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Etude de fonctionnelles géométriques dépendant de la courbure par des méthodes d'optimisation de formes. Applications aux fonctionnelles de Willmore et Canham-Helfrich

Thèse

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Part I

Introduction

Chapter 1

Introduction (en français)

Au sein de la nature, de nombreux phénomènes physiques sont gouvernés par la géométrie de leur environnement. Le principe les régissant est souvent modélisé sous forme d'une minimisation d'énergie. Certains problèmes comme les bulles de savon font intervenir les propriétés d'ordre un des surfaces (l'aire, la normale, la première forme fondamentale), tandis que d'autres comme les formes prises par les globules rouges au repos concernent également leurs propriétés d'ordre deux (les courbures principales, la seconde forme fondamentale).

Dans cette thèse, on s'intéresse à l'existence de solutions pour de tels problèmes d'optimisation de formes, ainsi qu'à la détermination d'une classe adéquate de formes admissibles. En effet, bien que la plupart du temps, la théorie géométrique de la mesure [33, 87] fournisse un cadre assez général pour comprendre ces questions précisément, le minimiseur possède souvent une régularité plus faible que celle escomptée, et il est difficile de comprendre (et de prouver) en quel sens il l'est, puisque des singularités peuvent parfois apparaître.

La motivation de départ pour cette thèse vient de la biologie. Dans un milieu acqueux, des phospholipides au repos s'assemblent immédiatement par paires pour former une bicouche, plus communément appelée vésicule. C'est un sac de fluide lui-même plongé dans un fluide ainsi que la membrane de base des cellules de tout être vivant.

Dépourvus de noyau chez les mammifères, les globules rouges sont des exemples typiques de vésicules équipés d'une structure supplémentaire interne jouant le rôle de squelette au sein de la membrane. Un des principaux travaux de la thèse fut d'introduire et étudier une condition de boule uniforme, notamment pour modéliser l'effet du squelette.

En effet, si la déformation locale est faible, alors le squelette ne joue aucun rôle et le globule rouge se comporte comme une vésicule. Dans le cas contraire, le squelette redistribue l'excès de contraintes locales sur toute la surface du vésicule. Ainsi, ce squelette agit comme si une borne uniforme est imposée aux courbures du vésicule.

Dans les années 1970, Canham [16] puis Helfrich [45] proposèrent un modèle simple pour décrire les vésicules. Si on impose l'aire de la bicouche et le volume de fluide qu'elle contient, la forme prise est un minimiseur pour l'énergie libre élastique suivante:

$$\mathcal{E} := \frac{k_b}{2} \int_{\text{membrane}} (H - H_0)^2 \, dA + k_G \int_{\text{membrane}} K dA, \tag{1.1}$$

où $H = \kappa_1 + \kappa_2$ désigne la courbure moyenne scalaire et $K = \kappa_1 \kappa_2$ celle de Gauss, où $H_0 \in \mathbb{R}$ (appelée courbure spontanée) mesure l'asymétrie entre les deux couches d'une vésicule, et où $k_b > 0$ ainsi que $k_G < 0$ sont deux autres constantes physiques.

Parmi la variété conséquente de problèmes soulevés par ce fascinant modèle, on peut mentionner: existence, unicité, propriétés, régularité des minimiseurs; simulations numériques précises par des méthodes de type *level-set* et *phase-field*; couplage de la structure avec la dynamique d'un fluide; rhéologie d'une multitude de vésicules dans un écoulement; contrôler la forme à partir d'une partie du bord, etc. Durant ces trois années de thèse, nous nous sommes principalement concentrés sur l'étude de trois axes de recherche qui se reflètent dans la structure de ce rapport.

Une première approche consiste à minimiser l'énergie de Canham-Helfrich (1.1) sans contrainte puis avec une contrainte d'aire. Le cas $H_0 = 0$ est connu sous le nom d'énergie de Willmore:

$$\mathcal{W}(\Sigma) := \frac{1}{4}\int_{\Sigma} H^2 dA. \tag{1.2}$$

Elle est beaucoup étudiée par les géomètres [79, 88, 92] notamment grâce à sa propriété d'invariance par transformations conformes. Cependant, cette particularité n'est plus vérifiée si $H_0 \neq 0$. Comme la sphère est le minimiseur de (1.2), c'est un bon candidat pour être le minimiseur de (1.1) parmi les surfaces d'aire fixée. Notre première contribution dans cette thèse a été d'étudier son optimalité (minimiseur local/global, point critique).

De plus, si on impose la topologie des formes admissibles, alors d'après le théorème de Gauss-Bonnet, l'énergie de Canham-Helfrich (1.1) est équivalente à l'énergie suivante dite de Helfrich:

$$\mathcal{H}(\Sigma) \ := \ \frac{1}{4}\int_{\Sigma} (H - H_0)^2\, dA \ = \ \frac{1}{4}\int_{\Sigma} H^2 dA \ - \ \frac{H_0}{2}\int_{\Sigma} H dA \ + \ \frac{H_0^2 A(\Sigma)}{4}. \tag{1.3}$$

Dans le cas spécifique des membranes à courbure spontanée négative $H_0 < 0$, on peut se demander si la minimisation de (1.3) sous contrainte d'aire peut être effectuée en minimisant individuellement chaque terme. Comme l'énergie de Willmore (1.2) est invariante par homothétie, et comme les sphères sont les seuls minimiseurs de (1.2), cette simplification n'a de sens que si la sphère est l'unique solution du problème suivant:

$$\inf_{A(\Sigma)=A_0} \frac{1}{2}\int_{\Sigma} H dA.$$

Par conséquent, notre deuxième travail dans cette thèse correspond à l'étude du problème ci-dessus, c'est-à-dire à la minimisation sous contrainte d'aire de la courbure moyenne totale parmi diverses classes de surfaces.

Ensuite, lorsqu'une contrainte d'aire et de volume sont considérées, le minimiseur ne peut alors pas être une sphère qui n'est plus admissible. En utilisant le point de vue de l'optimisation de formes, la troisième et plus importante contribution de cette thèse est d'introduire une classe plus raisonnable de surfaces, pour laquelle l'existence d'un minimiseur suffisamment régulier est assurée pour des fonctionnelles et des contraintes assez générales faisant intervenir les propriétés d'ordre un et d'ordre deux des surfaces:

$$\inf_{\Omega} \int_{\partial\Omega} F\left[\mathbf{x}, \mathbf{n}_{\partial\Omega}(\mathbf{x}), H_{\partial\Omega}(\mathbf{x}), K_{\partial\Omega}(\mathbf{x}), u_{\Omega}(\mathbf{x}), \nabla u_{\Omega}(\mathbf{x})\right] dA(\mathbf{x}).$$

En s'inspirant de ce que fit Chenais dans [20] quand elle considéra la propriété de cône uniforme, on considère les (hyper-)surfaces satisfaisant une condition de boule uniforme. On étudie d'abord des fonctionnelles purement géométriques puis nous autorisons la dépendance à travers la solution u_Ω de problèmes elliptiques aux limites d'ordre deux posés sur le domaine intérieur à la surface. On détaille maintenant chaque partie du rapport.

Première partie: sur la minimisation de l'énergie de Canham-Helfrich

Chapitre 3: un aperçu des modèles physiques associés aux vésicules

Dans ce chapitre, on explique d'abord ce qu'est une vésicule, puis on présente un modèle simplifié bidimensionnel pour caractériser leurs formes. Ensuite, on considère sa version tridimensionelle connue sous le nom d'énergie de Canham-Helfrich. Finalement, on détaille d'autres modèles de vésicules [85] et de globules rouges [59].

Chapitre 4: minimiser l'énergie de Helfrich sans contrainte

Dans ce chapitre, on étudie la minimisation de (1.3) parmi les C^2-surfaces compactes de $\mathbb{R}^3$:

$$\inf_{\Sigma} \frac{1}{4}\int_{\Sigma} (H - H_0)^2 dA.$$

On distingue trois cas selon le signe de la courbure spontanée $H_0 \in \mathbb{R}$. Ensuite, on montre que les même résultats sont vrais pour (1.1). Tout ceci est résumé dans le tableau 1.1. Finalement, en affaiblissant la régularité des formes admissibles, on étend le cas connu $H_0 = 0$ aux $C^{1,1}$-surfaces compactes simplement connexes de $\mathbb{R}^3$.

$k_G < 0 < k_b$	$H_0 < 0$	$H_0 = 0$	$H_0 > 0$
Existence	Pas de minimiseur global	Toute sphère [92]	La sphère de rayon $\frac{1}{H_0}$ [2]
$\inf_\Sigma \mathcal{H}(\Sigma)$	4π	4π	0
$\inf_\Sigma \mathcal{E}(\Sigma)$	$4\pi\,(2k_b + k_G)$	$4\pi\,(2k_b + k_G)$	$4\pi k_G$

Table 1.1: sur la minimisation de l'énergie Canham-Helfrich sans contrainte.

Chapitre 5: minimiser l'énergie de Helfrich sous contrainte d'aire

Dans ce chapitre, on s'intéresse à la minimisation sous contrainte d'aire $A_0 > 0$ de l'énergie de Helfrich (1.3) parmi les $C^{1,1}$-surfaces compactes simplement connexes de $\mathbb{R}^3$:

$$\inf_{A(\Sigma)=A_0} \frac{1}{4}\int_\Sigma (H - H_0)^2 dA. \tag{1.4}$$

On se réfère aux théorèmes 1.9 pour avoir des résultats d'existence associés au problème (1.4). A part exclure la sphère de la classe des formes admissibles, la contrainte de volume ne semble pas jouer un rôle spécifique dans le processus théorique utilisé en calcul des variations, c'est-à-dire ni pour la compacité de la suite minimisante, ni pour la (semi-)continuité (inférieure) de la fonctionnelle et des contraintes, ni pour la régularité des minimiseurs.

De plus, la sphère $\mathbb{S}_{A_0}$ d'aire A_0 semble être un bon candidat pour minimiser (1.4). Dans ce chapitre, on étudie en détail l'optimalité globale de cette sphère. Les résultats sont rassemblés dans la première ligne du tableau 1.2. Ils dépendent d'un paramètre adimensionnel spécifique:

$$c_0 := \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}, \tag{1.5}$$

et on prouve l'existence de deux nombres $c_- \approx -0.575$ et $c_+ \approx 1.46$ qui sont des valeurs de seuil.

Paramètre $c_0 = \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}$	$-\infty$ c_- 0 1 c_+ c_{++} $+\infty$
$\mathbb{S}_{A_0}$ est-elle minimiseur global ?	NON [? [OUI] ?] NON
$\mathbb{S}_{A_0}$ est-elle minimiseur local ?	OUI] ?] NON
$\mathbb{S}_{A_0}$ est-elle point critique ?	OUI

Table 1.2: résultats obtenus concernant l'optimalité pour (1.4) de la sphère $\mathbb{S}_{A_0}$ d'aire A_0.

Tout d'abord, pour tout $c_0 \in [0, 1]$, on déduit de l'inégalité de Cauchy-Schwarz que $\mathbb{S}_{A_0}$ est l'unique minimiseur global de (1.4). Puis, pour tout $c_0 > c_+$, on établit que $\mathbb{S}_{A_0}$ n'est pas un minimiseur de (1.4) parmi la classe des cigares. En particulier, dans ce cas, on déduit que $\mathbb{S}_{A_0}$ n'est plus un minimiseur global, même dans une sous-classe plus petite de formes admissibles (convexe, axisymétrique, condition de boule uniforme).

Theorem 1.1. *Soient $A_0 > 0$, $H_0 \in \mathbb{R}$, c_0 donné par* (1.5) *et $c_+ := \frac{1}{4}(1+\sqrt{2})^2 \approx 1.46$. On appelle cigare tout cylindre de longueur $L \geqslant 0$ sur lequel est recollé de manière $C^{1,1}$ deux demi-sphères de rayon $R > 0$. Si $c_0 < c_+$, alors la sphère $\mathbb{S}_{A_0}$ d'aire A_0 est l'unique minimiseur global de* (1.4) *parmi la classe des cigares. De plus, si $c_0 > c_+$, alors c'est le cigare $\mathbb{C}_{A_0}$ d'aire A_0 et de rayon:*

$$R_- := \sqrt{\frac{A_0}{3\pi}} \cos\left[\frac{1}{3}\arccos\left(\frac{-3}{H_0}\sqrt{\frac{3\pi}{A_0}}\right) + \frac{4\pi}{3}\right].$$

Pour finir, si $c_0 = c_+$, alors $\mathbb{S}_{A_0}$ et $\mathbb{C}_{A_0}$ sont les deux seuls minimiseurs de (1.4) *parmi les cigares.*

Finalement, pour tout réel $c_0 < c_-$, on prouve qu'une suite de $C^{1,1}$-surfaces axisymétriques non-convexes d'aire constante A_0 (convergeant vers une double-sphère) ont une énergie de Helfrich (1.3) strictement plus petite que celle de $\mathbb{S}_{A_0}$, qui n'est donc pas un minimiseur global de (1.4). Plus précisément, le résultat s'énonce ainsi.

Theorem 1.2. *Soient $A_0 > 0$, $H_0 \in \mathbb{R}$, c_0 donné par (1.5) et $c_- := \frac{1}{8\cos\theta} \approx -0.575$, où $\theta \approx 4.4934$ est l'unique solution de $\tan x = x$ sur intervalle $]\pi, \frac{3\pi}{2}[$. Alors, il existe une suite $(\Sigma_r)_{r>0}$ de $C^{1,1}$-surfaces de $\mathbb{R}^3$ compactes, simplement connexes, non convexes et axisymétriques, telle que:*

$$\frac{1}{4}\int_{\Sigma_r}(H-H_0)^2 dA - \frac{1}{4}\int_{\mathbb{S}_{A_0}}(H-H_0)^2 dA \quad \underset{r\to 0^+}{\longrightarrow} \quad 8\pi\,(c_0 - c_-)\,.$$

Cependant, et c'est le but de la deuxième partie de ce rapport, si on restreint la classe des formes admissibles à celles entourant un domaine intérieur convexe, ou à celles délimitant un domaine intérieur *axiconvexe*, c'est-à-dire un domaine axisymétrique dont l'intersection avec n'importe quel plan orthogonal à l'axe de symétrie est soit un disque soit vide, alors la sphère $\mathbb{S}_{A_0}$ d'aire A_0 est l'unique minimiseur global de (1.4) (cf. l'inégalité (1.14) et le théorème 1.5).

Chapitre 6: la sphère en tant que minimiseur local pour l'énergie de Helfrich sous contrainte d'aire

Dans ce chapitre, on étudie l'optimalité de la sphère $\mathbb{S}_{A_0}$ d'aire A_0 en tant que minimiseur local de (1.4). Les résultats sont rassemblés dans la deuxième ligne du tableau 1.2. Dans le cas $c_0 < 0$, en combinant l'observation faite dans le paragraphe sous (1.3) avec les résultats de la deuxième partie (remarque 11.1), on obtient que $\mathbb{S}_{A_0}$ est un minimiseur local de (1.4). De plus, comme $\mathbb{S}_{A_0}$ est un minimiseur global de (1.4) pour tout $c_0 \in [0,1]$, c'est en particulier un minimiseur local de (1.4).

Ensuite, en supposant $c_0 > 1$, on étudie des perturbations locales axisymétriques et régulières de la sphère. On effectue une homothétie afin d'étudier seulement les perturbations de la sphère unité. En effet, on a:

$$\inf_{A(\Sigma)=A_0} \frac{1}{4}\int_{\Sigma}(H-H_0)^2 dA = \inf_{A(\widetilde{\Sigma})=4\pi} \frac{1}{4}\int_{\widetilde{\Sigma}}(H-2c_0)^2 dA, \tag{1.6}$$

où c_0 est donné par (1.5). On prouve l'existence d'une valeur de seuil au dessus de laquelle la dérivée seconde de forme de (1.3) associée à cette famille de perturbations est négative. Plus précisément:

Proposition 1.3. *On considère une C^∞-fonction $\varphi : \mathbb{R} \to \mathbb{R}$ non identiquement nulle à support compact et des fonctions de la forme $\theta_\varepsilon : s \in [0, L(\varepsilon)] \mapsto s + \varepsilon\varphi(s)$. On suppose que chaque θ_ε génère une surface axisymétrique $\Sigma_\varepsilon \subset \mathbb{R}^3$ régulière (compacte, plongée, simplement connexe) via la courbe $s \in [0, L(\varepsilon)] \mapsto (\int_0^s \cos\theta_\varepsilon(s)ds, \int_0^s \sin\theta_\varepsilon(s)ds)$ paramétrée par la longueur d'arc. Alors en introduisant:*

$$F_{c_0} : \varepsilon \in \mathbb{R} \longmapsto F_{c_0}(\varepsilon) = \frac{1}{4\pi}\int_{\Sigma_\varepsilon}(H-2c_0)^2 dA, \tag{1.7}$$

on a $\dot{\mathbf{F}}_{\mathbf{c_0}}(\mathbf{0}) = \mathbf{0}$ et $\ddot{\mathbf{F}}_{\mathbf{c_0}}(\mathbf{0}) < \mathbf{0} \Leftrightarrow \mathbf{R}(\varphi) < \mathbf{c_0}$ ainsi que $\ddot{\mathbf{F}}_{\mathbf{c_0}}(\mathbf{0}) > \mathbf{0} \Leftrightarrow \mathbf{R}(\varphi) > \mathbf{c_0}$, où on a posé:

$$R(\varphi) = 1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \dot\varphi(s)^2 \sin s ds + \int_0^\pi \frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s \varphi(t)\sin t dt\right)^2 ds}{\displaystyle\int_0^\pi \varphi(s)^2 \sin s ds - \varphi(\pi)^2}, \tag{1.8}$$

qui est bien définie pour tout $\varphi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ satisfaisant:

$$\int_0^\pi \dot\varphi(s)^2 \sin s ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \frac{\left(\int_0^s \varphi(t)\sin t dt\right)^2}{\sin s} ds < +\infty. \tag{1.9}$$

De plus, l'application φ vérifie nécessairement les contraintes suivantes:

$$\begin{cases} \varphi(0) = 0, \quad \varphi(\pi) = -\dot{L}(0) \\[2mm] \varphi(\pi) = \displaystyle\int_0^\pi \varphi(s)\sin s ds = \frac{1}{\pi}\int_0^\pi s\varphi(s)\sin s ds \\[2mm] \varphi(\pi)^2 = \displaystyle\int_0^\pi (\pi - s)\,\varphi(s)^2 \cos s ds. \end{cases} \tag{1.10}$$

On introduit donc la valeur critique:

$$c_{++} := \inf R(\phi), \tag{1.11}$$

où R est définie par (1.8) et où l'infimum est pris parmi toutes les fonctions non identiquement nulles $\phi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ satisfaisant (1.9) et les contraintes (1.10).

Si $c_0 < c_{++}$, alors $c_0 < R(\varphi)$, i.e. $\ddot{F}_{c_0}(0) > 0$, pour tout $\varphi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ vérifiant (1.9)–(1.10). En particulier, si ε est choisi suffisamment petit, toute perturbation Σ_ε de la forme donnée dans la proposition 1.3 a une énergie de Helfrich (1.7) strictement plus grande que celle de la sphère unité, qui est donc un minimiseur local de (1.7) parmi cette classe de perturbations:

$$\begin{aligned}\frac{1}{4\pi}\int_{\Sigma_\varepsilon}(H-2c_0)^2dA - \frac{1}{4\pi}\int_{\mathbb{S}^2}(H-2c_0)^2dA &= F_{c_0}(\varepsilon)-F_{c_0}(0) = \dot{F}_{c_0}(0)\varepsilon + \ddot{F}_{c_0}(0)\frac{\varepsilon^2}{2} + o(\varepsilon^2)\\ &= \frac{\varepsilon^2}{2}\left(\ddot{F}_{c_0}(0) + \frac{o(\varepsilon^2)}{\varepsilon^2}\right) \quad (\ >0 \ \text{ pour } \varepsilon \text{ petit}).\end{aligned}$$

Réciproquement, si $c_0 > c_{++}$, il existe $\varphi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ vérifiant (1.9)–(1.10) tel que $c_0 > R(\varphi)$. On a donc $\ddot{F}_{c_0}(0) < 0$, sous réserve qu'on puisse construire une extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ de φ à support compact telle que la famille d'applications $\theta_\varepsilon : s \in [0, L(\varepsilon)] \to s + \varepsilon\tilde{\varphi}(s)$ soit bien définie au voisinage de $\varepsilon = 0$ et admissible au sens de la définition 8.1, c'est-à-dire qu'elle génère des surfaces axisymétriques $\Sigma_\varepsilon \subset \mathbb{R}^3$. Si c'est le cas, alors pour ε suffisamment petit, Σ_ε est une perturbation d'énergie de Helfrich (1.7) strictement plus petite que celle de la sphère unité, qui n'est donc pas un minimum local de (1.6). Après homothétie, $\mathbb{S}_{A_0}$ n'est donc pas un minimiseur local de (1.4).

