the fact that the stabilizer of $\omega \in \Omega'$ is the same as that of ω regarded as a point in Ω. See the proof of the Theorem 8.39 in [63]. $\square$

We shall need the following corollary.

Corollary 3.4.2. *If the space Ω of Proposition 3.4.1 is a union $\Omega = \cup_{\alpha\in I}\Omega_\alpha$ of closed, invariant subsets ($j_\alpha : \Omega_\alpha \to \Omega$ the inclusion), then $\mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G)$ is the disjoint union*

$$\mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G) \ = \ \cup_{\alpha\in I}(j_\alpha \rtimes G)^*(\mathrm{Prim}(\mathcal{C}_0(\Omega_\alpha) \rtimes G)) \, .$$

Proof. This follows from Proposition 3.4.1 using the fact that $\Omega \times \hat{G} = \cup_{\alpha\in I}\Omega_\alpha \times \hat{G}$. $\square$

Let ϕ be a $*$-morphism between two C^*-algebras A and B. Recall that the *support* $\mathrm{supp}(\phi) = \mathrm{Prim}_{\ker(\phi)}(A)$ of ϕ is the set of primitive ideals containing $\ker(\phi)$. If ϕ is surjective, its support is the image of $\phi^* : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ (which is defined in this particular case). Recall that the concept of "exhaustive family" is introduced in Definition 2.5.1.

We thus obtain the following corollary.

Corollary 3.4.3. *Let us use the notation of Corollary 3.4.2. Then the family of morphisms $\{j_\alpha \rtimes G \mid \alpha \in I\}$ is exhaustive.*

Proof. The support of $j_\alpha \rtimes G$ is the image of $\mathrm{Prim}(\mathcal{C}_0(\Omega_\alpha) \rtimes G)$. The result then follows from Corollary 3.4.2. $\square$

Let $\Omega := \widehat{\mathcal{E}_{\mathcal{S}}(X)} \smallsetminus X$. We now proceed to study $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$ and $\mathrm{Prim}(\mathcal{C}(\Omega) \rtimes X)$ using the results in [20, 63]. We first establish that the orbits are locally closed, using the results of the previous sections.

First of all, recall that, by Theorem 3.1.9, the set $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ identifies with the set of augmented $\mathcal{S}$-chains $(a, \overrightarrow{\alpha})$, with X acting only on $a \in X/Z(\overrightarrow{\alpha})$ by translations, see (3.10). Hence the set of all orbits of X acting on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ is in bijection with the set of all $\mathcal{S}$-chains. In particular, each of the sets $\mathcal{X}_{\mathcal{Z}}$ introduced in Equation (3.15) is X invariant and has *closed* orbits. Here, of course, $\mathcal{Z}$ is the $\mathcal{S}$-flag associated to any augmented $\mathcal{S}$-chain in an orbit of $\mathcal{X}_{\mathcal{Z}}$. Corollary 3.2.6 then yields the following result.

Lemma 3.4.4. *The orbits of X acting on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ and $\Omega := \widehat{\mathcal{E}_{\mathcal{S}}(X)} \smallsetminus X$ are locally closed.*

Recalling that the set of all orbits of X acting on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ is in natural bijection with the set of all $\mathcal{S}$-chains, Equation (3.10), gives that the stabilizer of the orbit associated to the $\mathcal{S}$-chain $\overrightarrow{\alpha}$ is $Z(\overrightarrow{\alpha})$. Therefore $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$ identifies the set of pairs $(\overrightarrow{\alpha}, \zeta)$, where $\overrightarrow{\alpha}$ is an $\mathcal{S}$-chain and ζ a character of $Z(\overrightarrow{\alpha})$.

In particular, since X is a single orbit in $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ (corresponding to the empty chain) and has stabilizer 0, it will contribute a single point to $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$, by Proposition 3.4.1 applied to the open inclusion $X \subset \widehat{\mathcal{E}_{\mathcal{S}}(X)}$. Moreover, this point is the spectrum of the algebra $\mathcal{C}_0(X) \rtimes X \cong \mathcal{K}(X)$. The orbits of X on $\Omega := \widehat{\mathcal{E}_{\mathcal{S}}(X)} \smallsetminus X$ will thus correspond to the non-empty $\mathcal{S}$-chains and we have a canonical isomorphism

$$\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X) \simeq \mathcal{C}(\Omega) \rtimes X. \tag{3.21}$$

(This isomorphism is also simply a consequence of the exact sequence obtained by taking the crossed product by X of the exact sequence $0 \to \mathcal{C}_0(X) \to \mathcal{E}_{\mathcal{S}}(X) \to \mathcal{C}(\Omega) \to 0$.)

We now study $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$, which is our primary interest. Let $\alpha \in \mathbb{S}_X$ and let $\Omega_\alpha := \{\chi \in \widehat{\mathcal{E}_{\mathcal{S}}(X)} |\ \chi|_{\mathcal{C}(\overline{X})} = \chi_\alpha\}$ be the set of characters of $\mathcal{E}_{\mathcal{S}}(X)$ that restrict to χ_α on $\mathcal{C}(\overline{X})$, as in Lemma 3.1.4. We obtain that

$$\Omega \;=\; \cup_{\alpha \in \mathbb{S}_X} \Omega_\alpha \,, \tag{3.22}$$

a disjoint union of closed subsets, with Ω_α the image of τ_α^* acting on the primitive ideal spectrum of $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$, see Lemma 3.1.5.

Theorem 3.4.5. *Let $\mathcal{S}$ be a finite semilattice of subspaces of X such that $0, X \in \mathcal{S}$. For each $\alpha \in \mathbb{S}_X$, we consider the map $\tau_\alpha \rtimes X : \mathcal{E}_{\mathcal{S}}(X) \rtimes X \to \mathcal{E}_{\mathcal{S}_\alpha}(X) \rtimes X$. Then the family $\{\tau_\alpha \rtimes X\}_{\alpha \in \mathbb{S}_X}$ is an exhausting family of morphisms of the C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$.*

Proof. We have $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X) \cong \mathcal{C}(\Omega) \rtimes X$, see Equation 3.21. The morphisms $\tau_\alpha \rtimes X$ then correspond to $j_\alpha \rtimes X$, where j_α is the restriction morphism from $\mathcal{C}(\Omega)$ to $\mathcal{C}(\Omega_\alpha)$. The result then follows from Corollary 3.4.3. (Note that we are in position to use this corollary in view of Lemma 3.4.4.) □

Let us notice that the morphisms $\tau_\alpha \rtimes X$ considered in the previous theorem were denoted simply τ_α before. Reverting to the original notation, for simplicity, we obtain

$$\sigma_{\mathrm{ess}}(P) \;:=\; \sigma_{\mathcal{E}_{\mathcal{S}}(X)/\mathcal{K}(X)}(P) \;=\; \cup_{\alpha \in \mathbb{S}_X} \sigma(\tau_\alpha(P)) \,,$$

where the first equality is valid by our definition of the essential spectrum and the second one is valid since the family τ_α is an exhausting family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$, which allows us to use the results of [46]. It was proved in [26] (see also [46]) that we can extend this property to affiliated operators. Therefore

$$\sigma_{\mathrm{ess}}(H) \;=\; \cup_{\alpha \in \mathbb{S}_X} \sigma(\tau_\alpha(H))$$

for H as in Theorem 1.3.1. This completes the proof of that theorem.

4

Exhaustive families of representations of C^*-algebras associated to the N-body Hamiltonians with asymptotically homogeneous interactions

We refer the reader to Section 1.4 for a detailed introduction of this Chapter.

4.1 The Melrose-Singer question

This chapter is based on the article "Exhaustive families of representations of C^*-algebras associated to the N-body Hamiltonians with asymptotically homogeneous interactions", joint work with Prudhon, C.R. Maths Ac. Sci. Paris 2019, 5 pages. Georgescu and Nistor in [21, 22, 27] have initiated a new approach in the study of Hamiltonians of N-body type with interactions that are asymptotically homogeneous at infinity on a finite dimensional Euclidean space X.

Recall from the Introduction that for any finite real vector space Z, $\overline{Z}$ denotes its spherical compactification.

For any subspace $Y \subset X$, $\pi_Y : X \to X/Y$ denotes the canonical projection. We recall H the Halmitonian of Equation 1.6.

$$H = -\Delta + \sum_{Y \in \mathcal{S}} v_Y \ , \tag{4.1}$$

where $v_Y \in C(\overline{X/Y})$ is seen as a bounded continuous function on X via the projection $\pi_Y : X \to X/Y$. The sum is over *all* subspaces $Y \subset X$, $Y \in \mathcal{S}$ and is assumed to be uniformly convergent. One of the main results of [27, 40] describe the essential spectrum of H extending the celebrated HVZ theorem [52]. The goal of this paper is to explain how these results can be extended to any family of subspaces that contains $\{0\}$ and to more general operators using C^*-algebras techniques.

Let $\mathcal{S}$ be a family of subspaces of X with $0 \in \mathcal{S}$. As before, we consider the commutative C^*-subalgebra $\mathcal{E}_{\mathcal{S}}(X)$ of the commutative C^*-algebra $C_b(X)$ of bounded uniformly continous functions on X by

$$\mathcal{E}_{\mathcal{S}}(X) = \langle C(\overline{X/Y}) \ , \ Y \in \mathcal{S} \ \rangle \subset \mathcal{C}_b(X). \tag{4.2}$$

The algebras $\mathcal{E}_{\mathcal{S}}(X)$ can be used to give an answer to a question of Melrose and Singer [39]. We shall need to reformulate Equation (1.5) to precise the question of Melrose and Singer : Let $n \in \mathbb{N}$ and

$X = \mathbb{R}^3$. We consider

$$\mathcal{P}_i^n = \{(x_1, \ldots, x_n) \in X^n \, ; \, x_i = 0\}$$
$$\mathcal{P}_{ij}^n = \{(x_1, \ldots, x_n) \in X^n \, ; \, x_i = x_j\}$$

Theorem 4.1.1. *Let n be an integer. Let $\mathcal{P}^n$ be the semilattice of subspaces of X^n generated by $\mathcal{P}_i^n$ and $\mathcal{P}_{ij}^n$. Then the spectrum $\Omega_{\mathcal{S}^n}$ of $\mathcal{E}_{\mathcal{S}^n}(X^n)$ is a compactification of X^n satisfying the following properties :*

1. *$\Omega_{\mathcal{S}^1}$ is the spherical compactification $\overline{X}$,*
2. *The action of the symmetric group $\mathfrak{S}_n$ on X^n extends continuously to $\Omega_{\mathcal{S}^n}$,*
3. *The projections $p_I^{n,k} \colon X^n \to X^k$, $p_I^{n,k}(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k})$ extend continuously to $p_I^{n,k} \colon \Omega_{\mathcal{S}_n} \to \Omega_{\mathcal{S}_k}$,*
4. *The difference maps $\delta_{ij}(x_1, \ldots, x_n) = x_i - x_j$ from X^n to X extend continuously to the compactifications.*

To prove this Theorem see Theorem 3.2.4 for convenient description of $\Omega_{\mathcal{S}}$.

The spectrum $\Omega_{\mathcal{S}^n}$ have very strong connection with the space built by Vasy in [59] and generalized by Kottke in the last section of [31], as we will see in Chapter 6.

4.2 Exhaustive families of representations and proofs

We recall results and notation of Section 2.3 The additive group X acts by translation on $C_b(X)$ and the subalgebra $\mathcal{E}_{\mathcal{S}}(X)$ is invariant. So a crossed product C^*-algebra is obtained

$$\mathcal{E}_{\mathcal{S}}(X) \rtimes X \, , \tag{4.3}$$

which can be regarded as an algebra of operators on $L^2(X)$. Thanks to the assumption $0 \in \mathcal{S}$, the algebra $C_0(X)$ belongs $\mathcal{E}_{\mathcal{S}}(X)$. Hence $C_0(X) \rtimes X$ is contained in $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. It follows from the definition of crossed products algebras that the C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ is generated by two kinds of operators : multiplication operators $f(q)$ associated to functions $f \in \mathcal{E}_{\mathcal{S}}(X)$, and convolution operators

$$\phi(p)u(x) := \int_X \phi(y)u(x-y)dy$$

with $\phi \in C_c(X)$, a continuous compactly supported function. An immediate computation shows that $f(q)\phi(p)$ (resp. $\phi(p)f(q)$) is a kernel operator with kernel

$$K(x,y) = f(x)\phi(y-x) \, , \qquad \text{(resp. } K(x,y) = f(y)\phi(y-x)). \tag{4.4}$$

Proposition 4.2.1. *For $f \in C(\overline{X})$ and $\phi \in C_c(X)$ the commutator $[f(q), \phi(p)]$ is compact.*

Thanks to equation (4.4), one sees that the commutator is a kernel operator with kernel

$$K(x,y) = \phi(y-x)(f(x) - f(y)) \, .$$

Hence, in view of $\phi \in C_c(X)$, the support of K is contained in a band around the diagonal. The distance between the border of the band and the diagonal is bounded. Moreover, K goes to 0 at infinity because f has radial limits. So the commutator is a limit of Hilbert-Schmidt operators, and hence is compact. Following Connes [12] and Baaj [4] we introduce the algebra $\Psi^\infty(\mathcal{E}_{\mathcal{S}}(X); X)$ of pseudodifferential operators associated to the action of X on $\mathcal{E}_{\mathcal{S}}(X)$. We shall need the C^*-algebra of $\Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$ given by the norm closure of $\Psi^0(\mathcal{E}_{\mathcal{S}}(X); X)$ and the exact sequence

$$0 \to \mathcal{E}_{\mathcal{S}}(X) \rtimes X \to \Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X) \xrightarrow{\sigma_0} C(\mathbb{S}_X \times \widehat{\mathcal{E}_{\mathcal{S}}(X)}) \to 0\,, \tag{4.5}$$

where σ_0 is the principal symbol map. Positive order pseudodifferential operators are examples of operators affiliated to the algebra of non positive order pseudodifferential operators $\Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$.

Let $\alpha \in \mathbb{S}_X$. For each $x \in X$, we let $(T_x f)(y) = f(y - x)$ denote the translation on $L^2(X)$. For any operator P on $L^2(X)$, we let

$$\tau_\alpha(P) = \lim_{r\to+\infty} T^*_{ra} P T_{ra}\,, \quad \text{if } \alpha = \hat{a} \in \mathbb{S}_X\,, \tag{4.6}$$

whenever the *strong* limit exists.

Lemma 4.2.2. *For $f \in C(\overline{X/Y})$ one has*

$$\tau_\alpha(f)(x) = \begin{cases} f(x) & \text{if } Y \supset \alpha\,, \\ f(\pi_Y(\alpha)) & \text{else.} \end{cases}$$

We define $\mathcal{S}_\alpha = \{Y \in \mathcal{S}\,;\, \alpha \subset Y\}$. It follows from the previous lemma that on $\mathcal{E}_{\mathcal{S}}(X)$, τ_α is the projection on the subalgebra $\mathcal{E}_{\mathcal{S}_\alpha}(X)$,

$$\tau_\alpha \colon \mathcal{E}_{\mathcal{S}}(X) \to \mathcal{E}_{\mathcal{S}_\alpha}(X)\,.$$

Theorem 4.2.3. *1. Let P be a self-adjoint operator affiliated to $\Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$ and $\alpha = \hat{a} \in \mathbb{S}_X$. Then the limit $\tau_\alpha(P) := \lim_{r\to+\infty} T^*_{ra} P T_{ra}$ exists.*

2. *Let $P \in \Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$. Then P is a Fredholm operator if and only if P is elliptic (i.e. $\sigma_0(P)$ is invertible) and for all $\alpha \in \mathbb{S}_X$, $\tau_\alpha(P)$ is invertible.*
3. *If $P \in \Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$,*

$$\sigma_{\mathrm{ess}}(P) = \cup_{\alpha\in\mathbb{S}_X} \sigma(\tau_\alpha(P)) \cup Im(\sigma_0(P))\,.$$

4. *If $P \in \Psi^m(\mathcal{E}_{\mathcal{S}}(X); X)$, $m > 0$, is elliptic, then,*

$$\sigma_{\mathrm{ess}}(P) = \cup_{\alpha\in\mathbb{S}_X} \sigma(\tau_\alpha(P))\,.$$

Note that in classical results on the N-body problem, one usually has the closure of the union in the spectral decomposition. See however [22, 40]. See also [27, 40] for related results, where operators affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ were considered.

Remark 4.2.4. It was noticed and explained in [23] that the terms $\sigma_0(P)(\xi)$ appearing in $Im(\sigma_0(P))$ are also some sort of "limit operators," in the sense that they can be realized by conjugation with some multiplication operators and passage to the limit. We owe this comment to Georgescu.

In [40] only finite semilattice $\mathcal{S}$ are considered. The closure of the union means that the family (τ_α) is a *faithful* family of morphism of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. The stronger result of [40] is obtained by showing that the family $(\tau_\alpha \rtimes X)_{\alpha\in\mathbb{S}_X}$ is actually an *exhaustive* family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$, when $\mathcal{S}$ is a finite semilattice. In [25], pseudodifferential operators on $\mathbb{R}$ were considered (see Remark 3.23 of that paper). In the framework of admissible locally compact group, decomposition of essential spectrum involving *exhaustive* families can be found in [41] [44].

