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Observateurs dynamiques et commande des systèmes, application aux systèmes de grande dimension

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To my family. . .

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Notation and acronyms

Sets and Norms

$\Re^n$	Set of n-dimensional real vectors
$\Re^{n\times m}$	Set of n×m dimensional real matrices
$\mathbb{F}_c$	Field of complex numbers
$\Vert\cdot\Vert_2$	The Euclidean vector norm
$\Vert\cdot\Vert_\infty$	The H_∞ norm
$Re(\lambda)$	The real part of eigenvalue λ

Matrices and Operators

$A > 0$	Real symmetric positive-definite matrix A
I	Identity matrix of appropriate dimension
I_n	Identity matrix of dimension $n \times n$
0	Null matrix of appropriate dimension
0_n	Null matrix of dimension $n \times n$
A^{-1}	Inverse of matrix $A \in \mathbb{R}^{n\times n}$, $\det A \neq 0$
A^T	Transpose of matrix A
$A^\perp$	Any matrix such that $A^\perp A = 0$ and $A^\perp A^{\perp^T} > 0$
A^+	Generalized inverse of matrix A satisfying $AA^+A = A$
$(\star)$	Block induced by symmetry
$rank\ A$	Rank of matrix A
$diag(a_i)$	A diagonal matrix with a_i as its i-th diagonal element

Acronyms

BMI	Bilinear Matrix inequality
DGCS	Decentralized Guaranteed Cost Stabilization
DO	Dynamic Observer
HDO	H_∞ dynamic observer
IGBT	Insulated Gate Bipolar Transistor
LMI	Linear Matrix Inequality
LSM	Linear Stepping Motor
LSS	Large-scale System
LTI	Linear Time Invariant
LTV	Linear Time Varying
PO	Proportional Observer
PI	Proportional Integral
PID	Proportional Integral Derivative
PIO	Proportional Integral Observer
SISO	Single-input Single-output
T-S	Takagi-Sugeno
UIO	Unknown Input Observer

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CHAPTER

1

Introduction

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1.1 Introduction

The focus of this dissertation is to develop and design a new class of observers called dynamic observer (DO) and to use it in control design. In the past several decades, the observer design problem has gained constantly high interest in the literature, due to the fact that the state variable cannot be measured frequently either because of the unavailability of appropriate sensors or due to high costs and long analysis times involved in the measurement processes.
The state variable is important in the state space representation, which is a mathematical model of a physical system as a set of inputs, outputs and state variables related by first-order differential equations. The general state space representation of linear continuous-time systems is expressed as

$$\dot{x} = Ax + Bu, \tag{1.1a}$$
$$y = Cx + Du, \tag{1.1b}$$

where $x \in \Re^n$, $u \in \Re^m$ and $y \in \Re^p$ are the state vector, the input vector and the output vector, respectively. Matrices A, B, C and D are known constant and of appropriate dimensions.
The state space representation of linear discrete-time systems is expressed as

$$x(k+1) = Ax(k) + Bu(k), \tag{1.2a}$$
$$y(k) = Cx(k) + Du(k), \tag{1.2b}$$

where $x(k) \in \Re^n$, $u(k) \in \Re^m$ and $y(k) \in \Re^p$ are the state vector, the input vector and the output vector, respectively.

In the state space representation, the state variable can show the full characteristics of system, therefore it is widely applied in the control process, such as state feedback control ([53] and [110]). State feedback control is to place the closed-loop poles of a system in the pre-determined locations of complex plane, by controlling the system state.
Take system (1.1) for example. The state feedback control law is represented by

$$u = K_{sf}x, \tag{1.3}$$

where K_{sf} is unknown matrix and of appropriate dimension to be determined.
By inserting state feedback control (1.3) into system (1.1a) it follows that:

$$\dot{x} = (A + BK_{sf})x. \tag{1.4}$$

In this case, the gain matrix K_{sf} can be derived from the solution of the following equation:

$$A + BK_{sf} = F_{sf}, \tag{1.5}$$

where F_{sf} represents the matrix of the pre-determined location of system poles.

Although state variables are of great importance, most of the time, state variables are not all available due to the package technology, or costly and difficulty to measure. Take the precision power source VX2600 for example. The VX2600 is a high precision DC source of VX Instruments company. The output voltage and current of VX2600 are $\pm 10V_{DC}$ and $\pm 5mA_{DC}$, respectively. In order to guarantee the accuracy of precision source, it must be packaged in a sealed box so that to isolate the outside disturbance. In this case, it is impossible to measure the states of some internal variables directly.

Even if the source is not packaged in a black box, it is still difficult to measure the states. Let us see the following figure of VX2600:

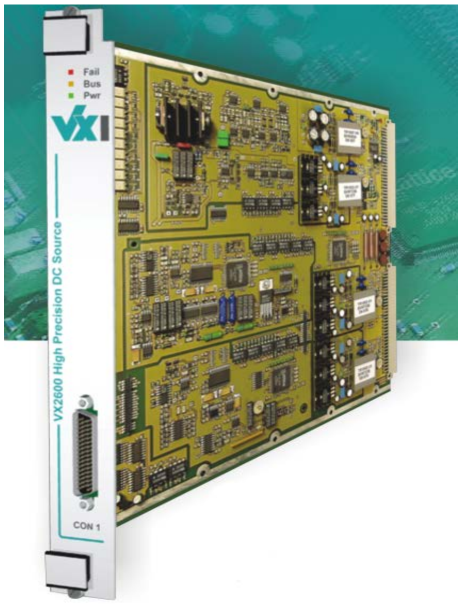

Figure 1.1: VX2600 High Precision DC Source

One can see from Figure 1.1 that VX2600 is made up of many electronic elements, such as capacitors, resistors, digital chips and so on. With the development of manufacture technology, now the digital chips can be made as small as possible. For example, the distance between two pins of chip ADA4870 is 1.27 millimeters, as shown in Figure 1.2. In this case, it is difficult to measure the value of one pin without touching other pins.

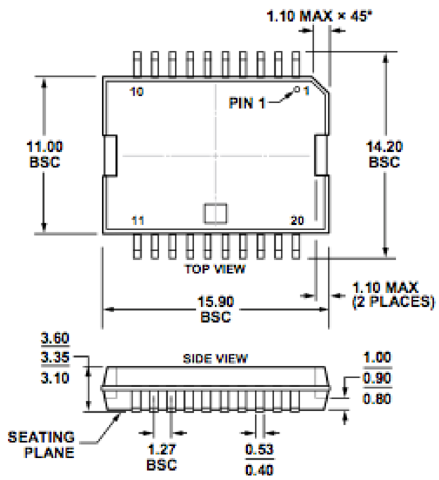

Figure 1.2: ADA4870 Package

On the other hand, the sensors and their associated cables are among the most expensive components in the system. The accuracy of sensors also reduces the reliability of control procedure, since sensors usually produce errors, such as noise and limited responsiveness.

With all the above discussions, it is necessary to use the observer to estimate the state in the case where the state cannot be determined by direct measurement. The concept of observer was firstly introduced by D.G. Luenberger [80] and [81], based on the theory that a system S_2 is served as an observer of the system S_1 if the available inputs and outputs of system S_1 are used as inputs to drive system S_2.

In control theory, an observer is a dynamic system that reconstructs the state of a given real system, from the measurements of inputs and outputs of the real system. Let us consider the following system:

$$\dot{x} = Ax + Bu, \tag{1.6a}$$
$$y = Cx. \tag{1.6b}$$

For system (1.6), an observer is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + K_o(y - \hat{y}), \tag{1.7a}$$
$$\hat{y} = C\hat{x}, \tag{1.7b}$$

where $\hat{x} \in \Re^n$ is the state of observer, called also the state estimate. $\hat{y} \in \Re^p$ is the output of observer. K_o is an unknown gain matrix and of appropriate dimension be determined.

One can see from system (1.7) more clearly that the observer is a reconstruction of the system states by using the inputs and outputs of original system. The only difference is that there is an additional term $K_o(y - \hat{y})$ in the observer, which is used to ensure that the state estimate $\hat{x}$ converges to the system state x, on receiving successive measured values of system inputs and outputs.

The structure of observer (1.7) is shown in the following figure:

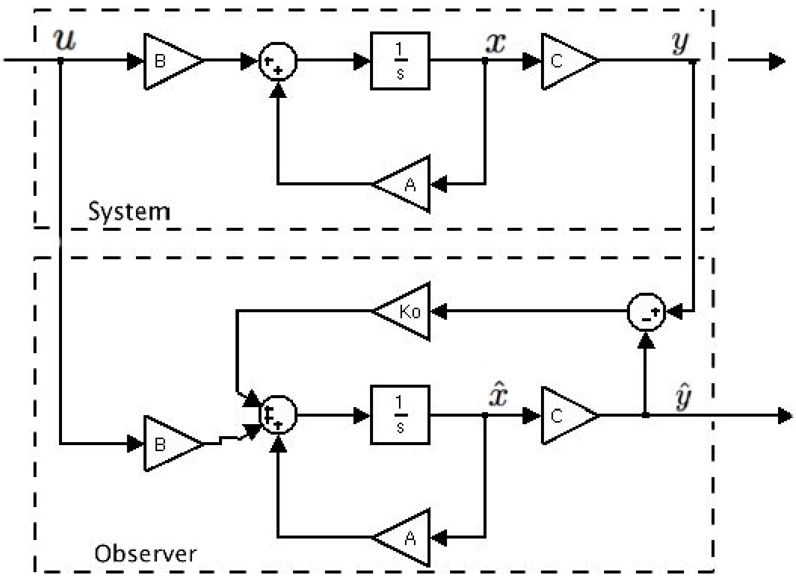

Figure 1.3: Structure of observer

1.2 Observers for linear systems

Ever since Luenburger presented the first result on the observer, the observers design have been greatly investigated and widely applied for linear systems (see [21, 43, 52, 61, 123] and references therein). The observer aims to estimate the system state x. Therefore we must study the estimation error e ($e = \hat{x} - x$). From system (1.6) and observer (1.7), we obtain the following estimation error dynamic:

$$\begin{aligned}
\dot{e} &= \dot{\hat{x}} - \dot{x}, \\
&= A\hat{x} + Bu + K_o(y - \hat{y}) - Ax - Bu, \\
&= (A - K_oC)\hat{x} - (A - K_oC)x, \\
&= (A - K_oC)e. \qquad (1.8)
\end{aligned}$$

By using Lyapunov function approach, we can determine the parameter matrix K_o such that system (1.8) is asymptotically stable, i.e, $e \to 0$ when $t \to \infty$. In this case, system (1.7) is an asymptotic observer for system (1.6).
The observer can be designed for either a continuous-time system or a discrete-time system. The characteristics are the same, and the design processes are at least very similar and in some cases they are identical. For the continuous-time system, the observer is asymptotically stable when the system matrix "$A - K_oC$" has all the eigenvalues on the left side of complex plane, while for the discrete-time system the observer is asymptotically stable when the system matrix has all the eigenvalues inside the unit circle.

1.2.1 Full-order observers and observer-based control

1.2.1.1 Full-order observers

During the past several years, many kinds of observers have been developed. One popularly used is the full-order observer [41]. Let us consider observer (1.7) for example. Notice that the dimension of observer state $\hat{x}$ is n, which is equal to the order of original system state. We call this kind of observers full-order observers, which can estimate all the system states.
A full-order observer accomplishes the estimation purpose by calculating the residual "$(y - \hat{y})$", which is the difference between the measured output and the corresponding quantity generated by the observer. In real world there are many formulations of observers, and most of them can be transformed into observer (1.7). For example, for system (1.6), the author of [82] proposed the following observer:

$$\dot{\hat{x}} = \hat{A}\hat{x} + Ly + Hu, \qquad (1.9a)$$

$$\hat{y} = C\hat{x}. \qquad (1.9b)$$

Matrices $\hat{A}$, L and H are unknown and of appropriate dimensions to be determined. Notice that if we take

$$\begin{aligned}
\hat{A} &= A - K_oC, \\
L &= K_o, \\
H &= B,
\end{aligned}$$

we can obtain the form of observer (1.7).

The applications of full-order observer can be found in many fields. For example, a distributed full-order observer type consensus protocol was proposed for inear multi-agent systems in [73]. In [82], the author designed a full-order observer for the state estimation of an aircraft motion. The authors of [36] and [97] solved the design problem of full-order observers for sensorless induction motor drives. In [124], a full-order observer based insulated gate bipolar transistor (IGBT) temperature estimation was presented.

1.2.1.2 Observer-based control

As it is discussed previously, the appearance of observer is due to the importance of system states in the feedback control, which is the most important among the various applications of observers. In the case where some states cannot be measured directly, we use their estimates in the control process, which is called observer-based control.
The observer-based control algorithm is designed in two parts: a "full-state feedback" part based on the assumption that all the state variables can be measured; and an observer to estimate the state of the process. The concept of separating the control design into these two parts is known as the separation principle, which is often a practical solution to many design problems.
Take system (1.6) for example, let us consider the following observer-based control law:

$$u = K_c\hat{x}, \tag{1.10}$$

where K_c is the control gain matrix of appropriate dimension to be determined.
By inserting control (1.10) into system (1.6), we obtain

$$\begin{aligned}\dot{x} &= Ax + BK_c\hat{x},\\ &= Ax + BK_c(\hat{x} - x) + BK_c x,\\ &= (A + BK_c)x + BK_c e.\end{aligned} \tag{1.11}$$

From equations (1.8) and (1.11), we obtain the following system:

$$\dot{\eta} = A_c\eta, \tag{1.12}$$

where

$$\begin{aligned}\eta &= \begin{bmatrix} x \\ e \end{bmatrix},\\ A_c &= \begin{bmatrix} A + BK_c & BK_c \\ 0 & A - K_oC \end{bmatrix}.\end{aligned}$$

Then we can determine matrices K_c and K_o separately.
In the past decades, the observer-based control has been introduced into many fields. In [78], the authors investigated the problem of observer-based control for linear systems with limited communication capacity. The separation theorem for robust pole placement of discrete-time linear control systems with full-order observers was derived in [114]. The authors of [117] introduced an observer-based control approach for linear stepping motor (LSM) drive systems. In [129], the observer-driven switching stabilization problem for a class of switched linear systems was solved. The paper [142] concerned the problem of event-driven observer-based feedback control for linear systems. Other applications of observer-based control can be found in [103, 105, 150] and references therein.

1.2.2 Reduced-order observers

Generally, in many systems, some state variables can be either measured directly, or calculated from the output. For system (1.6) we can see that in equation (1.6b), system output y has a relation with system state x, i.e.,

$$y = Cx.$$

In this case, we can calculate some system states and it is not needed to estimate all the system states. Instead, by taking "$y = Cx$" into account, one just needs to design an observer to estimate the unmeasured states. This kind of observer, which has lower dimension than the original system or the full-order observer, is called the reduced-order observer [80].
In order to design the reduced-order observer, we should perform some matrix transformations first [122]. For system (1.6), assume that C is a full rank matrix, that is, $rank\ C = p$. Then select a $(n-p) \times n$ matrix R such that the following $n \times n$ matrix P is nonsingular:

$$P = \begin{bmatrix} C \\ R \end{bmatrix}. \tag{1.13}$$

Let

$$x = P^{-1}\overline{x},$$

where $\overline{x}$ is a new state variable, then we obtain

$$\begin{aligned} \overline{A} &= PAP^{-1} \\ &= \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}, \\ \overline{B} &= PB = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}, \\ \overline{C} &= CP^{-1} = \begin{bmatrix} I_p & 0 \end{bmatrix}, \end{aligned}$$

In this case, from system (1.6) we obtain the following equivalent system:

$$\begin{bmatrix} \dot{\overline{x}}_1 \\ \dot{\overline{x}}_2 \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} u, \tag{1.14a}$$

$$y = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = \overline{x}_1. \tag{1.14b}$$

One can see from equation (1.14b) that state $\overline{x}_1$ can be obtained from output y and we just need to design an observer for the following system of dimension $n-p$:

$$\dot{\overline{x}}_2 = \overline{A}_{21}\overline{x}_1 + \overline{A}_{22}\overline{x}_2 + \overline{B}_2 u \tag{1.15}$$

The observer for system (1.15) is given by

$$\dot{z} = (\overline{A}_{22} + \overline{L}\overline{A}_{12})z + [(\overline{A}_{22} + \overline{L}\overline{A}_{12})\overline{L} + \overline{A}_{21} + \overline{L}\overline{A}_{11}]y + (\overline{B}_2 + \overline{L}\overline{B}_1)u, \tag{1.16a}$$

$$\hat{\overline{x}}_2 = z - \overline{L}y, \tag{1.16b}$$

where $z \in \Re^{n-p}$ is the state of observer and $\hat{\overline{x}}_2 \in \Re^{n-p}$ is the estimate of $\overline{x}_2$. $\overline{L}$ is the observer gain matrix and of appropriate dimension to be determined such that:

$$\lim_{t\to\infty} [\overline{x}_2(t) - \hat{\overline{x}}_2(t)] = 0.$$

By using this method, the authors of [130] proposed a type of reduced-order observer for matrix second-order linear systems. In [147], the design problem of positive real control via reduced-order observer was solved. The applications of this method can be also found in [143], [151] and references therein.
Another method to design the reduced-order observer is a direct one without doing the transformation to original system. For system (1.6), the authors of [64] proposed the following reduced-order observer:

$$\dot{z}(t) = Dz(t) + Gu(t) + Hy(t), \tag{1.17a}$$
$$\hat{x}(t) = Mz(t) + Ny(t), \tag{1.17b}$$

with the observer vector $z \in \Re^{n-p}$. Matrices $D \in \Re^{(n-p)\times(n-p)}$, $G \in \Re^{(n-p)\times m}$, $H \in \Re^{(n-p)\times p}$, $M \in \Re^{n\times(n-p)}$ and $N \in \Re^{n\times p}$ are unknown to be determined. In this case, one can determine all the matrices through the analysis of estimation error dynamic directly.
The applications of this method can be found in [50] where the reduced-order observer was designed for rectangular descriptor systems and in [131] where the problem of reduced-order-observer-based decentralized guaranteed cost stabilization (DGCS) of large systems was addressed.

Remark 1.2.1. *From the above results, we can see that the minimum order of reduced-order observer is $n - p$, which is a necessary condition of the existence of reduced-order observer.*

1.2.3 Functional observers

The design problem of a linear functional observer has gain considerable attention recently. Linear functional observers estimate linear functions of states without estimating all the individual states. Such functional estimates are useful in feedback control system. It is often that a state feedback control law does not necessarily require the availability of the complete states. Take control (1.3) for example. One can see that the control

$$u = K_{sf}x$$

is a linear function of state variables.
Furthermore, in the process of fault detection via residual signal generation, only the estimate of the output is required (y), which is a linear function of states $(y = Cx)$. In this case, it is more logical to estimate the desired output directly using a functional observer rather than estimating all the individual states.
The functional observer is essentially a reduced-order observer, because we only estimate a desired linear function of states by using functional observer. Besides, it can be seen that the condition for the existence of the functional observer is weaker than the detectability condition which is required in full- and reduced-order observers design.
Generally, in order to solve the functional observer design problem, many authors have proposed to transform the initial system to an equivalent one (by using some regular transformations) of reduced-order and to design an observer for this system. Recently, a straightforward method for functional observer design has been developed in [25]. For example, let us consider the following linear system:

$$\dot{x} = Ax + Bu, \tag{1.18a}$$
$$y = Cx, \tag{1.18b}$$
$$z_f = L_f x, \tag{1.18c}$$

where $z_f \in \Re^r$ is the vector to be estimated with $r \leq n$. Matrix L_f is known constant and of appropriate dimension. Without loss of generality, we assume that $rank\ C = p$ and $rank\ L_f = r$. For system (1.18), the functional observer is given by

$$\begin{aligned} \dot{w}_f &= N_f w_f + J_f y + H_f u, && (1.19a) \\ \hat{z}_f &= w_f + E_f y, && (1.19b) \end{aligned}$$

where $w_f \in \Re^r$ is the state of functional observer and $\hat{z}_f$ is the estimate of z_f. Matrices N_f, J_f, H_f and E_f are unknown and of appropriate dimensions to be designed.
In this case, we have the following results on the estimation error:

$$\begin{aligned} e_f &= \hat{z}_f - z_f, \\ &= w_f + E_f Cx - L_f x, \\ &= w_f - P_f x, \end{aligned} \tag{1.20}$$

where $P_f = L_f - E_f C$.
From equation (1.20), we obtain the dynamic of estimation error e_f:

$$\begin{aligned} \dot{e}_f &= \dot{w}_f - P_f \dot{x}, \\ &= N_f w_f + J_f y + H_f u - P_f A x - P_f B u, \\ &= N_f (w_f - P_f x) + (N_f P_f + J_f C - P_f A) x + (H_f - P_f B) u, \\ &= N_f e_f + (N_f P_f + J_f C - P_f A) x + (H_f - P_f B) u. \end{aligned} \tag{1.21}$$

Notice that if the following conditions are satisfied:

$$\begin{aligned} N_f P_f + J_f C - P_f A &= 0, && (1.22a) \\ H_f &= P_f B, && (1.22b) \end{aligned}$$

the dynamic of estimation error e_f becomes

$$\dot{e}_f = N_f e_f. \tag{1.23}$$

By using Lyapunov function approach to make matrix N_f to be a Hurwitz, $\hat{z}_f$ is an asymptotic estimate of the linear functional z_f for any $x(0)$, $w(0)$ and $u(0)$. The design problem of the functional observer (1.19) is reduced to determine all the parameter matrices such that conditions (1.22) are satisfied and matrix N_f is Hurwitz.
The applications of this approach can be found in many fields. The paper [29] concerned the design of functional observers for linear time-invariant (LTI) descriptor systems. The minimum order linear functional observer was designed in [40]. In [100], a finite time functional observer was proposed for linear systems. The authors of [101] designed a single linear functional observer for LTI systems.

1.2.4 Unknown input observers

In the observer design problem, it is necessary to estimate the state of systems in the presence of completely unknown inputs. This kind of observers, which are able to estimate the state of systems in the presence of unknown inputs, is called unknown input observer (UIO).
One method to design UIO depends on the matrix transformation [70, 99, 121]. The main step of this approach is to find a transformation to decompose the system in two subsystems. In one subsystem, the states are not effected by the unknown input and can be reconstructed by using the observer. In the other subsystem, the rest of the state can be expressed by using the system output and the state estimate of the first subsystem.
Let us consider the following linear system in the presence of unknown inputs:

$$\dot{x} = Ax + Bu + Fv, \tag{1.24a}$$
$$y = Cx, \tag{1.24b}$$

with $v \in \Re^l$ is the unknown input. Matrix F is known constant and of appropriate dimension. We assume that $rank\ C = p$, $rank\ F = l$ and $p \geq l$. Under the assumption that $rank\ F = l$, we select a matrix $N \in \Re^{n\times(n-l)}$ such that the following matrix is nonsingular:

$$T = \begin{bmatrix} N & F \end{bmatrix}. \tag{1.25}$$

Let

$$x = T\widetilde{x},$$

where $\widetilde{x}$ is the new state variable, then we obtain

$$\begin{aligned} \widetilde{A} &= T^{-1}AT \\ &= \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \\ \widetilde{B} &= T^{-1}B = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix}, \\ \widetilde{F} &= T^{-1}F = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\ \widetilde{C} &= CT = \begin{bmatrix} CN & CF \end{bmatrix}. \end{aligned}$$

In this case, system (1.24) becomes

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}u + \widetilde{F}v, \tag{1.26a}$$
$$y = \widetilde{C}\widetilde{x}, \tag{1.26b}$$

where states $\widetilde{x} = \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{bmatrix}$, $\widetilde{x}_1 \in \Re^{n-l}$ and $\widetilde{x}_2 \in \Re^l$.
Furthermore, we select a matrix $Q \in \Re^{p\times(p-l)}$ such that the following matrix is nonsingular:

$$U = \begin{bmatrix} CF & Q \end{bmatrix}.$$

Then we have the following results:

$$U^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

$$\begin{aligned} U_1CF &= I_l, && (1.27a)\\ U_2CF &= 0. && (1.27b)\end{aligned}$$

In this case, by using the following output transformation:

$$\begin{aligned} \widetilde{y} &= \begin{bmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{bmatrix}, \\ &= U^{-1}y, \\ &= \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} CN & CF \end{bmatrix} \widetilde{x}, \\ &= \begin{bmatrix} U_1CN & U_1CF \\ U_2CN & U_2CF \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{bmatrix}, \end{aligned}$$

and equations (1.27a) and (1.27b), we obtain

$$\widetilde{y}_1 = U_1y = U_1CN\widetilde{x}_1 + \widetilde{x}_2, \tag{1.28}$$

$$\widetilde{y}_2 = U_2y = U_2CN\widetilde{x}_1. \tag{1.29}$$

From equation (1.28) we have the following result on $\widetilde{x}_2$:

$$\widetilde{x}_2 = U_1y - U_1CN\widetilde{x}_1. \tag{1.30}$$

By substituting equation (1.30) into system (1.26), we obtain

$$\begin{aligned} \dot{\widetilde{x}}_1 &= \widetilde{A}_1\widetilde{x}_1 + \widetilde{B}_1u + E_1y, && (1.31a)\\ \widetilde{y}_2 &= \widetilde{C}_1\widetilde{x}_1, && (1.31b)\end{aligned}$$

with

$$\begin{aligned} \widetilde{A}_1 &= \widetilde{A}_{11} - \widetilde{A}_{12}U_1CN, \\ E_1 &= \widetilde{A}_{12}U_1, \\ \widetilde{C}_1 &= U_2CN. \end{aligned}$$

One can see from system (1.31) that state $\widetilde{x}_1$ is not effected by the unknown input v and it can be reconstructed by using observer (1.7). Besides, state $\widetilde{x}_2$ can be obtained from output y and state estimate $\hat{\widetilde{x}}_1$.

In this case, the complete estimate of original system state can be expressed by

$$\hat{x} = T \begin{bmatrix} \hat{\widetilde{x}}_1 \\ U_1y - U_1CN\hat{\widetilde{x}}_1 \end{bmatrix}. \tag{1.32}$$

Notice that one should do matrix transformation by using the above method. In order to simply the design procedure, another approach named the algebraic method has been developed. For system (1.24), the authors of [33] proposed the following UIO:

$$\begin{aligned} \dot{z} &= Nz + Ly + Gu, && (1.33a)\\ \hat{x} &= z - Ey && (1.33b)\end{aligned}$$

where z is the state vector. Matrices N, L, G and E are unknown and of appropriate dimensions to be determined.

In this case, from system (1.24) and UIO (1.33), we obtain the following estimation error dynamic:

$$\begin{aligned}
\dot{e} &= \dot{\hat{x}} - \dot{x}, \\
&= \dot{z} - P\dot{x}, \\
&= N(z - Px) + NPx + LCx + Gu - PAx - PBu - PFv, \\
&= Ne + (NP + LC - PA)x + (G - PB)u - PFv, \qquad (1.34)
\end{aligned}$$

where matrix $P = I_n + EC$.
One can see from equation (1.34) that if the following constraints are satisfied:

$$\begin{aligned}
NP + LC - PA &= 0, \qquad (1.35a) \\
G - PB &= 0, \qquad (1.35b) \\
PF &= 0, \qquad (1.35c)
\end{aligned}$$

the estimation error dynamic $\dot{e}$ will be independent of state x, input u and unknown input v. In this case, the estimation error dynamic becomes

$$\dot{e} = Ne. \qquad (1.36)$$

Then, the design problem is reduced to determine all the parameter matrices such that constrains (1.35) are satisfied and matrix N is Hurwitz.
The applications of this method to design UIO can be found in [24] for structure estimation of a moving object, in [62] for Takagi-Sugeno (T-S) descriptor system, in [66] for discrete-time switched descriptor systems and in [115, 136, 138] for fault detection.

1.2.5 Observers for uncertain systems

As it is introduced previously, a system can be represented as a group of differential equations characterized by a finite number of state variables. In most practical situations, system models suffer from model parameter variation which is represented by the system uncertainty. When the uncertainties in the system vary within given bounds, it is necessary to find a control law that guarantees an upper bound for the performance. In the observer design, the problems of systems with uncertain parameters have been treated in several categories according to different assumptions:

1. Let us consider the following system in [119]:

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} + \mathbf{PD_n}, \qquad (1.37a) \\
\mathbf{y} &= \mathbf{Cx}, \qquad (1.37b)
\end{aligned}$$

   where $\mathbf{P}$ is known matrix of appropriate dimension. Vector $\mathbf{D_n}$ represents the uncertainty, which is assumed to be upper bounded by

$$\|\mathbf{D_n}\| \le D_{max}, \qquad (1.38)$$

   with D_{max} is a positive scalar.

2. In [34] and [56], for the following system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\xi(\mathbf{t}, \mathbf{x}, \mathbf{u}), & \text{(1.39a)} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, & \text{(1.39b)} \end{aligned}$$

where $\mathbf{G}$ is known matrix of appropriate dimension, vector $\xi(\mathbf{t}, \mathbf{x}, \mathbf{u})$ is used to model the uncertainty which is bounded by

$$\|\xi(\mathbf{t}, \mathbf{x}, \mathbf{u})\| \leq \alpha, \tag{1.40}$$

where α is a positive scalar.

3. In [96], the uncertain linear system under consideration is described by the following representation:

$$\begin{aligned} \dot{\mathbf{x}} &= \left[\mathbf{A_0} + \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{A_i}\mathbf{r_i}\right]\mathbf{x} + \left[\mathbf{B_0} + \sum_{\mathbf{i=1}}^{\mathbf{l}} \mathbf{B_i}\mathbf{s_i}\right]\mathbf{u}, & \text{(1.41a)} \\ \mathbf{y} &= \left[\mathbf{C_0} + \sum_{\mathbf{i=1}}^{\mathbf{p}} \mathbf{C_i}\mathbf{v_i}\right] & \text{(1.41b)} \end{aligned}$$

where matrices $\mathbf{A_0}$, $\mathbf{A_i}$, $\mathbf{B_0}$, $\mathbf{B_i}$, $\mathbf{C_0}$ and $\mathbf{C_i}$ are known and of appropriate dimensions.

$\mathbf{r}$, $\mathbf{s}$ and $\mathbf{v}$ are used to represent uncertain parameters, which are assumed to be bounded in the following sets:

$$\begin{aligned} &\mathscr{R} = \{\mathbf{r} \in \mathfrak{R}^k : |r_i| \leq \overline{r} \quad \text{for } i = 1, 2, \ldots, k\}; \quad \overline{r} \geq 0; \\ &\mathscr{S} = \{\mathbf{s} \in \mathfrak{R}^l : |s_i| \leq \overline{s} \quad \text{for } i = 1, 2, \ldots, l\}; \quad \overline{s} \geq 0; \\ &\mathscr{V} = \{\mathbf{v} \in \mathfrak{R}^p : |v_i| \leq \overline{v} \quad \text{for } i = 1, 2, \ldots, p\}; \quad \overline{v} \geq 0. \end{aligned}$$

4. In [58], for the following linear uncertain system:

$$\begin{aligned} \dot{\mathbf{x}} &= [\mathbf{A} + \mathbf{\Delta A}(\mathbf{r})]\,\mathbf{x} + \mathbf{B}\mathbf{u}, & \text{(1.42a)} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, & \text{(1.42b)} \end{aligned}$$

$\mathbf{\Delta A}(\mathbf{r})$ has been introduced to represent the uncertainty which has the following structure:

$$\mathbf{\Delta A}(\mathbf{r}) = \sum_{\mathbf{i=1}}^{\mathbf{l}} \mathbf{D_i}\mathbf{F_i}(\mathbf{r})\mathbf{E_i}, \tag{1.43}$$

where $\mathbf{D_i}$ and $\mathbf{E_i}$ are constant matrices and matrices $\mathbf{F_i}(\mathbf{r})$ are assumed to be bounded by

$$\mathbf{F_i^T}(\mathbf{r})\mathbf{F_i}(\mathbf{r}) \leq \mathbf{\bar{r}^2 I}, \ \mathbf{i = 1, 2, \ldots, l}. \tag{1.44}$$

with $\mathbf{\bar{r}}$ is a known scalar.

No matter by which constraint the uncertainty is assumed to be bounded, the method to deal with the uncertainty is based on Lyapunov function approach. Take the system of [119] for example. The uncertain system is given by

$$\dot{x} = Ax + Bu + P_n D_n, \quad (1.45a)$$
$$y = Cx, \quad (1.45b)$$

where matrix P_n is known and of appropriate dimension. D_n is bounded by constraint (1.38). For system (1.45), by using observer (1.7), we obtain the following estimation error dynamic:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x}, \\ &= (A - K_o C)e - P_n D_n. \end{aligned} \quad (1.46)$$

In order to determine matrix K_o, we choose the following Lyapunov function:

$$V = e^T X e, \quad (1.47)$$

where matrix $X > 0$.
Then the derivative of V along the solution of (1.46) is given by

$$\begin{aligned} \dot{V} &= \dot{e}^T X e + e^T X \dot{e}, \\ &= [(A - K_o C)e - P_n D_n]^T X e + e^T X[(A - K_o C)e - P_n D_n], \\ &= e^T[(A - K_o C)^T X + X(A - K_o C)]e - 2e^T X P_n D_n. \end{aligned} \quad (1.48)$$

In this case, we can determine matrix K_o such that $\dot{V} < 0$.

1.2.6 H_∞ observers

In the daily life, the disturbance is very common. For example, the resistance of wind for a moving car. In control process, not only the reference input can affect the output, but also the disturbance can result in negative influences on the output. Therefore, in the observer design, one should pay more attention to the disturbance.
One method to deal with the disturbance is the disturbance observer, which estimates the disturbance directly. The frequency-domain method to design the disturbance observer can be found in [135] and references therein. The applications of disturbance observer can be also found in [63, 76, 113] for both continuous-time and discrete-time cases.

However, when we use the disturbance observer, it is necessary to add an extra observer, which makes the structure more complicated. Besides, due to the estimation error of the disturbance observer, the estimate of disturbances will affect the accuracy of performance. Therefore, another method named the H_∞ observer, which can directly minimize the influence of disturbances on the estimation error, has been developed.
The H_∞ observer combines the H_∞ theory ([18] and [46]) with the observer design, and the H_∞ theory is based on the H_∞ norm. Let us return to system (1.6). The H_∞ norm of the transfer function of system (1.6) is given by

$$\begin{aligned} \|T_{uy}(jw)\|_\infty &= \sup_{w \in \Re} \overline{\sigma}(T_{uy}(jw)) \\ &= \max_{u \in L_2} \frac{\|y\|_2}{\|u\|_2} \end{aligned} \quad (1.49)$$

where

$$T_{uy}(jw) = T_{uy} = C(sI - A)^{-1}B$$

is the transfer function matrix from the input u to the output y.
One can see from equation (1.49) that the H_∞ norm $\|T_{uy}\|_\infty$ represents the influence of the input u on the output y.
Now let us consider the H_∞ observer design for the following linear system in the presence of disturbances:

$$\dot{x} = Ax + Bu + B_w w, \tag{1.50a}$$
$$y = Cx, \tag{1.50b}$$

where $w \in \Re^f$ is the disturbance of finite energy. Matrix B_w is known constant and of appropriate dimension.
From system (1.50) and observer (1.7), we obtain the estimation error dynamic as follows:

$$\dot{e} = (A - K_o C)e - B_w w. \tag{1.51}$$

According to the H_∞ theory, we should minimize the effect of disturbance w on the estimation error e. In other words, we must find a positive scaler γ such that:

$$\|T_{we}\|_\infty < \gamma, \tag{1.52}$$

where

$$T_{we} = -[sI - (A - K_o C)]^{-1}B_w$$

represents the transfer function matrix from the disturbance w to the estimation error e.

The methods to design the H_∞ observer are mainly based on bounded real lemma and linear matrix inequality (LMI) optimization approach. The optimal H_∞ reduced-order filtering problems was solved in [13]. The authors of [37] designed an H_∞ observer to estimate the state variables of the vertical car. The authors of [126] designed an H_∞ observer-based robust fault detection for linear switched systems with external disturbances. In [79], the design problem of H_∞ control for T-S fuzzy systems was studied, based on fuzzy observers.

1.2.7 Proportional integral observers

Notice that in observer (1.7), there is only one proportional gain K_o. This kind of observers is also called proportional observer (PO). It is well known that the precision of estimation provided by the PO is directly affected by system disturbances. In order to provide a better estimation, the proportional integral observer (PIO) has been developed by the duality of proportional-integral (PI) control. The PI control is one special case of proportional-integral-derivative (PID) control in control theory [5], which is given by

$$u(t) = L_p\rho(t) + L_i\int_0^t \rho(\tau)d\tau, \tag{1.53}$$

where

τ is the variable of integration and takes on values from time 0 to the present time t.
ρ is the error between the reference signal and the output signal.
Parameters L_p and L_i are proportional gain and integral gain, respectively.

The idea of PI control is to incorporate an additional correction term that is proportional to the integral of output error $y - \hat{y}$. By the duality of the PI control, the PIO has been introduced to offer certain degrees of freedom for the observer parameter selection to eliminate the static estimation error. Let us consider the following linear system:

$$\begin{aligned} \dot{x} &= Ax + Bu + d(t), & \text{(1.54a)} \\ y &= Cx. & \text{(1.54b)} \end{aligned}$$

where $d(t)$ is the disturbance.
For system (1.54), by using the PO (1.7) we obtain the following estimation error dynamic:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x}, \\ &= (A - K_o C)e - d(t). \end{aligned} \tag{1.55}$$

Matrix K_o can be determined such that system (1.55) is asymptotically stable. Then by applying Laplace transform to equation (1.55), we have

$$E(s) = [sI - (A - K_o C)]^{-1} [e(0) - D(s)], \tag{1.56}$$

where $e(0)$ is the initial condition. $E(s)$ and $D(s)$ are the Laplace transforms of $e(t)$ and $d(t)$. Assume that $d(t)$ is a step function, which is expressed as

$$d(t) = \mathbf{d}, \forall\, t \geq 0, \tag{1.57}$$

where $\mathbf{d}$ is the amplitude of step function $d(t)$.
In this case, the Laplace transform of $d(s)$ is

$$D(s) = \frac{\mathbf{d}}{s}.$$

Since matrix $(A - K_o C)$ is invertible, by applying the final value theorem, the steady state error is expressed as

$$\begin{aligned} e_{po}(\infty) &= \lim_{s \to 0} sE(s), \\ &= \lim_{s \to 0} s\, [sI - (A - K_o C)]^{-1} \left[e(0) - \frac{\mathbf{d}}{s} \right], \\ &= \lim_{s \to 0} [sI - (A - K_o C)]^{-1} [se(0) - \mathbf{d}], \\ &= (A - K_o C)^{-1} \mathbf{d}. \end{aligned} \tag{1.58}$$

One can see from equation (1.58) that in the presence of disturbance $d(t)$ (1.57), there is a static error $(A - K_oC)^{-1}\mathbf{d}$ in the estimation of the PO.
Now, for system (1.54) let us consider the following PIO:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_p(y - \hat{y}) + K_{in}v, \quad (1.59a)$$
$$\dot{v} = y - \hat{y}, \quad (1.59b)$$
$$\hat{y} = C\hat{x}, \quad (1.59c)$$

where $v \in \Re^p$ is an auxiliary integral vector of output error. K_p and K_{in} are unknown matrices and of appropriate dimensions to be determined. One can see from equations (1.59) more clearly that there is another gain K_{in}, which is proportional to the integral term of output error in the PIO, compared with the PO (1.7).
By using the PIO, we obtain the following system for the estimation error e:

$$\dot{e} = (A - K_pC)e + K_{in}v - d(t), \quad (1.60a)$$
$$\dot{v} = -Ce. \quad (1.60b)$$

Then, matrices K_p and K_{in} can be determined such that system (1.60a) is asymptotically stable. Furthermore, by applying Laplace transforms to system (1.60), we have

$$sE(s) = (A - K_pC)E(s) + K_{in}V(s) - D(s) + e(0), \quad (1.61a)$$
$$V(s) = \frac{v(0) - CE(s)}{s}, \quad (1.61b)$$

where $V(s)$ is the Laplace transform of $v(t)$ and $v(0)$ is the initial condition.
From equation (1.61a), we obtain

$$E(s) = [sI - (A - K_pC)]^{-1}[K_{in}V(s) - D(s) + e(0)]. \quad (1.62)$$

In this case, by applying the final value theorem, the steady state error is given by

$$\begin{aligned} e_{pio}(\infty) &= \lim_{s\to 0} sE(s), \\ &= \lim_{s\to 0} s[sI - (A - K_pC)]^{-1}[K_{in}V(s) - D(s) + e(0)], \\ &= \lim_{s\to 0} [sI - (A - K_pC)]^{-1}[sK_{in}V(s) - sD(s) + se(0)], \\ &= -(A - K_pC)^{-1}\left\{\left[\lim_{s\to 0} sK_{in}V(s)\right] - \mathbf{d}\right\}. \end{aligned} \quad (1.63)$$

From equation (1.63), it is obvious that the steady state error equals to zero if the following condition is satisfied

$$\lim_{s\to 0} sK_{in}V(s) = \mathbf{d}. \quad (1.64)$$

With the above results, one can see that the PIO can eliminate the influence of disturbance $d(t)$ (1.57), by adding an integral term v.

Ever since the PIO was firstly proposed by Wojciechowski for single-input single-output (SISO) linear systems [128], the effectiveness of the PIO has been proved in many fields. For example, the PIO was developed for multivariable systems with step disturbances in [11, 104]. In [19], the authors proved the better performance of the PIO to attenuate both measurement noise and modeling errors for linear systems, compared with the PO. In [2], a PIO was designed to estimate simultaneously the sensor fault and the unknown input for linear descriptor systems. Through the pole placement algorithm, both full- and reduced-order PIOs were designed for unknown inputs descriptor systems in [65]. Based on Lyapunov technique, the PIO design problem was solved for T-S fuzzy systems subject to unmeasurable decision variables in [137].

1.2.8 Dynamic observers

From the previous introduction of observers, one can see that the observer design procedure is dual to the control design. For example, the determination of matrix K_o in PO (1.7) is dual to the design of static state feedback control (1.4). In the past decades, several observers design have been proposed, which are dual to the proptional control.
Recently, an alternative observer structure has been proposed by the duality of dynamic control. To distinguish the proposed observer from the classical observer, the new observer is named DO. For system (1.6), let us consider the following dynamic control [109]:

$$\begin{aligned} \dot{\lambda} &= \mathbf{\Psi_a} y + \mathbf{\Psi_b} \lambda, && (1.65a) \\ u &= \mathbf{\Phi_a} y + \mathbf{\Phi_b} \lambda, && (1.65b) \end{aligned}$$

where $\lambda \in \Re^n$ is the controller state. Matrices $\mathbf{\Psi_a}$, $\mathbf{\Psi_b}$, $\mathbf{\Phi_a}$ and $\mathbf{\Phi_b}$ are unknown and of appropriate dimensions to be determined.
By inserting the control (1.65) into system (1.6), we obtain the following closed-loop system:

$$\begin{aligned} \dot{x} &= (A + B\mathbf{\Phi_a} C)x + B\mathbf{\Phi_b}\lambda, && (1.66a) \\ \dot{\lambda} &= \mathbf{\Psi_a} Cx + \mathbf{\Psi_b}\lambda. && (1.66b) \end{aligned}$$

In this case, the design problem of dynamic control (1.65) is reduced to determine the parameter matrices $\mathbf{\Psi_a}$, $\mathbf{\Psi_b}$, $\mathbf{\Phi_a}$ and $\mathbf{\Phi_b}$ such that system (1.66) is asymptotically stable.

