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Présentée par

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Sur

Etudes des solutions de quelques équations aux dérivées partielles non linéaires via l'indice de Morse.

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Résumé

Cette thèse porte principalement sur l'étude des solutions de certaines équations aux dérivées partielles elliptiques via l'indice de Morse, y compris des solutions stables, i.e. quand l'indice de Morse est égal à zéro. Elle comporte deux parties indépendantes.

Dans la première partie, sous des hypothèses surlinéaires et sous-critiques sur f, on établit d'abord une estimation explicite de la norme L^∞ des solutions de $-\Delta u = f(u)$ avec $u = 0$ sur le bord, via leurs indices de Morse. On propose une approche plus transparente et plus souple que le travail de Yang [1998], ce qui nous permet de traiter des nonlinéarités très proches de la croissance critique.

Les résultats obtenus nous ont motivé de travailler sur des équations polyharmoniques $(-\Delta)^k u = f(x, u)$ avec notamment $k = 2$ et 3. Avec des hypothèses semblables à Yang [1998] sur f et des conditions au bord convenables, on obtient pour la première fois des estimations explicites de solution des équations polyhamoniques, via l'indice de Morse.

Dans la seconde partie, on considère un système de Lane-Emden

$$-\Delta u = \rho(x)v^p, \quad -\Delta v = \rho(x)u^\theta, \quad u, v > 0, \quad \text{dans } \mathbb{R}^N,$$

avec $1 < p \leq \theta$ et un poids radial ρ strictement positif. Nous montrons la non-existence de solution stable en petites dimensions N. Nos résultats améliorent les travaux précédents de Cowan & Fazly [2012]; Fazly [2012]; Hu [2015], et fournissent notamment des résultats du type Liouville pour solution stable, en petites dimensions N, valables pour tout $1 < p \leq \min(\frac{4}{3}, \theta)$.

Mots-clés : Indice de Morse, Estimation elliptique, Identité de Pohozaev, Équations polyharmoniques, Système de Lane-Emden avec poids, Théorème du type Liouville.

Abstract

The main concern of this thesis deals with the study of solutions of several elliptic partial differential equations via the Morse index, including the stable solutions, i.e. when the Morse index is zero. The thesis has two independent parts.

In the first part, under suplinear and subcritical assumptions on f, we establish firstly some explicit estimation for the L^∞ norms of solutions to $-\Delta u = f(u)$ with $u = 0$ on the boundary, via its Morse index. We propose an approach more transparent and easier than the work of Yang [1998], which allow us to treat some nonlinearities very close to the critical growth.

These results motivated us to consider the polyharmonic equations $(-\Delta)^k u = f(x, u)$ with especially $k = 2$ and 3. With the hypothesis on f similar to Yang [1998] and appropriate boundary conditions, we obtain for the first time some explicit estimations of solution via its Morse index, for the polyharmonic equations.

In the second part, we consider a Lane-Emden system

$$-\Delta u = \rho(x)v^p, \ \ -\Delta v = \rho(x)u^\theta, \ \ u, v > 0, \ \ \text{in } \mathbb{R}^N,$$

with $1 < p \le \theta$ and a radial positive weight ρ. We prove the non-existence of stable solution in small dimension case. Our results improve the previous works Cowan & Fazly [2012]; Fazly [2012]; Hu [2015], especially we prove some general Liouville type results for stable solutions in small dimension which hold true for any $1 < p \le \min(\frac{4}{3}, \theta)$.

Keywords: Morse index, Elliptic estimate, Pohozaev identity, Polyharmonic equation, Weighted Lane-Emden system, Stable solution, Liouville type theorem.

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Chapitre 1

Introduction Générale

Sommaire

Dans cette thèse, nous nous sommes intéressés à l'étude des solutions de certaines équations aux dérivées partielles, via leurs indices de Morse. Plus concrètement, nous avons travaillé sur les estimations explicites des solutions et sur la classification des solutions stables.

- On établit d'abord des estimations explicites de norme L^∞ de solution pour $-\Delta u = f(u)$ pour une large famille de nonlinéarité surlinéaire et sous-critique. On a notamment amélioré la technique de Yang [1998], ce qui nous a permis d'une part de trouver une approche plus transparente que Yang, d'autre part de traiter des nonlinéarités très proches de la croissance critique. Ce travail a été publié dans *Nonlineat Analysis*.

- On a ensuite considéré le cas polyharmonique: $(-\Delta)^k u = f(x, u)$. Sous des hypothèses semblables à Yang [1998] sur f et des conditions au bord convenables, on a réussi à établir pour $k = 2$ et 3, des estimations explicites de la norme infini des solutions en utilisant l'indice de Morse. C'est la première fois que de telles estimations soient établies.

- On a considéré dans un troisième travail des résultats du type Liouville pour les solutions stables du système de Lane-Emden $-\Delta u = \rho(x)v^p$, $-\Delta v = \rho(x)u^\theta$, $u, v > 0$ avec $1 < p \leq \theta$, et un poids ρ strictement positif admettant une croissance polynomiale à l'infini. En établissant une inégalité inverse que celle de Souplet, on a réussi notamment à traiter des cas où $1 < p \leq \frac{4}{3}$, qui n'était pas traité jusque là.

1 Estimation explicite des solutions via l'indice de Morse: Cas du Laplacien

Dans la première partie de cette thèse, on étudie les solutions de certains problèmes elliptiques surlinéaires et sous critiques via leurs indices de Morse.

On s'intéresse d'abord à l'équation semilinéaire d'ordre deux:

$$\begin{cases} -\Delta u = f(x,u) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{E_1}$$

où Ω est un ouvert borné, régulier de $\mathbb{R}^N$, $f(x,t)$ est une fonction de $C^1(\overline{\Omega}\times\mathbb{R})$.

1.1 Résultats d'existence et de multiplicité

En dimension $N=1$, et avec $f(x,s)=g(s)+h(x)$, Ehrman [1957] et Fucik & Lovicar [1975] ont montré que le problème (E_1) admet une infinité de solutions distinctes sous l'unique hypothèse

$$\lim_{|s|\to\infty}\frac{g(s)}{s}=\infty.$$

Pour $N=2$, Turner [1974] a montré que si f satisfait

$$C_1u^p \le f(x,u) \le C_2(1+u^p) \quad \text{avec } \; p<3, \;\; C_1, C_2>0,$$

alors toute solution classique et positive de (E_1) satisfait l'estimation à priori

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Ensuite, Nussbaum [1975] a établi la même estimation pour f satisfaisant

$$|f(x,u)| \le C(1+|u|^p) \quad \text{avec } \; p<\frac{N+1}{N-1}. \tag{$*$}$$

Son argument se base sur l'inégalité de Sobolev, et l'estimation de la norme L^1 pour $f(x,u)\varphi_1$, où φ_1 est une fonction propre qui correspond à la première valeur propre de $(-\Delta)$ dans $H_0^1(\Omega)$. De plus, si Ω est une boule de $\mathbb{R}^N$, Nussbaum a établi des estimations *a priori* pour toute solution positive à symétrie radiale de l'équation (E_1) avec $|f(x,u)|\le C(1+|u|^p)$ et $p<\frac{N+2}{N-2}$.

Il est clair que c'est difficile d'avoir une conclusion pour le cas général des nonlinéarités f, des résultats de multiplicité sont donc établis souvent sous des contraintes sur la croissance et la forme de f. Par exemple, en dimension $N\ge 2$, si $f(x,s)=f(s)$ est une fonction impaire, Coffman [1969], Hempel [1971], Ambrosetti [1973] et Rabinowitz [1974] ont prouvé que le problème (E_1) admet une infinité de solutions. Une question qui se pose est alors de savoir si ce résultat reste vrai si on admet une perturbation pour f.

Dans cet esprit, Bahri & Berestycki [1981] ont considéré (E_1) avec $f(x,u)=|u|^{p-1}u+h(x)$. Ils ont montré que pour $N\ge 2$ et $p_N\in\ [1,\frac{N+2}{N-2}[$, désigne la plus grande racine de

$$(2N-2)p^2-(N+2)p-N=0,$$

alors le problème (E_1) admet une infinité de solutions, si $1<p<p_N$ et $h\in L^2(\Omega)$. En fait, ils ont construit une suite de solutions (u_k), associées aux valeurs critiques de la fonctionnelle

$$I(u)=\int_\Omega\left[\frac{1}{2}|\nabla u|^2-F(x,u)\right]dx$$

telle que $\lim_{k\to+\infty} i(u_k) = +\infty$. Ici $i(u)$ désigne l'indice de Morse de la solution u, qui est défini comme le maximum des dimensions d'un sous-espace X de $C_c^1(\Omega)$ qui vérifie $\Lambda_u(\zeta) < 0$ sur $X \setminus \{0\}$ où Λ_u est juste la différentielle d'ordre deux de la fonctionnelle I, c'est à dire

$$\Lambda_u(\zeta) := \int_\Omega |\nabla\zeta|^2 dx - \int_\Omega f'(x,u)\zeta^2 dx, \quad \forall\, \zeta \in C_c^1(\Omega).$$

Ici $f'(x,t) := \frac{\partial f}{\partial t}(x,t)$. Cette méthode perturbative utilise essentiellement des estimations sur la croissance des valeurs critiques de I.

Pour le même type de non-linéarité, Bahri [1981] a établi l'existence d'une infinité de solutions pour (E_1) si la fonction h appartient à un ensemble qui est une intersection dénombrable d'ouverts denses dans $H^{-1}(\Omega)$. Ce résultat a résolu une conjecture développée dans l'article de Bahri & Berestycki [1981]. D'autre part, Bahri & Lions [1988] ont amélioré les estimations obtenues par Bahri-Berestycki; ils ont remplacé le terme forcing h par un terme nonlinéaire qui n'est pas nécessairement une fonction impaire. Ils ont montré que pour $p \in [1, \frac{N}{N-2}[$ avec $N \geq 2$, le problème $-\Delta u = |u|^{p-1}u + h(x,u)$, $u \in H_0^1(\Omega)$ où Ω est un domaine borné de $\mathbb{R}^N$, admet une infinité de solutions avec une restriction sur l'accroissement sous-critique de $h(x,u)$.

Récemment, ce résultat a été amélioré par Ramos *et al.* [2009a]. En effet, en utilisant l'indice de Morse des solutions pour le problème non perturbé, ils ont obtenu une suite de solutions changeant de signe et pour des nonlinéarités plus générales et sous des conditions d'accroissement sous-critique plus large.

1.2 Indice de Morse et son rôle

La compréhension des solutions de (E_1) via leurs indices de Morse s'est révélé très intéressante grâce au travail de Barhi-Lions, il semble naturel de relier l'indice de Morse à certaines propriétés qualitatives d'une solution. Par exemple, cela peut être utile pour montrer l'existence des solutions et pour établir une estimation uniforme des solutions.

En fait, si on revient à la démonstration du résultat d'existence d'une infinité de solutions dans Rabinowitz [1973] en dimension 1, on peut voir qu'il a classifié les solutions par leur nombre de zéros dans un diagramme bifurcatif. D'une part, on sait bien que le nombre de zéros de solutions nodales est lié naturellement à l'indice de Morse; D'autre part, un fait crucial de la preuve dans Rabinowitz [1973] est que les branches de bifurcations sont bornées et ne peuvent pas aller d'une fonction propre à une autre, puisqu'elles ont des nombres de zéros distincts. Or, cette dernière propriété est loin d'être vraie en dimension supérieure. L'étude du diagramme de bifurcation en utilisant l'indice de Morse au lieu du nombre de zéros est sans doute plus compliquée, mais possible dans certains cas particuliers.

Dans le travail de Zou [2001] (respectivement Yang [2004]), les auteurs ont utilisé l'information sur l'indice de Morse des solutions du problème perturbé (i.e. avec $f(x,s) = |s|^{p-1}s + h(x)$). Il est important de constater que dans ces deux travaux, il y a une légère ressemblance avec le diagramme bifurcatif de Rabinowitz [1973] dans le sens que la différence de l'indice de Morse des solutions du problème perturbé empêche les solutions limites de coïncider.

Comme l'indice de Morse permet d'avoir des résultats d'existence, de classification et de régularité (voir la sous-section 3.1.1), il est clairement utile d'essayer de la relier avec d'autres propriétés de solution, telle que la norme L^p, afin de mieux comprendre les équations elliptiques non linéaires. Dans leur travail pionnier, Bahri & Lions [1992] ont montré que si f satisfait :

$$f'(x,s)|s|^{-p+1} \to c(x) > 0 \quad \text{uniformément dans } \Omega, \quad \text{quand } s \to \pm\infty,$$

avec $c \in C(\overline{\Omega})$ et $1 < p < \frac{N+2}{N-2}$, alors toute suite de solutions (u_k) de (E_1) vérifie que

$$i(u_k) \to \infty \quad \text{si et seulement si} \quad \|u_k\|_{L^\infty(\Omega)} \to \infty.$$

C'est facile de voir que $i(u_k)$ reste bornée si $\|u_k\|_{L^\infty(\Omega)}$ est bornée. Pour le sens inverse, Barhi-Lions ont procédé par l'absurde et utilisent un argument de *blow-up*. Cette technique consiste à un changement d'échelle obtenu par dilatation, translation et normalisation, ce qui permet de définir une nouvelle suite $(\tilde{u}_k)$ définie sur de nouveaux domaines (Ω_k), telle que $(\tilde{u}_k)$ et son laplacien restent bornés dans $L^\infty(\Omega_k)$. De cette manière, on peut analyser le comportement *microlocal* concentré autour de l'extremum absolu à l'échelle *macroscopique*. En fait, la suite des domaines (Ω_k) converge vers l'espace entier $\mathbb{R}^N$ ou bien vers un demi-espace $\mathbb{R}^N_+$, suivant la distance du point de concentration par rapport au bord de Ω modulo le rapport de dilatation. On obtiendra alors des équations limites définies sur $\mathbb{R}^N$ ou bien $\mathbb{R}^N_+$, et on conclut par des résultats de classification des solutions avec l'indice de Morse fini pour le problème limite (voir plus de détails dans la sous-section 3.1.1)

1.3 D'autres résultats

Harrabi *et al.* [1998b] ont prouvé un résultat du type Bahri & Lions [1992] pour (E_1), mais avec une non-linéarité f n'ayant plus le même comportement asymptotique en $+\infty$ et $-\infty$. Ils supposent que :

$$(H') \qquad f'(x,t) \sim p^+ t^{p^+-1} \text{ en } +\infty, \qquad f'(x,s) \sim p^-|s|^{p^--1} \text{ en } -\infty, \text{ uniformément en } x,$$

avec $1 \leq p^-, p^+ < \frac{N+2}{N-2}[$ si $N \geq 4$ et $p^-, p^+ \in [2, 5[$ si $N = 3$. Il est facile de vérifier que (H') implique

$$(H) \qquad f(x,s) \sim s^{p^+} \text{ en } +\infty, \qquad f(x,s) \sim |s|^{p^--1}s \text{ en } -\infty.$$

Ils ont montré que, sous l'hypothèse (H'), pour toute suite (u_k) de solutions de (E_1), $\|u_k\|_{L^\infty(\Omega)}$ est bornée si et seulement si la suite des indices de Morse $i(u_k)$ est bornée. La preuve s'adapte de la méthode de Barhi-Lions, sauf que nous utilisons l'argument de blow-up autour du maximum et de minimum de u séparément, et que l'équation limite est maintenant de la forme $-\Delta u = u_+^p$ $(u_+ := \max(u, 0))$ sur demi-espace ou l'espace tout entier.

Pour l'équation de Neumann, i.e.

$$\begin{cases} -\Delta u = f(x,u) \text{ dans } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ sur } \partial\Omega \end{cases} \tag{1.1}$$

où Ω est un domaine borné régulier de $\mathbb{R}^N$, $N \geq 2$ et f est une nonlinéarité satisfaisant l'hypothèses (H'), Harrabi *et al.* [2011a] ont montré récemment qu'une suite de solutions (u_k) de (1.1) est uniformément bornée si et seulement si $i(u_k)$ reste bornée. Dans ce cas, ils ont besoin de la classification des solutions de l'équation $-\Delta u = u_+^p$ avec l'indice de Morse fini sur un demi-espace $\mathbb{R}^N_+$ et associée à la condition de Neumann au bord.

1.4 Nos résultats principaux

Dans le premier travail de cette thèse, on s'intéresse à obtenir des estimations de la norme L^p de solution du problème de Dirichlet, via l'indice de Morse. Notre approche s'inspire du travail de Yang [1998], où Yang a montré qu'il existe des relations entre les propriétés analytiques des solutions de (E_1) et les indices de Morse des solutions. Pour la première fois, Yang [1998] a réussi à établir des estimations explicites en montrant notamment que la norme L^q et L^∞ évoluent moins rapidement qu'un accroissement polynomial de l'indice. Il a obtenu ces résultats sous des hypothèses même plus faibles que celles dans Bahri & Lions [1992]. Sa méthode consiste à adapter les techniques de Bahri-Lions tout en considérant des fonctions tests locales bien choisies.

Introduisons d'abord les hypothèses de Yang sur la nonlinéarité f:

(H_1) (Sur-linéarité) Il existe $\mu > 0$ tel que

$$f'(x,s)s^2 \geq (1+\mu)f(x,s)s > 0, \quad \forall\, |s| > s_0, \quad x \in \Omega.$$

(H_2) (Croissante sous critique) il existe $0 < \theta < 1$ telle que

$$\frac{2N}{N-2}F(x,s) \geq (1+\theta)f(x,s)s, \quad \forall\, |s| > s_0, \quad x \in \Omega.$$

(H_3) Il existe une constante $C \geq 0$ satisfaisant

$$|\nabla_x F(x,s)| \leq C(F(x,s)+1), \quad \forall\, x \in \Omega.$$

Yang [1998] a montré que sous les hypothèses (H_1)–(H_3), alors il existe une constante positive $C = C(\Omega,\, f)$ telle que toute solution classique u de l'équation (E_1) vérifie

- $\displaystyle\int_\Omega |f(x,u)|^{p_0} \leq C(i(u)+1)^\alpha$ où $p_0 = 1 + \frac{(1+\theta)(N-2)}{(1-\theta)N+2(1+\theta)}$ et $\alpha = \left(\frac{3}{2} + \frac{3}{2+\mu}\right)\frac{(2+\mu)^2}{3\mu+\mu^2}$;
- Et il existe une constante $0 < \beta \leq \frac{2\alpha}{p_0 N(2-p_0)}\left[\frac{2}{N(2-p_0)} - \frac{1}{p_0}\right]^{-1}$ telle que

$$\|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^\beta.$$

Le premier but de notre travail est d'obtenir une estimation explicite de la norme L^∞ de la solution de l'équation (E_1) via l'indice de Morse sous des hypothèses plus faibles que (H_1)–(H_2). En plus on va proposer une démonstration plus transparente que celle développée par Yang. Finalement, notre approche s'adapte à des nonlinéarités très proches de l'accroissement critique, et on montre que dans ce cas, la norme L^∞ évolue moins rapidement qu'un accroissement exponentiel de l'indice. Ce dernier résultat est tout à fait nouveau.

Plus précisément, on traite le cas où $f(x,s) = f(s)$, i.e

$$-\Delta u = f(u) \;\text{ dans } \Omega \quad \text{et} \quad u = 0 \text{ sur } \partial\Omega. \tag{E_1'}$$

Ici Ω est un domaine régulier borné de $\mathbb{R}^N$ avec $N \geq 3$. Nos hypothèses sur f sont:

(f_0) Il existe $s_0 > 0$, $C > 0$ et $\gamma > 0$ tel que

$$|f(s)|^{\frac{2N}{N+2}} \geq C|s|^{2+\gamma}, \quad \forall\, |s| > s_0.$$

(f_1) Il existe $s_0 > 0$ et $C > 0$ tel que

$$C|f(s)|^{\frac{2N}{N+2}} \leq s^2 f'(s) - f(s)s, \text{ pour } |s| > s_0.$$

(f_2) Il existe $s_0 > 0$ et $C > 0$ tel que

$$C|f(s)|^{\frac{2N}{N+2}} \leq \frac{2N}{N-2}F(s) - f(s)s \text{ pour } |s| > s_0, \quad \text{où } F(s) = \int_0^t f(t)dt.$$

On remarque que si les hypothèses (H_1)–(H_2) de Yang sont vérifiées, alors il existe $C > 0$ tel que

$$C|f(s)|^{\frac{2N}{N+2}} \leq \mu s f(s) \leq s^2 f'(s) - s f(s), \quad \text{si } |s| > s_1,$$

et

$$C|f(s)|^{\frac{2N}{N+2}} \leq \theta f(s) s \leq \frac{2N}{N-2}F(s) - s f(s), \quad \text{si } |s| > s_1.$$

Cela signifie que les hypothèses (H_1)–(H_2) impliquent les conditions (f_1)–(f_2). On note aussi que l'hypothèse (f_2) a été introduite pour la première fois, dans le travail de Harrabi *et al.* [2011b] dans le but d'obtenir des estimations L^∞ au cas radial. On note également que (f_2) couvre une large classe de nonlinéarité proche de la croissance critique $\frac{N+2}{N-2}$. On peut citer l'exemple suivant mentionné dans Harrabi *et al.* [2011b] :

$$f(s) = \frac{|s|^{\frac{4}{N-2}} s}{(\ln(|s|+2))^q}.$$

Il est facile de voir que f satisfait (f_0) et (f_1). De plus, pour tout $q \geq \frac{N+2}{N-2}$, on a quand $|s| \to \infty$,

$$\frac{2N}{N-2}F(s) - s f(s) \sim \frac{(N-2)q}{2N} \times \frac{|s|^{\frac{2N}{N-2}}}{(\ln(2+|s|))^{q+1}} = o\left(s f(s)\right).$$

On observe alors que f ne vérifie pas l'hypothèse (H_2), mais elle satisfait (f_2).

Nos principaux résultats sont les suivants:

Théorème 1.1 *Supposons que f satisfait (f_0)–(f_2). Alors il existe une constante $C = C(\Omega, f)$ telle que si $u \in C^2(\Omega) \cap C(\overline{\Omega})$ est une solution de (E'_1), u vérifie*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C(1 + i(u))^\delta$$

où

$$\delta = \frac{N+2}{N} \times \max\left\{\frac{2(N+2)(\gamma+2)}{\gamma N} + 6, \frac{N}{N-2}\left[\frac{2(N+2)(\gamma+2)}{\gamma N}) + 4\right] + 1\right\}.$$

Théorème 1.2 *Supposons que f satisfait (f_0)–(f_2) et*

(f_3) *Il existe $q > 0$, $s_0 > 1$ et une constante positive C' tel que*

$$|f(s)| \leq C' \frac{|s|^{\frac{N+2}{N-2}}}{(\ln|s|)^q} \quad \forall\, |s| \geq s_0.$$

Alors il existe une constante $C = C(\Omega, f)$ telle que toute solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ de (E'_1) vérifie

$$\|u\|_{L^\infty} \leq C \exp\left[C_N (1 + i(u))^{\frac{2\delta}{q(N-2)}}\right].$$

Ici δ est celle définie dans le Théorème 3.1 et C_N est une constante qui ne dépend que de N.

Un point clé de notre approche est que l'on utilise une fonction test à support compact pour établir une version spéciale de l'identité de Pohozaev. Cela nous permet d'éviter les intégrales sphériques qui apparaissent dans l'approche de Yang [1998] pour l'estimation des intégrales près de $\partial\Omega$ et qui sont difficiles à contrôler.

Plus précisément, on décompose Ω comme dans la preuve de Yang [1998], et considère

$$\Omega_{1,R} := \left\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \frac{R}{2}\right\} \ \text{ et } \ \Omega_{2,R} := \left\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) \leq \frac{R}{3}\right\}.$$

La difficulté principale est l'estimation de u près du bord $\partial\Omega$, c'est à dire sur $\Omega_{2,R}$. Pour couvrir $\Omega_{2,R}$, on choisit comme Yang des boules dont les centres se situent dans

$$\Gamma(R) := \left\{x \in \mathbb{R}^N \backslash \overline{\Omega} : \ \mathrm{dist}(x, \partial\Omega) = \frac{R}{20}\right\}.$$

Puis on multiplie l'équation (E_1') par $\psi \nabla u \cdot n$ avec $n(x) := x - y$ et ψ une fonction positive $\in C_c^2(B_R(y))$, où $B_R(y)$ désigne la boule de centre y et de rayon R. En utilisant le fait que $\nu \cdot n \leq 0$ sur $\partial\Omega_R(y)$, où $\partial\Omega_R(y) := \partial\Omega \cap B_R(y)$ et ν désigne le vecteur unité normal extérieur à $\partial\Omega$, on a

$$\int_{\partial\Omega_R(y)} \left[(n \cdot \nabla u)(\nu \cdot \nabla u) - \frac{1}{2}\nu \cdot n |\nabla u|^2\right] \psi d\sigma \leq 0.$$

Ainsi l'identité de Pohozaev avec $\psi \nabla u \cdot n$ implique que

$$\begin{aligned} &\int_{\Omega} \left[\frac{2N}{N-2} F(u) - f(u)u\right] \psi dx \\ &\leq CR \|\nabla\psi\|_{\infty} \left[\|\nabla u\|^2_{L^2(A_{R,\psi}(y))} + \int_{A_{R,\psi}(y)} F(u) dx\right] + C\|\Delta\psi\|_{\infty} \|u\|^2_{L^2(A_{R,\psi}(y))}, \end{aligned} \tag{1.2}$$

avec

$$A_{R,\psi}(y) = B_R(y) \cap \Omega \cap \{\nabla\psi \neq 0\}.$$

En prenant ψ une focntion de troncature telle que $\psi \equiv 1$ sur $B_{R/2}(y)$, cela nous permet de contrôler l'intégrale sur $B_{R/2}(y) \cap \Omega$ par une intégrale sur $A_{R,\psi}(y)$, et d'obtenir des preuves plus faciles que Yang [1998]. Par contre, sous les hypothèses (f_0)–(f_2), l'estimation locale de la norme L^2 de ∇u via $i(u)$ est plus difficile à obtenir que sous les hypothèses (H_1)–(H_3). On a besoin d'établir une estimation intérieure afin d'avoir une dépendance explicite en fonction de $i(u)$. Pour cela, on considère les domaines suivants:

$$A := A_a^b = \{x \in \mathbb{R}^N ;\ a < |x - y| < b\}, \quad A_{R,\psi}(y) \subset A_\rho := A_{a+\rho}^{b-\rho} \ \text{pour } 0 < \rho < \frac{b-a}{4}.$$

Par l'estimation elliptique, il existe une constante $C > 0$ dépendant uniquement de N tel que

$$\|\nabla u\|^2_{L^2(A_\rho \cap \Omega)} \leq C \left(\|f\|^2_{L^{\frac{2N}{N+2}}(A \cap \Omega)} + \frac{1}{\rho^2}\|u\|^2_{L^2(A \cap \Omega)}\right). \tag{1.3}$$

Maintenant, en utilisant le faite que $i(u) < \infty$ et en travaillant avec des fonctions de troncature sur des anneaux disjoints, on peut montrer qu'il existe a et b convenables tels que

$$\int_{\Omega \cap A_a^b} |f(u)|^{\frac{2N}{N+2}} dx \leq C(1 + i(u))^{2(\frac{\gamma+2}{\gamma})}.$$

Associant ceci avec les inégalités (1.2) et (1.3), on peut conclure que

$$\|f(u)\|^2_{L^{\frac{2N}{N+2}}(\Omega)} \leq C(1+i(u))^{\delta},$$

avec δ du Théorème 3.1. Ensuite, par l'inégalité de Sobolev-Gagliardo-Nirenberg, on déduit que

$$\|\nabla u\|_{L^2(\Omega)} \leq C(1+i(u))^{\delta}.$$

Si la nonlinéarité f satisfait de plus la condition (f_3), l'estimation de $\|\nabla u\|_{L^2(\Omega)}$ permet d'appliquer alors la technique de Brezis & Kato [1978] pour obtenir une estimation explicite de la norme L^∞ de u, via son indice de Morse.

Dans ce premier travail, on a révisé également le Théorème 1.2 dans Yang [1998], et on obtient une estimation de la norme L^∞ des solutions de (E_1) sous les hypothèses (H_1)–(H_3).

Théorème 1.3 *Supposons que f satisfait (H_1)–(H_3). Alors il existe une constante $C = C(\Omega, f)$ telle que si $u \in C^2(\Omega) \cap C(\overline{\Omega})$ est une solution de (E_1), u vérifie*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C(i(u)+1)^{\alpha'} \quad et \quad \|u\|_{L^\infty} \leq C(i(u)+1)^{\beta'}$$

où

$$\alpha' = \frac{4}{\mu} + 3 \quad et \quad \beta' = \frac{3\mu+4}{3\mu\theta}\left(\frac{3N^2(1-\theta)+N(7\theta-4)-2\theta+12}{N(N-2)^2}\right).$$

Notre preuve est encore une fois plus transparente et plus simple que celle de Yang, qui donne une légère amélioration de l'estimation de la norme L^∞ donnée par Yang [1998] quand N devient large.

2 Estimation explicite pour les équations polyharmoniques.

Dans cette section, on considère les équations polyharmoniques $(E_k) : (-\Delta)^k u = f(x,u)$ dans Ω, $k \geq 2$, avec soit les conditions de Dirichlet au bord

$$u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{sur} \quad \partial\Omega; \tag{2.1}$$

soit les conditions de Navier au bord

$$u = \Delta u = \ldots = \Delta^{k-1} u = 0 \quad \text{sur} \quad \partial\Omega. \tag{2.2}$$

Ici Ω est un ouvert borné régulier de $\mathbb{R}^N$ avec $N > 2k$ et f est une fonction $C^1(\overline{\Omega} \times \mathbb{R})$.