Theorem 4.2.5. *Let $\mathcal{S}$ be a family of subspaces of X with $0 \in \mathcal{S}$. Then the family $(\tau_\alpha \rtimes X)_{\alpha \in \mathbb{S}_X}$ is an exhaustive family of $(\mathcal{E}_\mathcal{S}(X) \rtimes X)/\mathcal{K}(X)$.*

Let us prove this result. Let π be an irreducible representation of $\mathcal{E}_\mathcal{S}(X) \rtimes X/\mathcal{K}(X)$. It extends to an irreducible representation of $\mathcal{E}_\mathcal{S}(X) \rtimes X$ as well as to their multipliers algebras $\mathcal{M}(\mathcal{E}_\mathcal{S}(X) \rtimes X/\mathcal{K}(X))$ and $\mathcal{M}(\mathcal{E}_\mathcal{S}(X) \rtimes X)$. By proposition 4.2.1(i), one obtains the following commutative diagram :

$$\begin{array}{ccccccc} C(\overline{X}) & \hookrightarrow & \mathcal{E}_\mathcal{S}(X) & \longrightarrow & \mathcal{M}(\mathcal{E}_\mathcal{S}(X) \rtimes X) & & \\ \downarrow p & & & & \downarrow & & \\ C(\mathbb{S}_X) & & \xrightarrow{\phi} & & \mathcal{M}(\mathcal{E}_\mathcal{S}(X) \rtimes X/\mathcal{K}(X)) & \xrightarrow{\pi} & \mathcal{B}(\mathcal{H}_\pi) \end{array} \tag{4.7}$$

Lemma 4.2.6. *The image $\phi(C(\mathbb{S}_X))$ is central in $\mathcal{M}(\mathcal{E}_\mathcal{S}(X) \rtimes X/\mathcal{K}(X))$.*

In fact it is enough to show that any $f \in C(\overline{X})$ commutes with any element of $\mathcal{E}_\mathcal{S}(X) \rtimes X$ modulo a compact operator. But the result is true on the generators by Proposition 4.2.1, so the lemma follows by density.

By the Schur Lemma, we deduce that $\pi \circ \phi$ is a character of $C(\mathbb{S}_X)$. Hence there exists some $\alpha \in \mathbb{S}_X$ such that $\pi|_{C(\overline{X})} = \chi_\alpha I$, where χ_α is the character of $C(\overline{X})$ given by the evaluation at $\alpha \in \mathbb{S}_X$.

Proposition 4.2.7. *One has* $\ker \tau_\alpha = (\ker \chi_\alpha)\mathcal{E}_\mathcal{S}(X)$.

Proof. We need to show that $\mathcal{E}_\mathcal{S}(X)/\ker \tau_\alpha = \mathcal{E}_{\mathcal{S}_\alpha}(X)$ and $\mathcal{E}_\mathcal{S}(X)/(\ker \chi_\alpha)\mathcal{E}_\mathcal{S}(X)$ have the same characters. By definition, for any character χ of $\mathcal{E}_{\mathcal{S}_\alpha}(X)$, there exists a unique character χ' of $\mathcal{E}_\mathcal{S}(X)$ such that $\chi' = \chi \circ \tau_\alpha$. In view of lemma 4.2.2, this is equivalent to the following :

$$(\forall Y \in \mathcal{S}, \alpha \not\subset Y, \forall u \in C(\overline{X/Y})) \quad \chi(u) = u(\pi_Y(\alpha)). \tag{4.8}$$

In particular, for $Y = 0$, we see that $\chi_{|C(\overline{X})} = \chi_\alpha$. Reciprocally it follows from [27, Lemma 6.7] that if $\chi_{|C(\overline{X})} = \chi_\alpha$ then relation (4.8) is true. On the other hand, the characters of $\mathcal{E}_\mathcal{S}(X)/(\ker \chi_\alpha)\mathcal{E}_\mathcal{S}(X)$ are precisely the characters χ of $\mathcal{E}_\mathcal{S}(X)$ such that $\chi_{C(\overline{X})} = \chi_\alpha$. So $\ker \tau_\alpha = (\ker \chi_\alpha)\mathcal{E}_\mathcal{S}(X)$ as claimed. □

Now if $\pi_{|C(\overline{X})} = \chi_\alpha$, one has $\ker \pi \supset (\ker \chi_\alpha)\mathcal{E}_\mathcal{S}(X) = \ker \tau_\alpha$. Finally,

$$\ker(\tau_\alpha \rtimes X) = (\ker \tau_\alpha) \rtimes X \subset \ker \pi \,.$$

It follows that $(\tau_\alpha \rtimes X)_{\alpha \in \mathbb{S}_X}$ is an exhaustive family of morphisms.

Remark 4.2.8. The results presented here can easily be extended to pseudodifferential operators with matrix coefficients. For example, Dirac operators $D_V = D + V$, with potentials V as in (1.6) may be considered and satisfy the condition of Theorem 4.2.3.

See also [26, Example 6.35] for others physical interesting operators.

5

Essential spectrum, quasi-orbits and compactifications : application to the Heisenberg group

We refer the reader to Section 1.5 for a detailed introduction of this Chapter. Details on the definition of the Laplacian and its affiliation is discuss in Section 2.6

5.1 Quasi-regular dynamical system

Let (A, θ, G) be a C^*-dynamical system, where $A = \mathcal{C}(\Omega)$ is a commutative and a unital C^*-algebra. The quasi-regularity is a property of the C^*-dynamical system that links the quasi-orbits of the action G on Ω and the primitive spectrum of $A \rtimes G$. Topological conditions on the system leads to quasi-regularity and then to a convenient algebraic decomposition of the primitive spectrum of the crossed product.

Recall that for a compact dynamical system (Ω, θ, G), we denote by O^ω, the orbit of $\omega \in \Omega$. The quasi-orbit of ω, which is the closure of O^ω, will be denoted by Q^ω.

Definition 5.1.1. *Let (Ω, θ, G) be a compact dynamical system, this dynamical system is said to be* quasi-regular *if each irreducible representation of $C(\Omega) \rtimes G$ lives on a quasi-orbit. More precisely, let Π be an irreducible representation of $C(\Omega) \rtimes G$ and (π, U) the covariant pair that realizes Π (see Proposition 2.3.3). The representation Π lives on a quasi-orbit if there exists $\omega \in \Omega$ such that $Res(\ker(\Pi)) :=$ $\ker(\pi) = C^{\Omega^\omega}(\Omega) = \{f \in \mathcal{C}(\Omega), f_{|Q^\omega} = 0\}$.*

The following proposition gives a topological condition that implies the quasi-regularity.

Proposition 5.1.2 (Gootman-Rosenberg-Sauvageot)**.** *Let (A, θ, G) be a C^*-dynamical system. We suppose that A is separable and G is second countable. In this case, the C^*-dynamical system is quasi-regular.*

Proposition 5.1.2 is a consequence of Theorem 8.21 in [63]. It states that with the same assumptions, the C^*-dynamical system is *EH-regular*, which is a stronger condition than quasi-regularity. However, we only need quasi-regularity is this paper.

Definition 5.1.3. *A set $\{\omega_i, | i \in I\}$ of points of Ω is a called a* sufficient family *if the associated quasi-orbit $\{Q^{\omega_i} | i \in I\}$ form a covering of the space Ω, in other words, $\bigcup_{i \in I} Q^{\omega_i} = \Omega$.*

An important propriety is given by the following proposition, which is Proposition 6.3 in [44].

Proposition 5.1.4. *Let (Ω, θ, G) be a compact quasi-regular dynamical system, and $\{\omega_i, i \in I\}$ a sufficient family of points of Ω. We consider for every $i \in I$, the restriction morphism :*

$$p_i : \mathcal{C}(\Omega) \to C(Q^{\omega_i}), \quad p_i(f) = f|_{Q^{\omega_i}}$$

and then extend it to the crossed product

$$P_i := p_i^{\rtimes} : \mathcal{C}(\Omega) \rtimes G \to \mathcal{C}(Q^{\omega_i}) \rtimes G.$$

The family of morphims $\mathcal{P} := \{P_i, i \in I\}$ is then an exhaustive family of morphisms of the C^*-algebra $\mathcal{C}(\Omega) \rtimes G$.

5.2 Compactification of H.

In this subsection, we will consider two compactifications of the Heisenberg group and the induced action of H on each of these compactifications. We will denote by $\|.\|$ the usual euclidean norm.

5.2.1 The spherical compactification

We consider the spherical compactification of H induced by the canonical identification (2.20). We will denote by $\overline{H}$ this compactification, hence, we have $\overline{H} = H \cup \mathbb{S}_H$ with $\mathbb{S}_H \simeq (\mathbb{R}^3)^*/(\mathbb{R}_+^*) \simeq \mathbb{S}^2$. We shall need an explicit identification between $(\mathbb{R}^3)^*/(\mathbb{R}_+^*)$ and $\mathbb{S}^2$. Let $\alpha \in (\mathbb{R}^3)^*/(\mathbb{R}_+^*)$, then by definition, there exists $V \in H \setminus \{0\}$ such that the equivalence class α is the half-line $\mathbb{R}_+^* V$. We choose the point $v = (a, b, c) \in \mathbb{R}_+^* V$ such that $\|v\| = 1$, then the point $v \in \mathbb{S}^2$ characterizes α. Conversely to each point of the sphere, $z \in \mathbb{S}^2$, we can associate the half-line $\mathbb{R}_+^* z$ to obtain an element of $(\mathbb{R}^3)^*/(\mathbb{R}_+^*)$. A sequence $U_n = (a_n, b_n, c_n) \in H$ converges to the point $(a, b, c) \in \mathbb{S}_H$ if :

$$\lim_{n \to +\infty} \|U_n\| = +\infty, \quad \lim_{n \to +\infty} \frac{U_n}{\|U_n\|} = (a, b, c). \tag{5.1}$$

Translation at infinity

Let l_X be the left translation by the vector $X \in H$. The left translation l_X can be extended from $H \to H$ to $H \to \overline{H}$ that is, we want define $l_X(\alpha)$ for $\alpha \in \mathbb{S}_H$. It is convenient to have an explicit formula of $l_X(\alpha)$, with $X = (x, y, z) \in H$ and $\alpha = (a, b, c) \in \mathbb{S}_H$. Let U_n be as in equation (5.1), we have

$$l_X(U_n) = (x + a_n, y + b_n, z + c_n + xb_n)$$

Then

$$\begin{aligned} \|l_X(U_n)\|^2 &= x^2 + y^2 + z^2 + a_n^2 + b_n^2 + c_n^2 + x^2 b_n^2 \\ &\quad +2(xa_n + yb_n + zc_n + xb_nc_n + xzb_n) \\ &= \|U_n\|^2 \left(\frac{a_n^2 + b_n^2 + (c_n + xb_n)^2}{\|U_n\|^2} + \epsilon_n \right) \end{aligned}$$

with $\epsilon_n = \dfrac{\|X\|^2 + 2\langle X, U_n \rangle + 2xzb_n}{\|U_n\|^2} \to 0$. Moreover, $a^2 + b^2 + c^2 = 1$, hence for every $x \in \mathbb{R}$, the quantity $C := a^2 + b^2 + (c + xb)^2$ is always positive. Indeed, if $b \neq 0$ then $C \neq 0$, and if $b = 0$ then $C = a^2 + c^2 = 1$. This leads to :

$$\|l_X(U_n)\|^2 \sim \|U_n\|^2 \left(a^2 + b^2 + (c + xb)^2\right) = +\infty.$$

We obtain :

$$\frac{l_X(U_n)}{\|l_X(U_n)\|} \to \frac{1}{\sqrt{a^2+b^2+(c+xb)^2}}(a,b,c+xb).$$

The two points $(a,b,c+xb)$ and $\frac{1}{\sqrt{a^2+b^2+(c+xb)^2}}(a,b,c+xb)$ have the same equivalence class in $(\mathbb{R}^3)^*/(\mathbb{R}_+^*)$, hence we can drop the constant and only consider $(a,b,c+bx)$ for the limit of $l_X(U_n)$.

Characterization of the quasi-orbits

We can sum up the preceding subsection with this equality :

$$l_X((a,b,c)) = \lambda(a,b,c+bx) \tag{5.2}$$

where $X=(x,y,z)\in H, \alpha=(a,b,c)\in\mathbb{S}_H$ and $\lambda\in\mathbb{R}_+^*$. Using this relation, we can characterize the set of fixed points : $O_{fix} := \{(a,0,c), a^2+c^2=1\}$. For $\alpha\in\mathbb{S}_H\setminus O_{fix}$, that is $\alpha=(a,b,c)$ with $b\neq 0$. The orbit of α is :

$$O^\alpha = \{\mathbb{R}_+^*(a,b,x), x\in\mathbb{R}\}.$$

The set $Z_\alpha = \{(a,b,x), x\in\mathbb{R}\}$ is a line in $\mathbb{R}^3$ not containing the origin. The orbit O^α is the set of the half-lines starting at 0 that intersect Z_α. The pre-image in the projective space of O^α is a half-space. Then the intersection of this half-space and $\mathbb{S}^2$ is the open great half-circle of $\mathbb{S}^2$ with end points $(0,0,1),(0,0,-1)$ and passing a through the point $\frac{1}{\sqrt{a^2+b^2}}(a,b,0)$. The quasi-orbit is then the closed great half-circle. The difference between the orbit and the quasi-orbit is :

$$Q^\alpha = O^\alpha \sqcup \{(0,0,1),(0,0,-1)\}.$$

Let $\mathcal{F}$ be a family of points of $\mathbb{S}_H$ such that $\bigcup_{\alpha\in\mathcal{F}} Q^\alpha = \mathbb{S}_H$. We also assume that $\mathcal{F}$ is minimal is the sense that each points of $\mathcal{F}$ is necessary to make the cover of $\mathbb{S}_H$. With these choices, $\mathcal{F}$ is a sufficient family of the dynamical system $(\mathbb{S}_H, l, H)$, that is $\mathbb{S}_H = \bigcup_{\alpha\in\mathcal{F}} Q^\alpha$. An example of a minimal sufficient family is

$$\mathcal{F} = \{(a,b,0), a^2+b^2=1\}\cup\{(a,0,c), a^2+c^2=1, a\neq 0\}.$$

5.2.2 Spherical compactification via the exponential map

We consider another compactification of H. Let $\overline{\mathfrak{h}}$ be the spherical compactification of $\mathfrak{h}$ via the identification (2.21). We define $\widetilde{H}$ as the image of the exponential map of $\overline{\mathfrak{h}}$, formally $\widetilde{H} = \exp(\overline{\mathfrak{h}})$. Each point $s\in\widetilde{H}\setminus H$ is the limit of a sequence $\exp(U_n)$, where $U_n=(a_n,b_n,c_n)_0\in\mathfrak{h}$ and

$$\lim_{n\to+\infty} ||U_n|| = +\infty, \quad \lim_{n\to+\infty}\frac{U_n}{||U_n||} = (a,b,c)_0$$

with $||(a,b,c)_0|| = 1$.

Translation at infinity and the exponential map

As before, we want to extend the operator l_X to $\widetilde{H}$. That is, we want to find the point $(a',b',c')\in\widetilde{H}\setminus H$ such that

$$\lim_{n\to+\infty}\frac{V_n}{||V_n||} = (a',b',c')_0,$$

where $V_n \in \mathfrak{h}$ and checks $\exp(V_n) = l_X(\exp(U_n))$. We have :

$$X.\exp(U_n) = (a_n + x, b_n + y, c_n + z + xb_n + \frac{a_n b_n}{2}).$$

If we take $V_n = (a_n + x, b_n + y, c_n + z + \frac{1}{2}(xb_n - ya_n - xy))_0$, we obtain :

$$\|V_n\|^2 = \|U_n\|^2 \left(\frac{a_n^2 + b_n^2 + (c_n + \frac{xb_n}{2} - \frac{ya_n}{2})^2}{\|U_n\|^2} + \epsilon_n \right),$$

with

$$\epsilon_n = \frac{1}{||U_n||^2} \left(\|X\|^2 + 2\langle X, U_n \rangle + \frac{x^2 y^2}{4} - xyc_n + xzb_n - xyz - \frac{xy}{2}(xb_n - ya_n) \right).$$

Note that $\lim_{n \to +\infty} \epsilon_n = 0$. Moreover, $a^2 + b^2 + c^2 = 1$, hence for every $X \in H$, the quantity $C' := a^2 + b^2 + (c + \frac{xb}{2} - \frac{ya}{2})^2$ is always positive. To see this, suppose that $a = b = 0$ then $C' = c^2 = 1$. Otherwise, $0 < a^2 + b^2 \leq C'$. This leads to :

$$||V_n||^2 \sim ||U_n||^2 \left(a^2 + b^2 + (c + \frac{xb}{2} - \frac{ay}{2})^2 \right) \to +\infty.$$

Characterization of the orbit of exponential-radial compactification

We can sum up the preceding subsection with this equality :

$$l_X(\exp((a, b, c)_0)) = \exp(\lambda(a, b, c + \frac{1}{2}(xb - ay))_0) \tag{5.3}$$

where $X = (x, y, z) \in H, \alpha = (a, b, c)_0 \in \widetilde{H} \setminus H$ and $\lambda \in \mathbb{R}_+^*$. Using this relation, we note that $(0, 0, 1)$ and $(0, 0, -1)$ are the only fixed points. If $\alpha = (a, b, c)$ is not a fixed point then

$$O^{\exp(\alpha)} = \{(\mathbb{R}_+^*(a, b, x), x \in \mathbb{R}\}.$$

An example of a minimal sufficient family for $\exp(\overline{\mathfrak{h}})$ is

$$\mathcal{F} = \{(a, b, 0), a^2 + b^2 = 1\}.$$

Comparison of the (quasi)-orbits

We stress the differences between the two structure of orbits and quasi-orbits of the two compactifications $\overline{H}$ and $\widetilde{H} = \exp(\overline{\mathfrak{h}})$. In the two cases, we can identify the part at infinity with $\mathbb{S}^2$. For $\alpha \in \mathbb{S}^2$, we recall that the notation O^α is the orbit for the action of H on $\overline{H}$ and $O^{\exp(\alpha)}$ for the action of H on $\exp(\overline{\mathfrak{h}})$. By the equation (5.2) and (5.3), we obtain for $\alpha = (a, b, c) \in \mathbb{S}^2$:

— The two points $(0, 0, 1)$ and $(0, 0, -1)$ are fixed points in both cases.

— If $b \neq 0$ then $O^\alpha = O^{\exp(\alpha)}$.

— If $b = 0$ and $a \neq 0$, then $O^\alpha = \{\alpha\}$ and $O^{\exp(\alpha)} = \{(a, 0, x), x \in \mathbb{R} | a^2 + x^2 = 1\}$.

In other words, the main difference is that the great circle passing through the points $(0, 0, -1), (0, 0 - 1)$ and $(1, 0, 0)$ is the set of fixed points for $\overline{H}$. For $\exp(\overline{\mathfrak{h}})$, this great circle splits into two open half great circles : $O^{\exp(-1,0,0)}$ and $O^{\exp(1,0,0)}$ and two fixed point : $(0, 0, -1)$ and $(0, 0, 1)$.