By the duality of dynamic control (1.65), the author of [94] proposed the following DO structure:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + \theta, && (1.67a) \\ \hat{y} &= C\hat{x}, && (1.67b) \\ \dot{v} &= \overline{a}v + \overline{b}\eta, && (1.67c) \\ \theta &= \overline{c}v + \overline{d}\eta, && (1.67d) \\ \eta &= y - \hat{y}, && (1.67e) \end{aligned}$$

where v is an auxiliary vector and θ is the observer gain. Matrices $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d}$ are unknown and of appropriate dimensions to be determined.
One can see from the above equations that different from the static gains K_o in the PO, or K_p and K_{in} in the PIO, the proposed DO structure introduces the dynamic $L_d(s)$ in the observer gain, which is expressed as

$$L_d(s) = \theta = -[\overline{d} + \overline{c}(sI - \overline{a})^{-1}\overline{b}]Ce.$$

By using the DO (1.67) for system (1.54), we have

$$\begin{aligned} \dot{e} &= (A - \bar{d}C)e + \bar{c}v - d(t), && (1.68a) \\ \dot{v} &= -\bar{b}Ce + \bar{a}v. && (1.68b) \end{aligned}$$

Then matrices $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ can be determined such that system (1.68) is asymptotically stable. Subsequently, by applying Laplace transform to system (1.68), it follows that:

$$\begin{aligned} sE(s) &= (A - \bar{d}C)E(s) + \bar{c}V(s) - D(s) + e(0), && (1.69a) \\ sV(s) &= -\bar{b}CE(s) + \bar{a}V(s) + v(0). && (1.69b) \end{aligned}$$

From equation (1.69a), we have

$$E(s) = \left[sI - (A - \bar{d}C)\right]^{-1} \left[\bar{c}V(s) - D(s) + e(0)\right]. \tag{1.70}$$

In this case, by applying the final value theorem, the steady state error is given by

$$\begin{aligned} e_{do}(\infty) &= \lim_{s\to 0} sE(s), \\ &= \lim_{s\to 0} s\left[sI - (A - \bar{d}C)\right]^{-1} \left[\bar{c}V(s) - D(s) + e(0)\right], \\ &= \lim_{s\to 0} \left[sI - (A - \bar{d}C)\right]^{-1} \left[s\bar{c}V(s) - sD(s) + se(0)\right], \\ &= -(A - \bar{d}C)^{-1} \left\{\left[\lim_{s\to 0} s\bar{c}V(s)\right] - \mathbf{d}\right\}. && (1.71) \end{aligned}$$

One can see from equation (1.71) that the steady state error equal to zero if the following condition is satisfied

$$\lim_{s\to 0} s\bar{c}V(s) = \mathbf{d}. \tag{1.72}$$

From the above discussion, one can see that similar to the PIO, the DO can eliminate the effect of disturbance $d(t)$ (1.57), by adding an integral term v.

Furthermore, due to that there are four parameter matrices in the structure, the DO offers more extra degrees of freedom over the PIO, which can be shown to be useful in many cases, such as observer design by pole placement in some region of the complex space and by adding other performances.

The concept of including additional dynamics in observers was attempted in [48], where the authors put stable dynamics into the classical observer. Then, the preliminary version of this structure was presented by Park [94, 93]. In [22], the design problem of dynamic observer-based control for a class of uncertain linear systems were solved by using LMI optimization approach. The study of [72] was concerned with the design of dynamic observer-based robust control for linear systems by introducing a weighting matrix.

To the best of the author's knowledge, the design problem of DO for linear systems has not been fully investigated so far, and still remains an open and unsolved problem. This topic will be investigated further in this thesis.

1.3 Large scale system and decentralized observers

In the past several years, the study of large systems has become a hot topic in the science research (see [6, 14, 23, 47, 51, 60, 69, 87, 102, 108, 120, 134] and references therein). This growing interest comes quite naturally from the relatively rapid expansion of our societal needs. In order to meet this need, it is natural that a great many systems are combined into a large one to realize more functions and enhance the competitive ability.
Take the electrical grid for example ([12] and [91]). An electric power system is a network of electrical components, which is used to generate, transmit and consume electric power. Let us consider an electric power system that supplies a region's homes and industries with power, which is shown in Figure 1.4.

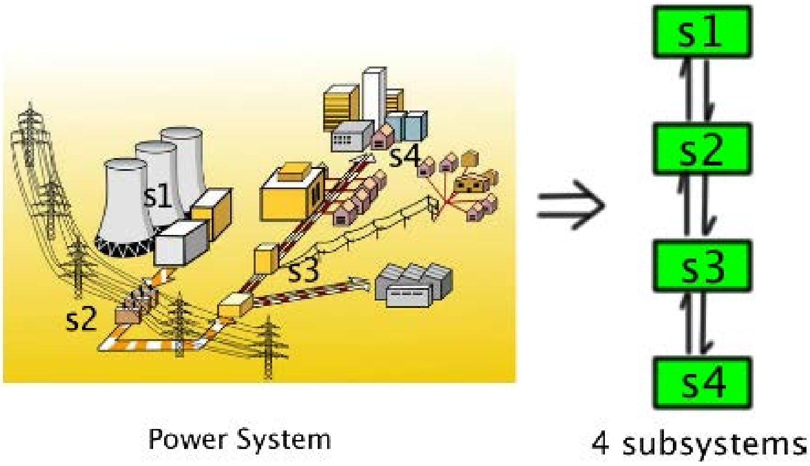

Figure 1.4: Electric Power System

One can see from Figure 1.4 that this power system is composed of four systems: the generators that supply the power (s_1), the transmission system that carries the power from the generating centers to the load centers (s_2), the distribution system that feeds the power to nearby loads (s_3) and the loads such as homes and industries (s_4).
The combination of different systems results in the expansion of system scale and then there appears the conception of large-scale system (LSS). The notation of LSS is applied to indicate such a kind of systems which have large dimensions and variables, complex structures (more links, more hierarchical or complicated relationship), target diverse, and can be considered as a group of small subsystems.

1.3.1 Large-scale system

Generally speaking, there are three essential factors of LSS:

1. Decomposing;
2. Complexity;
3. Loss of centrality.

We will give detailed descriptions of the three factors in the following sections.

1.3.1.1 Decomposing

First of all, one can see from the previous introductions that an LSS is of large dimension, i.e., it contains a large number of variables, or subsystems. Furthermore, these subsystems can be always coupled to many sub-subsystems. Thus, when we study an LSS, it is hard to consider all the subsystems as a whole system and design a controller for this system, due to the huge computation amount resulting from the large number of variables. In this case, in order to solve the control problem of LSS, we can firstly decompose an LSS into a set of small subsystems of lower dimensions.

Therefore, the first factor of LSS is decomposing, that is, the LSS can be decomposed into several subsystems. It can be illustrated through the following example. In Figure 1.5, we can see that an LSS has been decomposed into 7 interconnected subsystems $(s1, s2, \ldots, s7)$.

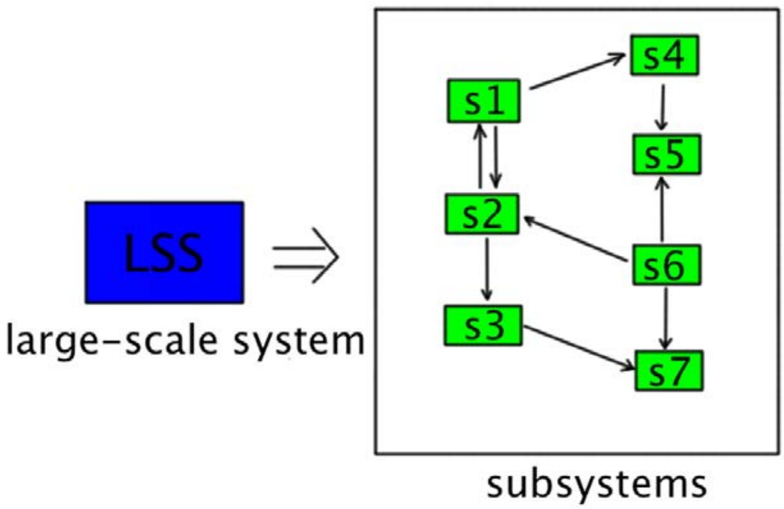

Figure 1.5: Large-Scale System

1.3.1.2 Complexity

On intuitive grounds it seems evident that without a connective structure, there would be no system at all, due to that the very essence of system concept relates to the notion of "something" being connected to "something" else. In the system analysis literature, there can be little doubt that the most overworked and least precise is the description "complexity", which is caused by the connectivity. Therefore, the complexity is another essence of LSS.

Let us return to Figure 1.5. One can see from Figure 1.5 that the output of subsystem s_3 has connections with subsystem s_7, but subsystem s_3 cannot obtain informations from subsystem s_7, since this connection is one-direction. On the other hand, although some subsystems do not have direct connections between each other, they can also communicate with other subsystems. For example, subsystems s_1 and s_6 can both influence subsystem s_3 through subsystem s_2. Subsystem s_1 can influence subsystem s_7 through subsystems s_2 and s_3.

1.3.1.3 Loss of centrality

The third factor is based on the notion of centrality. Given a system, the normal way is to design a central control by considering the system as a whole one. Unfortunately, as mentioned previously, it is difficult to design a central control for LSS due to the complexity, although the capability of data processing has been improved a lot in recent years.
In order to deal with the complexity, we can firstly decompose the LSS into a group of interconnected small subsystems according to the decomposition method. Then we can control each subsystem. In this case, it is observed that the overall system is no longer controlled by a single control but by several controls that all together represent a decentralized control, which means the failure of central concept. In summary, the third factor of LSS is that the concept of centrality fails in the study of LSS.
By considering the three factors of LSS, a large-scale linear system can be represented by the following state space representation:

$$\mathbf{S}: \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{1.73a}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}. \tag{1.73b}$$

Assume that system $\mathbf{S}$ is composed of r interconnected subsystems, and the i-th subsystem $\mathbf{S_i}$ is represented by

$$\mathbf{S_i}: \quad \dot{\mathbf{x}}_\mathbf{i} = \mathbf{A_i}\mathbf{x_i} + \mathbf{B}\mathbf{u_i} + \sum_{\substack{j \neq i \\ j=1}}^{\mathbf{r}} \mathbf{A_{ij}}\mathbf{x_j}, \tag{1.74a}$$
$$\mathbf{y_i} = \mathbf{C_i}\mathbf{x_i}, \tag{1.74b}$$

where $\mathbf{x_i} \in \Re^{n_i}$, $\mathbf{u_i} \in \Re^{m_i}$, $\mathbf{y_i} \in \Re^{p_i}$ are the state, the input and the output of the i-th subsystem, respectively. Matrices $\mathbf{A_i}$, $\mathbf{A_{ij}}$, $\mathbf{B_i}$ and $\mathbf{C_i}$ are known and of appropriate dimensions. The interconnection matrices $\mathbf{A_{ij}}$ represent how subsystem $\mathbf{S_j}$ affects subsystem $\mathbf{S_i}$.

1.3.2 Decentralized observers

One can see from the previous sections that due to the three factors of LSS, the traditional modeling method, control method, and optimization method fail to give reasonable solutions with reasonable efforts in study of LSS. Consequently, the decentralized control has been developed to solve the control problem of LSS.
The decentralized control is derived from the decentralization, which is one method used to treat the interconnection of LSS [7, 20, 45, 68, 86, 111, 132]. By using decentralization method, we can decompose the LSS into several subsystems which have week interconnections, then the interconnections of subsystems can be eliminated. In the decentralized control system, each subsystem can only gain part of the information of the whole system, and operate and manage only a subset of system variables.
Let us consider the following LSS, which is composed of N subsystems and the i-th subsystem is expressed as

$$\dot{x}_i = A_i x_i + B_i u_i + h_i(t, x), \tag{1.75a}$$
$$y_i = C_i x_i, \qquad i = 1, \ldots, N, \tag{1.75b}$$

where $x_i \in \Re^{n_i}$, $u_i \in \Re^{m_i}$ and $y_i \in \Re^{p_i}$ are the system state, the control input and the output, respectively. A_i, B_i and C_i are known constant matrices and of appropriate dimensions.

$h_i(t,x)$ represent the interconnections of other $N-1$ subsystems with the i-th subsystem, which is assumed to satisfy the following quadratic constraint:

$$h_i^T(t,x)h_i(t,x) \leq a_i^2 x^T H_i^T H_i x, \tag{1.76}$$

where scalers a_i are bounding parameters and matrices $H_i \in \Re^{l_i \times n}$ are constant bounding matrices.
With the representation of subsystems (1.75), the global system is characterized by the following state representation:

$$\dot{x} = Ax + Bu + h(t,x), \tag{1.77a}$$
$$y = Cx, \tag{1.77b}$$

where

$x \in \Re^n (n = \Sigma_{i=1}^N n_i, x = [x_1^T, \ldots, x_N^T]^T)$,
$u \in \Re^m (m = \Sigma_{i=1}^N m_i, u = [u_1^T, \ldots, u_N^T]^T)$,
$y \in \Re^p (p = \Sigma_{i=1}^N p_i, y = [y_1^T, \ldots, y_N^T]^T)$,
$h(t,x) = [h_1^T(t,x), \ldots, h_N^T(t,x)]^T$.
Matrices $A = diag(A_i)$, $B = diag(B_i)$ and $C = diag(C_i)$.

The interconnections $h(t,x)$ are assumed to satisfy the following quadratic constraint:

$$h^T(t,x)h(t,x) \leq x^T H^T \Psi^{-1} H x \tag{1.78}$$

where $H^T = \begin{bmatrix} H_1^T, \ldots, H_N^T \end{bmatrix}$ is a $l \times n$ matrix ($l = \sum_{i=1}^N l_i$), $\Psi = diag(\psi_1 I_{l_1}, \ldots, \psi_N I_{l_N})$ and $\psi_i = \alpha_i^{-2}$.
Next for system (1.77), let us consider the flowing decentralized observer-based control:

$$u = K_{dc}\hat{x}, \tag{1.79a}$$
$$\dot{\hat{x}} = A\hat{x} + Bu + K_{do}(y - \hat{y}), \tag{1.79b}$$
$$\hat{y} = C\hat{x}, \tag{1.79c}$$

where $\hat{x} = [\hat{x}_1^T, \ldots, \hat{x}_n^T]^T$ and $\hat{y} = [\hat{y}_1^T, \ldots, \hat{y}_N^T]^T$. $K_{dc} = diag(K_{dc_i})$ and $K_{do} = diag(K_{do_i})$ are the control and observer gain matrices to be designed and of appropriate dimensions.
By inserting control (1.79a) and observer (1.79b) into system(1.77a), we obtain the following system:

$$\dot{\eta}_{dc} = A_{dc}\eta_{dc} + h_{dc}, \tag{1.80a}$$
$$y = C_{dc}\eta_{dc}, \tag{1.80b}$$

where

$$\begin{aligned}
\eta_{dc} &= \begin{bmatrix} x \\ e \end{bmatrix}, \\
h_{dc} &= \begin{bmatrix} h(t,x) \\ -h(t,x) \end{bmatrix}, \\
A_{dc} &= \begin{bmatrix} A + BK_{dc} & BK_{dc} \\ 0 & A - BK_{do} \end{bmatrix}, \\
C_{dc} &= \begin{bmatrix} C & 0 \end{bmatrix}.
\end{aligned}$$

In this case, the decentralized observer-based control design problem is reduced to study system (1.80). One can determine matrices K_{dc} and K_{do} by making the closed-loop system (1.80) stable, based on the separation principle.
The applications of decentralized observer-based control can be found in many fields. For example, the authors designed a decentralized observer-based control for linear plants, via a shared communication network in [10]. The authors of [38] presented an observer-based decentralized control scheme for stability analysis of networked systems. In [67], a decentralized fuzzy observer-based control for large systems was presented. The reduced-order observer-based decentralized control design problem was solved by using LMI approach in [74]. The focus of paper [118] was on the design of a decentralized observer-based tracking control for a class of uncertain systems.

1.4 Background

In this section, we give some useful lemmas and theorems which will be used later.

1.4.1 Controllability and Stabilizability

Firstly, let us turn to some very important concepts in linear system theory [148].

Definition 1.4.1. *System* (1.6)*, or the pair* (A, B) *is said to be controllable, if for any initial state* $x(0) = x_0$*,* $t_1 > 0$ *and final state* x_1*, there exists an input* $u(.)$ *such that the solution of* (1.6) *satisfies* $x(t_1) = x_1$*. Otherwise, the system is said to be uncontrollable.*

Theorem 1.4.1. *The following are equivalent:*

(1) System (1.6) *is controllable or the pair* (A, B) *is controllable;*

(2) The controllability matrix $\mathcal{M}_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ *has full row rank;*

(3) The matrix $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ *has full row rank for all* $\lambda \in \mathbb{F}_c$*.*

Definition 1.4.2. *An continuous-time system* $\dot{x} = Ax$ *is said to be stable if all the eigenvalues of* A *are in the open left half plane., i.e., Re*$\lambda(A) < 0$*. A matrix* A *with such a property is said to be stable or Hurwitz.*

Definition 1.4.3. *System* (1.6) *or the pair* (A, B) *is stabilizable if* $A + BF$ *is stable for some* F*.*

Theorem 1.4.2. *The following are equivalent:*

(1) System (1.6) *is stabilizable or the pair* (A, B) *is stabilizable;*

(2) The matrix $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ *has full row rank for all Re*$\lambda \geq 0$*;*

(3) There exists a matrix F *such that* $A + BF$ *is Hurwitz.*

1.4.2 Observability and Detectability

Definition 1.4.4. *The dynamical system* (1.6) *or the pair* (C, A) *is said to be observable if for any time* $t_1 > 0$*, the initial state* $x(0) = x_0$ *can be determined from the time history of the input* $u(t)$ *and the output* $y(t)$ *in the interval of* $[0, t_1]$*. Otherwise, system* (1.6) *is said to be unobservable.*

Theorem 1.4.3. *The following are equivalent:*

(1) System (1.6) *is observable or the pair* (C, A) *is observable;*

(2) The observability matrix $\mathcal{M}_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ *has full column rank;*

(3) The matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ *has full column rank for all* $\lambda \in \mathbb{F}_c$*.*

Definition 1.4.5. *System* (1.6) *or the pair* (C, A) *is detectable if* $A + LC$ *is stable for some* L*.*

Theorem 1.4.4. *The following are equivalent:*

(1) System (1.6) *is detectable or the pair* (C, A) *is detectable;*

(2) The matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ *has full column rank for all* $Re\lambda \geq 0$*;*

(3) There exists a matrix L *such that* $A + LC$ *is Hurwitz.*

1.4.3 Schur complement lemma

[42] For any symmetric matrix M of the following form:

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

1. If C is invertible, then the following statements hold

   (1) $M > 0$, if and only if $C > 0$ and $A - BC^{-1}B^T > 0$.

   (2) If $C > 0$, then $M \geq 0$ if and only if $A - BC^{-1}B^T \geq 0$.

2. If A is invertible, then the following statements hold

   (1) $M > 0$ if and only if $A > 0$ and $C - B^T A^{-1} B > 0$.

   (2) If $A > 0$, then $M \geq 0$ if and only if $C - B^T A^{-1} B \geq 0$.

1.4.4 Lyapunov stability of linear systems

1. [107] For the following linear continuous-time system:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}, && (1.81a)\\ \mathbf{y} &= \mathbf{C}\mathbf{x}, && (1.81b)\end{aligned}$$

   let us consider the following Lyapunov function:

$$\mathbf{V} = \mathbf{x}^{\mathbf{T}}\mathbf{P}\mathbf{x}, \tag{1.82}$$

   where matrix $\mathbf{P} > \mathbf{0}$.

   Then the derivative of $\mathbf{V}$ is given by

$$\begin{aligned}\dot{\mathbf{V}} &= \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{P}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\mathbf{P}\dot{\mathbf{x}},\\ &= \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}. && (1.83)\end{aligned}$$

   In this case, system (1.81) is said to be asymptotically stable in the sense of Lyapunov if and only if there exists a matrix $\mathbf{P} > \mathbf{0}$ satisfying $\dot{\mathbf{V}} < \mathbf{0}$, i.e.,

$$\mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} < \mathbf{0}. \tag{1.84}$$

2. For the following linear discrete-time system:

$$\begin{aligned}\mathbf{x}(\mathbf{k}+\mathbf{1}) &= \mathbf{A}\mathbf{x}(\mathbf{k}), && (1.85a)\\ \mathbf{y}(\mathbf{k}) &= \mathbf{C}\mathbf{x}(\mathbf{k}), && (1.85b)\end{aligned}$$

   let us consider the following Lyapunov function:

$$\mathbf{V}(\mathbf{k}) = \mathbf{x}^{\mathbf{T}}(\mathbf{k})\mathbf{P}\mathbf{x}(\mathbf{k}), \tag{1.86}$$

   where matrix $\mathbf{P} > \mathbf{0}$.

   Then the discrete evolution of $\mathbf{V}$ is given by

$$\begin{aligned}\boldsymbol{\Delta}\mathbf{V}(\mathbf{k}) &= \mathbf{V}(\mathbf{k}+\mathbf{1}) - \mathbf{V}(\mathbf{k}),\\ &= \mathbf{x}^{\mathbf{T}}(\mathbf{k})(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A} - \mathbf{P})\mathbf{x}(\mathbf{k}). && (1.87)\end{aligned}$$

   In this case, system (1.85) is said to be asymptotically stable in the sense of Lyapunov if and only if there exists a matrix $\mathbf{P} > \mathbf{0}$ satisfying

$$-\mathbf{P} + \mathbf{A}\mathbf{P}\mathbf{A}^{\mathbf{T}} < \mathbf{0}. \tag{1.88}$$

1.4.5 Bounded real lemma

Lemma 1.4.1. *[16]. The continuous-time ystem* (1.1) *is asymptotically stable for* $u = 0$ *and* $\|Tuy\|_\infty < \gamma$ *for* $u \neq 0$*, if and only if there exist a matrix* $X > 0$ *and a positive scalar* γ *such that the following LMI holds*

$$\begin{bmatrix} A^TX + XA + C^TC & C^TD + XB \\ B^TX + D^TC & D^TD - \gamma^2 I \end{bmatrix} < 0. \tag{1.89}$$

Lemma 1.4.2. *[16]. The discrete-time ystem* (1.2) *is asymptotically stable for* $u = 0$ *and* $\|Tuy\|_\infty < \gamma$ *for* $u \neq 0$*, if and only if there exist a matrix* $X > 0$ *and a positive scalar* γ *such that the following LMI holds*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} < \begin{bmatrix} X & 0 \\ 0 & -\gamma^2 I \end{bmatrix}. \tag{1.90}$$

1.5 Structure of the thesis

The objectives of this dissertation are:

1. Propose a new form of H_∞ DO (HDO) for both continuous-time and discrete-time linear systems in the presence of disturbances and unknown inputs;

2. By inserting the HDO into closed-loop, an observer-based control is proposed for uncertain systems in the presence of disturbances;

3. By extending the obtained results to LSS, solve the decentralized observer-based control design problem of large-scale uncertain systems in the presence of disturbances.

The thesis is organized as follows: Chapter 1 is devoted to a presentation of background philosophy of our research work. In particular, we give an introduction to the observer firstly. Several kinds of observers for linear systems are presented. Then, we introduce the LSS and decentralized observer-based control. Thereafter, we give some lemmas and theorems that will be used in the following chapters.

Chapter 2 concerns the observer design problem for linear systems with unknown inputs and disturbances. For these systems, we propose a new form of HDO, which generalizes the results on the existing observers such as PO and PIO. Based on the parameterization of algebraic constraints obtained from the analysis of the estimation error, the observer design problem is reduced to the determination of one parameter matrix. Then the determination problem is formulated as an optimization problem in terms of LMIs. Both continuous-time and discrete-time systems are considered. Numerical examples are presented to illustrate the design procedure and the performances of the proposed HDO.
Furthermore, it is shown that the HDO design problem of uncertain systems in the presence of unknown inputs and disturbances can be transformed into an observer design problem of descriptor systems.

In Chapter 3, an observer-based control is proposed for uncertain system in the presence of disturbances, based on the HDO presented in Chapter 2. The uncertainty is time varying and assumed to be norm bounded. By using Lyapunov function approach, the control design problem is formulated as a bilinear matrix inequality (BMI) optimization problem. A two-step algorithm is proposed to transform the BMI into an LMI. A numerical example is presented to show the performance of the proposed observer-based control.

Both the results of Chapters 2 and 3 are extended to LSS in Chapter 4. For the large-scale uncertain systems in the presence of disturbances, we propose an H_∞ decentralized dynamic-observer-based control. These systems are composed of a set of subsystems, where the interconnections are assumed to be nonlinear and satisfy quadratic constraints. The control design is derived from the solution of BMI. A numerical example is presented to illustrate the performance of the proposed control.

In Chapter 5, the conclusions of our research work are presented and some perspectives for the future work are proposed.

CHAPTER

2

New H_∞ dynamic observer design

Contents

2.1 Introduction

In this chapter, we will study the observer design problem for linear systems in the presence of disturbances and unknown inputs. By investigating the results on the existing observer such as PO, PIO and DO, we propose a new form of HDO. The popularly used PO and PIO can be considered as the particular cases of this observer. The observer design problem is solved by using the LMI optimization method. Both continuous-time and discrete-time systems are considered.
This chapter is organized as follows: Section 2.2 is dedicated to investigate the observer design problem for continuous-time systems. The observer design problem for discrete-time systems is solved in section 2.3. Section 2.4 is dedicated to address the observer design problem for uncertain systems. In section 2.5 some conclusions are given.

2.2 A new H_∞ dynamic observer design for continuous-time systems

In this section, we will discuss the observer design problem for continuous-time systems. Let us consider the following linear system in the presence of unknown inputs and disturbances:

$$\dot{x} = Ax + Bu + Ld + Dw, \tag{2.1a}$$
$$y = Cx + Ew, \tag{2.1b}$$

with the initial state $x(0) = x_0$. $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the input vector, $d \in \Re^l$ is the unknown input vector, $w \in \Re^f$ is the disturbance vector of finite energy and $y \in \Re^p$ is the output vector, with $l \leq p$. Matrices A, B, C, D, E and L are known constant and of appropriate dimensions.

Remark 2.2.1. *The description of system* (2.1) *in the presence of unknown inputs and disturbances is general. In fact when the unknown input also affects the measurement y, we have the following description:*

$$\dot{x} = Ax + Bu + Ld + Dw, \tag{2.2a}$$
$$y = Cx + \overline{L}d + Ew, \tag{2.2b}$$

where matrix $\overline{L}$ is known constant and of appropriate dimension.
Assume that $rank\ \overline{L} = l_1 \leqslant l$, in this case there exist two nonsingular matrices U and V such that:

$$U\overline{L} = \begin{bmatrix} I_{l_1} & 0 \\ 0 & 0 \end{bmatrix} V.$$

Then by pre-multiplying equation (2.2b) *with matrix U, we obtain*

$$Uy = UCx + U\overline{L}d + UEw. \tag{2.3}$$

Assume that

$$Uy = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$
$$UC = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

$$\begin{aligned} Vd &= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \\ UE &= \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \end{aligned}$$

then equation (2.3) *can be rewritten as*

$$y_1 = C_1x + d_1 + E_1w, \tag{2.4a}$$
$$y_2 = C_2x + E_2w, \tag{2.4b}$$

From equation (2.4a) *we derive the following equation of d_1:*

$$d_1 = y_1 - C_1x - E_1w. \tag{2.5}$$

On the other hand, since matrix V is nonsingular, let also

$$LV^{-1} = \begin{bmatrix} L_1 & L_2 \end{bmatrix},$$

then we obtain the following result about equation (2.2a)*:*

$$\begin{aligned} \dot{x} &= Ax + Bu + LV^{-1}Vd + Dw, \\ &= Ax + Bu + L_1d_1 + L_2d_2 + Dw. \end{aligned} \tag{2.6}$$

Inserting equation (2.5) *into equation* (2.6) *yields*

$$\begin{aligned} \dot{x} &= Ax + Bu + L_1y_1 - L_1C_1x - L_1E_1w + L_2d_2 + Dw, \\ &= (A - L_1C_1)x + \begin{bmatrix} B & L_1 \end{bmatrix} \begin{bmatrix} u \\ y_1 \end{bmatrix} + L_2d_2 + (D - L_1E_1)w. \end{aligned} \tag{2.7}$$

Consequently, system (2.2) *can be written as*

$$\dot{x} = A_1x + B_1u_1 + L_2d_2 + D_1w, \tag{2.8a}$$
$$y_2 = C_2x + E_2w, \tag{2.8b}$$

where matrices

$$\begin{aligned} A_1 &= A - L_1C_1, \\ B_1 &= \begin{bmatrix} B & L_1 \end{bmatrix}, \\ u_1 &= \begin{bmatrix} u \\ y_1 \end{bmatrix}, \\ D_1 &= D - L_1E_1. \end{aligned}$$

One can see that system (2.8) *is in the form* (2.1). ■

In our thesis, for system (2.1), we propose a new form of HDO. Let us consider the following observer:

$$\dot{z} = Fz + Jy + Tu + Mv, \tag{2.9a}$$
$$\dot{v} = Pz + Qy + Gv, \tag{2.9b}$$
$$\hat{x} = Rz + Sy, \tag{2.9c}$$

where $z \in \Re^q$ is the state vector of the observer, with $n - p \leq q \leq n$, $v \in \Re^t$ is an auxiliary vector and $\hat{x} \in \Re^n$ is the state estimate. Matrices F, J, T, M, P, Q, G, R and S are unknown and of appropriate dimensions to be determined. Here the auxiliary vector v is similar to the additional term $w(t)$ in PIO [11], which is used to realize the steady state performance.

Remark 2.2.2. *Note that observer* (2.9) *is in a general form and generalizes the existing ones such as PO, PIO and DO:*

1. *In fact, for matrices $M = 0, P = 0, Q = 0$ and $G = 0$, we obtain the following PO:*

$$\begin{aligned} \dot{z} &= Fz + Jy + Tu, && \text{(2.10a)} \\ \hat{x} &= Rz + Sy. && \text{(2.10b)} \end{aligned}$$

2. *For matrices $P = -CR, Q = I_p - CS$ and $G = 0$, we obtain the following PIO:*

$$\begin{aligned} \dot{z} &= Fz + Jy + Tu + Mv, && \text{(2.11a)} \\ \dot{v} &= y - C\hat{x}, && \text{(2.11b)} \\ \hat{x} &= Rz + Sy. && \text{(2.11c)} \end{aligned}$$

3. *For matrices $F = A - \overline{D}C$, $J = \overline{D}$, $T = B$, $M = \overline{C}$, $P = \overline{B}C$, $Q = \overline{B}$ and $G = \overline{A}$, we obtain the following DO:*

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + \xi, && \text{(2.12a)} \\ \dot{v} &= \overline{A}v + \overline{B}\mu, && \text{(2.12b)} \\ \xi &= \overline{C}v + \overline{D}\mu, && \text{(2.12c)} \\ \mu &= y - \hat{y}. && \text{(2.12d)} \end{aligned}$$

■

The observer (2.9) aims to estimate the state of system (2.1), therefore our research is focused on the estimation error e ($e = \hat{x} - x$) and the observer design problem is to determine all the parameter matrices F, J, T, M, P, Q, G, R and S of observer (2.9) such that:

(1) For the disturbance $w = 0$, the estimation error $e(t) \to 0 (e(t) = e)$ for time $t \to \infty$;

(2) For the disturbance $w \neq 0$, the H_∞ norm $\|T_{we}\|_\infty < \gamma$;

(3) Decoupled the estimation error e from the unknown input d.

where T_{we} represents the transfer function matrix from the disturbance w to the estimation error e and γ is a positive scalar.

2.2.1 Observer design for systems without disturbances

In order to solve the problem, we first solve the problem for the system without disturbances ($w = 0$), then we will extend the result to the system in the presence of disturbances ($w \neq 0$). Before solving the problem, we define an error variable as:

$$\varepsilon = z - \Phi x, \tag{2.13}$$

where $\Phi \in \Re^{q \times n}$ is an arbitrary matrix. Then we can give the following lemma under the assumption that $w = 0$:

Lemma 2.2.1. *For $w = 0$, observer (2.9) is an observer for system (2.1) if there exists a matrix Φ such that:*

$$
\begin{aligned}
F\Phi - \Phi A + JC &= 0, && (2.14a)\\
\Phi L &= 0, && (2.14b)\\
T - \Phi B &= 0, && (2.14c)\\
P\Phi + QC &= 0, && (2.14d)\\
R\Phi + SC &= I_n, && (2.14e)
\end{aligned}
$$

and matrix

$$
\mathbb{A}_1 = \begin{pmatrix} F & M \\ P & G \end{pmatrix}
$$

is Hurwitz.

Proof. *In the case where $w = 0$, from the definition of error ε we obtain*

$$
\begin{aligned}
\dot{\varepsilon} &= \dot{z} - \Phi\dot{x},\\
&= Fz + Jy + Tu + Mv - \Phi Ax - \Phi Bu - \Phi Ld,\\
&= F(z - \Phi x) + F\Phi x + JCx + Tu + Mv - \Phi Ax - \Phi Bu - \Phi Ld,\\
&= F\varepsilon + (F\Phi - \Phi A + JC)x - \Phi Ld + (T - \Phi B)u + Mv. && (2.15)
\end{aligned}
$$

Furthermore, from equations (2.9b) and (2.9c) it follows that:

$$
\begin{aligned}
\dot{v} &= Pz + Qy + Gv,\\
&= P(z - \Phi x) + P\Phi x + QCx + Gv,\\
&= P\varepsilon + (P\Phi + QC)x + Gv, && (2.16)\\
\hat{x} &= Rz + Sy,\\
&= R(z - \Phi x) + R\Phi x + SCx,\\
&= R\varepsilon + (R\Phi + SC)x. && (2.17)
\end{aligned}
$$

One can see from equations (2.15) and (2.16) that the derivatives of error ε and auxiliary state v are independent of the system state x, the input u and the unknown input d, if the following conditions hold

$$
\begin{aligned}
F\Phi - \Phi A + JC &= 0,\\
\Phi L &= 0,\\
T - \Phi B &= 0,\\
P\Phi + QC &= 0.
\end{aligned}
$$

On the other hand, if the following condition holds

$$
R\Phi + SC = I_n,
$$

from equation (2.17) we obtain the following expression of estimation error e:

$$
e = R\varepsilon. \qquad (2.18)
$$

With these results, we have the following system:

$$\dot{\eta} = \mathbb{A}_1\eta, \tag{2.19a}$$
$$e = \mathbb{C}_1\eta, \tag{2.19b}$$

where

$$\eta = \begin{pmatrix} \varepsilon \\ v \end{pmatrix},$$
$$\mathbb{A}_1 = \begin{pmatrix} F & M \\ P & G \end{pmatrix},$$
$$\mathbb{C}_1 = \begin{pmatrix} R & 0 \end{pmatrix}.$$

From equation (2.18) *it is obvious that the estimation error* $e \to 0$ *if the error* $\varepsilon \to 0$*, when time* $t \to \infty$*. Furthermore, from system* (2.19) *we can see that the error* $\varepsilon \to 0$ *and the vector* $v \to 0$ *when time* $t \to \infty$*, if and only if matrix* $\mathbb{A}_1$ *is Hurwitz, which completes the proof.* □

Remark 2.2.3. *Notice that through **Lemma 2.2.1**, the estimation error* e *has been decoupled from the unknown input* d*.* ■

Subsequently, the observer design problem for systems without disturbances is reduced to determine all parameter matrices of observer (2.9) such that algebraic constraints (2.14) are satisfied and matrix $\mathbb{A}_1$ is Hurwitz.

2.2.2 Parameterization of algebraic constraints

In this section, we shall present the parameterization for the algebraic constraints (2.14) in order to reduce the number of parameter matrices needed to be determined. First of all, from equations (2.14d) and (2.14e) we have the following equation:

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \Phi \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{2.20}$$

The necessary and sufficient condition for equation (2.20) to have a solution is that the following rank condition must be satisfied:

$$rank \begin{pmatrix} \Phi \\ C \\ 0 \\ I_n \end{pmatrix} = rank \begin{pmatrix} \Phi \\ C \end{pmatrix} = n. \tag{2.21}$$

Assume that the rank condition (2.21) is satisfied, and let $\Lambda \in \Re^{q\times n}$ be an arbitrary matrix of full row rank such that:

$$rank \begin{pmatrix} \Lambda \\ C \end{pmatrix} = rank \begin{pmatrix} \Phi \\ C \end{pmatrix} = n, \tag{2.22}$$

then there always exist two parameter matrices Φ and $\overline{K}$ such that:

$$\begin{pmatrix} \Phi \\ C \end{pmatrix} = \begin{pmatrix} I_q & -\overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix}, \tag{2.23}$$

or equivalently

$$\Phi = \Lambda - \overline{K}C, \tag{2.24}$$

since matrix $\begin{pmatrix} \Lambda \\ C \end{pmatrix}$ is of full column rank.

By inserting equation (2.23) into equation (2.20), we have

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I_q & -\overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{2.25}$$

The general solution of equation (2.25) is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \left[\begin{pmatrix} 0 \\ I_n \end{pmatrix} \Upsilon^+ - Z(I_{q+p} - \Upsilon\Upsilon^+) \right] \begin{pmatrix} I_q & \overline{K} \\ 0 & I_p \end{pmatrix}, \tag{2.26}$$

where matrices $\Upsilon = \begin{pmatrix} \Lambda \\ C \end{pmatrix}$, and Z is an arbitrary matrix of appropriate dimension.

From equation (2.26), one can easily deduce matrices P, Q, R and S as:

$$\begin{aligned} P &= \begin{pmatrix} I_t & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} I_t & 0 \end{pmatrix} \left[\begin{pmatrix} 0 \\ I_n \end{pmatrix} \Upsilon^+ - Z(I_{q+p} - \Upsilon\Upsilon^+) \right] \begin{pmatrix} I_q & \overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ &= -Z_2\beta_1, \end{aligned} \tag{2.27a}$$

$$\begin{aligned} Q &= \begin{pmatrix} I_t & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ &= \begin{pmatrix} I_t & 0 \end{pmatrix} \left[\begin{pmatrix} 0 \\ I_n \end{pmatrix} \Upsilon^+ - Z(I_{q+p} - \Upsilon\Upsilon^+) \right] \begin{pmatrix} I_q & \overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ &= -Z_2\beta_2, \end{aligned} \tag{2.27b}$$

$$\begin{aligned} R &= \begin{pmatrix} 0 & I_n \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & I_n \end{pmatrix} \left[\begin{pmatrix} 0 \\ I_n \end{pmatrix} \Upsilon^+ - Z(I_{q+p} - \Upsilon\Upsilon^+) \right] \begin{pmatrix} I_q & \overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ &= \alpha_1 - Z_3\beta_1, \end{aligned} \tag{2.27c}$$

$$\begin{aligned} S &= \begin{pmatrix} 0 & I_n \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ &= \begin{pmatrix} 0 & I_n \end{pmatrix} \left[\begin{pmatrix} 0 \\ I_n \end{pmatrix} \Upsilon^+ - Z(I_{q+p} - \Upsilon\Upsilon^+) \right] \begin{pmatrix} I_q & \overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} 0 \\ I_p \end{pmatrix} \\ &= \alpha_2 - Z_3\beta_2, \end{aligned} \tag{2.27d}$$

where matrices

$$\begin{cases} Z_2 = \begin{pmatrix} I_t & 0 \end{pmatrix} Z, \\ Z_3 = \begin{pmatrix} 0 & I_n \end{pmatrix} Z, \\ \alpha_1 = \Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_2 = \Upsilon^+ \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix}, \\ \beta_1 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \beta_2 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix}. \end{cases} \tag{2.28}$$

Remark 2.2.4. *From the expression of the estimation error* (2.18) *one can see that the estimation error $e \to 0$ when the error $\varepsilon \to 0$. Also we can show that the estimation error e is independent of matrix R. Then we can take matrix $Z_3 = 0$ and obtain matrices R and S as:*

$$R = \alpha_1, \tag{2.29a}$$

$$S = \alpha_2. \tag{2.29b}$$

■

Secondly, from equations (2.14b) and (2.24), we have the following equation:

$$\overline{K}CL = \Lambda L, \tag{2.30}$$

which has a solution if and only if:

$$rank \begin{pmatrix} \Lambda L \\ CL \end{pmatrix} = rank \left[\begin{pmatrix} \Lambda \\ C \end{pmatrix} L \right] = rank\ L = rank\ CL, \tag{2.31}$$

which is the condition presented in [27].
In this case, one solution of equation (2.30) is given by

$$\overline{K} = \Lambda L(CL)^+. \tag{2.32}$$

One can deduce matrix $\overline{K}$ from equation (2.32) when matrix Λ is chosen. Then matrix Φ can be determined from equation (2.24).