De plus, si on pose $u(s) = \int_0^s \varphi(t)\sin t dt$, on peut exprimer c_{++} par un problème d'optimisation équivalent posé sur un espace de Sobolev à poids. Le résultat s'énonce ainsi.

Theorem 1.4. *Soit c_{++} la valeur critique donnée par* (1.11). *Alors on a:*

$$c_{++} = \inf_{\substack{u\in H^2_{\sin}(0,\pi)\\ u\neq 0\\ \int_0^\pi u(s)ds=0\\ \int_0^\pi(\pi-s)\cos s(\frac{\dot{u}(s)}{\sin s})^2ds=0}} \frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{\ddot{u}(s)^2}{\sin s}ds + 2\int_0^\pi \frac{u(s)^2}{\sin^3 s}ds}{\displaystyle\int_0^\pi \frac{\dot{u}(s)^2}{\sin s}ds}, \tag{1.12}$$

où on a posé $H^2_{\sin}(0,\pi) = \{u \in H^2_0(0,\pi),\ \ \int_0^\pi \frac{u(s)^2}{\sin^3 s}ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin s}ds + \int_0^\pi \frac{\ddot{u}(s)^2}{\sin s}ds < +\infty\}$. De plus, il existe un minimiseur à ce problème (1.12) *et également à* (1.11).

Finalement, on essaye d'évaluer la valeur exacte de c_{++} mais nous n'avons pas su traiter la contrainte non-linéaire $\int_0^s(\pi-s)\cos s(\frac{\dot{u}(s)}{\sin s})^2ds = 0$. On a donc décidé d'évaluer les points critiques de (1.12) sans cette contrainte. Ils sont donnés par un problème de valeurs propres associées à l'équation différentielle ordinaire unidimensionnelle non-linéaire du quatrième ordre suivante:

$$\forall s \in]0,\pi[, \quad \frac{d^2}{ds^2}\left(\frac{\ddot{u}(s)}{\sin s}\right) + 2\lambda\frac{d}{ds}\left(\frac{\dot{u}(s)}{\sin s}\right) + \frac{2u(s)}{\sin^3 s} = \mu, \quad u(0)=u(\pi)=\dot{u}(0)=\dot{u}(\pi)=0, \tag{1.13}$$

où λ est une valeur propre et μ le multiplicateur de Lagrange associé à la contrainte $\int_0^\pi u(s)ds = 0$. En particulier, d'après (1.12), une estimation par le bas de c_{++} est donnée par la plus petite valeur propre strictement positive λ pour laquelle la solution $u : [0,\pi] \to \mathbb{R}$ de (1.13) avec $\mu = 0$ n'est pas identiquement nulle et vérifie $\int_0^\pi u(s)ds = 0$.

Malheureusement, on n'a pas réussi à résoudre complètement le problème de valeur propre. Toutefois, on obtient une suite explicite de valeurs propres $(\lambda_{2i})_{i\geqslant 1} = \frac{1}{2}(2i+1)(2i+2)$ pour lesquelles la solution correspondante u_i à (1.13) avec $\mu = 0$ n'est pas identiquement nulle et vérifie $\int_0^\pi u_i(s)ds = 0$. Il y a également des raisons, notamment numériques, de penser que $\lambda_2 = 6$ est la plus petite valeur propre mais nous n'avons pas été en mesure de le prouver.

Deuxième partie: sur la minimisation de la courbure totale

Cette partie est la reproduction d'un article soumis intitulé *on the minimization of total mean curvature* [23], réalisé en collaboration avec Simon Masnou, Antoine Henrot et Takéo Takahashi. On a ajouté un exposé détaillé sur les propriétés du réarrangement croissant, ainsi que la preuve de l'inégalité de Minkowski (1.14) ci-dessous avec un traitement complet du cas d'égalité.

En 1901, Minkowski prouve que l'inégalité suivante est vraie pour tout ouvert non-vide convexe borné $\Omega \subset \mathbb{R}^3$ dont le bord $\partial\Omega$ est une C^2-surface:

$$\frac{1}{2}\int_{\partial\Omega} HdA \geqslant \sqrt{4\pi A(\partial\Omega)}, \tag{1.14}$$

où l'intégration de la courbure moyenne scalaire $H = \kappa_1 + \kappa_2$ est effectuée par rapport à la mesure de Hausdorff bidimensionnelle usuelle notée $A(\bullet)$.

Annoncée dans [69], l'inégalité (1.14) est prouvée dans [70, §7] en supposant une régularité C^2. La preuve se trouve également dans [73, Chapitre 6, Exercice (10)] pour le cas des ovaloïdes, c'est-à-dire des C^∞-surfaces compactes connexes de $\mathbb{R}^3$ dont la courbure de Gauss est partout strictement positive.

La preuve originelle de Minkowski est basée sur l'inégalité isopérimétrique qui est combinée aux formules de Steiner-Minkowski. L'inégalité (1.14) reste donc vraie si $\partial\Omega$ est seulement une surface de classe $C^{1,1}$ (ou de manière équivalente, si $\partial\Omega$ est une surface de *reach* strictement positif, cf. Théorèmes 16.5-16.6). Si aucune régularité est supposée sur le bord, la même inégalité reste valide mais il faut alors remplacer la courbure moyenne totale par la largeur moyenne du convexe.

L'égalité a lieu dans (1.14) si et seulement si Ω est une boule ouverte. Ceci fut énoncé sans preuve par Minkowski dans [70, §7]. Une démonstration de Favard se trouve dans [31, Section 19]. Elle est basée sur une inégalité de type Bonnesen faisant intervenir la notion de volume mixte. Dans le chapitre 13, on donne une preuve de l'inégalité (1.14), avec un traitement complet du cas d'égalité, et on considère aussi spécifiquement le cas axisymétrique, en s'inspirant des travaux de Bonnesen [10, Section VI, §35 (74)].

De plus, l'inégalité (1.14) est en fait une conséquence d'une généralisation due à Minkowski de l'inégalité isopérimétrique. Cette généralisation fait intervenir la notion de volume mixte associé à plusieurs convexes. On renvoie à [83, Théorème 6.2.1, Notes de la Section 6.2] et également à [11, Sections 49,52,56] pour un exposé plus détaillé sur cette question.

Dans cette partie, on s'intéresse principalement à la validité de (1.14) sous d'autres hypothèses, ainsi qu'au problème associé de minimisation de la courbure moyenne totale sous contrainte d'aire:

$$\inf_{\substack{\Sigma\in\mathfrak{C} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_{\Sigma} HdA, \tag{1.15}$$

pour certaines classes $\mathfrak{C}$ de surfaces dans $\mathbb{R}^3$. On rappelle que la motivation de départ pour le problème (1.15) est l'étude du problème (1.4) dans le cas particulier $H_0 < 0$. En effet, on peut se demander si le problème (1.4) peut être résolu en minimisant individuellement chaque terme. Puisque l'énergie de Willmore (1.2) est invariante par homothétie et que les sphères sont les seuls minimiseurs globaux de (1.2), cette simplification ne fait sens que si la sphère $\mathbb{S}_{A_0}$ est également la seule solution du problème (1.15). On prouve dans cette partie que c'est vrai si on considère une classe particulière de surfaces.

Tout d'abord, on introduit deux classes de 2-surfaces plongées dans $\mathbb{R}^3$: la classe $\mathcal{A}_{1,1}$ de toute les surfaces compactes qui sont le bord d'un domaine intérieur axisymétrique (c'est-à-dire un ensemble ayant une symétrie de révolution autour d'un axe), et la sous-classe $\mathcal{A}_{1,1}^+$ des surfaces *axiconvexes*, c'est-à-dire celles délimitant un domaine intérieur axisymétrique dont l'intersection avec n'importe quel plan orthogonal à l'axe de symétrie est soit un disque soit vide. On prouve d'abord la chose suivante:

Theorem 1.5. *On considère la classe $\mathcal{A}_{1,1}^{+}$ des $C^{1,1}$-surfaces de $\mathbb{R}^3$ axiconvexes. Alors on a:*

$$\forall \Sigma \in \mathcal{A}_{1,1}^{+}, \quad \frac{1}{2}\int_{\Sigma} H dA \geqslant \sqrt{4\pi A(\Sigma)},$$

où l'égalité a lieu si et seulement si Σ est une sphère. En particulier, pour tout $A_0 > 0$, on a:

$$\frac{1}{2}\int_{\mathbb{S}_{A_0}} H dA = \min_{\substack{\Sigma \in \mathcal{A}_{1,1}^{+} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_{\Sigma} H dA = \sqrt{4\pi A_0},$$

et la sphère $\mathbb{S}_{A_0}$ d'aire A_0 est l'unique minimiseur global de ce problème.

Ensuite, on montre que ce résultat ne peut s'étendre à la classe plus générale des $C^{1,1}$-surfaces compactes simplement connexes de $\mathbb{R}^3$ et on fournit même une piste de réponse négative pour une extension à la classe $\mathcal{A}_{1,1}$. Plus précisément:

Theorem 1.6. *Soit $A_0 > 0$. Il existe une suite de $C^{1,1}$-surfaces $(\Sigma_i)_{i\in\mathbb{N}}$ ainsi qu'une suite de $C^{1,1}$-surfaces axisymétriques $(\widetilde{\Sigma}_i)_{i\in\mathbb{N}} \subset \mathcal{A}_{1,1}$ satisfaisant $A(\Sigma_i) = A(\widetilde{\Sigma}_i) = A_0$ pour tout $i \in \mathbb{N}$ et telles que:*

$$\lim_{i\to+\infty} \frac{1}{2}\int_{\Sigma_i} H dA = -\infty \qquad \text{et} \qquad \lim_{i\to+\infty} \frac{1}{2}\int_{\widetilde{\Sigma}_i} H dA = 0^{+}.$$

Il s'ensuit évidemment que:

$$\inf_{\substack{\Sigma \in C^{1,1} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_{\Sigma} H dA = -\infty \qquad \text{et} \qquad \inf_{\substack{\Sigma \in \mathcal{A}_{1,1} \\ A(\Sigma)=A_0}} \left|\frac{1}{2}\int_{\Sigma} H dA\right| = 0.$$

Par conséquent, le problème (1.15) n'a pas de solution dans la classe des $C^{1,1}$-surfaces (compactes simplement connexes) et il y a de bonnes raisons de penser que c'est aussi le cas pour la classe $\mathcal{A}_{1,1}$ mais nous n'avons pas été en mesure de le prouver.

Toutefois, bien que le problème (1.15) n'admet pas de minimiseur global, on peut facilement se convaincre que la sphère $\mathbb{S}_{A_0}$ d'aire A_0 est un minimum local de (1.15) dans la classe des C^2-surfaces (Remarque 11.1) et on peut aussi montrer que $\mathbb{S}_{A_0}$ est l'unique point critique de (1.15) dans la classe des C^3-surfaces (Théorème 11.3) en calculant la variation première de la courbure moyenne totale et celle de l'aire (Proposition 11.2).

En particulier, comme les sphères sont les seuls minimiseurs globaux de (1.2), on déduit que $\mathbb{S}_{A_0}$ est toujours un point critique de (1.4) parmi les C^3-surfaces (compactes simplement connexes). C'est aussi un minimiseur local (1.4) parmi les C^2-surfaces de $\mathbb{R}^3$ pour tout $H_0 < 0$. Tous ces résultats sont mentionnés dans le tableau 1.2.

Ainsi, nous avons été naturellement conduit à considérer un autre problème:

$$\inf_{\substack{\Sigma \in \mathcal{A}_{1,1} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_{\Sigma} |H| dA, \tag{1.16}$$

pour lequel on a prouvé la chose suivante.

Theorem 1.7. *Soit $\mathcal{A}_{1,1}$ la classe des $C^{1,1}$-surfaces axisymétriques de $\mathbb{R}^3$. Alors on a:*

$$\forall \Sigma \in \mathcal{A}_{1,1}, \quad \frac{1}{2}\int_{\Sigma} |H| dA \geqslant \sqrt{4\pi A(\Sigma)},$$

où l'égalité a lieu si et seulement si Σ est une sphère. En particulier, pour tout $A_0 > 0$, on a:

$$\frac{1}{2}\int_{\mathbb{S}_{A_0}} |H| dA = \min_{\substack{\Sigma \in \mathcal{A}_{1,1} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_{\Sigma} |H| dA = \sqrt{4\pi A_0},$$

et la sphère $\mathbb{S}_{A_0}$ d'aire A_0 est l'unique minimiseur global de ce problème.

On mentionne qu'en 1973, Michael et Simon établissent dans [68] une inégalité de type Sobolev pour des C^2-variétés m-dimensionelle de $\mathbb{R}^n$, pour laquelle le cas $m = 2$ et $n = 3$ avec $f \equiv 1$ donne l'inégalité suivante:

$$\frac{1}{2}\int_\Sigma |H| dA \geqslant c_0 \sqrt{A(\Sigma)}.$$

Plus précisément, la constante de l'inégalité ci-dessus est $c_0 = \frac{1}{4^3}\sqrt{4\pi}$ [68, Théorème 2.1]. Une meilleure constante $c_0 = \frac{1}{2}\sqrt{2\pi}$ est obtenue par Topping dans [91, Lemme 2.1] mais ne semble pas optimale. D'après le théorème 1.7, nous pensons que la constante optimale est $c_0 = \sqrt{4\pi}$.

On renvoie à l'appendice de [91] pour une preuve concise de l'inégalité ci-dessus utilisant des idées de Simon. On mentionne également [19, Théorème 3.1–3.2] pour une version pondérée de cette inégalité. Cependant, la constante obtenue est moins fine comme cela est mentionné dans le dernier paragraphe de [19, Section 3.2].

Classe de surfaces Σ considérée	**Enoncé**	**Preuve**
$C^{1,1}$ compactes d'intérieur convexe	$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)}$ (égalité ssi sphère)	[31, 70]
$C^{1,1}$ axisymétriques d'int. convexe	$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)}$ (égalité ssi sphère)	[10]
$C^{1,1}$ axiconvexes	$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)}$ (égalité ssi sphère)	Th 1.5
$C^{1,1}$ axisymétriques	$\inf_{A(\Sigma)=A_0} \left\lvert \frac{1}{2}\int_\Sigma HdA \right\rvert = 0$	Th 1.6
$C^{1,1}$ axisymétriques	$\frac{1}{2}\int_\Sigma HdA > 0$	OUVERT
$C^{1,1}$compactes simplement connexes	$\inf_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma HdA = -\infty$	Th 1.6
C^2 compactes simplement connexes	$\mathbb{S}_{A_0}$ minimiseur local de $\inf_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma HdA$	Rm 11.1
C^3 compactes simplement connexes	$\mathbb{S}_{A_0}$ seul point critique de $\inf_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma HdA$	Th 11.3
$C^{1,1}$ axisymétriques	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{4\pi A(\Sigma)}$ (égalité ssi sphère)	Th 1.7
C^2 compactes simplement connexes	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{\frac{\pi}{2}A(\Sigma)}$	[68, 91]
$C^{1,1}$compactes simplement connexes	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{4\pi A(\Sigma)}$ (égalité ssi sphère)	OUVERT

Table 1.3: minimiser $\int H$ ou $\int |H|$ avec une contrainte d'aire.

Finalement, on a résumé dans le tableau 1.3 ci-dessus plusieurs résultats et questions ouvertes concernant les problèmes (1.15) et (1.16) (le terme *d'intérieur convexe* renvoie à une surface fermée qui délimite un domaine convexe). La partie s'organise de la façon suivante. Dans le chapitre 8, on rappelle les notations et les définitions de base d'une surface, d'axisymétrie et d'axiconvexité. Dans les chapitres 9 et 10, on prouve respectivement les théorèmes 1.5 et 1.6. Le chapitre 11 étudie l'optimalité de la sphère pour le problème (1.15) puis le théorème 1.7 est démontré dans le chapitre 12. Finalement, l'inégalité de Minkowski (1.14) est établie au chapitre 13 et nous obtenons quelques propriétés des réarrangements croissants dans le chapitre 14.

Troisième partie: condition d'ε-boule et existence de formes optimales pour une large classe de fonctionnelles géométriques

En utilisant le point de vue de l'optimisation de formes, le but de cette partie est d'introduire une classe raisonnable de surfaces, pour laquelle l'existence d'un minimiseur suffisamment régulier est assurée pour des fonctionnelles et des contraintes assez générales faisant intervenir les propriétés d'ordre un et deux des surfaces. En s'inspirant de ce que fit Chenais dans [20] quand elle considéra la propriété de cône uniforme, nous introduisons ici les (hyper-)surfaces satisfaisant une condition de boule uniforme dans le sens suivant.

Definition 1.8. *Soient $\varepsilon > 0$ et $B \subseteq \mathbb{R}^n$ un ouvert, $n \geqslant 2$. On dit qu'un ouvert $\Omega \subseteq B$ vérifie la condition ε-boule et on écrit $\Omega \in \mathcal{O}_\varepsilon(B)$ si pour tout $\mathbf{x} \in \partial\Omega$, il existe un vecteur unitaire $\mathbf{d_x}$ de $\mathbb{R}^n$ tel que:*

$$\begin{cases} B_\varepsilon(\mathbf{x} - \varepsilon \mathbf{d_x}) \subseteq \Omega \\ \\ B_\varepsilon(\mathbf{x} + \varepsilon \mathbf{d_x}) \subseteq B \backslash \overline{\Omega}, \end{cases}$$

où $B_r(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^n,\ \|\mathbf{y} - \mathbf{z}\| < r\}$ est la boule ouverte de $\mathbb{R}^n$ centrée en $\mathbf{z}$ de rayon r, et où $\overline{\Omega}$ désigne l'adhérence de Ω, $\partial\Omega = \overline{\Omega} \backslash \Omega$ sa frontière.

La condition de boule uniforme (extérieure/intérieure) est déjà considérée par Poincaré en 1890 [78]. Comme l'illustre la figure 1.1, elle empêche la formation de singularités telles que les coins, les fractures ou les auto-intersections. En fait, elle est connue pour caractériser la régularité $C^{1,1}$ des hypersurfaces depuis longtemps par tradition orale, et également la stricte positivité du *reach*, une notion introduite par Federer dans [32]. Nous n'avons pas trouvé de référence précise où ces deux caractérisations étaient clairement énoncées, prouvées et rassemblées. Elles sont donc établies dans le chapitre 16, reproduisant un proceeding accepté intitulé *some characterizations of a uniform ball property* [22]. On renvoie aux théorèmes 16.5–16.6 pour des énoncés précis.

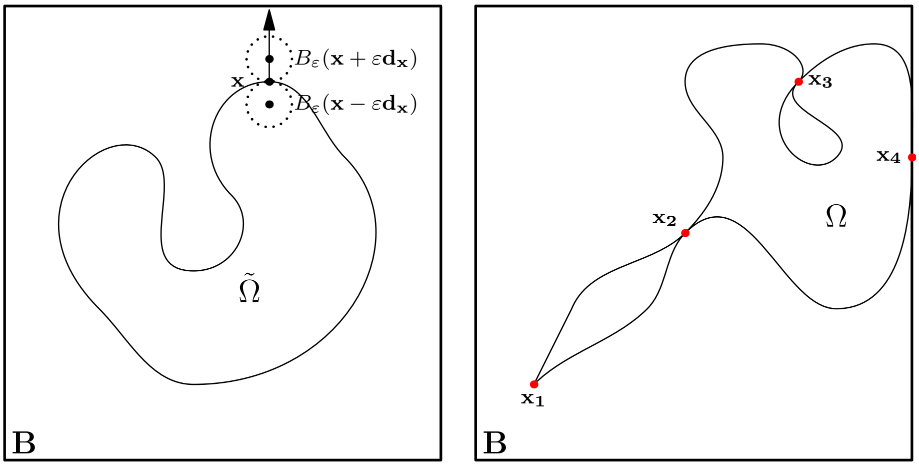

Figure 1.1: exemple d'un ouvert $\tilde{\Omega}$ de $\mathbb{R}^2$ qui vérifie la condition d'ε-boule tandis que l'ouvert Ω ne la satisfait pas. En effet, il n'existe aucun cercle passant par le point $\mathbf{x_1}$ ou $\mathbf{x_2}$ (respectivement $\mathbf{x_3}$ ou $\mathbf{x_4}$) dont l'intérieur est inclus dans Ω (respectivement dans $B \backslash \overline{\Omega}$).

Muni de cette classe de formes admissibles, on peut maintenant énoncer notre principal résultat général d'existence dans l'espace euclidien tridimensionnel $\mathbb{R}^3$. On renvoie au théorème 18.28 pour sa forme la plus générale dans $\mathbb{R}^n$, mais celle-ci est suffisante pour les trois applications physiques que nous présentons ci-après (d'autres exemples sont aussi détaillés dans la section 18.5).