5.3 An exhaustive family for $\mathcal{C}(\mathbb{S}_H) \rtimes H$.

For each function $f : H \to \mathbb{C}$, as in Subsection 2.3.2, we will denote by $f(q)$ the operator of multiplication acting on $L^2(X)$. This kind of operator is well defined for instance when $f \in \mathcal{C}_b(H)$. Let $\alpha \in \mathbb{S}_H$ and U_n a sequence of elements of H that converges to α. We want to characterize the function f_α such that $f_\alpha(q)$ is "invariant by right translations at the infinity." Thus, we consider f_α, which is the limit of the operator $f_n(q) := R_{U_n} f(q) R^*_{U_n}$, with $f \in \mathcal{C}(\widetilde{H})$. For any element $X \in H$ and $\phi \in L^2(H)$, we write

$$(f_n(q)\phi)(X) = R_{U_n} f(q) R^*_{U_n}(\phi)(X) = f(R_{U_n}(X))\phi(X) = f(XU_n)\phi(X).$$

When n goes to infinity $XU_n \to X\alpha$ and the value of $L_X(\alpha) = X\alpha$ hence it is an element of the quasi-orbits of Q^α. Hence, the function $f_\alpha := \lim f_n$ associated to the limit operator of $f_n(q)$ can be view as an element of $\mathcal{C}(Q^\alpha)$. At the level of functions, the link between f and f_α is $R_\alpha(f) = f_\alpha$, where R_α is the pointwise limit of the right translation R_{U_n}. Moreover, using the translation R_α, we can build an exhaustive family.

Theorem 5.3.1. *Let $\widetilde{H} = H \cup \mathbb{S}_H$ be one of the two compactifications of H considered in Section 5.2. We consider a sufficient family $\mathcal{F}$ of points of $\mathbb{S}_H$ with the extended action of H. For each $\alpha \in \mathcal{F}$, we extend the translation R_α to the crossed product : $\mathcal{C}(\widetilde{H}) \rtimes H \to \mathcal{C}(Q^\alpha) \rtimes H$. With this assumption, the family $\{R_\alpha\}_{\alpha \in \mathcal{F}}$ is an exhaustive family of morphisms of $\mathcal{C}(\mathbb{S}_H) \rtimes H$.*

The two main tools of the proof are the following lemma and the well behavior of the product constructions.

Lemma 5.3.2. *Let R_α be the right translation at the infinity defined as before. We have :*

$$\ker(R_\alpha) \subset \mathcal{C}^{\Omega^\alpha}(\widetilde{H}).$$

Where $C^{Q^\alpha}(\widetilde{H}) = \{f \in \mathcal{C}(\widetilde{H}), f_{|Q^\alpha} = 0\}$.

Proof of Lemma 5.3.2. Let $f \in \mathcal{C}(\widetilde{H})$ such that $f(\beta) \neq 0$, for $\beta \in Q^\alpha$. We want to show that $R_\alpha(f) \neq 0$. By definition of β, there exists a sequence U_n of elements of H such that $l_{U_n}(\alpha) = U_n.\alpha \to \beta$. The continuity of f implies $R_\alpha(f)(U_n) = f(U_n.\alpha) \to f(\beta) \neq 0$. We conclude that $R_\alpha(f) \neq 0$ and its finish the proof. □

Proof of Theorem 5.3.1. For each $\alpha \in \mathbb{S}_H$, we have $R_\alpha(\mathcal{C}_0(H)) = 0$ hence the function R_α can be defined on $\mathcal{C}(\widetilde{H})/\mathcal{C}_0(H) \simeq \mathcal{C}(\mathbb{S}_H)$. We extend R_α to the crossed product $\mathcal{C}(\mathbb{S}_H) \rtimes H$. Now, we show that the family $(R_\alpha)_{\alpha \in \mathcal{F}}$ is exhaustive. Let J be a primitive ideal of $\mathcal{C}(\mathbb{S}_H) \rtimes H$ and Π be an irreducible representation of $\mathcal{C}(\mathbb{S}_H) \rtimes H$ with kernel J. In view of Proposition 5.1.2, the separability of the $\mathcal{C}(\mathbb{S}_H)$ implies the quasi-regularity of $(\mathcal{C}(\mathbb{S}_H), l, H)$. Quasi-regularity implies the existence of a covariant pair (π, U) and $\beta \in \mathbb{S}_H$ such that $\Pi = \pi \rtimes U$ and $\ker \pi = \mathcal{C}^{Q^\beta}(\widetilde{H})$. Moreover, the family $\mathcal{F}$ is a sufficient family hence, there exists $\alpha \in \mathcal{F}$ such that $\beta \in Q^\alpha$ then $Q^\beta \subset Q^\alpha$ or equivalently $\mathcal{C}^{Q^\alpha}(\mathbb{S}_H) \subset \mathcal{C}^{Q^\beta}(\mathbb{S}_H)$. In view of Lemma 5.3.2, we have

$$\ker(R_\alpha) \subset \mathcal{C}^{Q^\alpha}(\mathbb{S}_H) \subset \mathcal{C}^{Q^\beta}(\mathbb{S}_H).$$

Then the equality holds when we pass to the crossed product and $\ker(R_\alpha) \subset J$. □

Proposition 5.3.3. *Let T be an operator affiliated to $\mathcal{C}(\widetilde{H}) \rtimes H$. We have the following spectral decomposition :*

$$\sigma_{ess}(T) = \bigcup_{\alpha \in \mathcal{F}} \sigma(T_\alpha), \tag{5.4}$$

where T_α is the image of T through R_α.

Corollary 5.3.4. *In particular, when $T \in \mathcal{C}(\widetilde{H}) \rtimes H$ and the operator T is associated to $\phi \in L^1(H, \mathcal{C}(\widetilde{H}))$ via the formula $T = Sch(\phi)$, we obtain*

$$\sigma_{ess}(T) = \bigcup_{\alpha \in \mathcal{F}} \sigma(R_\alpha(Sch(\phi))).$$

Corollary 5.3.5. *In particular, for $T = -\Delta + V$, an operator of Schrödinger-type with a potential $V \in \mathcal{C}(\widetilde{H})$ such that V is a real potential, the decomposition* (5.4) *holds.*

Proof. By Corollary 2.6.2, the Laplacian is affiliated to $\mathcal{C}(H^+) \rtimes H \subset \mathcal{C}(\widetilde{H}) \rtimes H$. The identity

$$(T + i) = -\Delta + V + i = (-\Delta + i)[1 + (-\Delta + i)^{-1}V]$$

leads to $T \in' \mathcal{C}(\widetilde{H}) \rtimes H$ hence, we can apply Proposition 5.3.3 on T. □

Proof of proposition 5.3.3. The isomorphism $\mathcal{C}(\widetilde{H})/\mathcal{C}_0(H) \simeq \mathcal{C}(\mathbb{S}_H)$ can be extended to an isomorphism of $(\mathcal{C}(\widetilde{H}) \rtimes H)/(\mathcal{C}_0(H) \rtimes H) \simeq \mathcal{C}(\mathbb{S}_H) \rtimes H$, since H is amenable. Moreover $\mathcal{C}_0(H) \rtimes H \simeq \mathcal{K}(L^2(H))$ and it is well-known that the essential spectrum of an element coincides with the usual spectrum of its image in the quotient by the compact operators. In other words, for T an operator affiliated to $\mathcal{C}(\widetilde{H}) \rtimes H$, we have $\sigma_{ess}(T) = \sigma(\pi(T))$, where $\pi : \mathcal{C}(\widetilde{H}) \rtimes H \to \mathcal{C}(\mathbb{S}_H) \rtimes H$. By Proposition 5.3.1, the family $(R_\alpha)_{\alpha \in \mathcal{F}}$ is an exhaustive family of morphisms, which gives the following spectral decomposition

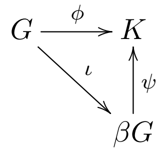
$$\sigma_{ess}(T) = \sigma(\pi(T)) = \bigcup_{\alpha \in \mathcal{F}} \sigma(R_\alpha(\pi(T))),$$

and hence the decomposition (5.4). □

5.4 Algebras generated by a family of compactifications

A similar approach could be used to study more complicated compactification of H (or more generally locally compact group). They often arise from continuous map $\phi : H \to K$ with, $\phi(H)$ is dense in a compact set K. The problem is to understand these compactifications. We describe a convenient approach in this section.

Let G be a locally compact group and K be a compact space with a continuous map $\phi : X \to K$. We also assume that $\phi(G)$ is dense in K. Let βG the Stone-Čech compaticfitcation of G and $\iota : G \to \beta G$ the embedding of G in βG. By the universal propriety of the Stone-Čech compactification, there exists a unique continuous map $\psi : \beta G \to K$ that makes the following diagram commutative :



Let $\psi_* : \mathcal{C}(K) \to \mathcal{C}(\beta G)$ be the pull-back induced by ψ. The commutativity of the preceding diagram and the density of $\phi(G)$ in K implies the injectivity of ψ_*. We can view $\mathcal{C}(K)$ as unital C^*-subalgebra of $\mathcal{C}(\beta G)$. The space βG is the spectrum of the C^*-algebra $\mathcal{C}_b(G)$, the continuous bounded functions on G equipped with the supremum norm. This leads to the isomorphism $\mathcal{C}(\beta G) \simeq \mathcal{C}_b(G)$. With all this identification, we can view $\mathcal{C}(K)$ as C^*-subalgebra of $\mathcal{C}_b(G)$.

We consider a family $\mathcal{K}$ of compact spaces and a continuous map $\phi_K : G \to K$ for each $K \in \mathcal{K}$ such that $\phi_K(X)$ is dense in K. We define the C^*-algebra generate by :

$$\mathcal{E}_{\mathcal{K}}(X) := \langle \mathcal{C}(K), \mathcal{C}_0(G), K \in \mathcal{K} \rangle.$$

The algebra $\mathcal{E}_{\mathcal{K}}(X)$ remains a C^*-algebra of $\mathcal{C}_b(G)$ because each generator is contained in $\mathcal{C}_b(G)$. Following Theorem 4.4 in [40], we will give a characterization of the spectrum of $\mathcal{E}_{\mathcal{K}}(G)$. We combine all the functions ϕ_K with the identity map on X to define Φ :

$$\Phi : id_G \times \prod_{K\in\mathcal{K}} \phi_k : G \to G \times \prod_{K\in\mathcal{K}} K, \quad \Phi(x) = (x, (\phi_K(x))_{K\in\mathcal{K}}).$$

For each $K \in \mathcal{K}$, we consider γ_K, the restriction map define by :

$$\gamma_K : \widehat{\mathcal{E}_{\mathcal{K}}(X)} \to \widehat{\mathcal{C}(K)} \simeq K, \quad \chi \mapsto \chi_{|\mathcal{C}(K)} = x_K.$$

As before, we combine all the map γ_K and add the restriction to $\mathcal{C}_0(X)$ to define $\Gamma : \widehat{\mathcal{E}_{\mathcal{K}}(X)} \to G \times \prod_{K\in\mathcal{K}} K$ with :

$$\chi \mapsto (x, x_K), \text{where } \chi(f) = f(x_K), f \in \mathcal{C}(K), K \in \mathcal{K}.$$

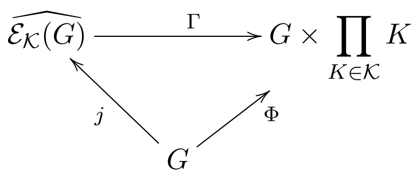
With this assumption, we can generalize the lemma 4.3 of [40].

Lemma 5.4.1. *The map Γ is continuous and a homeomorphism onto its image.*

Proof. We recall the argument in [40]. The continuity comes from the fact that the restriction of a character is continuous. The injectivity is a consequence of the construction of $\mathcal{E}_{\mathcal{K}}(G)$: the values of the character on the generators determines the character everywhere. We have a continuous map between two compact spaces hence, a homeomorphism on its image. □

Let $j : G \to \widehat{\mathcal{E}_{\mathcal{K}}(G)}$ be the extension of a character from the ideal $\mathcal{C}_0(G)$ to the algebra $\mathcal{E}_{\mathcal{K}}(G)$. We have all the notation to generalize Theorem 4.4 of [40].

Theorem 5.4.2. *The following diagram is commutative :*



Moreover, the diagram induces an homeomorphism between $\widehat{\mathcal{E}_{\mathcal{K}}(G)}$ and $\overline{\Phi(G)}$, the closure of the image of $\Phi(G)$.

Proof. For each $K \in \mathcal{K}$, the image of $\phi_K \circ j$ is given by the extension a character χ_x of $\mathcal{C}_0(G)$ to the algebra $\mathcal{E}_{\mathcal{K}}(G)$ and the restrict to $\mathcal{C}(K)$. This extension is unique and correspond to the character $\chi_{\phi_K(x)}$, that is, to the evaluation map at $\phi_K(x)$. Recall that $\mathcal{C}_0(G)$ is an essential ideal of $\mathcal{E}_{\mathcal{K}}(G)$ and then G is dense in the spectrum of $\mathcal{E}_{\mathcal{K}}(G)$. The continuity of j, Γ and Φ implies

$$\overline{j(G)} = \widehat{\mathcal{E}_{\mathcal{K}}(G)}, \quad \Gamma(\widehat{\mathcal{E}_{\mathcal{K}}(G)}) = \Gamma(\overline{j(G)}) = \overline{\Gamma(j(G))} = \overline{\Phi(G)}.$$

□

Example 5.4.3. Let G be a locally compact group and $\mathcal{F}$ a family of subgroups of G. For each $H \in \mathcal{F}$, we suppose that there exists a compactification of the quotient G/H such that the action of G on G/H extend to $\widetilde{G/H}$. The continuous map with dense image is given by the canonical map $\pi_H : G \to \widetilde{G/H}$.

Similar mixed algebras have been studied in [41] and by Power [48]. In particular, Power introduced the notion of *permanent point* to characterize the spectrum of the mixed (repeated compactification) algebras.

Definition 5.4.4. *Let A be a unital C^*-algebra and $(A_i)_{i\in I}$ a family of unital C^*-subalgebra of A. A point $\chi = (\chi_i)_{i\in I} \in \prod_{i\in I} \widehat{A_i}$ is called* permanent point *if for every Γ, a finite subset of I, the character χ verifies the following propriety :*
If, for every index $\gamma \in \Gamma$ and every contraction $a_\gamma \in A_\gamma$, $0 \leq a_\gamma \leq 1$, the equality $\chi_\gamma(a_\gamma) = 1$ is fulfilled, then $\| \prod_{\gamma\in\Gamma} a_\gamma\| = 1$.

The notion of "permanent point" is convenient to describe the spectrum of $\mathcal{A} := \langle A_i, i \in I\rangle$.

Proposition 5.4.5 (Power)**.** *The spectrum $\widehat{\mathcal{A}}$ can be embedded in $\prod_{i\in I} \widehat{A_i}$ via the restriction map. As subset of $\prod_{i\in I} \widehat{A_i}$, the spectrum $\widehat{\mathcal{A}}$ is exactly the set of permanent points.*

This provides a new way of looking at Theorem 5.4.2.

Deuxième partie

Manifolds with corners and N-body problem

This second part of the thesis uses some quite different techniques than the first part. Thus, while the first part of the thesis dealt mostly with C^*-algebras, this part is much more geometric. It serves the purpose of giving a better understanding of the spaces introduced in the first part as spectra of suitable C^*-algebras. In particular, while in this second part we do not use directly results from the first one, the motivation comes from there.

6

A comparison of the Georgescu and Vasy spaces associated to the N-body problems

We refer the reader to Section 1.6 for a detailed introduction of this Chapter.

6.1 Manifolds with corners and background material

We begin with some background material, mostly about manifolds with corners. This section contains few new results, but the presentation is new.

6.1.1 Charts and corner atlases

We now introduce manifolds with corners and their smooth structure. We also set up some important notation to be used throughout the paper. The terminology used for manifold with corners is not uniform, however, a good overview of the notion of manifolds with corners can be found in [30, 32, 33, 38, 45], to which we refer for the concepts not defined here and for further references. In this paper, we will mostly use the terminology introduced by Melrose and his coauthors, which predates most of the other ones.

Notation

Informally, a manifold with corners is a topological space locally modeled on the spaces

$$\mathbb{R}^n_k \;:=\; [0,\infty)^k \times \mathbb{R}^{n-k}. \tag{6.1}$$

Let $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ denote the sphere of radius one, as usual. We now set up an important piece of notation to be used throughout the paper. For $k, n \in \mathbb{N} = \{0, 1 \ldots\}$ with $k \leq n$, we let $\mathbb{S}^{n-1}_k \subset \mathbb{R}^n$ be

$$\mathbb{S}^{n-1}_k \;:=\; \mathbb{S}^{n-1} \cap \mathbb{R}^n_k \;=\; \{\phi = (\phi_1, \ldots \phi_n) \mid \|\phi\| = 1 \text{ and } \phi_i \geq 0 \text{ for } 1 \leq i \leq k\}, \tag{6.2}$$

where $\|.\|$ is the euclidean norm.

Let us write 0_V for the neutral element of a vector space V, when we want to stress the space to which it belongs. We will often use maps between subsets of euclidean spaces, and, as a rule, we will try not to permute the coordinates, and, moreover, our embedding will be "first components embedings." More precisely, let $k' \leq k$ and $n' - k' \leq n - k$, we shall then use with priority the canonical "first components" embedding given by :

$$\begin{aligned} \mathbb{R}^{n'}_{k'} \;\simeq\; [0,\infty)^{k'} \times \{0_{\mathbb{R}^{k-k'}}\} \times \mathbb{R}^{n'-k'} \times \{0_{\mathbb{R}^{n-n'}}\} \;\subseteq\; [0,\infty)^k \times \mathbb{R}^{n-k} \;=\; \mathbb{R}^n_k \\ (x', x'') \;\mapsto\; (x', 0_{\mathbb{R}^{k-k'}}, x'', 0_{\mathbb{R}^{n-n'}}) \end{aligned} \tag{6.3}$$

Other embeddings (involving permutations of the coordinates) between these sets will also be considered, and they will explained separately. For instance, we shall sometimes find it notationally convenient to use the *canonical permutation of coordinates* diffeomorphism

$$\begin{aligned} \mathrm{can} : \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \ &\simeq \ \mathbb{R}^{n+n'}_{k+k'} \\ (x', x'', y', y'') \ &\mapsto \ (x', y', x'', y'') \in [0, \infty)^{k+k'} \times \mathbb{R}^{n+n'-k-k'} \,, \end{aligned} \tag{6.4}$$

where $x' \in [0, \infty)^k$ and $y' \in [0, \infty)^{k'}$. (Compare with Equation (6.1).) However, if nothing else is mentioned, we consider the first components canonical embedding of Equation (6.3).