Remark 2.2.5. *Notice that the determination of matrices Φ and $\overline{K}$ is based on the condition where there are unknown inputs, i.e., matrix $L \neq 0$. Next we will give the determination in the condition where matrix $L = 0$. In this case, from equation* (2.24) *we have*

$$\begin{pmatrix} \Phi & \overline{K} \end{pmatrix} \begin{pmatrix} I_n \\ C \end{pmatrix} = \Lambda, \tag{2.33}$$

which always has a solution since

$$rank \begin{pmatrix} I_n \\ C \\ \Lambda \end{pmatrix} = rank \begin{pmatrix} I_n \\ C \end{pmatrix} = n. \tag{2.34}$$

In this case, one solution of equation (2.33) *is given by*

$$\begin{pmatrix}\Phi & \overline{K}\end{pmatrix} = \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+, \tag{2.35}$$

or equivalently

$$\begin{aligned}\Phi &= \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+ \begin{pmatrix} I_n \\ 0 \end{pmatrix}, &(2.36)\\ \overline{K} &= \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}. &(2.37)\end{aligned}$$

Obviously, one can also deduce matrices $\overline{K}$ and Φ directly according to equations (2.36) *and* (2.37). ■

Subsequently, equation (2.14a) can be rewritten as:

$$F(\Lambda - \overline{K}C) - (\Lambda - \overline{K}C)A + JC = 0,$$

or equivalently

$$F\Lambda + (J - F\overline{K})C = (\Lambda - \overline{K}C)A,$$

which leads to

$$\begin{pmatrix}F & \overline{K}_1\end{pmatrix} \Upsilon = \Theta, \tag{2.38}$$

with matrices

$$\begin{aligned}\overline{K}_1 &= J - F\overline{K}, &(2.39a)\\ \Theta &= (\Lambda - \overline{K}C)A. &(2.39b)\end{aligned}$$

In this case, the general solution of equation(2.38) is given by

$$\begin{pmatrix}F & \overline{K}_1\end{pmatrix} = \Theta\Upsilon^+ - Z_1(I_{q+p} - \Upsilon\Upsilon^+),$$

or equivalently

$$\begin{aligned}F &= [\Theta\Upsilon^+ - Z_1(I_{q+p} - \Upsilon\Upsilon^+)] \begin{bmatrix} I_q \\ 0 \end{bmatrix}, \\ &= \alpha_3 - Z_1\beta_1, &(2.40a)\\ \overline{K}_1 &= [\Theta\Upsilon^+ - Z_1(I_{q+p} - \Upsilon\Upsilon^+)] \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \\ &= \alpha_4 - Z_1\beta_3, &(2.40b)\end{aligned}$$

where Z_1 is an arbitrary matrix of appropriate dimension and matrices

$$\begin{cases} \alpha_3 = \Theta\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_4 = \Theta\Upsilon^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \beta_3 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} 0 \\ I_p \end{pmatrix}. \end{cases} \tag{2.41}$$

Then, from equations (2.39) and (2.40), we can deduce matrix J:

$$\begin{aligned} J &= \overline{K}_1 + F\overline{K}, \\ &= \alpha_4 - Z_1\beta_3 + \alpha_4\overline{K} - Z_1\beta_3\overline{K}, \\ &= \Theta\alpha_2 - Z_1\beta_2. \end{aligned} \tag{2.42}$$

From the above results one can see that matrix Λ is needed to be chosen according to rank condition (2.22). Then once matrices Z_1, Z_2, M and G are determined, all the parameter matrices of observer (2.9) can be deduced.
Furthermore, according to equations (2.27a) and (2.40a), the matrix $\mathbb{A}_1$ of system (2.19) becomes

$$\begin{aligned} \mathbb{A}_1 &= \begin{pmatrix} \alpha_3 - Z_1\beta_1 & M \\ -Z_2\beta_1 & G \end{pmatrix}, \\ &= \begin{pmatrix} \alpha_3 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} Z_1 & M \\ Z_2 & G \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & -I_t \end{pmatrix}, \end{aligned} \tag{2.43}$$

and system (2.19) becomes

$$\dot{\eta} = \mathbb{A}_1\eta, \tag{2.44a}$$
$$e = \mathbb{C}_1\eta, \tag{2.44b}$$

where matrices

$$\begin{aligned} \mathbb{A}_1 &= \mathbb{A}_{11} - \mathbb{Z}\mathbb{A}_{12}, \\ \mathbb{A}_{11} &= \begin{pmatrix} \alpha_3 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathbb{Z} &= \begin{pmatrix} Z_1 & M \\ Z_2 & G \end{pmatrix}, \\ \mathbb{A}_{12} &= \begin{pmatrix} \beta_1 & 0 \\ 0 & -I_t \end{pmatrix}. \end{aligned}$$

Consequently, all the four parameter matrices Z_1, Z_2, M and G are included in matrix $\mathbb{Z}$.
With all the above results we can see that in the case where the disturbance is missing $(w = 0)$, the observer design problem is reduced to study system (2.44), in other words, to determine matrix $\mathbb{Z}$ such that system (2.44) is stable.
Before solving the design problem, we give the stability condition of system (2.44) in the following lemma:

Lemma 2.2.2. *The necessary and sufficient condition for matrix $\mathbb{A}_1$ to be Hurwitz is that the pair $\left[\begin{pmatrix} \beta_1 & 0 \\ 0 & -I_t \end{pmatrix}, \begin{pmatrix} \alpha_3 & 0 \\ 0 & 0 \end{pmatrix}\right]$ is detectable, or equivalently, for all the unstable eigenvalues $\lambda(Re\lambda \geq 0)$ the following rank condition holds*

$$rank \begin{pmatrix} \lambda I_n - A & -L \\ C & 0 \end{pmatrix} = n + l. \tag{2.45}$$

Proof. *From equations* (2.22) *and* (2.30)*, it follows that:*

$$
\begin{aligned}
& rank\begin{pmatrix} \lambda I_n - A & -L \\ C & 0 \end{pmatrix} \\
= \;& rank\left[\begin{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix} & 0 \\ 0 & I_p \end{pmatrix}\begin{pmatrix} \lambda I_n - A & -L \\ C & 0 \end{pmatrix}\right] \\
= \;& rank\begin{pmatrix} \lambda\Lambda - \Lambda A & -\Lambda L \\ \lambda C - CA & -CL \\ C & 0 \end{pmatrix} \\
= \;& rank\left[\begin{pmatrix} I_q & 0 & 0 \\ 0 & I_p & -\lambda I_p \\ 0 & 0 & I_p \end{pmatrix}\begin{pmatrix} \lambda\Lambda - \Lambda A & -\Lambda L \\ \lambda C - CA & -CL \\ C & 0 \end{pmatrix}\right] \\
= \;& rank\begin{pmatrix} \lambda\Lambda - \Lambda A & -\Lambda L \\ -CA & -CL \\ C & 0 \end{pmatrix} \\
= \;& rank\left[\begin{pmatrix} I_q & -\Lambda L(CL)^+ & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix}\begin{pmatrix} \lambda\Lambda - \Lambda A & -\Lambda L \\ -CA & -CL \\ C & 0 \end{pmatrix}\right] \\
= \;& rank\begin{pmatrix} \lambda\Lambda - [\Lambda - \Lambda L(CL)^+C]A & 0 \\ -CA & -CL \\ C & 0 \end{pmatrix} \\
= \;& rank\begin{pmatrix} \lambda\begin{pmatrix} I_q & 0 \end{pmatrix}\Upsilon - \Theta\Upsilon^+\Upsilon & 0 \\ -CA & -CL \\ C & 0 \end{pmatrix} \\
= \;& rank\left[\begin{pmatrix} I_q & 0 & 0 \\ 0 & (CL)^+ & 0 \\ 0 & I_p - CL(CL)^+ & 0 \\ 0 & 0 & I_p \end{pmatrix}\begin{pmatrix} \lambda\begin{pmatrix} I_q & 0 \end{pmatrix}\Upsilon - \Theta\Upsilon^+\Upsilon & 0 \\ -CA & -CL \\ C & 0 \end{pmatrix}\right] \\
= \;& rank\begin{pmatrix} [\lambda\begin{pmatrix} I_q & 0 \end{pmatrix} - \Theta\Upsilon^+]\Upsilon & 0 \\ -(CL)^+CA & -I_l \\ -[I_p - CL(CL)^+]CA & 0 \\ C & 0 \end{pmatrix}.
\end{aligned}
$$

If the rank condition (2.45) *is satisfied, i.e.,*

$$
rank\begin{pmatrix} [\lambda\begin{pmatrix} I_q & 0 \end{pmatrix} - \Theta\Upsilon^+]\Upsilon & 0 \\ -(CL)^+CA & -I_l \\ -[I_p - CL(CL)^+]CA & 0 \\ C & 0 \end{pmatrix} = n + l,
$$

we obtain

$$
rank\begin{pmatrix} [\lambda\begin{pmatrix} I_q & 0 \end{pmatrix} - \Theta\Upsilon^+]\Upsilon & 0 \\ -(CL)^+CA & -I_l \\ -[I_p - CL(CL)^+]CA & 0 \\ C & 0 \end{pmatrix} = n + l,
$$

$$
\begin{aligned}
\Leftrightarrow \quad rank \begin{pmatrix} [\lambda\begin{pmatrix} I_q & 0 \end{pmatrix} - \Theta\Upsilon^+]\Upsilon \\ -[I_p - CL(CL)^+]CA \\ C \end{pmatrix} &= n \\
\Leftrightarrow \quad rank \left[\begin{pmatrix} \lambda\begin{pmatrix} I_q & 0 \end{pmatrix} - \Theta\Upsilon^+ \\ -[I_p - CL(CL)^+]CA\Upsilon^+ \\ C\Upsilon^+ \end{pmatrix} \Upsilon \right] &= rank\ \Upsilon.
\end{aligned}
$$

According to [26], the above rank condition is equivalent to

$$
\begin{aligned}
rank \begin{pmatrix} \lambda\begin{pmatrix} I_q & 0 \end{pmatrix} - \Theta\Upsilon^+ \\ -[I_p - CL(CL)^+]CA\Upsilon^+ \\ C\Upsilon^+ \\ I_{q+p} - \Upsilon\Upsilon^+ \end{pmatrix} &= q + p \\
\Leftrightarrow \quad rank \begin{pmatrix} \lambda I_q - \Theta\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix} \\ -[I_p - CL(CL)^+]CA\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix} \\ C\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix} \\ [I_{q+p} - \Upsilon\Upsilon^+] \begin{pmatrix} I_q \\ 0 \end{pmatrix} \end{pmatrix} &= q \\
\Leftrightarrow \quad rank \begin{pmatrix} \lambda I_q - \alpha_3 \\ \beta_1 \\ -[I_p - CL(CL)^+]CA\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix} \\ C\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix} \end{pmatrix} &= q \\
\Leftrightarrow \quad rank \begin{pmatrix} \lambda I_q - \alpha_3 \\ \beta_1 \end{pmatrix} &= q.
\end{aligned}
$$

Furthermore, we know that matrix $\mathbb{A}_1 (\mathbb{A}_1 = \mathbb{A}_{11} - \mathbb{Z}\mathbb{A}_{12})$ *is Hurwitz if and only if the pair* $[\mathbb{A}_{12}, \mathbb{A}_{12}]$ *is detectable, i.e., for all unstable eigenvalues* $\lambda (Re\lambda \geq 0)$

$$
\begin{aligned}
rank \begin{pmatrix} \lambda I_q - \alpha_3 & 0 \\ \beta_1 & 0 \\ 0 & -I_t \end{pmatrix} &= t + q \\
\Leftrightarrow \quad rank \begin{pmatrix} \lambda I_q - \alpha_3 \\ \beta_1 \end{pmatrix} &= q,
\end{aligned}
$$

which is equivalent to

$$
rank \begin{pmatrix} \lambda I_n - A & -L \\ C & 0 \end{pmatrix} = n + l.
$$

This completes the proof. □

Now we give the following theorem to determine matrix $\mathbb{Z}$:

Theorem 2.2.1. *There exists a parameter matrix $\mathbb{Z}$ such that system* (2.44) *is asymptotically stable if and only if there exist a symmetric positive definite matrix $\mathbb{P}$ and a matrix $\mathbb{Y}$ satisfying the following condition:*

$$\mathbb{A}_{11}^T\mathbb{P} + \mathbb{P}^T\mathbb{A}_{11} - \mathbb{Y}\mathbb{A}_{12} - \mathbb{A}_{12}^T\mathbb{Y}^T < 0. \tag{2.46}$$

In this case, matrix $\mathbb{Z}$ is determined by $\mathbb{Z} = \mathbb{P}^{-1}\mathbb{Y}$.

Proof. *For system* (2.44)*, let us choose the following Lyapunov function [107]:*

$$V(\eta) = \eta^T\mathbb{P}\eta, \tag{2.47}$$

where matrix $\mathbb{P} = \mathbb{P}^T > 0$.

Then the derivative of Lyapunov function $V(\eta)$ along the solution of system (2.44) *is given by*

$$\begin{aligned} \dot{V}(\eta) &= \dot{\eta}^T\mathbb{P}\eta + \eta^T\mathbb{P}\dot{\eta}, \\ &= \eta^T(\mathbb{A}_{11} - \mathbb{Z}\mathbb{A}_{12})^T\mathbb{P}\eta + \eta^T\mathbb{P}(\mathbb{A}_{11} - \mathbb{Z}\mathbb{A}_{12})\eta, \\ &= \eta^T(\mathbb{A}_{11}^T\mathbb{P} + \mathbb{P}^T\mathbb{A}_{11} - \mathbb{Y}\mathbb{A}_{12} - \mathbb{A}_{12}^T\mathbb{Y}^T)\eta, \end{aligned} \tag{2.48}$$

with matrix $\mathbb{Y} = \mathbb{P}\mathbb{Z}$.

One can see from equation (2.48) *that $\dot{V}(\eta) < 0$, in other words, system* (2.44) *is stable, if and only if the following inequality holds*

$$\mathbb{A}_{11}^T\mathbb{P} + \mathbb{P}^T\mathbb{A}_{11} - \mathbb{Y}\mathbb{A}_{12} - \mathbb{A}_{12}^T\mathbb{Y}^T < 0.$$

This completes the proof. □

Remark 2.2.6. *The determination of matrix $\mathbb{Z}$ is independent of the choice of the generalized inverse Υ^+. The proof can be found in [29].* ■

With all these results, the design procedure of the proposed observer for systems in the case where $w = 0$ can be obtained as follows:

step1. Choose matrix Λ according to rank condition (2.22);

step2. Compute matrices $\overline{K}, \Phi$ and Θ from equations (2.32), (2.24) and (2.39b);

step3. Compute matrices $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ and β_3 according to equations (2.28) and (2.41);

step4. Compute matrix $\mathbb{Z}$ according to **Theorem 2.2.1**;

step5. Deduce all the parameter matrices of observer (2.9) according to equations (2.27), (2.40), (2.42) and (2.14c).

2.2.3 Observer design for systems in the presence of disturbances

In this section, based on the previous results for systems without disturbances, we will design an HDO for systems in the presence of disturbances. Let us reconsider the linear system (2.1) and observer (2.9) in the case where $w \neq 0$.
According to **Lemma 2.2.1**, the derivative of error ε is given by

$$\dot{\varepsilon} = F\varepsilon + Mv + (JE - \Phi D)w. \tag{2.49}$$

Furthermore, the derivative of vector v becomes

$$\dot{v} = P\varepsilon + Gv + QEw, \tag{2.50}$$

and the expression of estimation error e becomes

$$e = R\varepsilon + SEw. \tag{2.51}$$

In this case, we obtain the following system:

$$\dot{\zeta}_1 = \mathbb{A}_1\zeta_1 + \mathbb{B}_1 w, \tag{2.52a}$$
$$e = \mathbb{C}_1\zeta_1 + \mathbb{D}_1 w, \tag{2.52b}$$

where

$$\begin{aligned} \zeta_1 &= \begin{pmatrix} \varepsilon \\ v \end{pmatrix}, \\ \mathbb{B}_1 &= \begin{pmatrix} JE - \Phi D \\ QE \end{pmatrix}, \\ \mathbb{D}_1 &= SE. \end{aligned}$$

Remark 2.2.7. *According to **Remark 2.2.4**, we take matrix $Z_3 = 0$ in the case where $w = 0$. In the case where $w \neq 0$, if we do not fix matrix $Z_3 = 0$, matrices R and S become*

$$\begin{aligned} R &= \alpha_1 - Z_3\beta_1, \\ S &= \alpha_2 - Z_3\beta_2. \end{aligned}$$

Consequently, matrices $\mathbb{C}_1$ and $\mathbb{D}_1$ become

$$\mathbb{C}_1 = \mathbb{C}_{11} - Z_3\mathbb{C}_{12}, \tag{2.54}$$

with

$$\begin{aligned} \mathbb{C}_{11} &= \begin{bmatrix} \alpha_1 & 0 \end{bmatrix}, \\ \mathbb{C}_{12} &= \begin{bmatrix} \beta_1 & 0 \end{bmatrix}, \end{aligned}$$

$$\mathbb{D}_1 = \mathbb{D}_{11} - Z_3\mathbb{D}_{12}, \tag{2.55}$$

with

$$\begin{aligned} \mathbb{D}_{11} &= \alpha_2 E, \\ \mathbb{D}_{12} &= \beta_2 E. \end{aligned}$$

According to bounded-real lemma [16], one can know that system (2.52) *is asymptotically stable for $w = 0$ and $\|T_{we}\|_\infty < \gamma$ for $w \neq 0$, if and only if there exist a symmetric positive definite matrix X and a given positive scalar γ such that the following LMI is satisfied*

$$\begin{pmatrix} \mathbb{A}_1^T X + X\mathbb{A}_1 + \mathbb{C}_1^T\mathbb{C}_1 & X\mathbb{B}_1 + \mathbb{C}_1^T\mathbb{D}_1 \\ (X\mathbb{B}_1)^T + \mathbb{D}_1^T\mathbb{C}_1 & \mathbb{D}_1^T\mathbb{D}_1 - \gamma^2 I_f \end{pmatrix} < 0. \tag{2.56}$$

By inserting matrices $\mathbb{C}_1$ and $\mathbb{D}_1$ into LMI (2.56)*, we obtain*

$$\mathbb{C}_1^T\mathbb{D}_1 = \mathbb{C}_{11}^T\mathbb{D}_{11} - \mathbb{C}_{11}^T Z_3\mathbb{D}_{12} - \mathbb{C}_{12}^T Z_3^T\mathbb{D}_{11} + \mathbb{C}_{12}^T Z_3^T Z_3\mathbb{D}_{12},$$

In this case, LMI (2.56) *becomes a BMI. In order to avoid this bilinearity and simplify the computation, in the case where $w \neq 0$ we fix matrix $Z_3 = 0$ and matrices R and S are given by equations* (2.29). ■

Remark 2.2.8. *According to the previous parameterization results, we can rewrite all the matrices $\mathbb{A}_1$, $\mathbb{B}_1$, $\mathbb{C}_1$ and $\mathbb{D}_1$ of system* (2.52) *as follows:*

$$\begin{aligned} \mathbb{A}_1 &= \mathbb{A}_{11} - \mathbb{Z}\mathbb{A}_{12}, \\ \mathbb{B}_1 &= \mathbb{B}_{11} - \mathbb{Z}\mathbb{B}_{12}, \\ \mathbb{B}_{11} &= \begin{pmatrix} -\Phi D + \Theta\Upsilon^+ \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix} E \\ 0 \end{pmatrix}, \\ \mathbb{B}_{12} &= \begin{pmatrix} \beta_2 E \\ 0 \end{pmatrix}, \\ \mathbb{C}_1 &= \begin{pmatrix} \alpha_1 & 0 \end{pmatrix}, \\ \mathbb{D}_1 &= \alpha_2 E. \end{aligned}$$

■

With all the above results, the design problem of the proposed HDO for the system in the presence of disturbances and unknown inputs is reduced to the analysis of system (2.52), in other words, to the determination of matrix $\mathbb{Z}$ such that:

1. For $w = 0$, system (2.52) is asymptotically stable;

2. For $w \neq 0$, the H_∞ norm $\|T_{we}\|_\infty$ is minimized.

Subsequently, we can give the following theorem to determine the parameter matrix $\mathbb{Z}$:

Theorem 2.2.2. *There exists a parameter matrix $\mathbb{Z}$ such that system* (2.52) *is asymptotically stable for $w = 0$ and $\|T_{we}\|_\infty < \gamma$ for $w \neq 0$, if and only if there exist a symmetric positive definite matrix X, a matrix Y and a positive scalar γ such that the following LMI is satisfied*

$$\begin{bmatrix} X\mathbb{A}_{11} + \mathbb{A}_{11}^T X - Y\mathbb{A}_{12} - (Y\mathbb{A}_{12})^T + \mathbb{C}_1^T\mathbb{C}_1 & X\mathbb{B}_{11} - Y\mathbb{B}_{12} + \mathbb{C}_1^T\mathbb{D}_1 \\ (X\mathbb{B}_{11} - Y\mathbb{B}_{12})^T + \mathbb{D}_1^T\mathbb{C}_1 & \mathbb{D}_1^T\mathbb{D}_1 - \gamma^2 I_f \end{bmatrix} < 0. \tag{2.57}$$

In this case, matrix $\mathbb{Z}$ is given by $\mathbb{Z} = X^{-1}Y$.

Proof. *Insert the matrices of **Remark 2.2.8** with their values into LMI* (2.56) *and define matrix $Y = X\mathbb{Z}$, then we can obtain LMI* (2.57)*, which completes the proof.* □

Occasionally, the solution of LMI (2.57) leads to matrices $M = 0, P = 0$ and $Q = 0$. In this case, the obtained observer is of the PO type. Considering the shortcoming of PO, we propose the following theorem to derive the general solution:

Theorem 2.2.3. *Let matrices* $\mathfrak{B} = \begin{pmatrix} -I_{q+t} \\ 0 \end{pmatrix}$ *and* $\mathfrak{C} = \begin{pmatrix} \mathbb{A}_{12} & \mathbb{B}_{12} \end{pmatrix}$ *, then there exists a parameter matrix* $\mathbb{Z}$ *such that system* (2.52) *is asymptotically stable for* $w = 0$ *and* $\|T_{we}\|_\infty < \gamma$ *for* $w \neq 0$*, if and only if there exist a symmetric positive definite matrix X and a positive scalar* γ *such that the following LMIs are satisfied*

$$\mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I < 0, \tag{2.58a}$$
$$\mathfrak{C}^{T\perp} \mathbb{Q} \mathfrak{C}^{T\perp T} < 0, \tag{2.58b}$$

where matrix

$$\mathbb{Q} = \begin{pmatrix} X\mathbb{A}_{11} + \mathbb{A}_{11}^T X + \mathbb{C}_1^T \mathbb{C}_1 & X\mathbb{B}_{11} + \mathbb{C}_1^T \mathbb{D}_1 \\ \mathbb{B}_{11}^T X + \mathbb{D}_1^T \mathbb{C}_1 & \mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I_f \end{pmatrix}. \tag{2.59}$$

Suppose that the above statements hold, let matrices $(\mathfrak{B}_l, \mathfrak{B}_r)$ *and* $(\mathfrak{C}_l, \mathfrak{C}_r)$ *be any full rank factors of matrices* $\mathfrak{B}$ *and* $\mathfrak{C}$*, i.e.* $\mathfrak{B} = \mathfrak{B}_l \mathfrak{B}_r$ *and* $\mathfrak{C} = \mathfrak{C}_l \mathfrak{C}_r$*.*
In this case, matrices $\mathbb{Z} = X^{-1}Y$ *and* Y *is given by*

$$Y = \mathfrak{B}_r^+ \mathbb{K} \mathfrak{C}_l^+ + \mathfrak{Z} - \mathfrak{Z} \mathfrak{C}_l \mathfrak{C}_l^+, \tag{2.60}$$

where $\mathfrak{Z}$ *is an arbitrary matrix and*

$$\mathbb{K} \triangleq -\mathbb{R}^{-1} \overline{\Phi} \mathfrak{C}_r^T (\mathfrak{C}_r \overline{\Phi} \mathfrak{C}_r^T)^{-1} + \mathbb{S}^{1/2} \mathbb{L} (\mathfrak{C}_r \overline{\Phi} \mathfrak{C}_r^T)^{-1/2}, \tag{2.61}$$

$$\mathbb{S} \triangleq \mathbb{R}^{-1} - \mathbb{R}^{-1} [\overline{\Phi} - \overline{\Phi} \mathfrak{C}_r^T (\mathfrak{C}_r \overline{\Phi} \mathfrak{C}_r^T)^{-1} \mathfrak{C}_r \overline{\Phi}] \mathbb{R}^{-1}, \tag{2.62}$$

$\mathbb{L}$ *is an arbitrary matrix such that* $\|\mathbb{L}\| < 1$ *and* $\mathbb{R}$ *is an arbitrary positive definite matrix such that:*

$$\overline{\Phi} \triangleq (\mathbb{R}^{-1} - \mathbb{Q})^{-1} > 0. \tag{2.63}$$

Proof. *We can rewrite LMI* (2.57) *as:*

$$\begin{aligned} &\begin{pmatrix} \Pi & X\mathbb{B}_{11} - Y\mathbb{B}_{12} + \mathbb{C}_1^T \mathbb{D}_1 \\ \mathbb{B}_{11}^T X - \mathbb{B}_{12}^T Y^T + \mathbb{D}_1^T \mathbb{C}_1 & \mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I_f \end{pmatrix} < 0, \\ \Leftrightarrow \quad & \mathbb{Q} + \mathfrak{B} Y \mathfrak{C} + (\mathfrak{B} Y \mathfrak{C})^T < 0, \end{aligned} \tag{2.64}$$

where matrices

$$\begin{aligned} \Pi &= X\mathbb{A}_{11} + \mathbb{A}_{11}^T X - Y\mathbb{A}_{12} - \mathbb{A}_{12}^T Y^T + \mathbb{C}_1^T \mathbb{C}_1, \\ \mathbb{Q} &= \begin{pmatrix} X\mathbb{A}_{11} + \mathbb{A}_{11}^T X + \mathbb{C}_1^T \mathbb{C}_1 & X\mathbb{B}_{11} + \mathbb{C}_1^T \mathbb{D}_1 \\ \mathbb{B}_{11}^T X + \mathbb{D}_1^T \mathbb{C}_1 & \mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I_f \end{pmatrix}, \\ \mathfrak{B} &= \begin{pmatrix} -I_{q+t} \\ 0 \end{pmatrix} \text{ and } \mathfrak{C} = \begin{pmatrix} \mathbb{A}_{12} & \mathbb{B}_{12} \end{pmatrix}. \end{aligned}$$

According to [106], matrix inequality (2.64) *is equivalent to*

$$\mathfrak{B}^\perp \mathbb{Q} \mathfrak{B}^{\perp T} < 0, \tag{2.65a}$$
$$\mathfrak{C}^{T\perp} \mathbb{Q} \mathfrak{C}^{T\perp T} < 0. \tag{2.65b}$$

In this case, matrix Y can be obtained by:

$$Y = \mathfrak{B}_r^+ \mathbb{K} \mathfrak{C}_l^+ + \mathfrak{Z} - \mathfrak{B}_r^+ \mathfrak{B}_r \mathfrak{Z} \mathfrak{C}_l \mathfrak{C}_l^+, \tag{2.66}$$

where $\mathfrak{Z}$ is an arbitrary matrix and

$$\mathbb{K} \triangleq -\mathbb{R}^{-1} \mathfrak{B}_l^T \overline{\Phi} \mathfrak{C}_r^T (\mathfrak{C}_r \overline{\Phi} \mathfrak{C}_r^T)^{-1} + \mathbb{S}^{1/2} \mathbb{L} (\mathfrak{C}_r \overline{\Phi} \mathfrak{C}_r^T)^{-1/2}, \tag{2.67}$$

$$\mathbb{S} \triangleq \mathbb{R}^{-1} - \mathbb{R}^{-1} \mathfrak{B}_l^T [\overline{\Phi} - \overline{\Phi} \mathfrak{C}_r^T (\mathfrak{C}_r \overline{\Phi} \mathfrak{C}_r^T)^{-1} \mathfrak{C}_r \overline{\Phi}] \mathfrak{B}_l \mathbb{R}^{-1}, \tag{2.68}$$

$\mathbb{L}$ is an arbitrary matrix such that $\|\mathbb{L}\| < 1$ and $\mathbb{R}$ is an arbitrary positive definite matrix such that:

$$\overline{\Phi} \triangleq (\mathfrak{B}_l \mathbb{R}^{-1} \mathfrak{B}_l^T - \mathbb{Q})^{-1} > 0. \tag{2.69}$$

Furthermore, since matrix $\mathfrak{B} = \begin{pmatrix} -I_{q+t} \\ 0 \end{pmatrix}$, we obtain the following results:

$$\begin{cases} \mathfrak{B}^\perp = \begin{pmatrix} 0 & -I_f \end{pmatrix}, \\ \mathfrak{B}_l = I_{q+t+f}, \\ \mathfrak{B}_r = \begin{pmatrix} -I_{q+t} \\ 0 \end{pmatrix}, \\ \mathfrak{B}_r^+ = \begin{pmatrix} -I_{q+t} & 0 \end{pmatrix}. \end{cases} \tag{2.70}$$

Replacing matrices (2.70) *into equations* (2.66)-(2.68) *and inequality* (2.69) *leads to equations* (2.60)-(2.62) *and inequality* (2.63). *In this case, matrix inequalities* (2.65) *become* (2.58). *The proof is completed.* □

Remark 2.2.9. *In* **Theorem 2.2.3***, matrices $\mathfrak{Z}$, $\mathbb{L}$, $\mathbb{R}$ can be chosen by adding additional constraints, such as the pole location of observer in special region of complex plane. If there is no extra additional constraints, matrices $\mathfrak{Z}$, $\mathbb{L}$, $\mathbb{R}$ can be chosen according to* **Theorem 2.2.3**.

Now, with all the above results, the design procedure of the HDO for systems in the presences of disturbances and unknown inputs can be summarized as the following algorithm:

step1. Choose matrix Λ according to rank condition (2.22);

step2. Compute matrices $\overline{K}, \Phi$ and Θ from equations (2.32), (2.24) and (2.39b);

step3. Compute matrices α_1, α_2, α_3, α_4, β_1, β_2 and β_3 according to equations (2.28) and (2.41);

step4. Compute matrices $\mathbb{A}_{11}$, $\mathbb{A}_{12}$, $\mathbb{B}_{11}$, $\mathbb{B}_{12}$, $\mathbb{C}_1$ and $\mathbb{D}_1$ according to **Remark 2.2.8**;

step5. Compute matrices $\mathbb{Q}$ and X from **Theorem 2.2.3**;

step6. Choose matrices $\mathbb{L}, \mathbb{R}$ and $\mathfrak{Z}$ according to **Theorem 2.2.3** and compute matrix Y.

step7. Compute matrix $\mathbb{Z}$ from $\mathbb{Z} = X^{-1}Y$;

step8. Deduce all the parameter matrices of observer (2.9) according to equations (2.27), (2.40), (2.42) and (2.14c).

2.2.4 Particular cases

As it is discussed previously, the proposed observer generalizes the results on the existing observers PO and PIO. Thus in this section, we shall show how to obtain the LMI formulations for the two particular cases:

2.2.4.1 Proportional observer

As shown in **Remark 2.2.2**, the PO corresponds to matrices $M = 0, P = 0, Q = 0$ and $G = 0$. In this case, we obtain the following PO:

$$\dot{z} = Fz + Jy + Tu, \tag{2.71a}$$
$$\hat{x} = Rz + Sy, \tag{2.71b}$$

and system (2.52) becomes

$$\dot{\zeta}_2 = \mathbb{A}_2\zeta_2 + \mathbb{B}_2 w, \tag{2.72a}$$
$$e = \mathbb{C}_2\zeta_2 + \mathbb{D}_2 w, \tag{2.72b}$$

where vector $\zeta_2 = \varepsilon$ and matrices

$$\mathbb{A}_2 = \mathbb{A}_{21} - \mathbb{Z}_2\mathbb{A}_{22},$$

$$\mathbb{A}_{21} = \alpha_3, \mathbb{A}_{22} = \beta_1,$$

$$\mathbb{B}_2 = \mathbb{B}_{21} - \mathbb{Z}_2\mathbb{B}_{22},$$

$$\mathbb{B}_{21} = \Theta\alpha_2 E - \Phi D, \mathbb{B}_{22} = \beta_2 E,$$

$$\mathbb{C}_2 = \alpha_1, \mathbb{D}_2 = \alpha_2 E,$$

$$\mathbb{Z}_2 = Z_1.$$

Consequently, matrices $\mathbb{Q}, \mathfrak{B}, \mathfrak{C}$ of **Theorem 2.2.3** become

$$\mathbb{Q} = \begin{pmatrix} X\mathbb{A}_{21} + \mathbb{A}_{21}^T X + \mathbb{C}_2^T\mathbb{C}_2 & X\mathbb{B}_{21} + \mathbb{C}_2^T\mathbb{D}_2 \\ (X\mathbb{B}_{21} + \mathbb{C}_2^T\mathbb{D}_2)^T & \mathbb{D}_2^T\mathbb{D}_2 - \gamma^2 I_f \end{pmatrix},$$
$$\mathfrak{B} = \begin{pmatrix} -I_q \\ 0 \end{pmatrix}, \mathfrak{C} = \begin{pmatrix} \mathbb{A}_{22} & \mathbb{B}_{22} \end{pmatrix}.$$

The parameter matrix $\mathbb{Z}_2$ can be determined by

$$\mathbb{Z}_2 = X^{-1}Y.$$

2.2.4.2 Proportional integral observer

As shown in **Remark 2.2.2**, the PIO corresponds to matrices $P = -CR, Q = I_p - CS$, and $G = 0$. In this case we obtain the following PIO:

$$\dot{z} = Fz + Jy + Tu + Mv, \tag{2.73a}$$
$$\dot{v} = y - C\hat{x}, \tag{2.73b}$$
$$\hat{x} = Rz + Sy, \tag{2.73c}$$

and system (2.52) becomes

$$\dot{\zeta}_3 = \mathbb{A}_3\zeta_3 + \mathbb{B}_3 w, \tag{2.74a}$$
$$e = \mathbb{C}_3\zeta_3 + \mathbb{D}_3 w, \tag{2.74b}$$

where vector $\zeta_3 = \begin{pmatrix} \varepsilon \\ v \end{pmatrix}$ and matrices

$$\mathbb{A}_3 = \mathbb{A}_{31} - \mathbb{A}_{32}\mathbb{Z}_3\mathbb{A}_{33},$$

$$\mathbb{A}_{31} = \begin{pmatrix} \alpha_3 & 0 \\ -C\alpha_1 & 0 \end{pmatrix}, \mathbb{A}_{32} = \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \mathbb{A}_{33} = \begin{pmatrix} \beta_1 & 0 \\ 0 & -I_p \end{pmatrix},$$

$$\mathbb{B}_3 = \mathbb{B}_{31} - \mathbb{B}_{32}\mathbb{Z}_3\mathbb{B}_{33},$$

$$\mathbb{B}_{31} = \begin{pmatrix} \Theta\alpha_2 E - \Phi D \\ D_2 - C\alpha_2 E \end{pmatrix}, \mathbb{B}_{32} = \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \mathbb{B}_{33} = \begin{pmatrix} \beta_2 E \\ 0 \end{pmatrix},$$

$$\mathbb{C}_3 = \begin{pmatrix} \alpha_1 & 0 \end{pmatrix}, \mathbb{D}_3 = \alpha_2 E$$

$$\mathbb{Z}_3 = \begin{pmatrix} Z_1 & M \end{pmatrix}.$$

Consequently, matrices $\mathbb{Q}, \mathfrak{B}, \mathfrak{C}$ of **Theorem 2.2.3** become

$$\mathbb{Q} = \begin{pmatrix} X\mathbb{A}_{31} + \mathbb{A}_{31}^T X + \mathbb{C}_3^T\mathbb{C}_3 & X\mathbb{B}_{31} + \mathbb{C}_3^T\mathbb{D}_3 \\ (X\mathbb{B}_{31} + \mathbb{C}_3^T\mathbb{D}_3)^T & \mathbb{D}_3^T\mathbb{D}_3 - \gamma^2 I_f \end{pmatrix},$$

$$\mathfrak{B} = \begin{pmatrix} -X\mathbb{A}_{32} \\ 0 \end{pmatrix}, \mathfrak{C} = \begin{pmatrix} \mathbb{A}_{33} & \mathbb{B}_{33} \end{pmatrix}.$$

The parameter matrix $\mathbb{Z}_3$ can be determined by

$$\mathbb{Z}_3 = Y.$$

Furthermore, we can see that the orthogonal complement of matrix $\mathfrak{B}$ is

$$\mathfrak{B}^\perp = \begin{pmatrix} \mathbb{A}_{32}^\perp X^{-1} & 0 \\ 0 & I_f \end{pmatrix}.$$

In this case, we can use equations (2.66) - (2.68) and inequality (2.69) to obtain the parameter matrices Y and $\mathbb{Z}_3$.

2.2.5 Numerical example

In the previous sections, we have solved the design problem of HDO for linear systems in the presence of disturbances and unknown inputs. In this section, a numerical example is presented to illustrate the design procedure and its performances of the proposed observer. Let us consider a system of the form (2.1) with:

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, F = \begin{bmatrix} 0.1 \\ 0.5 \\ 1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.1 \\ 1 \\ 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5 & 1 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 \end{bmatrix}.$$

The initial conditions are $x_1(0) = 1$, $x_2(0) = 1.1$ and $x_3(0) = 0.5$.

2.2.5.1 Observer design

For the above system, we design an HDO by using the previously proposed method. According to the design procedure, we first choose matrix

$$\Lambda = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

according to rank condition (2.22).
Then by choosing matrices

$$\begin{aligned}
\mathfrak{Z} &= 0, \\
\mathbb{R} &= 4.65 \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\mathbb{L} &= 0.1 \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},
\end{aligned}$$

we obtain the following results:

$$X = \begin{bmatrix} 0.6470 & 0.3359 & 0.0352 & 0 \\ 0.3359 & 0.6470 & -0.0352 & 0 \\ 0.0352 & -0.0352 & 0.2385 & 0 \\ 0 & 0 & 0 & 0.9829 \\ & 0 & 0 & \end{bmatrix},$$

$$Y = 1.0e^{+13} \times \begin{bmatrix} -0.3601 & 1.8232 & 0.7841 & 0.7315 & -0.0000 \\ 0.0523 & -0.2650 & -0.1140 & -0.1063 & -0.0000 \\ 0.1899 & -0.9616 & -0.4136 & -0.3858 & -0.0000 \\ -0.1539 & 0.7791 & 0.3351 & 0.3126 & -0.0000 \end{bmatrix},$$

$$\mathbb{Z} = 1.0e^{+13} \times \begin{bmatrix} -0.9358 & 4.7378 & 2.0376 & 1.9010 & -0.0000 \\ 0.6227 & -3.1524 & -1.3558 & -1.2649 & -0.0000 \\ 1.0265 & -5.1966 & -2.2349 & -2.0851 & -0.0000 \\ -0.1566 & 0.7927 & 0.3409 & 0.3181 & -0.0000 \end{bmatrix}$$

$$\gamma = 0.9.$$

Consequently, the obtained HDO is given by

$$\begin{aligned} \dot{z}_g &= \begin{bmatrix} -2.5340 & 0.8257 & 0.0 \\ 2.1018 & -1.2241 & -0.0 \\ -0.0627 & 1.4076 & -3.0 \end{bmatrix} z_g + \begin{bmatrix} -0.7815 \\ 0.7920 \\ 2.4943 \end{bmatrix} y \\ &+ \begin{bmatrix} -1.2581 \\ 1.2581 \\ 2.4194 \end{bmatrix} u + \begin{bmatrix} -0.0529 \\ -0.1506 \\ -0.4253 \end{bmatrix} v_g, \\ \dot{v}_g &= \begin{bmatrix} -0.1060 & -0.0889 & 0.0000 \end{bmatrix} z_g + 0.0053\, y \\ &+ 0.3205\, v_g, \\ \hat{x}_g &= \begin{bmatrix} 0.3333 & -0.3333 & 0.6667 \\ -0.6667 & 0.6667 & -0.3333 \\ 0.8333 & -0.1667 & 0.0000 \end{bmatrix} z_g + \begin{bmatrix} 0.0645 \\ 0.3226 \\ 0.6452 \end{bmatrix} y. \end{aligned}$$

Remark 2.2.10. *The obtained HDO is a full-order observer, for the dimension of observer $q = 3$. We can also design a reduced-order HDO by choosing $q = 2$, for the dimension of output is 1.*

In order to give a comparison with PO and PIO, we also design a PO and a PIO, which are designed under the same condition where $\gamma = 0.9$.
The obtained PO is given by

$$\begin{aligned} \dot{z}_p &= \begin{bmatrix} -2.2242 & 1.5515 & 0.5515 \\ -0.7573 & -4.5146 & 0.4854 \\ -0.7076 & -1.4151 & -2.4151 \end{bmatrix} z_p \\ &+ \begin{bmatrix} -1.2581 \\ 1.2581 \\ 2.4194 \end{bmatrix} u + \begin{bmatrix} 0.4485 \\ 1.5146 \\ 1.4151 \end{bmatrix} y. \\ \hat{x}_p &= \begin{bmatrix} 0.3333 & -0.3333 & 0.6667 \\ -0.6667 & 0.6667 & -0.3333 \\ 0.8333 & -0.1667 & 0.0000 \end{bmatrix} z_p + \begin{bmatrix} 0.0645 \\ 0.3226 \\ 0.6452 \end{bmatrix} y. \end{aligned}$$

The obtained PIO is given by

$$
\begin{aligned}
\dot{z}_{pi} &= \begin{bmatrix} -2.2244 & 1.0336 & -0.0 \\ 1.6071 & -1.6510 & -0.0 \\ -1.4666 & 0.1140 & -3.0 \end{bmatrix} z_{pi} + \begin{bmatrix} -0.8117 \\ 0.8117 \\ 2.5286 \end{bmatrix} y \\
&+ \begin{bmatrix} -1.2581 \\ 1.2581 \\ 2.4194 \end{bmatrix} u + \begin{bmatrix} -10.6338 \\ -11.9765 \\ -4.5172 \end{bmatrix} v_{pi}, \\
\dot{v}_{pi} &= y - C\hat{x}_{pi}, \\
\hat{x}_{pi} &= \begin{bmatrix} 0.3333 & -0.3333 & 0.6667 \\ -0.6667 & 0.6667 & -0.3333 \\ 0.8333 & -0.1667 & 0.0000 \end{bmatrix} z_{pi} + \begin{bmatrix} 0.0645 \\ 0.3226 \\ 0.6452 \end{bmatrix} y.
\end{aligned}
$$

2.2.5.2 Simulation results

In order to study the performances of our observer, we do some simulations with the obtained observer. Firstly, we investigate the convergence of the proposed observer.

The simulation results are shown in the following figures. Figure 2.1 represents the input added in the simulation. Figures 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7 represent the results with initial conditions $z_1(0) = 0.3$, $z_2(0) = 0.1$, $z_3(0) = 0.5$ and $v(0) = 0.1$.

Figures 2.2, 2.4, and 2.6 represent the states and theirs estimates obtained from HDO. The solid line represents the original state and the dashed line represents the estimates. Figures 2.3, 2.5, and 2.7 represent the estimation errors obtained from HDO.

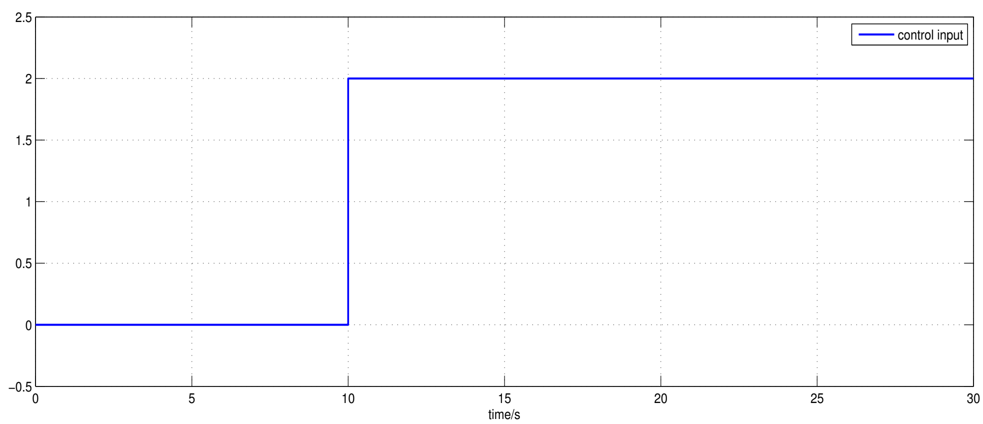

Figure 2.1: Control input

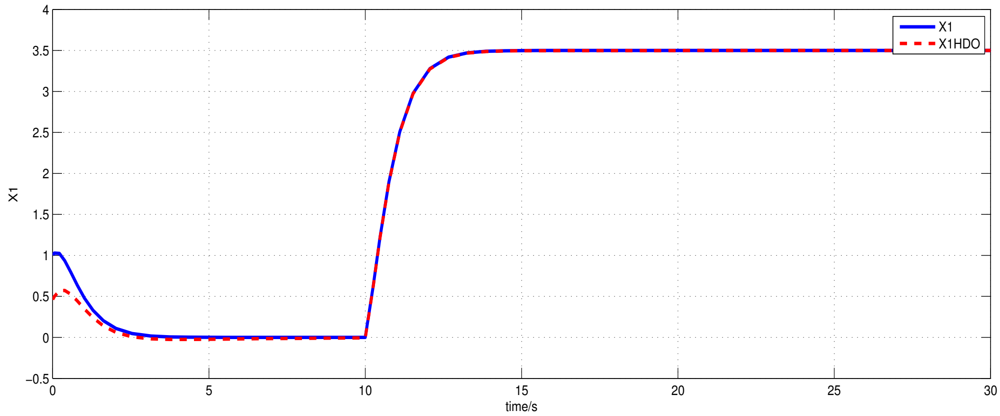

Figure 2.2: State x_1 and its estimate (solid line: original state; dashed line: estimate)

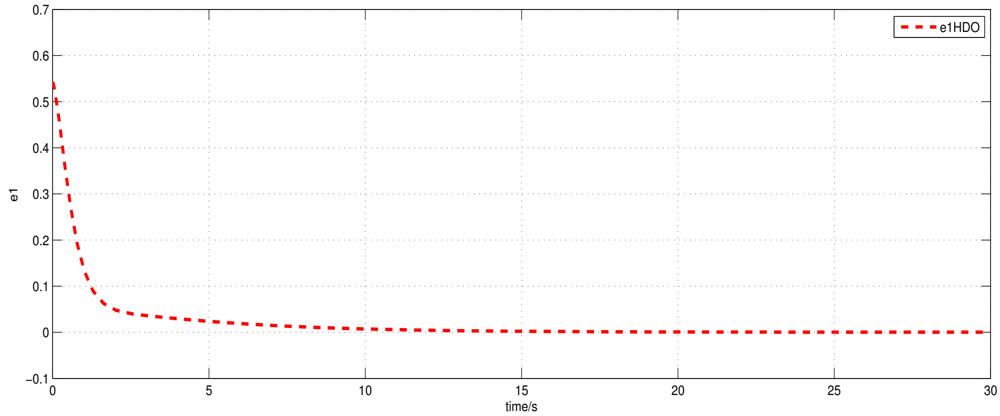

Figure 2.3: Estimation error e_1

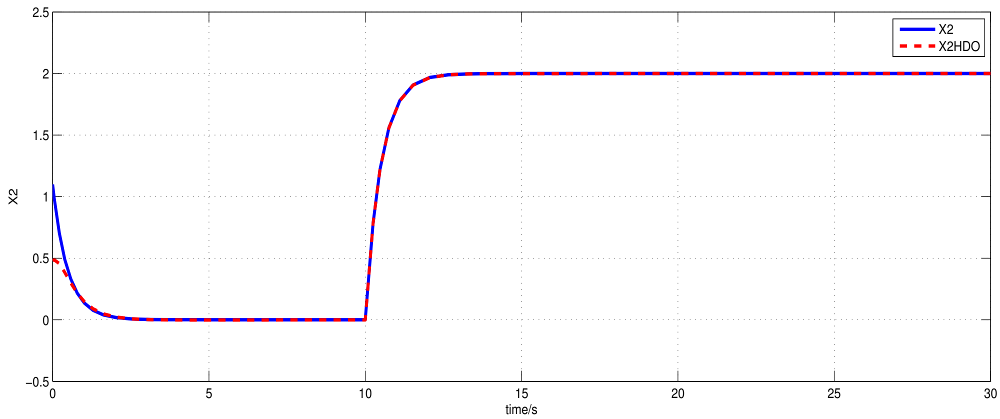

Figure 2.4: State x_2 and its estimate (solid line: original state; dashed line: estimate)

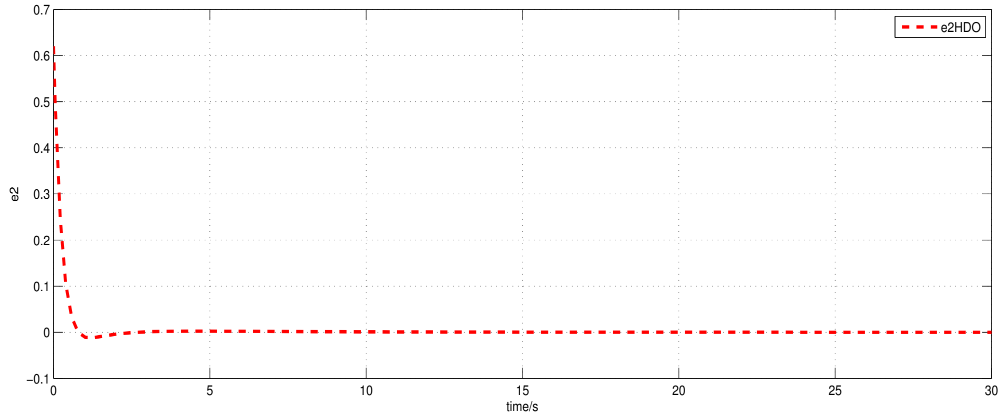

Figure 2.5: Estimation error e_2

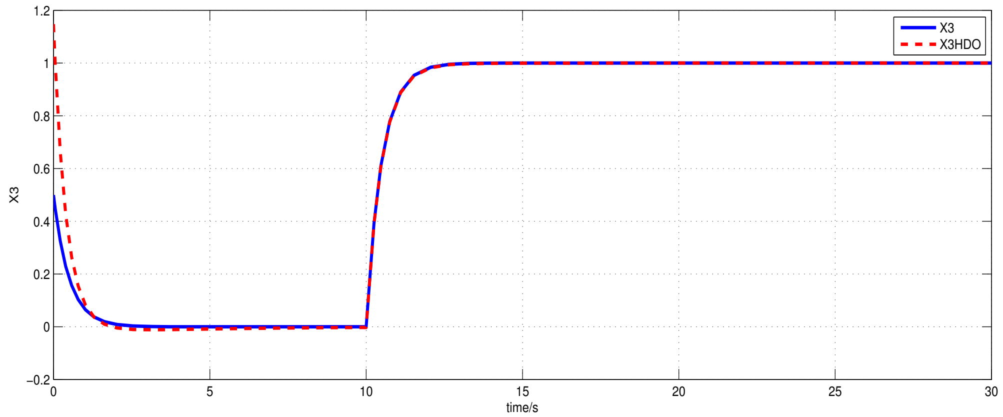

Figure 2.6: State x_3 and its estimate (solid line: original state; dashed line: estimate)

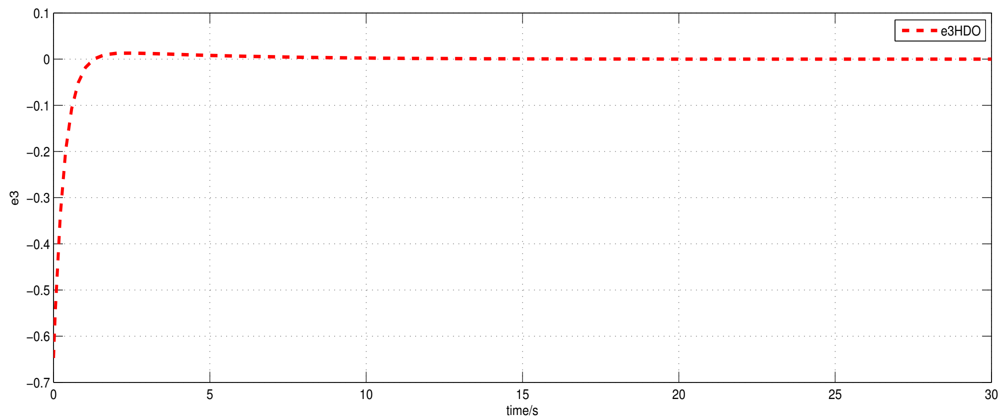

Figure 2.7: Estimation error e_3

From the above figures, we can see the convergence performances of the proposed observer. Although the initial conditions of observer are different from the initial conditions of system, the estimate converges to the system state.