Theorem 1.9. *Soit $\varepsilon > 0$ et $B \subset \mathbb{R}^3$ une boule ouverte de rayon suffisamment grand. On considère $(C, \widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, cinq applications continues j_0, f_0, g_0, g_1, $g_2 : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ et quatre applications continues j_1, j_2, f_1, $f_2 : \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ qui sont convexes en leur dernière variable. Alors, le problème suivant admet au moins une solution (voir les notations 1.10):*

$$\inf \int_{\partial\Omega} j_0 \left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1 \left[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_2 \left[\mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

où l'infimum est pris parmi tous les $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfaisant les contraintes:

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0 \left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1 \left[\mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_2 \left[\mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ \\ \displaystyle\int_{\partial\Omega} g_0 \left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1 \left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} K(\mathbf{x})\, g_2 \left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

La preuve du théorème 1.9 repose uniquement sur des outils basiques d'analyse et ne fait pas intervenir ceux de la théorie géométrique de la mesure. On mentionne également que le cas particulier $j_0 \geqslant 0$ et $j_1 = j_2 = 0$ sans contrainte a été obtenu en parallèle à nos travaux dans [40].

Notation 1.10. *On rappelle qu'on note $A(\bullet)$ (respectivement $V(\bullet)$) l'aire (resp. le volume), c'est-à-dire la mesure de Hausdorff bi(resp. tri-)dimensionnelle. L'intégration sur une surface est toujours effectuée par rapport à A. L'application de Gauss $\mathbf{n} : \mathbf{x} \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^2$ renvoie toujours au champ normal extérieur unitaire à la surface, tandis que $H = \kappa_1 + \kappa_2$ désigne la courbure moyenne scalaire et $K = \kappa_1\kappa_2$ celle de Gauss.*

Remark 1.11. *Dans le théorème ci-dessus, le rayon de B est pris suffisamment grand pour éviter que $\mathcal{O}_\varepsilon(B)$ ne soit vide. De plus, les hypothèses sur B peuvent être relaxées en supposant seulement que B soit un ouvert borné non-vide, suffisamment régulier (lipschitzien par exemple) pour que la mesure de Lebesgue tridimensionnelle de son bord soit nulle, et suffisamment gros pour contenir au moins une boule de rayon 3ε. Finalement, pour tout ensemble E, on rappelle qu'une application $j : E \times \mathbb{R} \to \mathbb{R}$ est qualifiée de convexe en sa dernière variable si pour tout $(\mathbf{x}, t, \tilde{t}) \in E \times \mathbb{R} \times \mathbb{R}$ et pour tout $\mu \in [0, 1]$, on a $j(\mathbf{x}, \mu t + (1-\mu)\tilde{t}) \leqslant \mu j(\mathbf{x}, t) + (1-\mu) j(\mathbf{x}, \tilde{t})$.*

Première application: minimiser l'énergie de Canham-Helfrich sous des contraintes d'aire et de volume

On rappelle que l'énergie de Canham-Helfrich (1.1) est un modèle simple pour décrire une vésicule. En imposant l'aire de la bicouche et le volume de fluide qu'elle contient, leur forme est un minimiseur pour l'énergie:

$$\mathcal{E}(\Sigma) = \frac{k_b}{2} \int_\Sigma (H - H_0)^2 \, dA + k_G \int_\Sigma K dA, \tag{1.17}$$

où la courbure spontanée $H_0 \in \mathbb{R}$ mesure l'asymétrie entre les deux couches, et où $k_b > 0$, $k_G < 0$ sont deux autres constantes physiques. Remarquons que si $k_G > 0$, pour tout $k_b, H_0 \in \mathbb{R}$, l'énergie de Canham-Helfrich (1.17) à aire A_0 et volume V_0 fixés n'est pas bornée inférieurement. En effet, dans ce cas, d'après le théorème de Gauss-Bonnet, le second terme $k_G \int K dA = 4\pi k_G(1-g)$ tend vers $-\infty$ quand le genre $g \to +\infty$, alors que le premier terme reste borné par $4|k_b|(12\pi + \frac{1}{4}H_0^2 A_0)$. Pour voir ce dernier point, il suffit d'utiliser [53, Remarque 1.7 (iii) (1.5)], [84, Théorème 1.1], et [88, Inégalité (0.2)] pour obtenir successivement:

$$\begin{aligned}
\inf_{\substack{A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \mathcal{E}(\partial\Omega) &\leqslant 4|k_b| \left(\inf_{\substack{A(\partial\Omega)=A_0\\ V(\Omega)=V_0\\ \text{genre}(\partial\Omega)=g}} \mathcal{W}(\partial\Omega) + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g) \\
&\leqslant 4|k_b| \left(\inf_{\text{genre}(\partial\Omega)=g} \mathcal{W}(\partial\Omega) + \inf_{\substack{A(\partial\Omega)=A_0\\ V(\Omega)=V_0\\ \text{genre}(\partial\Omega)=0}} \mathcal{W}(\partial\Omega) - 4\pi + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g) \\
&\leqslant 4|k_b| \left(8\pi + 8\pi - 4\pi + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g).
\end{aligned}$$

Le cas bidimensionnel de (1.17) est considéré par Bellettini, Dal Maso et Paolini dans [5]. Une partie de leurs résultats est retrouvée par Delladio [24] dans le cadre des graphes de Gauss spéciaux généralisés issus de la théorie des courants. Ensuite, Choksi et Veneroni [21] ont résolu le cas axisymétrique (1.17) en supposant $-2k_b < k_G < 0$. Dans le cas général, cette hypothèse assure une propriété de coercivité fondamentale [21, Lemme 2.1]: l'intégrande de (1.17) est standard au sens de [48, Définition 4.1.2]. Ainsi, il existe un minimiseur pour (1.17) dans la classe des 2-varifolds entiers rectifiables orientables de $\mathbb{R}^3$ ayant une seconde forme fondamentale généralisée L^2-bornée [48, Théorème 5.3.2] [72, Section 2] [6, Appendice]. Ces propriétés de compacité et semi-continuité inférieure sont déjà mises en évidence dans [6, Section 9.3].

Cependant, la régularité des minimiseurs reste un problème ouvert et des expériences *in vitro* montrent que des comportements singuliers de vésicules peuvent apparaître comme le phénomène de bourgeonnement [85, 86]. Quand la température augmente, une vésicule initialement sphérique devient un ellipsoïde allongé, puis elle prend la forme d'une poire avec une rupture de symétrie entre le haut et le bas. Finalement, le nez du rétrécissement se referme et il en résulte deux compartiments sphériques assis l'un sur l'autre mais toujours connectés par une étroite constriction [85, Section 1.1, Figure 1]. Cela ne peut se produire pour un globule rouge car son squelette empêche la membrane de trop se courber localement [59, Section 2.1]. Afin de prendre en compte cet aspect, la condition d'ε-boule est aussi motivée par la modélisation des formes d'équilibre des globules rouges. Nous avons même une idée de l'ordre de grandeur pour la valeur possible de ε [59, Section 2.1.5]. Notre résultat s'énonce ainsi.

Theorem 1.12. *Soit $H_0, k_G \in \mathbb{R}$ et $\varepsilon, k_b, A_0, V_0 > 0$ tels que $A_0^3 > 36\pi V_0^2$. Alors, le problème suivant admet au moins une solution (voir les notations 1.10):*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(\mathbb{R}^n)\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + k_G\int_{\partial\Omega} K dA.$$

Remark 1.13. *D'après l'inégalité isopérimétrique, si $A_0^3 < 36\pi V_0^2$, alors aucun $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ ne peut satisfaire les deux contraintes; et si l'égalité a lieu, la seule forme admissible est la boule d'aire A_0 et volume V_0. De plus, dans le théorème ci-dessus, remarquons qu'on n'a pas supposé $\Omega \in \mathcal{O}_\varepsilon(B)$ comme c'est le cas pour le théorème 1.9 car une borne uniforme sur le diamètre est déjà donnée par la fonctionnelle et la contrainte d'aire [88, Lemme 1.1]. Pour finir, le résultat ci-dessus reste vrai si H_0 est une fonction continue de la position et de la normale.*

Deuxième application: minimiser l'énergie de Helfrich à genre, aire et volume fixés

Comme le théorème de Gauss-Bonnet est vrai pour les ensembles de *reach* strictement positif [32, Théorème 5.19], on déduit des théorèmes 16.5–16.6 que $\int_\Sigma K dA = 4\pi(1-g)$ pour toute $C^{1,1}$-surface compacte connexe Σ (sans bord plongé dans $\mathbb{R}^3$) de genre $g \in \mathbb{N}$. Ainsi, au lieu de minimiser (1.17), on fixe souvent la topologie et on cherche un minimiseur pour l'énergie de Helfrich:

$$\mathcal{H}(\Sigma) = \frac{1}{4}\int_\Sigma (H - H_0)^2 \, dA, \tag{1.18}$$

à aire, genre et volume intérieur fixés. Comme (1.17), une telle fonctionnelle dépend de la surface mais aussi de son orientation. Toutefois, dans le cas $H_0 \neq 0$, l'énergie (1.18) n'est même pas continue inférieurement pour la convergence au sens des varifolds [6, Section 9.3]: le contrexemple est dû à Große-Brauckmann [38]. Dans le cadre de la condition de boule uniforme, on prouve:

Theorem 1.14. *Soient $H_0 \in \mathbb{R}$, $g \in \mathbb{N}$ et $\varepsilon, A_0, V_0 > 0$ tels que $A_0^3 > 36\pi V_0^2$. Alors, le problème suivant admet au moins une solution (voir les notations 1.10 et la remarque 1.13):*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(\mathbb{R}^n)\\ \mathrm{genre}(\partial\Omega)=g\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \int_{\partial\Omega}(H-H_0)^2 dA,$$

où la contrainte $\mathrm{genre}(\partial\Omega) = g$ *signifie $\partial\Omega$ est une $C^{1,1}$-surface compacte connexe de genre g.*

Troisième application: minimiser l'énergie de Willmore sous contraintes

Le cas particulier $H_0 = 0$ dans (1.18) est connu sous le nom de fonctionnelle de Willmore:

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_\Sigma H^2 dA. \tag{1.19}$$

Elle est beaucoup étudiée par les géomètres. Sans contrainte, Willmore [93, Théorème 7.2.2] a prouvé que les sphères sont les seuls minimiseurs globaux de (1.19). L'existence est établie par Simon [88] pour les surfaces de genre un, Bauer et Kuwert [4] pour celles de genre plus élevé. Récemment, Marques et Neves [66] ont résolu la conjecture dite de Willmore: les transformations conformes de la projection stéréographique du tore de Clifford sont les seuls minimiseurs globaux de (1.19) parmi les surfaces régulières de genre un.

Un des principaux ingrédients est l'invariance conforme de (1.19), à partir de laquelle on montre en particulier que minimiser (1.19) à ratio isopérimétrique fixé revient à imposer l'aire et le volume intérieur. Dans cette direction, Schygulla [84] établit l'existence d'un minimiseur pour (1.19) parmi les surfaces analytiques de genre zéro à ratio isopérimétrique fixé. Concernant des genres plus élevés, Keller, Mondino et Rivière [53] ont récemment obtenu des résultats similaires, en utilisant le point de vue des immersions développé par Rivière [79] pour caractériser précisément les points critiques de (1.19) ainsi que leur régularité. Notre résultat sur les ε-boules peut encore être ici utilisé pour prouver un résultat concernant (1.19). Il est connu sous le nom de modèle du couple-bicouche [85, Section 2.5.3] et il s'énonce ainsi.

Theorem 1.15. *Soient $M_0 \in \mathbb{R}$ et $\varepsilon, A_0, V_0 > 0$ tels que $A_0^3 > 36\pi V_0^2$. Alors, le problème suivant admet au moins une solution (voir les notations 1.10 et la remarque 1.13):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n) \\ \text{genre}(\partial\Omega)=g \\ A(\partial\Omega)=A_0 \\ V(\Omega)=V_0 \\ \int_{\partial\Omega} H dA = M_0}} \frac{1}{4}\int_{\partial\Omega} H^2 dA.$$

Cette partie s'organise de la manière suivante. Dans le chapitre 16, on énonce précisément les deux caractérisations associées à la condition de boule uniforme, en termes de *reach* strictement positif (théorème 16.5) et en termes de régularité $C^{1,1}$ (théorème 16.6). Puis on démontre les deux théorèmes, comme dans [22].

On suit alors la méthode classique issue du calcul des variations. On obtient d'abord dans la Section 17.1 la compacité de la classe $\mathcal{O}_\varepsilon(B)$ pour divers modes de convergence. Cela provient essentiellement du fait que la condition d'ε-boule implique une propriété de cône uniforme pour laquelle on a déjà des propriétés de compacité.

Puis, dans le chapitre 17, dans un repère local fixe, on effectue simultanément la paramétrisation par graphe des bords associés à une suite convergente dans $\mathcal{O}_\varepsilon(B)$ et on prouve la C^1-convergence forte ainsi que la $W^{2,\infty}$-convergence faible-étoile de ces graphes.

Finalement, au chapitre 18, on montre comment combiner ce résultat local avec une partition de l'unité adéquate afin d'obtenir la continuité globale de fonctionnelles géométriques générales. D'une manière générale, la preuve consiste toujours à exprimer l'intégrale dans la paramétrisation puis à montrer que l'intégrande est le produit d'un terme convergeant L^∞-faible-étoile avec un terme convergeant L^1-fortement. On conclut avec la Section 18.5 en donnant divers résultats d'existence et en y détaillant plusieurs applications.

Quatrième partie: résultats d'existence pour des fonctionnelles géométriques dépendant de la solution d'une équation d'état

Cette partie étend les résultats d'existence obtenus précédemment à des fonctionnelles géométriques générales dépendant également de la forme à travers les solutions de certains problèmes aux limites elliptiques d'ordre deux posés sur le domaine intérieur à la forme. On présente ici leurs versions tridimensionnelles et on renvoie au chapitre 23 pour des énoncés généraux dans $\mathbb{R}^n$.

Une dépendance à travers la solution du Laplacien Dirichlet

Pour tout domaine $\Omega \in \mathcal{O}_\varepsilon(B)$, le bord associé $\partial\Omega$ est de classe $C^{1,1}$ (cf. théorème 16.6). On peut donc considérer l'unique solution $u_\Omega \in H_0^1(\Omega,\mathbb{R}) \cap H^2(\Omega,\mathbb{R})$ du Laplacien Dirichlet posé sur un domaine $\Omega \in \mathcal{O}_\varepsilon(B)$ avec $f \in L^2(B,\mathbb{R})$ [37, Section 2.1 et Théorème 2.4.2.5]:

$$\begin{cases} -\Delta u_\Omega = f & \text{dans } \Omega \\ u_\Omega = 0 & \text{sur } \partial\Omega. \end{cases} \tag{1.20}$$

De plus, on dit que les applications $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ et $g : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ ont une croissance quadratique en leur première variable s'il existe une constante $c > 0$ telle que:

$$\forall (\mathbf{z},\mathbf{x},\mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \quad |f(\mathbf{z},\mathbf{x},\mathbf{y})| \leqslant c\left(1+\|\mathbf{z}\|^2\right) \tag{1.21}$$

$$\forall (\mathbf{z},\mathbf{x},\mathbf{y},t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}, \quad |g(\mathbf{z},\mathbf{x},\mathbf{y},t)| \leqslant c\left(1+\|\mathbf{z}\|^2\right). \tag{1.22}$$

Puis, on prouve l'extension suivante du théorème 1.9.

Theorem 1.16. *Soient $\varepsilon > 0$ et $B \subset \mathbb{R}^3$ une boule ouverte de rayon suffisamment grand. On considère $(C, \widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, cinq applications continues j_0, f_0, g_0, g_1, $g_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ ayant une croissance quadratique* (1.21) *en leur première variable, ainsi que quatre autres applications continues j_1, j_2, f_1, $f_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ ayant une croissance quadratique* (1.22) *en leur première variable et qui sont convexes en leur dernière variable. Alors, le problème suivant admet au moins une solution (voir les notations 1.10 et la remarque 1.11):*

$$\inf \int_{\partial\Omega} j_0 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_2 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

où $u_\Omega \in H^1_0(\Omega,\mathbb{R}) \cap H^2(\Omega,\mathbb{R})$ est l'unique solution de (1.20) *avec $f \in L^2(B,\mathbb{R})$, et où l'infimum est pris parmi tous les $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfaisant les contraintes:*

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} f_2 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ \displaystyle\int_{\partial\Omega} g_0 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} K(\mathbf{x})\, g_2 \left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

Dans le théorème ci-dessus, si on note $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ la fonctionnelle à minimiser, remarquons qu'elle est bien définie puisque d'après la croissance quadratique (1.21)–(1.22) des applications et d'après la continuité de l'opérateur trace $H^2(\Omega,\mathbb{R}) \to H^1(\partial\Omega,\mathbb{R})$, on a:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant c\left[A(\partial\Omega) + \|\nabla u_\Omega\|^2_{L^2(\partial\Omega,\mathbb{R}^3)}\right] \leqslant \tilde{c}\left[A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

On démontre le théorème 1.16 de la même manière que le théorème 1.9. Tout d'abord, on considère une suite minimisante $(\Omega_i)_{i\in\mathbb{N}}$ et par compacité, on obtient une sous-suite convergente. Puis, on effectue simultanément une paramétrisation par graphe d'applications $C^{1,1}$ notées $(\varphi_i)_{i\in\mathbb{N}}$ des bords associés à la sous-suite convergente de domaines. De plus, d'après la partie précédente, $(\varphi_i)_{i\in\mathbb{N}}$ converge C^1-fortement et $W^{2,\infty}$-faible-étoile. En utilisant une partition de l'unité adéquate, on exprime la fonctionnelle et les contraintes dans cette paramétrisation locale. Par conséquent, il reste à montrer qu'on peut faire correctement tendre $i \to +\infty$.

De manière générale, chaque intégrande obtenu est le produit d'un terme convergeant L^∞-faible-étoile avec un autre, pour lequel on veut appliquer le théorème de convergence dominée de Lebesgue afin d'obtenir sa L^1-convergence forte. Ainsi, pour pouvoir faire tendre $i \to +\infty$, on a besoin de la convergence presque partout et d'une borne uniforme intégrable pour chaque intégrande. Grâce aux hypothèses de croissance quadratique (1.21)–(1.22), ceci est vrai si l'application locale $\mathbf{x}' \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ converge L^2-fortement vers $\mathbf{x}' \mapsto \nabla u_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))$. Dans le chapitre 21, on démontre cette dernière assertion.

Une dépendance à travers la solution du Laplacien Neumann/Robin

Dans cette partie, il reste à étendre les résultats précédents pour des conditions de bord de type Neumann/Robin. Pour tout domaine $\Omega \in \mathcal{O}_\varepsilon(B)$, comme $\partial\Omega$ est de classe $C^{1,1}$ (théorème 16.6), il existe une unique solution $v_\Omega \in H^2(\Omega,\mathbb{R})$ au problème [37, Section 2.1 et Théorème 2.4.2.7]:

$$\begin{cases} -\Delta v_\Omega + \lambda v_\Omega = f & \text{dans } \Omega \\ \partial_n(v_\Omega) = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1.23}$$

où $\lambda > 0$ et $f \in L^2(B,\mathbb{R})$. De plus, il existe une unique solution $\tilde{v}_\Omega \in H^2(\Omega,\mathbb{R})$ au problème suivant [37, Section 2.1 et Théorème 2.4.2.6]:

$$\left\{\begin{array}{ll} -\Delta\tilde{v}_\Omega = f & \text{dans } \Omega \\ & \\ -\partial_n(\tilde{v}_\Omega) = \lambda\tilde{v}_\Omega & \text{sur } \partial\Omega. \end{array}\right. \tag{1.24}$$

où $\lambda > 0$ et $f \in L^2(B,\mathbb{R})$. Par ailleurs, si l'existence d'une solution unique dans $H^2(\Omega,\mathbb{R})$ est assurée, on est aussi capable de traiter dans (1.24) des conditions de bord non-linéaires de la forme $-\partial_n(\tilde{v}_\Omega) = \beta(\tilde{v}_\Omega)$, où $\beta : \mathbb{R} \to \mathbb{R}$ est une application lipschitzienne croissante satisfaisant $\beta(0) = 0$. Notons que si $\beta(x) = \lambda x$, on obtient (1.24) et que (1.23) est donnée par $\beta(x) = 0$.

De plus, on dit que les applications $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ et $g : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ ont une croissance quadratique en leurs deux premières variables s'il existe une constante $c > 0$ telle que:

$$\forall (s,\mathbf{z},\mathbf{x},\mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \quad |f(s,\mathbf{z},\mathbf{x},\mathbf{y})| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2\right) \tag{1.25}$$

$$\forall (s,\mathbf{z},\mathbf{x},\mathbf{y},t) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}, \quad |g(s,\mathbf{z},\mathbf{x},\mathbf{y},t)| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2\right). \tag{1.26}$$

On prouve alors le résultat suivant.