Smooth maps

We have the following standard definition.

Definition 6.1.1. *Let $U \subset \mathbb{R}^n_k$ and $V \subset \mathbb{R}^m_l$ be two open subsets and $f = (f_1, \dots, f_m) : U \to V$. We shall say that :*

(a) f is smooth *on U if there exists an open neighborhood W of U in $\mathbb{R}^n$ such that f extends to a smooth function $\tilde{f} : W \to \mathbb{R}^m$.*

(b) f is a diffeomorphism *between U and V if f is a bijection and both f and f^{-1} are smooth.*

Charts and corner atlases

We shall use suitable charts to define the smooth structure on manifolds with corners. Let M be a Hausdorff space. We proceed as in the case of smooth manifolds (without corners).

Definition 6.1.2. *A* chart on M with values in $\mathbb{R}^n_k$ *(simply,* "chart" *in what follows) is a couple (U, ϕ) with U an open subset of M and $\phi : U \to V$ a homeomorphism onto an open subset V of $\mathbb{R}^n_k$. Let (U, ϕ) and (U', ϕ') be two charts with values in $\mathbb{R}^n_k$ and in $\mathbb{R}^{n'}_{k'}$, respectively. Let $V := U \cap U'$. We shall say that the charts (U, ϕ) and (U', ϕ') are* compatible *if $V = \emptyset$ or if*

$$\phi' \circ \phi^{-1} : \phi(V) \to \phi'(V)$$

is a diffeomorphism (see Definition 6.1.1) between the open subsets $\phi(V) \subset \mathbb{R}^n_k$ and $\phi'(V) \subset \mathbb{R}^n_{k'}$.

Given a point $m \in M$ and a chart (U, ϕ) with $m \in U$, we can always find a chart (U', ϕ'), $\phi' : U' \to \mathbb{R}^{n'}_{k'}$, compatible with (U, ϕ) such that $\phi'(m) = 0$ and k' is minimal.

Definition 6.1.3. *A* corner atlas $\mathcal{A} = \{(U_a, \phi_a), a \in A\}$ *on M is a family of compatible charts such that $M = \bigcup_{a \in A} U_a$. A* manifold with corners *is a paracompact Hausdorff space M with a corner atlas (on M).*

A manifold with corners in the above sense is called a "t-manifold" in [37, Section 1.6].

Let $f : M \to M'$ be a map between two manifolds with corners. We will say that f is *smooth* if, for any two charts (U, ϕ) of M and (U', ϕ') of M', the map $\phi' \circ f \circ \phi^{-1}$ is smooth on its domain of definition $\phi(f^{-1}(U'))$. If f is a bijection and both f and f^{-1} are smooth, we shall say that f is a *diffeomorphism.*

The following are some examples of manifolds with corners that will be used in this paper.

Example 6.1.4. Using the notation from Subsection 6.1.1, we have the following :

(i) Any open subset of $\mathbb{R}^n_k := [0, \infty)^k \times \mathbb{R}^{n-k}$ is a manifold with corners.

(ii) The sphere octant $\mathbb{S}^{n-1}_k := \mathbb{S}^{n-1} \cap \mathbb{R}^n_k$ of Equation (6.2) is a manifold with corners.

(iii) Any smooth manifold is a manifold with corners (even if it doesn't have a boundary or any true corners).

6.1.2 The boundary and boundary faces

We now fix some standard terminology to be used in what follows. In particular, we need the intrinsic definition of the boundary of a manifold with corners. The *depth* of a point $p \in X$ is the number of non-negative coordinate functions vanishing at p in any local coordinate chart at p. It is the least k such that there exists a chart near U with values in $\mathbb{R}^n_k$. Let $(M)_k$ be the set of points of M of depth k. It is a smooth manifold (no corners). Its connected components are called the *open* boundary faces (or just the *open* faces) of codimension (or depth) k of M. A *boundary face* of depth k is the closure of an open boundary face of depth k. It is possible to construct a manifold with corners M that has a boundary face F such that F is not a manifold with corners for the induced smooth structure. More precisely, there are M and F such that $\{f|_F \mid f \in C^\infty(M)\}$ is not the set of smooth functions on F for some manifold-with-corners structure on F.

We will denote by $\mathcal{M}_k(M)$ the set of all *closed* boundary faces of codimension k. In particular, the *boundary* ∂M of M, defined as the set of all points of depth > 0, is given by

$$\partial M := \bigcup_{H \in \mathcal{M}_1(M)} H. \tag{6.5}$$

A boundary face of M of codimension one will be called a *hyperface* in what follows. Thus ∂M is the union of the hyperfaces of M. If H is a hyperface of M and $0 \leq x \in \mathcal{C}^\infty(M)$ is a function such that $H = x^{-1}(0)$ and $dx \neq 0$ on H, then x is called a *boundary defining function* of H. As above, there are examples of hypersurfaces, that do not have a boundary defining function. However, each boundary face F of codimension k can *locally* be represented as $F = \{x_1 = x_2 = \ldots = x_k = 0\}$, where x_j are boundary defining functions of the hyperfaces containing F. Here "locally" means that, given $p \in F$, there is an open neighborhood U of p in M such that the statement is true for M and F replaced with $M \cap U$ and, respectively, with $F \cap U$.

Remark 6.1.5. Note that in [37], by a "manifolds with corners" Melrose means a manifold with corners in our sense that has the further property that each of its hyperfaces (and hence each face) has a system of defining functions. Our definition is thus slightly more general. Furthermore, the submanifolds in [37] are sometimes required to be connected, which would be an inconvenient loss of generality for us. As the blow-up construction below is local, the results of [37] extend trivially to our framework.

It is also convenient to consider an alternative approach to the definition of manifolds with corners and of their smooth structure via embeddings, as in the next remark.

Remark 6.1.6. Every manifold with corners M is contained in a smooth manifold $\widetilde{M}$ of the same dimension [2, 30, 33, 38, 37, 45]. It is then convenient to define

$$TM := T\widetilde{M}|_M .$$

Up to a diffeomorphism, TM can be obtained by gluing the tangent spaces $T(\mathbb{R}^n_k) := \mathbb{R}^n_k \times \mathbb{R}^n$ using a corner atlas of M. We also let $T^+_x M$ be the set of tangent vectors of $T_x M$ that are inward-pointing or tangent to the boundary. It can be defined as the set of equivalence classes of curves starting at x and completely contained in M. We finally let $T^+M := \bigcup_{x \in M} T^+_x M$ with its projection map to M. Note that T^+M is not a fiber bundle, but a fiberwise conical closed subset of the tangent space. We note, however, that ∂M is intrinsically defined and sometimes it is *not* the *topological* boundary $\overline{M} \smallsetminus \overset{\circ}{M}$ of M, where the closure $\overline{M}$ and the interior $\overset{\circ}{M}$ are computed in $\widetilde{M}$. For instance, when $M := \{x \in \mathbb{R}^n \mid x_n \geq 0, \|x\| < 1\}$ and $\widetilde{M} = \mathbb{R}^n$, then $\partial M = \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| < 1\}$, whereas the topological boundary of M is $\partial M \cup \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| = 1\}$, a bigger set. In fact, we always have that ∂M is contained in the topological boundary of M in $\widetilde{M}$. Unlike ∂M, the topological boundary of M in $\widetilde{M}$ depends on $\widetilde{M}$.

6.1.3 Submanifolds of manifolds with corners

We now recall the central definition of a p-submanifold of a manifold with corners M [1, 31, 37, 59]. In our paper, p-submanifolds are important as we blow-up manifolds with corners only along closed p-submanifolds.

Let I be a subset of $\{1, \ldots, n\}$ and L be the subset of $\mathbb{R}^n_k$ defined by

$$L_I \ := \ \{ \, x = (x_1, \ldots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \text{ if } i \in I \, \} \, . \tag{6.6}$$

The number $b := \#\big(I \cap \{1, \ldots, k\}\big)$ will be called the *boundary depth* of L_I ; $c := \#I$ is the codimension of L_I and $d := n - c$ its dimension. After reordering the components, L_I is the first factor of $\mathbb{R}^n_k \cong \mathbb{R}^d_{k-b} \times \mathbb{R}^c_b$. The boundary depth of L_I is the boundary depth of interior point of L_I with respect to $\mathbb{R}^n_k$.

Definition and first properties of p-submanifolds

The sets L_I are the local models for p-submanifolds [37, Definition 1.7.4].

Definition 6.1.7. *A subset P of a manifold with corners M is a p-*submanifold *if, for every $x \in P$, there exists a chart (U, ϕ) with $x \in U$ and $I \subset \{1, 2, \ldots, n\}$ such that*

$$\phi(P \cap U) \ = \ L_I \cap \phi(U),$$

with L_I as defined in Equation (6.6)*. The number $n - \#I$ (respectively, $\#I$, respectively, $\#\big(I \cap \{1, \ldots, l\}\big)$) will be called the* dimension *(respectively, the* codimension*) of P in x, respectively, the* boundary depth *of P in x. These numbers are locally constant functions on P. For any interior point x in P and $\epsilon > 0$ small enough, these numbers are the dimension (respectively, the codimension, respectively, the boundary depth of x) of the intersection $B_\epsilon(x) \cap P$ in M. We allow p-submanifolds Y of non-constant dimension. We define $\dim Y$ as the maximum of the dimensions of the connected components of Y and* $\dim \emptyset = 0$.

This definition of a p-submanifold comes from [37]. Note that "p" is used as an abbreviation for "product," reflecting the fact that, locally in coordinate charts, p-submanifolds are a factor of the product $\mathbb{R}^n_k \cong \mathbb{R}^{n_1}_{k_1} \times \mathbb{R}^{n_2}_{k_2}$. A more general concept, that of an "interior binomial subvariety," was introduced and studied in [33].

We shall need the following lemma. Recall that a subset of a topological space is called *locally closed* if it is the intersection of a closed subset with an open subset.

Lemma 6.1.8. *Let $P \subset Q \subset M$ be manifolds with corners.*

(i) If P is a p-submanifold of M, then P is locally closed.

(ii) If both P and Q are p-submanifolds of M, then P is a p-submanifold of Q.

(iii) If P is a p-submanifold of Q and Q is a p-submanifold of M, then P is a p-submanifold of M.

Proof. Let us fix an atlas $\mathcal{A} := \{(U, \phi)\}$. The definition of a p-submanifold shows that it is a closed subset in every coordinate chart (U, ϕ). Hence it is locally closed. This proves (i).

In order to prove (ii) we consider functions $x^1, \ldots, x^\ell$ defining a p-submanifold P of codimension ℓ in M locally in a neighborhood of $x \in P$. Choose $I \subset \{1, \ldots, \ell\}$ such that $(dx^i|_p)_{i \in I}$ is a basis of $T^*_x Q$. Then in a possibly smaller neighborhood, the functions $(x^i)_{i \in I}$ define P as a p-submanifold of Q.

For (iii) we consider functions $x^1, \ldots, x^k$ locally defining P as a p-submanifold of Q. We extend these functions to locally defined functions on M. Then we choose functions $x^{k+1}, \ldots, x^l$ defining Q locally as a p-submanifold of M. Then $x^1, \ldots, x^l$ locally define P as a p-submanifold of M. □

Further notation and remarks

The following standard concepts will be important in the definition of the blow-up of a manifold with corners by a p-submanifold.

Definition 6.1.9. *Let $P \subset M$ be a p-submanifold of the manifold with corners M. Then $N^M P := TM|_P/TP$ is called the* normal bundle of P in M. *The image $N_+^M P$ of $T^+M|_P$ in $N^M P$ is called the* inward pointing normal fiber bundle of P in M. *In contrast to T^+M, which is not a fiber bundle, the projection map $N_+^M P \to P$ defines a fiber bundle structure over P on $N_+^M P$. Finally, the set $\mathbb{S}(N_+^M P)$ of unit vectors in $N_+^M P$ is called the set of* inward pointing spherical normal bundle of P in M. *The inward pointing spherical normal bundle of P in M comes equipped with a fiber bundle projection*

$$\mathbb{S}(N_+^M P) \; \to \; P \, .$$

We complete this section with a few remarks. We first notice the existence of suitable "tubular neighborhoods."

Remark 6.1.10. Let $P \subset M$ be a p-submanifold in the manifold with corners M. If M is compact, then P has a neighborhood $V_P \subset M$ such that V_P is diffeomorphic to the closed cone $N_+^M P$ via a diffeomorphism that sends P to the zero section of $N_+^M P \to M$ and induces the identity at the level of normal bundles. This was proved in [37, Proposition 2.10.1], under the additional assumption that P be closed. Moreover, the condition that M be compact is not necessary (since our p-manifolds are assumed to be locally closed). In this case, $N_+^M P$ is a cone with corners in $N^M P$. Generalizing Example 6.1.4, we obtain that all of the sets $N^M P$, $N_+^M P$, and $S^+(N^M P)$ introduced in the last definition are manifolds with corners. This is because the property of being a manifold with corners is a local property and the product of manifolds with corners is again a manifold with corners.

A related structure to p-submanifolds was considered in [45] :

Remark 6.1.11. The *tame* submanifolds considered in [2] require tubular neighborhods that are diffeomorphic to $N^M P$ (so $N_+^M P = N^M P$ for tame submanifolds). Hence the same is true of the A-tame submanifolds considered in [45], which are tame submanifolds plus a compatibility condition with respect to a Lie algebroid A on M.

We continue with a simple property of the boundary of a p-submanifold.

Remark 6.1.12. For L_I as in Equation (6.6), we have $\partial L_I \;=\; \bigcup_{1 \leq j \leq k, j \notin I} L_{\{j\} \cup I}$. Since $\partial \mathbb{R}^n_k \;=\; \bigcup_{1 \leq j \leq k} L_{\{j\}}$, we obtain

$$\begin{cases} L_I \subset L_{\{j\}} \subset \partial \mathbb{R}^n_k & \text{if } \; j \in I \cap \{1, 2, \dots k\} \\ \partial L_I \; = \; \partial \mathbb{R}^n_k \cap L_I & \text{if } \; I \cap \{1, 2, \dots k\} = \emptyset \, . \end{cases} \tag{6.7}$$

(the submanifolds of manifolds with corners considered in [2], called "tame submanifolds" in [45], fall thus into the second category). In any case, $\partial L_I \subset \partial \mathbb{R}^n_k$. We thus obtain that, if P is a p-submanifold of M, then

$$\partial P \; \subset \; \partial M \, . \tag{6.8}$$

In particular, $[0, \infty)^2$ is not a p-submanifold of $\mathbb{R}^2_1 = [0, \infty) \times \mathbb{R}$.

We finish our sequence of remarks with a note on our terminology.

Remark 6.1.13. By taking I to be an empty subset, we obtain that the open subsets of M are p-submanifolds. The empty set $\emptyset$ also satisfies the conditions defining a p-submanifold. The empty set and the p-submanifolds containing open subsets of M will be called *trivial p-submanifolds of M*. We allow the different connected components of a p-submanifold to have *different* dimensions. Therefore, if Y is a p-submanifold of M, we shall denote by $\dim(Y)$ the *largest* dimension of a connected component of Y. Thus, the *non-trivial* p-submanifolds of M are the p-submanifolds that are non-empty and of lower dimension than M.

6.2 The blow-up for manifolds with corners

We now introduce the blow-up of a manifold M with corners by a *closed* p-submanifold P with $\dim(P) < \dim(M)$ (see Remark 6.1.13 for the definition of the dimension of a p-submanifold). We also study some of the properties of the blow-up.

6.2.1 Definition of the blow-up and its smooth structure

Recall (Remark 6.1.13) that if $P \subset M$ is a p-submanifold, then $\dim(P)$ denotes the largest dimension of the connected components of P (these connected components are not necessarily all of the same dimension).

Definition of the blow-up as a set

We now define the underlying set of the blow-up of a manifold with corners M with respect to a p-submanifold. If A and B are disjoint, we sometimes denote $A \sqcup B \ := \ A \cup B$ their union.

Definition 6.2.1. *Let M be a manifold with corners and P be a* closed *p-submanifold of M. Let $\mathbb{S}(N_+^M P)$ be the inward pointing spherical normal bundle of P in M (Definition 6.1.9). The* blow-up of M along P *(or* with respect to P*) is the following union of disjoint sets :*

$$[M : P] \ := \ (M \setminus P) \sqcup \mathbb{S}(N_+^M P) .$$

In particular, $[M : \emptyset] = M$ and $[M : M] = \emptyset$. The blow-down map $\beta = \beta_{M,P} : [M : P] \to M$ is defined as the identity map on $M \setminus P$ and as the fiber bundle projection $\mathbb{S}(N_+^M P) \to P$ on the complement.

The blow-up $[M : P]$ is therefore not defined if P is not closed, but we allow P to consist of the disjoint union of several closed, connected p-submanifolds of M of different dimensions.

Assume $P \subset M$ to be a trivial p-submanifold. Then, by definition, there will be $x \in P$ such that P is of codimension 0 in a neighborhood of x, and hence the fiber of $\mathbb{S}(N_+^M P)$ over x is $\emptyset$. This is the reason why the case P trivial will often be excluded from consideration.

A general approach to smooth structures on the blow-up is contained in [33]. Here, we recall an approach that suffices for our needs. We begin with the case of open subsets of a model space $\mathbb{R}^n_l$.