L'indice de Morse d'une solution classique u de (E_k), notée par $i(u)$, est le maximum des dimensions de sous-espace Σ tel que $\Lambda_u(\phi) < 0$ sur $\Sigma \setminus \{0\}$ avec

$$\Lambda_u(\phi) := \int_\Omega \left[|D^k\phi|^2 - f'(x,u)\phi^2\right] dx \quad \forall\, \phi \in \Sigma,$$

où

$$D^k u := \begin{cases} \nabla(\Delta^{\frac{k-1}{2}} u) & \text{si } k \text{ est impair,} \\ \Delta^{\frac{k}{2}} u & \text{si } k \text{ est pair;} \end{cases}$$

et

$$\Sigma_k := \begin{cases} H^k_0(\Omega) & \text{si on travaille avec (2.1);} \\ \left\{\phi \in H^k(\Omega), \phi = \Delta\phi = \ldots = \Delta^{[\frac{k-1}{2}]}\phi = 0 \text{ on } \partial\Omega\right\} & \text{si on travaille avec (2.2).} \end{cases}$$

Une solution u de (E_k) est dite stable si $\Lambda_u(\phi) \geq 0$ pour tout $\phi \in \Sigma$, autrement dit si son indice de Morse est égale à zéro.

Pour le cas avec les conditions de Dirichlet au bord, Soranzo [1994] a établi des estimations a priori pour les solutions positives à symétrie radiale de $(-\Delta)^k u = |u|^{p-1}u$ avec $1 \leq p < \frac{N+2k}{N-2k}$ par une légère modification de la méthode de blow-up de Gidas & Spruck [1981]. Ce résultat a été généralisé par Reichel & Weth [2009] d'une part pour tout domaine borné et régulier Ω et solution u changeant de signe, d'autre part pour l'équation polyharmonique $Lu = f(x,u)$, où L est un opérateur uniformément elliptique d'ordre $2k$ donné par

$$L = \left(\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right)^k + \sum_{|\alpha| \leq 2m-1} b_\alpha(x) D^\alpha.$$

Ils ont utilisé également la méthode de blow-up, en se basant sur leur résultat de type Liouville pour l'équation sous critique $(-\Delta)^k u = u^p$ avec $1 \leq p < \frac{N+2k}{N-2k}$ dans le demi espace $\mathbb{R}^N_+$, avec des conditions de Dirichlet sur $\partial\mathbb{R}^N_+$.

Pour le cas avec les conditions de Navier au bord, en se basant sur l'identité de Pohozaev, la technique du plan mobile (*moving plane method* en anglais) et un argument de *boot-strap*, Soranzo [1994] a établi des estimations a priori L^∞ des solutions positives de $(-\Delta)^k u = |u|^{p-1}u$ avec un domaine borné, régulier et convexe Ω.

2.1 Estimation explicite des solutions via l'indice de Morse

Dans ce deuxième travail de cette thèse, on souhaite obtenir des estimations explicites de solutions aux équations polyharmoniques, via les indices de Morse, ce qui semble un thème de recherche complètement vierge. Comme la situation est bien plus complexe que le cas du laplacien, notre principal objectif est de nous concentrer sur les cas $k = 2$ et $k = 3$. Sous des conditions appropriées sur f, on a réussi à montrer que la norme L^∞ d'une solution classique évolue moins rapidement qu'un accroissement polynomial de son indice de Morse $i(u)$. Plus précisément, on a considéré les problèmes suivants:

$$(-\Delta)^2 u = f(x,u) \text{ dans } \Omega; \quad u = \Delta u = 0 \text{ sur } \partial\Omega. \tag{E_2}$$

et

$$(-\Delta)^3 u = f(x,u) \text{ dans } \Omega; \quad u = \frac{\partial u}{\partial \nu} = \frac{\partial^2 u}{\partial \nu^2} = 0 \text{ sur } \partial\Omega. \tag{E_3}$$

Voici nos hypothèses sur la nonlinéarité f qui reprennent celles de Yang [1998].

(H_1) Il existe $\mu > 0$ tel que $f'(x,s)s^2 \geq (1+\mu)f(x,s)s > 0$, $\forall\, |s| > s_0$ et $x \in \Omega$.

(H_2) Il existe $0 < \theta < 1$ telle que

$$\frac{2N}{N-2k} F(x,s) \geq (1+\theta) f(x,s)s, \quad \forall\, |s| > s_0, \quad x \in \Omega.$$

(H_3) Il existe une constante $C \geq 0$ satisfaisant $|\nabla_x F(x,s)| \leq C(F(x,s)+1)$ pour tout $x \in \Omega$.

Nos principaux résultats sont les suivants:

Théorème 1.4 *Si u est une solution classique de (E_2) avec $f \geq 0$ satisfaisant (H_1)–(H_3) dans $\mathbb{R}_+$ ou bien si u est une solution classique de (E_3) avec f satisfaisant (H_1)–(H_3), alors il existe une constante positive indépendante de u tel que*

$$\int_\Omega |f(x,u)|^{p_k} dx \leq C(i(u)+1)^{\alpha_k},$$

où $p_k = \frac{2N}{N(1-\theta)+2k(1+\theta)}$ et $\alpha_k = \frac{4k(\mu+1)}{\mu}$ pour $k = 2$ ou 3 respectivement.

Comme f admet une croissance sous-critique, En utilisant une itération standard de bootstrap, on peut prouver que

Théorème 1.5 *Sous les hypothèses du Thérème 1.4, il existe une constante positive C telle que toute solution de (E_k) vérifie, pour $k = 2$ ou 3 respectivement*

$$\|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^{\beta^k}, \quad avec \ \beta_k = \frac{2k\alpha_k}{p_k N(2-p_k)}\left[\frac{2k}{N(2-p_k)} - \frac{1}{p_k}\right]^{-1}, \ \alpha = \frac{4k(\mu+1)}{\mu},$$

et p_k défini dans le Théorème 1.4.

Notre approche s'inspire du travail de Hajlaoui *et al.* [2015], nous gardons les mêmes notations que dans le cas du laplacien, mais la considération est bien plus difficile que le cas de (E_1).

Pour l'étude de (E_2), on décompose le domaine Ω comme dans le cas du Laplacien, et on va établir des estimations L^p locale via l'indice de Morse. Voyons d'abord l'identité de Pohozaev avec $\psi\nabla u \cdot n$ où $n(x) := x - y$ et ψ est une fonction positive dans $C_c^2(B_R(y))$. Soit u une solution de (E_2), alors on obtient

$$\begin{aligned} &\frac{2N}{N-4}\int_\Omega F(x,u)\psi dx + \frac{2}{N-4}\int_\Omega \nabla_x F(x,u)\cdot n\psi dx - \int_\Omega (\Delta u)^2\psi dx \\ = &-\frac{4}{N-4}\int_\Omega \Delta u \nabla^2 u(\nabla\psi, n)dx + \frac{1}{N-4}\int_\Omega (\nabla\psi\cdot n)(\Delta u)^2 dx \\ &- \frac{4}{N-4}\int_\Omega (\nabla u\cdot\nabla\psi)\Delta u dx - \frac{2}{N-4}\int_\Omega (\nabla u \cdot n)\Delta u \Delta\psi dx \\ &- \frac{2}{N-4}\int_\Omega F(x,u)\nabla\psi\cdot n dx - \frac{2}{N-4}\int_{\partial\Omega_R(y)} \frac{\partial \Delta u}{\partial \nu}(\nabla u\cdot n)\psi d\sigma. \end{aligned} \tag{2.3}$$

Le choix de conditions de Navier était en fait motivé pour avoir le bon contrôle du terme au bord. Comme on suppose que $f \geq 0$, les conditions de Navier permettent de dire que $u \geq 0$ et $-\Delta u \geq 0$ dans Ω, cela implique que le dernier terme de (2.3) soit négatif.

En associant avec les hypothèses (H_1)–(H_3), on peut montrer que pour $R < R_0(\Omega)$,

$$\begin{aligned} &\int_\Omega f(x,u)u\psi dx + \int_\Omega (\Delta u)^2\psi dx \\ \leq\ & CR\|\nabla\psi\|_\infty \int_{A_{R,\psi}(y)} f(x,u)u dx + CR^2\int_{A_{R,\psi}(y)} |\nabla^2(u\nabla\psi)|^2 dx \\ &+ C\Big(1 + R\|\nabla\psi\|_\infty\Big)\|\Delta u\|^2_{L^2(A_{R,\psi}(y))} + C\Big(R^2\|\nabla(\Delta\psi)\|^2_\infty + \|\Delta\psi\|^2_\infty\Big)\|u\|^2_{L^2(A_{R,\psi}(y))} \\ &+ CR^2\Big(\|\Delta\psi\|^2_\infty + \frac{1}{R^2}\|\nabla\psi\|^2_\infty + \|\nabla^2\psi\|^2_\infty\Big)\|\nabla u\|^2_{L^2(A_{R,\psi}(y))} + CR^N. \end{aligned} \tag{2.4}$$

D'autre part, soit u une solution de l'équation (E_2) avec l'indice de Morse fini. En prenant les fonctions de test sous forme $u\phi^m$ où ϕ représente des fonctions de troncature sur des anneaux disjoints, on peut obtenir sous l'hypothèse (H_1) que il existe des constantes $a < b$ bien choisies telles que

$$\int_{A_a^b\cap\Omega}(\Delta u)^2dx + \int_{A_a^b\cap\Omega} f(x,u)udx \le C\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}}.$$

Comme déjà mentionné, le travail ici est plus complexe que le cas du Laplacien, car le développement de $|\Delta(u\phi^m)|^2$ génère beaucoup de termes d'intéraction et on doit les bien estimer tous.

On remarque aussi que (H_1) implique

$$\|u\|^2_{L^2(A\cap\Omega)} \le C\left(\int_{A\cap\Omega} f(x,u)udx\right)^{\frac{2}{2+\mu}} + C.$$

D'autre part, avec l'estimation elliptique, il existe $C > 0$ dependant que de N telle que pour toute fonction $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and $0 < \rho < \min(1, \frac{b-a}{4})$,

$$\|\nabla u\|^2_{L^2(A_\rho\cap\Omega)} \le C\left(\frac{1}{\rho^2}\|u\|^2_{L^2(A\cap\Omega)} + \|\Delta u\|^2_{L^2(A\cap\Omega)}\right).$$

En considérant les termes à droite dans (2.5), il reste le terme avec $\nabla^2(u\nabla\psi)$ à estimer. En remarquant que $u\nabla\psi = 0$ sur $\partial\Omega$, la théorie elliptique nous dit que $\|\nabla^2(u\nabla\psi)\|_{L^2(\Omega)} \le C_\Omega\|\Delta(u\nabla\psi)\|_{L^2(\Omega)}$, ce qui nous permet donc de conclure.

Pour l'équation (E_3), comparant à (E_2), en replaçant les conditions de Navier par les conditions de Dirichlet au bord, on n'impose plus la conditions du signe à f, ni à la solution u. Encore une fois, le choix des conditions de Dirichlet est motivé pour bien contrôler les termes aux bord qui apparaissent dans la formule de Pohozaev associée à $\psi\nabla u\cdot n$ avec $n(x) = x - y$. Même si on procède similairement comme pour (E_2) ou encore (E_1), il y a des difficultés supplémentaires qui surgissent dans chaque étape.

On peut voir d'abord qu'il y a beaucoup de termes nouveaux dans le développement de $|\nabla\Delta(u\phi^m)|^2$ que nous devons contrôler. Par exemple, nous avons eu besoin de montrer que pour tout $m \ge 3$ et tout $\epsilon > 0$, il existe $C > 0$ telle que $\forall\, u \in H_0^3(\Omega)$ et $\phi \in C^6(\overline{\Omega})$, on a

$$\begin{aligned}&\int_\Omega\Big[(\Delta u)^2|\nabla\phi^m|^2 + |\nabla u|^2|\nabla^2\phi^m|^2 + |\nabla u|^2(\Delta\phi^m)^2 + |\nabla^2 u|^2|\nabla\phi^m|^2\Big]dx\\ &\le \epsilon\int_\Omega|\nabla(\Delta u)|^2\phi^{2m}dx + C\int_\Omega u^2[\phi]_6\phi^{2m-6}dx,\end{aligned}$$

où

$$[\phi]_6(x) = \sum_{|\beta_1|+\ldots+|\beta_p|=6, |\beta_i|\ge 1}\prod_{i=1}^p|\partial_{\beta_i}\phi(x)|.$$

Dans le même registre, suite à l'identité de Pohozaev pour (E_3), on est amené à bien étudier des termes comme

$$\int_{A_{R,\psi}(y)}\Big|\nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi) + \nabla u\nabla\psi\Big]\Big|dx + \int_\Omega\Big|\Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)\Big|dx.$$

3 Théorème de type Liouville et système de Lane-Emden.

Dans la deuxième partie de cette thèse, on traite le problème de classification des solutions stables pour un système du type Lane-Emden non-autonome:

$$-\Delta u = \rho(x)v^p, \quad -\Delta v = \rho(x)u^\theta, \quad u, v > 0, \quad \text{dans}\, \mathbb{R}^N \tag{3.1}$$

où $1 < p \leq \theta$.

Le problème du type (3.1) trouve son origine dans la description de plusieurs phénomènes physiques, astrophysiques, chimiques et biologiques. Il a été traité par beaucoup d'auteurs. Le but de notre travail est d'établir des résultats de non-existence (du type Liouville) de solution stable pour (3.1) sous des conditions convenables sur ρ.

3.1 Théorème du type Liouville et indice de Morse

En analyse complexe, le théorème de Liouville affirme que toute fonction holomorphe et bornée sur $\mathbb{C}$ est constante. Cela se généralise comme suit: toute fonction harmonique $-\Delta u = 0$ sur $\mathbb{R}^N$ telle que $u \in L^\infty(\mathbb{R}^N)$ est forcément constante.

Dans le célèbre papier de Gidas & Spruck [1981a], les auteurs ont prouvé que l'équation

$$-\Delta u = |u|^{p-1}u \quad \text{dans} \quad \mathbb{R}^N \tag{3.2}$$

n'admet pas de solutions classiques positives non triviale si $1 < p < p_s := \frac{N+2}{N-2}$ et $N \geq 2$. Une preuve plus simple de ce résultat basée sur la méthode du plan mobile, a été donnée ensuite par Chen & Li [1991].

Pour le cas du demi-espace, Gidas & Spruck [1981a] ont montré que si u est une solution classique et positive de

$$-\Delta u = |u|^{p-1}u \quad \text{dans} \quad \mathbb{R}^N_+ \tag{3.3}$$

avec la condition de Dirichlet sur le bord $\partial\mathbb{R}^N_+$, alors $u \equiv 0$ si $p \leq p_s$. Ce résultat a été étendu par Dancer [1992] pour $1 < p < \frac{N+1}{N-3}$ si $N \geq 3$, ce qui montre déjà que l'exposant critique de Sobolev p_s n'est pas toujours une valeur critique pour les théorèmes du type Liouville. Tout récemment, Chen *et al.* [2014b] ont établi un résultat très général: Soit u une solution positive et bornée de $-\Delta u = f(u)$ sur $\mathbb{R}^N_+$ avec $u = 0$ sur $\partial\mathbb{R}^N_+$,

$$f \in C^1(\overline{\mathbb{R}_+}) \cap C^2(\mathbb{R}_+), \quad f(0) = 0 \quad \text{et } f \text{ convexe},$$

alors $u \equiv 0$.

Pour (3.2) ou (3.3), il est clair que des solutions changeant de signe existent pour tout $p > 1$. Donc la question qui se pose est de savoir sous quelle propriété qualitative peut-on espérer un théorème du type Liouville. Dans cette direction, Bahri & Lions [1992] ont réussi à relier la problématique à l'indice de Morse. En fait, ils ont montré que si u est une solution bornée de (3.2) avec $i(u) < \infty$ et $p < p_s$, alors $u \equiv 0$. De plus, pour $p = p_s$ et $N \geq 3$, Caffarelli *et al.* [1989] et Chen & Li [1991] ont montré que toute solution positive non triviale de (3.2) s'écrit sous la forme

$$u(x) = \left[\frac{\sqrt{N(N-2)}\lambda}{\lambda^2 + |x - x_0|^2}\right]^{\frac{N-2}{2}}, \quad \text{avec} \quad \lambda > 0, x_0 \in \mathbb{R}^N.$$

Donc ce sont des solutions avec l'indice de Morse fini grâce à la fameuse formule de Cwikel-Lieb-Rozenblum (voir Li & Yau [1983] et Farina [2007]).

En 2007, Farina a complètement classifié les solutions d'indice de Morse fini pour l'équation (3.2). Plus précisément, il met en évidence le rôle d'un nouvel exposant critique p_{JL} qui est plus grand que p_s, appelé l'exposant de Joseph-Lundgren (voir aussi Gui *et al.* [1992]), défini par

$$p_{JL} = \begin{cases} \infty & \text{si } N \leq 10 \\ \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{si } N \geq 11. \end{cases}$$

Parmi bien d'autres résultats, Farina [2007] a montré que l'unique solution stable de (3.2) est $u \equiv 0$ si $1 < p < p_{JL}$, et qu'une solution non triviale et avec l'indice de Morse fini existe pour (3.2) si et seulement si $p = p_s$ $(N \geq 3)$ ou bien $p \geq p_{JL}$ avec $N \geq 11$.

Tous ces résultats de classification sont fort utiles dans les études qualitatives et quantitatives des équations semilinéaires, pour des problèmes d'existence, de régularité, de comportement asymptotique ou encore des estimations *a priori* des solutions dans des domaines arbitraires de $\mathbb{R}^N$. Par exemple, avec ses thérorèmes du type Liouville, Farina a généralisé le fameux résultat de Barhi-Lions à des exposants sur-critiques, il a montré que si Ω est un domaine borné régulier de $\mathbb{R}^N$ et $p_s < p < p_{JL}$, alors une famille de solutions $(u_k) \in H_0^1(\Omega) \cap C(\overline{\Omega})$ de $-\Delta u = |u|^{p-1}u$ dans Ω est uniformément bornée si et seulement si $i(u_k)$ reste borné.

3.2 Équation polyharmonique et classification

La situation des équations polyhamoniques est clairement plus délicate: Soit $m \geq 2$ et $p > 1$,

$$\Delta^m u = |u|^{p-1}u, \quad \text{dans } \mathbb{R}^N, \tag{3.4}$$

Wei & Xu [1999] ont montré (voir aussi Lin [1998] si $m = 2$) que l'unique solution nonnégative classique de (3.4) est $u \equiv 0$ si $1 < p < \frac{N+2m}{(N-2m)_+}$ et que toute solution classique, positive non nulle de (3.4) quand $N > 2m$ et $p = \frac{N+2m}{N-2m}$ est de la forme

$$u(x) = c_{N,m}\left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{N-2m}{2}}, \quad \text{avec } x_0 \in \mathbb{R}^N, \ \lambda > 0.$$

Pour le cas de demi-espace, Reichel & Weth [2009] ont prouvé que si u est une solution classique positive et bornée de $(-\Delta)^m u = u^p$ dans $\mathbb{R}_+^N$ avec des conditions du Dirichlet au bord, alors $u \equiv 0$ si $1 < p < \frac{N+2m}{(N-2m)_+}$.

Le premier résultat de non-existence de solution changeant de signe avec l'indice de Morse fini de (3.4) a été établi par Ramos & Rodrigues [2001], pour $m = 2$ et p sous critique (i.e. $1 < p < p_{s,2} := \frac{N+4}{N-4}$.) Ensuite, Gazzola & Grunau [2006] et Karageorgis [2009] ont complètement classifié les solutions radiales et positives de l'équation (3.4) avec $m = 2$. Ils ont montré que si $p_{s,2} < p < p_{JL,2}$, il n'existe pas de solution radiale positive de (3.4) avec $m = 2$ et ayant un indice de Morse fini. Ici $p_{JL,2}$ désigne un nouveau exposant du type Joseph-Lundgren pour le bilaplacien:

$$p_{JL,2} = \begin{cases} +\infty & \text{si } N \leq 12 \\ \frac{N+2-\sqrt{N^2+4-N\sqrt{N^2-8N+32}}}{N-6-\sqrt{N^2+4-N\sqrt{N^2-8N+32}}} & \text{si } N \geq 13. \end{cases}$$

Réciproquement, toute solution radiale positive et globale de (3.4) avec $m = 2$ est stable si $p \geq p_{JL,2}$ et $N \geq 13$.

Comme déjà mentionné, Farina [2007] a complètement classifié les solutions avec l'indice de Morse fini (sans condition de signe ni de symétrie) pour le cas du laplacien (3.2). Ses preuves sont basées sur un usage délicat de l'itération du type Moser, grâce à l'identité suivante:

$$\int_{\mathbb{R}^N}(-\Delta\varphi)\varphi^q dx = \frac{4q}{(q+1)^2}\int_{\mathbb{R}^N}|\nabla\varphi^{\frac{q+1}{2}}|^2 dx, \quad \forall\, 0 \le \varphi \in C_0^2(\mathbb{R}^N).$$

Mais la technique de Farina s'adapte difficilement au cas polyharmonique. Soit $m = 2$, l'identité correspondante à la précédente devient

$$\int_{\mathbb{R}^N}\varphi^q(\Delta^2\varphi)dx = \frac{4q}{(q+1)^2}\int_{\mathbb{R}^N}|\Delta\varphi^{\frac{q+1}{2}}|^2 dx - q(q-1)^2\int_{\mathbb{R}^N}\varphi^{q-1}|\nabla u|^4,\ \forall\, 0 \le \varphi \in C_0^4(\mathbb{R}^N).$$

L'apparition du terme avec $\varphi^{q-1}|\nabla\varphi|^4$ rend l'itération de Moser difficile à réaliser.

Pour surmonter cette difficulté, pour solution $u > 0$, Wei & D.Ye [2013] ont proposé de réécrire l'équation biharmonique (3.4) (i.e. avec $m = 2$) en un système:

$$-\Delta u = v, \quad -\Delta v = u^p, \quad \text{dans } \mathbb{R}^N. \tag{3.5}$$

Ils ont introduit l'idée d'utiliser la comparaison entre u et v, prouvées par Souplet [2009]: Toute solution positive (u, v) de (3.5) avec u bornée vérifie $v^2 \ge \frac{2}{p+1}u^{p+1}$ dans $\mathbb{R}^N$. En utilisant la stabilité avec des fonctions de test $u\phi^k$ où $\phi \in C_0^4(\mathbb{R}^N)$, ils ont réussi à montrer que le problème (3.5) n'admet pas de solution classique stable si $N \le 8$ pour tout $p > 1$.

Ce résultat a été amélioré par Cowan [2013] qui a montré qu'il n'existe pas de solution positive stable de (3.5) pour $N \le 10$ et $p > 1$. Ensuite, en exploitant une idée de Cowan & Ghoussoub [2014] et Dupaigne *et al.* [2013], Harrabi *et al.* [2014] ont établi la non-existence de solution positive et stable de (3.5), pour tout $p > 1$ et $N < 2 + 2x_0$, où x_0 est la plus grande racine de l'équation:

$$H(x,p) = x^4 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4} = 0. \tag{3.6}$$

En particulier, comme $x_0 > 5$ pour tout $p > 1$, on obtient la non-existence de solution positive et stable de pour $N \le 12$ et $p > 1$.

Le dénouement pour le cas bilaplacien, i.e. $m = 2$ dans (3.4) a été donné récemment par Davila *et al.* [2014]. Ils ont montré finalement que l'équation (3.4) avec $m = 2$ n'admet pas de solution avec l'indice de Morse fini (sans hypothèse sur le signe ou la symétrie pour u) si $1 < p < p_{JL,2}$ et $p \ne p_{s,2}$ pour toute dimension N. Leur méthode se base sur la découverte d'une formule de monotonie et sur une analyse de blow-down, qui leur a permis de réduire le problème (quand $p_{s,2} < p < p_{JL,2}$) à la classification de solution homogène stable pour $\Delta^2 u = |u|^{p-1}u$ avec $u \in H^2_{loc} \cap L^{p+1}_{loc}$.

3.3 Système de Lane-Emden classique

On commence par définir la notion de la stabilité pour un système à deux équations. Soit

$$-\Delta u = f(x,v), \quad -\Delta v = g(x,u), \quad \text{dans } \mathbb{R}^N, \tag{3.7}$$

avec $f, g \in C^1(\mathbb{R}^{N+1}, \mathbb{R})$ satisfaisant $f_v := \frac{\partial f(x,v)}{\partial v}$, $g_u = \frac{\partial g(x,u)}{\partial u} \ge 0$ dans $\mathbb{R}$. Une solution régulière (u, v) de (3.7) est dite stable s'il existe deux fonctions positives non nulle et régulières ϕ, ψ telles que

$$-\Delta\phi = f_v(x,v)\psi, \qquad -\Delta\psi = g_u(x,v)\phi, \quad \text{dans}\, \mathbb{R}^N. \tag{3.8}$$

Cette définition est motivé par le travail de Montenegro [2005]. Ensuite, Cowan [2013] et Dupaigne *et al.* [2013] ont remarqué que toute solution stable vérifie

$$\int_{\mathbb{R}^N} \sqrt{f_v(x,v)g_u(x,u)}\phi^2 dx \le \int_{\mathbb{R}^N} |\nabla\phi|^2 dx, \quad \forall\, \phi \in C_c^\infty(\mathbb{R}^N). \tag{3.9}$$

Revenons d'abord sur le système de Lane-emden classique, i.e. $\rho(x) \equiv 1$ dans (3.1).

$$-\Delta u = v^p, \quad -\Delta v = u^\theta, \quad u, v > 0 \quad \text{dans } \mathbb{R}^N. \tag{3.10}$$

Ceci peut être considéré comme le plus simple système semilinéaire qui généralise l'équation $-\Delta u = u^p$, néanmoins son comportement est loin d'être bien compris, même pour l'existence de solution. Il est bien conjecturé que le système (3.10) n'a pas de solution classique si (p, θ) est sous critique, i.e.

$$\frac{1}{p+1} + \frac{1}{\theta+1} > 1 - \frac{2}{N}, \quad p,\, \theta > 0. \tag{3.11}$$

Cette conjecture a été résolu dans le cas radial par Serrin & Zou [1994]. Le cas général est résolu seulement en dimension $N \le 4$ par Souplet [2009].

Tout récemment, Chen *et al.* [2014a] ont caractérisé les solutions radiales stables de (3.10) pour $p, \theta \ge 1$. Ils ont donné naissance à une nouvelle hyperbole critique, appelée courbe de Joseph-Lundgren. Plus précisément, ils ont montré que si $\theta \ge p \ge 1$ alors une solution radiale de l'équation (3.10) est instable si et seulement si $N \le 10$ ou $N \ge 11$ et

$$\left[\frac{(N-2)^2 - (\gamma-\beta)^2}{4}\right]^2 < p\theta\gamma\beta(N-2-\alpha)(N-2-\beta). \tag{3.12}$$

avec

$$\gamma = \frac{2(p+1)}{p\theta - 1}, \quad \beta = \frac{2(\theta+1)}{p\theta-1}.$$

Le membre de gauche dans l'inégalité (3.12) est lié à la constante optimale de l'inégalité de Hardy-Rellich (voir le travail de Caldiroli & Musina [2012]) :

$$\int_{\mathbb{R}^N} |x|^{2-\gamma}|\Delta\varphi|^2 dx \ge \left[\frac{(N-2)^2-\gamma^2}{4}\right]^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^{2+\gamma}} dx, \quad \forall\, \varphi \in H^2(\mathbb{R}^N). \tag{3.13}$$

avec $\gamma = \alpha - \beta \in [0, 2[$.

Pour le cas général, en utilisant (3.9) sur (3.10) et un argument de l'itération, Cowan [2013] a montré non existence de solution stable pour (3.10) si $2 \le p \le \theta$ ou bien $\max(1, 2t_0^-) < p \le \theta$ et si

$$N < 2 + \frac{4(\theta+1)}{p\theta-1} t_0^+,$$

avec

$$t_0^\pm := \sqrt{\frac{p\theta(p+1)}{\theta+1}} \pm \sqrt{\frac{p\theta(p+1)}{\theta+1} - \sqrt{\frac{p\theta(p+1)}{\theta+1}}}. \tag{3.14}$$

En particulier, cela implique que si $2 \le p \le \theta$ et si $N \le 10$, alors (3.10) n'admet pas de solution stable.

3.4 Système de Lane Emden avec poids

Dans notre travail, on s'intéresse aux solutions stables de (3.1) avec un poids ρ continue et positive sur $\mathbb{R}^N$ qui vérifie:

$$\text{Il existe } A > 0, \text{ tel que } \rho(x) \geq A\rho_0(x) \text{ dans } \mathbb{R}^N, \text{ où } \rho_0(x) := (1+|x|^2)^{\frac{\alpha}{2}} \text{ avec } \alpha \geq 0. \tag{3.15}$$

Notre principal résultat est

Théorème 1.6 *Supposons que ρ est radiale et satisfait (3.15). Soit x_0 la plus grande racine du polynôme*

$$H(x) = x^4 - \frac{16p\theta(p+1)(\theta+1)}{(p\theta-1)^2}x^2 + \frac{16p\theta(p+1)(\theta+1)(p+\theta+2)}{(p\theta-1)^3}x - \frac{16p\theta(p+1)^2(\theta+1)^2}{(p\theta-1)^4}.$$

i) Si $\frac{4}{3} < p \leq \theta$ alors le système (3.1) n'admet pas de solutions classiques stables si $N < 2 + (2+\alpha)x_0$. En particulier, si $N \leq 10 + 4\alpha$, il n'existe pas de solution classique stable de (3.1) pour tout $\frac{4}{3} < p \leq \theta$.

ii) Si $1 < p \leq \min(\frac{4}{3}, \theta)$, (3.1) n'admet pas de solution classique stable et bornée si

$$N < 2 + \left[\frac{p}{2} + \frac{(2-p)(p\theta-1)}{(\theta+p-2)(\theta+1)}\right](\alpha+2)x_0.$$

En particulier, si $N \leq 6 + 2\alpha$, il n'existe pas de solution classique stable et bornée de (3.1) pour tout $1 < p \leq \min(\frac{4}{3}, \theta)$.