Theorem 1.17. *Soient $\varepsilon > 0$ et $B \subset \mathbb{R}^3$ une boule ouverte de rayon suffisamment grand. On considère $(C,\widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, cinq applications continues j_0, f_0, g_0, g_1, $g_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ ayant une croissance quadratique (1.25) en leurs deux premières variables, et quatre applications continues j_1, j_2, f_1, $f_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ ayant une croissance quadratique (1.26) en leurs deux premières variables et qui sont convexes en leur dernière variable. Alors, le problème suivant admet au moins une solution (voir les notations 1.10 et la remarque 1.11):*

$$\begin{aligned} \inf \int_{\partial\Omega} j_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ + \int_{\partial\Omega} j_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}), \end{aligned}$$

où $v_\Omega \in H^2(\Omega,\mathbb{R})$ est l'unique solution de soit (1.23) soit (1.24) avec $f \in L^2(B,\mathbb{R})$ ainsi que $\lambda > 0$, et où l'infimum est pris parmi tous les $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfaisant les contraintes:

$$\left\{\begin{aligned} &\int_{\partial\Omega} f_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ &\qquad + \int_{\partial\Omega} f_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ &\int_{\partial\Omega} g_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ &\qquad + \int_{\partial\Omega} K(\mathbf{x})\, g_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{aligned}\right.$$

Dans le théorème ci-dessus, si on note $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ la fonctionnelle à minimiser, remarquons que celle-ci est bien définie puisque d'après la croissance quadratique (1.25)–(1.26) des applications et d'après la continuité de l'opérateur trace $H^2(\Omega,\mathbb{R}) \to H^1(\partial\Omega,\mathbb{R})$, on a:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant c\left[A(\partial\Omega) + \|v_\Omega\|^2_{H^1(\partial\Omega,\mathbb{R})}\right] \leqslant \tilde{c}\left[A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

On démontre le théorème 1.17 avec la même méthode que celle utilisée pour le théorème 1.16 et décrite dans la section précédente. La principale tâche est de montrer que l'application locale $\mathbf{x}' \mapsto v_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ converge H^1-fortement vers $\mathbf{x}' \mapsto v_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))$. Dans le chapitre 22, on prouve cette dernière assertion.

Première application: des fonctionnelles quadratiques sur le domaine faisant intervenir la hessienne du Laplacien Dirichlet/Neumann/Robin

Dans cette thèse, notons que nous avons jusqu'à maintenant traité le cas de fonctionnelles contenant des intégrales de bord. En effet, la situation où le domaine d'intégration correspond à celui de (1.20) ou (1.23)-(1.24) comme:

$$\int_{\Omega} j\left[\mathbf{x}, u_{\Omega}(\mathbf{x}), \nabla u_{\Omega}(\mathbf{x})\right] dV(\mathbf{x}),$$

est standard dans le cadre de la propriété de cône uniforme [46, Section 4.3]. Comme la condition d'ε-boule implique une propriété de cône uniforme (cf. le point (i) du théorème 16.6), nous n'avons pas considéré de telles fonctionnelles pour le moment. Cependant, la classe $\mathcal{O}_{\varepsilon}(B)$ devient intéressante si des dérivées d'ordre deux de u_{Ω} apparaissent dans l'intégrande ci-dessus. Nos résultats s'énoncent de la façon suivante. On dit qu'une application $j : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2} \rightarrow \mathbb{R}$ a une croissance quadratique en ses trois dernières variables s'il existe une constante $c > 0$ telle que:

$$\forall(\mathbf{x}, s, \mathbf{z}, Y) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2}, \quad |j(\mathbf{x}, s, \mathbf{z}, Y)| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2 + \|Y\|^2\right), \tag{1.27}$$

où la norme considérée sur l'ensemble des (3×3)-matrices est celle de Frobenius, c'est-à-dire $\|Y\| = \sqrt{\mathrm{trace}([Y]^T Y)}$.

Theorem 1.18. *Soient $\varepsilon > 0$ et $B \subset \mathbb{R}^3$ une boule ouverte de rayon suffisamment grand. On considère $(C, \widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, trois applications mesurables $j_0, f_0, g_0 : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2} \rightarrow \mathbb{R}$ ayant une croissance quadratique (1.27) en leurs trois dernières variables, et continue en $(s, \mathbf{z}, Y)$ pour presque tout $\mathbf{x}$, cinq applications continues j_1, f_1, g_1, g_2, $g_3 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \rightarrow \mathbb{R}$ ayant une croissance quadratique (1.25) en leurs deux premières variables, et quatre applications continues j_2, j_3, f_2, $f_3 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ ayant une croissance quadratique (1.26) en leurs deux premières variables et qui sont convexes en leur dernière variable. Alors, le problème suivant admet au moins une solution (voir les notations 1.10 et la remarque 1.11):*

$$\inf \int_{\Omega} j_0\left[\mathbf{x}, v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathrm{Hess}\ v_{\Omega}(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} j_1\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \ + $$
$$\int_{\partial\Omega} j_2\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_3\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

où $v_{\Omega} \in H^2(\Omega, \mathbb{R})$ est l'unique solution de soit (1.20) soit (1.23) soit (1.24) avec $f \in L^2(B, \mathbb{R})$ ainsi que $\lambda > 0$, et où l'infimum est pris parmi tous les $\Omega \in \mathcal{O}_{\varepsilon}(B)$ satisfaisant les contraintes:

$$\begin{cases} C \geqslant \displaystyle\int_{\Omega} f_0\left[\mathbf{x}, v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathrm{Hess}\ v_{\Omega}(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} f_1\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \\ \displaystyle\int_{\partial\Omega} f_2\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_3\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \\ \\ \widetilde{C} = \displaystyle\int_{\Omega} g_0\left[\mathbf{x}, v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathrm{Hess}\ v_{\Omega}(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} g_1\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \\ \displaystyle\int_{\partial\Omega} H(\mathbf{x})\, g_2\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} K(\mathbf{x}) g_3\left[v_{\Omega}(\mathbf{x}), \nabla v_{\Omega}(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}). \end{cases}$$

Encore une fois, dans le théorème ci-dessus, si on note $J : \mathcal{O}_{\varepsilon}(B) \rightarrow \mathbb{R}$ la fonctionnelle à minimiser, remarquons que celle-ci est bien définie puisque d'après la croissance quadratique (1.25)–(1.27) des applications et d'après la continuité de l'opérateur trace $H^2(\Omega, \mathbb{R}) \rightarrow H^1(\partial\Omega, \mathbb{R})$, on a:

$$\forall \Omega \in \mathcal{O}_{\varepsilon}(B), \quad |J(\Omega)| \ \leqslant\ \tilde{c}\left[V(\Omega) + A(\partial\Omega) + \|u\|^2_{H^2(\Omega, \mathbb{R})}\right] < +\infty.$$

On observe également que l'énoncé ci-dessus traite le cas où l'intégration n'est pas effectuée sur tout le domaine Ω mais sur seulement une partie mesurable $\widetilde{\Omega} \subseteq \Omega$. En effet, il suffit d'introduire la fonction caractéristiques $\mathbf{1}_{\widetilde{\Omega}}$ dans l'intégrande j_0. Ceci ne peut être fait pour les intégrales de bord mais des fonctions plateaux continues peuvent toujours être utilisées. Finalement, la formulation adoptée ci-dessus permet de considérer des contraintes de la forme $K \subset \Omega$ pour un compact donné $K \subset B$ en posant $\widetilde{C} = V(K)$, $g_0 = \mathbf{1}_K$ et $g_1 = g_2 = g_3 = 0$.

Deuxième application: des problèmes d'identification de bord

Soient $\varepsilon > 0$ et B un ouvert comme dans la remarque 1.11. On considère $\Omega_0 \in \mathcal{O}_\varepsilon(B)$, un sous-ensemble $\Gamma_0 \subseteq \partial\Omega_0$ et $g_0 \in L^2(\Gamma_0, \mathbb{R})$. On imagine qu'on a de bonnes raisons de penser que g_0 est la restriction à Γ_0 de la dérivée normale d'une solution u_Ω au Laplacien Dirichlet (1.20) posé sur un domaine inconnu $\Omega \in \mathcal{O}_\varepsilon(B)$ tel que $\Gamma_0 \subseteq \partial\Omega$. Afin de trouver le *meilleur* $\Omega \in \mathcal{O}_\varepsilon(B)$ tel que $\partial_n(u_\Omega)|_{\partial\Omega_0} = g_0$, une possibilité est de résoudre le problème suivant:

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(B)\\ \Gamma_0\subseteq\partial\Omega}} \int_{\Gamma_0} \left[\partial_n\left(u_\Omega\right) - g_0\right]^2 dA. \tag{1.28}$$

De la même manière, si on suspecte $f_0 \in L^2(\Gamma_0, \mathbb{R})$ d'être la restriction à Γ_0 d'une solution v_Ω au Laplacien Neumann/Robin (1.23)-(1.24) posé sur une domaine inconnu $\Omega \in \mathcal{O}_\varepsilon(B)$ tel que $\Gamma_0 \subseteq \partial\Omega$, alors on doit résoudre:

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(B)\\ \Gamma_0\subseteq\partial\Omega}} \int_{\Gamma_0} \left(v_\Omega - f_0\right)^2 dA \tag{1.29}$$

On peut évidemment construire des fonctionnelles plus compliquées mais la principale difficulté ici est que le domaine d'intégration n'est pas toute la surface. On prouve le résultat suivant.

Proposition 1.19. *Soient $\Omega_0 \in \mathcal{O}_\varepsilon(B)$ et Γ_0 un sous-ensemble mesurable de $\partial\Omega_0$. Alors, le théorème 1.18 reste vrai si on ajoute la contrainte $\Gamma_0 \subseteq \partial\Omega$ et si le domaine d'intégration $\partial\Omega$ de la fonctionnelle et des contraintes est restreint à Γ_0. En particulier, les problèmes* (1.28)–(1.29) *possède un minimiseur.*

L'identification d'une forme par son bord comme dans (1.28)–(1.29) apparaît souvent dans les problèmes inverses et de contrôle optimal. Par exemple, on peut essayer de détecter une tumeur dans le cerveau. On place des électrodes sur le tête Γ_0 du patient. On mesure une certaine activité électrique g_0, puis on résout le problème (1.29). S'il n'y a pas de tumeur, alors le minimum est nul et la forme optimale correspond à Γ_0, autrement c'est $\Gamma_0 \cup \Gamma_1$, où Γ_1 est le bord de la tumeur.

Troisième application: le modèle de sac du MIT en physique quantique relativiste

Durant la conférence MODE 2014 à l'INSA-Rennes, Le Treust a fait un exposé sur les travaux de sa thèse [56]. Il a étudié des problèmes d'optimisation de formes provenant de la physique quantique relativiste. En particulier, les modèles de sac sont introduits pour étudier la structure interne des hadrons. L'énergie de ces particules est obtenue en sommant celle des quarks et des anti-quarks présents à l'intérieur du sac.

Dans le modèle du sac du MIT, les fonctions d'onde des quarks sont les vecteurs propres de l'opérateur de Dirac. Ainsi, le problème de l'état fondamental correspond à la minimisation à volume fixé de la première valeur propre strictement positive associée à l'opérateur de Dirac parmi les ouverts non-vides bornés de $\mathbb{R}^3$ ayant un bord de classe C^2. L'existence d'une forme optimale est actuellement ouverte.

Nous n'avons pas étudié ce problème mais il semble que le cadre de la condition d'ε-boule pourrait être utilisée afin d'approcher l'état fondamental du modèle de sac du MIT:

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(B)\\ V(\Omega)=V_0}} \lambda_1^{\mathrm{MIT}}(\Omega)$$

avec

$$\lambda_1^{\mathrm{MIT}}(\Omega) = \inf_{\substack{u\in H^1(\Omega,\mathbb{C}^2)\\ \int_\Omega |u|^2=1\\ -(\sigma.\mathbf{n}_{\partial\Omega})u=u \text{ sur } \partial\Omega}} \sqrt{m^2 + \int_\Omega \|\nabla u\|^2 + \int_{\partial\Omega}\left(m + \frac{H_{\partial\Omega}}{2}\right)|u|^2 dA},$$

où $m > 0$ est un paramètre donné fixé (la masse de la particule) et où $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ est un vecteur formé par les trois (2×2)-matrices de Pauli:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{et} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

La principale difficulté vient de la condition de bord non-linéaire associée au problème de valeurs propres $-(\sigma.\mathbf{n}_{\partial\Omega})u = u$ qui doit être comprise comme $-(\sigma_1\mathbf{n}_1 + \sigma_2\mathbf{n}_2 + \sigma_3\mathbf{n}_3)u = u$ sur $\partial\Omega$.

Pour conclure cette introduction, la dernière partie s'organise de la façon suivante. Dans le chapitre 20, on établit une estimation a priori de type H^2 pour les solutions de (1.20)-(1.24) dans la classe $\mathcal{O}_\varepsilon(B)$, où la constante obtenue ne dépend que de ε, du diamètre de B, et de la dimension n de l'espace. On suit essentiellement la méthode proposée par Grisvard [37, Sections 3.1.1-3.1.2]. Puis, dans les chapitres 21 et 22, on traite respectivement le cas Dirichlet et le cas Neumann/Robin. Finalement, dans le chapitre 23, on donne des résultats d'existence généraux dans $\mathbb{R}^n$ et on détaille plusieurs applications.

Chapter 2

Introduction (in English)

In the universe, many physical phenomena are governed by the geometry of their environment. The governing principle is usually modelled by some kind of energy minimization. Some problems such as soap films involve the first-order properties of surfaces (the area, the normal, the first fundamental form), while others such as the equilibrium shapes of red blood cells also concern the second-order ones (the principal curvatures, the second fundamental form).

In this thesis, we are interested in the existence of solutions to such shape optimization problems and in the determination of an accurate class of admissible shapes. Indeed, although geometric measure theory [33, 87] often provides a general framework for understanding these questions precisely, the minimizer usually comes with a poorer regularity than the one expected, and it is difficult to understand (and to prove) in what sense it is, since singularities may occur.

The original motivation of this thesis comes from biology. In aqueous media, phospholipids at rest immediately gather in pairs to form bilayers also called vesicles. It is a bag of fluid contained in a fluid and the basic membrane of any living cell.

Devoid of nucleus among mammals, red blood cells are typical examples of vesicles equipped with an additional internal structure playing the role of a skeleton inside the membrane. One of the main work of this thesis is to introduce and study a uniform ball condition, in particular to model the effects of the skeleton.

Indeed, if the local deformations are small, then the skeleton does not play any role and the red blood cell behaves like a vesicle. Otherwise, the skeleton redistributes the excess of local stress on the whole surface of the red blood cell. Therefore, this skeleton acts as if a uniform bound on the curvatures is imposed everywhere on the vesicle.

In the 70s, Canham [16] then Helfrich [45] suggested a simple model to characterize vesicles. Imposing the area of the bilayer and the volume of fluid it contains, their shape is a minimizer for the following free-bending energy:

$$\mathcal{E} := \frac{k_b}{2} \int_{\text{membrane}} (H - H_0)^2 \, dA + k_G \int_{\text{membrane}} K dA, \tag{2.1}$$

where $H = \kappa_1 + \kappa_2$ refers to the scalar mean curvature and $K = \kappa_1 \kappa_2$ to the Gaussian curvature, where $H_0 \in \mathbb{R}$ (called the spontaneous curvature) measures the asymmetry between the two layers, and where $k_b > 0$, $k_G < 0$ are two other physical constants.

Among the rich variety of problems arising from this exciting model, let us mention: existence, uniqueness, properties, regularity of minimizers; accurate numerical simulations by level-set and phase-field methods; coupling the structure with some fluid dynamics; rheology of many vesicles in a flow; controlling the shape from a piece of boundary. This three-year thesis leads us to mainly concentrate on the study of three axes of research reflected in the structure of this report.

A first approach consists in minimizing the Canham-Helfrich energy (2.1) without constraint then with an area constraint. The case $H_0 = 0$ is known as the Willmore energy:

$$\mathcal{W}(\Sigma) := \frac{1}{4} \int_{\Sigma} H^2 dA. \tag{2.2}$$

It has been widely studied by geometers [79, 88, 92] due to its conformal invariance property. However, this particularity does not hold if $H_0 \neq 0$. Since the sphere is the minimizer of (2.2), it is a good candidate to be the minimizer of (2.1) among surfaces of prescribed area. Our first main contribution in this thesis was to study its optimality (global/local minimizer, critical point).

Moreover, if we impose the topology of the admissible surfaces, then from the Gauss-Bonnet Theorem, the Canham-Helfrich energy (2.1) is equivalent to the so-called Helfrich energy:

$$\mathcal{H}(\Sigma) := \frac{1}{4}\int_{\Sigma}(H-H_0)^2\,dA = \frac{1}{4}\int_{\Sigma}H^2 dA - \frac{H_0}{2}\int_{\Sigma}H dA + \frac{H_0^2 A(\Sigma)}{4}. \tag{2.3}$$

In the specific case of membranes with a negative spontaneous curvature $H_0 < 0$, one can wonder whether the minimization of (2.3) with an area constraint can be done by minimizing individually each term. Since the Willmore energy (2.2) is invariant with respect to rescaling, and spheres are the only global minimizers of (2.2), this reduction makes sense only if the sphere is also the only solution of the following problem:

$$\inf_{A(\Sigma)=A_0} \frac{1}{2}\int_{\Sigma} H dA.$$

Therefore, our second main work in this thesis corresponds to the study of the above problem i.e. the minimization of total mean curvature with prescribed area among various class of surfaces.

Then, considering both an area and volume constraints, the minimizer cannot be the sphere, which is no more admissible. Using the shape optimization point of view, the third main and most important contribution of this thesis is to introduce a more reasonable class of surfaces, in which the existence of an enough regular minimizer is ensured for general functionals and constraints involving the first- and second-order geometric properties of surfaces:

$$\inf_{\Omega}\int_{\partial\Omega} F\left[\mathbf{x}, \mathbf{n}_{\partial\Omega}(\mathbf{x}), H_{\partial\Omega}(\mathbf{x}), K_{\partial\Omega}(\mathbf{x}), u_{\Omega}(\mathbf{x}), \nabla u_{\Omega}(\mathbf{x})\right] dA(\mathbf{x}).$$

Inspired by what Chenais did in [20] when she considered the uniform cone property, we consider the (hyper-)surfaces satisfying a uniform ball condition. We first study purely geometric functionals then we allow a dependence through the solution u_{Ω} of some second-order elliptic boundary value problems posed on the domain enclosed by the shape. Let us now detail each part of this document.

First part: on the minimization of the Canham-Helfrich energy

Chapter 3: an overview of the physical models associated with vesicles

In this chapter, we first explain what is a vesicle, then present a simplified two-dimensional model to characterize their shapes. Next, we consider its three dimensional version, known as the Canham-Helfrich energy. Finally, we give some other models of vesicles [85] and red blood cells [59].

Chapter 4: minimizing the Helfrich energy without constraint

In this chapter, we study the minimization of (2.3) among compact C^2-surfaces of $\mathbb{R}^3$:

$$\inf_{\Sigma} \frac{1}{4}\int_{\Sigma}(H-H_0)^2 dA.$$

We distinguish three cases depending on the sign of the spontaneous curvature $H_0 \in \mathbb{R}$. Then, we show the same results hold for (2.1), summarized in Table 2.1. Finally, weakening the regularity of admissible shapes, we extend the known case $H_0 = 0$ to compact simply-connected $C^{1,1}$-surfaces.

$k_G < 0 < k_b$	$H_0 < 0$	$H_0 = 0$	$H_0 > 0$
Existence	no global minimizer	any sphere [92]	the sphere of radius $\frac{1}{H_0}$ [2]
$\inf_{\Sigma}\mathcal{H}(\Sigma)$	4π	4π	0
$\inf_{\Sigma}\mathcal{E}(\Sigma)$	$4\pi\,(2k_b + k_G)$	$4\pi\,(2k_b + k_G)$	$4\pi k_G$

Table 2.1: On the minimization of the Canham-Helfrich energy without constraint.

Chapter 5: minimizing the Helfrich energy under area constraint

In this chapter, we are interested in minimizing the Helfrich energy (2.3) among compact simply-connected $C^{1,1}$-surfaces of $\mathbb{R}^3$ with prescribed area $A_0 > 0$:

$$\inf_{A(\Sigma)=A_0} \frac{1}{4}\int_{\Sigma}(H-H_0)^2 dA. \tag{2.4}$$

We refer to Theorems 2.9 to get some existence results associated with Problem (2.4) above. Except from excluding the sphere from the class of admissible shapes, the volume constraint does not seem to play a specific role in the theoretical process used in calculus of variations i.e. neither in the compactness of the minimizing sequence, nor in the (lower semi-)continuity of the functional and constraints, nor in the regularity of minimizers.

Moreover, the sphere $\mathbb{S}_{A_0}$ of area A_0 seems a good candidate for being the minimizer of (2.4). In this chapter, we study in detail the global optimality of this sphere. The results are summarized in the first row of Table 2.2 below. They depend on a specific adimensional parameter:

$$c_0 := \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}, \tag{2.5}$$

and we prove the existence of two numbers $c_- \approx -0.575$ and $c_+ \approx 1.46$ which are threshold values.

Parameter $c_0 = \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}$	$-\infty$	c_-		0		1		c_+		c_{++}	$+\infty$
Is the sphere a global minimizer ?	NO	[	?	[	YES	]	?	]		NO	
Is the sphere a local minimizer ?			YES			]		?		]	NO
Is the sphere a critical point ?						YES					

Table 2.2: Results obtained concerning the optimality in (2.4) of the sphere $\mathbb{S}_{A_0}$ with area A_0.