The blow-up of the local models

To start with, the blow-up $[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$ of $\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \simeq \mathbb{R}^{n+n'}_{l+l'}$ along its p-submanifold $\mathbb{R}^n_l \times \{0\} = \mathbb{R}^n_l \times \{0_{\mathbb{R}^{n'}}\}$ is, by Definition 6.2.1, the set

$$\begin{aligned} \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\} t &:= \left(\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \smallsetminus \mathbb{R}^n_l \times \{0\}\right) \sqcup \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \qquad (6.9) \\ &= \mathbb{R}^n_l \times \left(\mathbb{S}^{n'-1}_{l'} \sqcup \left(\mathbb{R}^{n'}_{l'} \setminus \{0\}\right)\right) . \end{aligned}$$

Let us consider the map

$$\begin{aligned} \kappa : \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty) \ &\to\ \mathbb{R}^n_l \times \left(\mathbb{S}^{n'-1}_{l'} \sqcup (\mathbb{R}^{n'}_{l'} \setminus \{0\})\right) , \\ \kappa(x,\xi,r) \ &:= \begin{cases} (x,\xi) \in \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} & \text{if } r = 0 \\ (x, r\xi) \in \mathbb{R}^n_l \times (\mathbb{R}^{n'}_{l'} \setminus \{0\}) & \text{if } r > 0 \,. \end{cases} \end{aligned} \tag{6.10}$$

The map κ is immediately seen to be a bijection and we will use it to endow $[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$ with the structure of a manifold with corners induced from $\mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty)$. Under this diffeomorphism, the blow-down map becomes

$$\beta : \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty) \to \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \,, \quad \beta(x,\xi,r) \ :=\ (x, r\xi) \,. \tag{6.11}$$

The blown-up space $[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$ is thus a space of "generalized spherical coordinates."

If $U \subset \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'}$ is an open subset, we endow

$$[U : U \cap (\mathbb{R}^n_l \times \{0\})] \ =\ \beta^{-1}(U) \subset [\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}] \tag{6.12}$$

with the induced structure of a manifold with corners.

The smooth structure of the blow-up

The following lemmas will allow us to define a manifolds with corners structure on blow-ups.

Lemma 6.2.2. *Let $P_i \subset M_i$, $i = 1, 2$, be closed p-submanifolds and let $\phi : M_1 \to M_2$ be a diffeomorphism such that $\phi(P_1) = P_2$. Then there exists a unique map $\phi^\beta : [M_1 : P_1] \to [M_2 : P_2]$ that is bijective and makes the following diagram commute*

$$\begin{array}{ccc} [M_1 : P_1] & \xrightarrow{\ \phi^\beta\ } & [M_2 : P_2] \\ {\scriptstyle \beta_{M_1,P_1}} \big\downarrow & & \big\downarrow {\scriptstyle \beta_{M_2,P_2}} \\ M_1 & \xrightarrow{\ \ \phi\ \ } & M_2 . \end{array}$$

This construction is functorial, in the sense that $(\phi \circ \psi)^\beta = \phi^\beta \circ \psi^\beta$. If M_i are open subsets of $\mathbb{R}^n_k$, then ϕ^β is a diffeomorphism.

Proof. The existence, uniqueness, and the functorial character of ϕ^β follows from the definition of the blow-up. The fact that ϕ^β is smooth if M_i are open subsets of the model space $\mathbb{R}^n_k$ is the content of Lemma 2.2 of [1]. □

Lemma 6.2.3. *Let $\mathcal{A} = \{(U_a, \phi_a) \mid a \in A\}$ be a* corner atlas *on a manifold with corners M, see Definition 6.1.3. Let $P \subset M$ be a closed p-submanifold and $\beta = \beta_{M,P} : [M : P] \to M$ be the blow-down map. We endow $[M : P]$ with the smallest topology that makes all the maps ϕ^β_a, $a \in A$, continuous (ϕ^β_a is defined on $\beta^{-1}(U_a)$). Then*

$$\beta^*(\mathcal{A}) \ :=\ \{(\beta^{-1}(U_a), \phi^\beta_a) \mid a \in A\}$$

is a corner atlas on $[M; P]$, where ϕ^β_a are the maps obtained from ϕ_a using Lemma 6.2.2. If we take another corner atlas $\mathcal{A}'$ of M that is compatible with $\mathcal{A}$, then $\beta^*(\mathcal{A})$ and $\beta^*(\mathcal{A}')$ will be compatible corner atlases on $[M : P]$.

Proof. This follows from Equation (6.12) and Lemma 6.2.2. □

Lemma 6.2.3 thus yields the desired smooth structure on $[M : P]$ that is moreover canonical (independent of any choices).

Definition 6.2.4. *Let M be a manifold with corners and $P \subset M$ be a closed p-submanifold. We endow $[M : P]$ with the smooth structure defined by the corner atlas $\beta^*(\mathcal{A})$ obtained from Lemma 6.2.3, for any corner atlas $\mathcal{A}$ on M.*

The smooth structure on $[M : P]$ is natural in the following strong sense.

Proposition 6.2.5. *With the notation of Lemma 6.2.2, we have that the map ϕ^β is a diffeomorphism (in general, not just in the case of open subsets of Euclidean spaces).*

Proof. If $\mathcal{A}$ is a corner atlas on M_2, then the pull-back of $\beta^*(\mathcal{A})$ to $[M_1 : P_1]$ is a corner atlas. □

The functoriality property of Lemma 6.2.2 then gives the following.

Corollary 6.2.6. *Let G be a discrete group acting smoothly on the manifold with corners M and let $P \subset M$ be a closed p-submanifold such that $g(P) = P$ for all $g \in G$. Then G acts smoothly on $[M : P]$.*

Proof. The action of every $g \in G$ on M defines a smooth action on $[M : P]$ by Proposition 6.2.5. It is a group action by the last part of Lemma 6.2.2 (the functoriality of the assignement $\phi \to \phi^\beta$. □

The blow-up $[M : P]$ of a manifold with corners is thus again a manifold with corners.

Remark 6.2.7. Let us now describe the faces of $[M : P]$, where P is a closed p-submanifold of constant codimension ℓ and constant boundary depth b of the manifold with corners M. Recall that every closed boundary face is by definition the closure of precisely one open (connected, non-empty) boundary face. Let us consider the fibration $\mathbb{S}N_+^M P \to P$ (the inward pointing spherical normal bundle of Definition 6.1.9). Recall that there exists a bijection between the open and closed faces of a manifold with corners. We shall distinguish two cases.

(i) *P is not a closed face of M, i.e. $b < \ell$.* Then the set $\mathcal{M}_\ell([M : P])$ of faces of codimension ℓ of $[M : P]$ are in bijection with the disjoint union $\mathcal{M}_\ell(M) \sqcup \mathcal{M}_{\ell-1}(\mathbb{S}N_+^M P)$. More precisely, if, on the one hand, $F \subset \mathbb{S}N_+^M P$ is an open face of codimension $\ell - 1$, then it will define a face of $[M : P] \supset \mathbb{S}N_+^M P$, but of codimension ℓ. On the other hand, if F is an open face of M, then $\beta_{M,P}^*(F) := \overline{\beta_{M,P}^{-1}(F \smallsetminus P)}$ will be a closed face of $[M : P]$ of the same codimension.

(ii) *P is a face of M, i.e. $b = \ell$.* In this case, we have almost the same result, except that we have to remove from $\mathcal{M}_\ell(M)$ the faces that are contained in P. For instance, if $H \subset \partial M$ is a closed face of codimension $\ell = 1$ (a hyperface) and a closed p-submanifold of M, there is a bijection between the faces of $[M : H]$ and those of M, which is consistent with the fact that $\beta_{M,H} : [M : H] \to M$ is a diffeomorphism.

The following proposition will not be used in the sequel, but might be helpful for the reader to imagine a geometric model for the blow-up.

Proposition 6.2.8. *We keep the previous notation, thus M is a manifold with corners and $P \subset M$ be a closed p-submanifold. There exists a metric g on M and $\epsilon_0 > 0$ such that, for any $0 < \epsilon < \epsilon_0$, there exists a diffeomorphism*

$$\Theta_\epsilon : [M : P]_\epsilon \; := \; \{x \in M \mid \operatorname{dist}_g(x, P) \geq \epsilon\} \; \to \; [M : P]$$

such that $\Theta_\epsilon(x) = x$ if $\operatorname{dist}_g(x, P) \geq 2\epsilon$. If M is compact, then for any metric g there is an $\epsilon_0 > 0$ with this property.

Proof. Let V_p be a tubular neighborhood of P. We choose the metric g and λ such that $\{x \mid \mathrm{dist}_g(x, P) < \lambda\} \subset V_p$. We then identify V_P with an open subset of N_+^P using the normal exponential map. This reduces our statement to the case when $M = N_+^M P$, in which case it is trivial. If M is compact, we can find a λ with the desired properties for any metric. □

6.2.2 Exploiting the local structure of the blow-up

The local character of the definition of the smooth structure of the blow-up $[M : P]$ of the manifold with corners M along a p-submanifold P means that most of the proofs involving blow-ups can be conveniently treated by first treating the model case $P := \mathbb{R}_k^n = \mathbb{R}_k^n \times \{0\} \subset \mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'} = M$. To simplify notation, we shall often omit factors of the form $\{0\}$ when there is no danger of confusion. This is the case with the following results.

The blow-down map is proper

We shall need to prove that certain maps are closed. This will be conveniently done by proving that they are proper, since a proper map between manifolds with corners is closed. In particular, we will show that the blow-down map is proper.

Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. For instance, the map β of Equation (6.11) is immediately seen to be proper. We shall say that f is *locally proper* if, for every $y \in Y$, there exists an open neighborhood V_y of y in Y such that the map $f^{-1}(V_y) \to V_y$ induced by f is proper.

Lemma 6.2.9. *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. Then f is proper if, and only if, it is locally proper.*

Proof. Clearly, every proper map is locally proper, by definition. Let us assume that f is locally proper and let $K \subset Y$ be a compact subset. For any $y \in K$ we choose the open neighborhood V_y as above (in the definition of a locally proper map). As Y is locally compact, there is an open neighborhood W_y of y in V_y such that its closure $\overline{W}_y$ in Y is a compact subset of V_y. The local properness of f together with the choice of V_y implies that $f^{-1}(\overline{W}_y \cap K)$ is compact. By the compactness of K we can coose $y_1, \dots, y_N$ such that K is covered by $\big(W_{y_j}\big)_{1 \le j \le N}$. Then $K = \bigcup_{j=1}^N \big(\overline{W}_{y_j} \cap K\big)$. Then

$$f^{-1}(K) \;=\; \bigcup_{j=1}^{N} f^{-1}(\overline{W}_{y_j} \cap K)$$

is also compact. This completes the proof. □

Corollary 6.2.10. *Let P be a closed p-submanifold of a manifold with corners M. The blow-down map $\beta_{M,P} : [M : P] \to M$ is proper.*

Proof. Using Lemma 6.2.9, we see that we can treat the problem in local coordinates. Then, in local coordinates, the blow-down map is given by Equation (6.11), which is a proper map, as we have already pointed out. □

Blow-ups and products

We have a simple, convenient behavior of the blow-up with respect to products.

Lemma 6.2.11. *Let M and M_1 be two manifolds with corners and P be a closed p-submanifold of M. Then $P \times M_1$ is a closed p-submanifold of $M \times M_1$ and the following diagram with smooth maps commutes :*

$$\begin{array}{ccc} [M \times M_1 : P \times M_1] & \xrightarrow{\simeq} & [M : P] \times M_1 \\ \beta_{M \times M_1, P \times M_1} \downarrow & & \downarrow \beta_{M,P} \times id \\ M \times M_1 & \xrightarrow{id} & M \times M_1 \,. \end{array} \tag{6.13}$$

Proof. Since the result is a local one and P is a p-submanifold of M, it is enough to treat the case

$$\begin{aligned} M &:= \mathbb{R}^{m+p}_{k_m+k_p} \simeq \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P &:= \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p} \subset M \\ M_1 &:= \mathbb{R}^l_{k_l} \,, \end{aligned}$$

where the first diffeomorphism is the canonical permutation of coordinates diffeomorphism of Equation 6.4.

With this choice, we see that $P \times M_1$ is p-submanifold of $M \times M_1$. By the definition, see Equation (6.9), we have natural diffeomorphisms (induced by suitable permutations of coordinates)

$$\begin{aligned} [M \times M_1 : P \times M_1] &\simeq [\mathbb{R}^{m+p+l}_{k_m+k_p+k_l} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{p+l}_{k_p+k_l}] \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l}_{k_p+k_l} \sqcup \left(\mathbb{R}^{m+p+l}_{k_m+k_p+k_l} \setminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{p+l}_{k_p+k_l})\right) \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^{p+l}_{k_p+k_l} \simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1} \end{aligned}$$

and

$$\begin{aligned} [M : P] &\simeq [\mathbb{R}^{m+p}_{k_m+k_p} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}] \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^p_{k_p} \sqcup \mathbb{R}^{m+p}_{k_m+k_p} \setminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}) \simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \,. \end{aligned}$$

The desired diffeomorphism $[M \times M_1 : P \times M_1] \simeq [M : P] \times M_1$ is then induced by the above diffeomorphisms and the canonical permutation of coordinates diffeomorphism $\mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{R}^l_{k_l} \to \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1}$ of Equation (6.4). □

6.2.3 Cleanly intersecting families and liftings

Clean intersections

We continue to exploit the local structure of the blow-up. Recall the following standard definition.

Definition 6.2.12. *Let M be a manifold with corners and $X_1, X_2, \ldots, X_k \subset M$ be p-submanifolds. We shall say that $X_1, X_2, \ldots, X_k$* have a clean intersection *if*

(i) $Y := X_1 \cap X_2 \cap \ldots \cap X_k$ is a p-submanifold of M (possibly empty),

(ii) for all $x \in Y$, $T_xY = T_xX_1 \cap T_xX_2 \cap \ldots \cap T_xX_k$.

We consider the conditions (i) and (ii) of the Definition 6.2.12 to be automatically satisfied if $Y := X_1 \cap X_2 \cap \ldots \cap X_k = \emptyset$. Similar conditions appear in Definition 2.7, [1]. They were used to define a *weakly transversal family* of connected submanifolds with corners. We shall need also the notion of a "cleanly intersecting family" (Definition 6.4.4), which roughly states that every subfamily intersects cleanly.

Lemma 6.2.13. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Then $P \cap Q$ is a p-submanifold of Q.*

Proof. According Definition 6.2.12 (i) $P \cap Q$ is a p-submanifold of M. Then Lemma 6.1.8 (ii) states that $P \cap Q$ is also a p-submanifold of Q. □

Liftings of submanifolds to blowups

We now consider the lifting of suitable submanifolds in M to $[M : P]$ as in [33, 37].

The local model for such lifts is given by the following lemma. (See Lemma 6.2.2 for the definition of j^β.)

Lemma 6.2.14. *If $l'' \geq l'$ and $n'' - l'' \geq n' - l'$, so that the canonical (first components) inclusion $j : \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \to \mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''}$ is defined, then there is a map j^β such that the diagram*

$$\begin{array}{ccc} [\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}] & \xrightarrow{j^\beta} & [\mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''} : \mathbb{R}^n_l \times \{0\}] \\ {\scriptstyle \beta_{\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}}} \big\downarrow & & \big\downarrow {\scriptstyle \beta_{\mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''} : \mathbb{R}^n_l \times \{0\}}} \\ \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} & \xrightarrow{j} & \mathbb{R}^n_l \times \mathbb{R}^{n''}_{l''}\,. \end{array} \tag{6.14}$$

commutes.

In fact the diagram (6.14) is obtained from

$$\begin{array}{ccc} [\mathbb{R}^{n'}_{l'} : \{0\}] & \xrightarrow{j_0^\beta} & [\mathbb{R}^{n''}_{l''} : \{0\}] \\ {\scriptstyle \beta_{\mathbb{R}^{n'}_{l'} : \{0\}}} \big\downarrow & & \big\downarrow {\scriptstyle \beta_{\mathbb{R}^{n''}_{l''} : \{0\}}} \\ \mathbb{R}^{n'}_{l'} & \xrightarrow{j_0} & \mathbb{R}^{n''}_{l''}\,. \end{array} \tag{6.15}$$

by taking for each space the product with $\mathbb{R}^n_l$ and extending the maps as a product with the identity map $\mathrm{id} : \mathbb{R}^n_l \to \mathbb{R}^n_l$, using the linear version of Lemma 6.2.11.

The lift j_0^β is given by

$$[\mathbb{R}^{n'}_{l'} : \{0\}] \simeq \mathbb{S}^{n'-1}_{l'} \times [0, \infty) \xrightarrow{i \times \mathrm{id}} \mathbb{S}^{n''-1}_{l''} \times [0, \infty) \simeq [\mathbb{R}^{n''}_{l''} : \{0\}]\,, \tag{6.16}$$

where $i : \mathbb{S}^{n'-1}_{l'} \to \mathbb{S}^{n''-1}_{l''}$ is the restriction of j_0. In particular, j_0^β and thus j^β are smooth.

Definition 6.2.15. *Let P be a p-submanifold of M and Q be a closed subset of M. The* lifting *$\beta^*(Q)$ of Q in $[M : P]$ is defined by*

$$\beta^*(Q) \;:=\; \overline{\beta^{-1}(Q \setminus P)} \;\; (\textit{closure in } [M : P])\,.$$

A more general version of the lifting β^* was defined in [37, Chap 5, Section 7] (for $Q \subset P$, in which case, with the notation of Definition 6.2.15, $\beta^*(Q) := \beta^{-1}(Q)$), but that version will not be needed in this paper.

We have the following result on the blow-up of submanifolds, due, in part, to Melrose [37, Chapter 5, Section 7].