Furthermore, we add a parametric uncertainty ΔA_1 to system matrix A. In this case, the disturbed matrix becomes $A + \Delta A_1$, where

$$\Delta A_1 = [0.5 + 0.1 \sin(\pi/10)t]\delta,$$

and

$$\delta = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 1 & 0.1 & 0.5 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}.$$

Then we compare the proposed observer with the designed PO and PIO, in the presence of disturbances, unknown inputs and uncertainties.

Figure 2.8 represents the control input, the unknown input and the disturbance used in the simulation. The control input is added from 15s to 35s. The unknown input is added in the beginning and ends at 100s. The disturbance appears from 18s to 30s.

Figures 2.9, 2.10, 2.13, 2.14, 2.17 and 2.18 represent the original system states and their estimates obtained from HDO, PO and PIO, respectively. Figures 2.9, 2.13 and 2.17 represent the time interval from the time 0s to 200s, while figures 2.10, 2.14 and 2.18 represent the time interval from the time 0s to 40s. The solid line represents the original system state. The dashed line, dotted line and dash-dotted line represent the state estimates obtained from HDO, PO and PIO, respectively.

Figures 2.11, 2.12, 2.15, 2.16, 2.19 and 2.20 represent the estimation errors obtained from HDO, PO and PIO, respectively. Figures 2.11, 2.15 and 2.19 represent the time interval from the time 0s to 200s, while figures 2.12, 2.16 and 2.20 represent the time interval from the time 0s to 40s. The dashed line, dotted line and dash-dotted line represent the estimation errors obtained from HDO, PO and PIO, respectively.

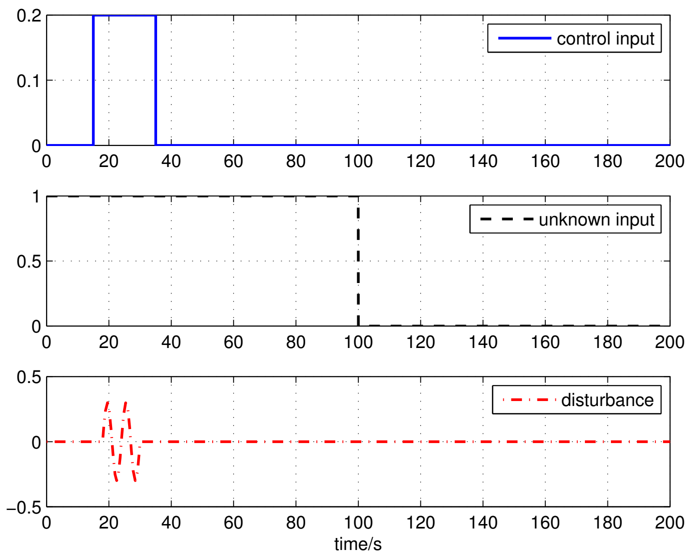

Figure 2.8: All inputs (solid line: control input; dashed line: unknown input; dash-dotted line: disturbance)

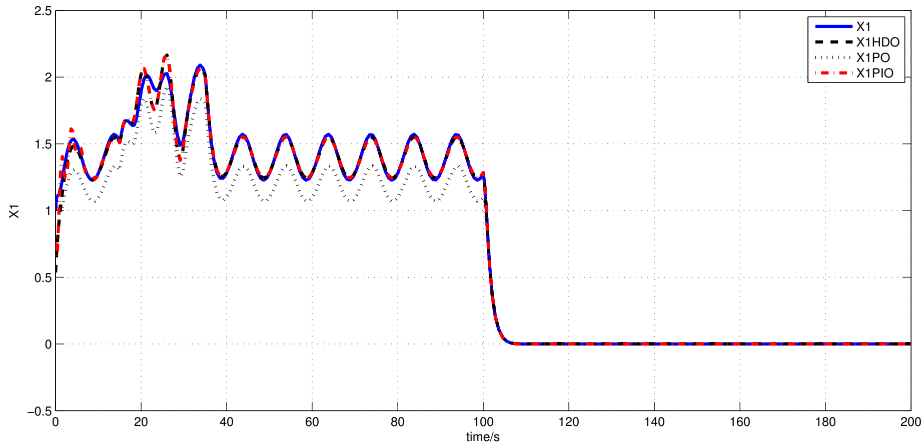

Figure 2.9: State x_1 and its estimates for 200s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

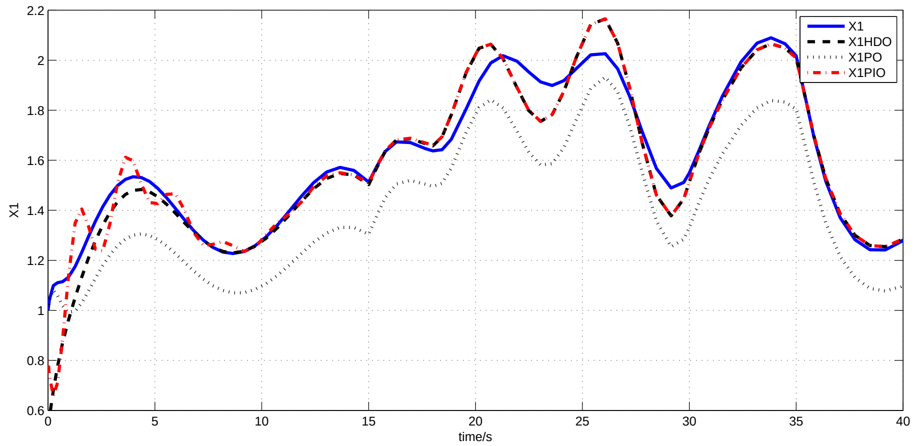

Figure 2.10: State x_1 and its estimates for 40s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

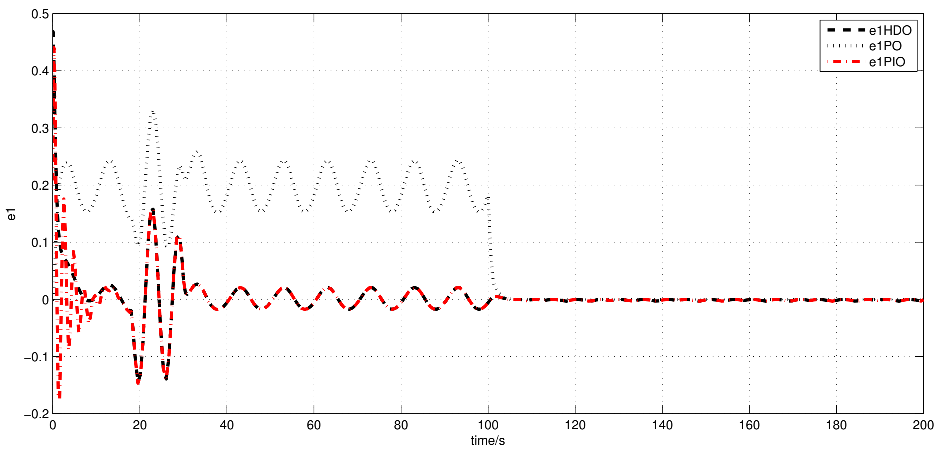

Figure 2.11: Estimation error e_1 for 200s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

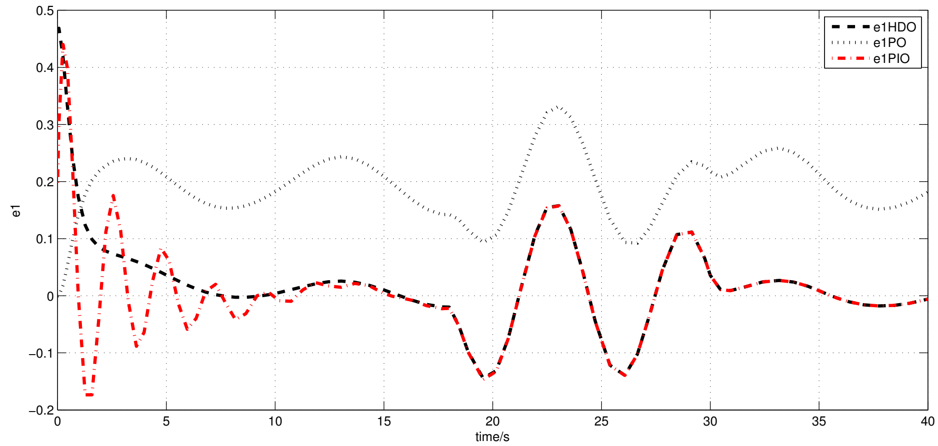

Figure 2.12: Estimation error e_1 for 40s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

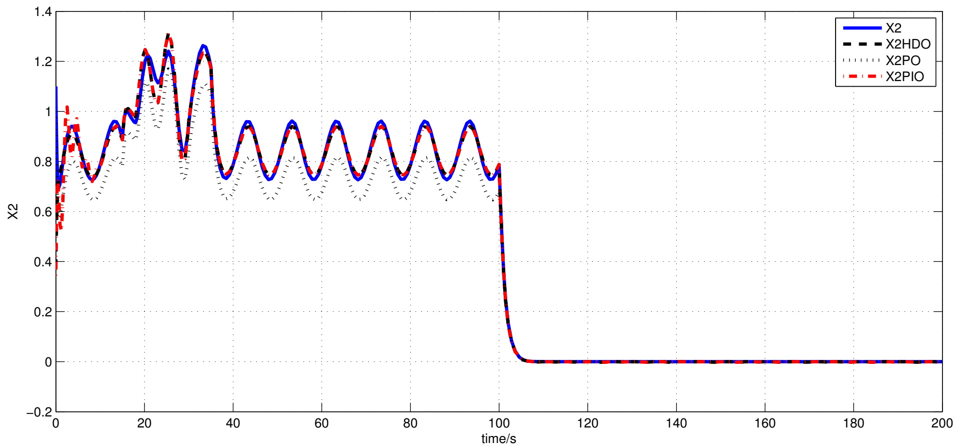

Figure 2.13: State x_2 and its estimates for 200s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

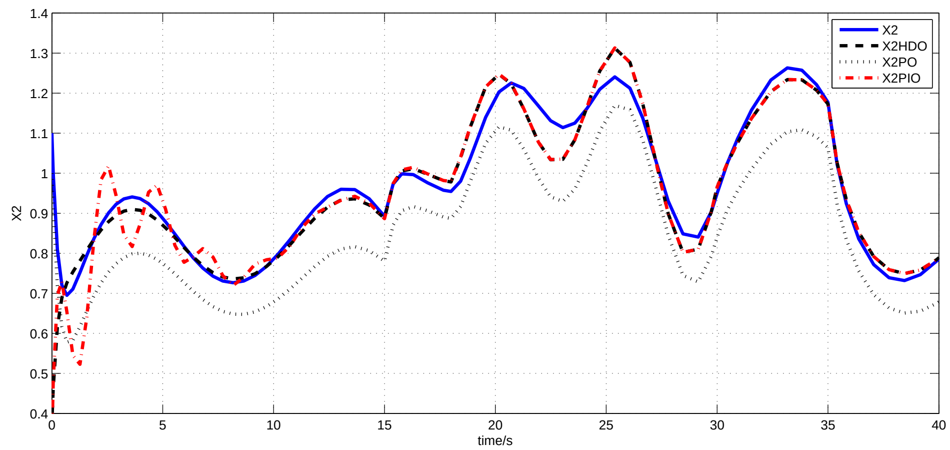

Figure 2.14: State x_2 and its estimates for 40s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

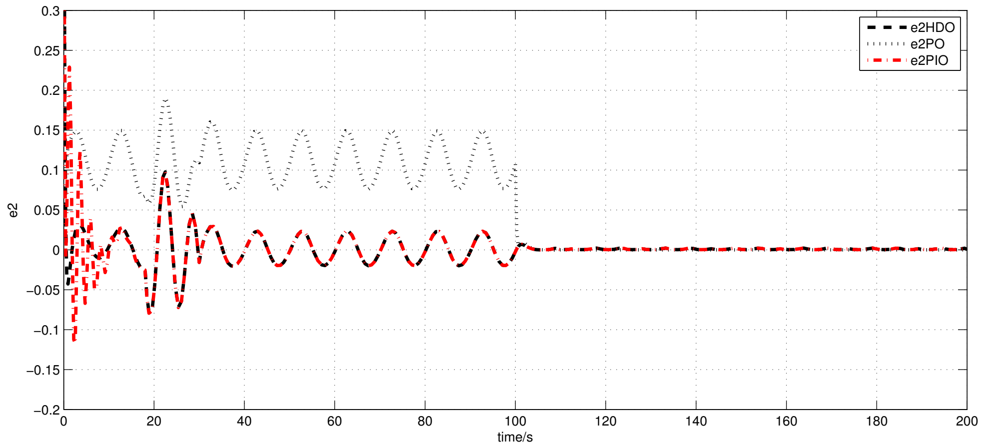

Figure 2.15: Estimation error e_2 for 200s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

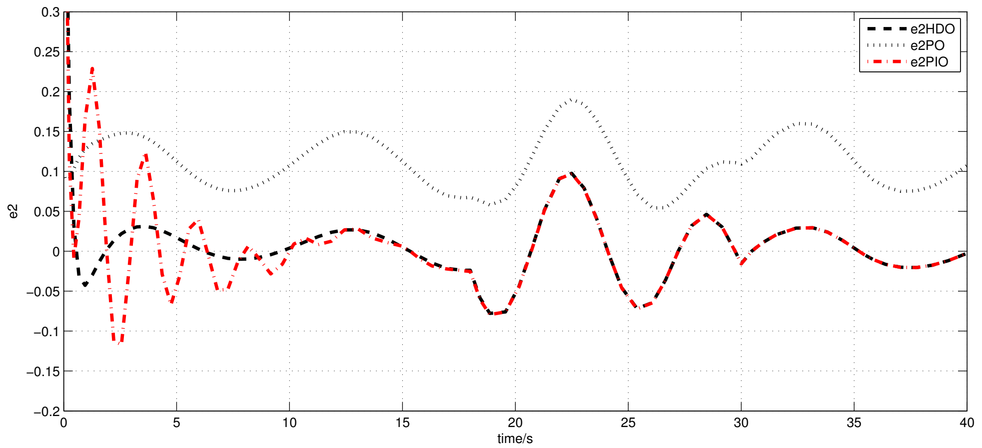

Figure 2.16: Estimation error e_2 for 40s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

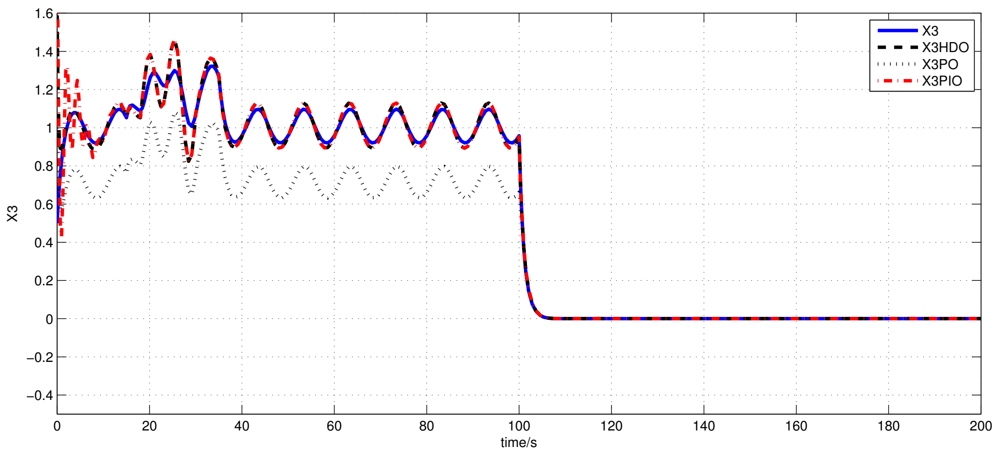

Figure 2.17: State x_3 and its estimates for 200s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

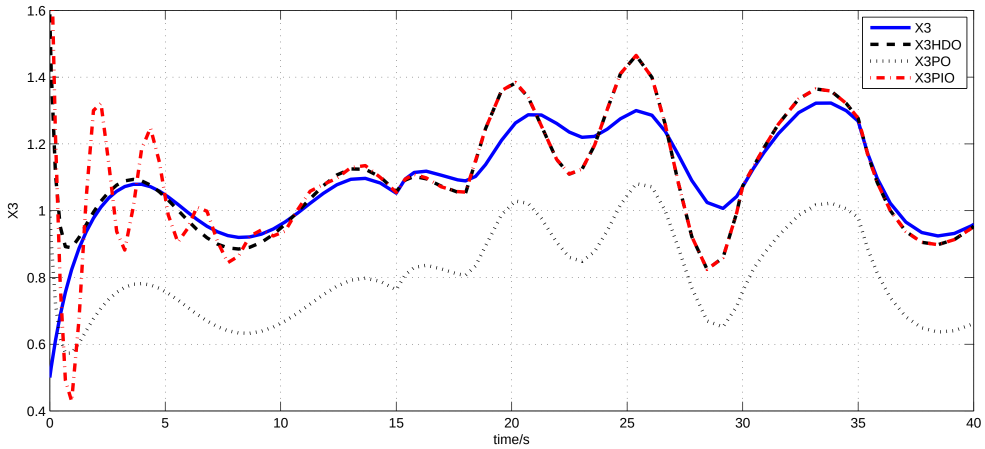

Figure 2.18: State x_3 and its estimates for 40s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

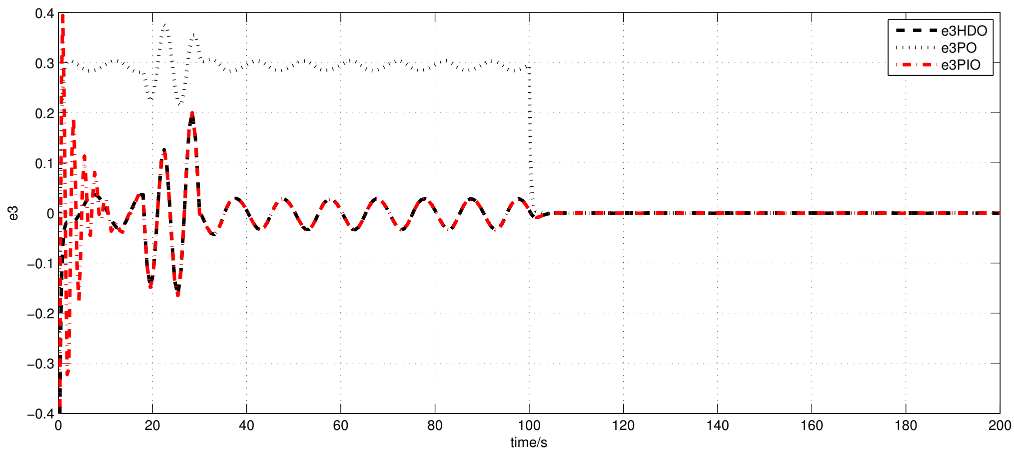

Figure 2.19: Estimation error e_3 for 200s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

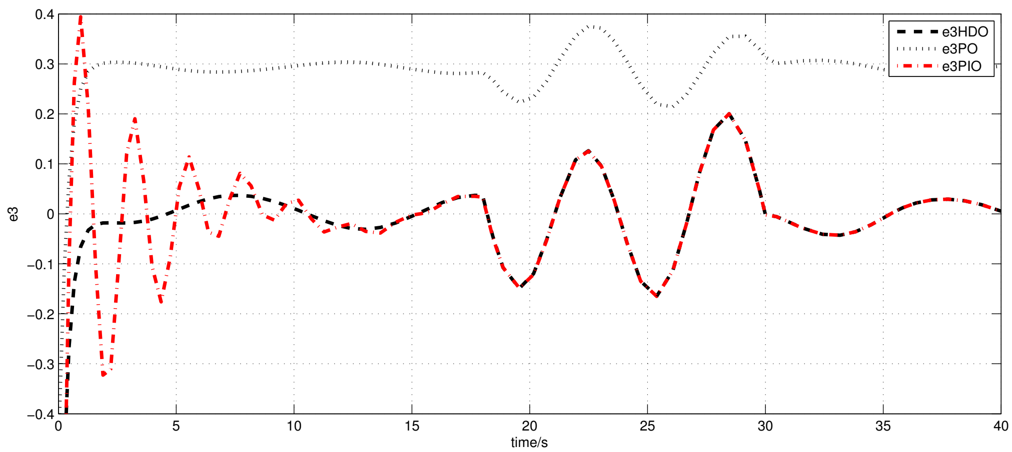

Figure 2.20: Estimation error e_3 for 40s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

From these figures, we can see the performances of the proposed observer and its behavior in the presence of an additional parameter uncertainty ΔA_1, compared with PO and PIO. The influence of the disturbance on the estimate is minimized directly by using the H_∞ theory (see Figure 2.9 and Figure 2.10 more clearly).
Take Figure 2.11 for example. One can see clearly that there is a static error in the estimate obtained from PO, while in the estimates obtained from both HDO and PIO, there are no static errors.
One can also see from these figures that the behavior of the proposed HDO is nearly the same as that of PIO in the steady state regime. The difference is in the transient regime. Take Figure 2.10 and Figure 2.12 for example. One can see that the estimate obtained from PIO presents some oscillations in the transient regime, compared with the HDO, while in the case of HDO, there are no oscillations.
Globally, we can see that the proposed HDO generalizes the existing observers PO and PIO and has better behavior than these observers in both transient regime and steady state regime.

2.3 A new H_∞ dynamic observer design for discrete-time systems

In section 2.2, we proposed a new form of HDO and solved the observer design problem for continuous-time systems. In this section, we shall investigate the observer design problem for discrete-time systems. Let us consider the following linear discrete-time system with unknown inputs and disturbances:

$$\begin{aligned} x_d(t+1) &= A_d x_d(t) + B_d u_d(t) + L_d d_d(t) + D_d w_d(t), && (2.75a) \\ y_d(t) &= C_d x_d(t) + E_d w_d(t), && (2.75b) \end{aligned}$$

with the initial state $x_d(0) = x_{d0}$. $x_d(t) \in \Re^n$, $u_d(t) \in \Re^m$, $d_d(t) \in \Re^l$, $w_d(t) \in \Re^f$ and $y_d(t) \in \Re^p$ are the state, the input, the unknown input, the disturbance of finite energy and the output, respectively. Matrices A_d, B_d, C_d, D_d, E_d and L_d are known to be constant and of appropriate dimensions. The description of system (2.75) is general according to **Remark 2.2.1** of section 2.2.
Let us consider the following HDO:

$$\begin{aligned} z_d(t+1) &= F_d z_d(t) + J_d y_d(t) + T_d u_d(t) + M_d v_d(t), && (2.76a) \\ v_d(t+1) &= P_d z_d(t) + Q_d y_d(t) + G_d v_d(t), && (2.76b) \\ \hat{x}_d(t) &= R_d z_d(t) + S_d y_d(t), && (2.76c) \end{aligned}$$

where $z_d(t) \in \Re^q$ is the state vector of the observer, $v_d(t) \in \Re^r$ is an auxiliary vector and $\hat{x}_d(t) \in \Re^n$ is the state estimate. Matrices F_d, J_d, T_d, M_d, P_d, Q_d, G_d, R_d and S_d are unknown and of appropriate dimensions to be determined. The auxiliary vector $v_d(t)$ is similar to the additional term α_{k+1} in PIO [15].

Remark 2.3.1. *Similar to the continuous-time case, we can obtain the PO, the PIO and the DO from the form* (2.76)*, in fact:*

1. *If matrices $M_d = 0$, $P_d = 0$, $Q_d = 0$ and $G_d = 0$, observer* (2.76) *becomes the PO one:*

$$\begin{aligned} z_d(t+1) &= F_d z_d(t) + J_d y_d(t) + T_d u_d(t), \\ \hat{x}_d(t) &= R_d z_d(t) + S_d y_d(t). \end{aligned}$$

2. *If matrices* $P_d = -C_d R_d$, $Q_d = I_p - C_d S_d$ *and* $G_d = 0$, *observer* (2.76) *becomes the PIO one:*

$$\begin{aligned} z_d(t+1) &= F_d z_d(t) + J_d y_d(t) + T_d u_d(t) + M_d v_d(t), \\ v_d(t+1) &= y_d(t) - C_d \hat{x}_d(t), \\ \hat{x}_d(t) &= R_d z_d(t) + S_d y_d(t). \end{aligned}$$

3. *If matrices* $F_d = A_d - \overline{D}_d C_d$, $J_d = \overline{D}_d$, $T_d = B_d$, $M_d = \overline{C}_d$, $P_d = \overline{B}_d C_d$, $Q_d = \overline{B}_d$ *and* $G_d = \overline{A}_d$, *observer* (2.76) *becomes the DO one:*

$$\begin{aligned} \hat{x}_d(t+1) &= A_d \hat{x}_d(t) + B_d u(t) + \xi_d(t), \\ v_d(t+1) &= \overline{A}_d v_d(t) + \overline{B}_d \mu_d(t), \\ \xi_d(t) &= \overline{C}_d v_d(t) + \overline{D}_d \mu_d(t), \\ \mu_d(t) &= y_d(t) - \hat{y}_d(t). \end{aligned}$$

■

The observer design problem is to determine all the parameter matrices F_d, J_d, T_d, M_d, P_d, Q_d, G_d, R_d and S_d of observer (2.76) such that:

1. For the disturbance $w_d(t) = 0$, the estimation error $(e_d(t) = \hat{x}_d(t) - x_d(t))$ $e_d(t) \to 0$ when $t \to \infty$;
2. For the disturbance $w_d(t) \neq 0$, minimize the effect of disturbance $w_d(t)$ on estimation error $e_d(t)$, i.e., find a scalar γ_{1d} such that $\|T_{dwe}\|_\infty < \gamma_{1d}$;
3. Decouple the estimation error $e_d(t)$ from the unknown input $d_d(t)$,

where T_{dwe} is the transfer function from $w_d(t)$ to $e_d(t)$ and γ_{1d} is a given positive scalar.

2.3.1 Observer design for systems without disturbances

In this section, we shall investigate the observer design problem of discrete-time systems without the disturbance $(w_d(t) = 0)$ and then extend the result to discrete-time systems with the disturbance $(w_d(t) \neq 0)$. Firstly, we define an error variable:

$$\varepsilon_d(t) = z_d(t) - \Phi_d x_d(t), \tag{2.78}$$

where $\Phi_d \in \Re^{q \times n}$ is an arbitrary matrix.
The following lemma gives the conditions of system (2.76) to be an observer for system (2.75) in the case where $w_d(t) = 0$:

Lemma 2.3.1. *For* $w_d(t) = 0$, *observer* (2.76) *will estimate the state* $x_d(t)$ *of system* (2.75) *if there exists a matrix* Φ_d *such that the following conditions hold*

$$\begin{aligned} F_d \Phi_d - \Phi_d A_d + J_d C_d &= 0, && \text{(2.79a)} \\ \Phi_d L_d &= 0, && \text{(2.79b)} \\ T_d - \Phi_d B_d &= 0, && \text{(2.79c)} \\ P_d \Phi_d + Q_d C_d &= 0, && \text{(2.79d)} \\ R_d \Phi_d + S_d C_d &= I_n, && \text{(2.79e)} \end{aligned}$$

and matrix

$$\mathbb{A}_{1d} = \begin{pmatrix} F_d & M_d \\ P_d & G_d \end{pmatrix}$$

is Schur.

Proof. *In the case where $w_d(t) = 0$, from the definition of error $\varepsilon_d(t)$, we obtain the discrete evolution of error ε_d as:*

$$\begin{aligned} \varepsilon_d(t+1) &= z_d(t+1) - \Phi_d x_d(t+1), \\ &= F_d z_d(t) + J_d y_d(t) + T_d u_d(t) + M_d v_d(t) \\ &\quad -\Phi_d A_d x_d(t) - \Phi_d B_d u_d(t) - \Phi_d L_d d_d(t), \\ &= F_d \varepsilon_d(t) + (F_d \Phi_d - \Phi_d A_d + J_d C_d) x_d(t) - \Phi_d L_d d_d(t) \\ &\quad +(T_d - \Phi_d B_d) u_d(t) + M_d v_d(t). \end{aligned} \tag{2.80}$$

Furthermore, from equations (2.76) *we obtain*

$$\begin{aligned} v_d(t+1) &= P_d z_d(t) + Q_d y_d(t) + G_d v_d(t), \\ &= P_d \varepsilon_d(t) + (P_d \Phi_d + Q_d C_d) x_d(t) + G_d v_d(t), \end{aligned} \tag{2.81}$$

and

$$\begin{aligned} \hat{x}_d(t) &= R_d z_d(t) + S_d y_d(t), \\ &= R_d \varepsilon_d(t) + (R_d \Phi_d + S_d C_d) x_d(t). \end{aligned} \tag{2.82}$$

One can see from equations (2.80) *and* (2.81) *that the discrete evolution $\varepsilon_d(t+1)$ and $v_d(t+1)$ are independent of the system state $x_d(t)$, the input $u_d(t)$ and the unknown input $d_d(t)$, if conditions* (2.79a) - (2.79d) *are satisfied.*
On the other hand, according to equation (2.82)*, if*

$$R_d \Phi_d + S_d C_d = I_n,$$

we obtain an expression of estimation error $e_d(t)$ as follows:

$$e_d(t) = R_d \varepsilon_d(t). \tag{2.83}$$

In this case, we obtain the following system:

$$\zeta_{1d}(t+1) = \mathbb{A}_{1d} \zeta_{1d}(t), \tag{2.84a}$$
$$e_d(t) = \mathbb{C}_{1d} \zeta_{1d}(t), \tag{2.84b}$$

where

$$\begin{aligned} \zeta_{1d}(t) &= \begin{bmatrix} \varepsilon_d(t) \\ v_d(t) \end{bmatrix}, \\ \mathbb{A}_{1d} &= \begin{pmatrix} F_d & M_d \\ P_d & G_d \end{pmatrix}, \\ \mathbb{C}_{1d} &= \begin{pmatrix} R_d & 0 \end{pmatrix}. \end{aligned}$$

From equations (2.84) *and* (2.83)*, it is obvious that if $\lim_{t\to\infty} \zeta_{1d}(t) \to 0$, then $\lim_{t\to\infty} e_d(t) \to 0$. This is satisfied if matrix $\mathbb{A}_{1d}$ is Schur, which completes the proof.* □

Remark 2.3.2. *From equations* (2.79b) *and* (2.84b)*, we can see that the estimation error* $e_d(t)$ *has been decoupled from the unknown input* $d_d(t)$. ■

Consequently, the observer design problem of discrete-time systems without the disturbance is reduced to determine all the parameter matrices such that conditions (2.79) are satisfied and matrix $\mathbb{A}_{1d}$ is Schur.

2.3.2 Parameterization of algebraic constraints

In this section, similar to the continuous-time systems, we will do some parameterizations to algebraic constraints (2.79). Firstly, from equations (2.79d) and (2.79e) we obtain the following equation:

$$\begin{pmatrix} P_d & Q_d \\ R_d & S_d \end{pmatrix} \begin{pmatrix} \Phi_d \\ C_d \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{2.85}$$

The necessary and sufficient condition for equation (2.85) to have a solution is the following rank condition must be satisfied:

$$rank \begin{pmatrix} \Phi_d \\ C_d \\ 0 \\ I_n \end{pmatrix} = rank \begin{pmatrix} \Phi_d \\ C_d \end{pmatrix} = n. \tag{2.86}$$

Assume that the above rank condition is satisfied, then let $\Lambda_d \in \Re^{q\times n}$ be an arbitrary matrix of full row rank such that:

$$rank \begin{pmatrix} \Lambda_d \\ C_d \end{pmatrix} = rank \begin{pmatrix} \Phi_d \\ C_d \end{pmatrix} = n. \tag{2.87}$$

Then there always exist two parameter matrices $\overline{K}_d \in \Re^{q\times p}$ and Φ_d such that

$$\begin{pmatrix} \Phi_d \\ C_d \end{pmatrix} = \begin{pmatrix} I_q & -\overline{K}_d \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda_d \\ C_d \end{pmatrix}, \tag{2.88}$$

or equivalently

$$\Phi_d = \Lambda_d - \overline{K}_d C_d. \tag{2.89}$$

According to equations (2.27), let matrix $\Upsilon_d = \begin{pmatrix} \Lambda_d \\ C_d \end{pmatrix}$ and we can deduce matrices P_d, Q_d, R_d and S_d as:

$$\begin{aligned} P_d &= -Z_{2d}\beta_{1d}, && \text{(2.90a)} \\ Q_d &= -Z_{2d}\beta_{2d}, && \text{(2.90b)} \\ R_d &= \alpha_{1d} - Z_{3d}\beta_{1d}, && \text{(2.90c)} \\ S_d &= \alpha_{2d} - Z_{3d}\beta_{2d}, && \text{(2.90d)} \end{aligned}$$

where matrices

$$\begin{cases} Z_{2d} = \begin{pmatrix} I_t & 0 \end{pmatrix} Z_d, \\ Z_{3d} = \begin{pmatrix} 0 & I_n \end{pmatrix} Z_d, \\ \alpha_{1d} = \Upsilon_d^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_{2d} = \Upsilon_d^+ \begin{pmatrix} \overline{K}_d \\ I_p \end{pmatrix}, \\ \beta_{1d} = (I_{q+p} - \Upsilon_d \Upsilon_d^+) \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \beta_{2d} = (I_{q+p} - \Upsilon_d \Upsilon_d^+) \begin{pmatrix} \overline{K}_d \\ I_p \end{pmatrix}, \end{cases} \tag{2.91}$$

and Z_d is an arbitrary matrix of appropriate dimension.

According to **Remark 2.2.4**, by taking matrix $Z_{3d} = 0$ we obtain

$$R_d = \alpha_{1d}, \tag{2.92a}$$
$$S_d = \alpha_{2d}. \tag{2.92b}$$

and matrix $\overline{K}_d$ is given by

$$\overline{K}_d = \Lambda_d L_d (C_d L_d)^+. \tag{2.93}$$

Next we give the parameterization in the condition where matrix $L_d = 0$.

Remark 2.3.3. *According to **Remark 2.2.5**, in the case where matrix $L_d = 0$, one can deduce matrices $\overline{K}_d$ and Φ_d according to the following equations:*

$$\Phi_d = \Lambda_d \begin{pmatrix} I_n \\ C_d \end{pmatrix}^+ \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \tag{2.94}$$
$$\overline{K}_d = \Lambda_d \begin{pmatrix} I_n \\ C_d \end{pmatrix}^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}. \tag{2.95}$$

■

Furthermore, matrices F_d, $\overline{K}_{1d}$ and J_d are given by

$$F_d = \alpha_{3d} - Z_{1d}\beta_{1d}, \tag{2.96}$$
$$\overline{K}_{1d} = \alpha_{4d} - Z_{1d}\beta_{3d}, \tag{2.97}$$
$$J_d = \Theta_d \alpha_{2d} - Z_{1d}\beta_{2d}. \tag{2.98}$$

with

$$\begin{cases} \alpha_{3d} = \Theta_d \Upsilon_d^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_{4d} = \Theta_d \Upsilon_d^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \beta_{3d} = (I_{q+p} - \Upsilon_d \Upsilon_d^+) \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \Theta_d = [\Lambda_d - \Lambda_d L_d (C_d L_d)^+ C_d] A_d. \end{cases} \tag{2.99}$$

From the above results, system (2.84) becomes

$$\begin{aligned} \zeta_{1d}(t+1) &= \mathbb{A}_{1d}\zeta_{1d}(t), && (2.100a)\\ e_d(t) &= \mathbb{C}_{1d}, && (2.100b)\end{aligned}$$

with matrices

$$\begin{aligned} \mathbb{A}_{1d} &= (\mathbb{A}_{11d} - \mathbb{Z}_{1d}\mathbb{A}_{12d}),\\ \mathbb{A}_{11d} &= \begin{pmatrix} \alpha_{3d} & 0\\ 0 & 0\end{pmatrix},\\ \mathbb{Z}_{1d} &= \begin{pmatrix} Z_{1d} & M_d\\ Z_{2d} & G_d\end{pmatrix},\\ \mathbb{A}_{12d} &= \begin{pmatrix} \beta_{1d} & 0\\ 0 & -I_t\end{pmatrix}.\end{aligned}$$

Consequently the observer design problem of discrete-time systems without the disturbance is reduced to determine the parameter matrix $\mathbb{Z}_{1d}$ such that system (2.100) is stable, i.e., matrix $\mathbb{A}_{1d}$ is Schur.

Now we give the following theorem to determine the parameter matrix $\mathbb{Z}_{1d}$:

Theorem 2.3.1. *There exists a parameter matrix $\mathbb{Z}_{1d}$ such that system (2.100) is asymptotically stable, if and only if there exist a symmetric positive definite matrix $\mathbb{X}_d$ and a matrix $\mathbb{Y}_d$ such that the following LMI holds*

$$\begin{bmatrix} -\mathbb{X}_d & \mathbb{A}_{11d}^T\mathbb{X}_d - \mathbb{A}_{12d}^T\mathbb{Y}_d^T\\ \mathbb{X}_d\mathbb{A}_{11d} - \mathbb{Y}_d\mathbb{A}_{12d} & -\mathbb{X}_d\end{bmatrix} < 0. \tag{2.101}$$

In this case, matrix $\mathbb{Z}_{1d}$ is determined by $\mathbb{Z}_{1d} = \mathbb{X}_d^{-1}\mathbb{Y}_d$.

Proof. *For system (2.100), we choose the following Lyapunov function candidate:*

$$V[\zeta_{1d}(t)] = \zeta_{1d}(t)^T\mathbb{X}_d\zeta_{1d}(t), \tag{2.102}$$

where matrix $\mathbb{X}_d = \mathbb{X}_d^T > 0$.
The discrete evolution of $V[\zeta_{1d}(t)]$ is $\Delta V[\zeta_{1d}(t)]$, which is given by

$$\begin{aligned} \Delta V[\zeta_{1d}(t)] &= V[\zeta_{1d}(t+1)] - V[\zeta_{1d}(t)],\\ &= \zeta_{1d}(t+1)^T\mathbb{X}_d\zeta_{1d}(t+1) - \zeta_{1d}(t)^T\mathbb{X}_d\zeta_{1d}(t),\\ &= \zeta_{1d}(t)^T[(\mathbb{A}_{11d} - \mathbb{Z}_{1d}\mathbb{A}_{12d})^T\mathbb{X}_d(\mathbb{A}_{11d} - \mathbb{Z}_{1d}\mathbb{A}_{12d}) - \mathbb{X}_d]\zeta_{1d}(t). \quad (2.103)\end{aligned}$$

From equation (2.103), it is easy to see that $\Delta V[\zeta_{1d}(t)] < 0$, i.e., system (2.100) is asymptotically stable, if and only if the following LMI is satisfied

$$(\mathbb{A}_{11d} - \mathbb{Z}_{1d}\mathbb{A}_{12d})^T\mathbb{X}_d(\mathbb{A}_{11d} - \mathbb{Z}\mathbb{A}_{12d}) - \mathbb{X}_d < 0. \tag{2.104}$$

In this case, by using Schur complement lemma, we obtain

$$\begin{bmatrix} -\mathbb{X}_d & \mathbb{A}_{11d}^T\mathbb{X}_d - \mathbb{A}_{12d}^T\mathbb{Y}_d^T\\ \mathbb{X}_d\mathbb{A}_{11d} - \mathbb{Y}_d\mathbb{A}_{12d} & -\mathbb{X}_d\end{bmatrix} < 0.$$

where matrix $\mathbb{Y}_d = \mathbb{X}_d\mathbb{Z}_{1d}$. This completes the proof. □

From the above results, the HDO design procedure for discrete-time systems without disturbances can be summarized as follows:

step1. Choose matrix Λ_d according to rank condition (2.87);

step2. Compute matrices $\overline{K}_d$, Φ_d and Θ_d from equations (2.93), (2.89) and (2.99);

step3. Compute matrices α_{1d}, α_{2d}, α_{3d}, α_{4d}, β_{1d}, β_{2d} and β_{3d} from equations (2.91) and (2.99);

step4. Compute matrix $\mathbb{Z}_{1d}$ from the solution of LMI (2.101);

step5. Deduce all the parameter matrices according to equations (2.90), (2.96), (2.98) and (2.79c).