First, for any $c_0 \in [0,1]$, we deduce from the Cauchy-Schwarz inequality that $\mathbb{S}_{A_0}$ is the unique global minimizer of (2.4). Then, for any $c_0 > c_+$, we establish that $\mathbb{S}_{A_0}$ is not a minimizer of (2.4) among cigars. In particular, in that case, we deduce that $\mathbb{S}_{A_0}$ is no longer a global minimizer, even in a smaller subclass of admissible shapes (convex, axisymmetric, uniform ball condition).

Theorem 2.1. *Let $A_0 > 0$, $H_0 \in \mathbb{R}$, c_0 as in (2.5), and $c_+ := \frac{1}{4}(1+\sqrt{2})^2 \approx 1.46$. We call cigar any cylinder of length $L \geqslant 0$ on which are glued in a $C^{1,1}$ way two half spheres of radius $R > 0$. If $c_0 < c_+$, then the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of (2.4) among the class of cigars. Moreover, if $c_0 > c_+$, then it is the cigar $\mathbb{C}_{A_0}$ of area A_0 and radius:*

$$R_- := \sqrt{\frac{A_0}{3\pi}}\cos\left[\frac{1}{3}\arccos\left(\frac{-3}{H_0}\sqrt{\frac{3\pi}{A_0}}\right)+\frac{4\pi}{3}\right].$$

At last, if $c_0 = c_+$, then $\mathbb{S}_{A_0}$ and $\mathbb{C}_{A_0}$ are the only two global minimizers of (2.4) among cigars.

Finally, for any $c_0 < c_-$, we prove that a sequence of non-convex axisymmetric $C^{1,1}$-surfaces of constant area (converging to a double-sphere) has a strictly lower Helfrich energy (2.3) than $\mathbb{S}_{A_0}$, which is thus not a global minimizer of (2.4). More precisely, the result states as follows.

Theorem 2.2. *Let $A_0 > 0$, $H_0 \in \mathbb{R}$, c_0 as in (2.5), and $c_- := \frac{1}{8\cos\theta} \approx -0.575$, where $\theta \approx 4.4934$ is the unique solution of $\tan x = x$ on the interval $]\pi, \frac{3\pi}{2}[$. Then, there exists a sequence $(\Sigma_r)_{r>0}$ of compact simply-connected non-convex axisymmetric $C^{1,1}$-surfaces of $\mathbb{R}^3$ such that:*

$$\frac{1}{4}\int_{\Sigma_r}(H-H_0)^2 dA - \frac{1}{4}\int_{\mathbb{S}_{A_0}}(H-H_0)^2 dA \underset{r\to 0^+}{\longrightarrow} 8\pi\,(c_0-c_-)\,.$$

However, and this is the purpose of the second part in this report, if we restrict the class of admissible shapes to the ones enclosing a convex inner domain, or to the one bounding an *axiconvex* domain, i.e. an axisymmetric domain whose intersection with any plane orthogonal to the symmetry axis is either a disk or empty, then the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of (2.4) (cf. Inequality (2.14) and Theorem 2.5).

Chapter 6: the sphere as a local minimizer of the Helfrich energy with prescribed area

In this chapter, we look at the optimality of the sphere $\mathbb{S}_{A_0}$ of area A_0 as a local minimizer of (2.4). The results are summarized in the second row of Table 2.2. In the case $c_0 < 0$, combining the observation made in the paragraph below (2.3) with a result of the second part (Remark 11.1), we get that $\mathbb{S}_{A_0}$ is a local minimizer of (2.4). Moreover, since $\mathbb{S}_{A_0}$ is a global minimizer of (2.4) for any $c_0 \in [0,1]$, it is in particular a local minimizer of (2.4).

Then, assuming $c_0 > 1$, we study some local smooth axisymmetric perturbations of the sphere. We make a rescaling in order to study only perturbations of the unit sphere. Indeed, we have:

$$\inf_{A(\Sigma)=A_0} \frac{1}{4}\int_\Sigma (H-H_0)^2 dA = \inf_{A(\widetilde{\Sigma})=4\pi} \frac{1}{4}\int_{\widetilde{\Sigma}} (H-2c_0)^2 dA. \tag{2.6}$$

We prove the existence of a threshold value above which the second-order shape derivative of the Helfrich energy associated with these families of perturbations is negative. More precisely:

Proposition 2.3. *Let us consider some well-defined maps of the form $\theta_\varepsilon : s \in [0, L(\varepsilon)] \mapsto s+\varepsilon\varphi(s)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-identically-zero smooth map with compact support. We assume that each θ_ε generates a (compact embedded simply-connected) axisymmetric smooth surface $\Sigma_\varepsilon \subset \mathbb{R}^3$ via the curve $s \in [0, L(\varepsilon)] \mapsto (\int_0^s \cos\theta_\varepsilon(s)ds, \int_0^s \sin\theta_\varepsilon(s)ds)$ parametrized by arc length. Then, introducing the Helfrich functional:*

$$F_{c_0} : \varepsilon \in \mathbb{R} \longmapsto F_{c_0}(\varepsilon) = \frac{1}{4\pi}\int_{\Sigma_\varepsilon} (H-2c_0)^2 dA, \tag{2.7}$$

we have $\dot{\mathbf{F}}_{\mathbf{c_0}}(\mathbf{0}) = \mathbf{0}$, *and* $\ddot{\mathbf{F}}_{\mathbf{c_0}}(\mathbf{0}) < \mathbf{0} \Leftrightarrow \mathbf{R}(\varphi) < \mathbf{c_0}$, *and* $\ddot{\mathbf{F}}_{\mathbf{c_0}}(\mathbf{0}) > \mathbf{0} \Leftrightarrow \mathbf{R}(\varphi) > \mathbf{c_0}$, *where we set:*

$$R(\varphi) = 1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \dot{\varphi}(s)^2 \sin s ds + \int_0^\pi \frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s \varphi(t)\sin t dt\right)^2 ds}{\displaystyle\int_0^\pi \varphi(s)^2 \sin s ds - \varphi(\pi)^2}, \tag{2.8}$$

which is well defined for any $\varphi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ satisfying the following growth condition:

$$\int_0^\pi \dot{\varphi}(s)^2 \sin s ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \frac{\left(\int_0^s \varphi(t)\sin t dt\right)^2}{\sin s} ds < +\infty. \tag{2.9}$$

Moreover, the map φ necessarily satisfies the following constraints:

$$\begin{cases} \varphi(0) = 0, \quad \varphi(\pi) = -\dot{L}(0) \\ \varphi(\pi) = \displaystyle\int_0^\pi \varphi(s)\sin s ds = \frac{1}{\pi}\int_0^\pi s\varphi(s)\sin s ds \\ \varphi(\pi)^2 = \displaystyle\int_0^\pi (\pi - s)\,\varphi(s)^2 \cos s ds. \end{cases} \tag{2.10}$$

Therefore, we consider the critical value:

$$c_{++} := \inf R(\phi), \tag{2.11}$$

where R is defined by (2.8) and where the infimum is taken among all non-zero maps $\phi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ satisfying the growth condition (2.9) and the constraints (2.10).

If $c_0 < c_{++}$, then $c_0 < R(\varphi)$ i.e. $\ddot{F}_{c_0}(0) > 0$ for any $\varphi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ satisfying (2.9)–(2.10). In particular, for ε small enough, any perturbation Σ_ε of the form given in Proposition 2.3 has a strictly greater Helfrich energy (2.7) than the one of the unit sphere, which is thus a local minimizer of (2.7) among this class of perturbations:

$$\begin{aligned} \frac{1}{4\pi}\int_{\Sigma_\varepsilon} (H-2c_0)^2 dA - \frac{1}{4\pi}\int_{\mathbb{S}^2} (H-2c_0)^2 dA &= F_{c_0}(\varepsilon) - F_{c_0}(0) = \dot{F}_{c_0}(0)\varepsilon + \ddot{F}_{c_0}(0)\frac{\varepsilon^2}{2} + o(\varepsilon^2) \\ &= \frac{\varepsilon^2}{2}\left(\ddot{F}_{c_0}(0) + \frac{o(\varepsilon^2)}{\varepsilon^2}\right) \quad (>0 \text{ for } \varepsilon \text{ small}). \end{aligned}$$

Conversely, if $c_0 > c_{++}$, there exists $\varphi \in W^{1,1}_{\mathrm{loc}}(0,\pi)$ satisfying (2.9)–(2.10) such that $c_0 > R(\varphi)$. We thus have $\ddot{F}_{c_0}(0) < 0$, provided we can build an extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of φ with compact support such that the family of maps $\theta_\varepsilon : s \in [0, L(\varepsilon)] \to s + \varepsilon\tilde{\varphi}(s)$ is well-defined around $\varepsilon = 0$ and admissible in the sense of Definition 8.1, i.e. generates some axisymmetric surfaces $\Sigma_\varepsilon \subset \mathbb{R}^3$. If this is the case, for ε small enough, Σ_ε is a perturbation with strictly lower Helfrich energy (2.7) than the one of the unit sphere, which is thus not a local minimizer of (2.6). With an appropriate rescaling, we deduce that $\mathbb{S}_{A_0}$ is not a local minimizer of (2.4).

Moreover, if we set $u(s) = \int_0^s \varphi(t)\sin t dt$, we can express c_{++} by an equivalent optimization problem posed in a weighted Sobolev space. The result states as follows.

Theorem 2.4. *Let c_{++} be the critical value given by* (2.11). *Then, we have:*

$$c_{++} = \inf_{\substack{u \in H^2_{\sin}(0,\pi) \\ u \neq 0 \\ \int_0^\pi u(s)ds = 0 \\ \int_0^\pi (\pi - s)\cos s (\frac{\dot{u}(s)}{\sin s})^2 ds = 0}} \frac{1}{2} \frac{\displaystyle\int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds + 2\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds}{\displaystyle\int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds}, \tag{2.12}$$

where we set $H^2_{\sin}(0,\pi) = \{u \in H^2_0(0,\pi), \ \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds + \int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds < +\infty\}$. Moreover, there exists a minimizer to this problem (2.12) *and also to Problem* (2.11).

Finally, we tried to evaluate the exact value of c_{++} but we did not manage to handle the non-linear constraint $\int_0^s (\pi - s)\cos s (\frac{\dot{u}(s)}{\sin s})^2 ds = 0$. Hence, we try to compute the critical value of (2.12) without this constraint. It becomes an eigenvalue problem associated with the following non-linear fourth-order one-dimensional ordinary differential equation:

$$\forall s \in]0,\pi[, \quad \frac{d^2}{ds^2}\left(\frac{\ddot{u}(s)}{\sin s}\right) + 2\lambda \frac{d}{ds}\left(\frac{\dot{u}(s)}{\sin s}\right) + \frac{2u(s)}{\sin^3 s} = \mu, \quad u(0) = u(\pi) = \dot{u}(0) = \dot{u}(\pi) = 0, \tag{2.13}$$

where λ is a positive eigenvalue and μ the Lagrange multiplier associated with the constraint $\int_0^\pi u(s)ds = 0$. In particular, from (2.12), an estimation from below for c_{++} is given by the lowest positive eigenvalue λ for which the solution $u : [0,\pi] \to \mathbb{R}$ of (2.13) with $\mu = 0$ is not identically zero and satisfies $\int_0^\pi u(s)ds = 0$.

Unfortunately, we did not manage to completely solve the eigenvalue problem. However, we obtain an explicit sequence of eigenvalues $(\lambda_{2i})_{i \geqslant 1} = \frac{1}{2}(2i+1)(2i+2)$, for which the corresponding solution u_i of (2.13) with $\mu = 0$ is not identically zero and satisfies $\int_0^\pi u_i(s)ds = 0$. There is good numerical evidence to think that $\lambda_2 = 6$ is the lowest but we were not able to prove it.

Second part: on the minimization of total mean curvature

This part is the reproduction of a submitted article entitled *on the minimization of total mean curvature* [23], done in collaboration with Simon Masnou, Antoine Henrot and Takéo Takahashi. We have added a more detailed exposition on the properties of non-decreasing rearrangements, and the proof of Minkowski's inequality (2.14) below with a complete treatment of the equality case.

In 1901, Minkowski proved that the following inequality holds for any non-empty bounded open convex subset $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is a C^2-surface:

$$\frac{1}{2}\int_{\partial\Omega} H dA \geqslant \sqrt{4\pi A(\partial\Omega)}, \tag{2.14}$$

where the integration of the scalar mean curvature $H = \kappa_1 + \kappa_2$ is done with respect to the usual two-dimensional Hausdorff measure referred to as $A(\bullet)$.

Announced in [69], Inequality (2.14) is proved in [70, §7] assuming C^2-regularity. The proof can also be found in [73, Chapter 6, Exercise (10)] in the case of ovaloids, i.e. compact connected smooth surfaces of $\mathbb{R}^3$ whose Gaussian curvature is positive everywhere.

The original proof of Minkowski is based on the isoperimetric inequality together with Steiner-Minkowski formulae. Therefore, Inequality (2.14) remains true if $\partial\Omega$ is only a surface of class $C^{1,1}$ (or equivalently, if $\partial\Omega$ has a positive reach, cf. Theorems 16.5-16.6). If we do not assume any regularity, the same inequality holds with the total mean curvature replaced by the mean width of the convex body.

Equality holds in (2.14) if and only if Ω is an open ball. This was stated by Minkowski in [70, §7] without proof. A proof due to Favard can be found in [31, Section 19] based on a Bonnesen-type inequality involving mixed volumes. In Chapter 13, we give a proof of inequality (2.14) with a complete treatment of the equality case, and also consider specifically the axisymmetric situation, inspired by Bonnesen [10, Section VI, §35 (74)].

Moreover, Inequality (2.14) is actually a consequence of a generalization due to Minkowski of the isoperimetric inequality. This generalization uses the notion of mixed volumes of convex bodies. We refer to [83, Theorem 6.2.1, Notes for Section 6.2] and [11, Sections 49,52,56] for a more detailed exposition on that question.

In this part, we are mainly interested in the validity of (2.14) under other various assumptions, and on the related problem of minimizing the total mean curvature with area constraint:

$$\inf_{\substack{\Sigma\in\mathfrak{C}\\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma H dA, \tag{2.15}$$

for a suitable class $\mathfrak{C}$ of surfaces in $\mathbb{R}^3$. We recall that the original motivation for Problem (2.15) is the study of Problem (2.4) in the particular case $H_0 < 0$. Indeed, one can wonder whether Problem (2.4) can be solved by minimizing individually each term in (2.3). Since the Willmore energy (2.2) is invariant with respect to rescaling, and spheres are the only global minimizers of (2.2), this reduction makes sense only if the sphere $\mathbb{S}_{A_0}$ is also the only solutions to Problem (2.15). We prove in this part that this is true if the problem is tackled in a particular class of surfaces.

Let us first introduce two classes of embedded 2-surfaces in $\mathbb{R}^3$: the class $\mathcal{A}_{1,1}$ of all compact surfaces which are boundaries of axisymmetric domains (i.e. sets with rotational invariance around an axis), and the subclass $\mathcal{A}_{1,1}^+$ of *axiconvex* surfaces, i.e. surfaces bounding an axisymmetric domain whose intersection with any plane orthogonal to the symmetry axis is either a disk or empty. We first prove the following:

Theorem 2.5. *Consider the class $\mathcal{A}_{1,1}^+$ of axiconvex $C^{1,1}$-surfaces in $\mathbb{R}^3$. Then, we have:*

$$\forall\Sigma\in\mathcal{A}_{1,1}^+,\quad \frac{1}{2}\int_\Sigma H dA \geqslant \sqrt{4\pi A(\Sigma)},$$

where the equality holds if and only if Σ is a sphere. In particular, for any $A_0 > 0$, we have:

$$\frac{1}{2}\int_{\mathbb{S}_{A_0}} H dA = \min_{\substack{\Sigma\in\mathcal{A}_{1,1}^+\\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma H dA = \sqrt{4\pi A_0},$$

and the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of this problem.

Then, we show this result cannot be extended to the general class of compact simply-connected $C^{1,1}$-surfaces of $\mathbb{R}^3$, and we even provide a negative clue for the extension to $\mathcal{A}_{1,1}$. More precisely:

Theorem 2.6. *Let $A_0 > 0$. There exists a sequence of $C^{1,1}$-surfaces $(\Sigma_i)_{i\in\mathbb{N}}$ and a sequence of axisymmetric $C^{1,1}$-surfaces $(\widetilde{\Sigma}_i)_{i\in\mathbb{N}} \subset \mathcal{A}_{1,1}$ such that $A(\Sigma_i) = A(\widetilde{\Sigma}_i) = A_0$ for any $i \in \mathbb{N}$ with:*

$$\lim_{i\to+\infty} \frac{1}{2}\int_{\Sigma_i} H dA = -\infty \qquad \text{and} \qquad \lim_{i\to+\infty} \frac{1}{2}\int_{\widetilde{\Sigma}_i} H dA = 0^+.$$

It follows obviously that:

$$\inf_{\substack{\Sigma\in C^{1,1}\\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma H dA = -\infty \qquad \text{and} \qquad \inf_{\substack{\Sigma\in\mathcal{A}_{1,1}\\ A(\Sigma)=A_0}} \left|\frac{1}{2}\int_\Sigma H dA\right| = 0.$$

Therefore, Problem (2.15) has no solution in the class of (compact simply-connected) $C^{1,1}$-surfaces, and there is good reason to think that it might be the same within the class $\mathcal{A}_{1,1}$, but we were not able to prove it.

However, although Problem (2.15) has no global minimizer, it is easily seen that the sphere $\mathbb{S}_{A_0}$ of area A_0 is a local minimizer of (2.15) in the class of C^2-surfaces (Remark 11.1) and it can also be proved that $\mathbb{S}_{A_0}$ is the unique critical point of (2.15) in the class of C^3-surfaces (Theorem 11.3) by computing the first variation of total mean curvature and of area (Proposition 11.2).

In particular, since spheres are the only global minimizers of (2.2), we deduce that $\mathbb{S}_{A_0}$ is always a critical point of (2.4) among (compact simply-connected) C^3-surfaces of $\mathbb{R}^3$. It is also a local minimizer of (2.4) among C^2-surfaces for any $H_0 < 0$. These results are mentionned in Table 2.2.

Hence, this leads us naturally to consider another problem:

$$\inf_{\substack{\Sigma\in\mathcal{A}_{1,1}\\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma |H| dA, \tag{2.16}$$

for which we can prove the following.

Theorem 2.7. *Let $\mathcal{A}_{1,1}$ denotes the class of axisymmetric $C^{1,1}$-surfaces in $\mathbb{R}^3$, then, we have:*

$$\forall \Sigma \in \mathcal{A}_{1,1}, \quad \frac{1}{2}\int_\Sigma |H| dA \geqslant \sqrt{4\pi A(\Sigma)},$$

where the equality holds if and only if Σ is a sphere. In particular, for any $A_0 > 0$, we have:

$$\frac{1}{2}\int_{\mathbb{S}_{A_0}} |H| dA = \min_{\substack{\Sigma\in\mathcal{A}_{1,1}\\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma |H| dA = \sqrt{4\pi A_0},$$

and the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of this problem.

Let us note that in 1973, Michael and Simon established in [68] a Sobolev-type inequality for m-dimensional C^2-submanifolds of $\mathbb{R}^n$, for which the case $m = 2$ and $n = 3$ with $f \equiv 1$ gives the following inequality:

$$\frac{1}{2}\int_\Sigma |H| dA \geqslant c_0\sqrt{A(\Sigma)}.$$

More precisely, the constant appearing in the above inequality is $c_0 = \frac{1}{4^3}\sqrt{4\pi}$ [68, Theorem 2.1]. The better constant $c_0 = \frac{1}{2}\sqrt{2\pi}$ was obtained by Topping in [91, Lemma 2.1] and does not seem optimal. From Theorem 2.7, we think that an optimal constant should be $c_0 = \sqrt{4\pi}$.

We refer to the appendix of [91] for a concise proof of the above inequality using Simon's ideas. We also mention [19, Theorems 3.1–3.2] for a weighted version of this inequality but less sharp as mentioned in the last paragraph of [19, Section 3.2].

Finally, we summarize in Table 2.3 several results and open questions related to Problems (2.15) and (2.16) (the term *inner-convex* refers to a closed surface which encloses a convex domain). The part is organized as follows. In Chapter 8, the notation and the basic definitions of surface, axisymmetry, and axiconvexity are recalled. In Chapter 9 and 10, we respectively give the proofs of Theorem 2.5 and 2.6. In Chapter 11, we study the optimality of the sphere for Problem (2.15) and Theorem 2.7 is proved in Chapter 12. Finally, Minkowski's inequality (2.14) is established in Chapter 13 and we show some properties of non-decreasing rearrangements in Chapter 14.