En suivant l'approche de Souplet [2009], on obtient une inégalité qui était connue pour (3.10): Si u et v résout (3.1) avec $\theta \geq p$, alors

$$u^{\theta+1} \leq \frac{\theta+1}{p+1}v^{p+1}. \tag{3.16}$$

Cela permet de voir que si $p = \theta$, on a $u \equiv v$, donc les solutions de (3.1) coïncide avec celle de

$$-\Delta u = \rho(x)u^p, \quad u > 0, \quad \text{dans } \mathbb{R}^N. \tag{3.17}$$

On obtient donc des conséquences suivantes:

Théorème 1.7 *Soient $p > 1$, ρ un poids radial vérifiant (3.15) et*

$$N < 2 + \frac{2(2+\alpha)}{p-1}\left(p + \sqrt{p^2 - p}\right). \tag{3.18}$$

1. *Si $\frac{4}{3} < p$ alors l'équation (3.17) n'admet pas de solution classique stable. En particulier, si $N \leq 10 + 4\alpha$, il n'existe pas de solution classique stable de (3.17).*
2. *Si $1 < p \leq \frac{4}{3}$, l'équation (3.17) n'admet de solution classique stable bornée.*

En se basant sur l'approche de Farina [2007], Fazly [2012] a montré le résultat précédent pour (3.17) si $p \geq 2$ et $\rho = \rho_0$, et Cowan & Fazly [2012] ont aussi montré ce résultat avec $p > 1$ et ρ radial qui satisfait $\lim_{|x|\to\infty} |x|^{-\alpha}\rho(x) = C > 0$. Au cas du système, en suivant les idées développées dans Cowan & Ghoussoub [2014]; Serrin & Zou [1994] et l'argument d'itération de

Cowan [2013], Hu [2015] a établi le résultat du type Liouville suivant: Soient $\rho = \rho_0$ avec $\alpha \geq 0$, et $2t_0^- < p \leq \theta$, alors (3.1) n'a pas de solution stable positive si

$$N < 2 + \frac{2(2+\alpha)(\theta+1)}{p\theta - 1} t_0^+.$$

Ici, t_0^- est celle définie par (3.14). Par conséquent, il n'existe pas de solution stable de (3.1) pour $N \leq 10 + 4\alpha$ et tout $2 < p \leq \theta$. Comme on a $2t_0^- < p$ pour tout $\frac{4}{3} < p \leq \theta$, les résultats cités restent valables en remplaçant 2 par $\frac{4}{3}$. On peut remarquer que $\sup_{1<p\leq\theta} t_0^- = 1$.

Notre travail améliorent donc tous ces résultats connus, sous des conditions moins contraignantes sur ρ, et fournit des résultats plus fort, car on peut montrer que $2t_0^+ \frac{\theta+1}{p\theta-1} < x_0$ quelque soit $1 < p \leq \theta$, surtout c'est la première fois que des résultats du type Liouville sont obtenus pour (3.1) ou (3.17) avec un poids strictement positive et valable en dimension petite, valable pour tout $1 < p \leq \frac{4}{3}$.

Notre preuve associe les techniques dans Cowan [2013]; Harrabi *et al.* [2014]; Serrin & Zou [1994]; Wei & D.Ye [2013]. Un point clé de notre approche pour traiter le cas $1 < p \leq \frac{4}{3}$ est une nouvelle estimation dans le sens inverse de l'inégalité (1.9) du Souplet. En fait, soit $\theta \geq p > 1$, (u, v) est une solution de l'équation (3.1) avec ρ satisfaisant (3.15) et u bornée, alors

$$v \leq \|u\|_\infty^{\frac{\theta-p}{p+1}} u. \tag{3.19}$$

Ce genre d'estimation semble être tout à fait nouvelle, et elle nous a permis de montrer que dans ce cas,

$$\int_{B_R} \rho(x) v^2 dx \leq C R^{N - \frac{2(\theta+1)p}{p\theta-1} - \frac{(p+1)\alpha}{p\theta-1} - \frac{4(2-p)}{\theta+p-2}}, \quad \forall\, R > 0.$$

Cette dernière inégalité est un point crucial qui nous a permis de démarrer l'argument d'itération de Cowan [2013] dans le cas $1 < p \leq \min(\frac{4}{3}, \theta)$ avec v^2.

Partie I

Estimation explicite des solutions via l'indice de Morse

Chapitre 2

Morse Indices of Solutions for Super-linear Elliptic PDE's

Sommaire

We establish L^∞ estimates for solutions of some general superlinear and subcritical elliptic equations via the Morse index. Our results generalize and improve the previous works of Yang [1998] and Bahri & Lions [1992].

1 Introduction

Consider the following elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u &= f(x,u) \quad \text{in } \Omega, \\ \quad u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a smooth bounded domain and f is a $\mathrm{C}^1(\Omega \times \mathbb{R})$ function that we will specify later. Let u be a classical solution, define

$$\Lambda_u(h) := \int_\Omega |\nabla h|^2 - \int_\Omega f'(x,u)h^2, \quad \forall\, h \in C_c^1(\Omega). \tag{1.2}$$

Here $f'(x,t) := \frac{\partial f}{\partial t}(x,t)$. The Morse index of u, $i(u)$ is defined as the maximal dimension of all subspaces X of $C_c^1(\Omega)$ such that $\Lambda_u(h) < 0$ for any $h \in X \setminus \{0\}$. A solution u is said stable if $i(u) = 0$.

By using symmetric Mountain Pass theorem, Ambrosetti & Rabinowitz [1973] showed that when f is super-linear, odd in u and has a subcritical growth, the equation (1.1) has infinitely many solutions u_k such that $\lim_{k\to\infty} i(u_k) = \lim_{k\to\infty} \|u_k\|_{L^\infty} = \infty$ (see also Chang [1993] and Schecher & Zou [2006] for more general results). In a celebrated paper, Bahri & Lions [1988], studied the perturbed equation i.e. $f(x,u) := |u|^{p-2}u + g(x,u)$ where $2 < p < \frac{2N-2}{N-2}$ and $g(x,u)$ is not assumed to be odd in u. They proved the existence of infinitely many solutions under appropriate growth restriction on g. This result was improved by Ramos *et al.* [2009b], using the Morse index of solutions for unperturbed problem, they obtained a sequence of sign-changing solutions to a more general problem. Figueiredo & Yang [2003] also considered problem (1.1) where the associated Euler-Lagrange functional does not satisfy the Palais-Smale condition. Variational and blow-up techniques were applied together with Morse's index to derive existence results (see also works of Ramos & Rodrigues [2001] and Yang [2004]).

The idea of using the Morse index of solutions to obtain further qualitative properties of solutions to a semilinear elliptic equations was first explored in the subcritical case by Bahri & Lions [1992]. They proved

Theorem 2.1 *Assume that f satisfies:*

$$f'(x,s)|s|^{-p+1} \to c(x) > 0 \text{ uniformly on } \Omega, \quad \text{as } s \to \pm\infty,$$

where $c \in \mathrm{C}(\overline{\Omega})$ and $1 < p < \frac{N+2}{N-2}$, then for any sequence of solutions u_n to (1.1),

$$i(u_n) \to \infty \quad \text{if and only if} \quad \|u_n\|_{L^\infty(\Omega)} \to \infty.$$

Suppose that $\|u_n\|_{L^\infty} \to \infty$ while $i(u_n)$ remains bounded, by blow-up technique, Bahri & Lions [1992] obtained a nontrivial bounded solution having finite Morse indices in the hole space or in the half space with Dirichlet conditions for the Lane-Emden equation:

$$-\Delta u = |u|^{p-1}u. \tag{1.3}$$

On the other hand, a spectral argument combined with the Pohozaev identity showed that $u \equiv 0$, hence the contradiction.

Harrabi *et al.* [1998b] extended Theorem 2.1 when f does not have the same asymptotic behavior at $+\infty$ and $-\infty$, namely f satisfies the following assumption

$$(H) \qquad f'(x,t) \sim p^+ t^{p^+-1} \text{ at } +\infty, \qquad f'(x,t) \sim p^-|t|^{p^--1} \text{ at } -\infty, \qquad \text{uniformly in } x,$$

with $p^- \neq p^+$ satisfying $p^-, p^+ \in \left(1, \frac{N+2}{N-2}\right)$, if $N \geq 4$ and $p^-, p^+ \in (2,5)$ if $N = 3$. The proof in Harrabi *et al.* [1998b] is harder than Bahri & Lions [1992], since the "blow-up" argument leads to deal with $-\Delta u = u_+^p$. They had to use the classification result in Harrabi *et al.* [1998a] together with Harnack's inequality and barrier functions estimates.

In the supercritical case, Theorem 2.1 was first extended by Dancer [2008] with the restriction $\frac{N+2}{N-2} < p < \frac{N}{N-4}$ if $N \geq 4$. In an elegant paper, Farina [2007] obtained a sharp classification for all finite Morse indices solutions of (1.3) (see also Dupaigne & Farina [2010]). This classification is useful to prove Theorem 2.1 when $1 < p < p_c(N)$, where $p_c(N)$ is the so-called Joseph-Lundgren exponent, which is much bigger than $\frac{N+2}{N-2}$. After that, using Harnack's inequality and combining with similar L^p-estimates derived by Farina [2007], Rebhi [2011] extended the result of Harrabi *et al.* [1998b] up to the optimal exponent $p_c(N)$. Davila *et al.* [2011] considered equation (1.3) in a general open set $\Omega \subset \mathbb{R}^N$, without any boundary conditions. By a boot-strap technique, the authors proved some regularity results for weak solutions in $H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ with finite Morse index, and they provided a universal estimate for classical solutions of $-\Delta u = f(u)$ in Ω, where f has an asymptotical behaviour like $|s|^{p-1}s$ at infinity.

Motivated by Bahri & Lions [1992], based on local interior estimates and careful boundary estimates, Yang obtained in Yang [1998] some explicit estimates of L^p or L^∞ norm for solutions to (1.1) via their Morse index. In particular, under weaker conditions than Bahri & Lions [1992], Yang controlled the L^p or L^∞ norm of solution by polynomial growth in Morse index. More precisely, consider the following conditions:

(H_1) (Super-linearity) There exists $\mu > 0$ such that

$$f'(x,s)s^2 \geq (1+\mu)f(x,s)s > 0, \quad \text{if } |s| > s_0, \quad x \in \Omega.$$

(H_2) (Subcritical growth) There exists $0 < \theta < 1$ such that

$$\frac{2N}{N-2}F(x,s) \geq (1+\theta)f(x,s)s, \quad \text{if } |s| > s_0, \quad x \in \Omega,$$

where $F(x,s) = \displaystyle\int_0^t f(x,t)dt$.

(H_3) There is a constant $C \geq 0$ such that

$$|\nabla_x F(x,s)| \leq C(F(x,s)+1), \quad x \in \Omega.$$

Yang proved then

Theorem 2.2 *(Theorems 2.1-2.2 in Yang [1998]). If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of (1.1) and f satisfies (H_1), (H_2) and (H_3), then there exists a positive constant $C = C(\Omega, f)$ such that*

- $\displaystyle\int_\Omega |f(x,u)|^{p_0} \leq C(i(u)+1)^\alpha$ *where* $p_0 = 1 + \frac{(1+\theta)(N-2)}{(1-\theta)N+2(1+\theta)}$ *and* $\alpha = \left(\frac{3}{2} + \frac{3}{2+\mu}\right)\frac{(2+\mu)^2}{3\mu+\mu^2}$.
- *There exists a constant β satisfying* $0 < \beta \leq \frac{2\alpha}{p_0 N(2-p_0)}\left[\frac{2}{N(2-p_0)} - \frac{1}{p_0}\right]^{-1}$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^\beta.$$

The results of Harrabi *et al.* [1998b] were generalised by Harrabi *et al.* [2011a] to similar equation with the Neumann boundary conditions. However, in that case, the techniques used in Yang [1998] cannot be adapted, due to the boundary estimates.

In this paper, we focus on the case for $f(x,s) = f(s)$ and consider

$$-\Delta u = f(u) \;\text{ in}\,\Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{1.1'}$$

Under hypotheses weaker than (H_1), (H_2) and (H_3), we obtain some L^∞ norm estimates of solution to equation (1.1′) via its Morse index. Moreover, we show that when the nonlinearity is close to the critical growth, the L^∞ norm evolves less rapidly than exponential growth of the Morse index of solutions. To state our results, we present first some assumptions on the nonlinearity f:

(f_0) There exist $s_0 > 0$, $C > 0$ and $\gamma > 0$ such that $|f(s)|^{\frac{2N}{N+2}} \geq C|s|^{2+\gamma}$, $\forall\, |s| > s_0$.

(f_1) There exist $s_0 > 0$ and $C > 0$ such that $C|f(s)|^{\frac{2N}{N+2}} \leq s^2 f'(s) - f(s)s$ for $|s| > s_0$.

(f_2) There exist $s_0 > 0$ and $C > 0$ such that $C|f(s)|^{\frac{2N}{N+2}} \leq \frac{2N}{N-2}F(s) - f(s)s$, for $|s| > s_0$.

Remark 2.1 *1. If (H_1) and (H_2) are satisfied, then there exits $C > 0$ such that*

$$C|f(s)|^{\frac{2N}{N+2}} \leq \mu s f(s) \leq s^2 f'(s) - sf(s), \quad \textit{if}\; |s| > s_1$$

and

$$C|f(s)|^{\frac{2N}{N+2}} \leq \theta f(s)s \leq \frac{2N}{N-2}F(s) - sf(s), \quad \textit{if}\; |s| > s_1,$$

hence (H_1) and (H_2) imply (f_1) and (f_2).

2. *Assumption* (f_2) *was first introduced in* Harrabi et al. *[2011b] to obtain a priori* L^∞-*estimates for radial solutions with prescribed number of nodes. In order to understand the improvement brought by* (f_2), *let us consider the following example mentioned in* Harrabi et al. *[2011b]:*

$$f(s) = \frac{|s|^{\frac{4}{N-2}}s}{(\ln(|s|+2))^q}, \quad q \geq \frac{N-2}{N+2}.$$

Clearly f *satisfies* (f_0) *and* (f_1) *conditions. Moreover, for* $|s| \to \infty$, *one has*

$$\frac{2N}{N-2}F(s) - sf(s) = q\int_0^{|s|} \frac{t^{\frac{N+2}{N-2}}}{(\ln(2+t))^{q+1}}dt \sim q\frac{N-2}{2N}\frac{|s|^{\frac{2N}{N-2}}}{(\ln(2+|s|))^{q+1}} = o\left(sf(s)\right).$$

Choosing $q \geq \frac{N-2}{N+2}$, *then* (f_2) *follows. Hence,* f *verifies* (f_2) *but never* (H_2).

Our results read as follows

Theorem 2.3 *Let* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *be a solution of equation* $(1.1')$. *If* f *satisfies* (f_0), (f_1) *and* (f_2), *then there exists* $C = C(\Omega,\, f)$ *such that*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C(1+i(u))^\delta,$$

where

$$\delta = \frac{N+2}{N}\max\left\{\left(\frac{2(\gamma+2)(N+2)}{\gamma N}\right)+2, \frac{N}{N-2}\left(\frac{2(\gamma+2)(N+2)}{\gamma N}\right)+1\right\}.$$

If, in addition, we assume that f satisfies the following condition:

(f_3) There exist $q > 0$, $s_0 > 1$ and a positive constant C such that

$$|f(s)| \leq C\frac{|s|^{\frac{N+2}{N-2}}}{(\ln|s|)^q} \quad \text{for } |s| \geq s_0,$$

then we have

Theorem 2.4 *Let* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *be a solution of* $(1.1')$. *If* f *satisfies* (f_0), (f_1), (f_2) *and* (f_3), *then there exists a positive constant* $C = C(\Omega, f)$ *such that*

$$\|u\|_{L^\infty} \leq C\exp\left[C_N(1+i(u))^{\frac{2\delta}{q(N-2)}}\right],$$

where δ *is as in Theorem 2.3.*

Remark 2.2 *If we suppose that* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *is a stable solution of* (1.1) *satisfying* (f_0) *and* (f_1), *then using* u *as a test function, there holds* $\|f(x,u)\|_{L^{\frac{2N}{N+2}}(\Omega)} \leq C$, *hence by elliptic regularity, one has* $\|\nabla u\|_{L^2(\Omega)} \leq C$. *If moreover* f *satisfies* $\lim_{|s|\to\infty}|s|^{-\frac{N+2}{N-2}}f(s) = 0$.

We can use a classical technique due to Brezis & Kato [1978] to derive that $\|u\|_{L^\infty} \leq C$.

As in works of Souto [1995] and Farina [2007], we will use test functions in the form φ^m (where m is a positive parameter) in order to prove L^p_{loc}-estimates which are **only** related to the Morse indices of the solutions (see Lemma 3.1). Moreover, we shall employ a cut-off function with compact support to derive a variant of the Pohozaev identity. This device allows us to avoid the spherical integrals raised in Lemma 2.1 Yang [1998] and which are very difficult to control.

Thus, we shall provide more easier and clearer proofs than those developed in Yang [1998] (see proof of Theorem 2.5, for instance). However, under (f_0), (f_1) and (f_2) the L^2-gradient local estimate is very difficult to derive. In fact, we need to establish an interior estimate in order to exhibit the explicit dependence on $i(u)$ (see Lemma 2.3 and Remark 3.2 below). Contrarily to Yang [1998] in which (H_2) implies the subcritical polynomial growth condition allowing to employ the bootstrap argument; here the nonlinearity f may be very close to the critical growth. We have to adapt the technique of Brezis & Kato [1978] to obtain an explicit estimate of L^∞ norm for solution to $(1.1')$ via its Morse index. To conclude, the main apport of our techniques consists in removing the spherical integrals raised in Yang [1998]. Therefore, giving attention to Damascelli *et al.* [2009] and Harrabi *et al.* [2014], we think that some similar L^∞ estimates via Morse index for solutions of the forth order subcritical elliptic equation and also for p-Laplace equations can be proved (work in preparation).
In this work, we also revise Theorem 2.2 in Yang [1998], where we obtain an estimate of L^∞ norm for solution to (1.1) via its Morse index, provided that (H_1), (H_2) and (H_3) are satisfied.

Theorem 2.5 *If $u \in C^2(\Omega)\cap C(\overline{\Omega})$ is a solution of (1.1) and f satisfies (H_1), (H_2) and (H_3), then there exists a positive constant $C = C(\Omega, f)$ such that*

1. $\displaystyle\int_\Omega |\nabla u|^2 \le C(i(u)+1)^{\alpha'}$, *where* $\alpha' = \frac{4}{\mu}+3$.
2. $\displaystyle\|u\|_{L^\infty} \le C(i(u)+1)^{\beta'}$, *where* $\displaystyle\beta' = \frac{3\mu+4}{3\mu\theta}\left(\frac{3N^2(1-\theta)+N(7\theta-4)-2\theta+12}{N(N-2)^2}\right)$.

Our proof of Theorem 2.5 is more transparent, and it allows us to improve the estimate of L^∞ norm given in Theorem 2.2 for large N. In fact, the exponent β' in Theorem 2.5 is smaller than the admissible constant for β in Theorem 2.2 when N is large.

This paper is organised as follows : In section 2, we prove a Pohozaev identity and we explain how to cover the domain Ω to derive a global estimate. We prove also some preliminary technical lemmas in section 2. The proofs of Theorems 2.3-2.4 and 2.5 are given respectively in sections 3 and 4.

2 Preliminary technical Lemmas

In the following, C denotes always a generic positive constant independent of u, even their value could be changed from one line to another one.

Let $y \in \mathbb{R}^N$ and $R > 0$. In all this work, $B_R(y)$ denote the open ball of center y and radius R. We denote by $\partial\Omega_R(y) := \partial\Omega \cap B_R(y)$, v the unit normal vector on $\partial\Omega$ and for $x \in B_R(y)\cap\Omega$, $n := x-y$. Then, we have the following version of the pohozaev identity

Lemme 2.1 *Let u be solution to (1.1). Let $\psi \in C_c^2(B_R(y))$. Then*

$$\begin{aligned}\int_{B_R(y)\cap\Omega}\frac{2N}{N-2}F(x,u)\psi \;&+\; \int_{B_R(y)\cap\Omega}\frac{2}{N-2}\nabla_x F(x,u)\cdot n\psi - \int_{B_R(y)\cap\Omega}|\nabla u|^2\psi\\ &= \int_{B_R(y)\cap\Omega}\frac{1}{N-2}|\nabla u|^2\nabla\psi\cdot n - \int_{B_R(y)\cap\Omega}\frac{2}{N-2}F(x,u)\nabla\psi\cdot n\\ &- \int_{B_R(y)\cap\Omega}\frac{2}{N-2}(n\cdot\nabla u)(\nabla u\cdot\nabla\psi)\\ &+ \int_{\partial\Omega_R(y)}\frac{2}{N-2}\left((n\cdot\nabla u)(v\cdot\nabla u)-\frac{1}{2}v\cdot n|\nabla u|^2\right)\psi.\end{aligned}$$

Proof. Multiplying equation (1.1) by $\nabla u\cdot n\psi$ and integrating by parts, we get

$$\int_{B_R(y)\cap\Omega}\nabla u\cdot\nabla(\nabla u\cdot n\psi)-\int_{\partial(B_R(y)\cap\Omega)}(\nabla u\cdot v)(\nabla u\cdot n)\psi=\int_{B_R(y)\cap\Omega}f(x,u)\nabla u\cdot n\psi. \tag{2.1}$$

For the left hand side of (2.1), we have

$$\begin{aligned}I_1&:=\int_{B_R(y)\cap\Omega}\nabla u\cdot\nabla(\nabla u\cdot n\psi)-\int_{\partial(B_R(y)\cap\Omega)}(\nabla u\cdot v)(\nabla u\cdot n)\psi\\&=\frac{1}{2}\int_{B_R(y)\cap\Omega}\nabla(|\nabla u|^2)\cdot n\psi+\int_{B_R(y)\cap\Omega}|\nabla u|^2\psi+\int_{B_R(y)\cap\Omega}(n\cdot\nabla u)(\nabla u\cdot\nabla\psi)\\&\quad-\int_{\partial(B_R(y)\cap\Omega)}(\nabla u\cdot v)(\nabla u\cdot n)\psi,\end{aligned}$$

hence

$$\begin{aligned}I_1=&\frac{2-N}{2}\int_{B_R(y)\cap\Omega}|\nabla u|^2\psi-\frac{1}{2}\int_{B_R(y)\cap\Omega}|\nabla u|^2 n\cdot\nabla\psi+\int_{B_R(y)\cap\Omega}(n\cdot\nabla u)(\nabla u\cdot\nabla\psi)\\&+\frac{1}{2}\int_{\partial\Omega_R(y)}v\cdot n|\nabla u|^2\psi-\int_{\partial\Omega_R(y)}(\nabla u\cdot v)(\nabla u\cdot n)\psi.\end{aligned}\tag{2.2}$$

For the right hand side of (2.1), we integrate by parts to get

$$\begin{aligned}I_2&:=\int_{B_R(y)\cap\Omega}f(x,u)\nabla u\cdot n\psi\\&=\int_{B_R(y)\cap\Omega}\psi n\cdot(F(x,u))'_x-\int_{B_R(y)\cap\Omega}\psi n\cdot\nabla_x F(x,u)\\&=-N\int_{B_R(y)\cap\Omega}F(x,u)\psi-\int_{B_R(y)\cap\Omega}F(x,u)n\cdot\nabla\psi-\int_{B_R(y)\cap\Omega}n\cdot\nabla_x F(x,u)\psi.\end{aligned}\tag{2.3}$$

Note that in the last equality, the Dirichlet boundary condition implies that $F(x,u)=0$ on $\partial\Omega_R(y)$. Therefore, the claim follows from (2.1), (2.1) and (2.3). As mentioned in the Introduction, to establish a global estimates, we need to cover the domain Ω by balls on which a local estimates, obtained after control of the boundary term contained in the above lemma, hold. To be more precise, consider the following sets

$$\Omega_1(R):=\left\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)>\frac{R}{2}\right\}\ \text{ and }\ \Omega_2(R):=\left\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)\le\frac{R}{3}\right\},\quad\forall\ R>0.$$

In our proofs of Theorems 2.3 and 2.5, $\Omega_2(R)$ will be covered by balls carefully chosen, as in the Yang's work Yang [1998], such that their centers lie in

$$\Gamma(R):=\left\{x\in\mathbb{R}^N\setminus\Omega:\ \operatorname{dist}(x,\partial\Omega)=\frac{R}{20}\right\},\tag{2.4}$$

while $\Omega\backslash\Omega_2(R)$ will be covered by balls such that their centers lie in $\Omega_1(R)$. The following lemma is devoted to the control of the boundary term for $y\in\Gamma(R)$, it is an adaptation of Lemma 2.2. in Yang [1998].

Lemme 2.2 *There exists $R_1(\Omega)<1$ such that for any $0<R<R_1(\Omega)$ and $y\in\Gamma(R)$, there holds*

$$\int_{\partial\Omega_R(y)}\left((n\cdot\nabla u)(v\cdot\nabla u)-\frac{1}{2}v\cdot n|\nabla u|^2\right)\psi\le 0,$$

for any positive function $\psi\in C_c^2(B_R(y))$.

Proof. We proceed similarly as in the proof of Lemma 2.2. in Yang [1998], to show that there exists $R_1 = R_1(\Omega) > 0$ such that, if $0 < R \le R_1$ and $y \in \Gamma(R)$ then $v \cdot n \le 0$, for $x \in \partial\Omega_R(y)$. If $\nabla u \neq 0$ for $x \in \partial\Omega_R(y)$, we have $v = \epsilon \frac{\nabla u}{|\nabla u|}$ with $\epsilon = \pm 1$. Therefore, there holds

$$\int_{\partial\Omega_R(y)} \left((n\cdot\nabla u)(v\cdot\nabla u) - \frac{1}{2} v\cdot n |\nabla u|^2 \right)\psi = \frac{1}{2}\int_{\partial\Omega_R(y)} v\cdot n|\nabla u|^2 \psi d\sigma \le 0,$$

for any positive function $\psi \in C_c^2(B_R(y))$. As a consequence, we obtain

Corollary 2.1 *Let $0 < R \le R_1(\Omega)$ and $y \in \Gamma(R)$. Let $\psi \in C_c^2(B_R(y))$ be positive. Suppose that u is a classical solution of $(1.1')$ and f satisfies (f_2), then*

$$\begin{aligned}\int_{B_R(y)\cap\Omega} |f(u)|^{\frac{2N}{N+2}}\psi \quad \le \quad & \frac{1}{2}\int_{B_R(y)\cap\Omega} u^2\Delta\psi + \frac{1}{N-2}\int_{B_R(y)\cap\Omega}|\nabla u|^2\nabla\psi\cdot n \\ & -\frac{2}{N-2}\int_{B_R(y)\cap\Omega} F(u)\nabla\psi\cdot n \\ & -\frac{2}{N-2}\int_{B_R(y)\cap\Omega}(n\cdot\nabla u)(\nabla u\cdot\nabla\psi) + C.\end{aligned}$$

Moreover, if $y \in \Omega_1(R)$ then the above inequality holds when we replace R by $\frac{R}{2}$.

Proof. Using Lemmas 2.1 and 2.2 there holds

$$\begin{aligned}\frac{2N}{N-2}\int_{B_R(y)\cap\Omega} F(u)\psi - \int_{B_R(y)\cap\Omega}|\nabla u|^2\psi \le & \frac{1}{N-2}\int_{B_R(y)\cap\Omega}|\nabla u|^2\nabla\psi\cdot n \\ & -\frac{2}{N-2}\int_{B_R(y)\cap\Omega} F(u)\nabla\psi\cdot n \\ & -\int_{B_R(y)\cap\Omega}\frac{2}{N-2}(n\cdot\nabla u)(\nabla u\cdot\nabla\psi).\end{aligned} \tag{2.5}$$

Note that if $y \in \Omega_1(R)$ then (2.5) holds when we replace R by $\frac{R}{2}$. Multiplying $(1.1')$ by $u\psi$ and integrating by parts, we get

$$\int_{B_R(y)\cap\Omega}|\nabla u|^2\psi - \int_{B_R(y)\cap\Omega} f(u)u\psi = \frac{1}{2}\int_{B_R(y)\cap\Omega} u^2\Delta\psi.$$

Therefore, the claim follows by combining the two last inequalities and using (f_2).

To prove Theorem 2.3, we need also to establish an interior estimate. More precisely. Let $R > 0$, $y \in \Omega_1(R)\cup\Gamma(R)$, $0 < a < b$ and

$$A := A_a^b = \{x \in \mathbb{R}^N;\ a < |x-y| < b\},$$

we denote by $A_\rho := A_{a+\rho}^{b-\rho}$, where $0 < \rho \le \min\{1, \frac{b-a}{20}\}$. Let u be a classical solution of the equation:

$$-\Delta u = g \quad \text{in } A\cap\Omega. \tag{2.6}$$

In the following Lemma, we establish an interior estimate for $\|\nabla u\|_{L^2(A_\rho\cap\Omega)}$, where we exhibit the dependence of the constant of this estimate with respect to ρ.