Proposition 6.2.16. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Then the inclusion $j : Q \to M$ lifts to a natural inclusion*

$$j^\beta : [Q : P \cap Q] \ := \ (Q \smallsetminus (P \cap Q)) \sqcup \mathbb{S}(N_+^Q(P \cap Q)) \to (M \smallsetminus P) \sqcup \mathbb{S}(N_+^M P) \ =: \ [M : P] \,.$$

The map j^β is smooth for the natural p-submanifold structures. In particular, the blow-down map $\beta : [M : P] \to M$ yields a natural diffeomorphism

$$\beta^*(Q) \xrightarrow{\cong} [Q : P \cap Q] \,.$$

Proof. The inclusion of Q into M restricts to a map $Q \smallsetminus (P \cap Q) \to M \smallsetminus P$. It also induces an inclusion $TQ \to TM$, extending the inclusion $T(P \cap Q) \to TP$. By taking quotients, we obtain a map

$$N^Q(P \cap Q) \ := \ TQ|_{P \cap Q}/T(P \cap Q) \ \to \ TM|_P/TP \ =: \ N^M P$$

This finally yields an inclusion $\mathbb{S}(N_+^Q P) \to \mathbb{S}(N_+^M P)$, which fits smoothly with a map $Q \smallsetminus (Q \cap P) \to M \smallsetminus P$ due to the fact that P and Q intersect smoothly. The result then follows from the definition of the blow-up, Definition 6.2.1. □

6.3 The graph family blow-up

We introduce also the blow-up with respect to more than one submanifold, called *graph blow-up*. Since the graph blow-up is defined *only* with respect to locally finite families of *closed, non-trivial* p-submanifolds, we shall assume from now on that all our p-submanifolds have these properties.

6.3.1 Definition of the graph blow-up

Let M be a manifold with corners and $\mathcal{F}$ be a locally finite set of *non-trivial* p-submanifolds of M. Then $\bigcup \mathcal{F} := \bigcup_{Y \in \mathcal{F}} Y$ is nowhere dense in M and $M \smallsetminus \bigcup \mathcal{F}$ is a dense, open subset of $[M : Y]$, for each $Y \in \mathcal{F}$. Motivated by the results of [27] and Theorem 3.2.4, we now introduce the following definition.

Definition 6.3.1. *Let $\mathcal{F}$ be a locally finite set of closed p-submanifolds of the manifold with corners M. Assume that all $Y \in \mathcal{F}$ satisfy* $\dim(Y) < \dim(M)$. *Then the* graph blow-up $\{M : \mathcal{F}\}$ *of M along $\mathcal{F}$ is defined by*

$$\{M : \mathcal{F}\} \ := \ \overline{\{(x, x, \ldots, x) \,|\, x \in M \smallsetminus \bigcup \mathcal{F}\}} \ \subset \ \prod_{Y \in \mathcal{F}} [M : Y] \,.$$

Let $\delta : M \smallsetminus \bigcup \mathcal{F} \to \prod_{Y \in \mathcal{F}} [M : Y]$ be the diagonal map of inclusions, $\delta(x) = (x, x, \ldots, x)$. Thus the graph blow-up $\{M : \mathcal{F}\}$ is the closure of the image through δ of the complement $M \smallsetminus \bigcup \mathcal{F}$ in the product $\prod_{Y \in \mathcal{F}} [M : Y]$ of all the blown-up spaces $[M : Y]$, $Y \in \mathcal{F}$:

$$\{M : \mathcal{F}\} \ := \ \overline{\delta(M \smallsetminus \bigcup \mathcal{F})} \ \subset \ \prod_{Y \in \mathcal{F}} [M : Y] \,,$$

$$M \smallsetminus \bigcup \mathcal{F} \ni x \to \delta(x) \ := \ (x, x, \ldots, x) \in \prod_{Y \in \mathcal{F}} [M : Y] \,.$$

Note that we have used here that $M \smallsetminus \bigcup \mathcal{F} \subset M \smallsetminus Y \subset [M : Y]$ for all $Y \in \mathcal{F}$. The graph blow-up will be compared in the next section to the iterated blow-up.

Definition 6.3.2. *If G is a Lie group acting smoothly on M and $\mathcal{F}$ is a locally finite set of closed p-submanifolds of M of dimensions $< \dim(M)$ such that, for every $Y \in \mathcal{F}$ and $g \in G$, we have $g(Y) \in \mathcal{F}$, then we shall say that $\mathcal{F}$* is a G-family of p-submanifolds of M.

Corollary 6.2.6 yields right away the following corollary

Corollary 6.3.3. *Let G be a discrete group and $\mathcal{F}$ be a G-family of p-submanifolds of M (see Definition 6.3.2). Then G acts continuously on $\{M : \mathcal{F}\}$. If $\{M : \mathcal{F}\}$ is a submanifold of $\prod_{Y\in\mathcal{F}}[M : Y]$, then the action of G is smooth.*

Proof. We have that each $g \in G$ acts on $M \smallsetminus \bigcup \mathcal{F}$ and on $\prod_{Y\in\mathcal{F}}[M : Y]$, with the action sending $[M : Y]$ to $[M : g(Y)]$, by Corollary 6.2.6, which also shows that this action is a smooth action of G on $\prod_{Y\in\mathcal{F}}[M : Y]$. The result follows since δ commutes with the action of G. □

6.3.2 Disjoint submanifolds

We are allowing our p-submanifolds to have components of different dimensions. Blowing-up with respect to such a manifold amounts, as we will see, to blowing up successively with respect to each component.

We need first to discuss the gluing of open subsests. Let us assume that we have two manifolds with corners M_1 and M_2 and that $U_i \subset M_i$ are open subsets ($i = 1, 2$). Let us also assume that we are given a diffeomorphism $\phi : U_1 \to U_2$. Then we define

$$\begin{aligned} M_1 \cup_\phi M_2 \ &:= \ (M_1 \sqcup M_2)/\{x \equiv \phi(x) \mid x \in U_1\}\,, \\ M_1 \cup_{\mathrm{id}} M_2 \ &=: \ M_1 \cup_{U_1} M_2\,, \quad \text{if } U_1 = U_2 \text{ and } \phi \text{ is the identity map id}\,. \end{aligned} \tag{6.17}$$

If ϕ is the identity, we shall call $M_1 \cup_{U_1} M_2$ the *union of M_1 and M_2 along $U_1 = U_2$*. Under favorable circumstances (but not always), $M_1 \cup_\phi M_2$ is also a manifold with corners. We have the following simple lemma.

Lemma 6.3.4. *Let M be a manifold with corners (and hence Hausdorff) and $M_i \subset M$, $i = 1, 2$, be open subsets with intersection U. Then there exists a unique structure of a manifols with corners on $M_1 \cup_U M_2$ that induces the given smooth structures on M_i, and hence we have a diffeomorphism $M_1 \cup_U M_2 \simeq M$.*

This allows us to "commute" the procedures of taking blow-ups with respect to disjoint manifolds.

Lemma 6.3.5. *Let us assume that P and Q are closed, non-trivial p-submanifolds of M such that $P \cap Q = \emptyset$. Let $\beta_{M,Q} : [M : Q] \to M$ be the blow-down map. Then $\beta^*(P) := \beta_{M,Q}^{-1}(P) = P$ and the iterated blow-up $[[M : Q] : P]$ is defined and diffeomorphic to $([M : Q] \smallsetminus P) \sqcup_{M\smallsetminus(P\cup Q)} ([M : P] \smallsetminus Q)$, the union of $[M : Q] \smallsetminus P$ and $[M : P] \smallsetminus Q$ along $M \smallsetminus (P \cup Q)$, a common open subset. In particular,*

$$[[M : P] : Q] \ = \ [[M : Q] : P] \ = \ [M : P \cup Q]\,,$$

with the same smooth structure.

Proof. Since $Q \subset M \smallsetminus P$ and $\beta_{M,P}$ is the identity on $M \smallsetminus P$, we obtain

$$\beta_{M,P}^*(Q) \ := \ \overline{\beta_{M,P}^{-1}(Q \smallsetminus P) \ = \ \overline{\beta_{M,P}^{-1}(Q)} \ = \ \overline{Q}} \ = \ Q\,,$$

which is a p-submanifold of $M \smallsetminus P$ and hence also of $[M : P]$, since the property of being a p-submanifold is a local one. It also follows that $\mathbb{S}(N_+^M Q) = \mathbb{S}(N_+^{[M:P]} Q)$, since $\mathbb{S}(N_+^M Q)$ is also defined locally. We thus obtain

$$\begin{aligned}[][M:P]:Q] &:= ([M:P]\smallsetminus Q)\sqcup \mathbb{S}(N_{+}^{[M:P]}Q)\\ &= (M\smallsetminus (P\cup Q))\sqcup \mathbb{S}(N_{+}^{M}P)\sqcup \mathbb{S}(N_{+}^{M}Q)\,,\end{aligned}$$

which is symmetric in Q and P, and hence $[[M:P]:Q]=[[M:Q]:P]$ as sets. The last part follows from Lemma 6.3.4 applied to the open subsets

$$\beta_{[M:P],Q}^{-1}([M:P]\smallsetminus Q) \;=\; [M:P]\smallsetminus Q \quad\text{and}$$
$$\beta_{[M:Q],P}^{-1}([M:Q]\smallsetminus P) \;=\; [M:Q]\smallsetminus P$$

of $[[M:P]:Q]=[[M:Q]:P]$. □

We shall need the following lemma.

Lemma 6.3.6. *Let us assume that P and Q are closed, non-trivial,* disjoint *p-submanifolds of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P}:[[M:Q]:P]\;\to\;[M:P]$$

that restricts to the identity on $M\smallsetminus(P\cup Q)$. Moreover, the product map

$$\mathcal{B}_{M,Q,P}\;:=\;(\zeta_{M,Q,P},\beta_{[M:Q],P}):[[M:Q]:P]\to[M:P]\times[M:Q]$$

is proper. Its image is a submanifold and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

Proof. Lemma 6.3.5 states that $[[M:Q]:P]=[M:P\cup Q]=[[M:P]:Q]$. This gives $\zeta_{M,Q,P}=\beta_{[M:P],Q}$. In particular, $\zeta_{M,Q,P}$ is proper, by Corollary 6.2.10. To prove that $\mathcal{B}_{M,Q,P}$ is proper, let us recall first notice that if $f_i:X\to Y_i$, $i=1,\ldots,N$, are continuous proper maps, then $\prod_{i=1}^{N}f_i:X\to\prod_{i=1}^{N}Y_i$ is also proper. This shows that $\mathcal{B}_{M,Q,P}=(\beta_{[M:P],Q},\beta_{[M:Q],P})$ is proper by Corollary 6.2.10. □

By iterating the above lemma, we obtain the following consequence.

Corollary 6.3.7. *Let $\mathcal{F}:=\{P_1,P_2,\ldots,P_k\}$ be a family of closed, non-trivial,* disjoint *p-submanifolds of a manifold with corners M. Then we have canonical diffeomorphisms inducing the identity on $M_0:=M\smallsetminus\bigcup_{j=1}^{k}P_j$ between the usual blow-ups and the graph blow-up (Definitions 6.2.1 and 6.3.1) :*

$$[[\ldots[[M:P_1]:P_2]:\,\ldots\,:P_{k-1}]:P_k]\;\simeq\;[M:\bigcup_{j=1}^{k}P_j]\;\simeq\;\{M:\mathcal{F}\}\,.$$

Proof. This follows by induction from Lemmas 6.3.5 and 6.3.6 since P_j identifies naturally with a p-submanifold of $[[\ldots[[M:P_1]:P_2]:\,\ldots\,:P_{j-2}]:P_{j-1}]$. □

6.4 Iterated blow-ups

The graph blow-up $\{M:\mathcal{F}\}$ introduced in the previous subsection has the advantage that it is defined in great generality and is obviously independent of the order on the family of p-submanifolds $\mathcal{F}$, up to an isomorphism. However, it is not clear what is the structure of the graph blow-up. To this end, in this section, we shall consider an iterated blow-up, which is defined under much more restrictiv conditions, but will be, by construction, a manifold with corners. The main result will be that the iterated blow-up and the graph blow-up are diffeomorphic.

6.4.1 Definition of the iterated blow-up

Recall the definition of the lifting $\beta^*(Q) = \beta^*_{M,P}(Q) := \overline{Q \smallsetminus P} \subset [M : P]$ (closure in $[M : P]$), Definition 6.2.15. We fix a manifold with corners M. We now introduce the *iterated version of the blow-up.*

Definition 6.4.1. *Let $(P_i)_{i=1}^k$, $P_i \subset M$, be a k-tuple of closed, non-trivial p-submanifolds of M and let $\beta_1 := \beta_{M,P_1} : [M : P_1] \to M$. (We do not assume any inclusion relations between the p-submanifolds P_i.) Whenever all the terms make sense, we define by induction on k the* iterated blow-up $[M : (P_i)_{i=1}^k]$ of M with respect to *or* along the ordered family $(P_i)_{i=1}^k$ *by*

$$[M : (P_i)_{i=1}^k] := \begin{cases} [M : P_1] & \text{if } k = 1\,, \\ \Big[[M : P_1] : \big(\beta_1^*(P_i)\big)_{i=2}^k\Big] & \text{if } k > 1\,. \end{cases}$$

We stress that we do not assume any inclusions among the manifolds P_i, but, on the other hand, $[M : (P_i)_{i=1}^k]$ is not always defined (unlike the graph blow-up !), as we need additional conditions in order to guarantee that $\beta_{j-1}^*\beta_{j-2}^* \cdots \beta_1^*(P_j)$ is a closed p-submanifold for all j. We shall also write

$$[M : (P_i)_{i=1}^k] =: [M : P_1, P_2, \ldots, P_k]\,,$$

and hence, using the pull-back by the map β_1, we have

$$[M : P_1, P_2, \ldots, P_k] := [[M : P_1] : \beta_1^*(P_2), \ldots, \beta_1^*(P_k)]\,.$$

We generalize this relation in the following remark.

Remark 6.4.2. Let $\gamma_1 := \beta_1^*$ and $\gamma_j := \beta_j^* \circ \gamma_{j-1} = \beta_j^* \circ ... \circ \beta_1^*$, where

$$\beta_k := \beta_{[M,P_1,P_2,\ldots,P_{k-1}],P_k} : [M : P_1, P_2, \ldots, P_k] \to [M : P_1, P_2, \ldots, P_{k-1}]\,.$$

We then have

$$\begin{aligned} [M : P_1, P_2, \ldots, P_j] &= [[M : P_1] : \gamma_1(P_2), \ldots, \gamma_1(P_j)] \\ &= [[[M : P_1] : \gamma_1(P_2)] : \gamma_2(P_3), \ldots, \gamma_2(P_j)] \\ &= \ \ldots \\ &= [[\ldots[[[M : P_1] : \gamma_1(P_2)] : \gamma_2(P_3)] \ \ldots] : \gamma_{j-1}(P_j)]\,. \end{aligned}$$

Note that $[M : P_1, P_2, \ldots, P_j]$ is always defined if $j = 1$. Then the condition that the iterated blow-up $[M : P_1, P_2, \ldots, P_j]$ be defined can then be formulated by induction as follows :

(i) the iterated blow-up $[M : P_1, \ldots, P_{j-1}]$ is defined, and

(ii) the lift $\gamma_{j-1}(P_j)$ is defined and is a closed p-submanifold of $[M : P_1, \ldots, P_{j-1}]$.

6.4.2 Clean semilattices

We now investigate the iterated blow-up $[M : (P_i)_{i=1}^k]$ of a manifold with corners M with respect to a (suitably) *ordered* family of non-trivial p-submanifolds of M.

Definition 6.4.3. *Let $\mathcal{F}$ be a locally finite (unordered) set of p-submanifolds of M. We shall say that $\mathcal{F}$ is a* cleanly intersecting family *if any $X_1, X_2, \ldots, X_j \in \mathcal{F}$ have a clean intersection (Definition 6.2.12).*

We consider the iterated blow-up mostly with respect to semilattices. Recall that a *meet semilattice* (or, simply, *semilattice* in what follows) is a partially ordered set $\mathcal{L}$ such that, for every two $x, y \in \mathcal{L}$, there is a greatest common lower bound $x \cap y \in \mathcal{L}$ of x and y. We shall consider only semilattices of subsets of a given set where the order is given by $\subset$ and where $x \cap y$ is the usual intersection of sets. We can now introduce the semilattices we are interested in. We let $\mathcal{P}(M)$ denote the set of all subsets of M.

Definition 6.4.4. *If $\mathcal{S} \subset \mathcal{P}(M)$ is a semilattice of closed p-submanifolds of M, then we shall say that $\mathcal{S}$ is a* clean semilattice of p-submanifolds of M *if every $Y \in \mathcal{S}$ satisfies $\dim(Y) < \dim(M)$ and $\mathcal{S}$ is a cleanly intersecting family of p-submanifolds of M.*

For the simplicity of the notation, we shall consider only semilattices $\mathcal{S} \subset \mathcal{P}(M)$ with $\emptyset \in \mathcal{S}$. This changes nothing in our results, but avoids us treating separately the cases $\emptyset \in \mathcal{S}$ and $\emptyset \notin \mathcal{S}$ in proofs.

Remark 6.4.5. Cleanly intersecting lattices are useful for studying iterated blow-ups because, if P, Q are two p-submanifolds of a manifolds with corners M such that P and Q intersects cleanly, then the lifts of P and Q in $[M : P \cap Q]$ are disjoint p-submanifolds of $[M : P \cap Q]$.

Recall the following result from [1] :

Proposition 6.4.6. *Let $\mathcal{S} \ni \emptyset$ be a clean semilattice (of p-submanifolds) of M and let P be a minimal element of $\mathcal{S} \smallsetminus \{\emptyset\}$. Then*

$$\mathcal{S}' \;:=\; \{\, [Q : P \cap Q] \,|\, Q \in \mathcal{S}\} \,.$$

is a cleanly intersecting semilattice of $[M : P]$ (note that $\emptyset = [P : P \cap P] =: P' \in \mathcal{S}'$).

Recall that minimality and the semilattice property imply for any $Q \in \mathcal{S}$ that we have either $P \subset Q$ or $P \cap Q = \emptyset$. In the first case, we have $[Q : P \cap Q] = [Q : P]$ and in the second case we have $[Q : P \cap Q] = Q$. Thus

$$\mathcal{S}' \;:=\; \{\, [Q : P] \,|\, P \subset Q \in \mathcal{S}\} \,\cup\, \{\, Q \,|\, Q \in \mathcal{S}, Q \cap P = \emptyset\} \,.$$

Let us also notice that $P' := [P : P \cap P] = \emptyset = [\emptyset : \emptyset \cap P] = \emptyset'$, whereas all the other manifolds Q' ($Q \in \mathcal{S} \smallsetminus \{\emptyset, P\}$) are different to each other and nonempty. Therefore, $\#(\mathcal{S}') = \#(\mathcal{S}) - 1$ (i.e. $\mathcal{S}'$ one element less than $\mathcal{S}$).