2.3.3 Observer design for systems in the presence of disturbances

In this section, the results of previous section for systems without disturbances will be applied to design an HDO for linear system (2.75) in the presence of disturbance ($w_d(t) \neq 0$). By using the algebraic constraints of **lemma 2.3.1**, we obtain

$$\varepsilon_d(t+1) = F_d\varepsilon_d(t) + M_d v_d(t) + (J_d E_d - \Phi_d D_d)w_d(t), \tag{2.105}$$

$$v_d(t+1) = P_d\varepsilon_d(t) + G_d v_d(t) + Q_d E_d w_d(t), \tag{2.106}$$

$$e_d(t) = R_d\varepsilon_d(t) + S_d E_d w_d(t). \tag{2.107}$$

In this case, we obtain the following system:

$$\begin{aligned} \zeta_{1d}(t+1) &= \mathbb{A}_{1d}\zeta_{1d}(t) + \mathbb{B}_{1d}w_d(t), && \text{(2.108a)}\\ e_d(t) &= \mathbb{C}_{1d}\zeta_{1d}(t) + \mathbb{D}_{1d}w_d(t), && \text{(2.108b)} \end{aligned}$$

where matrices

$$\begin{aligned} \mathbb{B}_{1d} &= \begin{pmatrix} J_d E_d - \Phi_d D_d \\ Q_d E_d \end{pmatrix}, \\ \mathbb{D}_{1d} &= S_d E_d. \end{aligned}$$

Remark 2.3.4. *According to the parametrization results of previous section and* ***Remark 2.2.7****, we can rewrite the matrices $\mathbb{A}_{1d}$, $\mathbb{B}_{1d}$, $\mathbb{C}_{1d}$ and $\mathbb{D}_{1d}$ of system* (2.108) *as follows:*

$$\begin{aligned} \mathbb{A}_{1d} &= \mathbb{A}_{11d} - \mathbb{Z}_{1d}\mathbb{A}_{12d}, \\ \mathbb{B}_{1d} &= \mathbb{B}_{11d} - \mathbb{Z}_{1d}\mathbb{B}_{12d}, \\ \mathbb{B}_{11d} &= \begin{pmatrix} -\Phi_d D_d + \Theta_d \Upsilon_d^{+} \begin{pmatrix} \overline{K}_d \\ I_p \end{pmatrix} E_d \\ 0 \end{pmatrix}, \\ \mathbb{B}_{12d} &= \begin{pmatrix} \beta_{2d} E_d \\ 0 \end{pmatrix}, \\ \mathbb{C}_{1d} &= \begin{pmatrix} \alpha_{1d} & 0 \end{pmatrix}, \\ \mathbb{D}_{1d} &= \alpha_{2d} E_d. \end{aligned}$$

■

Consequently, the design problem of the proposed observer for discrete-time systems in the presence of disturbances and unknown inputs is reduced to the analysis of system (2.108), i.e., the determination of the parameter matrix $\mathbb{Z}_{1d}$ such that:

1. For $w_d(t) = 0$, system (2.108) is stable;

2. For $w_d(t) \neq 0$, the H_∞ norm $\|T_{dwe}\|_\infty < \gamma_{1d}$;

According to **Theorem 2.2.3** for continuous-time systems, a new form of LMI is proposed to derive a more general solution of matrix $\mathbb{Z}_{1d}$, which is presented in the following theorem:

Theorem 2.3.2. *Let matrices* $\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ -I_{q+t} \end{pmatrix}$ *and* $\mathcal{C} = \begin{pmatrix} \mathbb{A}_{12d} & \mathbb{B}_{12d} & 0 \end{pmatrix}$*, then there exists a parameter matrix* $\mathbb{Z}_{1d}$ *such that system* (2.108) *is asymptotically stable for* $w_d(t) = 0$ *and* $\|T_{dwe}\|_\infty < \gamma_{1d}$ *for* $w_d(t) \neq 0$*, if and only if there exist a symmetric positive definite matrix* $\mathcal{X}$ *and a positive scalar* γ_{1d} *such that the following LMIs are satisfied*

$$\begin{aligned} \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{1d}^T\mathbb{C}_{1d} & \mathbb{C}_{1d}^T\mathbb{D}_{1d} \\ (\mathbb{C}_{1d}^T\mathbb{D}_{1d})^T & \mathbb{D}_{1d}^T\mathbb{D}_{1d} - \gamma_{1d}^2 I_f \end{bmatrix} &< 0, &\quad (2.109a)\\ \mathcal{C}^{T\perp}\mathcal{Q}\mathcal{C}^{T\perp T} &< 0, &\quad (2.109b) \end{aligned}$$

with

$$\mathcal{Q} = \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{1d}^T\mathbb{C}_{1d} & \mathbb{C}_{1d}^T\mathbb{D}_{1d} & \mathbb{A}_{11d}^T\mathcal{X} \\ (\mathbb{C}_{1d}^T\mathbb{D}_{1d})^T & \mathbb{D}_{1d}^T\mathbb{D}_{1d} - \gamma_{1d}^2 I_f & \mathbb{B}_{11d}^T\mathcal{X} \\ (\mathbb{A}_{11d}^T\mathcal{X})^T & (\mathbb{B}_{11d}^T\mathcal{X})^T & -\mathcal{X} \end{bmatrix}. \tag{2.110}$$

Assume that the above LMIs hold, let matrices $(\mathcal{B}_l, \mathcal{B}_r)$ *and* $(\mathcal{C}_l, \mathcal{C}_r)$ *be any full rank factors of matrices* $\mathcal{B}$ *and* $\mathcal{C}$*, i.e.,* $\mathcal{B} = \mathcal{B}_l\mathcal{B}_r, \mathcal{C} = \mathcal{C}_l\mathcal{C}_r$*.*
In this case, matrices $\mathbb{Z}_{1d}$ *can be determined by* $\mathbb{Z}_{1d} = \mathcal{X}^{-1}\mathcal{Y}$ *and* $\mathcal{Y}$ *is given by*

$$\mathcal{Y} = \mathcal{B}_r^+\mathcal{K}\mathcal{C}_l^+ + \mathcal{Z} - \mathcal{Z}\mathcal{C}_l\mathcal{C}_l^+, \tag{2.111}$$

where $\mathcal{Z}$ *is an arbitrary matrix and*

$$\mathcal{K} \triangleq -\mathcal{R}^{-1}\overline{\Phi}_d\mathcal{C}_r^T(\mathcal{C}_r\overline{\Phi}_d\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\overline{\Phi}_d\mathcal{C}_r^T)^{-1/2}, \tag{2.112}$$

$$\mathcal{S} \triangleq \mathcal{R}^{-1} - \mathcal{R}^{-1}[\overline{\Phi}_d - \overline{\Phi}_d\mathcal{C}_r^T(\mathcal{C}_r\overline{\Phi}_d\mathcal{C}_r^T)^{-1}\mathcal{C}_r\overline{\Phi}_d]\mathcal{R}^{-1}, \tag{2.113}$$

where $\mathcal{L}$ *is an arbitrary matrix such that* $\|\mathcal{L}\| < 1$ *and* $\mathcal{R}$ *is an arbitrary positive definite matrix such that*

$$\overline{\Phi}_d \triangleq (\mathcal{R}^{-1} - \mathcal{Q})^{-1} > 0. \tag{2.114}$$

Proof. *According to bounded-real lemma, system* (2.108) *is asymptotically stable for* $w_d(t) = 0$ *and* $\|T_{dwe}\|_\infty < \gamma_{1d}$ *for* $w_d(t) \neq 0$*, if and only if there exist a symmetric positive definite matrix* $\mathcal{X}$ *and a positive scalar* γ_{1d} *such that the following LMI is satisfied*

$$\begin{bmatrix} \mathbb{A}_{1d} & \mathbb{B}_{1d} \\ \mathbb{C}_{1d} & \mathbb{D}_{1d} \end{bmatrix}^T \begin{bmatrix} \mathcal{X} & 0 \\ 0 & I_{q+t} \end{bmatrix} \begin{bmatrix} \mathbb{A}_{1d} & \mathbb{B}_{1d} \\ \mathbb{C}_{1d} & \mathbb{D}_{1d} \end{bmatrix} < \begin{bmatrix} \mathcal{X} & 0 \\ 0 & \gamma_{1d}^2 I_f \end{bmatrix}, \tag{2.115}$$

which can be written as:

$$\begin{bmatrix} \mathbb{A}_{1d}^T \mathcal{X} \mathbb{A}_{1d} + \mathbb{C}_{1d}^T \mathbb{C}_{1d} - \mathcal{X} & \mathbb{A}_{1d}^T \mathcal{X} \mathbb{B}_{1d} + \mathbb{C}_{1d}^T \mathbb{D}_{1d} \\ (\mathbb{A}_{1d}^T \mathcal{X} \mathbb{B}_{1d} + \mathbb{C}_{1d}^T \mathbb{D}_{1d})^T & \mathbb{B}_{1d}^T \mathcal{X} \mathbb{B}_{1d} + \mathbb{D}_{1d}^T \mathbb{D}_{1d} - \gamma_{1d}^2 I_f \end{bmatrix} < 0. \tag{2.116}$$

By using Schur complement lemma and ***Remark 2.3.4****, we obtain the following inequality:*

$$\begin{aligned} & \begin{bmatrix} \mathbb{A}_{1d}^T \mathcal{X} \mathbb{A}_{1d} + \mathbb{C}_{1d}^T \mathbb{C}_{1d} - \mathcal{X} & \mathbb{A}_{1d}^T \mathcal{X} \mathbb{B}_{1d} + \mathbb{C}_{1d}^T \mathbb{D}_{1d} \\ (\mathbb{A}_{1d}^T \mathcal{X} \mathbb{B}_{1d} + \mathbb{C}_{1d}^T \mathbb{D}_{1d})^T & \mathbb{B}_{1d}^T \mathcal{X} \mathbb{B}_{1d} + \mathbb{D}_{1d}^T \mathbb{D}_{1d} - \gamma_{1d}^2 I_f \end{bmatrix} < 0, \\ \Leftrightarrow \quad & \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{1d}^T \mathbb{C}_{1d} & \mathbb{C}_{1d}^T \mathbb{D}_{1d} & \mathbb{A}_{1d}^T \mathcal{X} \\ (\mathbb{C}_{1d}^T \mathbb{D}_{1d})^T & \mathbb{D}_{1d}^T \mathbb{D}_{1d} - \gamma_{1d}^2 I_f & \mathbb{B}_{1d}^T \mathcal{X} \\ (\mathbb{A}_{1d}^T \mathcal{X})^T & (\mathbb{B}_{1d}^T \mathcal{X})^T & -\mathcal{X} \end{bmatrix} < 0, \\ \Leftrightarrow \quad & \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{1d}^T \mathbb{C}_{1d} & \mathbb{C}_{1d}^T \mathbb{D}_{1d} & \Omega_1 \\ (\mathbb{C}_{1d}^T \mathbb{D}_{1d})^T & \mathbb{D}_{1d}^T \mathbb{D}_{1d} - \gamma_{1d}^2 I_f & \Omega_2 \\ (\Omega_1)^T & (\Omega_2)^T & -\mathcal{X} \end{bmatrix} < 0, \\ \Leftrightarrow \quad & \mathcal{Q} + \mathcal{B}\mathcal{Y}\mathcal{C} + (\mathcal{B}\mathcal{Y}\mathcal{C})^T < 0, \end{aligned} \tag{2.117}$$

where

$$\Omega_1 = \mathbb{A}_{11d}^T \mathcal{X} - \mathbb{A}_{12d}^T \mathcal{Y}^T,$$

$$\Omega_2 = \mathbb{B}_{11d}^T \mathcal{X} - \mathbb{B}_{12d}^T \mathcal{Y}^T,$$

$$\mathcal{Q} = \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{1d}^T \mathbb{C}_{1d} & \mathbb{C}_{1d}^T \mathbb{D}_{1d} & \mathbb{A}_{11d}^T \mathcal{X} \\ (\mathbb{C}_{1d}^T \mathbb{D}_{1d})^T & \mathbb{D}_{1d}^T \mathbb{D}_{1d} - \gamma_{1d}^2 I_f & \mathbb{B}_{11d}^T \mathcal{X} \\ (\mathbb{A}_{11d}^T \mathcal{X})^T & (\mathbb{B}_{11d}^T \mathcal{X})^T & -\mathcal{X} \end{bmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ -I_{q+t} \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} \mathbb{A}_{12d} & \mathbb{B}_{12d} & 0 \end{pmatrix}$$

$$\mathcal{Y} = \mathcal{X}\mathbb{Z}_{1d}.$$

According to [106], matrix inequality (2.117) *is equivalent to*

$$\mathcal{B}^\perp \mathcal{Q} \mathcal{B}^{\perp T} < 0, \tag{2.118}$$

$$\mathcal{C}^{T\perp} \mathcal{Q} \mathcal{C}^{T\perp T} < 0. \tag{2.119}$$

In this case, matrix $\mathcal{Y}$ *can be obtained by*

$$\mathcal{Y} = \mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+, \tag{2.120}$$

where $\mathcal{Z}$ is an arbitrary matrix and

$$\mathcal{K} \triangleq -\mathcal{R}^{-1}\overline{\Phi}_d\mathcal{C}_r^T(\mathcal{C}_r\overline{\Phi}_d\mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2}\mathcal{L}(\mathcal{C}_r\overline{\Phi}_d\mathcal{C}_r^T)^{-1/2}, \tag{2.121}$$

$$\mathcal{S} \triangleq \mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{B}_l^T[\overline{\Phi}_d - \overline{\Phi}_d\mathcal{C}_r^T(\mathcal{C}_r\overline{\Phi}_d\mathcal{C}_r^T)^{-1}\mathcal{C}_r\overline{\Phi}_d]\mathcal{B}_l\mathcal{R}^{-1}, \tag{2.122}$$

where $\mathcal{L}$ is an arbitrary matrix such that the matrix $\|\mathcal{L}\| < 1$ and $\mathcal{R}$ is an arbitrary positive definite matrix such that:

$$\overline{\Phi}_d \triangleq (\mathcal{B}_l\mathcal{R}^{-1}\mathcal{B}_l^T - \mathcal{Q})^{-1} > 0. \tag{2.123}$$

Furthermore, since matrix $\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ -I_{q+t} \end{pmatrix}$, we have

$$\begin{cases} \mathcal{B}^{\perp} = \begin{pmatrix} I_{q+t+f} & 0 \end{pmatrix}, \\ \mathcal{B}_l = I_{q+t+f+q+t}, \\ \mathcal{B}_r = \begin{pmatrix} 0 \\ 0 \\ -I_{q+t} \end{pmatrix}, \\ \mathcal{B}_r^{+} = \begin{pmatrix} 0 & -I_{q+t} \end{pmatrix}. \end{cases} \tag{2.124}$$

In this case, by substituting matrices (2.124) *into equations* (2.120)-(2.122) *and inequality* (2.123), *we obtain equations* (2.111)-(2.113) *and inequality* (2.114). *The inequalities* (2.118) *and* (2.119) *become inequalities* (2.109a) *and* (2.109b) *respectively, which completes the proof.* □

From the above results, the HDO design problem for discrete-time systems in the presence of disturbances and unknown inputs can be summarized as the following design procedure:

step1. Choose matrix Λ_d according to rank condition (2.87);

step2. Compute matrices $\overline{K}_d$, Θ, $\overline{K}_{1d}$ and Φ_d from equations (2.93), (2.99) and (2.89);

step3. Compute matrices α_{1d}, α_{2d}, α_{3d}, α_{4d}, β_{1d}, β_{2d} and β_{3d} according to equations (2.91) and (2.99);

step4. Compute matrices $\mathbb{A}_{11d}$, $\mathbb{A}_{12d}$, $\mathbb{B}_{11d}$, $\mathbb{B}_{12d}$, $\mathbb{C}_{1d}$ and $\mathbb{D}_{1d}$ according to **Remark 2.3.4**;

step5. Compute matrices $\mathcal{Q}$ and $\mathcal{X}$ according to **Theorem 2.3.2**;

step6. Choose matrices $\mathcal{L}, \mathcal{R}$ and $\mathcal{Z}$ according to **Theorem 2.3.2** and compute matrix $\mathcal{Y}$.

step7. Compute matrix $\mathbb{Z}_{1d}$ from $\mathbb{Z}_{1d} = \mathcal{X}^{-1}\mathcal{Y}$;

step8. Deduce all matrices $F_d, J_d, T_d, M_d, P_d, Q_d, G_d, R_d$ and S_d according to equations (2.96), (2.90), (2.79c) and (2.98).

2.3.4 Particular cases

In the previous sections, we have solved the design problem of HDO for discrete-time systems. In this section, we shall show how to obtain the LMI formulations for these observers.

2.3.4.1 Proportional observer

As it is shown in **Remark 2.3.1**, the PO corresponds to matrices $M_d = 0, P_d = 0, Q_d = 0, G_d = 0$ and is given by

$$\begin{aligned} z_d(t+1) &= F_d z_d(t) + J_d y_d(t) + T_d u_d(t), && (2.125a)\\ \hat{x}_d(t) &= R_d z_d(t) + S_d y_d(t). && (2.125b) \end{aligned}$$

In this case, system (2.108) becomes

$$\begin{aligned} \zeta_{2d}(t+1) &= \mathbb{A}_{2d}\zeta_{2d}(t) + \mathbb{B}_{2d} w_d(t), && (2.126a)\\ e_d(t) &= \mathbb{C}_{2d}\zeta_{2d}(t) + \mathbb{D}_{2d} w_d(t), && (2.126b) \end{aligned}$$

where vector $\zeta_{2d}(t) = \varepsilon_d(t)$, and matrices

$$\mathbb{A}_{2d} = \mathbb{A}_{21d} - \mathbb{Z}_{2d}\mathbb{A}_{22d},$$

$$\mathbb{A}_{21d} = \alpha_{3d}, \mathbb{A}_{22d} = \beta_{1d},$$

$$\mathbb{B}_{2d} = \mathbb{B}_{21d} - \mathbb{Z}_{2d}\mathbb{B}_{22d},$$

$$\mathbb{B}_{21d} = \Theta_d\alpha_{2d}E_d - \Phi_d D_d, \mathbb{B}_{22d} = \beta_{2d}E_d,$$

$$\mathbb{C}_{2d} = \alpha_{1d}, \mathbb{D}_{2d} = \alpha_{2d}E_d$$

$$\mathbb{Z}_{2d} = Z_{1d}.$$

Consequently, matrices $\mathcal{Q}, \mathcal{B}, \mathcal{C}$ of **Theorem 2.3.2** become

$$\mathcal{Q} = \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{2d}^T\mathbb{C}_{2d} & \mathbb{C}_{2d}^T\mathbb{D}_{2d} & \mathbb{A}_{21d}^T\mathcal{X} \\ (\mathbb{C}_{2d}^T\mathbb{D}_{2d})^T & \mathbb{D}_{2d}^T\mathbb{D}_{2d} - \gamma_{1d}^2 I_f & \mathbb{B}_{21d}^T\mathcal{X} \\ (\mathbb{A}_{21d}^T\mathcal{X})^T & (\mathbb{B}_{21d}^T\mathcal{X})^T & -\mathcal{X} \end{bmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ -I_q \end{pmatrix}, \mathcal{C} = \begin{pmatrix} \mathbb{A}_{22d} & \mathbb{B}_{22d} & 0 \end{pmatrix}.$$

The design problem of PO is reduced to determine the parameter matrix $\mathbb{Z}_{2d}$, which is given by

$$\mathbb{Z}_{2d} = \mathcal{X}^{-1}\mathcal{Y}.$$

2.3.4.2 Proportional integral observer

According to **Remark 2.3.1**, the PIO corresponds to matrices $P_d = -C_d R_d$, $Q_d = I_p - C_d S_d$, $G_d = 0$ and is given by

$$\begin{aligned} z_d(t+1) &= F_d z_d(t) + J_d y_d(t) + T_d u_d(t) + M_d v_d(t), && (2.127a)\\ v_d(t+1) &= y_d(t) - C_d \hat{x}_d(t), && (2.127b)\\ \hat{x}_d(t) &= R_d z_d(t) + S_d y_d(t). && (2.127c) \end{aligned}$$

In this case, system (2.108) becomes

$$\begin{aligned} \zeta_{3d}(t+1) &= \mathbb{A}_{3d}\zeta_{3d}(t) + \mathbb{B}_{3d} w_d(t), && (2.128a)\\ e_d(t) &= \mathbb{C}_{3d}\zeta_{3d}(t) + \mathbb{D}_{3d} w_d(t), && (2.128b) \end{aligned}$$

where vector $\zeta_{3d}(t) = \begin{bmatrix} \varepsilon_d(t) \\ v_d(t) \end{bmatrix}$, and matrices

$$\mathbb{A}_{3d} = \mathbb{A}_{31d} - \mathbb{A}_{32d}\mathbb{Z}_{3d}\mathbb{A}_{33d},$$

$$\mathbb{A}_{31d} = \begin{pmatrix} \alpha_{3d} & 0 \\ -C_d\alpha_{1d} & 0 \end{pmatrix}, \mathbb{A}_{32d} = \begin{pmatrix} I_{q+p} \\ 0 \end{pmatrix}, \mathbb{A}_{33d} = \begin{pmatrix} \beta_{1d} & 0 \\ 0 & -I_t \end{pmatrix},$$

$$\mathbb{B}_{3d} = \mathbb{B}_{31d} - \mathbb{B}_{32d}\mathbb{Z}_{3d}\mathbb{B}_{33d},$$

$$\mathbb{B}_{31d} = \begin{pmatrix} \Theta_d\alpha_{2d}D_{2d} - T_d D_{1d} \\ D_{2d} - C_d\alpha_{2d}D_{2d} \end{pmatrix}, \mathbb{B}_{32d} = \begin{pmatrix} I_{q+p} \\ 0 \end{pmatrix}, \mathbb{B}_{33d} = \begin{pmatrix} \beta_{2d}D_{2d} \\ 0 \end{pmatrix},$$

$$\mathbb{C}_{3d} = \begin{pmatrix} \alpha_{1d} & 0 \end{pmatrix}, \mathbb{D}_{3d} = \alpha_{2d}D_{2d}$$

$$\mathbb{Z}_{3d} = \begin{pmatrix} Z_{1d} & M_d \end{pmatrix}.$$

Consequently, matrices $\mathcal{Q}, \mathcal{B}, \mathcal{C}$ of **Theorem 2.3.2** become

$$\mathcal{Q} = \begin{bmatrix} -\mathcal{X} + \mathbb{C}_{3d}^T\mathbb{C}_{3d} & \mathbb{C}_{3d}^T\mathbb{D}_{3d} & \mathbb{A}_{31d}^T\mathcal{X} \\ (\mathbb{C}_{3d}^T\mathbb{D}_{3d})^T & \mathbb{D}_{3d}^T\mathbb{D}_{3d} - \gamma_{1d}^2 I_f & \mathbb{B}_{31d}^T\mathcal{X} \\ (\mathbb{A}_{31d}^T\mathcal{X})^T & (\mathbb{B}_{31d}^T\mathcal{X})^T & -\mathcal{X} \end{bmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ -X\mathbb{A}_{32} \end{pmatrix}, \mathcal{C} = \begin{pmatrix} \mathbb{A}_{33d} & \mathbb{B}_{33d} & 0 \end{pmatrix}.$$

The design problem of PIO is to determine the parameter matrix $\mathbb{Z}_{3d}$, which is given by

$$\mathbb{Z}_{3d} = \mathcal{Y}.$$

Furthermore, the orthogonal complement of matrix $\mathcal{B}$ becomes

$$\mathcal{B}^\perp = \begin{pmatrix} I_{q+t} & 0 & 0 \\ 0 & I_f & 0 \\ 0 & 0 & \mathbb{A}_{32}^\perp X^{-1} \end{pmatrix}. \qquad (2.129)$$

In this case, we can determine matrices $\mathbb{Z}_{3d}$ and $\mathcal{Y}$ according to inequalities (2.118), (2.119) and (2.123) and equations (2.120) - (2.122).

2.3.5 Numerical example

So far, we have solved the HDO design problem for discrete-time systems in the presence of disturbances and unknown inputs. In this section, a numerical example is presented to illustrate the design procedure and the performance of the proposed observer. Let us consider the following linear discrete-time system of the form (2.75) with:

$$A_d = \begin{bmatrix} 0.1370 & 0.0990 & 0.2840 \\ 0.0118 & 0.2990 & 0.4700 \\ 0.8940 & 0.6610 & 0.0650 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.25 \\ 0.6 \\ 0.1 \end{bmatrix}, L_d = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$D_d = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_d = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

The initial conditions are $x_{d1}(0) = 0.1$, $x_{d2}(0) = 0.1$ and $x_{d3}(0) = 0.2$.

2.3.5.1 Observer design

For the above system, we design an HDO by using the previously proposed method. According to the design procedure proposed, we first choose matrix

$$\Lambda_d = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

according to rank condition (2.87).

Then, by choosing matrices:

$$\begin{aligned}
\mathcal{Z} &= 0, \\
\mathcal{R} &= 1000 \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
\mathcal{L} &= 0.1 \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},
\end{aligned}$$

we obtain the following results:

$$\gamma_{1d} = 1.8,$$

$$\mathcal{X} = \begin{bmatrix} 1.8979 & 0.0256 & -0.0256 & 0 & 0 \\ 0.0256 & 1.7184 & -0.0256 & 0 & 0 \\ -0.0256 & -0.0256 & 1.7249 & 0 & 0 \\ 0 & 0 & 0 & 1.6672 & 0 \\ 0 & 0 & 0 & 0 & 1.6672 \end{bmatrix},$$

$$\mathbb{Z}_{1d} = \begin{bmatrix} 0.0014 & -0.0014 & 0.0019 & -0.0025 & 0.0008 & -0.0022 & -0.0022 \\ 0.0015 & -0.0015 & 0.0021 & -0.0028 & 0.0009 & -0.0024 & -0.0024 \\ 0.0015 & -0.0015 & 0.0022 & -0.0028 & 0.0009 & -0.0024 & -0.0024 \\ 0.0016 & -0.0016 & 0.0022 & -0.0029 & 0.0009 & -0.0024 & -0.0024 \\ 0.0016 & -0.0016 & 0.0022 & -0.0029 & 0.0009 & -0.0024 & -0.0024 \end{bmatrix}.$$

Consequently, the obtained HDO is given by

$$\begin{aligned} z_d(t+1) &= \begin{bmatrix} -0.0860 & 0.1510 & 0.3065 \\ 0.1124 & 0.2366 & 0.3115 \\ 0.3955 & 0.1725 & 0.0082 \end{bmatrix} z_d(t) + \begin{bmatrix} 0.1300 & 0.661 \\ 0.1715 & 0.760 \\ 0.0851 & 0.198 \end{bmatrix} y_d(t) \\ &+ \begin{bmatrix} 0.1 \\ 0.35 \\ 0.5 \end{bmatrix} u_d(t) - \begin{bmatrix} 0.0022 & 0.0022 \\ 0.0024 & 0.0024 \\ 0.0024 & 0.0024 \end{bmatrix} v_d(t), \\ v_d(t+1) &= \begin{bmatrix} -0.0016 & 0.0016 & -0.0022 \\ -0.0016 & 0.0016 & -0.0022 \end{bmatrix} z_d(t) + \begin{bmatrix} 0.0029 & 0 \\ 0.0029 & 0 \end{bmatrix} y_d(t) \\ &- \begin{bmatrix} -0.0024 & -0.0024 \\ -0.0024 & -0.0024 \end{bmatrix} v_d(t), \\ \hat{x}_d &= \begin{bmatrix} 0.0769 & -0.0769 & 0.3846 \\ -0.3077 & 0.3077 & -0.0385 \\ 0.7692 & 0.2308 & -0.1538 \end{bmatrix} z_d(t) + \begin{bmatrix} 0.3077 & 0 \\ -0.2308 & 1 \\ 0.0769 & 0 \end{bmatrix} y_d(t). \end{aligned}$$

Similar to continuous-time systems, in order to provide a comparison with PO and PIO, we also design a PO and a PIO. The PO and PIO are designed under the same condition where $\gamma_{1d} = 1.8$. The obtained PO is given by

$$\begin{aligned} z_p(t+1) &= \begin{bmatrix} -0.1037 & 0.1569 & 0.2825 \\ 0.0922 & 0.2439 & 0.2849 \\ 0.3753 & 0.1797 & -0.0189 \end{bmatrix} z_p(t) \\ &+ \begin{bmatrix} 0.1642 & 0.6175 \\ 0.2091 & 0.7128 \\ 0.1242 & 0.1502 \end{bmatrix} y_d(t) + \begin{bmatrix} 0.1 \\ 0.35 \\ 0.5 \end{bmatrix} u_d(t), \\ \hat{x}_p(t) &= \begin{bmatrix} 0.0769 & -0.0769 & 0.3846 \\ -0.3077 & 0.3077 & -0.0385 \\ 0.7692 & 0.2308 & -0.1538 \end{bmatrix} z_p(t) + \begin{bmatrix} 0.3077 & 0 \\ -0.2308 & 1 \\ 0.0769 & 0 \end{bmatrix} y_d(t). \end{aligned}$$

The obtained PIO is given by

$$
\begin{aligned}
z_{pi}(t+1) &= \begin{bmatrix} -0.0923 & 0.1573 & 0.2966 \\ 0.1061 & 0.2429 & 0.3018 \\ 0.3894 & 0.1786 & -0.0013 \end{bmatrix} z_{pi}(t) \\
&+ \begin{bmatrix} 0.1435 & 0.661 \\ 0.1845 & 0.76 \\ 0.0980 & 0.198 \end{bmatrix} y_d(t) + \begin{bmatrix} 0.1 \\ 0.35 \\ 0.5 \end{bmatrix} u_d(t) \\
&+ \begin{bmatrix} -1.7657 & -0.0129 \\ 3.1576 & -0.0129 \\ -0.5348 & -0.0129 \end{bmatrix} v_{pi}(t), \\
v_{pi}(t+1) &= y_d(t) - C\hat{x}_{pi}, \\
\hat{x}_{pi}(t) &= \begin{bmatrix} 0.0769 & -0.0769 & 0.3846 \\ -0.3077 & 0.3077 & -0.0385 \\ 0.7692 & 0.2308 & -0.1538 \end{bmatrix} z_{pi}(t) + \begin{bmatrix} 0.3077 & 0 \\ -0.2308 & 1 \\ 0.0769 & 0 \end{bmatrix} y_d(t).
\end{aligned}
$$

2.3.5.2 Simulation results

In order to show the performances of the proposed observer, we do some simulations with the obtained observer. Furthermore, in the simulation we have considered the system with uncertainty ΔA_{1d} in the state matrix A_d. In this case the system matrix becomes $A_d + \Delta A_{1d}$ where:

$$\Delta A_{1d} = [0.1 + 0.01\sin(20t)]\Delta$$

and

$$\Delta = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 1 & 0.1 & 0.5 \\ 0.2 & 0.3 & 0.1 \end{bmatrix}.$$

The simulation results are shown in the following figures. Figure 2.21 shows all the inputs used in the simulation. The control input is zero. The dashed line represents the unknown input which is used in the simulation, and the dash-dotted line represents the disturbance of finite energy in the time interval from the time 2s to 3s.

Figures 2.22, 2.25 and 2.28 represent the original system states and their estimates obtained from HDO, PO and PIO, respectively. The solid line represents the original state. The dashed line represents the state estimate obtained from HDO, the dotted line represents the state estimate obtained from PO and the dash-dotted line represents the state estimate obtained from PIO.

Figures 2.23, 2.24, 2.26, 2.27, 2.29 and 2.30 represent the estimation errors obtained from HDO, PO and PIO, respectively. Figures 2.23, 2.26 and 2.29 are the steady behavior of estimation errors. Figures 2.24, 2.27 and 2.30 represent the estimation errors for 1s. The dashed line represents the estimation error of HDO. The dotted line represents the estimation error of PO and the dash-dotted line represents the estimation error of PIO.

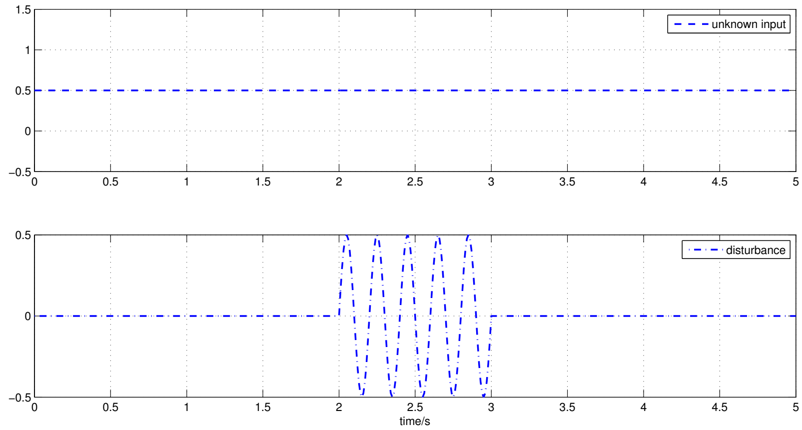

Figure 2.21: All inputs (dashed line: unknown input; dash-dotted line: disturbance)

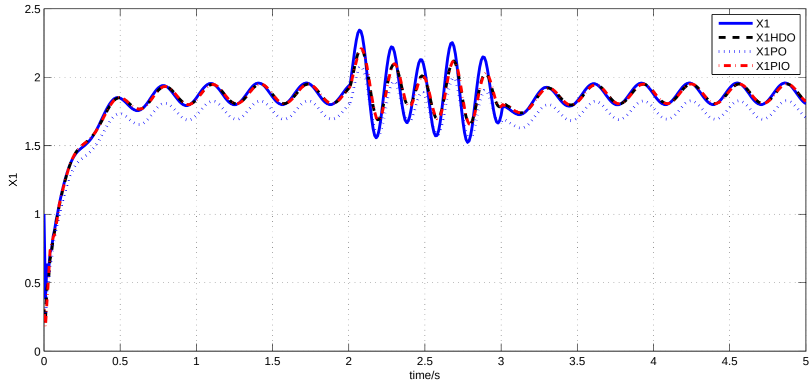

Figure 2.22: State x_1 and its estimates for 5s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

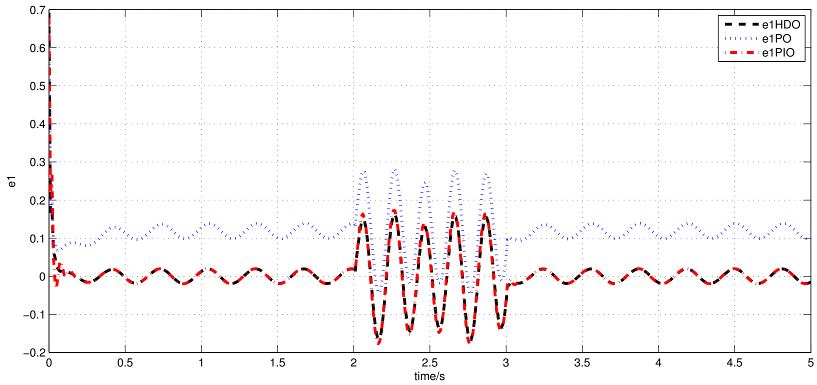

Figure 2.23: Estimation error e_1 for 5s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

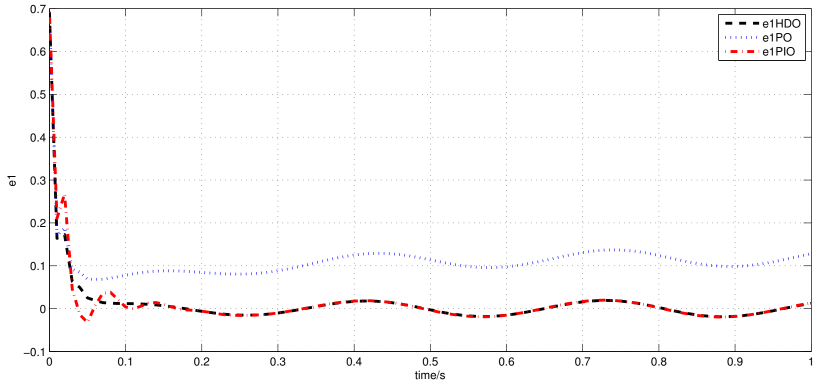

Figure 2.24: Estimation error e_1 for 1s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

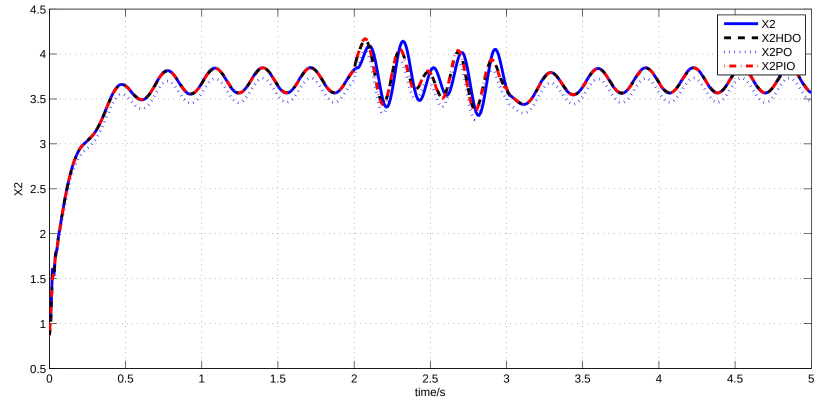

Figure 2.25: State x_2 and its estimates for 5s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

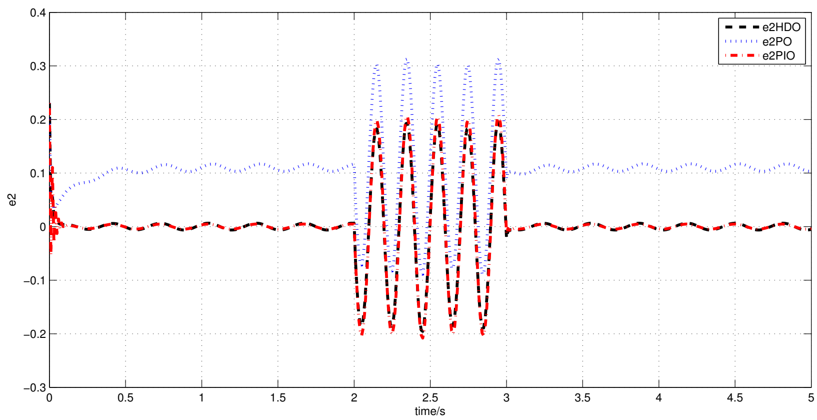

Figure 2.26: Estimation error e_2 for 5s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

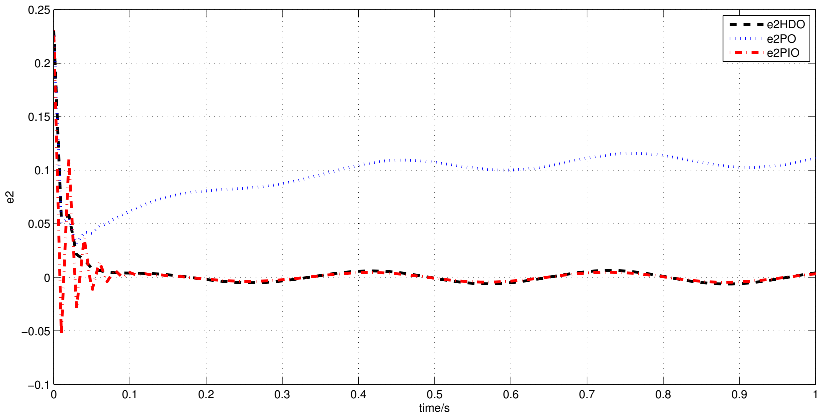

Figure 2.27: Estimation error e_2 for 1s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

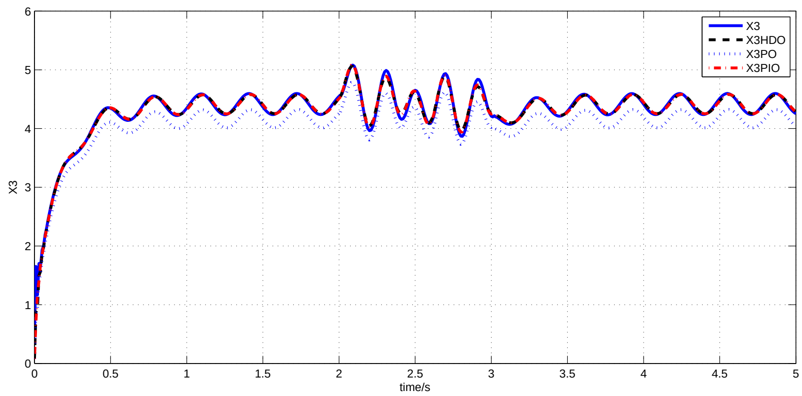

Figure 2.28: State x_3 and its estimates for 5s (solid line: original state; dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

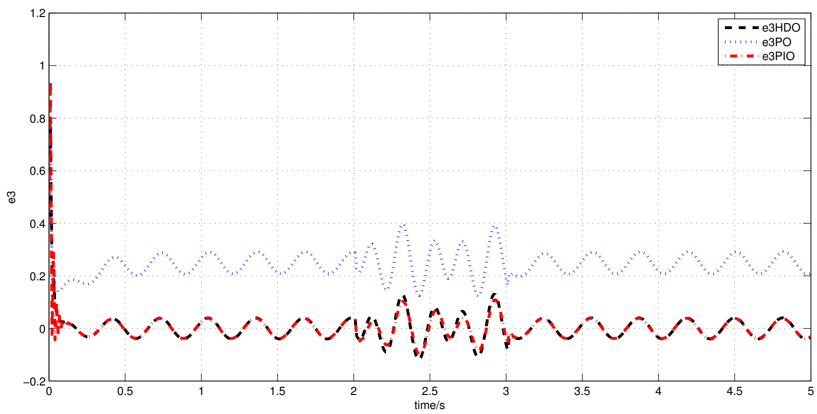

Figure 2.29: Estimation error e_3 for 5s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

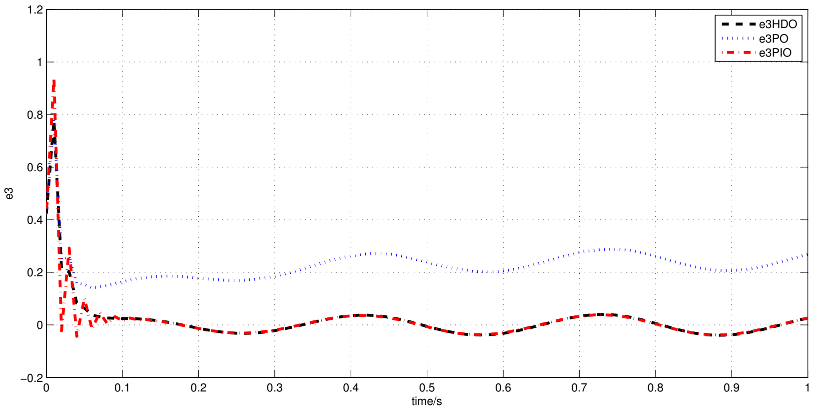

Figure 2.30: Estimation error e_3 for 1s (dashed line: HDO; dotted line: PO; dash-dotted line: PIO)

From the above figures one can see the performances of the proposed HDO in the presence of an additional parameter uncertainty ΔA_{1d}, compared with PO and PIO. The influence of the disturbance on the estimate is minimized by using H_∞ theory (see Figure 2.29 from the time 2s to 3s).
In the steady state regime, the HDO and PIO have the same better behaviors than PO. Take Figure 2.23 for example. One can see that there is a static error in the estimation error of PO, while in the estimation errors of HDO and PIO, there are no static errors.
Furthermore, in the transient regime, the estimation error of PIO presents greater oscillations (see Figure 2.27 for example), compared with HDO, while in the case of HDO, there are no oscillations.
In general, one can see that the proposed HDO generalizes the existing observers PO and PIO and has better behavior than these observers, in both transient regime and steady state regime.

2.4 A new H_∞ dynamic observer design for uncertain systems

In the previous sections, we have investigated the observer design problem for systems in the presence of disturbances and unknown inputs. In this section, we will investigate the observer design problem for uncertain systems.

One can see from previous sections that the observer design of discrete-time systems is similar to that of continuous-time systems. The difference is the formulation of LMIs. Therefore, in this section we only discuss the observer design problem for uncertain continuous-time systems.

Let us consider the following linear uncertain system in the presence of unknown inputs and disturbances:

$$\dot{x} = (A+\Delta A)x + Ld + Dw, \tag{2.130a}$$
$$y = Cx + Ew, \tag{2.130b}$$

with the initial state $x(0) = x_0$. Assume that the rank of matrix L is $rank\ L = r$. In order to simplify, we also assume that control input $u = 0$.

ΔA is an unknown matrix representing the time-varying parameter uncertainties [1], assumed to be of the following form:

$$\Delta A = \mathscr{M}\mathscr{F}(t)\mathscr{N}, \tag{2.131}$$

where $\mathscr{M}$ and $\mathscr{N}$ are known real constant matrices and of appropriate dimensions. $\mathscr{F}(t)$ is an unknown real-valued time-varying matrix satisfying

$$\mathscr{F}^T(t)\mathscr{F}(t) \leq I. \tag{2.132}$$

In order to decouple the unknown input, by pre-multiplying equation (2.130a) with matrix $L^\perp(L^\perp \in \Re^{(n-r)\times n})$, we obtain the following descriptor system:

$$\mathscr{E}\dot{x} = \mathscr{A}x + \Delta\mathscr{A}x + \mathscr{B}w, \tag{2.133a}$$

$$y = Cx + Ew, \tag{2.133b}$$

where matrices

$$\begin{aligned} \mathscr{E} &= L^\perp, \\ \mathscr{A} &= L^\perp A, \\ \mathscr{B} &= L^\perp D. \end{aligned}$$

$\Delta\mathscr{A}$ is bounded by

$$\Delta\mathscr{A} = \overline{\mathscr{M}}\mathscr{F}(t)\overline{\mathscr{N}}, \tag{2.134}$$

where $\overline{\mathscr{M}} = L^\perp\mathscr{M}$ and $\overline{\mathscr{N}} = \mathscr{N}$

One can see from system (2.133) that the observer design problem of uncertain systems (2.130) in the presence of unknown inputs and disturbances is transformed into the observer design problem of descriptor systems.
Next we give the following assumptions which is necessary for the observer design problem of descriptor systems:

Assumption 2.4.1. *For descriptor system* (2.133)*, we assume*

$$rank \begin{bmatrix} \mathscr{E} \\ C \end{bmatrix} = n \tag{2.135}$$

which is equivalent to

$$rank(CL) = rank\ L \tag{2.136}$$

according to [26] and [31]. □

Assumption 2.4.2. *Descriptor system* (2.133) *is impulse free, in other words,*

$$\det(s\mathscr{E} - \mathscr{A}) \neq 0.$$

□

The following lemma will be used in the design procedure:

Lemma 2.4.1. *[125] Let $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ be real matrices of appropriate dimensions and $\mathcal{F}$ satisfies $\mathcal{F}^T\mathcal{F} \leqslant I$. Then for any scalar $\rho > 0$ and vectors $x, y \in \Re^n$, we have*

$$2x^T\mathcal{D}\mathcal{F}\mathcal{E}y \leqslant \rho^{-1}x^T\mathcal{D}\mathcal{D}^Tx + \rho y^T\mathcal{E}^T\mathcal{E}y.$$

Now for descriptor system (2.133), let us consider the following HDO:

$$\begin{aligned} \dot{z} &= Fz + Jy + Mv, && (2.137a)\\ \dot{v} &= Pz + Qy + Gv, && (2.137b)\\ \hat{x} &= Rz + Sy, && (2.137c) \end{aligned}$$

where matrix $T = 0$ since there is no control inputs $(u = 0)$.