Lemme 2.3 *Let $g \in L^{\frac{2N}{N+2}}(A\cap\Omega)$ and suppose that u is a classical solution of (2.1). Then there exists a constant $C > 0$ not depend only on N such that*

$$\|\nabla u\|^2_{L^2(A_\rho\cap\Omega)} + \|\ u\|^2_{L^{\frac{2N}{N-2}}(A_\rho\cap\Omega)} \le C\left(\|g\|^2_{L^{\frac{2N}{N+2}}(A\cap\Omega)} + \frac{1}{\rho^2}\|u\|^2_{L^2(A\cap\Omega)}\right).$$

Proof. For use later, we recall that, using Gagliardo-Nirenberg-Sobolev inequality, if $w \in H_0^1(A\cap\Omega)$ then there exists a positive constant C not depend only on N such that

$$\|w\|_{L^{\frac{2N}{N-2}}(\Omega\cap A)} \leq C\|\nabla w\|_{L^2(\Omega\cap A)}. \tag{2.7}$$

Let $\eta_\rho \in C_0^2(\Omega\cap A)$ be a cut-off function satisfying $0 \leq \eta_\rho \leq 1$ in $\Omega\cap A$. Let u be a classical solution of (2.1). Multiplying (2.1) by $u\eta_\rho^2$, integrating by parts and applying Cauchy-Schwartz inequality there holds

$$\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 \leq 2\int_{\Omega\cap A}(|u||\nabla\eta_\rho|)(|\nabla u|\eta_\rho) + \int_{\Omega\cap A} gu\eta_\rho^2,$$

by Young and Hölder inequalities, we derive that for any $\epsilon > 0$

$$\begin{aligned}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 &\leq \frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 + C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 \\ &\quad + \left(\int_{\Omega\cap A}(g\eta_\rho)^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}\left(\int_{\Omega\cap A}(u\eta_\rho)^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}} \\ &\leq \frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 + C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + \frac{\epsilon}{3}\left(\int_{\Omega\cap A}(u\eta_\rho)^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \\ &\quad + C_\epsilon\left(\int_{\Omega\cap A}g^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}.\end{aligned}$$

Applying (2.7) with $w := u\eta_\rho \in H_0^1(\Omega\cap A)$, then there exists a positive constant C not depend only on N such that

$$\begin{aligned}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 &\leq \frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 + C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + \frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla(u\eta_\rho)|^2 \\ &\quad + C_\epsilon\left(\int_{\Omega\cap A}g^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} \\ &\leq \epsilon\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 + C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + C_\epsilon\left(\int_{\Omega\cap A}g^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}.\end{aligned}$$

Choose ϵ small enough, we get

$$\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 \leq C\left(\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + \left(\int_{\Omega\cap A}g^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}\right). \tag{2.8}$$

Consequently,

$$\begin{aligned}\int_{\Omega\cap A}|\nabla(u\eta_\rho)|^2 &\leq 2\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^2 + 2\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 \\ &\leq C\left[\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + \left(\int_{\Omega\cap A}g^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}\right].\end{aligned}$$

Take now $\eta_\rho := h(\frac{|x-y|}{\rho})$, where $h \in C_0^2(\mathbb{R})$ satisfying $0 \leq h \leq 1$, $h = 1$ in $[\frac{a}{\rho}+1, \frac{b}{\rho}-1]$ and $h = 0$ in $\mathbb{R}\backslash[\frac{a}{\rho}+\frac{1}{2}, \frac{b}{\rho}-\frac{1}{2}]$. We get easily our result with (2.8) and (2.7) for $w := u\eta_\rho$.

Remark 2.3 *If $g = f(u)$ and f satisfies (f_0), then there holds*

$$\|u\|^2_{L^2(A\cap\Omega)} \leq C\|u\|^2_{L^{2+\gamma}(A\cap\Omega)} \leq C\left(\|f(u)\|^{\frac{4N}{(N+2)(\gamma+2)}}_{L^{\frac{2N}{N+2}}(A\cap\Omega)} + 1\right).$$

3 Proof of Theorems 2.3 and 2.4

In this section, we will prove Theorems 2.3 and 2.4. For this, we need the following notations which are used throughout the rest of this paper.

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of equation (1.1′), $0 < R_0 < R_1(\Omega)$ and $y \in \Gamma(R_0)$, where $\Gamma(R_0)$ and $R_1(\Omega)$ are as in Lemma 2.2. For $j = 1, \cdots, 1 + i(u)$, using the notation A_a^b in Section 2, we denote

$$A_j =: A_{a_j}^{b_j} \quad \text{with} \ \ a_j = R_0 \frac{2(j + i(u))}{4(i(u)+1)}, \ \ b_j = R_0 \frac{2(j+i(u))+1}{4(i(u)+1)}. \tag{3.1}$$

Let φ_j be a family of C^2 functions satisfying the following conditions $(*)$:

- $\varphi_j = 1$ for $2(j + i(u)) < |x| < 2(j + i(u)) + 1$;
- $\mathrm{supp}(\varphi_j) \subset \{2(j + i(u)) - \frac{1}{2} \leq |x| \leq 2(j + i(u)) + \frac{3}{2}\}$;
- $\|\nabla \varphi_j\|_\infty + \|\Delta \varphi_j\|_\infty \leq C$.

Define the cut-off function ϕ_j supported in $B_{R_0}(y)$ by

$$\phi_j(x) := \varphi_j \left(\frac{4(i(u)+1)(x-y)}{R_0} \right),$$

then

$$\phi_j(x) = 1 \quad \text{in} \quad A_j, \quad \|\nabla \phi_j\|_\infty \leq \frac{C}{R_0}(1 + i(u)) \quad \text{and} \quad \|\Delta \phi_j\|_\infty \leq \frac{C}{R_0^2}(1+i(u))^2.$$

First, we prove the following lemma

Lemme 3.1 *If f satisfies (f_0) and (f_1), then there exist a positive constant $C = C(\Omega, f)$ and $j_0 \in \{1, 2, \cdots, 1 + i(u)\}$ such that*

$$\int_{\Omega \cap A_{j_0}} |f(u)|^{\frac{2N}{N+2}} \leq C(1 + i(u))^{2(\frac{\gamma+2}{\gamma})}.$$

Proof. Let $m \geq 1$. It is easy to see that $\{u\phi_j^m\}$ are mutually orthogonal in $L^2(\Omega)$ and for the quadratic form Λ_u defined by (3.7). By the definition of $i(u)$, there exists $j_0 \in \{1, 2, ..., 1 + i(u)\}$ such that

$$\int_\Omega |\nabla(u\phi_{j_0}^m)|^2 - \int_\Omega f'(u) u^2 \phi_{j_0}^{2m} \geq 0,$$

hence

$$\int_\Omega f'(u) u^2 \phi_{j_0}^{2m} - \int_\Omega |\nabla u \phi_{j_0}^m + u \nabla(\phi_{j_0}^m)|^2 \leq 0.$$

Then

$$\int_\Omega f'(u) u^2 \phi_{j_0}^{2m} - \int_\Omega |\nabla u|^2 \phi_{j_0}^{2m} \leq C \int_\Omega u^2 \left(|\Delta \phi_{j_0}| + |\nabla \phi_{j_0}|^2 \right) \phi_{j_0}^{2m-2}. \tag{3.2}$$

Multiplying (1.1′) by $u\phi_{j_0}^{2m}$ and integrating by parts, we obtain

$$\int_\Omega |\nabla u|^2 \phi_{j_0}^{2m} - \int_\Omega f(u) u \phi_{j_0}^{2m} \leq \frac{C}{R_0^2}(1 + i(u))^2 \int_\Omega u^2 \phi_{j_0}^{2m-2}. \tag{3.3}$$

From (3.5) and (3.6), we get

$$\int_\Omega f'(u)u^2\phi_{j_0}^{2m} - \int_\Omega f(u)u\phi_{j_0}^{2m} \le \frac{C}{R_0^2}(1+i(u))^2\int_\Omega u^2\phi_{j_0}^{2m-2}.$$

Using (f_1), there holds

$$\int_\Omega |f(u)|^{\frac{2N}{N+2}}\phi_{j_0}^{2m} \le \frac{C}{R_0^2}(1+i(u))^2\int_\Omega u^2\phi_{j_0}^{2m-2}.$$

By Young's inequality, we obtain for $\epsilon > 0$

$$\int_\Omega |f(u)|^{\frac{2N}{N+2}}\phi_{j_0}^{2m} \le C_{\epsilon,R_0}(1+i(u))^{2(\frac{\gamma+2}{\gamma})} + C\epsilon\int_\Omega u^{\gamma+2}\phi_{j_0}^{(m-1)(\gamma+2)}.$$

Choose $m = \frac{\gamma+2}{\gamma}$ so that $(m-1)(\gamma+2) = 2m$, we get

$$\int_\Omega |f(u)|^{\frac{2N}{N+2}}\phi_{j_0}^{2m} \le C_{\epsilon,R_0}(1+i(u))^{2(\frac{\gamma+2}{\gamma})} + C\epsilon\int_\Omega u^{\gamma+2}\phi_{j_0}^{2m}.$$

From (f_0), we deduce that

$$(1-C\epsilon)\int_\Omega |f(u)|^{\frac{2N}{N+2}}\phi_{j_0}^{2m} \le C_{\epsilon,R_0}(1+i(u))^{2(\frac{\gamma+2}{\gamma})}.$$

Choose $\epsilon > 0$ sufficiently small, we obtain the claim.

Now we are in a position to prove Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Let $\rho := \frac{R_0}{40(i(u)+1)}$ and denote by $A_{j_0,\rho} := A_{a_{j_0}+\rho}^{b_{j_0}-\rho}$. According to Lemma 2.3 and Remark 3.2 for $A = A_{j_0,\rho}$, and applying Lemma 3.1, then there exists a positive constant C independent of ρ such that

$$\|\nabla(u)\|^2_{L^2(A_{j_0,\rho})} + \|u\|^2_{L^{\frac{2N}{N-2}}(A_{j_0,\rho})} \le C(1+i(u))^{2\frac{(\gamma+2)(N+2)}{\gamma N}}. \tag{3.4}$$

Now choose a function $\psi_{j_0} \in C_c^2(\mathbb{R}^N)$ satisfying

- $\psi_{j_0} \equiv 1$ if $|x-y| \le a_{j_0}+\rho$, $\psi_{j_0}(x) \equiv 0$ if $|x-y| \ge b_{j_0}-\rho$,
- $\|\nabla\psi_{j_0}\|_{L^\infty} \le \frac{C}{\rho}$, $\|D^2\psi_{j_0}\|_{L^\infty} \le \frac{C}{\rho^2}$.

Applying Corollary 2.1 with $\psi = \psi_{j_0}$ and $R = R_0$, we find

$$\int_{B_{\frac{R_0}{2}}(y)\cap\Omega} |f(u)|^{\frac{2N}{N+2}} \le \frac{C}{\rho}\int_{A_{j_0,\rho}\cap\Omega}|\nabla u|^2 + \frac{C}{\rho^2}\int_{A_{j_0,\rho}\cap\Omega}u^2 + \frac{C}{\rho}\int_{A_{j_0,\rho}\cap\Omega}|F(u)|.$$

Observe that (f_2) implies $|F(s)| \le C_0(1+|s|^{\frac{2N}{N-2}})$, then by Hölder inequality, we deduce that

$$\int_{B_{\frac{R_0}{2}}(y)\cap\Omega} |f(u)|^{\frac{2N}{N+2}} \le \frac{C}{\rho}\int_{A_{j_0,\rho}\cap\Omega}|\nabla u|^2 + \frac{C}{\rho^2}\left(\int_{A_{j_0,\rho}\cap\Omega}u^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} + \frac{C}{\rho}\int_{A_{j_0,\rho}\cap\Omega}u^{\frac{2N}{N-2}} + C$$

Using (2.19), there holds

$$\left(\int_{B_{\frac{R_0}{2}}(y)\cap\Omega} |f(u)|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} \le C(1+i(u))^\delta,$$

where $\delta = \frac{N+2}{N} \max\left\{\left(\frac{2(\gamma+2)(N+2)}{\gamma N}\right) + 2, \frac{N}{N-2}\left(\frac{2(\gamma+2)(N+2)}{\gamma N}\right) + 1\right\}$. Note that if $y \in \Omega_1(R_0)$ we can proceed as previously replacing R_0 by $\frac{R_0}{2}$ to show that

$$\left(\int_{B_{\frac{R_0}{4}}(y)\cap\Omega} |f(u)|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} \leq C(1+i(u))^{\delta}.$$

By covering argument (see Section 2 for more details), we get finally

$$\left(\int_{\Omega} |f(u)|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}} \leq C(1+i(u))^{\delta}.$$

On the other hand, multiplying equation (1.1′) by u, integrating by part and using Hölder together with Gagliardo-Nirenberg-Sobolev inequalities, we obtain

$$\int_{\Omega} |\nabla u|^2 \leq C\left(\int_{\Omega} |f(u)|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}},$$

which implies that

$$\|\nabla u\|_{L^2}^2 \leq C(1+i(u))^{\delta}.$$

The proof of Theorem 2.3is well completed.

As noted in Remark 2.2, we can apply the method of Brezis & Kato [1978] to obtain the following result.

Corollary 3.1 *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of equation (1.1′) satisfying $i(u) \leq m$, for some $m \in \mathbb{N}$. If f satisfies (f_0), (f_1), (f_2) and*

$$\lim_{\pm\infty} \frac{f(s)}{|s|^{\frac{N+2}{N-2}}} = 0,$$

then there exists $C = C(\Omega, N, m)$ such that

$$\|u\|_{L^{\infty}(\Omega)} \leq C.$$

Proof of Theorem 2.4. By Theorem 2.3 and Sobolev's inequality, we obtain

$$\left(\int_{\Omega} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}} \leq \left(\int_{\Omega} |\nabla u|^2\right)^{\frac{1}{2}} \leq C(1+i(u))^{\frac{\delta}{2}}, \tag{3.5}$$

where δ is defined in Theorem 2.3.
Multiplying (1.1′) by $|u|^{p-1}u$, where $p > 1$ and integrating over Ω, we find

$$\frac{4p}{(1+p)^2}\int_{\Omega} |\nabla(|u|^{\frac{p-1}{2}}u)|^2 = \int_{\Omega} f(u)|u|^{p-1}u.$$

Since $u \in C^1(\overline{\Omega})$ then $v := |u|^{\frac{p-1}{2}}u \in H_0^1(\Omega)$. By Sobolev embedding theorem we get

$$\left(\int_{\Omega} |u|^{\frac{N(p+1)}{N-2}}\right)^{\frac{N-2}{N}} \leq C\int_{\Omega} |f(u)||u|^p. \tag{3.6}$$

For $\epsilon > 0$, we set

$$a_\epsilon := \epsilon\,(1+i(u))^{\frac{-2\delta}{N-2}} \quad \text{and} \quad s_\epsilon := \exp(a_\epsilon^{-\frac{1}{q}})$$

where q is as in (f_3). So $\lim\limits_{\epsilon\to 0} s_\epsilon = \infty$ and $(\ln|u|)^{-q} < a_\epsilon$ for $|u| > s_\epsilon$. Let now

$$\Lambda_\epsilon = \left\{ x \in \Omega;\ \ |u|(\ln|u|)^{-\frac{q(N-2)}{N+2+p(N-2)}} > s_\epsilon \right\}.$$

Note that if $x \in \Lambda_\epsilon$ then $|u| > s_\epsilon$. Therefore, from (f_3) we deduce that there exists $C > 0$ such that

$$\begin{aligned}\int_\Omega |f(u)||u|^p &\leq C\left(\int_{\Lambda_\epsilon} \frac{|u|^{\frac{N+2}{N-2}+p}}{(\ln|u|)^q} + \int_{^C\Lambda_\epsilon} \frac{|u|^{\frac{N+2}{N-2}+p}}{(\ln|u|)^q}\right) + C \\ &\leq Ca_\epsilon\left(\int_{\Lambda_\epsilon} |u|^{\frac{4}{N-2}}|u|^{p+1}\right) + Cs_\epsilon^{\frac{N+2}{N-2}+p} + C.\end{aligned}$$

By Hôlder inequality and (3.9), we get

$$\begin{aligned}\int_\Omega |f(u)||u|^p &\leq Ca_\epsilon\left(\int_{\Lambda_\epsilon} |u|^{\frac{2N}{N-2}}\right)^{\frac{2}{N}}\left(\int_{\Lambda_\epsilon} |u|^{\frac{N(p+1)}{N-2}}\right)^{\frac{N-2}{N}} + Cs_\epsilon^{\frac{N+2}{N-2}+p} + C \\ &\leq C\epsilon\left(\int_{\Lambda_\epsilon} |u|^{\frac{N(p+1)}{N-2}}\right)^{\frac{N-2}{N}} + Cs_\epsilon^{\frac{N+2}{N-2}+p} + C.\end{aligned}$$

Using (3.6) and choose ϵ sufficiently small, we obtain

$$\left(\int_\Omega |u|^{\frac{N(p+1)}{N-2}}\right)^{\frac{N-2}{N}} \leq Cs_\epsilon^{\frac{N+2}{N-2}+p} + C. \tag{3.7}$$

On the other hand, by L^p-elliptic regularity, Sobolev embedding and the fact that $\frac{3}{4}N > \frac{N}{2}$, we find

$$\|u\|_{L^\infty(\Omega)} \leq \left(\int_\Omega |f(u)|^{\frac{3N}{4}}\right)^{\frac{4}{3N}}.$$

Using (f_3), we deduce that there exists $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)}^{\frac{3N}{4}} \leq C\left(\int_\Omega |u|^{\frac{3N}{4}\frac{N+2}{N-2}} + 1\right).$$

Choose $p = \frac{3N+2}{4}$ so that $\frac{3N}{4}\frac{N+2}{N-2} = \frac{N(p+1)}{N-2}$, then we get

$$\|u\|_{L^\infty(\Omega)}^{\frac{3(N-2)}{4}} \leq C\left(\int_\Omega |u|^{\frac{(q+1)N}{N-2}}\right)^{\frac{N-2}{N}} + C.$$

Combining with (3.7), we are done.

4 Proof of Theorem 2.5.

Proof of Theorem 2.5. First, remark that conditions (H_1) and (H_2) imply that there exist two positive constants C_1 and C_2 such that

$$\frac{(N-2)(1+\theta)}{2N} f(x,s)s - C_1 \leq F(x,s) \leq (2+\mu)^{-1} f(x,s)s + C_1, \tag{4.1}$$

$$f(x,s)s \geq C_1 \left(|s|^{2+\mu} - 1\right), \tag{4.2}$$

$$|f(x,s)| \leq C_2 \left(|s|^{\frac{N(1-\theta)+2(1+\theta))}{(N-2)(1+\theta)}} + 1\right). \tag{4.3}$$

We split the proof into two steps.
Step 1: We prove first the estimate for $\|\nabla u\|_{L^2}$. Let $0 < R_0 < R_1(\Omega)$ and $y \in \Gamma(R_0)$, where $R_1(\Omega)$ is given in Lemma 2.2 and $\Gamma(R_0)$ is defined in (2.4). Let

$$\phi_j(x) = \varphi_j\left(\frac{4(i(u)+1)(x-y)}{R_0}\right), \quad x \in B_{R_0}(y), \quad j = 1, ..., i(u)+1,$$

where φ_j is defined by $(*)$ under (2.11). Let $m \geq 1$. As in the proof of Lemma 3.1, we show that there exist $j_0 \in \{1, ..., i(u)+1\}$ and $C > 0$ such that

$$\int_\Omega f'(x,u)u^2\phi_{j_0}^{2m} - \int_\Omega |\nabla u|^2\phi_{j_0}^{2m} \leq \frac{C}{R_0^2}(1+i(u))^2 \int_\Omega u^2\phi_{j_0}^{2m-2}. \tag{4.4}$$

Multiplying equation (1.1) by $u\phi_{j_0}^{2m}$ and integrating by parts, we obtain

$$\int_\Omega |\nabla u|^2\phi_{j_0}^{2m} - \int_\Omega f(x,u)u\phi_{j_0}^{2m} \leq \frac{C}{R_0^2}(1+i(u))^2 \int_\Omega u^2\phi_{j_0}^{2m-2}. \tag{4.5}$$

Let $0 < \mu' < \mu$. Combining (4.4)-(4.5) and using (H_1), we deduce that

$$\mu' \int_\Omega |\nabla u|^2\phi_{j_0}^{2m} + (\mu - \mu') \int_\Omega uf(x,u)\phi_{j_0}^{2m} \leq \frac{C}{R_0^2}(1+i(u))^2 \int_\Omega u^2\phi_{j_0}^{2m-2} + C.$$

By Young's inequality, we obtain that for any $\epsilon > 0$, there holds

$$\mu' \int_\Omega |\nabla u|^2\phi_{j_0}^{2m} + (\mu - \mu') \int_\Omega uf(x,u)\phi_{j_0}^{2m} \leq C_\epsilon \left(\frac{1+i(u)}{R_0}\right)^{2(\frac{\mu+2}{\mu})} + C\epsilon \int_\Omega u^{\mu+2}\phi_{j_0}^{(m-1)(\mu+2)}.$$

Choose m so that $2m = (m-1)(\mu+2)$ and using (4.2), we get

$$\mu' \int_\Omega |\nabla u|^2\phi_{j_0}^{2m} + (\mu - \mu') \int_\Omega uf(x,u)\phi_{j_0}^{2m} \leq C_\epsilon \left(\frac{1+i(u)}{R_0}\right)^{2(\frac{\mu+2}{\mu})} + C\epsilon \int_\Omega uf(x,u)\phi_{j_0}^{2m}.$$

Take ϵ small enough, we deduce that

$$\int_{A_{j_0}\cap\Omega} |\nabla u|^2 + \int_{A_{j_0}\cap\Omega} f(x,u)u \leq C(1+i(u))^{2(1+\frac{2}{\mu})}. \tag{4.6}$$

Now, define the cut-off function $\psi_{j_0} \in C_c^2(B_{b_{j_0}}(y))$ verifying $\psi_{j_0}(x) \equiv 1$ in $B_{a_{j_0}}(y)$, with

$$\|\nabla\psi_{j_0}\|_\infty \leq \frac{C}{R_0}(1+i(u)), \quad \|\Delta\psi_{j_0}\|_\infty \leq \frac{C}{R_0^2}(1+i(u))^2.$$

Here a_{j_0} and b_{j_0} are defined in (2.11). Applying Lemmas 2.1 and 2.2 with $\psi = \psi_{j_0}$ and $R = R_0$, using (H_2), (H_3) and (4.1), we derive that

$$\begin{aligned}(1+\theta)\int_\Omega f(x,u)u\psi_{j_0} - \int_\Omega |\nabla u|^2\psi_{j_0} &\leq C(1+i(u))\int_{A_{j_0}\cap\Omega}(|\nabla u|^2 + f(x,u)u) \\ &\quad + CR_0\int_\Omega f(x,u)u\psi_{j_0} + C.\end{aligned} \tag{4.7}$$

Multiplying the equation (1.1) by $u\psi_{j_0}$ and integrating by parts, we obtain

$$\int_\Omega |\nabla u|^2\psi_{j_0} - \int_\Omega f(x,u)u\psi_{j_0} \le \frac{C}{R_0^2}(1+i(u))^2\int_{A_{j_0}\cap\Omega} u^2. \tag{4.8}$$

Let $0<\theta'<\theta$. Combining (4.6)-(4.8), there holds

$$\theta'\int_\Omega |\nabla u|^2\psi_{j_0} + (\theta-\theta')\int_\Omega f(x,u)u\psi_{j_0} \le C((1+i(u))^{\frac{2(2+\mu)}{\mu}+1} + \frac{C}{R_0^2}(1+i(u))^2\int_{A_{j_0}\cap\Omega} u^2 + CR_0\int_\Omega f(x,u)u\psi_{j_0}.$$

By Hölder inequality, (4.2) and (4.6), we derive that

$$\begin{aligned}\theta'\int_\Omega |\nabla u|^2\psi_{j_0} + (\theta-\theta')\int_\Omega f(x,u)u\psi_{j_0} &\le C((1+i(u))^{\frac{2(2+\mu)}{\mu}+1} + CR_0\int_\Omega f(x,u)u\psi_{j_0}\\ &\quad +\frac{C}{R_0^2}(1+i(u))^2\left(\int_{A_{j_0}\cap\Omega} u^{2+\mu}\right)^{\frac{2}{2+\mu}}\\ &\le C((1+i(u))^{\frac{2(2+\mu)}{\mu}+1} + \frac{C}{R_0^2}(1+i(u))^{2+\frac{4}{\mu}}\\ &\quad +CR_0\int_\Omega f(x,u)u\psi_{j_0}.\end{aligned} \tag{4.9}$$

Take R_0 sufficiently small, as $\frac{R_0}{2}<a_{j_0}$, we get

$$\begin{aligned}&\int_{B_{\frac{R_0}{2}}(y)\cap\Omega} |\nabla u|^2 + \int_{B_{\frac{R_0}{2}}(y)\cap\Omega} f(x,u)u\\ &\le \int_\Omega |\nabla u|^2\psi_{j_0} + \int_\Omega f(x,u)u\psi_{j_0} \le C(1+i(u))^{\frac{4}{\mu}+3}.\end{aligned}$$

Proceeding exactly as in the proof of Theorem 2.3, using $\frac{R_0}{2}$ instead of R_0 and the covering argument, we can conclude that

$$2\int_\Omega |\nabla u|^2 = \int_\Omega (|\nabla u|^2 + uf(x,u)) \le C(1+i(u))^{\frac{4}{\mu}+3}. \tag{4.10}$$

Step 2: We prove here the L^∞ estimate. Using (4.10) and Sobolev's embedding theorem,

$$\left(\int_\Omega |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}} \le C(1+i(u))^{\frac{4}{\mu}+3}. \tag{4.11}$$

For $\epsilon>0$, set

$$a_\epsilon := \epsilon(1+i(u))^{\frac{-4}{N-2}(\frac{3}{2}+\frac{2}{\mu})} \quad \text{and} \quad s_\epsilon := a_\epsilon^{-\frac{(N-2)(1+\theta)}{2N\theta}}.$$

Hence

$$|u|^{\frac{2N}{(N-2)(1+\theta)}-1-\frac{N+2}{N-2}} < a_\epsilon \quad \text{for } |u|>s_\epsilon. \tag{4.12}$$

Define $A_\epsilon=\{x\in\Omega;\ |u|>s_\epsilon\}$. For $q>0$, from (4.3) and (4.12), there exists $C>0$ such that

$$\begin{aligned}\int_\Omega |f(x,u)||u|^q &\le C\left(\int_{A_\epsilon} |u|^{\frac{2N}{(N-2)(1+\theta)}+q-1} + \int_{{}^C A_\epsilon} |u|^{\frac{2N}{(N-2)(1+\theta)}+q-1}\right) + C\\ &\le Ca_\epsilon\left(\int_{A_\epsilon} |u|^{\frac{4}{N-2}}|u|^{q+1}\right) + Cs_\epsilon^{\frac{2N}{(N-2)(1+\theta)}-1+q} + C.\end{aligned} \tag{4.13}$$

By Hölder inequality and (4.11), we obtain

$$\int_\Omega |f(x,u)||u|^q \leq C\epsilon \left(\int_\Omega |u|^{\frac{N(q+1)}{N-2}}\right)^{\frac{N-2}{N}} + Cs_\epsilon^{\frac{2N}{(N-2)(1+\theta)}-1+q} + C.$$

Using (3.6) and ϵ small enough, we deduce that

$$\left(\int_\Omega |u|^{\frac{N(q+1)}{N-2}}\right)^{\frac{N-2}{N}} \leq Cs_\epsilon^{\frac{2N}{(N-2)(1+\theta)}-1+q} + C. \tag{4.14}$$

Proceeding exactly as in the proof of Theorem 2.4, there exists $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)}^{\frac{3N}{4}} \leq C\left(\int_\Omega |u|^{\frac{3N}{4}\frac{N(1-\theta)+2(1+\theta))}{(N-2)(1+\theta)}} + 1\right).$$

Choose $q = \frac{3N(1-\theta)+5\theta-2}{4(1+\theta)}$ so that $\frac{3N}{4}\frac{N(1-\theta)+2(1+\theta)}{(N-2)(1+\theta)} = \frac{N(q+1)}{N-2}$. Using (4.14), we have

$$\|u\|_{L^\infty(\Omega)}^{\frac{3(N-2)}{4}} \leq \left(\int_\Omega |u|^{\frac{N(q+1)}{N-2}}\right)^{\frac{N-2}{N}} + C \leq Cs_\epsilon^{\frac{2N}{(N-2)(1+\theta)}-1+q} + C.$$

By the definition of s_ϵ, there holds $\|u\|_{L^\infty(\Omega)} \leq C(1+i(u))^{\beta'}$, with β' given in Theorem 2.5.

Chapitre 3

Explicit estimates for solutions to higher order elliptic PDEs via Morse index

Sommaire

In this chapter, we establish L^∞ and L^p estimates for solutions of some polyharmonic elliptic equations via the Morse index. As far as we know, it seems to be the first time that such explicit estimates are obtained for polyharmonic problems.