Proof. The proof is the same as the one of Theorem 2.8 in [1]. $\square$

Let $\mathcal{S}$ be cleanly intersecting semilattice (of p-submanifolds) of M and let us arrange $\mathcal{S} \smallsetminus \{\emptyset\}$ in a sequence $(P_i)_{i=1}^k = (P_1, P_2, \ldots, P_k)$. Recall that $(P_i)_{i=1}^k$ is an ordering of $\mathcal{S} \smallsetminus \{\emptyset\}$ compatible with the inclusion if $i \leq j$ whenever $P_i \subset P_j$, see Equation (1.15).

Proposition 6.4.7. *Let $\mathcal{S}$ be cleanly intersecting semilattice (of closed p-submanifolds) and $(P_i)_{i=1}^k = (P_1, P_2, \ldots, P_k)$ be an ordering of $\mathcal{S} \smallsetminus \{\emptyset\}$ compatible with the inclusion ($P_i \subset P_j \Rightarrow i \leq j$). Then $[M : (P_i)_{i=1}^k]$ is defined.*

Proof. We shall write $[M : \mathcal{S}] := [M : (P_i)_{i=1}^k]$. (This definition of $[M : \mathcal{S}]$ is implicitly assuming that a compatible order was chosen on $\mathcal{S}$. The notation is nevertheless justified since Theorem 6.4.10 will show that the result is independent of the order.) To prove that $[M : \mathcal{S}] := [M : (P_i)_{i=1}^k]$ is defined, we shall proceed by induction on the number of elements of $\mathcal{S}$. As $\emptyset \in \mathcal{S}$, let us assume, for the initial verification step, that $|\mathcal{S}| = 2$ and, more precisely, that $\mathcal{S} = \{\emptyset, P\}$, for some non-trivial p-submanifold P of M. Then $[M : (P_i)_{i=1}^1] := [M : P]$ is defined.

Let us assume that the result is true for semilattices $\mathcal{S}$ with j elements and prove it for lattices with $j + 1$ elements. Then the semilattice $\mathcal{S}'$ obtained from $\mathcal{S}$ via Proposition 6.4.6 is a cleanly intersecting semilattice with j elements of $[M : P_1]$ by that same proposition. Therefore $[[M : P_1] : \mathcal{S}']$ is defined by the induction hypothesis, and hence, using also Remark 6.4.2, we have that

$$[M : \mathcal{S}] \ := \ [[M : P_1] : \mathcal{S}'] \tag{6.18}$$

is also defined. □

6.4.3 The pair blow-up lemma

We now perform some essential calculations in local coordinates that will be needed for our main result. Recall that $\mathbb{S}^n_k := \mathbb{S}^n \cap \mathbb{R}^{n+1}_k$. For $\psi \in \mathbb{S}^{n'+1}_{k'+1} := \mathbb{S}^{n'+1} \cap \mathbb{R}^{n'+2}_{k'+1}$, we shall write $\psi =: (\psi_1, \tilde{\psi})$, with $\psi_1 \in [0,1]$ and $\tilde{\psi} \in \mathbb{R}^{n'+1}_{k'}$, and we define the map

$$\begin{gathered}\Upsilon : \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \to \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right), \\ (\phi, \psi) \mapsto (\psi_1 \phi, \tilde{\psi}).\end{gathered} \tag{6.19}$$

We embed the sphere octant $\{0\} \times \mathbb{S}^{n'}_{k'} = \{0_{\mathbb{R}^n}\} \times \mathbb{S}^{n'}_{k'} \subset \mathbb{R}^{n+n'+1}$ into $\mathbb{R}^{n+n'+1}$ by mapping the sphere octant onto the *last* components of $\mathbb{R}^{n+n'+1}$. Of course, $\mathbb{S}^{n,n'}_{k,k'} := \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right) \simeq \mathbb{S}^{n+n'}_{k+k'} = \mathbb{S}^{n+n'} \cap \mathbb{R}^{n+n'+1}_{k+k'}$ by the canonical permutation of coordinates diffeomorphism of Equation (6.4). By abuse of notation, we shall let $\mathbb{S}^{n'}_{k'}$ denote the image of $\{0\} \times \mathbb{S}^{n'}_{k'}$ in $\mathbb{S}^{n+n'}_{k+k'}$ under this permutation of coordinates.

Lemma 6.4.8 (Melrose)**.** *We use the notation in the last paragraph. Let* $\mathbb{S}^{n,n'}_{k,k'} := \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right) \simeq \mathbb{S}^{n+n'}_{k+k'}$*. The map*

$$\begin{aligned}\Psi : \mathbb{S}^{n,n'}_{k,k'} \smallsetminus \left(\{0\} \times \mathbb{S}^{n'}_{k'}\right) \ &\to \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \\ (\eta, \mu) \ &\mapsto \ \left(\frac{\eta}{|\eta|}, \ (|\eta|, \mu)\right)\end{aligned}$$

satisfies $\Upsilon \circ \Psi = id$ *and extends to a diffeomorphism*

$$\widetilde{\Psi} : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \ \xrightarrow{\sim} \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$$

such that $\beta_{\mathbb{S}^{n,n'}_{k,k'}, \{0\} \times \mathbb{S}^{n'}_{k'}} = \Upsilon \circ \widetilde{\Psi}$.

In particular, we obtain a diffeomorphism

$$[\mathbb{S}^{n+n'}_{k+k'} : \mathbb{S}^{n'}_{k'}] \ \xrightarrow{\sim} \ \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$$

Proof. By Proposition 6.2.16, the blow-up of $\mathbb{S}^{n,n'}_{k,k'}$ along $\{0\} \times \mathbb{S}^{n'+1}_{k'} = \mathbb{S}^{n,n'}_{k,k'} \cap \{0\} \times \mathbb{R}^{n'+1}_{k'} \simeq \mathbb{S}^{n'+1}_{k'}$ is diffeomorphic to the lifting of $\mathbb{S}^{n,n'}_{k,k'}$ to $[\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}]$ via the blow-down map

$$\beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\} \times \mathbb{R}^{n'+1}_{k'}} : [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}] \to \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}.$$

We have a canonical diffeomorphism $\kappa^{-1} : [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}] \simeq \mathbb{S}^{n-1}_k \times [0, +\infty) \times \mathbb{R}^{n'+1}_{k'}$ (see Equation (6.10)) with blow-down map :

$$\begin{gathered}\beta : \mathbb{S}^{n-1}_k \times [0, +\infty) \times \mathbb{R}^{n'+1}_{k'} \to \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \\ \beta(z, r, x) \ := \ (rz, x).\end{gathered}$$

We have that $(z, r, x) \in \beta^{-1}(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'}))$ if, and only if $\|\beta(z,r,x)\| = 1$ and $r > 0$. Assume that $\|\beta(z,r,x)\| = 1$ and $r > 0$. Hence $\|rz\|^2 + \|x\|^2 = 1$. Note that $z \in \mathbb{S}^{n-1}_k$, and hence $\|x\|^2 + |r|^2 = 1$. This leads to $(r, x) \in \mathbb{S}^{n'+1}_{k'+1} \subset \mathbb{R}^{n'+2}_{k'+1} = [0, +\infty) \times \mathbb{R}^{n'+1}_k$. We have

$$\beta^{-1}\big(\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'})\big) \;=\; (\mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}) \smallsetminus (\{0\} \times \mathbb{R}^{n'+1}_{k'}).$$

The closure of this set is $\mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$, and hence the result. The relation $\Upsilon \circ \Psi = id$ follows from the defining formulas. The relation $\beta_{\mathbb{S}^{n,n'}_{k,k'}, \{0\}\times\mathbb{S}^{n'}_{k'}} = \Upsilon \circ \widetilde{\Psi}$ follows from the fact that they coincide on the dense, open subset $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus (\{0\} \times \mathbb{S}^{n'}_{k'})$. □

We now treat the basic case when the blow-up is defined, namely the simplest case when we blow up by exactly two submanifolds P and Q. The case when these two submanifolds are disjoint was already treated in Lemma 6.3.6, so now we treat the remaining case, that is, the one submanifold is contained in the other.

Lemma 6.4.9. *Let us assume that Q is a p-submanifold of P and that P is a p-submanifold of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : [M : Q, P] \;:=\; [[M : Q] : [P : Q]] \;\to\; [M : P]$$

that restricts to the identity on $M \smallsetminus P$. Moreover, the product map

$$\mathcal{B}_{M,Q,P} \;:=\; (\zeta_{M,Q,P}, \beta_{[M:Q],[P:Q]}) : [M : Q, P] \to [M : P] \times [M : Q]$$

is proper and is a diffeomorphism onto its image.

Proof. The uniqueness of the map $\zeta_{M,Q,P}$ follows from the fact that it is the identity on the dense subset $M \smallsetminus (P \cup Q)$. The statement is local, so, in view of lemma 6.2.11, we can assume that $Q = \{0\}$. That is, we can assume that

$$\begin{cases} M & := \; \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P & := \; \{0\} \times \mathbb{R}^p_{k_p} \\ Q & := \; \{0\} \end{cases}$$

We have

$$\begin{aligned} [M : P] \;&=\; [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0\} \times \mathbb{R}^p_{k_p}] \\ &=\; [\mathbb{R}^m_{k_m} : \{0\}] \times \mathbb{R}^p_{k_p} \\ &=\; \mathbb{S}^{m-1}_{k_m} \times [0, +\infty) \times \mathbb{R}^p_{k_p} \\ &=\; \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1}\,. \end{aligned}$$

Its blow-down map is $\beta_{M,P}(x, t, y) = (tx, y)$.

On the other hand, we have (using the notation of Lemma 6.4.8) :

$$[M : Q] \;=\; [\mathbb{R}^{m+p}_{k_m+k_p} : \{0\}] \;=\; \mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0, \infty).$$

Its blow-down map is $\beta_{M,Q}(x, t) = tx$. Lemma 6.2.16 gives that the lift of P to $[M : Q]$ is $P' := [P : Q] = \{0\} \times \mathbb{S}^{p-1}_{k_p} \times [0, +\infty)$. Lemmas 6.2.11 and 6.4.8 (in this order) then give canonical isomorphisms

$$\begin{aligned} [[M : Q] : P'] \;&\simeq\; [\mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0, \infty) : \mathbb{S}^{p-1}_{k_p} \times [0, +\infty)] \\ &=\; [\mathbb{S}^{m,p-1}_{k_m,\,k_p} : \mathbb{S}^{p-1}_{k_p}] \times [0, +\infty) \\ &\simeq\; \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^{p}_{k_p+1} \times [0, +\infty)\,. \end{aligned}$$

The blow-down map, up to the canonical isomorphisms, is given by the map $\Upsilon \times id$, where Υ is as defined in Equation (6.19). Hence $\Upsilon \times id(\phi, \psi, t) = (\psi_1 \phi, \tilde{\psi}, t)$.

The desired map $\zeta_{M,Q,P}$ is then obtained from the blow-down map $\mathbb{S}^p_{k_p+1} \times [0, +\infty) \to \mathbb{R}^{p+1}_{k_p+1} = [0, \infty) \times \mathbb{R}^p_{k_p}$, that is $\zeta_{M,Q,P}(x, y, t) = (x, ty)$. In particular, it is proper. It remains to check that this map is the identity of the set of regular points. To this end, it is enough to check that $\beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ (\Upsilon \times id)$ on $\mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0, +\infty)$. Indeed

$$\beta_{M,P} \circ \zeta_{M,Q,P}(x, y, t) \;=\; \beta_{M,P}(x, ty) \;=\; \beta_{M,P}(x, ty) \;=\; (ty_1 x, t\tilde{y})\,.$$

On the other hand,

$$\beta_{M,Q} \circ (\Upsilon \times id)(x, y, t) \;=\; \beta_{M,Q}(y_1 x, \tilde{y}, t) \;=\; (ty_1 x, t\tilde{y})\,.$$

This shows that $\beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ (G \times id)$.

Finally, the map $\mathcal{B} = \mathcal{B}_{M,Q,P} := (\zeta_{M,Q,P}, \beta_{[M:Q],[P:Q]})$

$$\mathcal{B} : \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0, +\infty) \to \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0, \infty)$$

is proper since each of its two components is proper by the discussion above and by the Corollary 6.2.10. It is given in local coordinates by $\mathcal{B}(x, y, t) = \big(x, ty, (y_1 x, \tilde{y}), t\big)$ with differentiable left inverse $(x, z, (w_1, w_2), t) \to (x, (\|w_1\|, w_2), t)$. Hence $\mathcal{B}$ is a diffeomorphism onto its image. □

Using the similar result for disjoint manifolds, Lemma 6.3.6, we obtain the following result. (Recall that our semilattices contain the empty set, but do not contain the ambient manifold M.)

Theorem 6.4.10. *Let $\mathcal{S}$ be a cleanly intersecting semilattice of M (so $\emptyset \in \mathcal{S}$ and $M \notin \mathcal{S}$). Then for each $P \in \mathcal{S}$ there exists a unique smooth map $\phi_{\mathcal{S},P} : [M : \mathcal{S}] \to [M : P]$ that is the identity on $M \smallsetminus \bigcup_{Q \in \mathcal{S}, Q \neq M} Q$. These maps are such that the induced map*

$$\mathcal{B}_{\mathcal{S}} \;:=\; \prod_{P \in \mathcal{S}} \phi_{\mathcal{S},P} : [M : \mathcal{S}] \to \prod_{P \in \mathcal{S}} [M : P]$$

is proper and is a diffeomorphism onto its image.

Proof. We shall proceed by induction on the number of elements of $\mathcal{S}$. If $\mathcal{S}$ has two elements, we have $\mathcal{S} = \{\emptyset, P\}$ and $\mathcal{B}_{\mathcal{S}} = (id, \beta_{M,P})$, so the claim is trivially satisfied, since the blow-down map is proper (Corollary 6.2.10).

If $\mathcal{S}$ has three elements, we have $\mathcal{S} = \{\emptyset, Q, P\}$ and there are two possibilities : $Q \subset P$ or $Q \cap P = \emptyset$. If $Q \subset P$, the result was already proved in Lemma 6.4.9, with $\mathcal{B}_{\mathcal{S}} = \mathcal{B}_{M,Q,P}$ (and $\phi_{\mathcal{S},Q} := \beta_{[M:Q],P}$, $\phi_{\mathcal{S},P} := \zeta_{M,Q,P}$). Similarly, if $Q \cap P = \emptyset$, the result was already proved in Lemma 6.3.6, again with $\mathcal{B}_{\mathcal{S}} = \mathcal{B}_{M,Q,P}$ (and with all the ϕ maps being given by blow-down maps β).

Let us now proceed with the induction step. For $P \in \mathcal{S}$, $P \neq P_1$, we have either $P_1 \subset P$ or $P_1 \cap P = \emptyset$, by the minimality of P_1. Let $P' := [P : P_1]$, if $P_1 \subset P$, and $P' := P$, otherwise (if $P_1 \cap P = \emptyset$). We shall use the notation of Equation (6.18), in particular, the semilattice $\mathcal{S}'$ of Proposition 6.4.6 is $\mathcal{S}' = \{P' \mid P \in \mathcal{S}\}$. By the induction hypothesis, the map $\mathcal{B}_{\mathcal{S}'}$ is a diffeomorphism onto its image. The same property is shared by the maps $\mathcal{B}_{M,P_1,P} : [[M : P_1] : P'] \to [M : P_1] \times [M : P]$ of the lemmas 6.3.6 and 6.4.9. Let us consider the composition

$$
\begin{aligned}
[M:\mathcal{S}] \;:=\; [[M:P_1]:\mathcal{S}'] \xrightarrow{\mathcal{B}_{\mathcal{S}'}} & \prod_{P'\in\mathcal{S}'} [[M:P_1]:P'] \\
\xrightarrow{\prod \mathcal{B}_{M,P_1,P}} & \prod_{P'\in\mathcal{S}'} [M:P_1]\times[M:P] \\
= & \; M\times[M:P_1]\times \prod_{P\in\mathcal{S},P\neq P_1,P\neq\emptyset} [M:P_1]\times[M:P]\,. \qquad (6.20)
\end{aligned}
$$

(The factor $M\times[M:P_1] = M\times M'$ corresponds to $P' = P = \emptyset$ and $P' = P = P_1$.) The two maps of the composition are both injective immersions, and hence their composition is again an injective immersion, that is, a diffeomorphism onto its image. The desired map $\phi_{\mathcal{S},P}$ is the projection onto the component. The projection of the composite map onto one of the factors is the identity on $M \smallsetminus \bigcup_{Q\in\mathcal{S},Q\neq M} Q$ and yields the desired map $\phi_{\mathcal{S},P}$. Note that all components with factors of the form $[M:P_1]$ (which are repeated), yield the same projection, again because this projection is the identity map on $M\smallsetminus\bigcup_{Q\in\mathcal{S},Q\neq M} Q$. By removing these repetitions, we obtain the desired map $\mathcal{B}_{\mathcal{S}}$. The resulting composition is proper since it is the composition of two maps with proper factors. □

We obtain the following corollary.

Corollary 6.4.11. *Let $\mathcal{S}$ be a cleanly intersecting semilattice of M. Then the map $\mathcal{B}_{\mathcal{S}}$ of Theorem 6.4.10 induces a natural homeomorphism*

$$\mathcal{B}_{\mathcal{S}} : [M:\mathcal{S}] \xrightarrow{\sim} \{M:\mathcal{S}\}\,.$$

If G is a discrete acting smoothly on M such that $g(\mathcal{S}) = \mathcal{S}$ for $g\in G$, then G acts smoothly on $[M:\mathcal{S}]$ and the action commutes with the above diffeomorphism $\mathcal{B}_{\mathcal{S}}$.