Firstly, we define an error variable as:

$$\varepsilon = z - \Phi\mathscr{E}x. \tag{2.138}$$

Similar to the previous sections, we obtain the following results:

$$\begin{aligned} \dot{\varepsilon} &= \dot{z} - \Phi\mathscr{E}\dot{x}, \\ &= Fz + Jy + Mv - \Phi\mathscr{A}x - \Phi\Delta\mathscr{A}x - \Phi\mathscr{B}w, \\ &= F\varepsilon + Mv + (JE - \Phi\mathscr{B})w + (F\Phi\mathscr{E} - \Phi\mathscr{A} + JC)x - \Phi\Delta\mathscr{A}x, && (2.139)\\ \dot{v} &= P\varepsilon + Gv + QEw + (P\Phi\mathscr{E} + QC)x, && (2.140)\\ \hat{x} &= R\varepsilon + SEw + (R\Phi\mathscr{E} + SC)x. && (2.141) \end{aligned}$$

Then, the algebraic constraints (2.14) of **Lemma 2.2.1** become

$$\begin{aligned} F\Phi\mathscr{E} - \Phi\mathscr{A} + JC &= 0, && (2.142a)\\ P\Phi\mathscr{E} + QC &= 0, && (2.142b)\\ R\Phi\mathscr{E} + SC &= I_n, && (2.142c) \end{aligned}$$

and equation (2.20) becomes

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \Phi\mathscr{E} \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{2.143}$$

The necessary and sufficient condition for equation (2.143) to have a solution is that the following rank condition must be satisfied

$$rank \begin{pmatrix} \Phi\mathscr{E} \\ C \\ 0 \\ I_n \end{pmatrix} = rank \begin{pmatrix} \Phi\mathscr{E} \\ C \end{pmatrix} = n. \tag{2.144}$$

Assume that above rank condition holds, and let Λ be an arbitrary matrix of full row rank such that:

$$rank \begin{pmatrix} \Lambda \\ C \end{pmatrix} = rank \begin{pmatrix} \Phi\mathscr{E} \\ C \end{pmatrix} = n. \tag{2.145}$$

In this case, equation (2.24) becomes

$$\Phi\mathscr{E} + \overline{K}C = \Lambda, \tag{2.146}$$

or equivalently

$$\begin{pmatrix}\Phi & \overline{K}\end{pmatrix}\begin{pmatrix}\mathscr{E}\\ C\end{pmatrix} = \Lambda. \tag{2.147}$$

One solution of equation (2.147) is given by

$$\Phi = \Lambda\begin{pmatrix}\mathscr{E}\\ C\end{pmatrix}^{+}\begin{pmatrix}I_{n-r}\\ 0\end{pmatrix}, \tag{2.148}$$

$$\overline{K} = \Lambda\begin{pmatrix}\mathscr{E}\\ C\end{pmatrix}^{+}\begin{pmatrix}0\\ I_p\end{pmatrix}. \tag{2.149}$$

Furthermore, matrices P and Q are given by equations (2.27) and (2.28). Matrices R and S are given by equations (2.29). Matrices F, $\overline{K}_1$ and J are given by equations (2.40), (2.41) and (2.42), where matrix $\Theta = \Phi\mathscr{A}$.

With the above parameterization results, system (2.52) becomes

$$\dot{\zeta}_1 = \mathbb{A}_1\zeta_1 + \mathbb{B}_1 w + \mathbb{E}_1\Delta\mathscr{A}x, \tag{2.150a}$$

$$e = \mathbb{C}_1\zeta_1 + \mathbb{D}_1 w, \tag{2.150b}$$

with matrices

$$\begin{cases} \mathbb{A}_1 = \mathbb{A}_{11} - \mathbb{Z}\mathbb{A}_{12}, \\ \mathbb{B}_1 = \mathbb{B}_{11} - \mathbb{Z}\mathbb{B}_{12}, \\ \mathbb{B}_{11} = \begin{pmatrix} -\Phi\mathscr{B} + \Theta\Upsilon^{+}\begin{pmatrix}\overline{K}\\ I_p\end{pmatrix}E \\ 0 \end{pmatrix}, \\ \mathbb{B}_{12} = \begin{pmatrix}\beta_2 E\\ 0\end{pmatrix}, \\ \mathbb{C}_1 = \begin{pmatrix}\alpha_1 & 0\end{pmatrix}, \\ \mathbb{D}_1 = \alpha_2 E, \\ \mathbb{E}_1 = \begin{pmatrix}-\Phi\\ 0\end{pmatrix}. \end{cases} \tag{2.151}$$

With all the above results, we can see that the observer design problem of uncertain systems (2.130) in the presence of disturbances and unknown inputs is reduced to study system (2.150), in other words, to determine matrix $\mathbb{Z}$ such that:

1. For $w = 0$, matrix $\mathbb{A}_1$ is a stability matrix;
2. For $w \neq 0$, the H_∞ norm $\|T_{we}\|_\infty$ is minimized.

The parameter matrix $\mathbb{Z}$ can be determined by using the following theorem:

Theorem 2.4.1. *System* (2.150) *is asymptotically stable for $w = 0$ and $\|T_{we}\|_\infty < \gamma$ for $w \neq 0$, if there exist two symmetric positive definite matrices P_1 and P_2, a matrix $\mathbb{Y}$ and a given positive scalar γ such that the following matrix inequality holds*

$$\Omega = \begin{bmatrix} \Omega_1 & \mathscr{E}^T P_1 \overline{\mathscr{M}} & 0 & 0 & \mathscr{E}^T P_1 \mathscr{B} \\ \star & -I & 0 & 0 & 0 \\ \star & \star & \Omega_2 & P_2 \mathbb{E}_1 \overline{\mathscr{M}} & \Omega_3 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I_f \end{bmatrix} < 0 \tag{2.152}$$

where

$$\begin{aligned} \Omega_1 &= \mathscr{A}^T P_1 \mathscr{E} + \mathscr{E}^T P_1 \mathscr{A} + 2\overline{\mathscr{N}}^T \overline{\mathscr{N}}, \\ \Omega_2 &= \mathbb{A}_{11}^T P_2 + P_2 \mathbb{A}_{11} - A_{12}^T \mathbb{Y}^T - \mathbb{Y} \mathbb{A}_{12} + \mathbb{C}_1^T \mathbb{C}_1, \\ \Omega_3 &= P_2 \mathbb{B}_{11} - \mathbb{Y} \mathbb{B}_{12} + \mathbb{C}_1^T \mathbb{D}_1. \end{aligned}$$

In this case, matrix $\mathbb{Z}$ is given by $\mathbb{Z} = P_2^{-1} \mathbb{Y}$.

Proof. *For system* (2.150)*, let us choose a Lyapunov function as follows:*

$$V = V_1 + V_2, \tag{2.153}$$

where Lyapunov functions $V_1 = (\mathscr{E}x)^T P_1 (\mathscr{E}x)$ and $V_2 = \zeta_1^T P_2 \zeta_1$. Matrices $P_1 > 0$ and $P_2 > 0$.

Subsequently, the derivative of V_1 along the solution of system (2.133) *is given by*

$$\begin{aligned} \dot{V}_1 &= (\mathscr{E}\dot{x})^T P_1 \mathscr{E} x + (\mathscr{E}x)^T P_1 (\mathscr{E}\dot{x}), \\ &= x^T (\mathscr{A}^T P_1 \mathscr{E} + \mathscr{E}^T P_1 \mathscr{A}) x \\ &\quad + 2x^T \mathscr{E}^T P_1 \Delta\mathscr{A} x + x^T \mathscr{E}^T P_1 \mathscr{B} w + w^T \mathscr{B}^T P_1 \mathscr{E} x. \end{aligned} \tag{2.154}$$

*According to **Lemma 2.4.1** and conditions* (2.131) *and* (2.132)*, by choosing $\rho = 1$, we obtain the following inequality:*

$$\begin{aligned} \dot{V}_1 &\leqslant x^T (\mathscr{A}^T P_1 \mathscr{E} + \mathscr{E}^T P_1 \mathscr{A}) x + x^T \mathscr{E}^T P_1 \mathscr{B} w + w^T \mathscr{B}^T P_1 \mathscr{E} x \\ &\quad + x^T \mathscr{E}^T P_1 \overline{\mathscr{M}} (\mathscr{E}^T P_1 \overline{\mathscr{M}})^T x + x^T \overline{\mathscr{N}}^T \overline{\mathscr{N}} x. \end{aligned} \tag{2.155}$$

On the other hand, the derivative of V_2 along the solution of system (2.150) *is given by*

$$\begin{aligned} \dot{V}_2 &= \dot{\zeta}_1^T P_2 \zeta_1 + \zeta_1^T P_2 \dot{\zeta}_1 \\ &= \zeta_1^T (\mathbb{A}_1^T P_2 + P_2 \mathbb{A}_1) \zeta_1 + \zeta_1^T P_2 \mathbb{B}_1 w + w^T \mathbb{B}_1^T P_2 \zeta_1 + 2\zeta_1^T P_2 \mathbb{E}_1 \Delta\mathscr{A} x. \end{aligned} \tag{2.156}$$

*According to **Lemma 2.4.1** and conditions* (2.131) *and* (2.132)*, and by choosing $\rho = 1$, we have the following inequality:*

$$\begin{aligned} \dot{V}_2 &\leqslant \zeta_1^T [\mathbb{A}_1^T P_2 + P_2 \mathbb{A}_1 + P_2 \mathbb{E}_1 \overline{\mathscr{M}} (P_2 \mathbb{E}_1 \overline{\mathscr{M}})^T] \zeta_1 \\ &\quad + x^T \overline{\mathscr{N}}^T \overline{\mathscr{N}} x + \zeta_1^T P_2 \mathbb{B}_1 w + w^T \mathbb{B}_1^T P_2 \zeta_1. \end{aligned} \tag{2.157}$$

Finally, we obtain the following results on $\dot{V}$:

$$
\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2, \\
&\leqslant x^T[\mathscr{A}^T P_1 \mathscr{E} + \mathscr{E}^T P_1 \mathscr{A} + \mathscr{E}^T P_1 \overline{\mathscr{M}}(\mathscr{E}^T P_1 \overline{\mathscr{M}})^T + 2\overline{\mathscr{N}}^T \overline{\mathscr{N}}]x \\
&\quad + x^T \mathscr{E}^T P_1 \mathscr{B} w + w^T \mathscr{B}^T P_1 \mathscr{E} x \\
&\quad + \zeta_1^T [\mathbb{A}_1^T P_2 + P_2 \mathbb{A}_1 + P_2 \mathbb{E}_1 \overline{\mathscr{M}}(P_2 \mathbb{E}_1 \overline{\mathscr{M}})^T]\zeta_1 \\
&\quad + \zeta_1^T P_2 \mathbb{B}_1 w + w^T \mathbb{B}_1^T P_2 \zeta_1, \\
&\leqslant \begin{bmatrix} x^T & \zeta_1^T & w^T \end{bmatrix} \Omega' \begin{bmatrix} x^T & \zeta_1^T & w^T \end{bmatrix}^T,
\end{aligned} \tag{2.158}
$$

where matrices

$$
\begin{aligned}
\Omega' &= \begin{bmatrix} \Omega_1' & 0 & \mathscr{E}^T P_1 \mathscr{B} \\ \star & \Omega_2' & P_2 \mathbb{B}_1 \\ \star & \star & 0 \end{bmatrix}, \\
\Omega_1' &= \mathscr{A}^T P_1 \mathscr{E} + \mathscr{E}^T P_1 \mathscr{A} + \mathscr{E}^T P_1 \overline{\mathscr{M}}(\mathscr{E}^T P_1 \overline{\mathscr{M}})^T + 2\overline{\mathscr{N}}^T \overline{\mathscr{N}}, \\
\Omega_2' &= \mathbb{A}_1^T P_2 + P_2 \mathbb{A}_1 + P_2 \mathbb{E}_1 \overline{\mathscr{M}}(P_2 \mathbb{E}_1 \overline{\mathscr{M}})^T.
\end{aligned}
$$

From equation (2.150b) *and inequality* (2.158)*, we obtain*

$$
\begin{aligned}
& \dot{V} + e^T e - \gamma^2 w^T w \\
\leq\ & x^T[\mathscr{A}^T P_1 \mathscr{E} + \mathscr{E}^T P_1 \mathscr{A} + \mathscr{E}^T P_1 \overline{\mathscr{M}}(\mathscr{E}^T P_1 \overline{\mathscr{M}})^T + 2\overline{\mathscr{N}}^T \overline{\mathscr{N}}]x \\
& + x^T \mathscr{E}^T P_1 \mathscr{B} w + w^T \mathscr{B}^T P_1 \mathscr{E} x \\
& + \zeta_1^T [\mathbb{A}_1^T P_2 + P_2 \mathbb{A}_1 + P_2 \mathbb{E}_1 \overline{\mathscr{M}}(P_2 \mathbb{E}_1 \overline{\mathscr{M}})^T + \mathbb{C}_1^T \mathbb{C}_1]\zeta_1 \\
& + \zeta_1^T (P_2 \mathbb{B}_1 + \mathbb{C}_1^T \mathbb{D}_1) w + w^T (\mathbb{B}_1^T P_2 + \mathbb{D}_1^T \mathbb{C}_1)\zeta_1 \\
& + w^T (\mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I_f) w, \\
\leq\ & \begin{bmatrix} x^T & \zeta_1^T & w^T \end{bmatrix} \overline{\Omega} \begin{bmatrix} x^T & \zeta_1^T & w^T \end{bmatrix}^T,
\end{aligned} \tag{2.159}
$$

where matices

$$
\begin{aligned}
\overline{\Omega} &= \begin{bmatrix} \Omega_1' & 0 & \mathscr{E}^T P_1 \mathscr{B} \\ \star & \Omega_2'' & P_2 \mathbb{B}_1 + \mathbb{C}_1^T \mathbb{D}_1 \\ \star & \star & \mathbb{D}_1^T \mathbb{D}_1 - \gamma^2 I_f \end{bmatrix}, \\
\Omega_2'' &= \mathbb{A}_1^T P_2 + P_2 \mathbb{A}_1 + P_2 \mathbb{E}_1 \overline{\mathscr{M}}(P_2 \mathbb{E}_1 \overline{\mathscr{M}})^T + \mathbb{C}_1^T \mathbb{C}_1.
\end{aligned}
$$

Notice that if $\overline{\Omega} < 0$, we have

$$
\dot{V} + e^T e - \gamma^2 w^T w < 0. \tag{2.160}
$$

By integrating inequality (2.160) *it follows*

$$
V(\infty) - V(0) < \gamma^2 \int w^T w dt - \int e^T e dt. \tag{2.161}
$$

Under the zero initial state, we obtain

$$\begin{aligned} & V(\infty) < \gamma^2 \int w^T w dt - \int e^T e dt \\ \Leftrightarrow \quad & 0 < \gamma^2 \int w^T w dt - \int e^T e dt \\ \Leftrightarrow \quad & \int e^T e dt < \gamma^2 \int w^T w dt, \end{aligned}$$

or equivalently

$$\|T_{we}\|_\infty < \gamma.$$

With the above results, it is obvious that for system (2.150)*, the* H_∞ *norm* $\|T_{we}\|_\infty < \gamma$ *if inequality* $\overline{\Omega} < 0$ *holds.*

On the other hand, in the case where $w = 0$*, inequality* (2.159) *becomes*

$$\dot{V} \leq \begin{bmatrix} x^T & \zeta_1^T & 0 \end{bmatrix} \overline{\Omega} \begin{bmatrix} x^T & \zeta_1^T & 0 \end{bmatrix}^T - e^T e. \tag{2.162}$$

One can see from inequality (2.162) *that if* $\overline{\Omega} < 0$*, then* $\dot{V} < 0$*, since it is alway satisfied that* $e^T e \geq 0$*.*

In summary, we know that a sufficient condition for system (2.150) *to be stable when* $w = 0$ *and* $\|T_{we}\|_\infty < \gamma$ *when* $w \neq 0$ *is that inequality* $\overline{\Omega} < 0$ *must hold.*

Subsequently, by inserting the matrices $\mathbb{A}$ *and* $\mathbb{B}$ *of equation* (2.151) *into inequality* $\overline{\Omega} < 0$*, the latter becomes:*

$$\begin{bmatrix} \Omega_1' & 0 & \mathscr{E}^T P_1 \mathscr{B} \\ \star & \Omega_2''' & P_2\mathbb{B}_{11} - P_2\mathbb{Z}\mathbb{B}_{12} + \mathbb{C}_1^T\mathbb{D}_1 \\ \star & \star & \mathbb{D}_1^T\mathbb{D}_1 - \gamma^2 I_f \end{bmatrix} < 0 \tag{2.163}$$

where

$$\Omega_2''' = \mathbb{A}_{11}^T P_2 - A_{12}^T \mathbb{Z}^T P_2 + P_2\mathbb{A}_{11} - P_2\mathbb{Z}\mathbb{A}_{12} + P_2\mathbb{E}_1\overline{\mathscr{M}}(P_2\mathbb{E}_1\overline{\mathscr{M}})^T + \mathbb{C}_1^T\mathbb{C}_1.$$

In this case, by using Schur complement lemma and defining matrix $\mathbb{Y} = P_2\mathbb{Z}$*, inequality* (2.163) *becomes* (2.152)*, which completes the proof.*

□

Notice that matrix inequality (2.152) is bilinear. In order to solve the matrix inequality, we can rewrite matrix inequality (2.152) as:

$$\mathfrak{Q} + \overline{\mathfrak{B}} P_1 \overline{\mathfrak{C}} + (\overline{\mathfrak{B}} P_1 \overline{\mathfrak{C}})^T < 0, \tag{2.164}$$

where matrices

$$\mathfrak{Q} = \begin{bmatrix} 2\overline{\mathscr{N}}^T\overline{\mathscr{N}} & 0 & 0 & 0 & 0 \\ \star & -I & 0 & 0 & 0 \\ \star & \star & \Omega_2 & P_2\mathbb{E}_1\overline{\mathscr{M}} & \Omega_3 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \mathbb{D}_1^T\mathbb{D}_1 - \gamma^2 I_f \end{bmatrix}, \tag{2.165}$$

$$\overline{\mathfrak{B}} = \begin{bmatrix} \mathscr{E}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \overline{\mathfrak{C}} = \begin{bmatrix} A & \overline{\mathscr{M}} & 0 & 0 & \mathscr{B} \end{bmatrix}. \tag{2.166}$$

Then we can use **Theorem 2.2.3** to solve the matrix inequality (2.164).

2.5 Conclusions

In this chapter, a new form of HDO has been proposed for linear systems in the presence of disturbances and unknown inputs. The proposed observer generalizes the results on the existing observers such as PO and PIO.

Firstly, we investigated the observer design problem for continuous-time systems. Based on the parameterization of the algebraic constraints obtained from the analysis of the estimation error, the observer design problem was reduced to the determination of one parameter matrix. Then the determination problem was formulated as an optimization problem in terms of LMIs, based on bounded real lemma. Subsequently, the results of continuous-time systems were extended to discrete-time systems. The observer design was derived from the solution of LMIs, based on bounded-real lemma and Schur complement lemma.

Numerical examples were presented to illustrate the design procedure and the performances of the proposed HDO for both continuous-time systems and discrete-time systems. The performances of the proposed observer were shown to be better in both transient regime and steady state regime, compared with PO and PIO.

In the end, we discussed the observer design problem of uncertain systems. One can see that the design problem of uncertain systems in the presence of disturbances and unknown inputs can be transformed into the observer design problem of descriptor systems.

CHAPTER

3

H_∞ dynamic-observer-based control design

Contents

3.1 Introduction

As a challenging problem, observer-based control has been an active research area in the past decades. The important use of observer is to realize observer-based control design, in which a state feedback law has to be implemented when the states of system are not accessible. A subsequent development of this is the requirement of making the observer optimal and robust due to inevitable presence of uncertainties in the system. Several approaches have been presented in [57, 85, 54, 149].
It is well known that the observer-based control design can be realized by using the separation principle. This principle permits to design the observer independently of the control, and the eigenvalues of the closed-loop system are formed by those of the observer and those of the control. The most existing results take the use of PO [39, 75, 84, 114] or PIO [44].
In Chapter 2, we have proposed a new form of HDO. The performances of the proposed observer were shown to be better than other observers such as PO and PIO. Therefore, in this chapter, the new HDO will be inserted into the closed-loop and an H_∞ dynamic-observer-based control design is proposed for uncertain systems in the presence of disturbances. The parameter uncertainties are time-varying and norm-bounded. The design problem is formulated as a BMI optimization problem.
This chapter is organized as follows. In section 3.2, the observer-based control design problem of uncertain systems in the presence of disturbances is presented. The algebraic constraints are derived through the analysis of the estimation error and the parameterization results are presented in section 3.3. In section 3.4, the control design problem is formulated as a BMI optimization problem and a two-step algorithm is proposed to solve this problem. A numerical example is provided to illustrate the performance of the proposed control in section 3.5. Finally, some conclusions are given in section 3.6.

3.2 Problem formulation

Let us consider the following uncertain system in the presence of disturbances:

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + Bu + Dw, && \text{(3.1a)} \\ y &= Cx + Ew, && \text{(3.1b)} \end{aligned}$$

with initial state $x(0) = x_0$. $x \in \Re^n$, $u \in \Re^m$, $w \in \Re^f$ and $y \in \Re^p$ are the system state, the control input, the exogenous disturbance of finite energy and the output, respectively. Matrices A, B, C, D and E are known constant and of appropriate dimensions.

ΔA is an unknown matrix representing the time-varying parameter uncertainty, which is assumed to be norm bounded and has the following structure [1]:

$$\Delta A = \mathcal{M}\mathcal{F}(t)\mathcal{N}, \tag{3.2}$$

where $\mathcal{M}$ and $\mathcal{N}$ are known real constant matrices and of appropriate dimensions. $\mathcal{F}(t)$ is an unknown matrix function satisfying

$$\mathcal{F}^T(t)\mathcal{F}(t) \leq I. \tag{3.3}$$

The observer we consider is the HDO proposed in Chapter 2, which is given by

$$\begin{aligned} \dot{z} &= Fz + Jy + Tu + Mv, && (3.4a)\\ \dot{v} &= Pz + Qy + Gv, && (3.4b)\\ \hat{x} &= Rz + Sy, && (3.4c) \end{aligned}$$

where $z \in \Re^q$ is the state of observer, $v \in \Re^v$ is an auxiliary vector and $\hat{x} \in \Re^n$ is the estimate of system states. Matrices F, J, T, M, P, Q, G, R and S are unknown and of appropriate dimensions to be determined.

The control law for system (3.1) is expressed as

$$u = K\hat{x}, \tag{3.5}$$

where K is an unknown matrix and of appropriate dimension. The control we use is an observer-based control where K is the gain matrix to be determined.

With all the presentations, the design problem of H_∞ dynamic-observer-based control for uncertain systems (3.1) in the presence of disturbances is reduced to determine the parameter matrices F, J, T, M, P, Q, G, R and S of observer (3.4) and the matrix K of control (3.5) such that the following two conditions are satisfied:

1. For $w = 0$, the estimation error $e(t) \to 0$ when $t \to \infty$, and the closed-loop system obtained from the system (3.1), the observer (3.4) and the control (3.5) is stable;

2. For $w \neq 0$, minimize the effect of w on the output y.

3.3 Algebraic constraints and parameterizations

Similar to Chapter 2, in this section we shall present some algebraic constraints and parameterizations before solving the control design problem, based on a new defined error variable and the analysis of the estimation error.

3.3.1 Algebraic constraints

Firstly, we define an error variable:

$$\varepsilon = z - \Phi x, \tag{3.6}$$

where vector $\varepsilon \in \Re^q$ and $\Phi \in \Re^{q\times n}$ is an arbitrary matrix.
Different from the analysis in Chapter 2, there is an uncertain term ΔA in system (3.1). In this case, the derivative of error ε is given by

$$\begin{aligned} \dot{\varepsilon} &= \dot{z} - \Phi\dot{x},\\ &= Fz + JCx + JEw + Tu + Mv\\ &\quad -\Phi(A+\Delta A)x - \Phi Bu - \Phi Dw,\\ &= F(z-\Phi x) + F\Phi x + JCx + JEw + Tu + Mv\\ &\quad -\Phi Ax - \Phi\Delta Ax - \Phi Bu - \Phi Dw,\\ &= F\varepsilon + Mv + (JE - \Phi D)w - \Phi\Delta Ax\\ &\quad +(F\Phi + JC - \Phi A)x + (T - \Phi B)u. \end{aligned} \tag{3.7}$$

Furthermore, from observer (3.4) it follows that:

$$\begin{aligned} \dot{v} &= Pz + Qy + Gv, \\ &= P\varepsilon + Gv + QEw + (P\Phi + QC)x. \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \hat{x} &= Rz + Sy, \\ &= R\varepsilon + SEw + (R\Phi + SC)x. \end{aligned} \tag{3.9}$$

On the other hand, inserting control (3.5) into system (3.1) yields

$$\begin{aligned} \dot{x} &= Ax + \Delta Ax + BK\hat{x} + Dw, \\ &= Ax + \Delta Ax + BK(Rz + Sy) + Dw, \\ &= Ax + \Delta Ax + BK[R(z - \Phi x) + R\Phi x + SCx + SEw] + Dw, \\ &= [A + BK(R\Phi + SC)]x + BKR\varepsilon + (D + BKSE)w + \Delta Ax. \end{aligned} \tag{3.10}$$

Notice that if the following algebraic constraints hold

$$F\Phi + JC - \Phi A = 0, \tag{3.11a}$$
$$T - \Phi B = 0, \tag{3.11b}$$
$$P\Phi + QC = 0, \tag{3.11c}$$
$$R\Phi + SC = I_n, \tag{3.11d}$$

equations (3.10), (3.7) and (3.8) become

$$\dot{x} = (A + BK)x + BKR\varepsilon + (D + BKSE)w + \Delta Ax, \tag{3.12}$$

$$\dot{\varepsilon} = F\varepsilon + Mv + (JE - \Phi D)w - \Phi\Delta Ax, \tag{3.13}$$

$$\dot{v} = P\varepsilon + Gv + QEw, \tag{3.14}$$

and we obtain an expression of estimation error e as follows:

$$e = R\varepsilon + SEw. \tag{3.15}$$

In this case, we obtain the following system:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\varepsilon} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} A + BK & BKR & 0 \\ 0 & F & M \\ 0 & P & G \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix} + \begin{bmatrix} D + BKSE \\ JE - \Phi D \\ QE \end{bmatrix} w \\ &\quad + \begin{bmatrix} \Delta A & 0 & 0 \\ -\Phi\Delta A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix}, \\ y &= \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix} + Ew, \end{aligned}$$

or equivalently

$$\begin{aligned} \dot{\xi} &= \mathbb{A}\xi + \mathbb{B}w + \varDelta\mathbb{A}\xi, && (3.16a)\\ y &= \mathbb{C}\xi + \mathbb{D}w, && (3.16b) \end{aligned}$$

where

$$\begin{cases} \xi = \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix} = \xi(t), \\ \mathbb{A} = \begin{bmatrix} A+BK & BKR & 0 \\ 0 & F & M \\ 0 & P & G \end{bmatrix}, \\ \mathbb{B} = \begin{bmatrix} D+BKSE \\ JE-\Phi D \\ QE \end{bmatrix}, \\ \mathbb{C} = \begin{bmatrix} C & 0 & 0 \end{bmatrix}, \\ \mathbb{D} = E. \end{cases} \tag{3.17}$$

$\varDelta\mathbb{A}$ is given by

$$\begin{aligned} \varDelta\mathbb{A} &= \begin{bmatrix} \Delta A & 0 & 0 \\ -\Phi\Delta A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} \mathcal{M} \\ -\Phi\mathcal{M} \\ 0 \end{bmatrix} \mathcal{F}(t) \begin{bmatrix} \mathcal{N} & 0 & 0 \end{bmatrix}, \\ &= \mathbb{M}\mathcal{F}(t)\mathbb{N}, && (3.18) \end{aligned}$$

where matrices $\mathbb{M} = \begin{bmatrix} \mathcal{M} \\ -\Phi\mathcal{M} \\ 0 \end{bmatrix}$ and $\mathbb{N} = \begin{bmatrix} \mathcal{N} & 0 & 0 \end{bmatrix}$.

With all the above results, the H_∞ dynamic-observer-based control design problem is reduced to study system (3.16) in the presence of disturbances and uncertainties. In detail, we must determine all the parameter matrices of system (3.16) such that:

1. Conditions (3.11) are satisfied;

2. For $w = 0$, the augmented system (3.16) must be asymptotically stable;

3. For $w \neq 0$, minimize the effect of w on y, i.e., the H_∞ norm $\|T_{wy}\|_\infty < \gamma$,

where T_{wy} is the transfer function matrix from w to y and γ is a given positive scalar.

3.3.2 Parameterizations

In this section, in order to simplify the problem, we will do some parameterizations to algebraic constraints (3.11). The procedure is similar to the parametrization presented in Chapter 2. Firstly, from equations (3.11c) and (3.11d) we obtain the following equation:

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \Phi \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{3.19}$$

The necessary and sufficient condition for equation (3.19) to have a solution is that the following rank condition must be satisfied:

$$rank \begin{pmatrix} \Phi \\ C \\ 0 \\ I_n \end{pmatrix} = rank \begin{pmatrix} \Phi \\ C \end{pmatrix} = n. \tag{3.20}$$

Assume that the rank condition (3.20) holds, and let Λ ($\Lambda \in \Re^{q\times n}$) be an arbitrary full row rank matrix satisfying

$$rank \begin{pmatrix} \Lambda \\ C \end{pmatrix} = rank \begin{pmatrix} \Phi \\ C \end{pmatrix} = n, \tag{3.21}$$

then there always exist two parameter matrices $\overline{K}$ ($\overline{K} \in \Re^{q\times p}$) and Φ such that:

$$\begin{pmatrix} \Phi \\ C \end{pmatrix} = \begin{pmatrix} I_q & -\overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix}, \tag{3.22}$$

which leads to

$$\Phi = \Lambda - \overline{K}C, \tag{3.23}$$

also equivalently,

$$\begin{pmatrix} \Phi & \overline{K} \end{pmatrix} \begin{pmatrix} I_n \\ C \end{pmatrix} = \Lambda. \tag{3.24}$$

In this case, a solution of equation (3.24) is given by

$$\begin{pmatrix} \Phi & \overline{K} \end{pmatrix} = \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+, \tag{3.25}$$

or equivalently,

$$\Phi = \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+ \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \tag{3.26}$$

$$\overline{K} = \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}. \tag{3.27}$$

Consequently, by inserting equation (3.22) into equation (3.19) we obtain

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I_q & -\overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \tag{3.28}$$

and matrices P, Q, R and S are given as follows:

$$\begin{aligned} P &= -Z_2\beta_1, && (3.29a)\\ Q &= -Z_2\beta_2, && (3.29b)\\ R &= \alpha_1 - Z_3\beta_1, && (3.29c)\\ S &= \alpha_2 - Z_3\beta_2, && (3.29d) \end{aligned}$$

with

$$\begin{cases} Z_2 = \begin{pmatrix} I_t & 0 \end{pmatrix} Z, \\ Z_3 = \begin{pmatrix} 0 & I_n \end{pmatrix} Z, \\ \alpha_1 = \Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_2 = \Upsilon^+ \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix}, \\ \beta_1 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \beta_2 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix}, \\ \Upsilon = \begin{pmatrix} \Lambda \\ C \end{pmatrix}, \end{cases} \tag{3.30}$$

and Z is an arbitrary matrix of appropriate dimension.

Remark 3.3.1. *According to **Remark 2.2.7**, we know that if we take matrix $Z_3 = 0$, we can avoid the bilinearity. In this case, matrices R and S are given by*

$$\begin{aligned} R &= \alpha_1, && (3.31)\\ S &= \alpha_2. && (3.32) \end{aligned}$$

■

On the other hand, matrices F and $\overline{K}_1$ are given by:

$$\begin{aligned} F &= \alpha_3 - Z_1\beta_1, && (3.33a)\\ \overline{K}_1 &= \alpha_4 - Z_1\beta_3, && (3.33b) \end{aligned}$$

with matrices

$$\begin{cases} \alpha_3 = \Theta\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_4 = \Theta\Upsilon^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \beta_3 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \Theta = \Phi A. \end{cases} \tag{3.34}$$

Furthermore, matrix J is given by

$$J = \Theta\alpha_2 - Z_1\beta_2. \tag{3.35}$$

Based on the above developments, we obtain the following results on the matrices $\mathbb{A}$ and $\mathbb{B}$ of system (3.16):

$$\begin{aligned}
\mathbb{A} &= \begin{bmatrix} A+BK & BKR & 0 \\ 0 & F & M \\ 0 & P & G \end{bmatrix}, \\
&= \begin{bmatrix} A+BK & BKR & 0 \\ 0 & \alpha_3 - Z_1\beta_1 & M \\ 0 & -Z_2\beta_1 & G \end{bmatrix}, \\
&= \begin{bmatrix} A+BK & \begin{pmatrix} BKR & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{A}_1 - \mathbb{Z}_1\mathbb{A}_2 \end{bmatrix},
\end{aligned} \tag{3.36}$$

where matrices

$$\mathbb{A}_1 = \begin{bmatrix} \alpha_3 & 0 \\ 0 & 0 \end{bmatrix}, \mathbb{Z}_1 = \begin{bmatrix} Z_1 & M \\ Z_2 & G \end{bmatrix} \text{ and } \mathbb{A}_2 = \begin{bmatrix} \beta_1 & 0 \\ 0 & -I_t \end{bmatrix}, \tag{3.37}$$

and

$$\begin{aligned}
\mathbb{B} &= \begin{bmatrix} D+BKSE \\ JE - \Phi D \\ QE \end{bmatrix}, \\
&= \begin{bmatrix} D+BKSE \\ \Theta\alpha_2 E - Z_1\beta_2 E - \Phi D \\ -Z_2\beta_2 E \end{bmatrix}, \\
&= \begin{bmatrix} D+BKSE \\ \mathbb{B}_1 - \mathbb{Z}_1\mathbb{B}_2 \end{bmatrix},
\end{aligned} \tag{3.38}$$

where matrices

$$\mathbb{B}_1 = \begin{bmatrix} \Theta\alpha_2 E - \Phi D \\ 0 \end{bmatrix} \text{ and } \mathbb{B}_2 = \begin{bmatrix} \beta_2 E \\ 0 \end{bmatrix}. \tag{3.39}$$

Remark 3.3.2. *The determination of matrix $\mathbb{Z}_1$ is independent of the choice of the generalized inverse Υ^+. The proof can be found in [29].* ■

Notice that once matrices K, Z_1, Z_2, M and G, i.e., matrices K and $\mathbb{Z}_1$ are determined, we can deduce all the parameter matrices of observer (3.4) and control (3.5) according to equations (3.29), (3.30), (3.33), (3.34) and (3.35). Therefore the control design problem is reduced to determine parameter matrices K and $\mathbb{Z}_1$ such that:

1. For $w = 0$, the augmented state $\xi(t) \to 0$ when $t \to \infty$;

2. For $w \neq 0$, the H_∞ norm $\|T_{wy}\|_\infty < \gamma$.

3.4 H_∞ dynamic-observer-based control design

With the help of the previous works, the observer-based control design problem is reduced to determine two parameter matrices K and $\mathbb{Z}_1$. In this section, the method to determine these matrices will be investigated. Firstly, we give the following theorem:

Theorem 3.4.1. *System* (3.16) *is asymptotically stable for $w = 0$ and $\|T_{wy}\|_\infty < \gamma$ for $w \neq 0$, if there exist two symmetric positive definite matrices $\mathbb{X}_1$, $\mathbb{X}_2$, and a given positive scalar γ such that the following matrix inequality holds*

$$\begin{bmatrix} \mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N} + \mathbb{C}^T\mathbb{C} & \mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D} \\ (\mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D})^T & \mathbb{D}^T\mathbb{D} - \gamma^2 I_f \end{bmatrix} < 0. \tag{3.40}$$

Proof. *For system* (3.16)*, we choose the flollowing Lyapunov function:*

$$V(\xi) = \xi^T\mathbb{X}\xi, \tag{3.41}$$

where matices $\mathbb{X} = diag(\mathbb{X}_1, \mathbb{X}_2)$ and $\mathbb{X}_j^T = \mathbb{X}_j > 0$ $(j = 1, 2)$.
Then the derivative of Lyapunov function $V(\xi)$ along the solution of system (3.16) *is given by*

$$\begin{aligned} \dot{V}(\xi) &= \dot{\xi}^T\mathbb{X}\xi + \xi^T\mathbb{X}\dot{\xi}, \\ &= [\mathbb{A}\xi + \mathbb{B}w + \Delta\mathbb{A}\xi]^T\mathbb{X}\xi + \xi^T\mathbb{X}[\mathbb{A}\xi + \mathbb{B}w + \Delta\mathbb{A}\xi], \\ &= \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi + 2\xi^T\mathbb{X}\Delta\mathbb{A}\xi. \end{aligned} \tag{3.42}$$

*According to **Lemma 2.4.1** and inequality* (3.18)*, and by choosing $\rho = 1$, we obtain the following result:*

$$\begin{aligned} \dot{V}(\xi) &= \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi + 2\xi^T\mathbb{X}\mathbb{M}\mathcal{F}(t)\mathbb{N}\xi, \\ &\leq \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi + \xi^T[\mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N}]\xi, \\ &\leq \xi^T[\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N}]\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi. \end{aligned} \tag{3.43}$$

Subsequently, from inequality (3.43) *and equation* (3.16b)*, we obtain*

$$\begin{aligned} \dot{V}(\xi) + y^Ty - \gamma^2w^Tw \leq\ & \xi^T[\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N}]\xi \\ & +\xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi \\ & +(\mathbb{C}\xi + \mathbb{D}w)^T(\mathbb{C}\xi + \mathbb{D}w) - \gamma^2w^Tw, \\ \leq\ & \xi^T[\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N} + \mathbb{C}^T\mathbb{C}]\xi \\ & +\xi^T(\mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D})w + [\xi^T(\mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D})w]^T \\ & +w^T(\mathbb{D}^T\mathbb{D} - \gamma^2I_f)w, \end{aligned} \tag{3.44}$$

or equivalently

$$\dot{V}(\xi) + y^Ty - \gamma^2w^Tw \leq \begin{bmatrix}\xi^T & w^T\end{bmatrix} \Omega \begin{bmatrix}\xi^T & w^T\end{bmatrix}^T, \tag{3.45}$$

where matrix

$$\Omega = \begin{bmatrix} \mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N} + \mathbb{C}^T\mathbb{C} & \mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D} \\ (\mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D})^T & \mathbb{D}^T\mathbb{D} - \gamma^2 I_f \end{bmatrix}.$$

From inequality (3.45) *we can see that if* $\Omega < 0$*, we have*

$$\dot{V}(\xi) + y^T y - \gamma^2 w^T w < 0. \tag{3.46}$$

Then by integrating inequality (3.46) *it follows that:*

$$V(\infty) - V(0) < \gamma^2 \int w^T w dt - \int y^T y dt.$$

It is easy to see that under the zero initial state, the above inequality is equivalent to

$$\|T_{wy}\|_\infty < \gamma.$$

One can see from the above results that in the case where $w \neq 0$*, the* H_∞ *norm* $\|T_{wy}\|_\infty < \gamma$*, if inequality* $\Omega < 0$ *holds.*

Next let us consider the case where $w = 0$*. If* $w = 0$*, inequalities* (3.45) *becomes*

$$\dot{V}(\xi) \leq \begin{bmatrix}\xi^T & 0\end{bmatrix} \Omega \begin{bmatrix}\xi^T & 0\end{bmatrix}^T - y^T y, \tag{3.47}$$

In this case, it is obvious that if $\Omega < 0$*, then* $\dot{V}(\xi) < 0$ *since it is always satisfied that* $y^T y \geq 0$*. In other words, system* (3.16) *is stable, if inequality* $\Omega < 0$ *holds.*

With all the above results, one can know that system (3.16) *is asymptotically stable for* $w = 0$ *and* $\|T_{wy}\|_\infty < \gamma$ *for* $w \neq 0$*, if there exist two symmetric positive definite matrices* $\mathbb{X}_1$*,* $\mathbb{X}_2$*, and a given positive scalar* γ *such that the matrix inequality* $\Omega < 0$ *holds, i.e., the matrix inequality* (3.40) *holds. The proof is completed.* □

According to **Theorem 3.4.1**, we obtain the following theorem to determine the two matrices $\mathbb{Z}_1$ and K:

Theorem 3.4.2. *System* (3.16) *is asymptotically stable for* $w = 0$ *and* $\|T_{wy}\|_\infty < \gamma$ *for* $w \neq 0$*, if there exist two symmetric positive definite matrices* $\mathbb{X}_1$ *and* $\mathbb{X}_2$*, three matrices* $\mathbb{Q}_1$*,* $\mathbb{Q}_2$ *and* K *of appropriate dimensions, and a given positive scalar* γ *such that the following matrix inequality holds*

$$\begin{bmatrix} \Omega_1 & \begin{pmatrix} BKR & 0 \end{pmatrix} & \Omega_3 & \mathbb{X}_1^{-1}\mathcal{N}^T & \mathcal{M} & \mathbb{X}_1^{-1}C^T \\ \star & \Omega_2 & \mathbb{X}_2\mathbb{B}_1 - \mathbb{Q}_2\mathbb{B}_2 & 0 & \mathbb{X}_2\begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} & 0 \\ \star & \star & E^T E - \gamma^2 I_f & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, \tag{3.48}$$

where

$$\begin{aligned} \Omega_1 &= A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1 + [A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1]^T, \mathbb{Q}_1 = K(\mathbb{X}_1^T)^{-1}, \\ \Omega_2 &= \mathbb{X}_2\mathbb{A}_1 - \mathbb{Q}_2\mathbb{A}_2 + (\mathbb{X}_2\mathbb{A}_1 - \mathbb{Q}_2\mathbb{A}_2)^T, \\ \Omega_3 &= D + BKSE + \mathbb{X}_1^{-1}C^T E. \end{aligned}$$

In this case, matrices $K = \mathbb{Q}_1\mathbb{X}_1$ *and* $\mathbb{Z}_1 = \mathbb{X}_2^{-1}\mathbb{Q}_2$*.*

Proof. *By using Schur complement lemma, equality* (3.40) *can be rewritten as:*

$$\begin{bmatrix} \mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} & \mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D} & \mathbb{N}^T & \mathbb{X}\mathbb{M} & \mathbb{C}^T \\ \star & \mathbb{D}^T\mathbb{D} - \gamma^2 I_f & 0 & 0 & 0 \\ \star & \star & -I & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0. \tag{3.49}$$

In this case, substituting matrices of equations (3.17)*,* (3.29)*,* (3.30)*,* (3.33)*,* (3.34) *and* (3.36) *into matrix inequality* (3.49) *yields*

$$\left\{\begin{matrix} \begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \times \begin{bmatrix} A+BK & \begin{pmatrix} BKR & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{A}_1 - \mathbb{Z}_1\mathbb{A}_2 \end{bmatrix} + \begin{bmatrix} A+BK & \begin{pmatrix} BKR & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{A}_1 - \mathbb{Z}_1\mathbb{A}_2 \end{bmatrix}^T \times \begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} & \begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \times \begin{pmatrix} D+BKSE \\ \mathbb{B}_1 - \mathbb{Z}_1\mathbb{B}_2 \end{pmatrix} + \begin{bmatrix} C^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} \times E & \begin{bmatrix} \mathcal{N}^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} & \begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \times \begin{bmatrix} \mathcal{M} \\ \begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} \end{bmatrix} & \begin{bmatrix} C^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} \\ \star & E^TE - \gamma^2 I_f & 0 & 0 & 0 \\ \star & \star & -I & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -I \end{matrix}\right\} < 0, \tag{3.50}$$

or equivalently

$$\begin{bmatrix} \Omega_1' & \begin{pmatrix} \mathbb{X}_1 BKR & 0 \end{pmatrix} & \Omega_3' & \mathcal{N}^T & \mathbb{X}_1\mathcal{M} & C^T \\ \star & \Omega_2 & \mathbb{X}_2\mathbb{B}_1 - \mathbb{Q}_2\mathbb{B}_2 & 0 & \mathbb{X}_2\begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} & 0 \\ \star & \star & E^TE - \gamma^2 I_f & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, \tag{3.51}$$

where

$$\begin{aligned} \Omega_1' &= \mathbb{X}_1 A + \mathbb{X}_1 BK + (\mathbb{X}_1 A + \mathbb{X}_1 BK)^T, \\ \Omega_2 &= \mathbb{X}_2\mathbb{A}_1 - \mathbb{Q}_2\mathbb{A}_2 + (\mathbb{X}_2\mathbb{A}_1 - \mathbb{Q}_2\mathbb{A}_2)^T, \mathbb{Q}_2 = \mathbb{X}_2\mathbb{Z}_1, \\ \Omega_3' &= \mathbb{X}_1(D + BKSE) + C^T E. \end{aligned}$$

Then, by pre- and post-multiplying matrix inequality (3.51) *with matrices*

$$\begin{bmatrix} \mathbb{X}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} (\mathbb{X}_1^T)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

matrix equality (3.51) *becomes*

$$\begin{bmatrix} \Omega_1'' & \begin{pmatrix} BKR & 0 \end{pmatrix} & \Omega_3 & \mathbb{X}_1^{-1}\mathcal{N}^T & \mathcal{M} & \mathbb{X}_1^{-1}C^T \\ \star & \Omega_2 & \mathbb{X}_2\mathbb{B}_1 - \mathbb{Q}_2\mathbb{B}_2 & 0 & \mathbb{X}_2\begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} & 0 \\ \star & \star & E^TE - \gamma^2 I_f & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, \tag{3.52}$$

where

$$\begin{aligned} \Omega_1'' &= A(\mathbb{X}_1^T)^{-1} + BK(\mathbb{X}_1^T)^{-1} + [A(\mathbb{X}_1^T)^{-1} + BK(\mathbb{X}_1^T)^{-1}]^T, \\ \Omega_3 &= D + BKSE + \mathbb{X}_1^{-1}C^TE. \end{aligned}$$

In this case, by defining matrix

$$\mathbb{Q}_1 = K(\mathbb{X}_1^T)^{-1},$$

we can obtain matrix inequality (3.48)*, which completes the proof.* □

With all the above results, one can see that the H_∞ dynamic-observer-based control design problem for uncertain systems in the presence of disturbances is reduced to solve matrix inequality (3.48). Unfortunately, matrix inequality (3.48) is a BMI so that the problem is not convex. In order to solve it, we propose the following two-step algorithm:

s1: Determine matrices $\mathbb{Q}_1$ and $\mathbb{X}_1$ from solving the following LMI:

$$A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1 + [A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1]^T < 0 \tag{3.53}$$

which corresponds to the first block of matrix inequality (3.48), i.e., matrix inequality

$$\Omega_1 < 0.$$

In this case, matrix K can be deduced by

$$K = \mathbb{Q}_1\mathbb{X}_1.$$

s2: By inserting the matrices K and $\mathbb{X}_1$ obtained from the first step into matrix inequality (3.48), then matrix inequality (3.48) becomes an LMI and the solution of this LMI permits to deduce matrix $\mathbb{Z}_1$ by

$$\mathbb{Z}_1 = \mathbb{X}_2^{-1}\mathbb{X}_3 \tag{3.54}$$

So far, we have investigated the H_∞ dynamic-observer-based control design problem for uncertain systems in the presence of disturbances. The control design can be summarized as the following design procedure:

step1. Choose matrix Λ according to condition (3.21);

step2. Compute matrices Φ, $\overline{K}$ and Θ from equations (3.26), (3.27) and (3.34);

step3. Compute matrices $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$ and β_3 according to equations (3.30) and (3.34);

step4. Compute matrices $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{C}$ and $\mathbb{D}$ according to equations (3.17), (3.36), (3.37), (3.38) and (3.39);

step5. Determine matrices K and $\mathbb{X}_1$ by solving LMI (3.53);

step6. By inserting matrices K and $\mathbb{X}_1$ into matrix inequality (3.48) and solving the deduced LMI, compute matrix $\mathbb{Z}_1$;

step7. Deduce all the parameter matrices of observer (3.4) according to equations (3.29), (3.33), (3.35) and (3.11b).