1 Introduction

Consider the following polyharmonic equations $(P_k) : (-\Delta)^k u = f(x, u)$ in Ω with the Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{on} \quad \partial\Omega; \tag{1.1}$$

or the Navier boundary conditions

$$u = \Delta u = \ldots = \Delta^{k-1} u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^N$ $(N > 2k)$ is a bounded domain with smooth boundary and f is a $C^1(\Omega \times \mathbb{R})$ function that we will specify later. Define

$$\Lambda_u(\phi) := \int_\Omega |D^k \phi|^2 - f'(x, u)\phi^2 \quad \text{for} \quad \phi \in \Sigma_k \tag{1.3}$$

where

$$D^k = \begin{cases} \nabla\Delta^{\frac{k-1}{2}} & \text{for } k \text{ odd;} \\ \Delta^{\frac{k}{2}} & \text{for } k \text{ even} \end{cases}$$

and

$$\Sigma_k := \begin{cases} H_0^k(\Omega) & \text{if we work with (1.1);} \\ \left\{\phi \in H^k(\Omega), \phi = \Delta\phi = ... = \Delta^{[\frac{k-1}{2}]}\phi = 0 \text{ on } \partial\Omega\right\} & \text{if we work with (1.2).} \end{cases}$$

The Morse index of a classical solution u of (P_k), denoted by $i(u)$ is defined as the maximal dimension of all subspaces of Σ_k such that $\Lambda_u(\phi) < 0$ in $\Sigma \setminus \{0\}$. We say that u is stable if its Morse index is equal to zero. Our aim here is to get some explicit estimates of u using its Morse index $i(u)$.

We begin by presenting some assumptions on the nonlinearity f:

(H_1) (superlinearity) There exists $\mu > 0$ such that

$$f'(x,s)s^2 \geq (1+\mu)f(x,s)s > 0, \quad \text{for } |s| > s_0, \ \ x \in \Omega.$$

(H_2) (subcritical growth) There exists $0 < \theta < 1$ such that

$$\frac{2N}{N-2k}F(x,s) \geq (1+\theta)f(x,s)s, \quad \text{for } |s| > s_0, \ \ x \in \Omega,$$

where $F(x,s) = \displaystyle\int_0^t f(x,t)dt$.

(H_3) There is a constant $C \geq 0$ such that

$$|\nabla_x F(x,s)| \leq C(F(x,s)+1), \quad x \in \Omega.$$

We say that f satisfies (H_i) in $\mathbb{R}_+$, if we have the assumption (H_i) only for s large enough.

For the second order case, i.e. $k = 1$, Bahri & Lions [1992] obtained the estimates of solutions in $H_0^1(\Omega)$ for superlinear and subcritical growth f, by using the blow-up technique and the Morse index of the solutions. Motivated by Bahri & Lions [1992], based on some local interior estimates and careful boundary estimates, Yang obtained in Yang [1998] the first explicit estimates of L^p or L^∞ norm for solutions to (P_1) via the Morse index. More precisely, Yang proved that

Theorem A *Let f satisfy (H_1)-(H_3), then there exist positive constant C, α and β such that any $u \in C^2(\Omega) \cap C(\overline{\Omega})$, solution of (P_1) satisfies*

$$\int_\Omega |f(x,u)|^{p_0}dx \leq C(i(u)+1)^\alpha, \quad \|u\|_{L^\infty(\Omega)} \leq C(i(u)+1)^\beta,$$

where

$$p_0 = 1 + \frac{(1+\theta)(N-2)}{(1-\theta)N + 2(1+\theta)}, \quad \alpha = \left(\frac{3}{2} + \frac{3}{2+\mu}\right)\frac{(2+\mu)^2}{3\mu+\mu^2}$$

and

$$\beta = \frac{2\alpha}{p_0 N(2-p_0)}\left[\frac{2}{N(2-p_0)} - \frac{1}{p_0}\right]^{-1}.$$

Hajlaoui *et al.* [2015] revised the results of Yang [1998], they obtained similar L^∞-estimate for solution to (P_1). The proof in Hajlaoui *et al.* [2015] is more transparent, and it allows them to get a slightly better estimate for large dimension N :

Theorem B *Let f satisfy (H_1)-(H_3), then there exist positive constant C, α' and β' such that any classical solution u of (P_1) satisfies*

$$\int_\Omega |\nabla u|^2 dx \le C(i(u)+1)^{\alpha'}, \quad \|u\|_{L^\infty} \le C(i(u)+1)^{\beta'},$$

where

$$\alpha' = \frac{4}{\mu} + 3 \quad \text{and} \quad \beta' = \frac{3\mu+4}{3\mu\theta} \times \frac{3N^2(1-\theta)+N(7\theta-4)-2\theta+12}{N(N-2)^2}.$$

In this chapter, we will try to handle the polyharmonic equations. Let

$$(E_k) \qquad \begin{cases} (-\Delta)^k u = f(x,u) & \text{in } \Omega; \\ u \text{ satisfies } (1.1), & \text{if } k \text{ is odd;} \\ u \text{ satisfies } (1.2), & \text{if } k \text{ is even.} \end{cases}$$

To simplify the presentation, we will concentrate on the cases $k = 2$ and $k = 3$, even we believe that the results should hold true for general $k \in \mathbb{N}$. We will provide some L^p and L^∞ estimates in polynomial growth function of the Morse index, for classical solutions of (E_2) and (E_3), provided suitable conditions on f. As far as we know, it seems to be the first time that some explicit estimates are obtained for polyharmonic problems via the Morse index.

As in Hajlaoui *et al.* [2015], we shall employ a cut-off function with compact support to derive a variant of the Pohozaev identity. This device allows us to avoid the spherical integrals raised in Yang [1998], which are very difficult to control, especially for the polyharmonic situations. Furthermore, under (H_1)-(H_3), the local L^2-estimate of ∇u and Δu via the Morse index seem also difficult to derive for the polyharmonic equation than for (P_1) the second order case. As in Hajlaoui *et al.* [2015], we need to exhibit the explicit dependence on $i(u)$ (see Lemma 2.3 and lemma 3.3 below). The following are our main results.

Theorem 3.1 *If u is a classical solution of (E_2) with $f \ge 0$ satisfying (H_1)-(H_3) in $\mathbb{R}_+$; or if u is a classical solution of (E_3) with f satisfying (H_1)-(H_3), then there exists a positive constant C independent of u such that*

$$\int_\Omega |f(x,u)|^{p_k} dx \le C(i(u)+1)^{\alpha_k}$$

where

$$p_k = \frac{2N}{N(1-\theta)+2k(1+\theta)} \quad \text{and} \quad \alpha_k = \frac{4k(\mu+1)}{\mu} \quad \text{where } k = 2 \text{ or } 3 \text{ respectively.}$$

By setting up a standard boot-strap iteration, as f has subcritical growth, we can proceed similarly as in the proof of Theorem 2.2 in Yang [1998] and claim that

Theorem 3.2 *If u is a classical solution of (E_2) with $f \ge 0$ satisfying (H_1)-(H_3) in $\mathbb{R}_+$; or if u is a classical solution of (E_3) with f satisfying (H_1)-(H_3), then there exists a positive constant C independent of u such that (for $k = 2$ or 3 respectively),*

$$\|u\|_{L^\infty(\Omega)} \le C(i(u)+1)^{\beta_k}, \quad \text{where } \beta_k = \frac{2k\alpha_k}{p_k N(2-p_k)} \left[\frac{2k}{N(2-p_k)} - \frac{1}{p_k}\right]^{-1}, \quad \alpha_k = \frac{4k(\mu+1)}{\mu},$$

and p_k is defined in Theorem 3.1.

By assumptions (H_1) and (H_2) in $\mathbb{R}$ (resp. in $\mathbb{R}_+$), there exist two positive constants C_1 and C_2 such that for $|s|$ large enough (resp. for s large enough),

$$\frac{(N-2k)(1+\theta)}{2N} f(x,s)s - C_1 \leq F(x,s) \leq \frac{1}{2+\mu} f(x,s)s + C_1, \tag{1.4}$$

$$f(x,s)s \geq C_1(|s|^{2+\mu} - 1) \tag{1.5}$$

and

$$|f(x,s)| \leq C_2 \left(|s|^{\frac{N(1-\theta)+2k(1+\theta))}{(N-2k)(1+\theta)}} + 1 \right). \tag{1.6}$$

This chapter is organized as follows : We give the proof of Theorem 3.1 for $k=2$ and $k=3$ respectively in sections 2 and 3. In the following, C denotes always a generic positive constant independent of the solution u, even their value could be changed from one line to another one.

2 Proof for $k=2$

Here we will prove Theorem 3.1 for $k=2$.

2.1 Preliminaries

Let $y \in \mathbb{R}^N$ and $R > 0$. Throughout the chapter, we denote by $B_R(y)$ the open ball of center y and radius R and $\partial\Omega_R(y) := \partial\Omega \cap B_R(y)$. For $x \in B_R(y) \cap \Omega$, let $n := x - y$. We denote also

$$u_{j_i \cdots j_k} := \frac{\partial^k u}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}}.$$

First of all, we have the following Pohozaev identity.

Lemme 2.1 *Let u be a classical solution to (E_2). Let $\psi \in C_c^2(B_R(y))$. Then*

$$\begin{aligned}
&\frac{2N}{N-4}\int_\Omega F(x,u)\psi dx + \frac{2}{N-4}\int_\Omega \nabla_x F(x,u)\cdot n\psi dx - \int_\Omega (\Delta u)^2\psi dx \\
= &-\frac{4}{N-4}\int_\Omega \Delta u \nabla^2 u(\nabla\psi, n) dx + \frac{1}{N-4}\int_\Omega (\nabla\psi\cdot n)(\Delta u)^2 dx \\
&-\frac{4}{N-4}\int_\Omega (\nabla u\cdot\nabla\psi)\Delta u dx - \frac{2}{N-4}\int_\Omega (\nabla u\cdot n)\Delta u \Delta\psi dx \\
&-\frac{2}{N-4}\int_\Omega F(x,u)\nabla\psi\cdot n dx - \frac{2}{N-4}\int_{\partial\Omega_R(y)} \frac{\partial \Delta u}{\partial\nu}(\nabla u\cdot n)\psi d\sigma.
\end{aligned}$$

Proof. Multiplying equation (E_2) by $\nabla u \cdot n\psi$ and integrating by parts, we get

$$-\int_\Omega \nabla(\Delta u)\cdot\nabla(\nabla u\cdot n\psi) + \int_{\partial\Omega_R(y)} (\nabla(\Delta u)\cdot v)\cdot(\nabla u\cdot n)\psi = \int_\Omega f(x,u)\nabla u\cdot n\psi. \tag{2.1}$$

For the left hand side of (2.1), we have:

$$\begin{aligned} I_2 &= \int_\Omega \Delta u \cdot \Delta(\nabla u \cdot n\psi) + \int_{\partial\Omega_R(y)} (\nabla(\Delta u)\cdot v)\cdot(\nabla u\cdot n)\psi \\ &= \int_\Omega \Delta u[\psi\Delta(\nabla u\cdot n) + 2\nabla(\nabla u\cdot n)\cdot\nabla\psi] \\ &\quad + \int_\Omega \Delta u(\nabla u\cdot n)\Delta\psi + \int_{\partial\Omega_R(y)} (\nabla(\Delta u)\cdot v)\cdot(\nabla u\cdot n)\psi \\ &= \int_\Omega \Delta u\psi\Delta(\nabla u\cdot n) + \int_\Omega \Delta u(\nabla u\cdot n)\Delta\psi \\ &\quad +2\int_\Omega \Delta u\,[\nabla(\nabla u\cdot n)\cdot\nabla\psi] + \int_{\partial\Omega_R(y)} (\nabla(\Delta u)\cdot v)\cdot(\nabla u\cdot n)\psi. \end{aligned}$$

We now may need to deal with the first and third term on the right hand side of the above equality.
For the first term, one obtains

$$\begin{aligned} \int_\Omega \Delta u\psi\Delta(\nabla u\cdot n) &= \int_\Omega \Delta u\psi\left[\sum_j \frac{\partial}{\partial^2 x_j}\left(\sum_i \frac{\partial u}{\partial x_i}(x_i - y_i)\right)\right] \\ &= \int_\Omega \Delta u\psi\left[\sum_{i,j} \frac{\partial}{\partial^2 x_j}\left(\frac{\partial u}{\partial x_i}(x_i - y_i)\right)\right] \\ &= \int_\Omega \Delta u\psi\left[\sum_{i,j}(u_{jj})_i(x_i - y_i) + 2\sum_{j,i} u_{ji}\delta_{j,i}\right] \\ &= \int_\Omega \Delta u\psi\left[\sum_{i,j}(u_{jj})_i(x_i - y_i) + 2\sum_i u_{ii}\right] \\ &= \int_\Omega \Delta u\psi\,(\nabla(\Delta u)\cdot n + 2\Delta u) \\ &= \int_\Omega \frac{\nabla((\Delta u)^2)}{2}\cdot n\psi + 2\int_\Omega (\Delta u)^2\psi \\ &= -\int_\Omega \frac{N}{2}(\Delta u)^2\psi - \int_\Omega \frac{1}{2}(\Delta u)^2(\nabla\psi\cdot n) + 2\int_\Omega (\Delta u)^2\psi \\ &= \frac{4-N}{2}\int_\Omega (\Delta u)^2\psi - \frac{1}{2}\int_\Omega (\Delta u)^2(\nabla\psi\cdot n). \end{aligned}$$

For the third term, we have

$$\begin{aligned}
2\int_\Omega \Delta u\,[\nabla(\nabla u\cdot n)\cdot\nabla\psi] &= 2\int_\Omega \Delta u\left[\sum_i \frac{\partial}{\partial x_i}\left(\sum_j \frac{\partial u}{\partial x_j}(x_j-y_j)\right)\frac{\partial\psi}{\partial x_i}\right]\\
&= 2\int_\Omega \Delta u\left[\sum_{ij}\frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_j}(x_j-y_j))\frac{\partial\psi}{\partial x_i}\right]\\
&= 2\int_\Omega \Delta u\left[\sum_{ij}\partial_{ij}u(x_j-y_j)\frac{\partial\psi}{\partial x_i}+\sum_{ij}\frac{\partial u}{\partial x_j}\delta_{ij}\frac{\partial\psi}{\partial x_i}\right]\\
&= 2\int_\Omega \Delta u\left(\sum_{ij}\partial_{ij}u(x_j-y_j)\frac{\partial\psi}{\partial x_i}\right)+2\int_\Omega \Delta u(\nabla u\cdot\nabla\psi)\\
&= 2\int_\Omega \Delta u\nabla^2 u(n\cdot\nabla\psi)+2\int_\Omega \Delta u(\nabla u\cdot\nabla\psi).
\end{aligned}$$

We can regroup the terms then we have

$$\begin{aligned}
I_1 : =& \frac{4-N}{2}\int_\Omega(\Delta u)^2\psi+2\int_\Omega \Delta u\nabla^2u(n\cdot\nabla\psi)+2\int_\Omega\Delta u(\nabla u\cdot\nabla\psi)\\
&+\int_\Omega \Delta u(\nabla u\cdot n)\Delta\psi-\frac{1}{2}\int_\Omega(\Delta u)^2(\nabla\psi\cdot n)\\
&+\int_{\partial\Omega_R(y)}(\nabla(\Delta u)\cdot v)\cdot(\nabla u\cdot n)\psi.
\end{aligned}\tag{2.2}$$

For the right hand side of (2.1), we integrate by parts to get

$$\begin{aligned}
I_2 :=& \int_\Omega f(x,u)\nabla u\cdot n\psi\\
=& \int_\Omega \psi n\cdot(F(x,u))'_x-\int_\Omega\psi n\cdot\nabla_x F(x,u)\\
=& -N\int_\Omega F(x,u)\psi-\int_\Omega F(x,u)n\cdot\nabla\psi-\int_\Omega n\cdot\nabla_x F(x,u)\psi.
\end{aligned}\tag{2.3}$$

Note that in the last equality, the Dirichlet boundary condition implies that $F(x,u)=0$ on $\partial\Omega_R(y)$. Therefore, the claim follows from (2.1),(2.2) and (2.3).

To establish a global estimate, we will cover the domain Ω by small balls and obtain local estimates. To be more precise, consider

$$\Omega_{1,R} := \left\{x\in\Omega:\ \mathrm{dist}(x,\partial\Omega)>\frac{R}{2}\right\}\ \text{ and }\ \Omega_{2,R} := \left\{x\in\Omega:\ \mathrm{dist}(x,\partial\Omega)\le\frac{R}{3}\right\},\quad\forall\, R>0.$$

The main difficulty is the estimates of u near the boundary, that is, in $\Omega_{2,R}$. We need to choose carefully the balls as in Yang [1998]. Indeed, we will take balls with center lying in

$$\Gamma(R) := \left\{x\in\mathbb{R}^N\backslash\Omega:\ \mathrm{dist}(x,\partial\Omega)=\frac{R}{20}\right\},\tag{2.4}$$

The domain $\Omega\backslash\Omega_{2,R}$ will be covered by balls with center lying in $\Omega_{1,R}$. The following lemma is devoted to the control of the boundary term for $y\in\Gamma(R)$ in the above Pohozaev identity.

Lemme 2.2 *There exists $R_1 > 0$ depending on Ω such that if $f(x,u) \geq 0$ and u is a classical solution of (E_2), then for any $0 < R \leq R_1$ and $y \in \Gamma(R)$, there holds*

$$\int_{\partial\Omega_R(y)} \frac{\partial \Delta u}{\partial \nu}(\nabla u \cdot n)\psi d\sigma \geq 0,$$

for any nonnegative function $\psi \in C_c^2(B_R(y))$.

Proof. As in the proof of Lemma 2.2 of Yang [1998], there exists $R_1 > 0$ such that if $0 < R \leq R_1$ and $y \in \Gamma(R)$ then $\nu \cdot n \leq 0$ for any $x \in \partial\Omega_R(y)$.

As $f(x,u) \geq 0$, the maximum principle implies that $-\Delta u \geq 0$ in Ω as $\Delta u = 0$ on $\partial\Omega$, hence $u \geq 0$. Therefore $\frac{\partial \Delta u}{\partial \nu} \geq 0$ on $\partial\Omega$ and $\nabla u \cdot n = (n \cdot v)\frac{\partial u}{\partial \nu} \geq 0$ on $\partial\Omega$, so we obtain the claim.

Consequently, we get

Proposition 2.1 *There exists $R_0 > 0$ small who satisfies the following property: Let u be a classical solution of (E_2) with $f \geq 0$ verifying (H_1)-(H_3) in $\mathbb{R}_+$. Then for any $0 < R \leq R_0$, $y \in \Gamma(R)$ and $0 \leq \psi \in C_c^4(B_R(y))$, there holds*

$$\begin{aligned} &\int_\Omega f(x,u)u\psi dx + \int_\Omega (\Delta u)^2 \psi dx \\ &\leq CR\|\nabla\psi\|_\infty \int_{A_{R,\psi}(y)} f(x,u)udx + CR^2 \int_{A_{R,\psi}(y)} |\nabla^2(u\nabla\psi)|^2 dx \\ &\quad + C\Big(1 + R\|\nabla\psi\|_\infty\Big)\|\Delta u\|^2_{L^2(A_{R,\psi}(y))} + C\Big(R^2\|\nabla(\Delta\psi)\|^2_\infty + \|\Delta\psi\|^2_\infty\Big)\|u\|^2_{L^2(A_{R,\psi}(y))} \\ &\quad + CR^2\Big(\|\Delta\psi\|^2_\infty + \frac{1}{R^2}\|\nabla\psi\|^2_\infty + \|\nabla^2\psi\|^2_\infty\Big)\|\nabla u\|^2_{L^2(A_{R,\psi}(y))} + CR^N, \end{aligned} \tag{2.5}$$

where

$$A_{R,\psi}(y) = B_R(y) \cap \Omega \cap \{\nabla\psi \neq 0\}.$$

Moreover, for $y \in \Omega_{1,R}$, the above inequality holds true if we replace R by $\frac{R}{2}$.

Proof. Let $y \in \Gamma(R)$ with $R < R_1$ and $0 \leq \psi \in C_c^4(B_R(y))$. Using Lemmas 2.1–2.2, (H_1)-(H_3) and (1.4), we obtain

$$\begin{aligned} &(1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega (\Delta u)^2\psi dx \\ &\leq \frac{4}{N-4}\int_{A_{R,\psi}(y)} |\Delta u||\nabla^2 u(\nabla\psi, n)|dx + \frac{1}{N-4}\int_{A_{R,\psi}(y)} (\Delta u)^2|\nabla\psi \cdot n|dx \\ &\quad + \frac{4}{N-4}\int_{A_{R,\psi}(y)} |\Delta u||\nabla u \cdot \nabla\psi|dx + \frac{2}{N-4}\int_{A_{R,\psi}(y)} |\Delta u||\nabla u \cdot n||\Delta\psi|dx \\ &\quad + \frac{1}{(N-4)}\int_{A_{R,\psi}(y)} f(x,u)u|\nabla\psi \cdot n|dx + CR\int_{B_R(y)\cap\Omega} f(x,u)u\psi dx + CR^N. \end{aligned} \tag{2.6}$$

A direct calculation implies that

$$\nabla^2 u(\nabla\psi, n) = \sum_{ij} u_{ij}\psi_i n_j = \sum_{ij}(u\psi_i)_{ij}n_j - u\nabla(\Delta\psi)\cdot n - \Delta\psi(\nabla u \cdot n) - \nabla^2\psi(\nabla u, n).$$

By the Cauchy-Schwarz inequality, there exists $C > 0$ such that

$$\begin{aligned}\int_{A_{R,\psi}(y)} |\Delta u||\nabla^2 u(\nabla\psi, n)|dx &\leq C\int_{A_{R,\psi}(y)} |\Delta u|^2 dx + CR^2\int_{A_{R,\psi}(y)} u^2|\nabla(\Delta\psi)|^2 dx \\ &+ CR^2\int_{A_{R,\psi}(y)} |\nabla^2(u\nabla\psi)|^2 dx \\ &+ CR^2\int_{A_{R,\psi}(y)} |\nabla u|^2\Big(\|\Delta\psi\|_\infty^2 + \|\nabla^2\psi\|_\infty^2\Big)dx.\end{aligned} \tag{2.7}$$

On the other hand, recall that $u = \Delta u = 0$ on $\partial\Omega$ and $\psi \in C_c^4(B_R(y))$, multiplying the equation (E_2) by $u\psi$ and integrating by parts, we get readily

$$\begin{aligned}\int_\Omega (\Delta u)^2\psi dx - \int_\Omega f(x,u)u\psi dx &\leq C\int_{A_{R,\psi}(y)} |\Delta u|\Big[|\nabla u\cdot\nabla\psi| + |u||\Delta\psi|\Big]dx \\ &\leq C\int_{A_{R,\psi}(y)} \Big[(\Delta u)^2 + |\nabla u\cdot\nabla\psi|^2 + (\Delta\psi)^2u^2\Big]dx.\end{aligned} \tag{2.8}$$

Remark that

$$\begin{aligned}\frac{\theta}{2}\int_\Omega (\Delta u)^2\psi dx + \frac{\theta}{2}\int_\Omega f(x,u)u\psi dx &= (1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega (\Delta u)^2\psi dx \\ &+ \left(1+\frac{\theta}{2}\right)\left[\int_\Omega (\Delta u)^2\psi dx - \int_\Omega f(x,u)u\psi dx\right].\end{aligned}$$

Fix $R_0 \in (0, R_1)$ such that $CR_0 < 1$. Combining (2.6)-(2.8), using again Cauchy-Schwarz inequality, there holds clearly (2.5). The proof for $y \in \Omega_{1,R}$ is completely similar , so we omit it.

Remark 3.1 *The key point in* (2.5) *is that the integral over the ball $B_R(y)\cap\Omega$ is now controlled by the integrals over the annuli type domain $A_{R,\psi}(y)$ when we work with suitable cut-off function ψ.*

Let $R > 0$, $y \in \Omega_{1,R} \cup \Gamma(R)$, $0 < a < b$. Denote

$$A := A_a^b = \{x \in \mathbb{R}^N;\ a < |x-y| < b\},\quad A_\rho := A_{a+\rho}^{b-\rho}\ \text{ for }\ 0 < \rho < \frac{b-a}{4}. \tag{$*$}$$

We will use also the following classical estimates.

Lemme 2.3 *There exists a constant $C > 0$ depending only on N such that for any $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and $0 < \rho < \min(1, \frac{b-a}{4})$, we have*

$$\|\nabla u\|_{L^2(A_\rho\cap\Omega)}^2 \leq C\left(\frac{1}{\rho^2}\|u\|_{L^2(A\cap\Omega)}^2 + \|\Delta u\|_{L^2(A\cap\Omega)}^2\right).$$

Proof. First, since $u\eta\ \in H_0^1(A\cap\Omega)$, then Gagliardo-Nirenberg-Sobolev inequality implies:

$$\|u\eta\|_{L^{\frac{2N}{N-2}}(\Omega\cap A)} \leq C_N\|\nabla(u\eta)\|_{L^2(\Omega\cap A)}, \tag{2.9}$$

where C_N is positive constant depending only on N. Let $\eta_\rho \in C_0^2(\Omega\cap A)$ be a cut-off function satisfying $0 \leq \eta_\rho \leq 1$ in $\Omega\cap A$. Multiplying the equation $-\Delta u = \vartheta$, by $u\eta_\rho^{2m}$ and integrating by parts, there holds

$$\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}\le 2m\int_{\Omega\cap A}(|u||\nabla\eta_\rho|)(|\nabla u|\eta_\rho^{2m-1})+\int_{\Omega\cap A}|\vartheta u|\eta_\rho^{2m}.$$

By Young and Hölder's inequalities, we derive that for any $\epsilon>0$

$$\begin{aligned}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}&\le\frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}+C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2\eta_\rho^{2m-2}\\&\quad+\left(\int_{\Omega\cap A}(|\vartheta|\eta_\rho^{2m})^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}\left(\int_{\Omega\cap A}(|u|\eta_\rho^{2m})^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}\\&\le\frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}+C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2+\frac{\epsilon}{3}\left(\int_{\Omega\cap A}(|u|\eta_\rho^{2m})^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}\\&\quad+C_\epsilon\left(\int_{\Omega\cap A}(|\vartheta|\eta_\rho^{2m})^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}.\end{aligned}$$

Using (2.9), we see that, there exists a positive constant C depending only on N such that

$$\begin{aligned}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}&\le\frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}+C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2\\&\quad+\frac{\epsilon}{3}\int_{\Omega\cap A}|\nabla(u\eta_\rho^{2m})|^2+C_\epsilon\left(\int_{\Omega\cap A}(|\vartheta|\eta_\rho^{2m})^{\frac{2N}{N+2}}\right)^{\frac{N+2}{N}}\\&\le\epsilon\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}+C_\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2+C_\epsilon\int_{\Omega\cap A}\vartheta^2\eta_\rho^{4m}.\end{aligned}$$

Choose ϵ small enough, we get

$$\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}\le C\left(\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2+\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m}\right).\tag{2.10}$$

Consequently,

$$\int_{\Omega\cap A}|\nabla(u\eta_\rho^{2m})|^2\le2\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m}+2\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2\le C\left[\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2+\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m}\right].$$

Taking $\eta_\rho:=h(\frac{|x-y|}{\rho})$, where $h\in C_0^2(\mathbb{R})$ satisfying $0\le h\le1$, $h=1$ in $[\frac{a}{\rho}+1,\frac{b}{\rho}-1]$ and $h=0$ in $\mathbb{R}\backslash[\frac{a}{\rho}+\frac12,\frac{b}{\rho}-\frac12]$, we get the claimed estimate of $\|\nabla u\|_{L^2(A_\rho\cap\Omega)}$.

Remark 3.2 *If f satisfies (H_1), using (1.5), there holds*

$$\|u\|^2_{L^2(A\cap\Omega)}\le C\left(\int_{A\cap\Omega}f(x,u)udx\right)^{\frac{2}{2+\mu}}+C.$$

2.2 Estimation via Morse index

Let u be a solution to (E_2) with $f\ge0$ and finite Morse index $i(u)$. For $y\in\Gamma(R)\cup\Omega_{1,R}$, denote

$$A_j=:A_{a_j}^{b_j}\quad\text{with}\ \ a_j=\frac{2(j+i(u))}{4(i(u)+1)}R,\ \ b_j=\frac{2(j+i(u))+1}{4(i(u)+1)}R,\quad 1\le j\le i(u)+1.\tag{2.11}$$

Fix a cut-off function $\Phi\in C^\infty(\mathbb{R})$ such that $\Phi=1$ in $[0,1]$ and $\mathrm{supp}(\Phi)\subset(-\frac12,\frac32)$. Let

$$\phi_j(x):=\Phi\left(\frac{4(i(u)+1)|x-y|}{R}-2j-2i(u)\right).$$

Then for any $1 \leq j \leq i(u)+1$, $\phi_j \in C_c^\infty(B_R(y))$,

$$\phi_j(x) = 1 \text{ in } A_j, \quad \|\nabla\phi_j\|_\infty \leq \frac{C}{R}(1+i(u)) \quad \text{and} \quad \|\Delta\phi_j\|_\infty \leq \frac{C}{R^2}(1+i(u))^2. \tag{2.12}$$

We prove the following lemma.