Third part: uniform ball property and existence of optimal shapes for a wide class of geometric functionals

Using the shape optimization point of view, the aim of this part is to introduce a reasonable class of surfaces, in which the existence of an enough regular minimizer is ensured for general functionals and constraints involving the first- and second-order geometric properties of surfaces. Inspired by what Chenais did in [20] when she considered the uniform cone property, we introduce here the (hyper-)surfaces satisfying a uniform ball condition in the following sense.

Class of surfaces Σ	Assertion	Proof
$C^{1,1}$ compact inner-convex	$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	[31, 70]
$C^{1,1}$ axisymmetric inner-convex	$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	[10]
$C^{1,1}$ axiconvex	$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	Thrm 2.5
$C^{1,1}$ axisymmetric	$\inf\limits_{A(\Sigma)=A_0} \left\lvert \frac{1}{2}\int_\Sigma HdA \right\rvert = 0$	Thrm 2.6
$C^{1,1}$ axisymmetric	$\frac{1}{2}\int_\Sigma HdA > 0$	**OPEN**
$C^{1,1}$ compact simply-connected	$\inf\limits_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma HdA = -\infty$	Thrm 2.6
C^2 compact simply-connected	$\mathbb{S}_{A_0}$ is a local minimizer of $\inf\limits_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma HdA$	Rmrk 11.1
C^3 compact simply-connected	$\mathbb{S}_{A_0}$ unique critical point of $\inf\limits_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma HdA$	Thrm 11.3
$C^{1,1}$ axisymmetric	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	Thrm 2.7
C^2 compact simply-connected	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{\frac{\pi}{2}A(\Sigma)}$	[68, 91]
$C^{1,1}$ compact simply-connected	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	**OPEN**

Table 2.3: Minimizing $\int H$ or $\int |H|$ with area constraint.

Definition 2.8. *Let $\varepsilon > 0$ and $B \subseteq \mathbb{R}^n$ be open, $n \geqslant 2$. We say that an open set $\Omega \subseteq B$ satisfies the ε-ball condition and we write $\Omega \in \mathcal{O}_\varepsilon(B)$ if for any $\mathbf{x} \in \partial\Omega$, there exits a unit vector $\mathbf{d_x}$ of $\mathbb{R}^n$ such that:*

$$\begin{cases} B_\varepsilon(\mathbf{x} - \varepsilon\mathbf{d_x}) \subseteq \Omega \\ B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d_x}) \subseteq B\backslash\overline{\Omega}, \end{cases}$$

where $B_r(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^n,\ \|\mathbf{y} - \mathbf{z}\| < r\}$ denotes the open ball of $\mathbb{R}^n$ centred at $\mathbf{z}$ and of radius r, where $\overline{\Omega}$ is the closure of Ω, and where $\partial\Omega = \overline{\Omega}\backslash\Omega$ refers to its boundary.

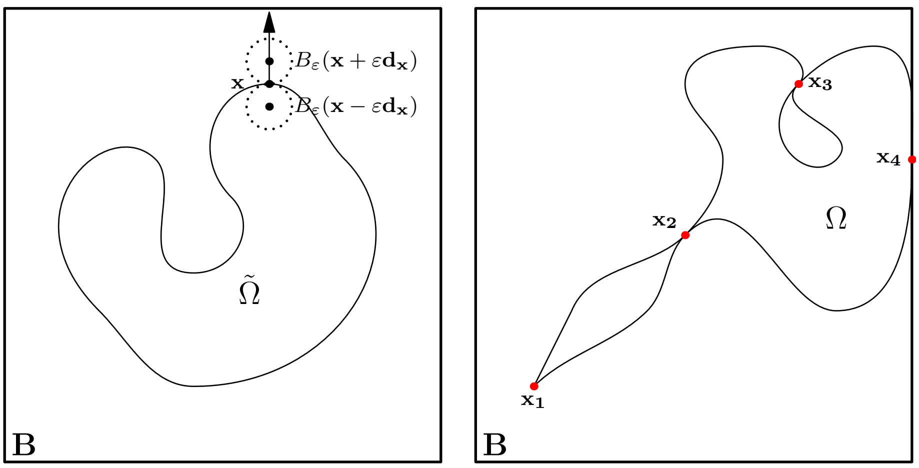

Figure 2.1: Example of an open set $\tilde{\Omega}$ in $\mathbb{R}^2$ satisfying the ε-ball condition, whereas Ω does not. Indeed, there is no circle passing through $\mathbf{x_1}$ or $\mathbf{x_2}$ (respectively $\mathbf{x_3}$ or $\mathbf{x_4}$) whose enclosed inner domain is included in Ω (respectively in $B\backslash\overline{\Omega}$).

The uniform (exterior/interior) ball condition was already considered by Poincaré in 1890 [78]. As illustrated in Figure 2.1, it avoids the formation of singularities such as corners, cuts, or self-intersections. In fact, it has been known to characterize the $C^{1,1}$-regularity of hypersurfaces for a long time by oral tradition, and also the positiveness of their reach, a notion introduced by Federer in [32]. We did not find any reference where these two characterizations were gathered. Hence, they are established in Chapter 16, reproducing an accepted proceeding entitled *some characterizations of a uniform ball property* [22]. We refer to Theorems 16.5–16.6 for precise statements.

Equipped with this class of admissible shapes, we can now state our main general existence result in the three-dimensional Euclidean space $\mathbb{R}^3$. We refer to Theorem 18.28 for its most general form in $\mathbb{R}^n$, but the following one is enough for the three physical applications we are presenting hereafter (further examples are also detailed in Section 18.5).

Theorem 2.9. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C, \widetilde{C}) \in \mathbb{R}\times\mathbb{R}$, five continuous maps j_0, f_0, g_0, g_1, $g_2 : \mathbb{R}^3\times\mathbb{S}^2 \to \mathbb{R}$, and four maps j_1, j_2, f_1, $f_2 : \mathbb{R}^3\times\mathbb{S}^2\times\mathbb{R} \to \mathbb{R}$ which are continuous and convex in the last variable. Then, the following problem has at least one solution (see Notation 2.10):*

$$\inf \int_{\partial\Omega} j_0\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} j_1\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), H\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} j_2\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), K\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right),$$

where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} f_1\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), H\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} f_2\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), K\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) \leqslant C \\ \displaystyle\int_{\partial\Omega} g_0\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} H\left(\mathbf{x}\right) g_1\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} K\left(\mathbf{x}\right) g_2\left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) = \widetilde{C}. \end{cases}$$

The proof of Theorem 2.9 only relies on basic tools of analysis and does not use the ones of geometric measure theory. We also mention that the particular case $j_0 \geqslant 0$ and $j_1 = j_2 = 0$ without constraints was obtained in parallel to our work in [40].

Notation 2.10. *We recall that we denote by $A(\bullet)$ (respectively $V(\bullet)$) the area (resp. the volume) i.e. the two(resp. three)-dimensional Hausdorff measure, and the integration on a surface is done with respect to A. The Gauss map $\mathbf{n} : \mathbf{x} \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^2$ always refers to the unit outer normal field of the surface, while $H = \kappa_1 + \kappa_2$ is the scalar mean curvature and $K = \kappa_1\kappa_2$ is the Gaussian curvature.*

Remark 2.11. *In the above theorem, the radius of B is large enough to avoid $\mathcal{O}_\varepsilon(B)$ being empty. Moreover, the assumptions on B can be relaxed by requiring B to be a non-empty bounded open set, smooth enough (Lipschitz for example) such that its boundary has zero three-dimensional Lebesgue measure, and large enough to contain at least an open ball of radius 3ε. Finally, for any set E, we recall that a map $j : E\times\mathbb{R} \to \mathbb{R}$ is said to be convex in its last variable if for any $(\mathbf{x}, t, \tilde{t}) \in E\times\mathbb{R}\times\mathbb{R}$ and any $\mu \in [0, 1]$, we have $j(\mathbf{x}, \mu t + (1-\mu)\tilde{t}) \leqslant \mu j(\mathbf{x}, t) + (1-\mu) j(\mathbf{x}, \tilde{t})$.*

First application: minimizing the Canham-Helfrich energy with area and volume constraints

We recall that the Canham-Helfrich energy (2.1) is a simple model to describe vesicles. Imposing the area of the bilayer and the volume of fluid it contains, their shape is a minimizer for the energy:

$$\mathcal{E}(\Sigma) = \frac{k_b}{2}\int_\Sigma (H - H_0)^2\, dA + k_G \int_\Sigma K dA, \tag{2.17}$$

where the spontaneous curvature $H_0 \in \mathbb{R}$ measures the asymmetry between the two layers, and where $k_b > 0$, $k_G < 0$ are two other physical constants. Note that if $k_G > 0$, for any $k_b, H_0 \in \mathbb{R}$, the Canham-Helfrich energy (2.17) with given area A_0 and volume V_0 is not bounded from below. Indeed, in that case, from the Gauss-Bonnet Theorem, the second term $k_G \int K dA = 4\pi k_G (1 - g)$ tends to $-\infty$ as the genus $g \to +\infty$, while the first term remains bounded by $4|k_b|(12\pi + \frac{1}{4}H_0^2 A_0)$. To see this last point, use [53, Remark 1.7 (iii) (1.5)], [84, Theorem 1.1], and [88, Inequality (0.2)] in order to get successively:

$$\inf_{\substack{A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \mathcal{E}(\partial\Omega) \leqslant 4|k_b| \left(\inf_{\substack{A(\partial\Omega)=A_0\\ V(\Omega)=V_0\\ \mathrm{genus}(\partial\Omega)=g}} \mathcal{W}(\partial\Omega) + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g)$$

$$\leqslant 4|k_b| \left(\inf_{\mathrm{genus}(\partial\Omega)=g} \mathcal{W}(\partial\Omega) + \inf_{\substack{A(\partial\Omega)=A_0\\ V(\Omega)=V_0\\ \mathrm{genus}(\partial\Omega)=0}} \mathcal{W}(\partial\Omega) - 4\pi + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g)$$

$$\leqslant 4|k_b| \left(8\pi + 8\pi - 4\pi + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g).$$

The two-dimensional case of (2.17) is considered by Bellettini, Dal Maso, and Paolini in [5]. Some of their results are recovered by Delladio [24] in the framework of special generalized Gauss graphs from the theory of currents. Then, Choksi and Veneroni [21] solve the axisymmetric case of (2.17) assuming $-2k_b < k_G < 0$. In the general case, this hypothesis gives a fundamental coercivity property [21, Lemma 2.1]: the integrand of (2.17) is standard in the sense of [48, Definition 4.1.2]. Hence, we get a minimizer for (2.17) in the class of rectifiable integer oriented 2-varifold in $\mathbb{R}^3$ with L^2-bounded generalized second fundamental form [48, Theorem 5.3.2] [72, Section 2] [6, Appendix]. These compactness and lower semi-continuity properties were already noticed in [6, Section 9.3].

However, the regularity of minimizers remains an open problem and experiments show that singular behaviours can occur to vesicles such as the budding transition [85, 86]. As the temperature increases, an initially spherical vesicle becomes a prolate ellipsoid, then takes a pear shape with broken up/down symmetry, and finally the neck closes, resulting in two spherical compartments that are sitting on top of each other but still connected by a narrow constriction [85, Section 1.1, Figure 1]. This cannot happen to red blood cells because their skeleton prevents the membrane from bending too much locally [59, Section 2.1]. To take this aspect into account, the uniform ball condition is also motivated by the modelization of the equilibrium shapes of red blood cells. We even have a clue for its physical value [59, Section 2.1.5]. Our result states as follows.

Theorem 2.12. *Let $H_0, k_G \in \mathbb{R}$ and $\varepsilon, k_b, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Notation 2.10):*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(\mathbb{R}^n)\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + k_G\int_{\partial\Omega} K dA.$$

Remark 2.13. *From the isoperimetric inequality, if $A_0^3 < 36\pi V_0^2$, one cannot find any $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ satisfying the two constraints; and if equality holds, the only admissible shape is the ball of area A_0 and volume V_0. Moreover, in the above theorem, note that we did not assume the $\Omega \in \mathcal{O}_\varepsilon(B)$ as it is the case for Theorem 2.9 because a uniform bound on their diameter is already given by the functional and the area constraint [88, Lemma 1.1]. Finally, the result above also holds if H_0 is continuous function of the position and the normal.*

Second application: minimizing the Helfrich energy with given genus, area, and volume

Since the Gauss-Bonnet Theorem is valid for sets of positive reach [32, Theorem 5.19], we get from Theorems 16.5–16.6 that $\int_\Sigma K dA = 4\pi(1-g)$ for any compact connected $C^{1,1}$-surface Σ (without boundary embedded in $\mathbb{R}^3$) of genus $g \in \mathbb{N}$. Hence, instead of minimizing (2.17), people often fix the topology and search for a minimizer of the Helfrich energy:

$$\mathcal{H}(\Sigma) = \frac{1}{4}\int_\Sigma (H-H_0)^2\, dA, \tag{2.18}$$

with given genus, area and enclosed volume. Like (2.17), such a functional depends on the surface but also on its orientation. However, in the case $H_0 \neq 0$, Energy (2.18) is not even lower semi-continuous with respect to the varifold convergence [6, Section 9.3]: the counterexample is due to Große-Brauckmann [38]. Using the framework of the uniform ball condition, we prove the following.

Theorem 2.14. *Let $H_0 \in \mathbb{R}$, $g \in \mathbb{N}$, and $\varepsilon, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Notation 2.10 and Remark 2.13):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n) \\ \mathrm{genus}(\partial\Omega) = g \\ A(\partial\Omega) = A_0 \\ V(\Omega) = V_0}} \int_{\partial\Omega} (H - H_0)^2 dA,$$

where genus$(\partial\Omega) = g$ *has to be understood as $\partial\Omega$ is a compact connected $C^{1,1}$-surface of genus g.*

Third application: minimizing Willmore's energy with various constraints

The particular case $H_0 = 0$ in (2.18) is known as the Willmore functional:

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_\Sigma H^2 dA. \tag{2.19}$$

It has been widely studied by geometers. Without constraint, Willmore [93, Theorem 7.2.2] proved that spheres are the only global minimizers of (2.19). Existence was established by Simon [88] for genus-one surfaces, Bauer and Kuwert [4] for higher genus. Recently, Marques and Neves [66] solved the so-called Willmore conjecture: the conformal transformations of the stereographic projection of the Clifford torus are the only global minimizers of (2.19) among smooth genus-one surfaces.

A main ingredient is the conformal invariance of (2.19), from which we can in particular deduce that minimizing (2.19) with prescribed isoperimetric ratio is equivalent to impose the area and the enclosed volume. In this direction, Schygulla [84] established the existence of a minimizer for (2.19) among analytic surfaces of zero genus and given isoperimetric ratio. For higher genus, Keller, Mondino, and Rivière [53] recently obtained similar results, using the point of view of immersions developed by Rivière [79] to characterize precisely the critical points of (2.19) and their regularity. Our result on the uniform ball condition can again be used to prove results for (2.19). It is known as the bilayer-couple model [85, Section 2.5.3] and it states as follows.

Theorem 2.15. *Let $M_0 \in \mathbb{R}$ and $\varepsilon, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Notation 2.10 and Remark 2.13):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n) \\ \mathrm{genus}(\partial\Omega) = g \\ A(\partial\Omega) = A_0 \\ V(\Omega) = V_0 \\ \int_{\partial\Omega} H dA = M_0}} \frac{1}{4} \int_{\partial\Omega} H^2 dA,$$

where genus$(\partial\Omega) = g$ *has to be understood as $\partial\Omega$ is a compact connected $C^{1,1}$-surface of genus g.*

This part is organized as follows. In Chapter 16, we precisely state the two characterizations associated with the uniform ball condition, in terms of positive reach (Theorem 16.5) and in terms of $C^{1,1}$-regularity (Theorem 16.6). Then, we give the proofs of the theorems, as in [22].

Following the classical method from the calculus of variations, we first obtain in Section 17.1 the compactness of the class $\mathcal{O}_\varepsilon(B)$ for various modes of convergence. This essentially follows from the fact that the ε-ball condition implies a uniform cone property, for which we already have compactness results.

Then, in Chapter 17, we parametrize in a fixed local frame simultaneously all the graphs associated with the boundaries of a converging sequence in $\mathcal{O}_\varepsilon(B)$ and we prove the C^1-strong convergence and the $W^{2,\infty}$-weak-star convergence of these local graphs.

Finally, in Chapter 18, we show how to use this local result on a suitable partition of unity to get the global continuity of general geometric functionals. Merely speaking, the proof always consists in expressing the integral in the parametrization and show that the integrand is the product of a L^∞-weak-star converging term with an L^1-strong converging term. We conclude by giving in Section 18.5 some existence results and detail several applications.

Fourth part: existence of minimizers for functionals depending on the geometry and the solution of a state equation

This part is devoted to the extension of the existence results obtained in the previous part for general geometric functionals also depending on the shape through the solutions of some second-order elliptic boundary value problems posed on the inner domain enclosed by the shape. Here, we present their three-dimensional version and refer to Chapter 23 for their general form in $\mathbb{R}^n$.

A dependence through the solution of the Dirichlet Laplacian

For any domain $\Omega \in \mathcal{O}_\varepsilon(B)$, the associated boundary $\partial\Omega$ has $C^{1,1}$-regularity (cf. Theorem 16.6). Hence, we can consider the unique solution $u_\Omega \in H_0^1(\Omega,\mathbb{R}) \cap H^2(\Omega,\mathbb{R})$ of the Dirichlet Laplacian posed on a domain $\Omega \in \mathcal{O}_\varepsilon(B)$ with $f \in L^2(B,\mathbb{R})$ [37, Section 2.1 and Theorem 2.4.2.5]:

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega \\ \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Moreover, we say that the maps $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ and $g : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ have a quadratic growth in the first variable if there exists a constant $c > 0$ such that:

$$\forall (\mathbf{z},\mathbf{x},\mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \quad |f(\mathbf{z},\mathbf{x},\mathbf{y})| \leqslant c\left(1 + \|\mathbf{z}\|^2\right) \tag{2.21}$$

$$\forall (\mathbf{z},\mathbf{x},\mathbf{y},t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}, \quad |g(\mathbf{z},\mathbf{x},\mathbf{y},t)| \leqslant c\left(1 + \|\mathbf{z}\|^2\right). \tag{2.22}$$

Then, we prove the following extension of Theorem 2.9.

Theorem 2.16. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C,\widetilde{C}) \in \mathbb{R}^2$, five continuous maps j_0, f_0, g_0, g_1, $g_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ with quadratic growth* (2.21) *in the first variable, and four continuous maps j_1, j_2, f_1, $f_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth* (2.22) *in the first variable, and convex in the last variable. Then, the following problem has at least one solution (see Notation 2.10 and Remark 2.11):*

$$\inf \int_{\partial\Omega} j_0\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ + \int_{\partial\Omega} j_2\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

where $u_\Omega \in H_0^1(\Omega,\mathbb{R}) \cap H^2(\Omega,\mathbb{R})$ is the unique solution of (2.20) *with $f \in L^2(B,\mathbb{R})$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:*

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} f_2\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ \\ \displaystyle\int_{\partial\Omega} g_0\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} K(\mathbf{x})\, g_2\left[\nabla u_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

In the above theorem, if we denote by $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growth (2.21)–(2.22) of the maps and from the continuity of the trace operator $H^2(\Omega,\mathbb{R}) \to H^1(\partial\Omega,\mathbb{R})$, we have:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant c\left[A(\partial\Omega) + \|\nabla u_\Omega\|^2_{L^2(\partial\Omega,\mathbb{R}^3)}\right] \leqslant \tilde{c}\left[A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

We prove Theorem 2.16 with the same method used for Theorem 2.9. Considering a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}}$, we first get from compactness a converging subsequence. Then, we parametrize simultaneously by local graphs of $C^{1,1}$-maps $(\varphi_i)_{i\in\mathbb{N}}$ the boundaries associated with the converging subsequence of domains. Moreover, from the previous part, $(\varphi_i)_{i\in\mathbb{N}}$ converges strongly in C^1 and weakly in $W^{2,\infty}$. Using a suitable partition of unity, we express the functional and the constraints in this local parametrization. Therefore, it remains to show that we can correctly let $i \to +\infty$.

Merely speaking, each integrand obtained is the product of a L^∞-weak-star converging term with a remaining term, on which we want to apply Lebesgue Domination Convergence Theorem to get its L^1-strong convergence. Hence, to let $i \to +\infty$, we need the almost-everywhere convergence and a uniform integrable bound for each integrand. Due to the continuity and the quadratic growth (2.21)–(2.22) hypothesis, this is the case if the local map $\mathbf{x}' \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ strongly converges in L^2 to the local map $\mathbf{x}' \mapsto \nabla u_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))$. We prove in Chapter 21 this assertion holds true.