3.5 Numerical example

In the previous sections, we have solved the observer-based control design problem for uncertain system in the presence of disturbances. In this section, a numerical example is presented to illustrate the design procedure and the performances of the proposed control. Let us consider the following system of the form (3.1):

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, E = 0.1,$$

$$\mathcal{M} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathcal{N} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix},$$

$$\mathcal{F}(t) = 0.5 \sin 0.04\pi t,$$

The initial conditions are $x_1 = 1, x_2 = 1$ and $x_3 = 0.5$.

It is easy to see that the above system is unstable, since the first eigenvalue is 1, which is on the right side of the complex plane. The control aim is to stabilize the system, i.e., place all eigenvalues on the left complex plane, by using the control (3.5) based on the proposed HDO (3.4).

3.5.1 Observer-based control design

For the above system, we design an H_∞ dynamic-observer-based control by using the proposed method. According to the design procedure presented in section 3.4, we first choose matrix

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

according to rank condition (3.20).

Then, by following the design procedure, we obtain

$$\begin{aligned}
K &= \begin{bmatrix} -1.3263 & 0.5386 & -0.3288 \end{bmatrix}, \\
\mathbb{Z}_1 &= \begin{bmatrix} -23.4341 & 2.3202 & 0.1615 & 48.7380 & 0.3305 \\ -57.2720 & 5.9519 & 0.4177 & 119.3283 & 0.8779 \\ -11.4150 & 1.2071 & 0.0852 & 23.7986 & 0.1861 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.5000 \end{bmatrix}, \\
\gamma &= 0.9.
\end{aligned}$$

Consequently, the obtained HDO is given by:

$$\begin{aligned}
\dot{z} &= \begin{bmatrix} -26.5820 & 4 & -10 \\ -59.9857 & 2 & -6 \\ -11.8024 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 47.1640 \\ 117.9714 \\ 23.6048 \end{bmatrix} y \\
&\quad + \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.3305 \\ 0.8779 \\ 0.1861 \end{bmatrix} v, \\
\dot{v} &= \begin{bmatrix} 0.2014 & 0.001 & -0.001 \end{bmatrix} z - \begin{bmatrix} 0.4027 \end{bmatrix} y - \begin{bmatrix} 0.5 \end{bmatrix} v, \\
\hat{x} &= \begin{bmatrix} 0.4 & 0 & 0 \\ -0.8 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 0.2 \\ -0.4 \\ 0 \end{bmatrix} y.
\end{aligned}$$

The obtained observer is full-order observer. One can also design a reduced-order observer by choosing $q = 2$.
The obtained control is given by

$$u = K\hat{x} = \begin{bmatrix} -1.3263 & 0.5386 & -0.3288 \end{bmatrix} \hat{x}.$$

Based on the determined HDO and control, the eigenvalues of the closed-loop system are:

$$\begin{bmatrix}-0.5070 + 2.7492i, & -0.5070 - 2.7492i, & -1.5639\end{bmatrix}.$$

One can see from the results that all the eigenvalues are on the left side of the complex plane. In other words, the controlled system becomes stable by using the proposed H_∞ dynamic-observer-based control.

3.5.2 Simulation results

In order to show the performances of the proposed control, we have done some simulations with the determined observer and control. Furthermore, we consider another uncertainty $\Delta\overline{A}$ in the state matrix $(A + \Delta A + \Delta\overline{A})$ where

$$\Delta\overline{A} = \Delta \sin 2\pi t$$

and

$$\Delta = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0 & 0.2 & 0.1 \end{bmatrix}.$$

The simulation results are shown in the following figures.
Figure 3.1 represents the disturbance of finite energy used in the simulation.
Figures 3.2, 3.4 and 3.6 represent the original system states and their estimates obtained from the proposed HDO. The solid line represents the original system state. The dotted line represents the state estimate.
Figures 3.3, 3.5 and 3.7 represent the estimation errors of each system state obtained from the proposed HDO.
Figure 3.8 represents the output of system.

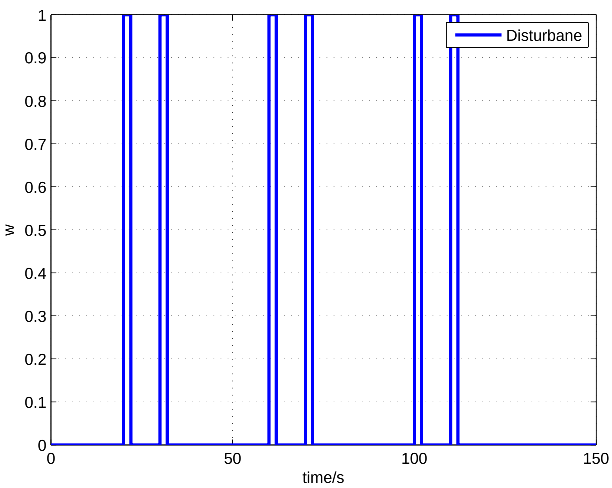

Figure 3.1: Disturbance

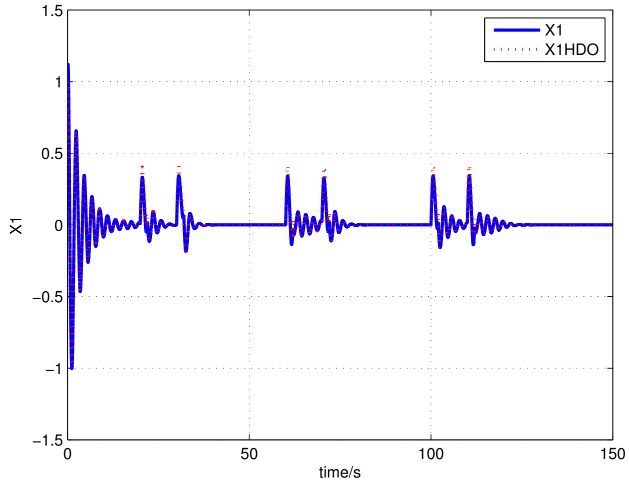

Figure 3.2: State x_1 and its estimate (solid line: original state; dotted line: HDO)

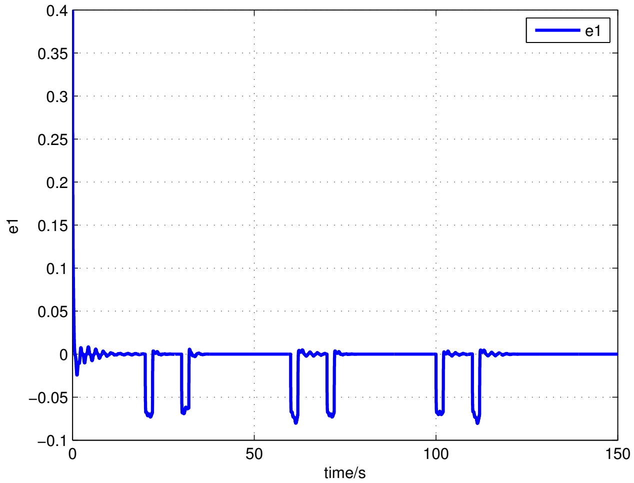

Figure 3.3: Estimation error e_1

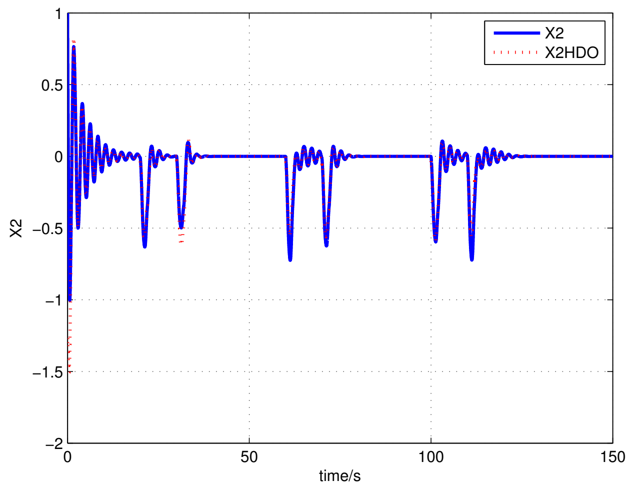

Figure 3.4: State x_2 and its estimate (solid line: original state; dotted line: HDO)

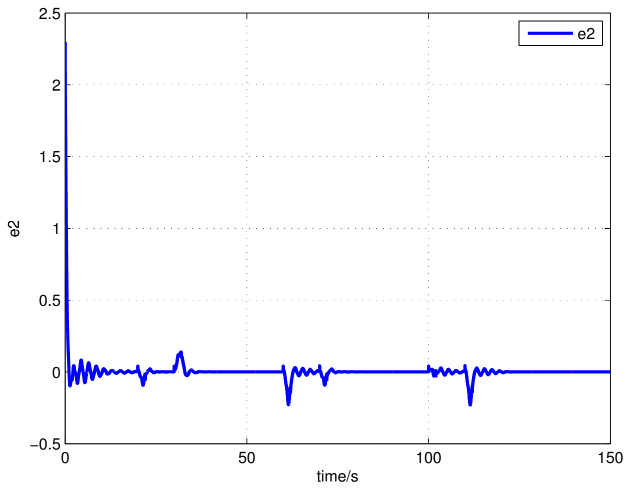

Figure 3.5: Estimation error e_2

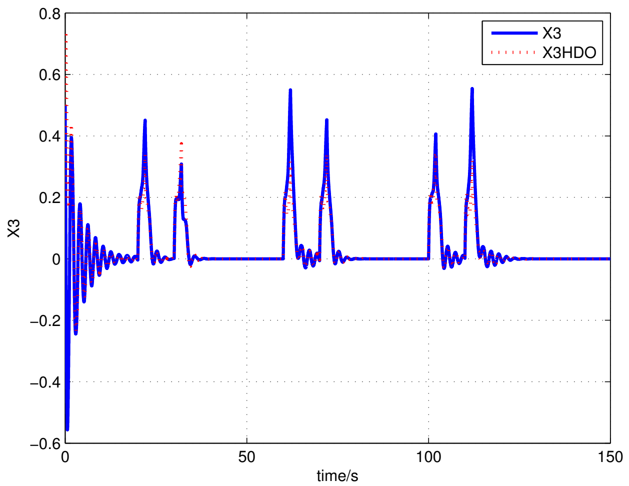

Figure 3.6: State x_3 and its estimate (solid line: original state; dotted line: HDO)

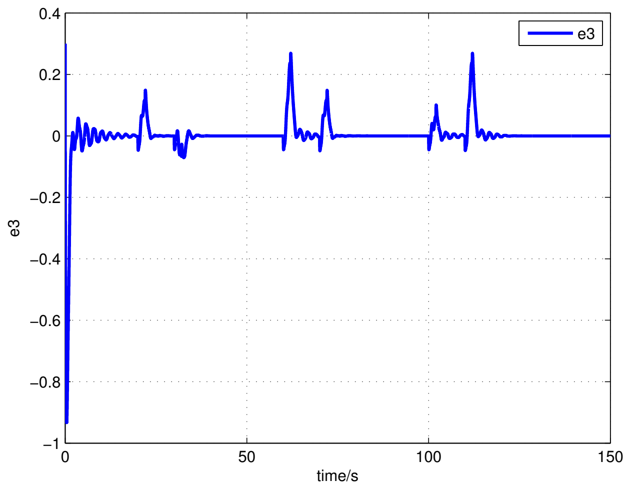

Figure 3.7: Estimation error e_3

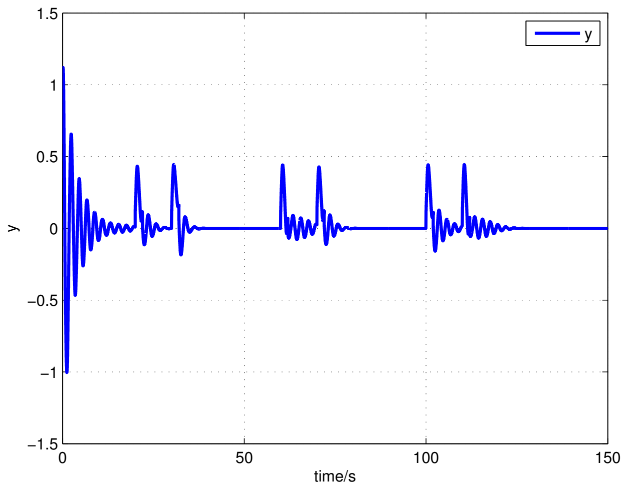

Figure 3.8: Output y

From the above figures one can see the performances of the proposed H_∞ dynamic-observer-based control, in the presence of an additional uncertainty.
Take Figure 3.8 for example. The control stabilizes the unstable system. After a period of oscillation, the controlled system becomes asymptotically stable. In the case when the disturbance appears, the control can guarantee the asymptotic stability of system and minimize the effect of disturbance on the controlled output.
Furthermore, we can see from Figures 3.2 and 3.3 that the influence of disturbances on the estimate is minimized. Besides, both the proposed control and observer are insensitive to the uncertainty (see Figures 3.8 and 3.3 from time 45s to 55s for example).
Generally, we can see that the proposed H_∞ dynamic-observer-based control can guarantee the asymptotic stability of the closed-loop system and handle the uncertainty and disturbance very well.

3.6 Conclusions

This chapter focused on the H_∞ dynamic-observer-based control design problem for uncertain systems in the presence of disturbances, based on the HDO proposed in Chapter 2. The uncertainty was time varying and assumed to be norm bounded.
Firstly, through the analysis of the estimation error, the algebraic constraints were obtained. Then the control design problem was reduced to the determination of two parameter matrices, based on the parameterizations of the algebraic constraints. Thereafter, the determination problem was formulated as a BMI optimization problem and a two-step algorithm was proposed to solve the problem.
A numerical example that contained an unstable system was presented to illustrate the design procedure of the H_∞ dynamic-observer-based control. One can see that the proposed control can guarantee the asymptotic stability of closed-loop system and was insensitive to uncertainties and disturbances.

Chapter 4

H_∞ decentralized dynamic-observer-based control for large-scale uncertain systems

Contents

4.1 Introduction

In the past several years, the system of large-scale has gained considerable attention. In order to solve the problem of LSS, the decentralized observer-based control has been developed. For example, the observer-based decentralized control scheme for stability analysis of networked systems was proposed in [38]. In [71], the observer-based decentralized control was designed to enhance the power system dynamic stability. The authors of [74] proposed a reduced-order observer-based control design, by using LMI approach. The usage of decentralized observer was extended to continuous moving web in [95].

In recent years, the decentralized control has been developed for large-scale uncertain systems. In [88], the observer-based decentralized control was designed for large uncertain multivariable plants. In [89], the authors solved the problem of decentralized stabilization and output tracking for large uncertain systems. On the other hand, in order to deal with the disturbance, the decentralized H_∞ filtering problem for linear systems was solved by using LMI technique in [133]. In [140], the authors proposed an observer-based decentralized H_∞ control for interconnected systems. Furthermore, the applications of decentralized control for uncertain systems with disturbances can be found in [8, 49, 118] and references therein.

In this chapter, by using the HDO proposed in Chapter 2, we propose an H_∞ decentralized dynamic-observer-based control for large-scale uncertain systems in the presence of disturbances. These systems are composed of a set of interconnected subsystems of lower dimensions, where the interconnections are assumed to be nonlinear and satisfy quadratic constraints. First we do parameterizations to the algebraic constraints obtained from the analysis of the estimation error. Then based on the parameterization, Lyapunov function approach and Schur complement lemma, the control design problem is formulated as an BMI optimization problem and a two-step algorithm is proposed to solve the problem.

The rest of this chapter is organized as follows. In section 4.2, the H_∞ decentralized dynamic-observer-based control design problem for large-scale uncertain systems in the presence of disturbances is presented. The algebraic constraints are derived through the analysis of the estimation error and parameterizations of these constraints are presented in section 4.3. In section 4.4, the design problem is reduced to a BMI optimization problem and a two-step algorithm is presented to solve the problem. A numerical example consisting of two unstable subsystems is presented to illustrate the design procedure and the performance of the proposed control in section 4.5. Finally, some conclusions are given in section 4.6.

4.2 Problem formulation

Let us consider the following large-scale uncertain system in the presence of disturbances, which is composed of N subsystems and the i-th subsystem is described by the following model:

$$\begin{aligned} \dot{x}_i &= (A_i + \Delta A_i)x_i + B_i u_i + D_i w_i + h_i(t,x), && (4.1a)\\ y_i &= C_i x_i + E_i w_i, \qquad\qquad i = 1,\dots,N, && (4.1b) \end{aligned}$$

where $x_i \in \Re^{n_i}$, $u_i \in \Re^{m_i}$, $w_i \in \Re^{f_i}$ and $y_i \in \Re^{p_i}$ are the system state, the control input, the exogenous disturbance of finite energy and the output, respectively. Matrices A_i, B_i, C_i, D_i and E_i are known constant and of appropriate dimensions.

$h_i(t,x)$ represent the interconnections of $N-1$ subsystems with the i-th subsystem. We take the representations of interconnections presented in [139], which are assumed to satisfy the following quadratic constraint:

$$h_i^T(t,x)h_i(t,x) \leq a_i^2 x^T H_i^T H_i x, \tag{4.2}$$

where scalars a_i are bounding parameters and matrices $H_i \in \Re^{l_i \times n}$ are constant bounding matrices.
ΔA_i are unknown matrices representing the time-varying parameter uncertainties, assumed to be norm bounded and has the following structure:

$$\Delta A_i = \mathcal{M}_i \mathcal{F}_i(t) \mathcal{N}_i, \tag{4.3}$$

where $\mathcal{M}_i$ and $\mathcal{N}_i$ are known real constant matrices and of appropriate dimensions. $\mathcal{F}_i(t)$ are unknown matrix functions satisfying

$$\mathcal{F}_i^T(t)\mathcal{F}_i(t) \leq I. \tag{4.4}$$

Next let us consider the proposed HDO for each subsystem, which is presented by:

$$\dot{z}_i = F_i z_i + J_i y_i + T_i u_i + M_i v_i, \tag{4.5a}$$
$$\dot{v}_i = P_i z_i + Q_i y_i + G_i v_i, \tag{4.5b}$$
$$\hat{x}_i = R_i z_i + S_i y_i, \tag{4.5c}$$

where $z_i \in \Re^{q_i}$ is the state of observer, $v_i \in \Re^{t_i}$ is an auxiliary vector, $\hat{x}_i \in \Re^{n_i}$ is the state estimate. Matrices F_i, J_i, T_i, M_i, P_i, Q_i, G_i, R_i and S_i are unknown and of appropriate dimensions to be determined.

The observer-based control of each subsystem is expressed as

$$u_i = K_i \hat{x}_i, \tag{4.6}$$

where K_i are unknown matrices and of appropriate dimensions to be determined.

From the representation of subsystems (4.1), the global system can be described by the following state representation:

$$\dot{x} = (A + \Delta A)x + Bu + Dw + h(t,x), \tag{4.7a}$$
$$y = Cx + Ew, \tag{4.7b}$$

where vectors

$x \in \Re^n (n = \Sigma_{i=1}^N n_i, x = [x_1^T, \ldots, x_N^T]^T)$,
$u \in \Re^m (m = \Sigma_{i=1}^N m_i, u = [u_1^T, \ldots, u_N^T]^T)$,
$w \in \Re^f (f = \Sigma_{i=1}^N f_i, w = [w_1^T, \ldots, w_N^T]^T)$,
$y \in \Re^p (p = \Sigma_{i=1}^N p_i, y = [y_1^T, \ldots, y_N^T]^T)$,
$h(t,x) = [h_1^T(t,x), \ldots, h_N^T(t,x)]^T$.
Matrices $A = diag(A_i)$, $B = diag(B_i)$, $C = diag(C_i)$, $D = diag(D_i)$ and $E = diag(E_i)$.

$h(t, x)$ represents the interconnection of the global system, which is assumed to meet the following quadratic constraint:

$$\begin{aligned} h^T(t,x)h(t,x) &\leq \alpha_1^2 x^T H_1^T H_1 x + \cdots + \alpha_N^2 x^T H_N^T H_N x, \\ &\leq x^T \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix}^T \begin{bmatrix} \alpha_1^2 & \dots & 0 \\ \star & \ddots & 0 \\ \star & \star & \alpha_N^2 \end{bmatrix} \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix} x, \\ &\leq x^T H^T \Psi^{-1} H x, \end{aligned} \tag{4.8}$$

where matrices $H = [H_1^T, \dots, H_N^T]^T$ is a $l \times n$ matrix ($l = \Sigma_{i=1}^N l_i$), $\Psi = diag(\Psi_i)$ with $\Psi_i = \lambda_i I$ and $\lambda_i = 1/a_i^2$.

ΔA represents the time-varying parameter uncertainty of the global system, defined in the following form:

$$\Delta A = \mathcal{M}\mathcal{F}(t)\mathcal{N}, \tag{4.9}$$

where $\Delta A = diag(\Delta A_i)$, matrices $\mathcal{M} = diag(\mathcal{M}_i)$ and $\mathcal{N} = diag(\mathcal{N}_i)$. $\mathcal{F}(t) = diag(\mathcal{F}_i(t))$ satisfies

$$\mathcal{F}^T(t)\mathcal{F}(t) \leq I. \tag{4.10}$$

Furthermore, the global observer composed of N local HDOs (4.5) is given by

$$\dot{z} = Fz + Jy + Tu + Mv, \tag{4.11a}$$
$$\dot{v} = Pz + Qy + Gv, \tag{4.11b}$$
$$\hat{x} = Rz + Sy, \tag{4.11c}$$

where vectors

$z \in \Re^q (q = \Sigma_{i=1}^N q_i, z = [z_1^T, \dots, z_N^T]^T)$,
$v \in \Re^t (t = \Sigma_{i=1}^N t_i, v = [v_1^T, \dots, v_N^T]^T)$,
$\hat{x} \in \Re^n, \hat{x} = [\hat{x}_1^T, \dots, \hat{x}_N^T]^T$.

Matrices $F = diag(F_i)$, $J = diag(J_i)$, $T = diag(T_i)$, $M = diag(M_i)$, $P = diag(P_i)$, $Q = diag(Q_i)$, $G = diag(G_i)$, $R = diag(R_i)$ and $S = diag(S_i)$ are unknown and of appropriate dimensions to be determined.

The observer-based control law of global system (4.7) is expressed as

$$u = K\hat{x}, \tag{4.12}$$

where $K = diag(K_i)(K_i \in \Re^{m_i \times n_i})$ is unknown matrix to be determined.

Consequently, with all the above presentations, the H_∞ decentralized dynamic-observer-based control for large-scale uncertain systems in the presence of disturbances (4.7) is reduced to determine all the parameter matrices of observer (4.11) and control (4.12) such that:

1. For $w = 0$, the estimation error $e(t) \to 0$ when $t \to \infty$ and the closed-loop system obtained from system(4.7), observer (4.11) and control (4.12) is stable;

2. For $w \neq 0$, minimize the effect of w on y, i.e., minimize the H_∞ norm $\|T_{wy}\|_\infty$.

The following lemma will be used in the control design procedure:

Lemma 4.2.1. *[35] For any matrices (or vectors) $\mathfrak{X}$ and $\mathfrak{Y}$ of appropriate dimensions, the following inequality holds*

$$\mathfrak{X}^T\mathfrak{Y} + \mathfrak{Y}^T\mathfrak{X} \leq \mathfrak{X}^T\Pi\mathfrak{X} + \mathfrak{Y}^T\Pi^{-1}\mathfrak{Y} \tag{4.13}$$

where Π is any positive definite symmetric matrix.

4.3 Algebraic constraints and parameterizations

In this section, similar to the previous chapters, we shall present some algebraic constraints and parameterization results.

4.3.1 Algebraic constraints

Firstly, we define an error variable:

$$\varepsilon = z - \Phi x, \tag{4.14}$$

where $\varepsilon \in \mathfrak{R}^q(\varepsilon = [\varepsilon_1^T, \ldots, \varepsilon_N^T]^T)$ and $\Phi \in \mathfrak{R}^{q\times n}(\Phi = diag(\Phi_i), \Phi_i \in \mathfrak{R}^{q_i\times n_i})$ is an arbitrary matrix.
Then the derivative of error ε is given by

$$\begin{aligned}
\dot{\varepsilon} &= \dot{z} - \Phi\dot{x}, \\
&= Fz + JCx + JEw + Tu + Mv, \\
&\quad -\Phi(A + \Delta A)x - \Phi Bu - \Phi Dw - \Phi h(t,x) \\
&= F(z - \Phi x) + F\Phi x + JCx + JEw + Tu + Mv, \\
&\quad -\Phi Ax - \Phi\Delta Ax - \Phi Bu - \Phi Dw - \Phi h(t,x) \\
&= F\varepsilon + Mv + (JE - \Phi D)w - \Phi h(t,x) - \Phi\Delta Ax \\
&\quad +(F\Phi + JC - \Phi A)x + (T - \Phi B)u.
\end{aligned} \tag{4.15}$$

From observer (4.11) it follows

$$\begin{aligned}
\dot{v} &= Pz + Qy + Gv, \\
&= P\varepsilon + Gv + QEw + (P\Phi + QC)x,
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\hat{x} &= Rz + Sy,\\ &= R\varepsilon + SEw + (R\Phi + SC)x. \end{aligned} \tag{4.17}$$

On the other hand, by inserting control (4.12) into system (4.7), the system equation (4.7a) becomes

$$\begin{aligned}\dot{x} &= (A + \Delta A)x + BK\hat{x} + Dw + h(t,x),\\ &= Ax + BK[R(z - \Phi x) + R\Phi x + SCx + SEw]\\ &\quad + \Delta Ax + Dw + h(t,x),\\ &= [A + BK(R\Phi + SC)]x + BKR\varepsilon + (D + BKSE)w\\ &\quad + \Delta Ax + h(t,x).\end{aligned} \tag{4.18}$$

Notice that if the following conditions hold

$$F\Phi + JC - \Phi A = 0, \tag{4.19a}$$
$$T - \Phi B = 0, \tag{4.19b}$$
$$P\Phi + QC = 0, \tag{4.19c}$$
$$R\Phi + SC = I_n, \tag{4.19d}$$

we obtain the following results on the derivative of state x, error ε and auxiliary vector v:

$$\dot{x} = (A + BK)x + BKR\varepsilon + (D + BKSE)w + h(t,x) + \Delta Ax, \tag{4.20}$$

$$\dot{\varepsilon} = F\varepsilon + Mv + (JE - \Phi D)w - \Phi h(t,x) - \Phi\Delta Ax, \tag{4.21}$$

$$\dot{v} = P\varepsilon + Gv + QEw, \tag{4.22}$$

Furthermore, the expression of the estimation error e is given by

$$e = R\varepsilon + SEw. \tag{4.23}$$

Consequently, we obtain the following system:

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{\varepsilon}\\ \dot{v}\end{bmatrix} &= \begin{bmatrix}A + BK & BKR & 0\\ 0 & F & M\\ 0 & P & G\end{bmatrix}\begin{bmatrix}x\\ \varepsilon\\ v\end{bmatrix} + \begin{bmatrix}D + BKSE\\ JE - \Phi D\\ QE\end{bmatrix} w\\ &\quad + \begin{bmatrix}I_n\\ -\Phi\\ 0\end{bmatrix} h(t,x) + \begin{bmatrix}I_n\\ -\Phi\\ 0\end{bmatrix}\Delta Ax,\end{aligned} \tag{4.24a}$$

$$y = \begin{bmatrix}C & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ \varepsilon\\ v\end{bmatrix} + Ew, \tag{4.24b}$$

or equivalently,

$$\dot{\xi} = \mathbb{A}\xi + \mathbb{B}w + h(t,\xi) + \Delta\mathbb{A}\xi, \tag{4.25a}$$
$$y = \mathbb{C}\xi + \mathbb{D}w, \tag{4.25b}$$

where

$$\begin{cases} \xi = \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix} = \xi(t), \\ \mathbb{A} = \begin{bmatrix} A + BK & BKR & 0 \\ 0 & F & M \\ 0 & P & G \end{bmatrix}, \\ \mathbb{B} = \begin{bmatrix} D + BKSE \\ JE - \Phi D \\ QE \end{bmatrix}, \\ h(t, \xi) = \Gamma h(t, x), \Gamma = \begin{bmatrix} I_n \\ -\Phi \\ 0 \end{bmatrix}, \\ \mathbb{C} = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \text{ and } \mathbb{D} = E. \end{cases} \tag{4.26}$$

$h(t, \xi)$ is assumed to satisfy the following constraint inequality:

$$\begin{aligned} h^T(t, \xi)h(t, \xi) &\leq x^T H^T \Gamma^T \Psi^{-1} \Gamma H x, \\ &\leq \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix}^T \begin{bmatrix} H^T \Gamma^T \Psi^{-1} \Gamma H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix}, \\ &\leq \xi^T \mathbb{H}^T \Psi^{-1} \mathbb{H} \xi, \end{aligned} \tag{4.27}$$

with matrix $\mathbb{H} = \begin{bmatrix} \Gamma H & 0 & 0 \end{bmatrix}$.

$\mathit{\Delta}\mathbb{A}$ is similar to the uncertainty ΔA in system (4.7). It is used to represent the uncertainty which is assumed to satisfy the following equation:

$$\mathit{\Delta}\mathbb{A} = \mathbb{M}\mathcal{F}(t)\mathbb{N}, \tag{4.28}$$

where matrices $\mathbb{M} = \begin{bmatrix} \mathcal{M} \\ -\Phi\mathcal{M} \\ 0 \end{bmatrix}$ and $\mathbb{N} = \begin{bmatrix} \mathcal{N} & 0 & 0 \end{bmatrix}$.

Consequently, one can see from the above developments that the H_∞ decentralized dynamic-observer-based control design problem is reduced to study the augmented uncertain system (4.25), in other words, to determine all the parameter matrices of observer (4.11) and control (4.12) such that:

p1. Constraint conditions (4.19) are satisfied;

p2. For $w = 0$, the augmented state $\xi(t) \to 0$ when $t \to \infty$;

p3. For $w \neq 0$, the H_∞ norm $\|T_{wy}\|_\infty < \gamma$.

4.3.2 Parameterizations

Subsequently, we shall do some parameterizations to the constraints condition (4.19). We only present the results, since the procedure is similar to the previous chapters. From equations (4.19c) and (4.19d) we obtain the following equation:

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \Phi \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{4.29}$$

The necessary and sufficient condition for equation (4.29) to have a solution is that the following rank condition must be satisfied:

$$rank \begin{pmatrix} \Phi \\ C \\ 0 \\ I_n \end{pmatrix} = rank \begin{pmatrix} \Phi \\ C \end{pmatrix} = n. \tag{4.30}$$

Assume that rank condition (4.30) holds, and let $\Lambda \in \Re^{q \times n}$ $(\Lambda = diag(\Lambda_i), \Lambda_i \in \Re^{q_i \times n_i})$ be an arbitrary full row rank matrix such that:

$$rank \begin{pmatrix} \Lambda \\ C \end{pmatrix} = rank \begin{pmatrix} \Phi \\ C \end{pmatrix} = n. \tag{4.31}$$

Then there always exist two parameter matrices $\overline{K} = diag(\overline{K}_i)$ $(\overline{K}_i \in \Re^{q_i \times p_i})$ and Φ such that

$$\begin{pmatrix} \Phi \\ C \end{pmatrix} = \begin{pmatrix} I_q & -\overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix}, \tag{4.32}$$

which leads to

$$\Phi = \Lambda - \overline{K}C, \tag{4.33}$$

or equivalently,

$$\begin{pmatrix} \Phi & \overline{K} \end{pmatrix} \begin{pmatrix} I_n \\ C \end{pmatrix} = \Lambda. \tag{4.34}$$

One solution of equation (4.34) is given by

$$\Phi = \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+ \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \tag{4.35}$$

$$\overline{K} = \Lambda \begin{pmatrix} I_n \\ C \end{pmatrix}^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}. \tag{4.36}$$

By inserting equation (4.32) into equation (4.29), we have

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I_q & -\overline{K} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \Lambda \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{4.37}$$

In this case, matrices P, Q, R and S are given by

$$\begin{aligned} P &= -Z_2\beta_1, && (4.38a)\\ Q &= -Z_2\beta_2, && (4.38b)\\ R &= \alpha_1 - Z_3\beta_1, && (4.38c)\\ S &= \alpha_2 - Z_3\beta_2, && (4.38d) \end{aligned}$$

where matrices

$$\begin{cases} Z_2 = \begin{pmatrix} I_t & 0 \end{pmatrix} Z, \\ Z_3 = \begin{pmatrix} 0 & I_n \end{pmatrix} Z, \\ \Upsilon = \begin{pmatrix} \Lambda \\ C \end{pmatrix}, \\ \alpha_1 = \Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_2 = \Upsilon^+ \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix}, \\ \beta_1 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \beta_2 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} \overline{K} \\ I_p \end{pmatrix}. \end{cases} \qquad (4.39)$$

and Z is an arbitrary matrix of appropriate dimension.

Remark 4.3.1. *According to **Remark 2.2.7**, in order to avoid the the bilineaity, we take matrix $Z_3 = 0$ and obtain*

$$\begin{aligned} R &= \alpha_1, && (4.40)\\ S &= \alpha_2. && (4.41) \end{aligned}$$

■

On the other hand, matrices F and $\overline{K}_1$ are given by

$$\begin{aligned} F &= \alpha_3 - Z_1\beta_1, && (4.42a)\\ \overline{K}_1 &= \alpha_4 - Z_1\beta_3, && (4.42b) \end{aligned}$$

where Z_1 is an arbitrary matrix of appropriate dimension, and matrices

$$\begin{cases} \alpha_3 = \Theta\Upsilon^+ \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \\ \alpha_4 = \Theta\Upsilon^+ \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \beta_3 = (I_{q+p} - \Upsilon\Upsilon^+) \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \\ \Theta = \Phi A. \end{cases} \qquad (4.43)$$

Furthermore, matrix J is given by

$$J = \Theta\alpha_2 - Z_1\beta_2. \qquad (4.44)$$

Consequently, based on the above parameterizations, the matrices $\mathbb{A}$ and $\mathbb{B}$ of augmented system (4.25) can be written as:

$$\begin{aligned}
\mathbb{A} &= \begin{bmatrix} A+BK & BKR & 0 \\ 0 & F & M \\ 0 & P & G \end{bmatrix}, \\
&= \begin{bmatrix} A+BK & BKR & 0 \\ 0 & \alpha_3 - Z_1\beta_1 & M \\ 0 & -Z_2\beta_1 & G \end{bmatrix}, \\
&= \begin{bmatrix} A+BK & \begin{pmatrix} BKR & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{A}_1 - \mathbb{Z}_1\mathbb{A}_2 \end{bmatrix}, \qquad (4.45)
\end{aligned}$$

where matrices

$$\mathbb{A}_1 = \begin{bmatrix} \alpha_3 & 0 \\ 0 & 0 \end{bmatrix}, \mathbb{Z}_1 = \begin{bmatrix} Z_1 & M \\ Z_2 & G \end{bmatrix} \text{ and } \mathbb{A}_2 = \begin{bmatrix} \beta_1 & 0 \\ 0 & -I_t \end{bmatrix}, \qquad (4.46)$$

and

$$\begin{aligned}
\mathbb{B} &= \begin{bmatrix} D+BKSE \\ JE - \Phi D \\ QE \end{bmatrix}, \\
&= \begin{bmatrix} D+BKSE \\ \Theta\alpha_2 E - Z_1\beta_2 E - \Phi D \\ -Z_2\beta_2 E \end{bmatrix}, \\
&= \begin{bmatrix} D+BKSE \\ \mathbb{B}_1 - \mathbb{Z}_1\mathbb{B}_2 \end{bmatrix}, \qquad (4.47)
\end{aligned}$$

where matrices

$$\mathbb{B}_1 = \begin{bmatrix} \Theta\alpha_2 E - \Phi D \\ 0 \end{bmatrix} \text{ and } \mathbb{B}_2 = \begin{bmatrix} \beta_2 E \\ 0 \end{bmatrix}. \qquad (4.48)$$

With all these results, one can see that matrix Λ is needed to be choosen according to rank condition (4.31). In this case, once matrices K, Z_1, Z_2, M and G, i.e., matrices K and $\mathbb{Z}_1$ are determined, we can deduce all the parameter matrices of the observer (4.11) and control (4.12) according to equations (4.38), (4.39), (4.40), (4.41), (4.42) and (4.44). Thus the H_∞ decentralized dynamic-observer-based control design problem is reduced to the determination of parameter matrices K and $\mathbb{Z}_1$ such that the purposes p2 and p3 are satisfied.

4.4 H_∞ decentralized dynamic-observer-based control design

In previous sections, based on the parameterization of algebraic constraints, the control design problem is reduced to the determination of two parameter matrices K and $\mathbb{Z}_1$. In this section, we will solve the determination problem. Firstly, we give the following theorem:

Theorem 4.4.1. *With $\Psi = diag(\Psi_i)$ ($\Psi_i = \lambda_i I$ and $\lambda_i = 1/a_i^2$), system* (4.25) *is asymptotically stable for $w = 0$ and $\|T_{wy}\|_\infty < \gamma$ for $w \neq 0$, if there exist two symmetric positive definite matrices $\mathbb{X}_1 > 0$, $\mathbb{X}_2 > 0$, and a given positive scalar γ such that the following matrix inequality holds*

$$\Omega = \begin{bmatrix} \mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} + \mathbb{H}^T\Psi^{-1}\mathbb{H} + \mathbb{X}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N} + \mathbb{C}^T\mathbb{C} & \mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D} \\ (\mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D})^T & \mathbb{D}^T\mathbb{D} - \gamma^2 I_f \end{bmatrix} < 0. \quad (4.49)$$

Proof. *For system* (4.25)*, we choose the following Lyapunov function:*

$$V(\xi) = \xi^T\mathbb{X}\xi, \quad (4.50)$$

where matrix $\mathbb{X} = diag(\mathbb{X}_1, \mathbb{X}_2)$ and $\mathbb{X}_j > 0 (j = 1, 2)$.