Lemme 2.4 *Let f satisfy (H_1) and let u be a smooth solution to (E_2) with Morse index $i(u) < \infty$. Then for any $0 < R \leq R_0$, $y \in \Gamma(R) \cup \Omega_{1,R}$, there exists $j_0 \in \{1, 2, ..., 1+i(u)\}$ verifying*

$$\int_{A_{j_0}\cap\Omega} (\Delta u)^2 dx + \int_{A_{j_0}\cap\Omega} f(x,u)u dx \leq C\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}}. \tag{2.13}$$

Proof. First, for $\epsilon \in (0,1)$ and $\eta \in C^2(\mathbb{R}^N)$,

$$\begin{aligned}\int_\Omega [\Delta(u\eta)]^2 dx &= \int_\Omega (u\Delta\eta + 2\nabla u\nabla\eta + \eta\Delta u)^2\, dx\\ &\leq \left(1+\frac{\epsilon}{2}\right)\int_\Omega (\Delta u)^2\eta^2 dx + \frac{C}{\varepsilon}\int_\Omega u^2(\Delta\eta)^2 dx + \frac{C}{\epsilon}\int_\Omega |\nabla u|^2|\nabla\eta|^2 dx.\end{aligned}$$

Using $\Delta(u^2) = 2|\nabla u|^2 + 2u\Delta u$, there holds

$$\int_\Omega |\nabla u|^2|\nabla\eta|^2 dx \leq \frac{1}{2}\int_\Omega u^2\Delta(|\nabla\eta|^2)dx + \int_\Omega |u||\Delta u||\nabla\eta|^2 dx. \tag{2.14}$$

Take $\eta = \zeta^m$ with $m > 2$, $\zeta \geq 0$ and apply Young's inequality, we get

$$\begin{aligned}\int_\Omega |u||\Delta u||\nabla\zeta^m|^2 dx &= m^2\int_\Omega |u||\Delta u||\nabla\zeta|^2\zeta^{2m-2}dx\\ &\leq \epsilon^2\int_\Omega (\Delta u)^2\zeta^{2m}dx + C_{\epsilon,m}\int_\Omega u^2|\nabla\zeta|^4\zeta^{2m-4}dx.\end{aligned} \tag{2.15}$$

Here $C_{\varepsilon,m}$ denotes a constant depending only on ε and m. Therefore

$$\begin{aligned}\int_\Omega [\Delta(u\zeta^m)]^2 dx &\leq C_{\epsilon,m}\int_\Omega u^2\Big[|\Delta\zeta|^2 + |\nabla\zeta|^4 + |\Delta(|\nabla\zeta|^2)|\Big]\zeta^{2m-4}dx\\ &\quad + (\epsilon+1)\int_\Omega (\Delta u)^2\zeta^{2m}dx.\end{aligned} \tag{2.16}$$

Consider now the family of functions $\{u\phi_j^m\}_{1\leq j\leq i(u)+1}$, $m > 2$. With the definition of ϕ_j, it's easy to see that different ϕ_j are supported by disjoint sets for different j, so they are linearly independent as $u > 0$ in Ω. Therefore, there must exist $j_0 \in \{1, 2, ..., 1+i(u)\}$ such that $\Lambda_u(u\phi_{j_0}^m) \geq 0$ where Λ is the quadratic form given by (1.3). Combining $\Lambda_u(u\phi_{j_0}^m) \geq 0$ with (2.12) and (2.16), we obtain

$$\int_\Omega f'(x,u)u^2\phi_{j_0}^{2m}dx - (1+\epsilon)\int_\Omega (\Delta u)^2\phi_{j_0}^{2m}dx \leq \frac{C_\varepsilon}{R^4}(1+i(u))^4\int_\Omega u^2\phi_{j_0}^{2m-4}dx. \tag{2.17}$$

Moreover, multiply the equation (E_2) by $u\eta^2$ and integrate by parts, we get, using (2.14)

$$\begin{aligned}&\int_\Omega \Big[(\Delta u)^2\eta^2 - f(x,u)u\eta^2\Big]dx\\ =&\; -4\int_\Omega \eta\Delta u\nabla u\cdot\nabla\eta dx - 2\int_\Omega \eta u\Delta u\Delta\eta dx - 2\int_\Omega u\Delta u|\nabla\eta|^2 dx\\ \leq&\; \varepsilon\int_\Omega (\Delta u)^2\eta^2 dx + C_\varepsilon\int_\Omega u^2(\Delta\eta)^2 dx + C_\varepsilon\int_\Omega |\nabla u|^2|\nabla\eta|^2 dx - 2\int_\Omega u\Delta u|\nabla\eta|^2 dx\\ \leq&\; \varepsilon\int_\Omega (\Delta u)^2\eta^2 dx + C_\varepsilon\int_\Omega u^2\Big[(\Delta\eta)^2 + |\Delta(|\nabla\eta|^2)\Big]dx + C_\varepsilon\int_\Omega |u\Delta u||\nabla\eta|^2 dx.\end{aligned}$$

Take now $\eta = \phi_{j_0}^m$ with $m = 2 + \frac{2}{\mu} > 2$, there holds as for (2.15),

$$\int_\Omega |u\Delta u||\nabla\eta|^2 dx \le \varepsilon \int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + C_\varepsilon \int_\Omega u^2 \phi_{j_0}^{2(m-2)} |\nabla \phi_{j_0}|^4 dx.$$

By (2.12), we deduce then

$$(1-2\epsilon)\int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx - \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^4}(1+i(u))^4 \int_\Omega u^2 \phi_{j_0}^{2m-4} dx. \tag{2.18}$$

Let $\varepsilon < \frac{1}{2}$, multiplying (2.18) by $\frac{1+2\varepsilon}{1-2\varepsilon}$, using (2.17) and (H_1), we get

$$\epsilon \int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + \left(\mu - \frac{4\epsilon}{1-2\epsilon}\right) \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^4}(1+i(u))^4 \int_\Omega u^2 \phi_{j_0}^{2m-4} dx + C_\varepsilon.$$

Fix now $\epsilon < \min(2, \frac{\mu}{4+2\mu})$, there holds

$$\int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx \le \frac{C}{R^4}(1+i(u))^4 \int_\Omega u^2 \phi_{j_0}^{2m-4} dx + C.$$

Therefore, using (1.5) and $R \le R_0$, for any $\epsilon' > 0$,

$$\begin{aligned}\int_\Omega (\Delta u)^2 \phi_{j_0}^{2m} dx + \int_\Omega uf(x,u)\phi_{j_0}^{2m} dx &\le C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}} + C + \varepsilon' \int_\Omega |u|^{\mu+2}\phi_{j_0}^{(m-2)(\mu+2)} dx \\ &\le C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}} + C\epsilon' \int_\Omega f(x,u)u\phi_{j_0}^{(m-2)(\mu+2)} dx \\ &= C_{\epsilon'}\left(\frac{1+i(u)}{R}\right)^{\frac{4\mu+8}{\mu}} + C\epsilon' \int_\Omega f(x,u)u\phi_{j_0}^{2m} dx.\end{aligned}$$

For the last line, we used $(m-2)(\mu+2) = 2m$. Take $\epsilon' > 0$ small enough, the estimate (2.13) is proved.

2.3 Proof of Theorem 3.1 completed

Now, we are in position to prove Theorem 3.1 for $k = 2$. Fix

$$R = R_0, \quad \rho := \frac{R}{10(i(u)+1)}, \quad A_{j_0,\rho} := A_{a_{j_0}+\rho}^{b_{j_0}-\rho} \subset A_{j_0} \text{ be as in } (*).$$

According to Lemmas 2.3, 2.3 and Remark 3.2, there exists a positive constant C independent of $y \in \Gamma(R) \cup \Omega_{1,R}$ such that

$$\|\Delta u\|_{L^2(A_{j_0,\rho}\cap\Omega)}^2 + \|\nabla u\|_{L^2(A_{j_0,\rho}\cap\Omega)}^2 \le C(1+i(u))^{\frac{4\mu+8}{\mu}}. \tag{2.19}$$

Here, a_{j_0} and b_{j_0} are defined in (2.11) with j_0 given by Lemma 2.3.

Consider a cut-off function $\xi_{j_0} \in C_c^4(B_{b_{j_0}-\rho}(y))$ verifying $\xi_{j_0}(x) \equiv 1$ in $B_{a_{j_0}+\rho}(y)$, with

$$\|\nabla \xi_{j_0}\|_\infty \le \frac{C}{R}(1+i(u)), \quad \|\Delta \xi_{j_0}\|_\infty \le \frac{C}{R^2}(1+i(u))^2.$$

Applying Proposition 2.1 with $\psi = \xi_{j_0}$, as $A_{R,\psi}(y) \subset A_{j_0,\rho} \cap \Omega$, we get

$$\begin{aligned}&\int_\Omega f(x,u)u\xi_{j_0}dx + \int_\Omega (\Delta u)^2\xi_{j_0}dx\\ \leq\ & C(1+i(u))\int_{A_{j_0,\rho}\cap\Omega}\Big[(\Delta u)^2 + f(x,u)u\Big]dx + C\int_{A_{j_0,\rho}\cap\Omega}|\nabla^2(u\nabla\xi_{j_0})|^2dx\\ &+ C(1+i(u))^6\|u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + C(1+i(u))^4\|\nabla u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + CR^N.\end{aligned} \tag{2.20}$$

Since $u\nabla\xi_{j_0} = 0$ on $\partial\Omega$, by standard elliptic theory, there exists $C_\Omega > 0$ depending only on Ω such that

$$\begin{aligned}\int_\Omega|\nabla^2(u\nabla\xi_{j_0})|^2dx &\leq C_\Omega\int_\Omega|\Delta(u\nabla\xi_{j_0})|^2dx\\ &= C_\Omega\int_{A_{j_0,\rho}\cap\Omega}|\Delta(u\nabla\xi_{j_0})|^2dx\\ &\leq C\int_{A_{j_0,\rho}\cap\Omega}\Big[u^2|\nabla(\Delta\xi_{j_0})|^2 + |\nabla u|^2|\nabla^2\xi_{j_0}|^2 + (\Delta u)^2|\nabla\xi_{j_0}|^2\Big]dx.\end{aligned} \tag{2.21}$$

From (2.20), (2.21), we get the following inequality

$$\begin{aligned}&\int_\Omega f(x,u)u\xi_{j_0}dx + \int_\Omega (\Delta u)^2\xi_{j_0}dx\\ \leq\ & C(1+i(u))\int_{A_{j_0,\rho}\cap\Omega}\Big[(\Delta u)^2 + f(x,u)u\Big]dx + C(1+i(u))^2\|\Delta u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)}\\ &+ C(1+i(u))^6\|u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + C(1+i(u))^4\|\nabla u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + CR^N.\end{aligned} \tag{2.22}$$

On the other hand, using Remark 3.2 and Lemma 2.3, there holds

$$\|u\|^2_{L^2(A_{j_0}\cap\Omega)} \leq C\left(\int_{A_{j_0}\cap\Omega} f(x,u)udx\right)^{\frac{2}{2+\mu}} + C \leq C(1+i(u))^{\frac{8}{\mu}}. \tag{2.23}$$

Combining (2.13), (2.19), (2.22) and (2.23), one obtains

$$\int_\Omega f(x,u)u\xi_{j_0}dx + \int_\Omega(\Delta u)^2\xi_{j_0}dx \leq C(1+i(u))^{\frac{8\mu+8}{\mu}}.$$

As $\frac{R}{2} < a_{j_0}$ and $R = R_0$, we get then for any $y \in \Gamma(R) \cup \Omega_{1,R}$,

$$\int_{B_{\frac{R_0}{2}}(y)\cap\Omega}\Big[|\Delta u|^2 + f(x,u)u\Big]dx \leq C(1+i(u))^{\frac{8\mu+8}{\mu}}.$$

By covering argument and (1.6), we get finally

$$\int_\Omega f(x,u)^{p_2}dx \leq C\int_\Omega f(u)udx + C \leq C(1+i(u))^{\alpha_2},$$

where $p_2 = \frac{2N}{N(1-\theta)+4(1+\theta)}$ and $\alpha_2 = \frac{8(\mu+1)}{\mu}$. So we are done.

3 Proof of Theorem 3.1 for $k = 3$

In this section, we consider the equation (E_3). We will proceed as for (E_2) and keep the same notations, but we replace the Navier boundary conditions by the Dirichlet boundary conditions and we have no more the sign condition for f.

3.1 Preliminaries

We make some preparations here. For $\psi \in C^m$ for $m \geq 1$, to simplify the notation, we define

$$[\psi]_m(x) = \sum_{|\beta_1|+\ldots+|\beta_p|=m, |\beta_i|\geq 1} \prod_{i=1}^{p} |\partial_{\beta_i}\psi(x)|$$

and the semi-norms

$$|\psi|_{m,\infty} = \sum_{\alpha_1+\ldots+\alpha_p=m, \alpha_i\geq 1} \prod_{i=1}^{p} \|\nabla^{\alpha_i}\psi\|_\infty, \forall\, m \geq 1.$$

Obviously, for any $\psi \in C^m$, we have $\|[\psi]_m\|_\infty \leq C_m|\psi|_{m,\infty}$.

Lemme 3.1 *Let $m \geq 3$. For any $\epsilon > 0$, there exists $C_{\varepsilon,m} > 0$ such that for any $u \in H_0^3(\Omega)$ and $\zeta \in C^6(\overline{\Omega})$, there holds*

$$\int_\Omega \Big[(\Delta u)^2|\nabla\zeta^m|^2 + |\nabla u|^2|\nabla^2\zeta^m|^2\Big]dx \leq \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\varepsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx. \quad (3.1)$$

Proof. Using the equality $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2$, we have

$$\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx \leq \frac{1}{2}\int_\Omega u^2\Delta\left(|\nabla\zeta|^4\zeta^{2m-4}\right)dx + \int_\Omega |u||\Delta u||\nabla\zeta|^4\zeta^{2m-4}dx.$$

Applying Young's inequality, we get, for any $\epsilon > 0$

$$\int_\Omega |u\Delta u||\nabla\zeta|^4\zeta^{2m-4}dx \leq \varepsilon\int_\Omega (\Delta u)^2|\nabla\zeta|^2\zeta^{2m-2}dx + C_\varepsilon\int_\Omega u^2|\nabla\zeta|^6\zeta^{2m-6}dx.$$

So we get

$$\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx \leq \varepsilon\int_\Omega (\Delta u)^2|\nabla\zeta|^2\zeta^{2m-2}dx + C_\varepsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx. \quad (3.2)$$

On the other hand, direct integrations by parts yield (recall that $u \in H_0^3(\Omega)$)

$$\begin{aligned}
\int_\Omega (\Delta u)^2|\nabla\eta|^2dx &= -\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx - 2\int_\Omega \Delta u\nabla^2\eta(\nabla\eta, \nabla u)dx\\
&= -\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta, \nabla(\Delta u))dx\\
&\quad + 2\int_\Omega u\Delta u|\nabla^2\eta|^2dx + 2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx\\
&= -\int_\Omega \nabla u\nabla(\Delta u)|\nabla\eta|^2dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta, \nabla(\Delta u))dx\\
&\quad + \int_\Omega \Big[\Delta(u^2) - 2|\nabla u|^2\Big]|\nabla^2\eta|^2dx + 2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_\Omega \Big[(\Delta u)^2|\nabla\eta|^2 + 2|\nabla u|^2|\nabla^2\eta|^2\Big]dx = &+2\int_\Omega \Big[u\Delta u\nabla\eta\cdot\nabla(\Delta\eta) + u\nabla^2\eta(\nabla\eta, \nabla(\Delta u))\Big]dx\\
&+\int_\Omega \Big[u^2\Delta(|\nabla^2\eta|^2) - \nabla u\nabla(\Delta u)|\nabla\eta|^2\Big]dx.
\end{aligned} \quad (3.3)$$

Consider $\eta = \zeta^m$. For any $\epsilon > 0$, by Cauchy-Schwarz inequality, we have

$$-\int_\Omega \nabla u \nabla(\Delta u)|\nabla\eta|^2 dx + 2\int_\Omega u\nabla^2\eta(\nabla\eta, \nabla(\Delta u))dx$$
$$\leq \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\varepsilon\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx$$

and

$$2\int_\Omega u\Delta u\nabla\eta\cdot\nabla(\Delta\eta)dx \leq \epsilon\int_\Omega |\Delta u|^2|\nabla\zeta^m|^2dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Inserting the two above estimates in (3.3), one gets

$$(1-\epsilon)\int_\Omega (\Delta u)^2|\nabla\zeta^m|^2dx + \int_\Omega |\nabla u|^2|\nabla^2\zeta^m|^2dx$$
$$\leq \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\varepsilon\int_\Omega |\nabla u|^2|\nabla\zeta|^4\zeta^{2m-4}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Take another small enough ε in (3.2), there holds

$$(1-2\epsilon)\int_\Omega (\Delta u)^2|\nabla\zeta^m|^2dx + \int_\Omega |\nabla u|^2|\nabla^2\zeta^m|^2dx \leq \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

The proof is completed.

Using Lemma 3.1, we obtain also

Lemme 3.2 *Let $m \geq 3$. For any $0 < \epsilon < 1$, there exists $C_\epsilon > 0$ such that for any $u \in H_0^3(\Omega)$ and $\zeta \in C^6(\overline{\Omega})$,*

$$\int_\Omega \Big[|\nabla u|^2(\Delta\zeta^m)^2 + |\nabla^2 u|^2|\nabla\zeta^m|^2\Big]dx \leq \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Proof. From (2.14), we obtain

$$\begin{aligned}\int_\Omega |\nabla^2 u|^2|\nabla\zeta^m|^2dx &\leq \frac{1}{2}\int_\Omega |\nabla u|^2\Delta(|\nabla\zeta^m|^2)dx + m^2\int_\Omega |\nabla u\cdot\nabla(\Delta u)||\nabla\zeta|^2\zeta^{2m-2}dx\\ &\leq \int_\Omega |\nabla u|^2\Big[C_\varepsilon|\nabla\zeta|^4\zeta^{2m-4} + \nabla\zeta^m\nabla(\Delta\zeta^m)\Big]dx + \int_\Omega |\nabla u|^2|\nabla^2\zeta^m|^2dx\\ &\quad + \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx.\end{aligned} \tag{3.4}$$

Rewrite

$$C_\varepsilon|\nabla\zeta|^4\zeta^{2m-4} + \nabla\zeta^m\nabla(\Delta\zeta^m) = \zeta^{2m-4}\nabla\zeta\cdot\Psi$$

with a smooth function Ψ. In the spirit of (2.14), we have

$$\begin{aligned}\int_\Omega |\nabla u|^2\zeta^{2m-4}\nabla\zeta\cdot\Psi dx &\leq \frac{1}{2}\int_\Omega u^2\Delta(\zeta^{2m-4}\nabla\zeta\cdot\Psi)dx + \int_\Omega |u||\Delta u|\zeta^{2m-4}\nabla\zeta\cdot\Psi dx\\ &\leq \int_\Omega u^2\Big[|\Delta(\zeta^{2m-4}\nabla\zeta\cdot\Psi)| + C_\varepsilon|\Psi|^2\zeta^{2m-6}\Big]dx + \varepsilon\int_\Omega (\Delta u)^2|\nabla\zeta|^2\zeta^{2m-2}dx.\end{aligned}$$

Hence

$$\int_\Omega |\nabla u|^2\zeta^{2m-4}\nabla\zeta\cdot\Psi dx \leq C_\varepsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx + \varepsilon\int_\Omega (\Delta u)^2|\nabla\zeta^m|^2dx. \tag{3.5}$$

Combining (3.1) and (3.4)-(3.5), there holds

$$\int_\Omega |\nabla^2 u|^2|\nabla\zeta^m|^2 dx \le \varepsilon\int_\Omega |\nabla(\Delta u)|^2\zeta^{2m}dx + C_\epsilon\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx.$$

Furthermore, integrating by parts,

$$\begin{aligned}\int_\Omega |\nabla u|^2(\Delta\zeta^m)^2 dx =& -2\int_\Omega \nabla^2 u(\nabla u, \nabla\zeta^m)\Delta\zeta^m dx - \int_\Omega |\nabla u|^2\nabla(\Delta\zeta^m)\nabla\zeta^m dx\\ \le& \frac{1}{2}\int_\Omega |\nabla u|^2(\Delta\zeta^m)^2 dx + C\int_\Omega |\nabla^2 u|^2|\nabla\zeta^m|^2 dx\\ &+\frac{1}{2}\int_\Omega u^2\Delta\Big[\nabla(\Delta\zeta^m)\nabla\zeta^m\Big]dx + \int_\Omega |u||\Delta u|\nabla(\Delta\zeta^m)\nabla\zeta^m dx.\end{aligned}$$

We deduce that

$$\int_\Omega |\nabla u|^2(\Delta\zeta^m)^2 dx \le C\int_\Omega \Big[|\nabla^2 u|^2|\nabla\zeta^m|^2 + |\Delta u|^2|\nabla\zeta^m|^2\Big]dx + C\int_\Omega u^2[\zeta]_6\zeta^{2m-6}dx,$$

so using the previous estimates, we are done.

Let $R > 0$, $y \in \Omega_{1,R}\cup\Gamma(R)$, $0 < a < b$. Denote $A := A_a^b$ and $A_\rho := A_{a+\rho}^{b-\rho}$, similar to Lemma 2.3, we have

Lemme 3.3 *There exists a constant $C > 0$ depending only on N such that for any $u \in H_0^3(\Omega)$ and $0 < \rho < \min(1, \frac{b-a}{4})$, we have*

$$\|\Delta u\|^2_{L^2(A_\rho\cap\Omega)} \le C\left(\frac{1}{\rho^4}\|u\|^2_{L^2(A\cap\Omega)} + \|\nabla(\Delta u)\|^2_{L^2(A\cap\Omega)}\right).$$

Proof. Multiplying the equation $-\Delta u = \vartheta$, by $\vartheta\eta_\rho^{2m}$ and integrating by parts, we obtain

$$\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m} \le \int_{\Omega\cap A}|\nabla u||\nabla(\vartheta)|\eta_\rho^{2m} + 2m\int_{\Omega\cap A}(|\nabla u||\nabla\eta_\rho|)(|\vartheta|\eta_\rho^{2m-1}).$$

By Young's inequality, we derive that for any $\epsilon > 0$

$$(1-C\epsilon)\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m} \le C\epsilon\int_{\Omega\cap A}|\nabla u|^2\eta_\rho^{2m} + C_\epsilon\int_{\Omega\cap A}|\nabla(\vartheta)|^2\eta_\rho^{2m} + \frac{C}{\epsilon}\int_{\Omega\cap A}|\nabla u|^2|\nabla\eta_\rho|^2\eta_\rho^{2m-2}.$$

From (2.10), we deduce that

$$\begin{aligned}(1-C\epsilon)\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m} \le& C\epsilon\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + C_\epsilon\int_{\Omega\cap A}|\nabla(\vartheta)|^2\eta_\rho^{2m}\\ &+\frac{C}{\epsilon}\int_{\Omega\cap A}|\nabla u|^2|\nabla\eta_\rho|^2\eta_\rho^{2m-2}.\end{aligned}\tag{3.6}$$

Recall that $-\Delta u = \vartheta$ and using the equality $\Delta(u^2) = 2|\nabla u|^2 - 2u\vartheta$, there holds

$$\frac{C}{\epsilon}\int_{\Omega\cap A}|\nabla u|^2|\nabla\eta_\rho|^2\eta_\rho^{2m-2} \le \frac{C}{2\epsilon}\int_{\Omega\cap A}u^2|\Delta(|\nabla\eta_\rho|^2)| + \frac{C}{\epsilon}\int_{\Omega\cap A}u\vartheta|\nabla\eta_\rho|^2\eta_\rho^{2m-2}.$$

By Young's inequality, we get

$$\frac{C}{\epsilon}\int_{\Omega\cap A}|\nabla u|^2|\nabla\eta_\rho|^2\eta_\rho^{2m-2} \le C_\epsilon\int_{\Omega\cap A}u^2\left(|\Delta(|\nabla\eta_\rho|^2)| + |\nabla\eta_\rho|^4\right) + C\epsilon\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m}.\tag{3.7}$$

Combining (3.6)-(3.7) and choose ϵ small enough, we obtain

$$\int_{\Omega\cap A}\vartheta^2\eta_\rho^{2m} \le C\left(\int_{\Omega\cap A}u^2|\nabla\eta_\rho|^2 + \int_{\Omega\cap A}|\nabla(\vartheta)|^2\eta_\rho^{2m} + \int_{\Omega\cap A}u^2\left(|\Delta(|\nabla\eta_\rho|^2)| + |\nabla\eta_\rho|^4\right)\right).$$

We Choose the same η_ρ as before, in order to conclude easily the estimate for $\|\Delta u\|_{L^2(A_\rho\cap\Omega)}$, we obtain the claim.

3.2 Explicit estimate via Morse index

Lemme 3.4 *Let f satisfies (H_1) and u be a solution to (E_3) with finite Morse index $i(u)$. Then for any $y \in \Gamma(R) \cup \Omega_{1,R}$ with $R > 0$, there exists $j_0 \in \{1, 2, ..., 1 + i(u)\}$ such that*

$$\int_{A_{j_0}\cap\Omega} |\nabla(\Delta u)|^2 dx + \int_{A_{j_0}\cap\Omega} f(x, u)u dx \le C \left(\frac{1 + i(u)}{R}\right)^{\frac{6\mu+12}{\mu}}.$$

Proof. Take $\eta \in C^6(\overline{\Omega})$. By direct calculations, we get, as $u \in H_0^3(\Omega)$,

$$\begin{aligned}\int_\Omega [\nabla(\Delta(u\eta))]^2 dx &= \int_\Omega \left(\nabla(\Delta u)\eta + \Delta u \nabla\eta + 2\nabla^2 u \nabla\eta + \nabla u \Delta\eta + 2\nabla u \nabla^2\eta + u\nabla(\Delta\eta)\right)^2 \\ &\le (1+\epsilon)\int_\Omega |\nabla(\Delta u)|^2 \eta^2 dx \\ &\quad + C_\varepsilon \int_\Omega \Big[|\Delta u|^2|\nabla\eta|^2 + |\nabla^2 u|^2|\nabla\eta|^2 + |\nabla u|^2\left(|\nabla^2\eta|^2 + |\Delta\eta|^2\right) + u^2|\nabla(\Delta\eta)|^2\Big] dx.\end{aligned}$$

Using Lemmas 3.1-3.2, let $\eta = \zeta^m$ with $m = 3 + \frac{6}{\mu} > 3$, we derive that

$$\int_\Omega |\nabla(\Delta(u\zeta^m))|^2 dx \le (1+\varepsilon)\int_\Omega |\nabla(\Delta u)|^2 \zeta^{2m} dx + C_\varepsilon \int_\Omega u^2 [\zeta]_6 \zeta^{2m-6} dx.$$

As in section 2, we can easily check that $\{u\phi_j^m\}_{1\le j\le i(u)+1}$ are linearly independent, so there exists $j_0 \in \{1, 2, ..., 1 + i(u)\}$ such that $\Lambda_u(u\phi_{j_0}^m) \ge 0$. The above estimate with $\zeta = \phi_{j_0}$ implies then

$$\int_\Omega f'(x, u)u^2\phi_{j_0}^{2m} dx - (1+\epsilon)\int_\Omega |\nabla(\Delta u)|^2 \phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^6}(1 + i(u))^6 \int_\Omega u^2 \phi_{j_0}^{2m-6} dx. \tag{3.8}$$

Now, take $u\phi_{j_0}^{2m}$ as the test function for (E_3), the integration by parts yields that

$$\int_\Omega |\nabla(\Delta u)|^2 \phi_{j_0}^{2m} dx - \int_\Omega f(x, u)u\phi_{j_0}^{2m} dx = \int_\Omega \nabla(\Delta u) \cdot \left[\nabla\left(\Delta(u\phi_{j_0}^{2m})\right) - \nabla(\Delta u)\phi_{j_0}^{2m}\right] dx.$$

Developing the right hand side, applying again Lemmas 3.1-3.2, we can conclude: For any $\epsilon > 0$, there exists C_ϵ such that

$$(1-\epsilon)\int_\Omega |\nabla(\Delta u)|^2 \phi_{j_0}^{2m} dx - \int_\Omega f(x, u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^6}(1 + i(u))^6 \int_\Omega u^2 \phi_{j_0}^{2m-6} dx. \tag{3.9}$$

Multiplying (3.9) by $\frac{1+2\epsilon}{1-\epsilon}$ adding it with (3.8), we obtain from (H_1) that

$$\epsilon\int_\Omega |\nabla(\Delta u)|^2 \phi_{j_0}^{2m} dx + \left(\mu - \frac{3\epsilon}{1-\epsilon}\right)\int_\Omega f(x, u)u\phi_{j_0}^{2m} dx \le \frac{C_\epsilon}{R^6}(1 + i(u))^6 \int_\Omega u^2 \phi_{j_0}^{2m-6} dx + C.$$

Fix $0 < \epsilon < \frac{\mu}{3+\mu}$, we get

$$\int_\Omega |\nabla(\Delta u)|^2 \phi_{j_0}^{2m} dx + \int_\Omega f(x, u)u\phi_{j_0}^{2m} dx \le \frac{C}{R^6}(1 + i(u))^6 \int_\Omega u^2 \phi_{j_0}^{2m-6} dx + C.$$

By Young's inequality, for any $\epsilon' > 0$, there holds

$$\begin{aligned}\int_\Omega |\nabla(\Delta u)|^2 \phi_{j_0}^{2m} dx + \int_\Omega uf(x, u)\phi_{j_0}^{2m} dx &\le C_{\epsilon'}\left(\frac{1 + i(u)}{R}\right)^{\frac{6\mu+12}{\mu}} + \epsilon'\int_\Omega |u|^{\mu+2}\phi_{j_0}^{(m-3)(\mu+2)} dx \\ &\le C_{\epsilon'}\left(\frac{1 + i(u)}{R}\right)^{\frac{6\mu+12}{\mu}} + C\epsilon'\int_\Omega f(x, u)u\phi_{j_0}^{2m} dx.\end{aligned}$$

We used (1.5) and $(m - 3)(2 + \mu) = 2m$ for the last line. Take ϵ' small enough, the claim follows.

3.3 Proof of Theorem 3.1 for $k=3$

We show firstly the Pohozaev identity for (E_3).