Proof. Theorem 6.4.10 gives that $\mathcal{B}_{\mathcal{S}} = \delta$ on the dense open subset $M\smallsetminus\bigcup_{Q\in\mathcal{S},Q\neq M} Q$, where δ is the diagonal embedding $\delta(x) = (x,x,\ldots,x)$ considered before. Hence $\mathcal{B}_{\mathcal{S}}$ maps to the graph blow-up $\{M:\mathcal{S}\}$, by the definition of the later. We know that $\mathcal{B}_{\mathcal{S}}$ is continuous and proper, and hence with closed image. This gives that $\mathcal{B}_{\mathcal{S}}([M:\mathcal{S}]) = \{M:\mathcal{S}\}$. In particular, $\{M:\mathcal{S}\}$ is a smooth manifold and $\mathcal{B}_{\mathcal{S}}$ is a diffeomorphism. This in conjunction with Corollary 6.2.6 gives that G acts smoothly on $\{M:\mathcal{S}\}$, and hence also on $[M:\mathcal{S}]$. □

6.5 Applications to the N-body problem

For any finite dimensional real vector space Z, recall that $\mathbb{S}_Z$ denotes the *set of vector directions* in Z, that is, the set of (non-zero) half-lines $\mathbb{R}_+^* v$, with $0\neq v\in Z$. The disjoint union and $\overline{Z} := Z\sqcup\mathbb{S}_Z$ is then called the *radial compactification* of Z. For example, if $Z=\mathbb{R}$, then $\overline{\mathbb{Z}} := [-\infty,\infty]$ with the usual topology (with a neighborhood system basis consisting of possibly infinite intervals). The action of the group $GL(Z)$ of linear automorphisms of Z extends, by definition, to an action on $\overline{Z}$. Similarly, if $Y\subset Z$, then $\overline{Y}\subset\overline{Z}$. In particular, $\overline{Z}$ is the union of all *closed lines* $\overline{\mathbb{R}v}$, $0\neq v\in Z$. It is possible to endow $\overline{Z}$ with a compact topology that makes the inclusions $\overline{\mathbb{R}v}\subset\overline{Z}$ continuous for all $0\neq v\in Z$, which explains why $\overline{Z}$ is called the radial compactification of Z. We will obtain the topology on $\overline{Z}$ from the next lemma, which also describes its smooth structure.

Let us regard $GL(\mathbb{R}^n)$ as a subgroup of $GL(\mathbb{R}^{n+1})$ with the action on the last n variables. There is a bijection between the set of vector directions in $\mathbb{R}^{n+1}$ and its unit sphere $\mathbb{S}^n$. This allows us to regard

$\mathbb{S}^n_1 := \{(x_1, x') \mid x_1 \geq 0\}$ as a subset of the set of vector directions in $\mathbb{R}^n$, where we used the usual the notation of Equation (6.1). This yields an action of $GL(\mathbb{R}^n)$ on $\mathbb{S}^n_1$. We have the following standard lemma, where we denote, as usual,

$$\langle x\rangle^2 \;:=\; 1 + \|x\|^2 \;=\; \|(1,x)\|^2.$$

See also [37, 59].

Lemma 6.5.1. *The map $\mathbb{R}^n \ni x \mapsto \frac{1}{\langle x\rangle}(1,x) \in \mathbb{S}^n_1$ extends by continuity on each closed line $\overline{\mathbb{R}v}$, $v \in \mathbb{R}^n$ to a bijection $\Theta_n : \overline{\mathbb{R}^n} \to \mathbb{S}^n_1$ that is equivariant for the actions of $GL(\mathbb{R}^n)$ on $\overline{\mathbb{R}^n}$ and on $\mathbb{S}^n_1$.*

Proof. There is a bijection between the set of vector directions in $\mathbb{R}^{n+1}$ and its unit sphere $\mathbb{S}^n$. The map Θ_n thus associates to each $(1, v) \in \{1\} \times \mathbb{R}^n$ its vector direction. Thus the map Θ_n commutes with the action on $GL(\mathbb{R}^n)$ on $\mathbb{R}^n$ and on $\mathbb{S}^n_1$. If we approach infinity in $\mathbb{R}^n$ in the direction of a vector $0 \neq v \in \mathbb{R}^n$, then $\Theta_n(tv) \to (0, \frac{v}{\|v\|})$ as $t \to \infty$. The resulting map defined on $\mathbb{S}^n_1$ also commutes with the action of $GL(\mathbb{R}^n)$. Thus the global map Θ_n, defined on $\overline{\mathbb{R}^n} := \mathbb{R}^n \sqcup \mathbb{S}_{\mathbb{R}^n}$ also commutes with the action of $GL(\mathbb{R}^n)$. ☐

This lemma gives us the desired smooth structure on radial compactifications.

Remark 6.5.2. Lemma 6.5.1 gives $\overline{\mathbb{R}^n}$ a natural structure of a manifold with boundary. For any n-dimensional vector space X and any linear isomorphism $X \simeq \mathbb{R}^n$, we obtain a bijection $\overline{X} \simeq \overline{\mathbb{R}^n}$ which we can use to endow $\overline{X}$ with the induced structure of a manifold with boundary. This structure on $\overline{X}$ is independent of the chosen isomorphism $X \simeq \mathbb{R}^n$ since $GL(\mathbb{R}^n)$ acts by diffeomorphisms on $\overline{\mathbb{R}^n}$. In particular, if $Y \subset X$ is a (linear) subspace, then $\overline{Y} \subset \overline{X}$ is p-submanifold and $\mathbb{S}_Y \simeq \mathbb{S}_X \cap \overline{Y}$.

Let X be a euclidean space and $\mathcal{F}$ be a finite semilattice of linear subspaces of X, $X \notin \mathcal{F}$, $\{0\} \in \mathcal{F}$. We shall be interested in the induced semilattice $\mathcal{S}$ on the boundary :

$$\mathcal{S} \;:=\; \{\, \mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F}\,\}\,. \tag{6.21}$$

Then $\emptyset \in \mathcal{S}$, as it corresponds to the subspace $\{0\} \subset X$. In what follows, the role played by M in the previous sections will be played by the spherical compactification $\overline{X}$ of X.

Proposition 6.5.3. *The canonical surjection $\pi_{X/Y} : X \to X/Y$ extends to a smooth map ψ_Y : $[\overline{X}, \mathbb{S}_Y] \to \overline{X/Y}$ such that the induced map $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$ is a diffeomorphism onto its image. Let $G = GL(X,Y) \subset GL(X)$ be the group of linear isomorphisms $X \to X$ that leave Y invariant. Then ψ_Y is G-equivariant.*

Proof. In view of Remark 6.5.2, we can assume $X = \mathbb{R}^n$ and $Y = \{0\} \times \mathbb{R}^q \simeq \mathbb{R}^q$. We shall write $\mathbb{S}^{q-1}$ instead of $\{0\} \times \mathbb{S}^{q-1}$, for simplicity. Recall that Lemma 6.4.8 yields a diffeomorphism $\widetilde{\Psi} : [\mathbb{S}^{r,r'}_{k,k'} : \{0\} \times \mathbb{S}^{r'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{r-1}_{k} \times \mathbb{S}^{r'+1}_{k'+1}$. We shall use this result for $r = n - q$, $r' = q - 1$, $k = 1$, and $k' = 0$. Since $\mathbb{S}^{q-1} = \mathbb{S}^{q-1}_0$, we obtain the diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^{n-q,q-1}_{1,0} : \{0\} \times \mathbb{S}^{q-1}] \xrightarrow{\sim} \mathbb{S}^{n-q-1}_1 \times \mathbb{S}^{q}_1\,.$$

Let $p_1 : \mathbb{S}^{n-q-1}_1 \times \mathbb{S}^{q+1}_1 \to \mathbb{S}^{n-q-1}_1$ be the projection onto the first compoenent. Since $\mathbb{S}^{n-q,q-1}_{1,0} = \mathbb{S}^{n-1}_1$ and $\mathbb{S}^q = \mathbb{S}^q_0$, the map Θ_n of Lemma 6.5.1 induces by Lemma 6.2.2 a map $\Theta^{\beta}_n : [\overline{X} : \mathbb{S}_Y] \to [\mathbb{S}^{n-q,q-1}_{1,0} : \mathbb{S}^{q-1}]$. We claim that the map

$$\psi_Y \;:=\; \Theta^{-1}_{n-q} \circ p_1 \circ \widetilde{\Psi} \circ \Theta^{\beta}_n \;:\; [\overline{X} : \mathbb{S}_Y] \;\to\; \overline{X/Y}$$

is the desired extension.

Indeed, the formula for Θ_m^{-1} is given by $\Theta_m^{-1}(y_0, y_1, \ldots, y_m) = \frac{1}{y_0}(y_1, \ldots, y_m)$, if $y_0 > 0$. Then, for $v = (v_{Y^\perp}, v_Y) \in Y^\perp \oplus Y \simeq X$ (here $Y^\perp = \mathbb{R}^{n-q}$), a straightforward calculation gives

$$\begin{aligned}
\Theta_{n-q}^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta_n(v) &= \Theta_{n-q}^{-1} \circ p_1 \circ \widetilde{\Psi}\Big(\frac{1}{\langle v \rangle}(1, v) \Big) \\
&= \Theta_{n-q}^{-1} \circ p_1 \Big(\frac{1}{\langle v_{Y^\perp} \rangle}(1, v_{Y^\perp}),\ \frac{1}{\langle v \rangle}(\langle v_{Y^\perp} \rangle, v_Y) \Big) \\
&= \Theta_{n-q}^{-1} \Big(\frac{1}{\langle v_{Y^\perp} \rangle}(1, v_{Y^\perp}) \Big) \\
&= v_{Y^\perp} \\
&= \pi_{X/Y}(v)\,.
\end{aligned}$$

So ψ_Y is indeed the desired extension of $\psi_{X/Y}$.

The restriction of the map $\beta_{\overline{X},\mathbb{S}_Y}$ to $\overline{X} \smallsetminus \mathbb{S}_Y$ is a diffeomorphism onto its image. The map $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$, when restricted to $\beta_{\overline{X},\mathbb{S}_Y}^{-1}(\mathbb{S}_Y) := \mathbb{S}N_+^{\overline{X}}\mathbb{S}_X \simeq \mathbb{S}_Y \times \overline{X/Y}$ becomes the inclusion map $\mathbb{S}_Y \times \overline{X/Y} \to \overline{X} \times \overline{X/Y}$. The result follows. $\square$

Remark 6.5.4. Let $\psi := p_1 \circ \widetilde{\Psi}$, using the notation of the proof of the last proposition. We thus have a commutative diagram

$$\begin{array}{ccc}
[\overline{X} : \mathbb{S}_Y] & \xrightarrow{\ \psi_Y\ } & \overline{X/Y} \\
{\scriptstyle \Theta_n^\beta}\big\downarrow & & \big\uparrow{\scriptstyle \Theta_{n-q}^{-1}} \\
[\mathbb{S}_{1,0}^{n-q,q-1} : \mathbb{S}^{q-1}] & \xrightarrow{\ \psi\ } & \mathbb{S}_1^{n-q}
\end{array} \tag{6.22}$$

Remark 6.5.5. The extension ψ_Y was also considered in [27, 31]. It satisfies the following property. Let Y be a linear subspace of X. If $x_n \in X$ converges to $\overline{x} \in \overline{X}$ and $\overline{x} \notin \mathbb{S}_Y$, then $x_n + Y$ converges in $\overline{X/Y}$ to $\psi_Y(\overline{x})$.

Remark 6.5.6. Let $\mathcal{E}_\mathcal{S}(X)$ be the norm closed algebra generated by all the spaces $\mathcal{C}(\overline{X/Y})$ as in (1.8). By Theorem 3.2.4, the spectrum of $\mathcal{E}_\mathcal{S}(X)$ is the closure of the image of X in the product $\prod_{Y \in \mathcal{F}} \overline{X/Y}$.

Recall the maps $\phi_{\mathcal{S},\mathbb{S}_Y} : [\overline{X} : \mathcal{S}] \to [\overline{X} : \mathbb{S}_Y]$ of Theorem 6.4.10 and the maps $\psi_Y : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$. We then have the following result.

Corollary 6.5.7. *The product map*

$$\Xi_\mathcal{S} : [\overline{X} : \mathcal{S}] \ \to \ \prod_{Y \in \mathcal{F}} \overline{X/Y} \tag{6.23}$$

of the composite maps $\psi_Y \circ \phi_{\mathcal{S},\mathbb{S}_Y} : [\overline{X} : \mathcal{S}] \to [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ is a diffeomorphism onto its image. Let G be a discrete group of linear autormorphisms of X that map a subspace in $\mathcal{F}$ to a subspace in $\mathcal{F}$ (thus $g(\mathcal{S}) = \mathcal{S}$ for all $g \in G$). Then G acts on $[\overline{X} : \mathcal{S}]$ and the map $\Xi_\mathcal{S}$ is G-equivariant.

Proof. For $Y = 0$, the map $\psi_Y \circ \phi_{\mathcal{S},\mathbb{S}_Y}$ is simply the blow-down map $[\overline{X} : \mathcal{S}]$. If we replace the maps ψ_Y with the maps $(\beta_{\overline{X},\mathbb{S}_Y}, \psi_Y) \circ \phi_{\mathcal{S},\mathbb{S}_Y}$ of Proposition 6.5.3. The result follows from that Proposition together with Theorem 6.4.10. The product of these maps has many repeated factors of $[\overline{X} : \mathcal{S}] \to \overline{X}$. The corollary is obtained by keeping one of these repeated factors. The action of G and the fact that $\Xi_\mathcal{S}$ is G-equivariant follow from Theorem 6.4.10. $\square$

Combining Corollary 6.5.7 with Remark 6.5.6, we obtain the following result.

Theorem 6.5.8. *There exists a natural homeomorphism*

$$\widehat{\mathcal{E}_{\mathcal{S}}(X)} \simeq [\overline{X} : \mathcal{S}]$$

that is the identity on X.

Proof. Let $\delta : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$ be the diagonal map. Remark 6.5.6 states that we have a homeomorphism $\widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \overline{\delta(X)}$. Corollary 6.5.7 states that the map $\Xi_{\mathcal{S}}$ defined on $[\overline{X}, \mathcal{S}]$ is a diffeomorphism onto its image. Since $[\overline{X}, \mathcal{S}]$ is compact, the image is closed. It moreover contains $\delta(X)$ as a dense open subset. Therefore $[\overline{X}, \mathcal{S}]$ is also homeomorphic to $\overline{\delta(X)}$. □

We obtain the following result for the spaces introduced in [6, 15, 24].

Remark 6.5.9. In [6, 15, 24], Georgescu and his collaborator has considered the norm closed subalgebra of functions $\mathfrak{A}_{\mathcal{S}}$ of $L^{\infty}(X)$ generated by all the algebras $\mathcal{C}_0(X/Y)$. This corresponds to potentials that have limit 0 on X/Y. The spectrum of this algebra (after we adjoin a unit) identifies with the closure of the image of X in $\prod_{Y \in \mathcal{S}}(X/Y)^+$, where Z^+ denotes the one point compactification of a locally compact space Z. Since $\mathfrak{A}_{\mathcal{S}} \subset \mathcal{E}_{\mathcal{S}}(X)$, we obtain that $\widehat{\mathfrak{A}_{\mathcal{S}}}$ is a quotient of $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$, and hence also a quotient of $[\overline{X} : \mathcal{S}]$. Generally, the topology on $\widehat{\mathfrak{A}_{\mathcal{S}}}$ is rather complicated and singular, see also [34, 35] for concrete examples when $\dim(X) = 2$.

Remark 6.5.10. In the concrete case of the N-body problem, we take $X := \mathbb{R}^{3N}$ and consider the subspaces $\mathcal{P}_j$ and $\mathcal{P}_{ij}$ of Equation (1.5) We let $\mathcal{F}$ be the semilattice generated by the subspaces $\mathcal{P}_i$ and $\mathcal{P}_{ij}$, $i, j \in \{1, 2, \ldots, N\}$. Let $\mathcal{S}$ be the finite semilattice of p-submanifolds of $\overline{X}$ as in Equation (6.21). Then $[\overline{X} : \mathcal{S}]$ will be endowed with natural smooth actions of $X := \mathbb{R}^{3N}$ by translation, of S_N, the symmetric group on N variables, by permutation, and $O(3)$ acting diagonally on the components of $X := \mathbb{R}^{3N}$.

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Résumé

Nous étudions l'opérateur $H = -\Delta + V$ qui représente l'énergie d'un système à N-électrons. Pour cela, nous utilisons les algèbres d'opérateurs. Nous commençons par définir une C^*-algèbres A qui contient le potentiel V du problème puis nous prenons son produit croisé $A \rtimes X$. Les résolvantes de H sont ainsi contenues dans cette C^*-algèbre dans $A \rtimes X$. Par une étude précise du spectre de $A \rtimes X$, nous obtenons une décomposition spectrale essentiel de H et donc un résultat qui étend le théorème HVZ dans la continuité des travaux de V. Georgescu. Nous étendons ce résultat en remplaçant l'espace euclidien X par le groupe de Heisenberg. Dans la seconde partie de la thèse, nous montrons que le spectre de la C^*-algèbre A et un espace introduit par A. Vasy dans les années 2000 sont les mêmes. L'espace construit par A. Vasy est construit par éclatements successifs d'une variété différentielle à coins. La preuve repose également sur des résultats d'éclatements de variétés. En particulier, nous avons introduit la notion de "graph blow-up" d'une variété par rapport à une familles assez général de sous-variétés.

Abstract

We study the operator $H = -\Delta + V$ that describes the energy of a system with N electrons. To do this, we use operator algebras. We thus first define a C^*-algebra A that contains the potentials V of the problem and then consider the crossed product $A \rtimes X$. The resolvents of H then belong to the C^*-algebra $A \rtimes X$. By a precise study of the spectrum of $A \rtimes X$, we obtain a decomposition of the essential spectrum of H, and hence of result that extends the HVZ theorem, in the spirit of Georgescu. We extend these results by replacing the underlying euclidean space X with the Heisenberg group. In the second part of the thesis, we show that the spectrum of A and the space introduce by A. Vasy around the year 2000 are the same. The space introduced by A. Vasy is defined using the blow-up of differentials manifolds with corners. The proofs are based on some differential geometric results on blow-ups of manifolds, in particular, we introduce the notion of "graph blow-up" of a manifold with respect to a rather general family of submanifolds.