A dependence through the solution of the Neumann/Robin Laplacian

The remaining work of this part is to extend the previous results to the Neumann/Robin boundary conditions. For any domain $\Omega \in \mathcal{O}_\varepsilon(B)$, since $\partial\Omega$ has $C^{1,1}$-regularity (Theorem 16.6), there exists a unique solution $v_\Omega \in H^2(\Omega, \mathbb{R})$ to the following problem [37, Section 2.1 and Theorem 2.4.2.7]:

$$\begin{cases} -\Delta v_\Omega + \lambda v_\Omega = f & \text{in } \Omega \\ \\ \partial_n(v_\Omega) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where $\lambda > 0$ and $f \in L^2(B, \mathbb{R})$. Moreover, there exists a unique solution $\tilde{v}_\Omega \in H^2(\Omega, \mathbb{R})$ to the following problem [37, Section 2.1 and Theorem 2.4.2.6]:

$$\begin{cases} -\Delta \tilde{v}_\Omega = f & \text{in } \Omega \\ \\ -\partial_n(\tilde{v}_\Omega) = \lambda \tilde{v}_\Omega & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

where $\lambda > 0$ and $f \in L^2(B, \mathbb{R})$. Furthermore, if the existence of a unique solution in $H^2(\Omega, \mathbb{R})$ is ensured, we are also able to treat in (2.24) some non-linear boundary conditions of the form $-\partial_n(\tilde{v}_\Omega) = \beta(\tilde{v}_\Omega)$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map with $\beta(0) = 0$. Note that if $\beta(x) = \lambda x$, we get (2.24) and (2.23) is given by $\beta(x) = 0$.

Moreover, we say that the maps $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ have a quadratic growth in their two first variables if there exists a constant $c > 0$ such that:

$$\forall (s, \mathbf{z}, \mathbf{x}, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \quad |f(s, \mathbf{z}, \mathbf{x}, \mathbf{y})| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2\right) \tag{2.25}$$

$$\forall (s, \mathbf{z}, \mathbf{x}, \mathbf{y}, t) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}, \quad |g(s, \mathbf{z}, \mathbf{x}, \mathbf{y}, t)| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2\right). \tag{2.26}$$

Then, we prove the following.

Theorem 2.17. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C, \widetilde{C}) \in \mathbb{R}^2$, five continuous maps j_0, f_0, g_0, g_1, $g_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ with quadratic growth* (2.25) *in the two first variables, and four continuous maps j_1, j_2, f_1, $f_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth* (2.26) *in the two first variables, and convex in the last variable. Then, the following problem has at least one solution (see Notation 2.10 and Remark 2.11):*

$$\inf \int_{\partial\Omega} j_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x})$$
$$+ \int_{\partial\Omega} j_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

where $v_\Omega \in H^2(\Omega, \mathbb{R})$ is the unique solution of either (2.23) *or* (2.24) *with $f \in L^2(B, \mathbb{R})$ and $\lambda > 0$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:*

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ \displaystyle\qquad + \int_{\partial\Omega} f_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ \\ \displaystyle\int_{\partial\Omega} g_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ \displaystyle\qquad + \int_{\partial\Omega} K(\mathbf{x})\, g_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

In the above theorem, if we denote by $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growth (2.25)–(2.26) of the maps and from the continuity of the trace operator $H^2(\Omega,\mathbb{R}) \to H^1(\partial\Omega,\mathbb{R})$, we have:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant c\left[A(\partial\Omega) + \|v_\Omega\|^2_{H^1(\partial\Omega,\mathbb{R})}\right] \leqslant \tilde{c}\left[A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

We prove Theorem 2.17 with the same method used for Theorem 2.16 and described in the previous section. The main task is to show that the local map $\mathbf{x}' \mapsto v_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ strongly converges in H^1 to the local map $\mathbf{x}' \mapsto v_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))$. It is the purpose of Chapter 22 to prove this holds true.

First application: some quadratic functionals on the domain involving the second-order derivatives of the Dirichlet/Neumann/Robin Laplacian

In this thesis, note that until now we only treat the case of functionals involving boundary integrals. Indeed, the case where the domain of integration corresponds to the one of (2.20) or (2.23)-(2.24) such as:

$$\int_\Omega j\left[\mathbf{x}, u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x})\right] dV(\mathbf{x}),$$

is standard with the framework of the uniform cone property [46, Section 4.3]. Since the ε-ball condition implies an $\alpha(\varepsilon)$-cone property (cf. Point (i) in Theorem 16.6), we have not considered such functionals for the time being. However, the class $\mathcal{O}_\varepsilon(B)$ becomes interesting if some second-order partial derivatives of u_Ω appear in the above integrand. Our result states as follows. We say that a map $j : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2} \to \mathbb{R}$ has a quadratic growth in its three last variables if there exists a constant $c > 0$ such that:

$$\forall (\mathbf{x}, s, \mathbf{z}, Y) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2}, \quad |j(\mathbf{x}, s, \mathbf{z}, Y)| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2 + \|Y\|^2\right), \tag{2.27}$$

where the Frobenius norm is considered on the set of (3×3)-matrices i.e. $\|Y\| = \sqrt{\operatorname{trace}([Y]^T Y)}$.

Theorem 2.18. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C, \widetilde{C}) \in \mathbb{R}^2$, three measurable maps $j_0, f_0, g_0 : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2} \to \mathbb{R}$ with quadratic growth (2.27) in their three last variables, and continuous in $(s, \mathbf{z}, Y)$ for almost every $\mathbf{x}$, five continuous maps j_1, f_1, g_1, g_2, $g_3 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ with quadratic growth (2.25) in the two first variables, and four continuous maps j_2, j_3, f_2, $f_3 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth (2.26) in the two first variables, and convex in the last variable. Then, the following problem has at least one solution (see Notation 2.10 and Remark 2.11):*

$$\inf \int_\Omega j_0\left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \operatorname{Hess} v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} j_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \;\; +$$
$$\int_{\partial\Omega} j_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_3\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

where $v_\Omega \in H^2(\Omega, \mathbb{R})$ is the unique solution of either (2.20) or (2.23) or (2.24) with $f \in L^2(B, \mathbb{R})$ and $\lambda > 0$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:

$$\begin{cases} C \geqslant \displaystyle\int_\Omega f_0\left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \operatorname{Hess} v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} f_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \\ \displaystyle\int_{\partial\Omega} f_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_3\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \\ \\ \widetilde{C} = \displaystyle\int_\Omega g_0\left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \operatorname{Hess} v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} g_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \\ \displaystyle\int_{\partial\Omega} H(\mathbf{x})\, g_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} K(\mathbf{x}) g_3\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}). \end{cases}$$

Again, in the above theorem, if we denote by $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growth (2.25)–(2.27) of the maps and from the continuity of the trace operator $H^2(\Omega,\mathbb{R}) \to H^1(\partial\Omega,\mathbb{R})$, we have:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \;\leqslant\; \tilde{c}\left[V(\Omega) + A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

Also observe that the above statement treats the case where the integration is not done on the whole domain Ω but only on a measurable part $\widetilde{\Omega} \subseteq \Omega$. Indeed, it suffices to introduce the characteristic function $\mathbf{1}_{\widetilde{\Omega}}$ in the integrand j_0. This cannot be done for the boundary integrals but continuous cutoff functions can still be considered. Finally, the formulation adopted above allows constraints of the form $K \subset \Omega$ for a given compact set $K \subset B$, by setting $\widetilde{C} = V(K)$, $g_0 = \mathbf{1}_K$, and $g_1 = g_2 = g_3 = 0$.

Second application: boundary shape identification problems

Let $\varepsilon > 0$ and B be an open set as in Remark 2.11. We consider $\Omega_0 \in \mathcal{O}_\varepsilon(B)$, a subset $\Gamma_0 \subseteq \partial\Omega_0$, and $g_0 \in L^2(\Gamma_0, \mathbb{R})$. Imagine there is good reason to think that g_0 is the restriction to Γ_0 of the normal derivative associated with the solution u_Ω of the Dirichlet Laplacian (2.20) posed on an unknown domain $\Omega \in \mathcal{O}_\varepsilon(B)$ such that $\Gamma_0 \subseteq \partial\Omega$. In order to to find the *best* $\Omega \in \mathcal{O}_\varepsilon(B)$ such that $\partial_n(u_\Omega)|_{\partial\Omega_0} = g_0$, one possibility is to solve the following problem:

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(B) \\ \Gamma_0 \subseteq \partial\Omega}} \int_{\Gamma_0} \left[\partial_n\left(u_\Omega\right) - g_0\right]^2 dA. \tag{2.28}$$

Similarly, if we suspect that $f_0 \in L^2(\Gamma_0, \mathbb{R})$ is the restrictions to Γ_0 of the solution v_Ω to the Neumann/Robin Laplacian (2.23)-(2.24) posed on an unknown domain $\Omega \in \mathcal{O}_\varepsilon(B)$ such that $\Gamma_0 \subseteq \partial\Omega$, then we have to solve:

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(B) \\ \Gamma_0 \subseteq \partial\Omega}} \int_{\Gamma_0} \left(v_\Omega - f_0\right)^2 dA. \tag{2.29}$$

Of course, we can build more complicated functionals but the main difficulty here is that the domain of integration is not the whole surface. We prove the following result.

Proposition 2.19. *Let $\Omega_0 \in \mathcal{O}_\varepsilon(B)$ and Γ_0 be a measurable subset of $\partial\Omega_0$. Then, Theorem 2.18 remains true if we add the constraint $\Gamma_0 \subseteq \partial\Omega$ and if the domain of integration $\partial\Omega$ in the functional and the constraints are restricted to Γ_0. In particular, Problems* (2.28)–(2.29) *have a minimizer.*

The identification of shape through its boundary like (2.28)–(2.29) often appear in inverse and optimal control problems. For example, let us try to detect a tumor in the brain. We put some electrods on the head Γ_0 of a patient, measure some electric activity g_0, and solve Problem (2.29). If no tumor exists, then the infimum is zero and the optimal shape is Γ_0, otherwise it is $\Gamma_0 \cup \Gamma_1$, where Γ_1 is the boundary of the tumor.

Third application: the MIT-bag model in relativistic quantum mechanics

During the conference MODE 2014 at the INSA-Rennes, Le Treust made a talk on his thesis [56]. He has studied some shape optimization problems coming from relativistic quantum mechanics. In particular, bag models are introduced to study the internal structure of hadrons. The energy of these particules is given by summing the energy of the quarks and anti-quarks living in the bag.

In the MIT-bag model, the wave functions of the quarks are the eigenvectors of the Dirac operator. Hence, the fundamental state problem corresponds to the minimization with prescribed volume of the first positive eigenvalue associated with this Dirac operator among non-empty open bounded subset of $\mathbb{R}^3$ with C^2-boundary. The existence of an optimal shape is actually open.

We did not study this problem but it seems that the framework of the uniform ball condition might be used again to approximate the fundamental state of the MIT-bag model:

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(B) \\ V(\Omega) = V_0}} \lambda_1^{\mathrm{MIT}}(\Omega)$$

with

$$\lambda_1^{\mathrm{MIT}}(\Omega) \quad = \quad \inf_{\substack{u \in H^1(\Omega, \mathbb{C}^2) \\ \int_\Omega |u|^2 = 1 \\ -(\sigma . \mathbf{n}_{\partial\Omega}) u = u \text{ on } \partial\Omega}} \sqrt{m^2 + \int_\Omega \|\nabla u\|^2 + \int_{\partial\Omega} \left(m + \frac{H_{\partial\Omega}}{2}\right) |u|^2 dA},$$

where $m > 0$ is a given fixed parameter (the mass of the particle) and where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector formed by the three Pauli (2×2)-matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The main difficulty comes from the boundary non-linear constraint associated with the eigenvalue problem $-(\sigma.\mathbf{n}_{\partial\Omega})u = u$, which has to be understood as $-(\sigma_1\mathbf{n}_1 + \sigma_2\mathbf{n}_2 + \sigma_3\mathbf{n}_3)u = u$ on $\partial\Omega$.

To conclude this introduction, the last part is organized as follows. In Chapter 20, we establish H^2-*a priori* estimates for the solutions of (2.20)-(2.24) in the class $\mathcal{O}_\varepsilon(B)$, where the constant obtained depend only on ε, the diameter of B, and the dimension n of the space. We essentially follow the method suggested by Grisvard [37, Sections 3.1.1-3.1.2]. Then, in Chapters 21 and 22, we respectively treat the Dirichlet and the Neumann/Robin case. Finally, in Chapter 23, we give very general existence results in $\mathbb{R}^n$ and detail several applications.

Part II

On the minimization of the Canham-Helfrich energy

Chapter 3

An overview of the physical models associated with vesicles

In this chapter, we first explain what is a vesicle, then present a simplified two-dimensional model to characterize their shapes. Next, we consider its three dimensional version, known as the Canham-Helfrich energy. Finally, we give some other models of vesicles and red blood cells. Concerning the biological and physical point of view, we refer to [59, 85] for further details on the subject.

3.1 The biological structure of vesicles

In biology, a phospholipid is a certain kind of lipid and the main ingredient constituting the membrane of any living cell. Its molecule structure consists of a hydrophilic head, on which are connected two hydrophobic tails. Hence, when a sufficiently large amount of phospholipids is inserted in a aqueous media, they immediately gather in pairs to form bilayers also called vesicles, as illustrated in Figure 3.1.

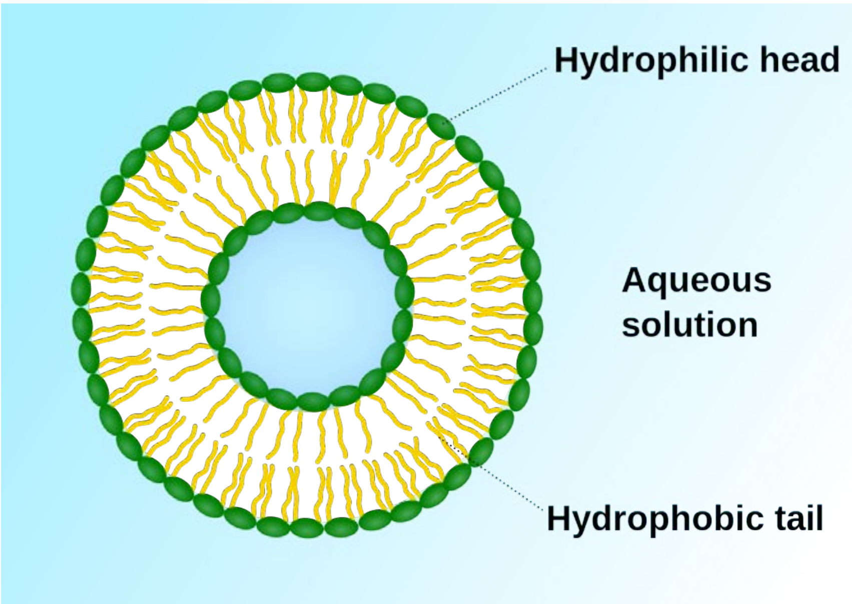

Figure 3.1: Scheme of a vesicle formed by phospholipids (source: article *vesicle* on Wikipedia).

Merely speaking, a vesicle is a bag of fluid itself contained in a fluid. It is the basic membrane of all living cells and understanding it well is a first fundamental step in the comprehension of general cells behaviour. Mammalian red blood cells are devoid of nucleus (cf. Figure 3.2) and convey the oxygen and the carbon dioxyde through the body via the blood. They are typical examples of vesicles, on which is fixed a network of proteins playing the role of a skeleton inside the membrane.

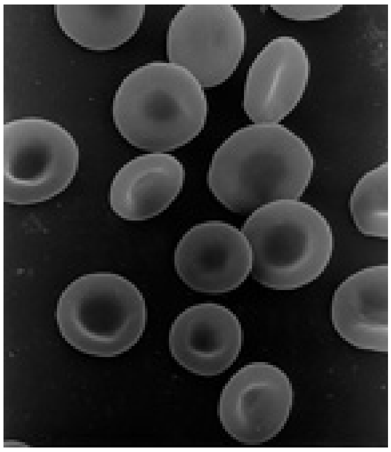

Figure 3.2: Scanning electron micrograph of human red blood cells: ca. $6 - 8$ μm in diameter (source: article *red blood cell* on Wikipedia).

In this thesis, we are mainly interested in the mathematical problems arising from the study of shapes associated with such vesicles. For example, Figure 3.3 illustrates the effects of osmotic pressure on the shapes of human red blood cells. From an optimization point of view, it follows from the least action principle that their shape at rest is minimizing a free bending energy under some constraints, such as the surface of the bilayer and the volume of fluid it contains.

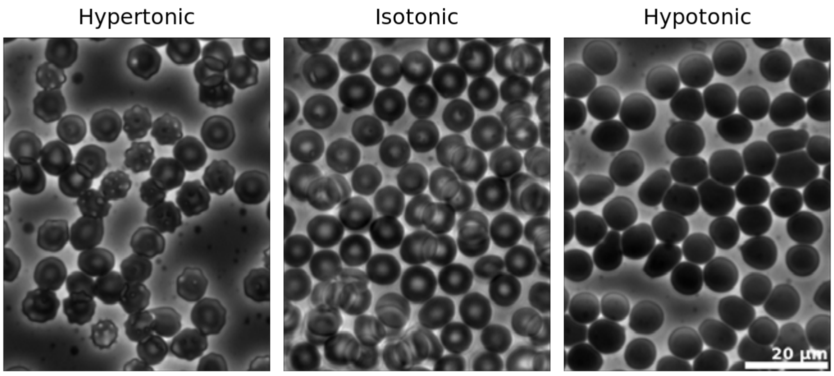

Figure 3.3: Human red blood cells viewed by phase contrast light microscopy. Three conditions are shown: hypertonic conditions, where they contract and appear *spiky*; isotonic conditions, where they show their normal discocyte shape; and hypotonic conditions, where they expand and become more round (source: article *red blood cell* on Wikipedia).

3.2 A two-dimensional simplified model for vesicles

In this section, we mainly reproduce the two-dimensional model described in [17]. First, we need to understand how the shape of a vesicle behaves once it is bent. In other words, we want to model the effect of curvature on the elastic energy associated with the bilayer. In a first simplified approach, we consider the two-dimensional curvature generated in a plane and forget about those generated in directions that are not in that plane.

In Figure 3.4, a small piece of rectilinear membrane is represented on the left. The red segments and the yellow one correspond to the space available respectively for the heads and the tails of the molecules. This same piece of membrane is represented on the right, once bent. If the length L_0 of the portion is chosen sufficiently small on the left, then on the right, the red and yellow segments become three arcs of circles having the same center and the same span θ.

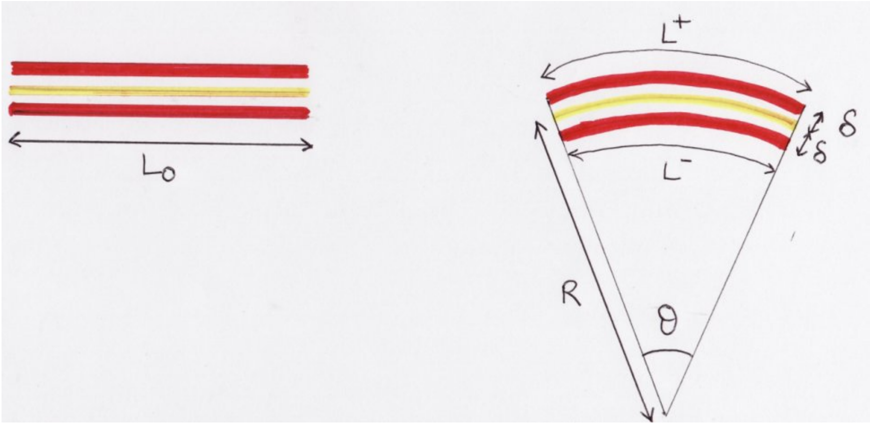

Figure 3.4: A plane portion of rectilinear membrane at rest and once bent (source: [17]).

Consequently, on the curved membrane, it is not possible for the three segments to have the same length. Indeed, the exterior red line has a greater length than the interior one. Let δ be the thickness of each layer and R the radius of curvature i.e. the radius of the circle associated with the yellow arc. Therefore, the (exterior/interior) heads have to fit in arcs of length $L^{\pm} = (R \pm \delta)\theta$ whereas the tails have a length $L_0 = R\theta$ at their disposal.

We assume that the mechanical energy of each layer is varying like an elastic one. Hence, we can define the intrinsic stiffness k of the membrane as follows: the force needed to lengthen/reduce of d a segment of initial length L_0 is $F := \frac{kd}{L_0}$. We deduce that the elastic energy of each layer is:

$$E^{\pm} := \frac{1}{2}F^{\pm}d^{\pm} = \frac{k(d^{\pm})^2}{2L_0} = \frac{k(L^{\pm} - L_0)^2}{2L_0} = \frac{k\delta^2\theta}{2R} = \frac{k\delta^2 L_0}{2R^2}.$$

We obtain that the energy of the piece of membrane is $E := E^+ + E^- = \frac{k\delta^2 L_0}{R^2}$. Introducing the constant $k_b := k\delta^2$ called the bending rigidity, and the curvature $\kappa = \frac{1}{R}$ i.e. the inverse of the radius associated with the best circle approximating the membrane at a point, we get the total energy of the membrane by summing all the contributions of infinitesimal length $ds = L_0$:

$$\mathbf{E} := \int_{\text{membrane}} E \; = \; k\delta^2 \int_{\text{membrane}} \frac{L_0}{R^2} \; = \; k_b \int_{\text{membrane}} \kappa^2 ds. \tag{3.1}$$

Finally, besides the convenience of establishing (3.1) to model the shapes of vesicles in a simpler way than its three-dimensional version presented thereafter, minimizing the elastic energy (3.1) among smooth curves with various constraints (such as the perimeter and/or the enclosed area [8]) is of great interest in itself because it appears in many fields of applied sciences.