Then the derivative of Lyapunov function $V(\xi)$ along the solution of system (4.25) *is given by*

$$\begin{aligned} \dot{V}(\xi) &= \dot{\xi}^T\mathbb{X}\xi + \xi^T\mathbb{X}\dot{\xi}, \\ &= [\mathbb{A}\xi + \mathbb{B}w + h(t,\xi) + \varDelta\mathbb{A}\xi]^T\mathbb{X}\xi \\ &\quad + \xi^T\mathbb{X}[\mathbb{A}\xi + \mathbb{B}w + h(t,\xi) + \varDelta\mathbb{A}\xi], \\ &= \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi \\ &\quad + \xi^T\mathbb{X}h(t,\xi) + h^T(t,\xi)\mathbb{X}\xi + 2\xi^T\mathbb{X}\varDelta\mathbb{A}\xi. \end{aligned} \quad (4.51)$$

*According to **Lemma 2.4.1** and the expression of uncertainty $\varDelta\mathbb{A}$* (4.28)*, by choosing $\rho = 1$, we obtain the following inequality:*

$$\begin{aligned} \dot{V}(\xi) &= \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi \\ &\quad + \xi^T\mathbb{X}h(t,\xi) + h^T(t,\xi)\mathbb{X}\xi + 2\xi^T\mathbb{X}\mathbb{M}\mathcal{F}(t)\mathbb{N}\xi, \\ &\leq \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi \\ &\quad + \xi^T\mathbb{X}h(t,\xi) + h^T(t,\xi)\mathbb{X}\xi + \xi^T[\mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N}]\xi. \end{aligned} \quad (4.52)$$

*According to **Lemma 4.2.1** and the constraint of interconnection $h(t,\xi)$* (4.27)*, by choosing $\Pi = I$, we obtain the following inequality:*

$$\begin{aligned} \dot{V}(\xi) &\leq \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi \\ &\quad + h^T(t,\xi)h(t,\xi) + \xi^T\mathbb{X}^T\mathbb{X}\xi + \xi[\mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N}]\xi, \\ &\leq \xi^T(\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X})\xi + \xi^T\mathbb{X}\mathbb{B}w + w^T\mathbb{B}^T\mathbb{X}\xi \\ &\quad + \xi^T[\mathbb{H}^T\Psi^{-1}\mathbb{H} + \mathbb{X}^T\mathbb{X} + \mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T + \mathbb{N}^T\mathbb{N}]\xi. \end{aligned} \quad (4.53)$$

From inequality (4.53) *and system equation* (4.25b)*, we have*

$$\begin{aligned}\dot{V}(\xi)+y^Ty-\gamma^2w^Tw &= \dot{V}(\xi)+(\mathbb{C}\xi+\mathbb{D}w)^T(\mathbb{C}\xi+\mathbb{D}w)-\gamma^2w^Tw,\\ &\leq \xi^T(\mathbb{X}\mathbb{A}+\mathbb{A}^T\mathbb{X})\xi+\xi^T\mathbb{X}\mathbb{B}w+w^T\mathbb{B}^T\mathbb{X}\xi\\ &\quad+\xi^T[\mathbb{H}^T\Psi^{-1}\mathbb{H}+\mathbb{X}^T\mathbb{X}+\mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T+\mathbb{N}^T\mathbb{N}]\xi\\ &\quad+\xi^T\mathbb{C}^T\mathbb{C}\xi+\xi^T\mathbb{C}^T\mathbb{D}w+w^T\mathbb{D}^T\mathbb{C}\xi+w^T\mathbb{D}^T\mathbb{D}w-\gamma^2w^Tw,\\ &\leq \xi^T[\mathbb{X}\mathbb{A}+\mathbb{A}^T\mathbb{X}+\mathbb{H}^T\Psi^{-1}\mathbb{H}+\mathbb{X}^T\mathbb{X}+\mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T\\ &\quad+\mathbb{N}^T\mathbb{N}+\mathbb{C}^T\mathbb{C}]\xi+\xi^T(\mathbb{X}\mathbb{B}+\mathbb{C}^T\mathbb{D})w\\ &\quad+[\xi^T(\mathbb{X}\mathbb{B}+\mathbb{C}^T\mathbb{D})w]^T+w^T(\mathbb{D}^T\mathbb{D}-\gamma^2I_f)w,\end{aligned} \tag{4.54}$$

or equivalently

$$\dot{V}(\xi)+y^Ty-\gamma^2w^Tw \leq \begin{bmatrix}\xi^T & w^T\end{bmatrix}\Omega\begin{bmatrix}\xi^T & w^T\end{bmatrix}^T, \tag{4.55}$$

where

$$\Omega=\begin{bmatrix}\mathbb{X}\mathbb{A}+\mathbb{A}^T\mathbb{X}+\mathbb{H}^T\Psi^{-1}\mathbb{H}+\mathbb{X}^T\mathbb{X}+\mathbb{X}\mathbb{M}(\mathbb{X}\mathbb{M})^T+\mathbb{N}^T\mathbb{N}+\mathbb{C}^T\mathbb{C} & \mathbb{X}\mathbb{B}+\mathbb{C}^T\mathbb{D}\\ (\mathbb{X}\mathbb{B}+\mathbb{C}^T\mathbb{D})^T & \mathbb{D}^T\mathbb{D}-\gamma^2I_f\end{bmatrix}. \tag{4.56}$$

Notice that if $\Omega<0$, we have

$$\dot{V}(\xi)+y^Ty-\gamma^2w^Tw<0, \tag{4.57}$$

which is equivalent to

$$\|T_{wy}\|_\infty<\gamma. \tag{4.58}$$

One can see from the above results that for system (4.25)*, the H_∞ norm $\|T_{wy}\|_\infty<\gamma$ in the case where $w\neq 0$, if inequality $\Omega<0$ holds.*

On the other hand, in the case where $w=0$, if $\Omega<0$ we have the following results:

$$\begin{aligned}&\dot{V}(\xi)+y^Ty<0,\\ \Leftrightarrow\quad &\dot{V}(\xi)<-y^Ty\leq 0,\end{aligned} \tag{4.59}$$

since it is always satisfied that $y^Ty\geq 0$. In this case, one can see that system (4.25) *is stable.*

With all the results, we can see that system (4.25) *is stable for $w=0$ and $\|T_{wy}\|_\infty<\gamma$ for $w\neq 0$, if matrix inequality* (4.49) *is satisfied. This completes the proof.* □

Subsequently, from **Theorem 4.4.2**, we obtain the following theorem to determine the parameter matrices K and $\mathbb{Z}_1$:

Theorem 4.4.2. *With $\Psi=diag(\Psi_i)$ ($\Psi_i=\lambda_iI$ and $\lambda_i=1/a_i^2$), system* (4.25) *is asymptotically stable for $w=0$ and $\|T_{wy}\|_\infty<\gamma$ for $w\neq 0$, if there exist two symmetric positive definite matrices $\mathbb{X}_1=diag(\mathbb{X}_{11},\mathbb{X}_{12})$ and $\mathbb{X}_2=diag(\mathbb{X}_{21},\mathbb{X}_{22})$, matrices $\mathbb{Q}_1=diag(\mathbb{Q}_{11},\mathbb{Q}_{12})$, $\mathbb{Q}_2=diag(\mathbb{Q}_{21},\mathbb{Q}_{22})$ and K of appropriate dimensions, and a given positive scalar γ such that the following matrix inequality holds*

$$
\begin{bmatrix}
\Omega_1 & \begin{pmatrix} BKR & 0 \end{pmatrix} & \Omega_3 & \mathbb{X}_1^{-1}\mathcal{N}^T & \mathcal{M} & \mathbb{X}_1^{-1}H^T\Gamma^T & I & 0 & \mathbb{X}_1^{-1}C^T \\
\star & \Omega_2 & \Omega_4 & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{X}_2\begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 & \mathbb{X}_2 & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\star & \star & E^TE-\gamma^2 I_f & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -\Psi & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & -I
\end{bmatrix} < 0 \tag{4.60}
$$

where

$$
\begin{aligned}
&\Omega_1 = A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1 + [A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1]^T, \mathbb{Q}_1 = K(\mathbb{X}_1^T)^{-1}, \\
&\Omega_2 = \mathbb{X}_2\mathbb{A}_1 - \mathbb{Q}_2\mathbb{A}_2 + (\mathbb{X}_2\mathbb{A}_1 - \mathbb{Q}_2\mathbb{A}_2)^T, \mathbb{Q}_2 = \mathbb{X}_2\mathbb{Z}_1, \\
&\Omega_3 = D + BKSE + \mathbb{X}_1^{-1}C^TE, \\
&\Omega_4 = \mathbb{X}_2\mathbb{B}_1 - \mathbb{Q}_2\mathbb{B}_2.
\end{aligned}
$$

In this case, matrices $K = \mathbb{Q}_1\mathbb{X}_1$ *and* $\mathbb{Z}_1 = \mathbb{X}_2^{-1}\mathbb{Q}_2$.

Proof. *Matrix inequality* (4.49) *can be rewritten as*

$$
\begin{bmatrix}
\mathbb{X}\mathbb{A} + \mathbb{A}^T\mathbb{X} & \mathbb{X}\mathbb{B} + \mathbb{C}^T\mathbb{D} & \mathbb{N}^T & \mathbb{X}\mathbb{M} & \mathbb{H}^T & \mathbb{X}^T & \mathbb{C}^T \\
\star & \mathbb{D}^T\mathbb{D} - \gamma^2 I_f & 0 & 0 & 0 & 0 & 0 \\
\star & \star & -I & 0 & 0 & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 \\
\star & \star & \star & \star & -\Psi & 0 & 0 \\
\star & \star & \star & \star & \star & -I & 0 \\
\star & \star & \star & \star & \star & \star & -I
\end{bmatrix} < 0, \tag{4.61}
$$

by using Schur complement lemma.
Subsequently, substituting the matrices of equations (4.45)*,* (4.46)*,* (4.47)*,* (4.48)*,* (4.38)*,* (4.39)*,* (4.42) *and* (4.43) *into matrix inequality* (4.61) *yields*

$$
\left\{
\begin{array}{c}
\begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \times \begin{bmatrix} A+BK & \begin{pmatrix} BKR & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{A}_1 - \mathbb{Z}_1\mathbb{A}_2 \end{bmatrix} + \begin{bmatrix} A+BK & \begin{pmatrix} BKR & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{A}_1 - \mathbb{Z}_1\mathbb{A}_2 \end{bmatrix}^T \times \begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star
\end{array}
\right.
$$

$$
\begin{array}{cc}
\begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \times \begin{pmatrix} D + BKSE \\ \mathbb{B}_1 - \mathbb{Z}_1\mathbb{B}_2 \end{pmatrix} + \begin{bmatrix} C^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} \times E & \begin{bmatrix} \mathcal{N}^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} \\
E^T E - \gamma^2 I_f & 0 \\
\star & -I \\
\star & \star \\
\star & \star \\
\star & \star \\
\star & \star
\end{array}
$$

$$
\left.
\begin{array}{cccc}
\begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} \times \begin{bmatrix} \mathcal{M} \\ \begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} \end{bmatrix} & \begin{bmatrix} \mathbb{H}^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} & \begin{pmatrix} \mathbb{X}_1 & 0 \\ 0 & \mathbb{X}_2 \end{pmatrix} & \begin{bmatrix} C^T \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-I & 0 & 0 & 0 \\
\star & -\Psi & 0 & 0 \\
\star & \star & -I & 0 \\
\star & \star & \star & -I
\end{array}
\right\} < 0, \quad (4.62)
$$

which is equivalent to

$$
\begin{bmatrix}
\Omega_1' & \begin{pmatrix} \mathbb{X}_1 BKR & 0 \end{pmatrix} & \Omega_3' & \mathcal{N}^T & \mathbb{X}_1\mathcal{M} & H^T\Gamma^T & \mathbb{X}_1 & 0 & C^T \\
\star & \Omega_2' & \Omega_4' & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{X}_2 \begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 & \mathbb{X}_2 & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\star & \star & E^T E - \gamma^2 I_f & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -\Psi & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -I & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -I & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & -I
\end{bmatrix} < 0, \tag{4.63}
$$

where

$$
\begin{aligned}
\Omega_1' &= \mathbb{X}_1 A + \mathbb{X}_1 BK + (\mathbb{X}_1 A + \mathbb{X}_1 BK)^T, \\
\Omega_2' &= \mathbb{X}_2\mathbb{A}_1 - \mathbb{X}_2\mathbb{Z}_1\mathbb{A}_2 + (\mathbb{X}_2\mathbb{A}_1 - \mathbb{X}_2\mathbb{Z}_1\mathbb{A}_2)^T, \\
\Omega_3' &= \mathbb{X}_1(D + BKSE) + C^T E, \\
\Omega_4' &= \mathbb{X}_2\mathbb{B}_1 - \mathbb{X}_2\mathbb{Z}_1\mathbb{B}_2.
\end{aligned}
$$

Then, by pre- and post-multiplying inequality (4.63) *with matices*

$$\begin{bmatrix} \mathbb{X}_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} (\mathbb{X}_1^T)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

we obtain

$$\begin{bmatrix} \Omega_1'' & \begin{pmatrix} BKR & 0 \end{pmatrix} & \Omega_3 & \mathbb{X}_1^{-1}\mathcal{N}^T & \mathcal{M} & \mathbb{X}_1^{-1}H^T\Gamma^T & I & 0 & \mathbb{X}_1^{-1}C^T \\ \star & \Omega_2' & \Omega_4' & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \mathbb{X}_2\begin{pmatrix} -\Phi\mathcal{M} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 & \mathbb{X}_2 & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \star & \star & E^TE-\gamma^2 I_f & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -I & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\Psi & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -I \end{bmatrix} < 0, \tag{4.64}$$

where

$$\Omega_1'' = A(\mathbb{X}_1^T)^{-1} + BK(\mathbb{X}_1^T)^{-1} + [A(\mathbb{X}_1^T)^{-1} + BK(\mathbb{X}_1^T)^{-1}]^T,$$
$$\Omega_3 = (D + BKSE) + \mathbb{X}_1^{-1}C^TE.$$

In this case, by defining matrices $\mathbb{Q}_1 = K(\mathbb{X}_1^T)^{-1}$ and $\mathbb{Q}_2 = \mathbb{X}_2\mathbb{Z}_1$, matrix inequality (4.64) *becomes matrix inequality* (4.60)*, which completes the proof.* □

Consequently, the H_∞ decentralized dynamic-observer-based control design problem of large-scale uncertain systems (4.7) is formulated as a matrix inequality optimization problem. Notice that matrix inequality (4.60) is a BMI and in order to solve the problem, we propose the following two-step algorithm:

s1: Determine matrices $\mathbb{Q}_1$ and $\mathbb{X}_1$ by solving the following LMI:

$$A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1 + [A(\mathbb{X}_1^T)^{-1} + B\mathbb{Q}_1]^T < 0, \tag{4.65}$$

which corresponds to the first block of inequality (4.60), i.e., matrix inequality $\Omega_1 < 0$. In this case, matrix K can be deduced by

$$K = \mathbb{Q}_1\mathbb{X}_1. \tag{4.66}$$

s2: By inserting the matrices K and $\mathbb{X}_1$ derived from the first step into inequality (4.60), the latter becomes an LMI and the solution of this LMI permits to deduce matrices $\mathbb{Z}_1$ by

$$\mathbb{Z}_1 = \mathbb{X}_2^{-1}\mathbb{Q}_2. \tag{4.67}$$

With all the above results, we have presented a method to solve the design problem of H_∞ decentralized dynamic-observer-based control for large-scale uncertain systems in the presence of disturbances. The above results can be summarized in the following design procedure:

step1: Choose matrix Λ according to rank condition (4.31);

step2: Compute matrices Φ, $\overline{K}$ and Θ according to equations (4.35), (4.36) and (4.43);

step3: Compute matrices α_1, α_2, α_3, α_4, β_1, β_1 and β_3 according to equations (4.39) and (4.43);

step4: Comput matrices R and S according to equation (4.40) and (4.41);

step5: Comput matrices $\mathbb{A}_1$, $\mathbb{A}_1$, $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{C}$ and $\mathbb{D}$ according to equations (4.46), (4.48), (4.26);

step6: Determine matrices K and $\mathbb{X}_1$ from solving the LMI (4.65);

step7: Insert the obtained values of matrices K and $\mathbb{X}_1$ into inequality (4.60), and solve the LMI (4.60) to determine parameters $\mathbb{Z}_1$, γ and a_i;

step8: Deduce all the parameter matrices of observer (4.11) according to equations (4.38), (4.42) and (4.44).

4.5 Numerical example

In the precedent sections, we have solved the H_∞ decentralized dynamic-observer-based control design problem for large-scale uncertain systems subject to disturbances. In this section, a numerical example is presented to illustrate the design procedure and the performances of the proposed control. Let us consider a large-scale uncertain system consisting of two subsystems which are described by

$$
\begin{aligned}
\text{Subsystem 1:} \quad & A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \\
& B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}, D_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\
& C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, E_1 = 0.1, \\
& \mathcal{M}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathcal{N}_1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}, \\
& F_1(t) = 0.5 \sin 0.04\pi t,
\end{aligned}
$$

$$
\begin{aligned}
\text{Subsystem 2:} \quad & A_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \\
& B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
& C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, E_2 = 0.2, \\
& \mathcal{M}_2 = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \mathcal{N}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
& F_2(t) = 0.5 \sin \pi t.
\end{aligned}
$$

The interconnections between the two subsystems are

$$
\begin{aligned}
\text{Subsystem 1:} \quad & h_1(t, x) = a_1 \sin(5t/\pi) H_1 x, \\
& H_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \\
\text{Subsystem 2:} \quad & h_2(t, x) = a_2 \sin(2.5t/\pi) H_2 x, \\
& H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

The initial conditions of the two subsystems are:

$$x_{11} = 1, x_{12} = 0, x_{13} = 0.5$$

and

$$x_{21} = 1.1, x_{22} = 1.$$

It is easy to see that both the two subsystems are unstable. The eigenvalues of subsystem 1 are 1, -1 and -1. The eigenvalues of subsystem 2 are 1 and -1.
Our control aim is to design an H_∞ decentralized dynamic-observer-based control in order to stabilize the large-scale uncertain system formed by the above two subsystems.

4.5.1 Observer-based control design

In this section, we will determine all the parameter matrices of the observer (4.11) and control (4.12) by using the proposed method. According to the design procedure, firstly we choose matrices Λ_1 and Λ_2 according to rank condition (4.31). The two matrices Λ_1 and Λ_2 we choose are:

$$\Lambda_1 = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

and

$$\Lambda_2 = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}.$$

By following the design procedure, we obtain

$$K_1 = \begin{bmatrix} -1.8126 & -1.2469 & 0.7254 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -1.5 & -2 \end{bmatrix},$$

$$a_1 = 0.3, a_2 = 0.4,$$

$$\gamma = 0.9.$$

Then, the obtained local HDOs for each subsystem are given by

$$\text{subsystem 1:} \quad \dot{z}_1 = \begin{bmatrix} -11.8038 & 4 & 6 \\ -12.5619 & 3 & 8 \\ -8.0209 & 2 & 2 \end{bmatrix} z_1 + \begin{bmatrix} 11.6077 \\ 13.1238 \\ 11.0419 \end{bmatrix} y_1$$
$$+ \begin{bmatrix} 2 \\ 2.5 \\ 2 \end{bmatrix} u_1 + 10^3 \begin{bmatrix} 0.5344 & 0.4276 & 0.3577 \\ 0.4616 & 1.1218 & 0.6230 \\ 0.2171 & 0.3686 & 0.7405 \end{bmatrix} v_1,$$
$$\dot{v}_1 = \begin{bmatrix} 0.0718 & 0 & 0 \\ 0.0524 & 0 & 0 \\ 0.0326 & 0 & 0 \end{bmatrix} z_1 + \begin{bmatrix} -0.1436 \\ -0.1048 \\ -0.0653 \end{bmatrix} y_1$$
$$+ \begin{bmatrix} -724.2775 & 0.5073 & 0.2197 \\ 0.5029 & -725.5666 & 0.3709 \\ 0.2131 & 0.3691 & -725.9822 \end{bmatrix} v_1,$$
$$\hat{x}_1 = \begin{bmatrix} 0.4 & 0 & 0 \\ -0.8 & 1 & 0 \\ -0.4 & 0 & 1 \end{bmatrix} z_1 + \begin{bmatrix} 0.2 \\ -0.4 \\ -0.2 \end{bmatrix} y_1.$$

$$\text{subsystem 2:} \quad \dot{z}_2 = \begin{bmatrix} -4.3399 & 4 \\ -3.0088 & 1 \end{bmatrix} z_2 + \begin{bmatrix} 6.6797 \\ 6.0176 \end{bmatrix} y_2$$

$$
\begin{aligned}
&+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 643.6592 & 361.0189 \\ 317.1734 & 731.9549 \end{bmatrix} v_2, \\
\dot{v}_2 = \; & \begin{bmatrix} -0.1299 & 0 \\ -0.0327 & 0 \end{bmatrix} z_2 + \begin{bmatrix} 0.2599 \\ 0.0655 \end{bmatrix} y_2 \\
&+ \begin{bmatrix} -726.1181 & -0.0082 \\ -0.0082 & -726.1654 \end{bmatrix} v_2, \\
\hat{x}_2 = \; & \begin{bmatrix} 0.4 & 0 \\ -0.4 & 1 \end{bmatrix} z_2 + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} y_2.
\end{aligned}
$$

The observers we designed are full-order observers. One can also design reduced-order observers, by choosing $q_1 = 2$, $t_1 = 1$, $q_2 = 1$ and $t_2 = 1$.

Subsequently, the obtained local controls for each subsystem are given by

$$
\begin{aligned}
\text{subsystem 1:} \quad u_1 = \; & \begin{bmatrix} -1.8126 & -1.2469 & 0.7254 \end{bmatrix} \hat{x}_1. \\
\text{subsystem 2:} \quad u_2 = \; & \begin{bmatrix} -1.5 & -2 \end{bmatrix} \hat{x}_2.
\end{aligned}
$$

With the determined observers and controls, the eigenvalues of the closed-loop subsystem 1 are

$$
\begin{bmatrix} -0.7429 + 3.1619i & -0.7429 - 3.1619i & -1.2248, \end{bmatrix}
$$

and the eigenvalues of the closed-loop subsystem 2 are

$$
\begin{bmatrix} -0.5 & -1 \end{bmatrix},
$$

which are all on the left side of complex plane. In other words, the large system composed of two subsystems become stable, by using the proposed H_∞ decentralized dynamic-observer-based control.

4.5.2 Simulation results

In this section, in order to show the performance of the proposed control, we have done some simulations with the obtained observer and control. Furthermore, we consider the system with another uncertainties $\Delta\overline{A}_i$ in each subsystem state matrix $(A_i + \Delta A_i + \Delta\overline{A}_i)$ where

$$
\Delta\overline{A}_1 = \Delta_1 \sin_1 2\pi t,
$$

$$
\Delta_1 = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0 & 0.2 & 0.1 \end{bmatrix},
$$

and

$$
\Delta\overline{A}_2 = \Delta_2 \sin_2 5\pi t,
$$

$$
\Delta_2 = \begin{bmatrix} 0.1 & 01 \\ 0.1 & 0.2 \end{bmatrix}.
$$

The simulation results are shown in the following figures. Figure 4.1 represent the disturbances of finite energy used in the simulation. For subsystem 1, the disturbance is added in the time interval from 30s to 40s, while for subsystem 2, the disturbance is added in the time interval from 70s to 80s.

Figures 4.2, 4.4, 4.6, 4.8 and 4.10 represent the original states and their estimates. Figures 4.2, 4.4 and 4.6 represent the original states of subsystem 1 and their estimates obtained from HDO, while Figures 4.8 and 4.10 represent the original states of subsystem 2 and their estimates obtained from HDO. The solid line represents the original system state. The dotted line represents the estimate.

Figures 4.3, 4.5, 4.7, 4.9 and 4.11 represent the estimation errors obtained from HD. Figures 4.3, 4.5 and 4.7 represent the estimation errors for subsystem 1. Figures 4.9 and 4.11 represent the estimation errors for subsystem 2.

Figures 4.12 and 4.13 represent the outputs of each controlled subsystem. Figure 4.12 represents the output of subsystem 1 and Figure 4.13 represents the output of subsystem 2.

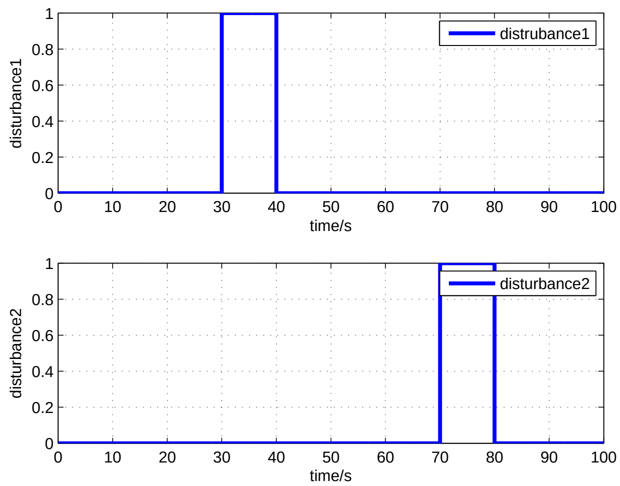

Figure 4.1: Disturbances

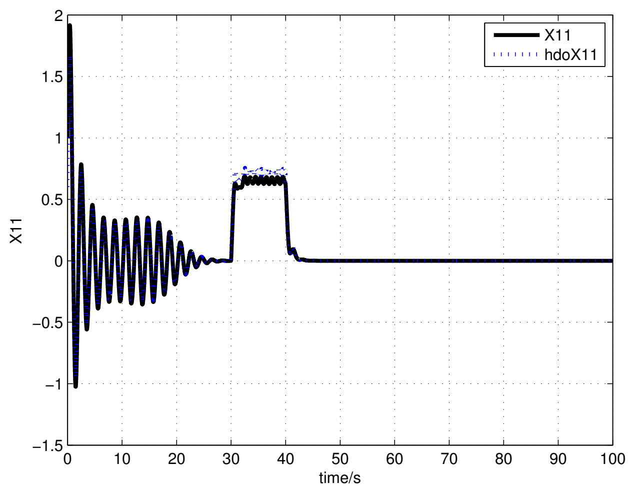

Figure 4.2: State x_{11} and its estimate (solid line: original state; dotted line: HDO)

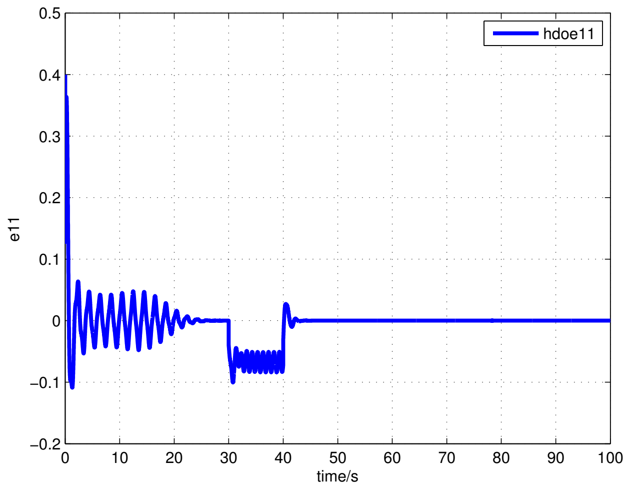

Figure 4.3: Estimation error e_{11}

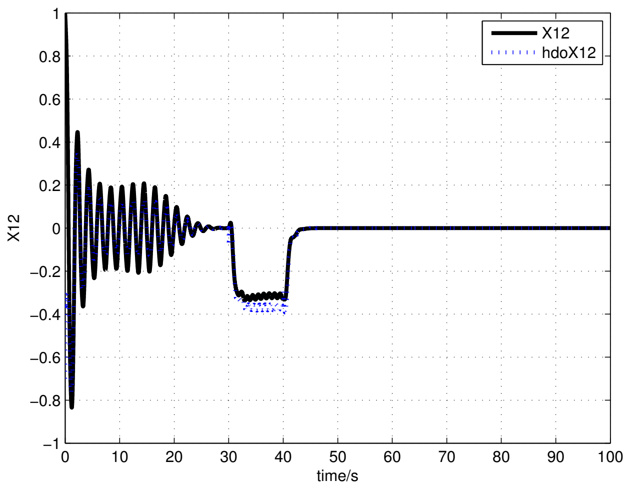

Figure 4.4: State x_{12} and its estimate (solid line: original state; dotted line: HDO)

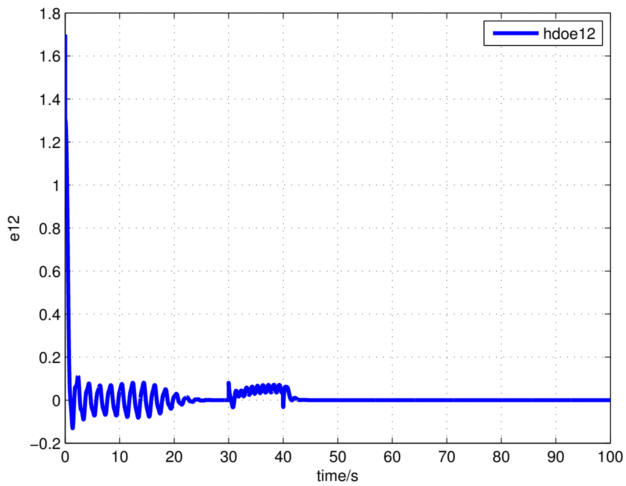

Figure 4.5: Estimation error e_{12}

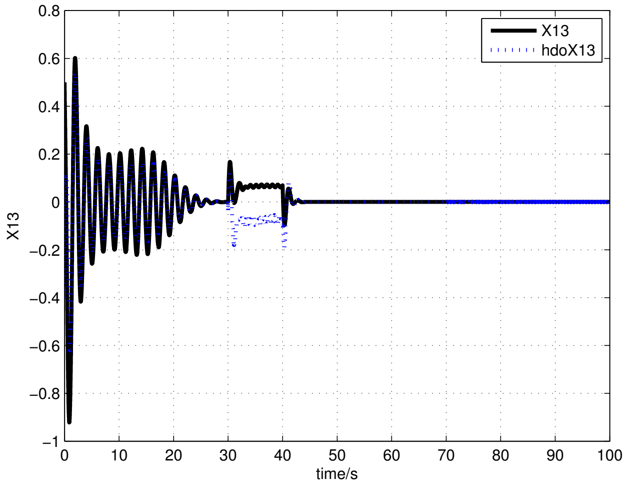

Figure 4.6: State x_{13} and its estimate (solid line: original state; dotted line: HDO)

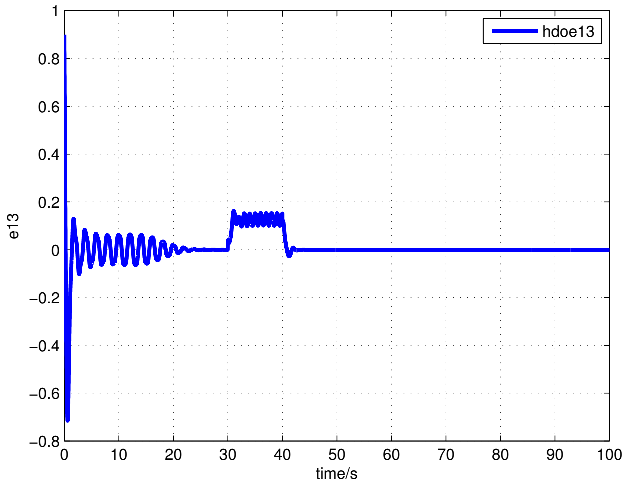

Figure 4.7: Estimation error e_{13}

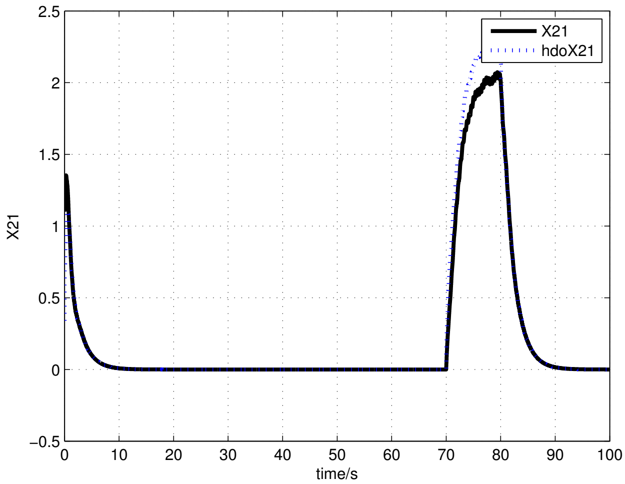

Figure 4.8: State x_{21} and its estimate (solid line: original state; dotted line: HDO)

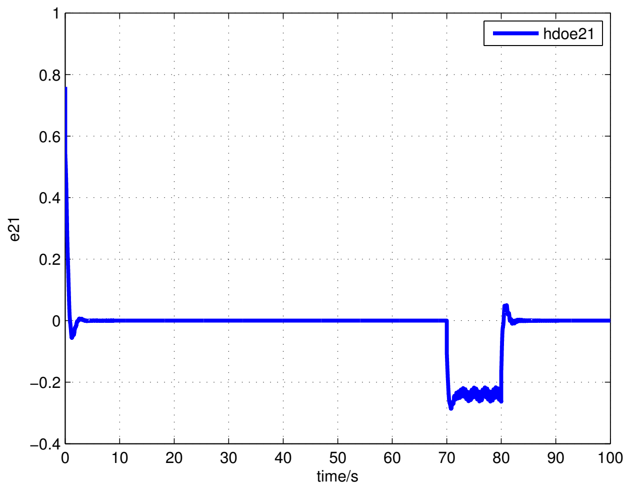

Figure 4.9: Estimation error e_{21}

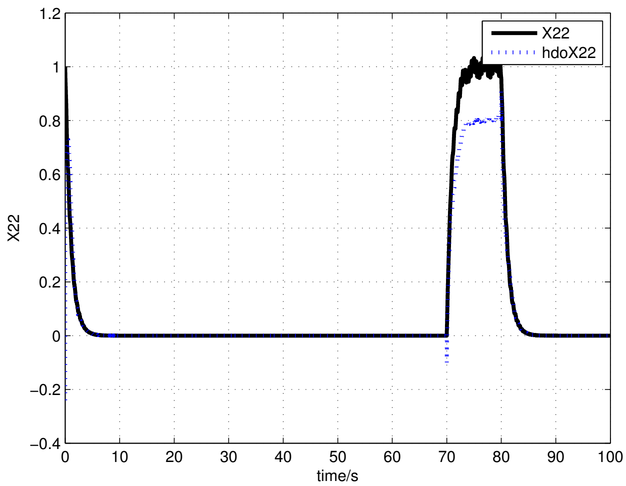

Figure 4.10: State x_{22} and its estimate (solid line: original state; dotted line: HDO)

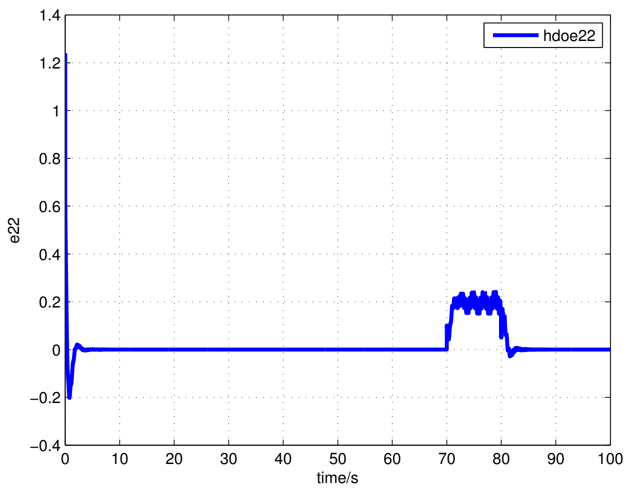

Figure 4.11: Estimation error e_{22}

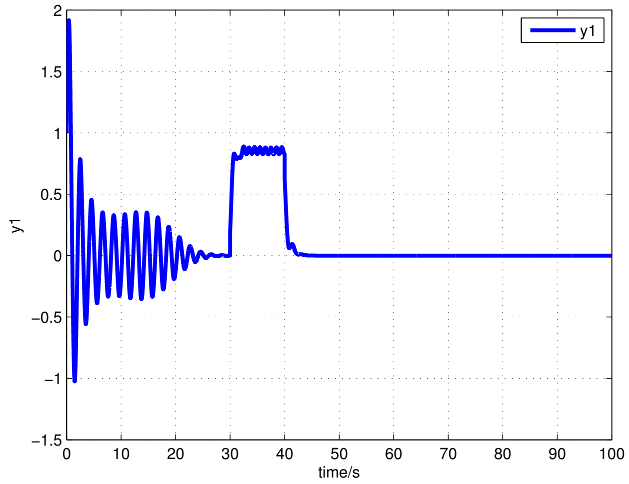

Figure 4.12: Output y_1

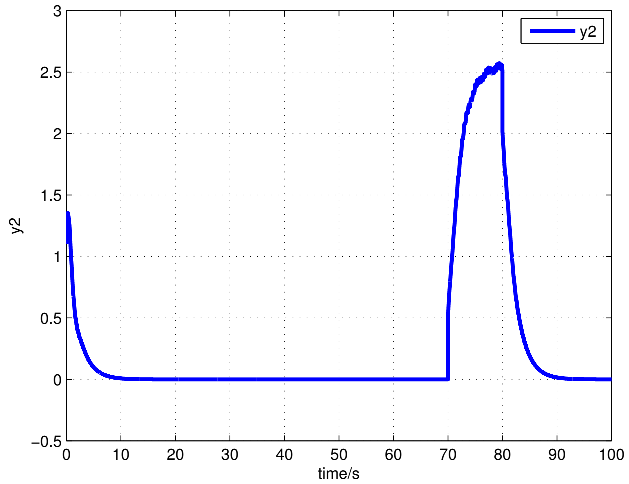

Figure 4.13: Output y_2

From the above figures, one can see the performances of the proposed H_∞ decentralized dynamic-observer-based control, in the presence of additional uncertainties.

Take Figure 4.12 for example. The control stabilizes subsystem 1. After a period of oscillation, subsystem 1 becomes asymptotically stable. For subsystem 2, one can see from Figure 4.13 that by using the proposed control, it becomes asymptotically stable.

In the case when disturbances appear, the control can guarantee the asymptotic stability and the influences of disturbances on each controlled output are minimized (see Figure 4.12 from 30s to 40s and Figure 4.13 from 70s to 90s). On the other hand, the effects of disturbances on estimation errors are minimized, see Figures 4.3 and 4.9 for example.

Furthermore, we can also see that the proposed control is insensitive to uncertainties.

Generally, the proposed H_∞ decentralized dynamic-observer-based control can guarantee the asymptotic stability of the controlled systems in the presence of disturbances and uncertainties.

4.6 Conclusions

This chapter focused on the H_∞ decentralized dynamic-observer-based control design problem for large-scale uncertain systems in the presence of disturbances. These systems were composed of a group of interconnected subsystems, where the interconnections were assumed to be nonlinear and satisfy quadratic constraints. The uncertainties were time varying and assumed to be norm bounded.

The results of Chapters 2 and 3 have been applied to solve the control design problem. Firstly, from the analysis of the estimation error, algebraic constraints were derived. Then the parameterizations have been done to the algebraic constraints. Based on the parameterizations, Lyapunov function approach and Schur complement lemma, the control design problem was formulated as a BMI optimization problem. A two-step algorithm was proposed to solve the problem.

A numerical example composed of two unstable subsystems was presented to illustrate the design procedure and the performances of the proposed control. One can see that the proposed control can guarantee the asymptotic stability of the controlled subsystems in the presence of disturbances and it is insensitive to uncertainties.

CHAPTER

5

Conclusions and perspectives

Contents

In our thesis, we have proposed a new form of HDO, which generalizes the existing observers such as PO and PIO. We also investigated the application of this HDO in the observer-based control of LSS. In this chapter a brief summary of the work discussed in the previous chapters will be given and some areas that merit future research will be described.

5.1 Conclusions

This thesis began with the HDO design for systems in the presence of disturbances and unknown inputs in Chapter 2. Firstly, algebraic constraints were derived from the analysis of the estimation error. Then, based on the parameterizations of the algebraic constraints, the design problem was reduced to the determination of one parameter matrix. Based on bounded real lemma and Schur complement lemma, the determination problem was formulated as an optimization problem in terms of LMIs. Both continuous-time and discrete-time systems were considered. The performances of the proposed HDO were proved to be better in both transient regime and steady state regime, compared with PO and PIO.
Furthermore, we studied the observer design problem of uncertain systems in the presence of unknown inputs and disturbances. It was shown that the problem can be transformed into an observer design problem of descriptor systems.

In Chapter 3, by inserting the proposed HDO into a closed-loop, an observer-based control was designed for uncertain systems in the presence of disturbances. The uncertainty was time varying and assumed to be norm bounded. Based on the parameterizations of the algebraic constraints and Lyapunov function approach, the design problem was formulated as a BMI optimization problem, and a two-step algorithm was proposed to solve this problem. The performances of the proposed control were illustrated through a numerical example, which contained an unstable system.

By extending the obtained results to LSS, an H_∞ decentralized dynamic-observer-based control was proposed for large-scale uncertain systems with disturbances in Chapter 4. These systems were composed of several interconnected subsystems, where the interconnections were assumed to be nonlinear and satisfy quadratic constraints. The control design problem was formulated as a BMI optimization problem, and the two-step algorithm was used to solve the problem. A numerical example that contained two unstable subsystems was presented to illustrate the design procedure and the performances of the proposed control.

5.2 Perspectives

In this thesis some topics were addressed and in the previous section we highlighted several contributions of this thesis. However, some new interesting questions also arose. These questions and other interesting problems are left for future research. We will now discuss some issues that warrant further investigations:

- Do more practice simulations with the proposed observer and observer-based control;
- In Chapter 2, we found that the design problem of uncertain systems with unknown inputs can be transformed into an observer design problem of descriptor systems. Based on the results on the observer design of descriptor systems [28, 30, 32, 92, 98], we can study the HDO design problem for uncertain systems in the presence of unknown inputs and disturbances;

- In Chapter 3, for the observer-based control design, only the effect of disturbances on the output is minimized ($\|T_{wy}\|_\infty < \gamma$). According to [103], we will minimize the effect of disturbances on both estimation error ($\|T_{we}\|_\infty < \gamma_o$) and control ($\|T_{wy}\|_\infty < \gamma_c$), in the observer-based control design.

- Due to the negative influence of time-delays, the design problem of observers and observer-based control for time-delay systems has been greatly investigated. In the case where the time-delay is assumed to satisfy some constraints, the design can be decoupled from the time-delay [55] and [116]. Otherwise, the problem can be solved by using Lyapunov function approach [17, 22, 77, 127, 146]. We can investigate the HDO and observer-based design problems for time-delay systems, and then extend the results to large-scale time-delay systems, such as network systems;

- The adaptive observer has become a hot topic in the research field, since it can estimate the unmeasured states as well as the unknown parameters simultaneously. The adaptive observer design problems of LTI systems were solved in [59, 90, 112, 141], while in [3, 4, 9, 83, 144, 145], the adaptive observers were designed for linear time varying (LTV) systems. Based on the existing results, the proposed HDO can be used to estimate the states and unknown parameters of small systems first, then the results can be extended to LSS.

Appendix A

Publication list

Journal papers

- **Nan Gao**, Mohamed Darouach, Holger Voos and Marouane Alma. New unified H_∞ dynamic observer design for linear systems with unknown inputs. Automatica, Provisionally accepted.
- **Nan Gao**, Mohamed Darouach, Marouane Alma and Holger Voos. H_∞ decentralized dynamic-observer-based control design. Transaction on IoT and Cloud Computing, 2(3):47-69, 2014.

International Conferences

- **Nan Gao**, Mohamed Darouach, Marouane Alma and Holger Voos. Decentralized dynamic-observer-based control for large scale nonlinear uncertain systems. 2015 American Control Conference (ACC). Accepted. July 2015.
- Marouane Alma, Harouna Souley Ali, Mohamed Darouach and **Nan Gao**. An Hinfinity observer design for linear descriptor systems. 2015 American Control Conference (ACC). Accepted. July 2015.
- **Nan Gao**, Mohamed Darouach, Holger Voos and Marouane Alma. H_∞ dynamic observer design for linear time invariant systems. 53rd IEEE Conference on Decision and Control (CDC), pages 1631-1636, December 2014.
- **Nan Gao**, Mohamed Darouach, Marouane Alma and Holger Voos. H_∞ decentralized dynamic-observer-based control for large-scale uncertain nonlinear systems. 2nd International Conference on Control, Decision and Information Technologies (CoDIT'14), pages 364-369, November 2014.
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Résumé

Cette thèse est le résultat de recherche effectuée à Longwy au sein du département CID ≪ Contrôle Identification et Diagnostic ≫ du Centre de Recherche en Automatique de Nancy (CRAN). Elle concerne, d'une part, la synthèse des observateurs dynamiques (d'ordre plein et d'ordre réduit) et la commande basée observateur d'une classe de systèmes linéaires incertains, d'autre part, l'application de ces résultats aux systèmes de grande dimension.

Dans une première partie, une nouvelle forme d'observateurs dynamiques H_∞ est conçue pour les systèmes linéaires en présence d'entrées inconnues et de perturbations, pour les systèmes continus et discrets. L'observateur proposé généralise ceux existants tels que les observateurs proportionnels et proportionnels-intégrales. La conception d'observateur est fondée sur la résolution des inégalités matricielles linéaires (LMI).

Ensuite, ces observateurs ont été utilisés dans la synthèse de contrôleurs basés observateur pour les systèmes incertains en présence de perturbations. Cette synthèse est basée sur le paramétrage des solutions des contraintes algébriques obtenues à partir des erreurs d'estimation. La solution est obtenue à partir de la résolution des inégalités matricielles bilinéaires en utilisant un algorithme à 2 étapes.

Dans la dernière partie, les résultats obtenus ont été étendus aux systèmes de grande dimension. Dans ce cadre, les systèmes considérés sont décomposés en plusieurs sous systèmes interconnectés de faible dimension, où les interconnections sont supposées non linéaires et satisfaire des contraintes quadratiques. Une commande décentralisée basée observateur dynamique est proposée pour les systèmes interconnectés incertains en présence de perturbations.

Mots clés: Observateur dynamique, commande basée observateur, système linéaire incertain, système de grande dimension, LMI, commande décentralisée.

Abstract

The present thesis is the result of research conducted in Longwy, within the department Control, Identification, Diagnosis (CID) of Research Center for Automatic Control of Nancy (CRAN). This thesis investigates the problem of dynamic observer (full- and reduced-order) and observer-based control design and their applications to large-scale systems.

Firstly, a new form of H_∞ dynamic observer is designed for linear systems in the presence of unknown inputs and disturbances. The proposed observer generalizes the existing results on proportional observer and proportional integral observer. The observer design is based on the solution of linear matrix inequalities (LMI). Both continuous-time and discrete-time systems are considered.

Thereafter, by inserting the proposed observer into a closed-loop, an observer-based control is presented for uncertain systems in the presence of disturbances. Based on the parameterization of algebraic constraints obtained from the analysis of the estimation error, the control design is derived from the solution of bilinear matrix inequality, by using a two-step algorithm.

Finally, the obtained results have been extended to large-scale systems. A decentralized observer-based control is proposed for large-scale uncertain systems in the presence of disturbances. These systems are composed of several interconnected subsystems of low dimensions, where the interconnections are assumed to be nonlinear and satisfy quadratic constraints.

Key words: Dynamic observer, observer-based control, linear uncertain system, large-scale system, LMI, decentralized control.