Lemme 3.5 *Let u be solution to (E_3). Let $\psi \in C_c^4(B_R(y))$. Then*

$$\begin{aligned}
&N\int_\Omega F(x,u)\psi dx + \int_\Omega \nabla_x F(x,u)\cdot n\psi dx - \frac{N-6}{2}\int_\Omega |\nabla(\Delta u)|^2\psi dx\\
&= \frac{1}{2}\int_\Omega |\nabla(\Delta u)|^2(\nabla\psi\cdot n)dx - \int_\Omega F(x,u)\nabla\psi\cdot ndx\\
&\quad - \int_\Omega \Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)dx - 2\int_\Omega \nabla(\Delta u)\nabla\Big[\nabla^2 u(n,\nabla\psi) + \nabla u\nabla\psi\Big]dx\\
&\quad + \int_{\partial\Omega_R(y)} \frac{\partial\Delta u}{\partial\nu}\left(\nabla(\Delta u)\cdot n\right)\psi d\sigma - \frac{1}{2}\int_{\partial\Omega_R(y)} |\nabla(\Delta u)|^2(\nu\cdot n)\psi d\sigma.
\end{aligned}$$

For the boundary terms, we have

Lemme 3.6 *There exists $R_1>0$ depending only on Ω such that for any u smooth function in $H_0^3(\Omega)$, any $0<R<R_1$, $y\in\Gamma(R)$ and any nonnegative function ψ, there holds*

$$\int_{\partial\Omega_R(y)} \frac{\partial\Delta u}{\partial\nu}(\nabla(\Delta u)\cdot n)\psi d\sigma - \frac{1}{2}\int_{\partial\Omega_R(y)} |\nabla(\Delta u)|^2\nu\cdot n\psi d\sigma \le 0.$$

Proof. Take $R_1>0$ such that $v\cdot n\le 0$ on $\partial\Omega_R(y)$ for any $0<R\le R_1$ and $y\in\Gamma(R)$. As $u\in H_0^3(\Omega)$, we know that $\nabla(\Delta u)$ is parallel to ν on $\partial\Omega$, in other words $\nabla(\Delta u)(x)=\lambda(x)\nu(x)$ on $\partial\Omega$. Therefore

$$\frac{\partial\Delta u}{\partial\nu}(\nabla(\Delta u)\cdot n) - \frac{1}{2}(\nu\cdot n)|\nabla(\Delta u)|^2 = \frac{\lambda^2}{2}(\nu\cdot n)\le 0,\quad \forall\, x\in\partial\Omega_R(y).$$

So we are done.

Similar to Proposition 2.1, we can claim

Proposition 3.1 *There exists $R_0>0$ small who satisfies the following property: Let u be a classical solution of (E_3) with f verifying (H_1)-(H_3). Then for any $0<R\le R_0$, $y\in\Gamma(R)$ and $\zeta\in C_c^6(B_R(y))$ verifying $0\le\zeta\le 1$ and $\psi=\zeta^{2m}$ with $m\ge 3$, there holds*

$$\begin{aligned}
&\int_\Omega f(x,u)u\psi dx + \int_\Omega |\nabla(\Delta u)|^2\psi dx\\
&\le CR\|\nabla\zeta\|_\infty \int_{A_{R,\psi}(y)} f(x,u)udx + C\Big(1+R\|\nabla\zeta\|_\infty + R^2|\zeta|_{2,\infty}\Big)\|\nabla(\Delta u)\|^2_{L^2(A_{R,\psi}(y))}\\
&\quad + CR^2|\zeta|_{6,\infty}\|\nabla u\|^2_{L^2(A_{R,\psi}(y))} + C\Big(|\zeta|_{6,\infty} + R^2|\zeta|_{8,\infty}\Big)\|u\|^2_{L^2(A_{R,\psi}(y))}.
\end{aligned}\tag{3.10}$$

Proof. Using Lemmas 3.5- 3.6, (H_1)–(H_3) and by (1.4), we obtain

$$\begin{aligned}
&\frac{N-6}{2}\left[(1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega |\nabla(\Delta u)|^2\psi dx\right]\\
&\le CR\|\nabla\psi\|_\infty\int_{A_{R,\psi}(y)} |\nabla(\Delta u)|^2dx + CR\|\nabla\psi\|_\infty\int_{A_{R,\psi}(y)} f(x,u)udx\\
&\quad + \int_{A_{R,\psi}(y)} \Big|\nabla(\Delta u)\nabla\Big[\nabla^2u(n,\nabla\psi)+\nabla u\nabla\psi\Big]\Big|dx + CR^N\\
&\quad + CR\int_\Omega f(x,u)u\psi dx + \int_\Omega \Big|\Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)\Big|dx.
\end{aligned}\tag{3.11}$$

We will use also the following lemma.

Lemme 3.7 *For any $R<1$, $\psi=\zeta^{2m}$ with $\zeta\in C_c^6(B_R(y))$ in Proposition 3.1, there exists a positive constant C such that*

$$\begin{aligned}&\int_{A_{R,\psi}(y)}\Big|\nabla(\Delta u)\nabla\Big[\nabla^2u(n,\nabla\psi)+\nabla u\nabla\psi\Big]\Big|dx+\int_\Omega\Big|\Delta\psi\nabla(\Delta u)\nabla(n\cdot\nabla u)\Big|dx\\&\le C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2dx+CR^2\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2[\zeta]_2dx\\&+CR^2\int_{A_{R,\psi}(y)}|\nabla u|^2[\zeta]_6dx+\int_{A_{R,\psi}(y)}u^2\left([\zeta]_6+R^2[\zeta]_8\right)dx.\end{aligned}\tag{3.12}$$

Proof. Indeed, in $B_R(y)\cap\Omega$,

$$\begin{aligned}\Big|\nabla(\Delta u)\nabla\Big[\nabla^2u(n,\nabla\psi)+\nabla u\nabla\psi\Big]\Big|&\le CR|\nabla(\Delta u)|\Big(|\nabla^3(u\nabla\psi)|+|\nabla^2u||\nabla^2\psi|+|\nabla u||\nabla^3\psi|+|u||\nabla^4\psi|\Big)\\&+C|\nabla(\Delta u)|\Big(|\nabla^2u||\nabla\psi|+|\nabla u||\nabla^2\psi|\Big).\end{aligned}$$

We get then

$$\begin{aligned}&\int_{A_{R,\psi}(y)}\Big|\nabla(\Delta u)\nabla\Big[\nabla^2u(n,\nabla\psi)+\nabla u\nabla\psi\Big]\Big|dx\\&\le C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2dx+CR^2\int_{A_{R,\psi}(y)}|\nabla^3(u\nabla\psi)|^2dx+CR^2\int_{A_{R,\psi}(y)}|\nabla^2u|^2|\nabla^2\psi|^2dx\\&+CR^2\int_{A_{R,\psi}(y)}|\nabla u|^2|\nabla^3\psi|^2dx+CR^2\int_{A_{R,\psi}(y)}u^2|\nabla^4\psi|^2dx\\&+C\int_{A_{R,\psi}(y)}|\nabla^2u|^2|\nabla\psi|^2dx+C\int_{A_{R,\psi}(y)}|\nabla u|^2|\nabla^2\psi|^2dx.\end{aligned}$$

First, using Lemmas 3.1-3.2 on $A_{R,\psi}(y)\cap\Omega$, the last two terms can be upper bounded by

$$C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2dx+C\int_{A_{R,\psi}(y)}u^2[\zeta]_6dx.$$

Moreover, as $u\nabla\psi\in H_0^3(\Omega)$, there exists $C>0$ depending only on Ω such that

$$\int_{A_{R,\psi}(y)}|\nabla^3(u\nabla\psi)|^2dx=\int_\Omega|\nabla^3(u\nabla\psi)|^2dx\le C\int_\Omega|\nabla\Delta(u\nabla\psi)|^2dx=C\int_{A_{R,\psi}(y)}|\nabla\Delta(u\nabla\psi)|^2dx.$$

Remark that (as $\psi=\zeta^{2m}$)

$$\begin{aligned}|\nabla\Delta(u\nabla\psi)|^2&\le C\Big(|\nabla(\Delta u)|^2|\nabla\psi)|^2+|\nabla^2u|^2|\nabla^2\psi|^2+|\nabla u|^2|\nabla^3\psi|^2+u^2|\nabla^4\psi|^2\Big)\\&\le C\Big(|\nabla(\Delta u)|^2[\zeta]_2+|\nabla u|^2[\zeta]_6+u^2[\zeta]_8\Big)+C|\nabla^2u|^2|\nabla^2\psi|^2.\end{aligned}$$

Using the equality $2|\nabla^2u|^2=\Delta(|\nabla u|^2)-2\nabla u\cdot\nabla(\Delta u)$, we obtain

$$\begin{aligned}\int_{A_{R,\psi}(y)}|\nabla^2u|^2|\nabla^2\psi|^2dx&\le\frac12\int_{A_{R,\psi}(y)}|\nabla u|^2\Delta(|\nabla^2\psi|^2)dx+\int_{A_{R,\psi}(y)}|\nabla u\cdot\nabla(\Delta u)||\nabla^2\psi|^2dx\\&\le\frac12\int_{A_{R,\psi}(y)}|\nabla u|^2|\Delta(|\nabla^2\psi|^2)|dx+C\int_{A_{R,\psi}(y)}|\nabla(\Delta u)|^2|\nabla^2\psi|dx\\&+C\int_{A_{R,\psi}(y)}|\nabla u|^2|\nabla^2\psi|^3dx.\end{aligned}$$

Hence
$$\int_{A_{R,\psi}(y)} |\nabla^2 u|^2 |\nabla^2 \psi|^2 dx \le \int_{A_{R,\psi}(y)} |\nabla u|^2 [\zeta]_6 dx + C \int_{A_{R,\psi}(y)} |\nabla(\Delta u)|^2 [\zeta]_2 dx. \tag{3.13}$$
Combining all these inequalities, we obtain the estimate for the first left term in (3.12).

On the other hand,
$$\begin{aligned}&\int_{A_{R,\psi}(y)} \Big|\Delta\psi \nabla(\Delta u)\nabla(n\cdot\nabla u)\Big| dx\\ \le& \int_{A_{R,\psi}(y)} |\nabla(\Delta u)| \Big[R|\nabla^2 u||\Delta\psi| + |\nabla u||\Delta\psi|\Big] dx\\ \le& \int_{A_{R,\psi}(y)} |\nabla(\Delta u)|^2 dx + C\int_{A_{R,\psi}(y)} \Big[R^2|\nabla^2 u|^2|\nabla^2\psi|^2 + |\nabla u|^2(\Delta\psi)^2\Big] dx.\end{aligned}$$
Applying (3.13) and Lemma 3.2, the proof is completed.

Coming back to the proof of (3.10) . Take $u\zeta^{2m}$ as the test function for (E_3), using Lemmas 3.1-3.2, for any $\epsilon > 0$ there exists C_ϵ such that
$$\int_\Omega |\nabla(\Delta u)|^2 \zeta^{2m} dx - \int_\Omega f(x,u)u\zeta^{2m} dx \le \varepsilon \int_\Omega |\nabla(\Delta u)|^2\zeta^{2m} dx + C_\varepsilon \int_\Omega u^2[\zeta]_6 \zeta^{2m-6} dx. \tag{3.14}$$
Remark that
$$\begin{aligned}\frac{\theta}{2}\int_\Omega |\nabla(\Delta u)|^2\psi dx + \frac{\theta}{2}\int_\Omega f(x,u)u\psi dx =& (1+\theta)\int_\Omega f(x,u)u\psi dx - \int_\Omega |\nabla(\Delta u)|^2\psi dx\\ &+\left(1+\frac{\theta}{2}\right)\left[\int_\Omega |\nabla(\Delta u)|^2\psi dx - \int_\Omega f(x,u)u\psi dx\right].\end{aligned}$$
Combining (3.11)-(3.12) and (3.14), for $\varepsilon, R > 0$ small enough, we have (3.10).

Proof of Theorem 3.1 for $k = 3$ completed.

Now, we are in position to prove Theorem 3.1 for $k = 3$. Fix
$$R = R_0, \quad m = 3 + \frac{6}{\mu}, \quad \rho := \frac{R}{10(i(u)+1)}, \quad A_{j_0,\rho} := A_{a_{j_0}+\rho}^{b_{j_0}-\rho} \subset A_{j_0} \text{ be as in } (*).$$
Using Remark 3.2 and lemma 3.4, there holds
$$\|u\|^2_{L^2(A_{j_0}\cap\Omega)} \le C\left(\int_{A_{j_0}\cap\Omega} f(x,u)u\right)^{\frac{2}{2+\mu}} + C \le C(1+i(u))^{\frac{12}{\mu}}. \tag{3.15}$$
According to Lemmas 2.3, 3.3, 3.4 and (3.15), there exists a positive constant C independent of $y \in \Gamma(R)\cup\Omega_{1,R}$ such that
$$\|\nabla(\Delta u)\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} + \|\nabla u\|^2_{L^2(A_{j_0,\rho}\cap\Omega)} \le C(1+i(u))^{\frac{6\mu+12}{\mu}}. \tag{3.16}$$
Combining (3.10), (3.15) and (3.16), one obtains
$$\int_\Omega f(x,u)u\xi_{j_0} dx + \int_\Omega (\Delta u)^2\xi_{j_0} dx \le C(1+i(u))^{\frac{12\mu+12}{\mu}}.$$
As $\frac{R}{2} < a_{j_0}$ and $R = R_0$, we get then for any $y \in \Gamma(R)\cup\Omega_{1,R}$,
$$\int_{B_{\frac{R_0}{2}}(y)\cap\Omega} \Big[|\Delta u|^2 + f(x,u)u\Big] dx \le C(1+i(u))^{\frac{12\mu+12}{\mu}}.$$
The proof is completed by the covering argument.

Partie II

Théorème de type Liouville et système de Lane-Emden

Chapitre 4

Liouville theorems for stable solutions of the weighted Lane-Emden system

Sommaire

We examine the general weighted Lane-Emden system

$$-\Delta u = \rho(x)v^p, \quad -\Delta v = \rho(x)u^\theta, \quad u, v > 0 \quad \text{in } \mathbb{R}^N$$

where $1 < p \leq \theta$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ is a radial continuous function satisfying $\rho(x) \geq A(1+|x|^2)^{\frac{\alpha}{2}}$ in $\mathbb{R}^N$ for some $\alpha \geq 0$ and $A > 0$. We prove some Liouville type results for stable solution and improve the previous works Cowan [2013], Fazly [2012] and Hu [2015]. In particular, we establish a new comparison property (see Proposition 1.1 below) which is crucial to handle the case $1 < p \leq \frac{4}{3}$. Our results can be applied also to the weighted Lane-Emden equation $-\Delta u = \rho(x)u^p$ in $\mathbb{R}^N$.

1 Introduction

We consider the following weighted Lane-Emden system

$$-\Delta u = \rho(x)v^p, \quad -\Delta v = \rho(x)u^\theta, \quad u, v > 0 \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $1 < p \leq \theta$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ is a radial continuous function satisfying the following assumption;

($\star$) There exists $\alpha \geq 0$ and $A > 0$ such that $\rho(x) \geq A\rho_0(x)$ in $\mathbb{R}^N$ where $\rho_0 := A(1+|x|^2)^{\frac{\alpha}{2}}$.

Remark that under the scaling transformation $u = \gamma^{\frac{1}{\theta+1}}\widetilde{u}$, $v = \widetilde{v}$ with $\gamma > 0$, the following system

$$-\Delta\widetilde{u} = \widetilde{\rho}(x)\widetilde{v}^p, \quad -\Delta\widetilde{v} = \gamma\widetilde{\rho}(x)\widetilde{u}^\theta, \quad \widetilde{u},\widetilde{v} > 0 \quad \text{in } \mathbb{R}^N$$

is equivalent to (1.1) with $\rho = \gamma^{\frac{1}{\theta+1}}\widetilde{\rho}$.

To define the notion of stability, we consider a general system given by

$$-\Delta u = f(x,v), \quad -\Delta v = g(x,u), \quad \text{in } \mathbb{R}^N \tag{1.2}$$

with $f, g \in C^1(\mathbb{R}^{N+1},\mathbb{R})$ satisfying $f_s := \frac{\partial f(x,s)}{\partial s}, g_s = \frac{\partial f(x,s)}{\partial s} \geq 0$ in $\mathbb{R}$. A smooth solution (u,v) of (1.2) is said stable if there exist positive smooth functions ξ, ζ verifying

$$-\Delta\xi = f_v(x,v)\zeta, \quad -\Delta\zeta = g_u(x,u)\xi, \quad \text{in } \mathbb{R}^N.$$

This definition is motivated by Montenegro [2005], Fazly [2012] and Cowan [2013].

In this chapter, we prove the following classification result:

Theorem 4.1 *Suppose that ρ satisfies $(\star)$ and let x_0 be the largest root of the polynomial $H(x) =$*

$$x^4 - \frac{16p\theta(p+1)(\theta+1)}{(p\theta-1)^2}x^2 + \frac{16p\theta(p+1)(\theta+1)(p+\theta+2)}{(p\theta-1)^3}x - \frac{16p\theta(p+1)^2(\theta+1)^2}{(p\theta-1)^4}. \tag{1.3}$$

i) If $\frac{4}{3} < p \leq \theta$ then (1.1) has no stable classical solution if $N < 2 + (2+\alpha)x_0$. In particular, if $N \leq 10 + 4\alpha$, then (1.1) has no classical stable solution for all $\frac{4}{3} < p \leq \theta$.

ii) If $1 < p \leq \min(\frac{4}{3}, \theta)$, then (1.1) has no bounded classical stable solution, if

$$N < 2 + \left[\frac{p}{2} + \frac{(2-p)(p\theta-1)}{(\theta+p-2)(\theta+1)}\right](\alpha+2)x_0.$$

Therefore, if $N \leq 6 + 2\alpha$, the system (1.1) has no bounded classical stable solution for all $\theta \geq p > 1$.

As a consequence of Theorem 4.1, we obtain the following classification result for stable solution of the Lane-Emden equation

$$-\Delta u = \rho(x)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

Corollaire 1.1 *Suppose that ρ satisfies $(\star)$ and let $p > 1$.*

i) If $\frac{4}{3} < p$ then (1.4) has no stable classical solution if

$$N < 2 + \frac{2(2+\alpha)}{p-1}\left(p + \sqrt{p^2 - p}\right). \tag{1.5}$$

In particular, if $N \leq 10 + 4\alpha$, then (1.4) has no stable classical solution for all $\frac{4}{3} < p$.

ii) If $1 < p \leq \frac{4}{3}$, (1.4) has no bounded stable classical solution for N verifying (1.2).

Therefore, there is no bounded stable classical solution of (1.4) for all $p > 1$ if $N \leq 10 + 4\alpha$.

Recalling that for the autonomous case, i.e. when $\rho \equiv 1$, the stable solutions of the corresponding Lane-Emden equation and system, or the biharmonic equation (corresponding to $p = 1$) have been widely studied by many authors. See for instance works Farina [2007], Wei & D.Ye [2013], Cowan [2013], Harrabi *et al.* [2014], Chen *et al.* [2014a] and Davila *et al.* [2014] and the references there in.

For the second order Lane-Emden equation ($p > 1$)

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

Farina [2007] classified completely all finite Morse index classical solutions for $1 < p < p_{JL}$, where p_{JL} stands for the Joseph-Lundgren exponent Joseph & Lundgren [1973] (see also Gui *et al.* [1992]). More precisely, the equation (1.6) admits nontrivial classical solutions with finite Morse index if and only if $N \geq 3$, $p = \frac{N+2}{N-2}$ or $N \geq 11$ and $p \geq p_{JL}$. For the biharmornic equation ($p > 1$)

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N \tag{1.7}$$

with $p > 1$. Davila *et al.* [2014] recently gave a complete classification of finite Morse index solutions. They derived a monotonicity formula for the solutions of (1.7) and reduced to problem to the nonexistence of stable homogeneous solutions.

It is worthy to mention that Chen *et al.* [2014a] proved an optimal Liouville type result for the radial stable solutions of (1.1) for $\theta \geq p > 1$ and $\rho \equiv 1$.

For the weighted equation or system with positive weights, the Liouville type results are less understood.

- Using Farina's approach, Fazly proved the nonexistence of classical stable solutions of (1.1) for $\rho = \rho_0$, N satisfying (1.2) and $p \geq 2$. See Theorem 2.3 in Fazly [2012].

- Using also Farina's approach, Cowan & Fazly [2012] established a Liouville type result for classical stable sub-solutions of (1.1) for N satisfying (1.2), $p > 1$ and

$$\lim_{|x|\to\infty} \frac{\rho(x)}{\rho_0(x)} = C \in (0, \infty). \tag{1.8}$$

  See Theorem 1.3-(3) with $\alpha = 0$ in Cowan & Fazly [2012].

- Adopting the new approach of Cowan & Ghoussoub [2014], Hu [2015] proved the following Liouville theorem for classical stable solutions of (1.1) for $\rho = \rho_0$ and $\theta \geq p \geq 2$ or $\theta = p > \frac{4}{3}$, obtaining a direct extension of Theorem 1 in Cowan [2013] for $\rho \equiv 1$. More precisely, let t_0^+ and t_0^- be the quantities used by Cowan [2013]:

$$t_0^{\pm} = \sqrt{\frac{p\theta(p+1)}{\theta+1}} \pm \sqrt{\frac{p\theta(p+1)}{\theta+1} - \sqrt{\frac{p\theta(p+1)}{\theta+1}}},$$

Hu [2015] proved:

Theorem A *Suppose that $\rho = \rho_0$ with $\alpha \geq 0$.*

i) If $2t_0^- < p \leq \theta$ and N satisfies

$$N < 2 + \frac{2(2+\alpha)(\theta+1)}{p\theta - 1} t_0^+,$$

then there is no classical stable solution of (1.1). In particular there is no classical stable solution of (1.1) for any $2 \leq p \leq \theta$ and $N \leq 10 + 4\alpha$.

ii) If $p > \frac{4}{3}$ and N satisfies (1.2), then there is no classical stable solution of (1.4).

Remark 4.1 *It is known that for $1 < p \le \theta$, there hold $t_0^- < 1 < t_0^+$, t_0^- is decreasing and t_0^+ is increasing in $z := \frac{p\theta(p+1)}{\theta+1}$. Moreover, $\lim_{z\to\infty} t_0^- = \frac{1}{2}$ and $\lim_{z\to\infty} t_0^+ = 1$.*

Remark 4.2 *We have $2t_0^- < p$ if $p > \frac{4}{3}$. Indeed, if $p > \frac{4}{3}$ then $\theta \ge p > \frac{4}{3}$ and $z > \frac{16}{9}$. Since $f(z) := \sqrt{z} - \sqrt{z - \sqrt{z}}$ is decreasing in z, there holds $2t_0^- = 2f(z) < 2f(\frac{16}{9}) = \frac{4}{3} < p$.*

Using the above remark, we see that Theorem A (hence Theorem 1 in Cowan [2013]) can be extended immediately for $\frac{4}{3} < p \le \theta$.

- We can show that $2t_0^+ \frac{\theta+1}{p\theta-1} < x_0$ for any $1 < p \le \theta$ (see Lemma 2.4 below), where x_0 is the largest root of the polynomial H given by (1.3). So Theorem 4.1 improves the bound given in Theorem A.
- In Theorem 4.1 and Corollary 1.1, we prove a classification result for (1.1) with ρ satisfying ($\star$) without the restriction $\rho = \rho_0$ in Theorem A; or the condition (1.8) used by Cowan & Fazly [2012].
- Our approach permits to prove a Liouville type result for $\theta \ge p > 1$. To the best of our knowledge, no general Liouville type result was known for stable solution of (1.1) with positive weight for $1 < p \le \frac{4}{3}$.

To prove Theorem 4.1, we will use the following Souplet type estimate in Souplet [2009]. Its proof is the same as for Lemma 2.3 of Hu [2015] where we replace just ρ_0 by ρ, so we omit the details.

Lemme 1.1 *Let $\theta \ge p > 1$ and ρ satisfy ($\star$). Then any classical solution of (1.1) verifies*

$$u^{\theta+1} \le \frac{\theta+1}{p+1} v^{p+1} \quad in \ \mathbb{R}^N. \tag{1.9}$$

However, to handle the case $1 < p \le \frac{4}{3}$, we need the following new comparison property between u and v. It is somehow an inverse version of Souplet's estimate (1.9), and has its own interest.

Proposition 1.1 *Let $\theta \ge p > 1$ and suppose that ρ satisfies ($\star$). Let (u, v) be a classical solution of (1.1) and assume that u is bounded, then*

$$v \le \|u\|_\infty^{\frac{\theta-p}{p+1}} u.$$

This chapter is organized as follows. In section 2, we prove some preliminaries results, in particular we give the proof of Proposition 1.1. The proofs of Theorem 4.1 and Corollary 1.1 are given in section 3.

2 Preliminaries

In order to prove our results, we need some technical lemmas. In the following, C denotes always a generic positive constant independent on (u, v), which could be changed from one line to another. The ball of center 0 and radius $r > 0$ will be denoted by B_r.

2.1 Comparison property

In this subsection, we give the proofs of Proposition 1.1. First, we can adapt the proof of Lemma 2.1 of Fazly [2012] (which was inspired by the previous works Serrin & Zou [1996] and Mitidier & Pokhozhaev [2001]), to obtain the following integral estimates for all classical solutions of (1.1).

Lemme 2.1 *Let $p \geq 1$, $\theta > 1$ and suppose that ρ satisfies $(\star)$. For any classical solution (u,v) of (1.1) there exists $C > 0$ such that for any $R \geq 1$, there holds*

$$\int_{B_R} \rho(x) v^p dx \leq C R^{N - \frac{2(\theta+1)p}{p\theta-1} - \frac{(p+1)\alpha}{p\theta-1}}, \quad \int_{B_R} \rho(x) u^\theta dx \leq C R^{N - \frac{2(p+1)\theta}{p\theta-1} - \frac{(\theta+1)\alpha}{p\theta-1}}.$$

Proof. Let $\varphi_0 \in C_c^\infty(B_2)$ be a cut-off function verifying $0 \leq \varphi_0 \leq 1$, $\varphi_0 = 1$ for $|x| < 1$. Take $\psi := \varphi_0(R^{-1}x)$ for $R \geq 1$. Multiplying the equation $-\Delta u = \rho(x) v^p$ by ψ^m and integrating by parts, there holds then

$$\int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx = -\int_{\mathbb{R}^N} u \Delta(\psi^m) dx \leq \frac{C}{R^2} \int_{B_{2R} \setminus B_R} u \psi^{m-2} dx.$$

By Hölder's inequality, we get

$$\int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx \leq \frac{C}{R^2} \left(\int_{B_{2R} \setminus B_R} \rho(x)^{-\frac{\theta'}{\theta}} dx \right)^{\frac{1}{\theta'}} \left(\int_{B_{2R} \setminus B_R} \rho(x) u^\theta \psi^{(m-2)\theta} dx \right)^{\frac{1}{\theta}},$$

where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. From $(\star)$ we deduce that for $R \geq 1$,

$$\int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx \leq C R^{\frac{N}{\theta'} - \frac{\alpha}{\theta} - 2} \left(\int_{B_{2R} \setminus B_R} \rho(x) u^\theta \psi^{(m-2)\theta} dx \right)^{\frac{1}{\theta}}.$$

Similarly, using $-\Delta v = \rho(x) u^p$, we obtain, for $k \geq 2$,

$$\int_{\mathbb{R}^N} \rho(x) u^\theta \psi^k dx \leq C R^{\frac{N}{p'} - \frac{\alpha}{p} - 2} \left(\int_{B_{2R} \setminus B_R} \rho(x) v^p \psi^{(k-2)p} dx \right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Take now k and m large verifying $m \leq (k-2)p$ and $k \leq (m-2)\theta$. Combining the two above inequalities, we get

$$\begin{aligned} \int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx &\leq C R^{\frac{N}{\theta'} - \frac{\alpha}{\theta} - 2} R^{\left(\frac{N}{p'} - \frac{\alpha}{p} - 2\right)\frac{1}{\theta}} \left(\int_{B_{2R} \setminus B_R} \rho(x) v^p \psi^{(k-2)p} dx \right)^{\frac{1}{p\theta}} \\ &\leq C R^{N - \frac{N}{p\theta} - \frac{\alpha(p+1)}{p\theta} - \frac{2(\theta+1)}{\theta}} \left(\int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx \right)^{\frac{1}{p\theta}}. \end{aligned}$$

Hence

$$\int_{B_R} \rho(x) v^p \psi^m dx \leq \int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx \leq C R^{N - \frac{2(\theta+1)p}{p\theta-1} - \frac{(p+1)\alpha}{p\theta-1}}.$$

Similarly, we obtain the estimate for u.

Now we are in position to prove the inverse comparison property.

Proof of Proposition 1.1. Let $w = v - \lambda u$, where $\lambda = \|u\|_\infty^{\frac{\theta-p}{p+1}}$. We have, as $\theta \geq p$,

$$\Delta w = \rho(x)\left(\lambda v^p - u^\theta\right) = \rho(x)\left[\lambda v^p - \left(\frac{u}{\|u\|_\infty}\right)^\theta \|u\|_\infty^\theta\right] \geq \rho(x)\left[\lambda v^p - \left(\frac{u}{\|u\|_\infty}\right)^p \|u\|_\infty^\theta\right]$$
$$= \rho(x)\|u\|_\infty^{\theta-p}\left(\frac{\lambda v^p}{\|u\|_\infty^{\theta-p}} - u^p\right)$$
$$= \rho(x)\|u\|_\infty^{\theta-p}\left(\lambda^{-p}v^p - u^p\right).$$

It follows that $\Delta w \geq 0$ in the set $\{w \geq 0\}$. Consider $w_+ := \max(w, 0)$. Next, we split the proof into two cases.

Case 1: $p \geq 2$. For any $R > 0$, there holds

$$\begin{aligned}(p-1)\int_{B_R} w_+^{p-2}|\nabla w_+|^2 dx &= -\int_{B_R} w_+^{p-1}\Delta w dx + \int_{\partial B_R} w_+^{p-1}\frac{\partial w}{\partial \nu}d\sigma \\ &\leq \int_{\partial B_R} w_+^{p-1}\frac{\partial w}{\partial \nu}d\sigma \\ &= \frac{R^{N-1}}{2}f'(R)\end{aligned} \tag{2.1}$$

where

$$f(R) := \int_{S^{N-1}} w_+^p(R\sigma)d\sigma \leq \int_{S^{N-1}} v^p(R\sigma)d\sigma =: g(R).$$

Hereafter, S^{N-1} denotes by the unit sphere in $\mathbb{R}^N$. By Lemma 2.1, we derive that

$$\begin{aligned}\int_0^R r^{N-1}\int_{S^{N-1}} \rho(r\sigma)v^p(r\sigma)d\sigma dr &= \int_0^R r^{N-1}\rho(r)\int_{S^{N-1}} v^p(r\sigma)d\sigma dr \\ &\leq CR^{N-\frac{2(\theta+1)p}{p\theta-1}-\frac{(p+1)\alpha}{p\theta-1}} = o(R^N) \quad \text{as } R \to \infty.\end{aligned}$$

Using $(\star)$, there holds

$$\int_0^R r^{N-1+\alpha}g(r)dr = o(R^N) \quad \text{as } R \to \infty.$$

This implies that $\liminf_{r\to\infty} g(r) = 0$, hence $\liminf_{r\to\infty} f(r) = 0$. Consequently, there exist $R_i \to \infty$ such that $f'(R_i) \leq 0$. Take (2.1) with $R = R_i$ and let $i \to \infty$, we conclude that w_+ is constant in $\mathbb{R}^N$. If $w \equiv C > 0$ then $v \geq C > 0$ in $\mathbb{R}^N$, which contradicts Lemma 2.1. Hence $w_+ \equiv 0$ in $\mathbb{R}^N$, i.e. $v - \lambda u \leq 0$ in $\mathbb{R}^N$.

Case 2: $1 < p < 2$. For any $R > 0$ and $\epsilon > 0$, we have

$$\begin{aligned}(p-1)\int_{B_R}(\epsilon + w_+)^{p-2}|\nabla w_+|^2 dx &= -\int_{B_R}(\epsilon + w_+)^{p-1}\Delta w dx + \int_{\partial B_R}(\epsilon + w_+)^{p-1}\frac{\partial w}{\partial \nu}d\sigma \\ &\leq \int_{\partial B_R}(\epsilon + w_+)^{p-1}\frac{\partial w}{\partial \nu}d\sigma.\end{aligned}$$

Letting $\epsilon \to 0$ (passing to limit in the l.h.s. via monotone convergence and use the dominated convergence on the r.h.s.), we get always the estimate (2.1), which will lead to the same conclusion: $w_+ \equiv 0$ in $\mathbb{R}^N$.

2.2 Consequence of stability

With the ideas of Cowan & Ghoussoub [2014] and Dupaigne *et al.* [2013], we can proceed similarly as the proof of Lemma 2.1 in Hu [2015] and claim

Lemme 2.2 *If (u,v) is a nonnegative classical stable solution of (1.1), then*

$$\sqrt{p\theta}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}\phi^2dx \le \int_{\mathbb{R}^N}|\nabla\phi|^2dx, \quad \forall\, \phi\in C_c^1(\mathbb{R}^N). \tag{2.2}$$

The following Lemma is a consequence of the stability inequality (2.2) and Proposition 1.1. It plays an crucial role to handle the case $1<p\le\frac{4}{3}$. Here we use also some ideas coming from works Wei & D.Ye [2013] and Harrabi *et al.* [2014].

Lemme 2.3 *Let (u,v) be a stable solution to (1.1) with $1<p\le\min(\frac{4}{3},\theta)$. Assume that u is bounded and ρ satisfies $(\star)$, there holds*

$$\int_{B_R}\rho(x)v^2dx \le CR^{N-\frac{2(\theta+1)p}{p\theta-1}-\frac{(p+1)\alpha}{p\theta-1}-\frac{2(2+\alpha)(2-p)}{\theta+p-2}}, \quad \forall\, R>0. \tag{2.3}$$

Proof. Let (u,v) be a stable solution of (1.1), where u is bounded. Take $\eta\in C_c^\infty(\mathbb{R}^N)$. Multiplying $-\Delta v=\rho(x)u^\theta$ by $v\eta^2$ and integrating by parts, there holds

$$\int_{\mathbb{R}^N}|\nabla v|^2\eta^2dx = \int_{\mathbb{R}^N}\rho(x)u^\theta v\eta^2dx+\frac{1}{2}\int_{\mathbb{R}^N}v^2\Delta(\eta^2)dx.$$

Using Lemma 1.1, we get

$$\int_{\mathbb{R}^N}|\nabla v|^2\eta^2dx \le \sqrt{\frac{\theta+1}{p+1}}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p+1}{2}}v\eta^2dx+\frac{1}{2}\int_{\mathbb{R}^N}v^2\Delta(\eta^2)dx.$$

Set $\phi=v\eta$ in (2.2) and integrating by parts, we deduce that

$$\sqrt{p\theta}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}v^2\eta^2dx \le \int_{\mathbb{R}^N}|\nabla v|^2\eta^2dx+\int_{\mathbb{R}^N}v^2|\nabla\eta|^2dx-\frac{1}{2}\int_{\mathbb{R}^N}v^2\Delta(\eta^2)dx.$$

Combining the two last inequalities, we obtain

$$\left(\sqrt{p\theta}-\sqrt{\frac{\theta+1}{p+1}}\right)\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p+3}{2}}\eta^2dx \le \int_{\mathbb{R}^N}v^2|\nabla\eta|^2dx.$$

Using Proposition 1.1, there exists a positive constant C such that

$$\int_{\mathbb{R}^N}\rho(x)v^{\frac{\theta+p+2}{2}}\eta^2dx \le C\int_{\mathbb{R}^N}v^2|\nabla\eta|^2dx.$$

Take φ_0 a cut-off function in $C_c^\infty(B_2)$ verifying $0\le\varphi_0\le 1$, $\varphi_0=1$ for $|x|<1$. Let $\eta=\varphi^m$ with $\varphi:=\varphi_0(R^{-1}x)$ for $R>0$, we arrive at

$$\int_{\mathbb{R}^N}\rho(x)v^{\frac{\theta+p+2}{2}}\varphi^{2m}dx \le \frac{C}{R^2}\int_{B_{2R}\setminus B_R}v^2\varphi^{2m-2}dx.$$

Using $(\star)$, there holds

$$\int_{\mathbb{R}^N}\rho(x)v^{\frac{\theta+p+2}{2}}\varphi^{2m}dx \le \frac{C}{R^{2+\alpha}}\int_{B_{2R}\setminus B_R}\rho(x)v^2\varphi^{2m-2}dx \le \frac{C}{R^{2+\alpha}}\int_{\mathbb{R}^N}\rho(x)v^2\varphi^{2m-2}dx. \tag{2.4}$$

Denote

$$J_1 := \int_{\mathbb{R}^N} \rho(x) v^{\frac{\theta+p+2}{2}} \varphi^{2m} dx, \quad J_2 := \int_{\mathbb{R}^N} \rho(x) v^2 \varphi^{2m-2} dx.$$

Remark that $p < 2 < \frac{\theta+p+2}{2}$ for $1 < p \leq \frac{4}{3}$ and $\theta \geq p$. A direct calculation yields

$$2 = p\lambda + \frac{\theta+p+2}{2}(1-\lambda) \quad \text{with } \lambda = \frac{\theta+p-2}{\theta+2-p} \in (0,1).$$

Take m large such that $m\lambda > 1$. By Hölder's inequality, Lemma 2.1 and (2.4), we get

$$\begin{aligned} J_2 \leq J_1^{1-\lambda} \left(\int_{\mathbb{R}^N} \rho(x) v^p \varphi^{2m\lambda-2} dx \right)^{\lambda} &\leq \left(\frac{CJ_2}{R^{2+\alpha}} \right)^{1-\lambda} \left(\int_{B_{2R}} \rho(x) v^p dx \right)^{\lambda} \\ &\leq C' J_2^{1-\lambda} R^{-(2+\alpha)(1-\lambda)} \left(R^{N - \frac{2(\theta+1)p}{p\theta-1} - \frac{(p+1)\alpha}{p\theta-1}} \right)^{\lambda}, \end{aligned}$$

which implies

$$J_2 \leq C R^{N - \frac{2(\theta+1)p}{p\theta-1} - \frac{(p+1)\alpha}{p\theta-1} - \frac{2(2+\alpha)(2-p)}{\theta+p-2}},$$

so we are done.

2.3 Property of the polynomial H

Consider the polynomial H given by (1.3). Performing the change of variables $x = \frac{\theta+1}{p\theta-1}s$, a direct computation yields

$$H(x) = \left(\frac{\theta+1}{p\theta-1} \right)^4 L(s)$$

where

$$L(s) := s^4 - \frac{16p\theta(p+1)}{\theta+1} s^2 + \frac{16p\theta(p+1)(p+\theta+2)}{(\theta+1)^2} s - \frac{16p\theta(p+1)^2}{(\theta+1)^2}. \tag{2.5}$$

Hence $H(x) < 0$ if and only if $L(s) < 0$.

Lemme 2.4 *Let $1 < p \leq \theta$, then $L(2) < 0$ and L has a unique root s_0 in $(2, \infty)$ and $2t_0^+ < s_0$. Moreover, if $p > \frac{4}{3}$, then $L(p) < 0$ and s_0 is the unique root of L in (p, ∞).*

Proof. Using $1 < p \leq \theta$,

$$\begin{aligned} L(2) &= 16 - \frac{64p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)(p+\theta+2)}{(\theta+1)^2} - \frac{16p\theta(p+1)^2}{(\theta+1)^2} \\ &= 16 - \frac{64p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)^2}{(\theta+1)^2} - \frac{16p\theta(p+1)^2}{(\theta+1)^2} \\ &= 16 - \frac{32p\theta(p+1)}{(\theta+1)} + \frac{16p\theta(p+1)^2}{(\theta+1)^2} \\ &\leq 16 - \frac{32p\theta(p+1)}{(\theta+1)} + \frac{16p\theta(p+1)}{(\theta+1)} \\ &= 16 \frac{(1-p^2)\theta + (1-p\theta)}{(\theta+1)} < 0. \end{aligned}$$

Very similarly, we can check that

$$L'(2) \leq 32 - \frac{32p\theta(p+1)}{(\theta+1)} < 0.$$

Furthermore, we have

$$L''(s) = 12s^2 - \frac{32p\theta(p+1)}{\theta+1},$$

then L'' can change at most once the sign from negative to positive for $s \geq 2$. As $\lim_{s\to\infty} L(s) = \infty$, it's clear that L admits a unique root in $(2, \infty)$. Moreover, we can check that

$$L(2t_0^+) = \frac{16p\theta(p+1)(\theta-p)}{(\theta+1)^2}(1-2t_0^+) < 0.$$

Hence, there holds $2t_0^+ < s_0$.

Now we consider $L(p)$. Rewrite

$$L(s) = s^4 - 16\frac{p\theta(p+1)}{\theta+1}\left(s^2 - \frac{p+\theta+2}{\theta+1}s + \frac{p+1}{\theta+1}\right).$$

For $s > 1$, we see that

$$\left(s^2 - \frac{p+\theta+2}{\theta+1}s + \frac{p+1}{\theta+1}\right)'_\theta = \frac{p+1}{(\theta+1)^2}(s-1) > 0,$$

Then for $s > 1$, as $\theta \geq p > 1$, there holds

$$s^2 - \frac{p+\theta+2}{\theta+1}s + \frac{p+1}{\theta+1} > s^2 - 2s + 1 = (s-1)^2 \quad \text{and} \quad \frac{p\theta(p+1)}{\theta+1} \geq p^2.$$

Finally, we get (for $p > 1$)

$$L(p) < p^4 - 16p^2(p-1)^2 = p^2(5p-4)(4-3p)$$

and

$$L'(p) = 4p^3 - 16\frac{p\theta(p+1)}{\theta+1}\left(2p - \frac{p+\theta+2}{\theta+1}\right) < 4p^3 - 16p^2(2p-2) = 4p^2(8-7p),$$

We check readily that for $p > \frac{4}{3}$, $L(p) < 0$ and $L'(p) < 0$, so we can conclude as above.

3 Proofs of Theorem 4.1 and Corollary 3.1.

The following lemma plays an important role in dealing with Theorems 4.1 and Corollary 3.1, where we use some ideas from Harrabi *et al.* [2014]. Here and in the following, we define $R_k = 2^k R$ for all $R > 0$ and integers $k \geq 1$.

Lemme 3.1 *Suppose that ρ satisfies $(\star)$ and let (u, v) be a stable solution of (1.1). Then for any $s > \frac{p+1}{2}$ verifying $L(s) < 0$, there exists $C < \infty$ such that*

$$\int_{B_R} \rho(x) u^\theta v^{s-1} dx \leq \frac{C}{R^2}\int_{B_{2R}} v^s dx, \quad \forall\, R > 0.$$

Proof. Take $\phi \in C_0^2(\mathbb{R}^N)$. Let (u, v) be a stable solution of (1.1), the integration by parts yields that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u^{\frac{q+1}{2}}|^2\phi^2 dx &= \frac{(q+1)^2}{4}\int_{\mathbb{R}^N} u^{q-1}|\nabla u|^2\phi^2 dx \\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} \phi^2\nabla(u^q)\nabla u dx \\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} \rho(x)u^q v^P\phi^2 dx + \frac{q+1}{4q}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)dx,\end{aligned} \tag{3.1}$$

and

$$(q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\nabla\phi dx = \frac{1}{2}\int_{\mathbb{R}^N}\nabla(u^{q+1})\nabla(\phi^2)dx = -\frac{1}{2}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)dx. \tag{3.2}$$

Take $\varphi = u^{\frac{q+1}{2}}\phi$ with $q > 0$ into the stability inequality (2.2) and using (3.1)-(3.2), we obtain

$$\begin{aligned}\sqrt{p\theta}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{q+1}\phi^2 dx &\le \int_{\mathbb{R}^N}|\nabla\varphi|^2 dx \\ &\le \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N}\rho(x)u^q v^p\phi^2 dx + C\int_{\mathbb{R}^N} u^{q+1}\Big[|\nabla\phi|^2 + \Delta(\phi^2)\Big]dx,\end{aligned}$$

so we get

$$a_1\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{q+1}\phi^2 dx \le \int_{\mathbb{R}^N}\rho(x)u^q v^p\phi^2 dx + C\int_{\mathbb{R}^N} u^{q+1}\Big[|\nabla\phi|^2 + \Delta(\phi^2)\Big]dx,$$

with $a_1 = \frac{4q\sqrt{p\theta}}{(q+1)^2}$. Choose now $\phi(x) = \varphi_0(R^{-1}x)$ where $\varphi_0 \in C_c^\infty(B_2)$ such that $\varphi_0 \equiv 1$ in B_1, there holds then

$$\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{q+1}\phi^2 dx \le \frac{1}{a_1}\int_{\mathbb{R}^N}\rho(x)u^q v^p\phi^2 dx + \frac{C}{R^2}\int_{B_{2R}} u^{q+1}dx. \tag{3.3}$$

Similarly, applying the stability inequality (2.2) with $\varphi = v^{\frac{r+1}{2}}\phi$, $r > 0$, we obtain

$$\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}v^{r+1}\phi^2 dx \le \frac{1}{a_2}\int_{\mathbb{R}^N}\rho(x)u^\theta v^r\phi^2 dx + \frac{C}{R^2}dx\int_{B_{2R}} v^{r+1}dx \tag{3.4}$$

with $a_2 = \frac{4r\sqrt{p\theta}}{(r+1)^2}$. Combining (3.3) and (3.4),

$$\begin{aligned}& I_1 + a_2^{\frac{2(r+1)}{p+1}} I_2 \\ &:= \int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{q+1}\phi^2 dx + a_2^{\frac{2(r+1)}{p+1}}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}v^{r+1}\phi^2 dx \\ &\le \frac{1}{a_1}\int_{\mathbb{R}^N}\rho(x)u^q v^p\phi^2 dx + a_2^{\frac{2r+1-p}{p+1}}\int_{\mathbb{R}^N}\rho(x)u^\theta v^r\phi^2 dx + \frac{C}{R^2}\int_{B_{2R}}\left(u^{q+1} + v^{r+1}\right)dx.\end{aligned} \tag{3.5}$$

Fix now

$$q = \frac{(\theta+1)r}{p+1} + \frac{\theta-p}{p+1}, \quad \text{or equivalently } q+1 = \frac{(\theta+1)(r+1)}{p+1}. \tag{3.6}$$

Let $r > \frac{p-1}{2}$, by Young's inequality, there holds

$$
\begin{aligned}
&\frac{1}{a_1}\int_{\mathbb{R}^N}\rho(x)u^q v^p\phi^2 dx\\
&=\frac{1}{a_1}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{\frac{(\theta+1)r}{p+1}dx+\frac{\theta+1}{p+1}\left(\frac{1-p}{2}\right)}v^{\frac{p+1}{2}}\phi^2 dx\\
&=\frac{1}{a_1}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{(q+1)\frac{2r+1-p}{2(r+1)}}v^{\frac{p+1}{2}}\phi^2 dx\\
&\le\frac{2r+1-p}{2(r+1)}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}u^{q+1}\phi^2 dx+\frac{p+1}{2(r+1)}a_1^{-\frac{2(r+1)}{p+1}}\int_{\mathbb{R}^N}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}v^{r+1}\phi^2 dx\\
&=\frac{2r+1-p}{2(r+1)}I_1+\frac{p+1}{2(r+1)}a_1^{-\frac{2(r+1)}{p+1}}I_2;
\end{aligned}
$$

and similarly we have

$$
a_2^{\frac{2r+1-p}{p+1}}\int_{\mathbb{R}^N}\rho(x)u^\theta v^r\phi^2 dx\le\frac{p+1}{2(r+1)}I_1+\frac{2r+1-p}{2(r+1)}a_2^{\frac{2(r+1)}{p+1}}I_2.
$$

Combining the above two estimates with (3.5), we derive that

$$
a_2^{\frac{2(r+1)}{p+1}}I_2\le\left[\frac{2r+1-p}{2(r+1)}a_2^{\frac{2(r+1)}{p+1}}+\frac{p+1}{2(r+1)}a_1^{\frac{-2(r+1)}{p+1}}\right]I_2+\frac{C}{R^2}\int_{B_{2R}}\left(u^{q+1}+v^{r+1}\right)dx,
$$

hence

$$
\frac{p+1}{2(r+1)}\left[(a_1a_2)^{\frac{2(r+1)}{p+1}}-1\right]I_2\le CR^{-2}a_1^{\frac{2(r+1)}{p+1}}\int_{B_{2R}}\left(u^{q+1}+v^{r+1}\right)dx.
$$

Thus, if $a_1a_2 > 1$, by the choice of ϕ,

$$
\int_{B_R}\rho(x)u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}v^{r+1}dx\le I_2\le\frac{C}{R^2}\int_{B_{2R}}\left(u^{q+1}+v^{r+1}\right)dx.
$$

Using (3.6) and (1.9), there hold $u^{q+1}\le Cv^{r+1}$ and $u^{\frac{\theta-1}{2}}v^{\frac{p-1}{2}}v^{r+1}\ge u^\theta v^r$. Denote $s = r+1$, we conclude that if $a_1a_2 > 1$ and $s > \frac{p+1}{2}$,

$$
\int_{B_R}\rho(x)u^\theta v^{s-1}dx\le\frac{C}{R^2}\int_{B_{2R}}v^s dx.
$$

Furthermore, we can check that $a_1a_2 > 1$ is equivalent to $L(s) < 0$, the proof is completed. We need also the following L^1 elliptic regularity result, see Lemma 5 in Cowan [2013].

Lemme 3.2 *For any $1 \le \beta < \frac{N}{N-2}$, there exists $C > 0$ such that for any smooth non-negative function w, we have*

$$
\left(\int_{B_{R_k}}w^\beta\right)^{\frac{1}{\beta}}dx\le CR^{N\left(\frac{1}{\beta}-1\right)+2}\int_{B_{R_{k+1}}}|w|dx+CR^{N\left(\frac{1}{\beta}-1\right)}\int_{B_{R_{k+1}}}wdx.
$$

Applying the above two lemmas, we establish the following result which plays an essential role in iteration process.

Lemme 3.3 *Suppose that ρ satisfies $(\star)$ and let (u,v) be a classical stable solution of (1.1), with $1 < p \le \theta$. Then for any $1 \le \lambda < \frac{N}{N-2}$, $2t_0^- < q < s_0$ and nonnegative integer $k \ge 1$, there holds*

$$\left(\int_{B_{R_k}} v^{q\lambda}\right)^{\frac{1}{\lambda}} dx \le CR^{N\left(\frac{1}{\lambda}-1\right)} \int_{B_{R_{k+2}}} v^q dx, \quad \textit{for all } R \ge 1. \tag{3.7}$$

Proof. A simple calculation gives

$$|\Delta(v^q)| \le q(q-1)v^{q-2}|\nabla v|^2 + q\rho(x)v^{q-1}u^\theta.$$

Using Lemma 3.2, ewe get

$$\begin{aligned}\left(\int_{B_{R_k}} v^{q\lambda}\right)^{\frac{1}{\lambda}} dx \le CR^{N\left(\frac{1}{\lambda}-1\right)+2} &\int_{B_{R_{k+1}}} v^{q-2}|\nabla v|^2 dx \\ + CR^{N\left(\frac{1}{\lambda}-1\right)+2} &\int_{B_{R_{k+1}}} \rho(x) v^{q-1}u^\theta dx \\ + CR^{N\left(\frac{1}{\lambda}-1\right)} &\int_{B_{R_{k+1}}} v^q dx.\end{aligned} \tag{3.8}$$

Now, take a cut-off function $\phi \in C_0^2(B_{R_{k+2}})$ verifying $\phi \equiv 1$ in $B_{R_{k+1}}$ and $|\nabla\phi| \le \frac{C}{R}$. Multiplying $-\Delta v = \rho(x)u^\theta$ by $v^{q-1}\phi^2$ and integrating by parts, we have

$$(q-1)\int_{\mathbb{R}^N} v^{q-2}|\nabla v|^2\phi^2 dx = -2\int_{\mathbb{R}^N} v^{q-1}\phi\nabla v\nabla\phi dx + \int_{\mathbb{R}^N} \rho(x)v^{q-1}u^\theta\phi^2 dx. \tag{3.9}$$

By Young's inequality,

$$2\int_{\mathbb{R}^N} v^{q-1}|\nabla v||\nabla\phi|\phi dx \le \frac{q-1}{2}\int_{\mathbb{R}^N} v^{q-2}|\nabla v|^2\phi^2 dx + C\int_{\mathbb{R}^N} v^q|\nabla\phi|^2 dx.$$

Inserting this into (3.9), using the properties of ϕ, we obtain

$$\int_{\mathbb{R}^N} v^{q-2}|\nabla v|^2\phi^2 dx \le C\int_{B_{R_{k+2}}} \rho(x)v^{q-1}u^\theta dx + \frac{C}{R^2}\int_{B_{R_{k+2}}} v^q dx.$$

Substituting the above inequality into (3.8), there holds

$$\left(\int_{B_{R_k}} v^{q\lambda}\right)^{\frac{1}{\lambda}} dx \le CR^{N\left(\frac{1}{\lambda}-1\right)+2}\int_{B_{R_{k+2}}} \rho(x)v^{q-1}u^\theta dx + CR^{N\left(\frac{1}{\lambda}-1\right)}\int_{B_{R_{k+2}}} v^q dx.$$

Since ρ satisfies $(\star)$, we can use Lemmas 3.1 to find (3.7). Now, we can follow exactly the iteration process as for Corollary 2 in Cowan [2013] (see also Proposition 3.1 in Hu [2015]) to obtain

Corollaire 3.1 *Suppose that $1 < p \le \theta$ and ρ satisfies $(\star)$. Let (u,v) be a classical stable solution of (1.1) and $q \in (2t_0^-, s_0)$, then for $q \le \beta < \frac{N}{N-2}s_0$, there are $\ell \in \mathbb{N}$ and $C < \infty$ such that for any $R > 0$,*

$$\left(\int_{B_R} v^\beta dx\right)^{\frac{1}{\beta}} \le CR^{N\left(\frac{1}{\beta}-\frac{1}{q}\right)}\left(\int_{B_{R_\ell}} v^q dx\right)^{\frac{1}{q}}, \quad \textit{with } R_\ell = 2^\ell R.$$

Now we are in position to complete the proof of Theorem 4.1.

Proof of Theorem 4.1 completed. Let (u,v) be a classical stable solution of (1.1) with ρ satisfying $(\star)$. We split the proof into two cases.

Case 1: $p > \frac{4}{3}$. Let $p > q > 0$. Using Hölder's inequality, there holds

$$\int_{B_R} v^q dx \le \left(\int_{B_R} \rho v^p dx\right)^{\frac{q}{p}} \left(\int_{B_R} \rho^{-\frac{p}{p-q}} dx\right)^{\frac{p-q}{p}}. \tag{3.10}$$

Applying Lemma 2.1, from $(\star)$ we get

$$\int_{B_R} v^q dx \le CR^{\left[N-\frac{2(\theta+1)p}{p\theta-1}-\frac{(p+1)\alpha}{p\theta-1}\right]\frac{q}{p}+\left(N-\frac{\alpha q}{p-q}\right)\frac{p-q}{p}} = CR^{N-\frac{(2+\alpha)(\theta+1)}{p\theta-1}q}.$$

By Remark 4.1, we known that $2t_0^- < p$, then applying Corollary 3.1 with $2t_0^- < q < p$ and combining with Lemma 2.1, we can claim that for any $p \le \beta < \frac{N}{N-2}s_0$, there exists $C > 0$ such that

$$\left(\int_{B_R} v^\beta dx\right)^{\frac{1}{\beta}} \le CR^{N\left(\frac{1}{\beta}-\frac{1}{q}\right)+\frac{1}{q}\left[N-\frac{(2+\alpha)(\theta+1)}{p\theta-1}q\right]}. \tag{3.11}$$

Note that

$$N\left(\frac{1}{\beta}-\frac{1}{q}\right)+\frac{1}{q}\left[N-\frac{(2+\alpha)(\theta+1)}{p\theta-1}q\right] < 0 \quad \Leftrightarrow \quad N < \frac{(2+\alpha)(\theta+1)}{p\theta-1}\beta.$$

Suppose now

$$N < 2 + \left(\frac{(2+\alpha)(\theta+1)}{p\theta-1}\right)s_0.$$

We can take β small but close to $\frac{N}{N-2}s_0$ such that $N < \frac{(2+\alpha)(\theta+1)}{p\theta-1}\beta$. With a such β, tending $R \to \infty$ in (3.11), we get $\|v\|_{L^\beta(\mathbb{R}^N)} = 0$, this is just impossible since v is positive. In other words, the equation (1.1) has no classical stable solution if $N < 2+(2+\alpha)x_0$ where $x_0 = \frac{\theta+1}{p\theta-1}s_0$.

Moreover, adopting the proof of Remark 2 in Cowan [2012], we can easily show that

$$2t_0^+\frac{\theta+1}{p\theta-1} > 4, \quad \forall\, \theta \ge p > 1.$$

By Lemma 2.4, $x_0 > 2t_0^+\frac{\theta+1}{p\theta-1} > 4$. This means that if $N \le 10+4\alpha$, (1.1) has no classical stable solution for any $\theta \ge p > \frac{4}{3}$.

Case 2: $1 < p \le \frac{4}{3}$ *and u is bounded.* Let now $2 > q > 0$, using (3.10), with p is replaced by 2 and applying Lemma 2.3, it follows that for any $R > 1$,

$$\begin{aligned}\int_{B_R} v^q dx &\le CR^{\left[N-\frac{2(\theta+1)p}{p\theta-1}-\frac{(p+1)\alpha}{p\theta-1}-\frac{2(2+\alpha)(2-p)}{\theta+p-2}\right]\frac{q}{2}+\left(N-\frac{\alpha q}{2-q}\right)\frac{2-q}{2}}\\ &= CR^{N-\left[\frac{(\theta+1)p}{p\theta-1}+\frac{(2+\alpha)(2-p)}{\theta+p-2}+\frac{p(\theta+1)\alpha}{2(p\theta-1)}\right]q},\end{aligned}$$

Proceeding as above, we can apply Corollary 3.1, with $2t_0^- < q < 2$ and $q < \beta < \frac{Ns_0}{N-2}$ to complete the proof of Theorem 1.6.

Proof of Corollary 3.1. Let u be a solution of the weighted Lane-Emden equation (1.4), then $v = u$ verify the system (1.1) with $p = \theta$. Remark that u is stable for (1.4) means just the estimate (2.2) holds true with $v = u$ and $p = \theta$, which is the departure point of our study. Moreover, we have

$$t_0^{\pm} = p \pm \sqrt{p^2 - p}$$

and

$$L(s) = s^4 - 16p^2 s^2 + 32p^2 s - 16p^2 = (s^2 + 4p(s-1))(s - 2t_0^-)(s - 2t_0^+).$$

Then, $2t_0^+$ is the largest root of L as $t_0^+ > p > 1$. Therefore

$$x_0 = \frac{2p + 2\sqrt{p^2 - p}}{p - 1}$$

is the largest root of H, and we can check easily that $x_0 > 4$ for all $p > 1$. The result follows immediately by applying Theorem 4.1.

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