Indeed, the problem has already been considered by Bernoulli [7] and Euler [27] to model the equilibrium shapes taken by a flexible elastic rod upon compression. They were interested in finding a curve of given length, with minimal elastic energy (3.1) joining two given points with two given tangents. The stationary configurations of this problem are called *elasticae*.

Elasticae have been studied for a long time [57]. We only mention that Euler [27] completely solved the problem, Saalschültz [80] parametrized it through elliptic functions, and Born [12] proved that *elasticae* without inflection points are stable. He also compared the model with experiments. More recently, Sachkov [81] studied *elasticae* as an optimal control problem and established the existence of minima. Let us give some applications of *elasticae* in the literature:

- one-dimensional elasticity theory [61], strength of materials (columns, beams, elastics rods), calculus of variations [8, 5, 75], optimal control theory [50, 81];
- shapes and size in biology such as tree-like structure [76] (maximal height of a tree, curvature of the spine, mechanics of insect wings) or the modelling of DNA molecules [64];

- ball rolling by the shortest path on a plane table without sliding [49], filament dynamics of vortices in incompressible flows [44], profile of capillarity surface between vertical planes [55];
- non linear splines in approximation theory [9], recovery of images (inpainting) in computer vision [74, 75], regularization of images enclosing some pixels [14].

3.3 The three-dimensional model of Canham and Helfrich

The previous reasoning can be generalized in the three-dimensional space $\mathbb{R}^3$ by replacing the curvature $\kappa = \frac{1}{R}$ by the scalar mean curvature $H = \kappa_1 + \kappa_2$. Geometrically speaking, as shown in Figure 3.5, the scalar mean curvature $H(p)$ is obtained by summing the two curvatures associated with the curves Γ formed by the intersection of the surface S with two orthogonal planes passing through the normal to the surface S at the point p.

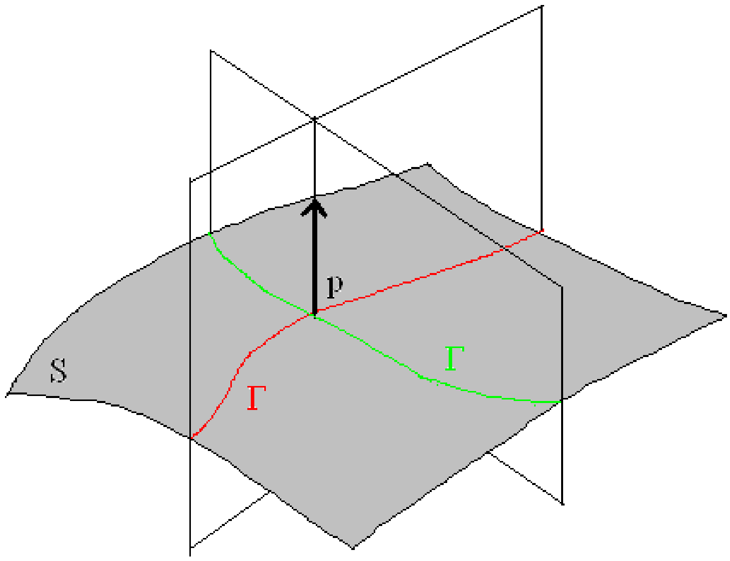

Figure 3.5: The scalar mean curvature of a surface in $\mathbb{R}^3$ is obtained by summing the curvatures of the yellow and red curves at the considered point (source: [77]).

We can prove that the value of $H(p)$ (i.e. the sum) does not depend on the choice of such pair of orthogonal planes (unlike the value of each curvature). Moreover, if we consider the two planes furnishing the highest and lowest curvature, then they are orthogonal. Their associated curvatures are called the principal curvatures and denoted by κ_1 and κ_2. From the foregoing, we have $H = \kappa_1 + \kappa_2$ and their product $K = \kappa_1 \kappa_2$ is referred to as the Gaussian curvature.

Theorem 3.1 (Gauss-Bonnet Theorem). *Let $\Sigma \subset \mathbb{R}^3$ be a compact connected $C^{1,1}$-surface (embedded without boundary) of genus $g \in \mathbb{N}$. Then, we have:*

$$\int_\Sigma K dA = 4\pi(1-g),$$

where dA is the infinitesimal area element corresponding to an integration with respect to the usual two-dimensional Hausdorff measure $A(\bullet)$.

Proof. We refer to [73, Chapter 8] for a proof on smooth surfaces in $\mathbb{R}^3$. Federer [32, Theorem 5.19] extended the result to the sets of positive reach, which is equivalent to require a $C^{1,1}$-regularity in the case of compact surfaces (cf. Theorems 16.5–16.6). We also mention [93, Section 4.7] to extend the result on smooth compact connected orientable two-dimensional Riemannian manifolds. $\square$

3.3.1 Minimizing the Canham-Helfrich energy with an area and volume constraints

During the 70s, Canham [16] then Helfrich [45] suggested a simple model to characterize vesicles. Imposing the area of the bilayer and the volume of fluid it contains, their shape is a minimizer for the following free-bending energy:

$$\mathcal{E} = \frac{k_b}{2} \int_{\text{membrane}} (H - H_0)^2 \, dA + k_G \int_{\text{membrane}} K dA, \tag{3.2}$$

where $H_0 \in \mathbb{R}$ (called the spontaneous curvature) measures the asymmetry between the two layers, and where $k_b > 0$, $k_G < 0$ are two other physical constants.

We recall that if $k_G > 0$, for any $k_b, H_0 \in \mathbb{R}$ the Canham Helfrich energy (3.2) with prescribed area and enclosed volume is not bounded from below. We refer to (1.17) or (2.17) in the introduction to get further details and known results with references about this minimization problem. These are quickly sum up in Table 3.1 below.

Some positive existence results	**A negative existence result**
The two-dimensional case [5, 24]	If $k_G > 0$, the infimum is $-\infty$
The axisymmetric case with $-2k_b < k_G < 0$ [21]	
The varifold case with $-2k_b < k_G < 0$ [6, 48, 72]	

Table 3.1: Minimizing the Canham-Helfrich energy (3.2) with prescribed area and volume.

However, the regularity of minimizers remains an open problem and experiments show that singular behaviours can occur to vesicles such as the budding transition [85, 86]. As the temperature increases, an initially spherical vesicle becomes a prolate ellipsoid, then takes a pear shape with broken up/down symmetry. Finally, the neck closes, resulting in two spherical compartments that are sitting on top of each other but still connected by a narrow constriction [85, Section 1.1].

It cannot happen to red blood cells because their skeleton prevents the membrane from bending too much locally [59, Section 2.1]. To take this aspect into account, the uniform ball condition introduced in this thesis (cf. Definition 15.1) is also strongly motivated by the modelization of the equilibrium shapes of red blood cells. Moreover, one application of our results is the existence of a minimizer to (3.2) in the class of sets satisfying the uniform ball condition (cf. Theorem 15.5).

3.3.2 Minimizing the Helfrich energy with given genus, area, and volume

Considering Theorem 3.1, instead of minimizing (3.2), people usually fix the topology and search for a minimizer of the following energy referred to as the Helfrich energy:

$$\mathcal{H}(\Sigma) = \frac{1}{4} \int_{\Sigma} (H - H_0)^2 \, dA, \tag{3.3}$$

with prescribed genus, area, and enclosed volume. Like (3.2), this functional depends on the surface but also on its orientation. However, in the case $H_0 \neq 0$, Energy (3.3) is not lower semi-continuous with respect to the varifold convergence [6, Section 9.3]: the counterexample is due to Große-Brauckmann [38]. Hence, we cannot directly use the tools of geometric measure theory but the existence of a minimizer is ensured among sets satisfying the ε-ball property (cf. Theorem 15.7).

3.3.3 Minimizing the Willmore functional with various constraints

The particular case $H_0 = 0$ in (3.3) is known as the Willmore functional:

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_{\Sigma} H^2 dA. \tag{3.4}$$

It has been widely studied by geometers due to its conformal invariance property. Some results and references about the minimization of (3.4) are summarized in Table 3.2. They were already given in the introduction. We refer to (1.19) or (2.19) for further details.

Existence for $\inf_\Sigma \mathcal{W}(\Sigma)$	Class	Constraint	Inequality
Willmore [93]	genus $g = 0$	none	$\frac{1}{4}\int H^2 dA \geqslant 4\pi$ [92]
Simon [88]	genus $g = 1$	none	$\frac{1}{4}\int H^2 dA \geqslant 2\pi^2$ [66]
Bauer and Kuwert [4]	genus $g \geqslant 1$	none	
Schygulla [84]	genus $g = 0$	isoperimetric ratio	
Keller, Mondino, and Rivière [53]	genus $g \geqslant 1$	isoperimetric ratio	

Table 3.2: Minimizing the Willmore energy (3.4) with various constraints.

An existence result related to (3.4) is the particular case $H_0 = 0$ of (3.3). Again, the difficulty with these kind of functionals is not to obtain a minimizer (compactness and lower semi-continuity in the class of varifolds for example) but to show it is regular in the usual sense (i.e. a smooth embedded surface). We now give a last application coming from the modelling of vesicles: the bilayer-couple model [85, Section 2.5.3]. For any $g \in \mathbb{N}, M_0 \in \mathbb{R}, A_0, V_0 > 0$, it states as follows:

$$\inf_{\substack{\Omega\subseteq\mathbb{R}^3,\text{open}\\ \text{genus}(\partial\Omega)=g\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0\\ \int_{\partial\Omega} H dA = M_0}} \frac{1}{4}\int_{\partial\Omega} H^2 dA, \tag{3.5}$$

where $V(\bullet)$ refers to the usual three-dimensional Hausdorff measure and where the constraint $\text{genus}(\partial\Omega) = g$ has to be understood as $\partial\Omega$ is a compact connected $C^{1,1}$-surface of genus g.

Finally, the framework of the uniform ball property can again be used to prove the existence of a minimizer to (3.5) among sets satisfying the ε-ball condition (cf. Theorem 15.8) if the isoperimetric inequality is satisfied for the constraints i.e. if $A_0^3 > 36\pi V_0^2$, otherwise the class of admissible sets is empty (or reduced to a ball in the case of equality).

3.4 From vesicles to the modelling of red blood cells

In his seminal paper, Helfrich introduced Energy (3.2) in which the spontaneous curvature is supposed to reflect a possible asymmetry of the membrane. The model becomes very popular because its simplicity still gathers the mathematical difficulties encountered in the modelling such as the budding transition [85, 86]. However, it turns out that the spontaneous curvature is a dynamical variable thus no longer constant over the vesicle. Indeed, its effective value has remained elusive since they is no measurements of this quantity for phospholipid vesicles [85, Section 2.5.2].

Then, another more general model has been proposed so far to model the equilibrium shapes of vesicles. It is called the area-difference-elasticity model [85, Section 2.4.5]. It consists in minimizing the following energy with prescribed area $A_0 > 0$ and enclosed volume $0 < V_0 < \frac{1}{6}(\frac{1}{\pi}A_0^3)^{\frac{1}{2}}$:

$$\mathcal{F} := \frac{k_b}{2}\int_{\text{membrane}} (H - H_0)^2 \, dA + k_G \int_{\text{membrane}} K dA + \frac{k_m \delta^2}{A_0}\left(\int_{\text{membrane}} H dA - \frac{\Delta A_0}{2\delta}\right)^2, \tag{3.6}$$

where the parameters of the model and their orders of magnitude are summarized in Table 3.3. Note that all the previous models can be obtained as a particular case of (3.6). Indeed, if we set $k_m = 0$, then we get (3.2). If $k_m = 0$ and $k_G = +\infty$, we deduce (3.3), and if in addition, $H_0 = 0$, we have (3.4). Finally, the case $H_0 = 0$ and $k_G = k_m = +\infty$ gives the bilayer-couple model (3.5).

We can also reduce the number of parameters by simplifying the expression of (3.6). Introducing the constant $h_0 := H_0 + \frac{\delta k_m \Delta A_0}{k_b A_0}$, called effective spontaneous curvature, (3.6) now takes the form:

$$\frac{k_b}{2}\int_{\text{membrane}} (H - h_0)^2 dA + k_G \int_{\text{membrane}} K dA + \frac{k_m \delta^2}{A_0}\left(\int_{\text{membrane}} H dA\right)^2 + \text{constants}.$$

Considering that $\frac{k_m \delta^2}{A_0}$ is negligible compared to $\frac{k_b}{2}$, the model (3.6) is again equivalent to the Canham-Helfrich's one (3.2). However, although H_0 and ΔA_0 are not readily accessible, the

Parameters	Notation	Order of magnitude
Area of the membrane	A_0	140 μm^2
Volume of fluid contained in the vesicle	V_0	100 μm^3
Thickness of a monolayer	δ	2 nm
Area modulus	k_A	0.5 J.m^{-2}
Osmotic volume modulus	k_V	7.23×10^5 J.m^{-3}
Bending modulus of the bilayer	k_b	2.0×10^{-19} J.m^{-2}
Elastic compression modulus of a monolayer	$k_m \sim \frac{3k_b}{2\delta^2}$	7.5×10^{-2} J.m^{-4}
Gaussian bending rigidity	k_G	Unknown
Spontaneous curvature	H_0	Unknown
Area difference between the two layers	$\Delta A_0 = A_{\text{out}} - A_{\text{in}}$	Unknown

Table 3.3: Parameters associated with the area-difference-elasticity model [85, Section 2.5] and their orders of magnitude [59, Sections 2.1 and 2.3, Table 2.2].

constant h_0 is calculable *a posteriori* from experiments. Indeed, we can easily evaluate the topology (i.e. the genus g) and measure the area A_0 and the volume V_0 of a real vesicle to which corresponds only one $h_0(A_0, V_0, g)$ possible.

Moreover, let us assume $H_0 = 0$ so that h_0 is proportional to the area difference $A_{out} - A_{in}$ between the outer and the inner layer of the vesicle. Hence, h_0 characterizes the asymmetry of the bilayer in a more physical way. Indeed, a positive effective spontaneous curvature promotes convex shapes whereas a negative one locally prefers concavity. In particular, with this interpretation, we recover the variety of shapes described in Figure 3.3.

Finally, although they behave the same, there is some differences between the shapes of vesicles and the ones of red blood cells. Indeed, red blood cells are equipped with an additional internal structure: a network of proteins playing the role of a skeleton inside the membrane. Its elasticity is characterized by a shear modulus $\mu \sim 2.5 \times 10^{-6}$J.m^2 and a stretch modulus $k_\alpha \sim 5 \times 10^{-6}$J.m^2. Note that both are negligible compared to the area modulus k_A (see Table 3.3).

We define the elastic length scale $\Lambda := (\frac{k_b}{\mu})^{\frac{1}{2}} \sim 0.3$ μm in order to measure the relative importance between the elastic energy associated with the vesicle and the one of the skeleton. If the local deformations imposed are negligible compared to Λ, then the skeleton does not play any role and the red blood cell behaves like a vesicle. Otherwise, the skeleton redistributes the excess of local stress on the whole surface of the red blood cell.

For example, large negative values of the effective spontaneous curvature lead to the formation of buds of characteristic radius $\frac{1}{h_0}$ connected to the vesicle by a narrow neck, whereas in the presence of a skeleton, the neck region of the red blood cell experiences an excess of shear so budding is replaced by spicule formation of typical length Λ [59, Sections 2.1.5–2.1.7, Figure 2.5].

To conclude this chapter, the skeleton acts as if a uniform bound on the curvature is imposed everywhere on the vesicle. We think this is a strong motivation for introducing the ε-ball condition (cf. Definition 15.1) as a possible model for the skeleton of red blood cells. We even have a clue on the physical value of ε. Indeed, from the foregoing, the elastic length scale Λ is a good order of magnitude for ε.

Chapter 4

Minimizing the Helfrich energy without constraint

In this chapter, we study the minimization of (3.3) among compact C^2-surfaces of $\mathbb{R}^3$:

$$\inf_{\Sigma} \frac{1}{4} \int_{\Sigma} (H - H_0)^2 dA. \tag{4.1}$$

We distinguish three cases depending on the sign of the spontaneous curvature $H_0 \in \mathbb{R}$. Then, we show the same results hold for (3.2), summarized in Table 4.1. Finally, weakening the regularity of admissible shapes, we extend the known case $H_0 = 0$ to compact simply-connected $C^{1,1}$-surfaces.

$k_G < 0 < k_b$	$H_0 < 0$	$H_0 = 0$	$H_0 > 0$
Existence to (4.1)	no global minimizer	any sphere [92]	the sphere of radius $\frac{1}{H_0}$ [2]
$\inf_\Sigma \mathcal{H}(\Sigma)$	4π	4π	0
$\inf_\Sigma \mathcal{E}(\Sigma)$	$4\pi\,(2k_b + k_G)$	$4\pi\,(2k_b + k_G)$	$4\pi k_G$

Table 4.1: On the minimization without constraint of the Canham-Helfrich energy (3.2) denoted by $\mathcal{E}$ and the Helfrich energy (3.3) referred to as $\mathcal{H}$.

4.1 The case of negative spontaneous curvature

To solve the situation $H_0 < 0$, the arguments are quite similar to the ones used in the known case $H_0 = 0$ [73, Chapter 5, Exercise (15)], whose proof is based on the Chern-Lashoff's Theorem [73, Theorem 5.29], itself depending on the surjectivity of Gauss map restricted to points whose Gaussian curvature is non-negative [73, Chapter 4, Exercise (5)]. Hence, we first establish other versions of these two results, to then treat the case of negative spontaneous curvature.

Proposition 4.1. *Let Σ be any compact C^2-surface of $\mathbb{R}^3$. Then, the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^2$ restricted to the set $\{\mathbf{x} \in \Sigma,\ H(\mathbf{x}) \geqslant 0 \text{ and } K(\mathbf{x}) \geqslant 0\}$ is surjective.*

Proof. Let $\mathbf{u} \in \mathbb{S}^2$ and Σ be a compact C^2-surface of $\mathbb{R}^3$. From the compactness of Σ, there exists a ball B of radius $r > 0$ centred at the origin $\mathbf{0}$ such that $\Sigma \subset B$. We introduce the point $\mathbf{x}_0 = \mathbf{0} + 2r\mathbf{u}$ and the function $h : \mathbf{x} \in \Sigma \mapsto \langle \mathbf{x}_0 - \mathbf{x} \mid \mathbf{u}\rangle$. First, with our choice of $\mathbf{x}_0$, note that h is positive on Σ:

$$\forall \mathbf{x} \in \Sigma, \quad h(\mathbf{x}) = \langle \mathbf{x}_0 - \mathbf{x} \mid \mathbf{u}\rangle = 2r - \langle \mathbf{x} - \mathbf{0} \mid \mathbf{u}\rangle \geqslant 2r - \|\mathbf{x} - \mathbf{0}\| \geqslant r > 0.$$

Then, the map h is continuous on the compact set Σ so there exists $\mathbf{y} \in \Sigma$ attaining its minimum. Moreover, h is differentiable on Σ thus its differential at $\mathbf{y}$ is zero. Considering any $\mathbf{v} \in T_{\mathbf{y}}\Sigma$, i.e. a curve $\alpha :]-\eta, \eta[\to \Sigma$ such that $\alpha(0) = \mathbf{y}$ and $\alpha'(0) = \mathbf{v}$, we have:

$$\forall \mathbf{v} \in T_{\mathbf{y}}\Sigma, \quad D_{\mathbf{y}}h(\mathbf{v}) := \left.\frac{d}{dt}\right|_{t=0} (h \circ \alpha)(t) = -\langle \mathbf{v} \mid \mathbf{u}\rangle = 0.$$

Consequently, the unit vector $\mathbf{u}$ is orthogonal to $T_{\mathbf{y}}\Sigma$ so we get $\mathbf{u} = \pm\mathbf{n}(\mathbf{y})$. We now show $\mathbf{u} = \mathbf{n}(\mathbf{y})$. Since the Gauss map $\mathbf{n}$ refers to the outer unit normal field to the surface, for $t > 0$ small enough, the point $\mathbf{y} - t\mathbf{n}(\mathbf{y})$ belongs to the inner domain of Σ denoted by Ω. Using the fact that $\mathbf{x}_0 \notin \overline{\Omega}$, there exists $\delta \in]0,1[$ such that $\mathbf{y}_\delta := \delta(\mathbf{y} - t\mathbf{n}(\mathbf{y})) + (1-\delta)\mathbf{x}_0$ belongs to $\partial\Omega = \Sigma$. We get:

$$h(\mathbf{y}_\delta) \geqslant h(\mathbf{y}) \quad \Longleftrightarrow \quad \delta t \langle \mathbf{n}(\mathbf{y}) \mid \mathbf{u}\rangle \geqslant (1-\delta) h(\mathbf{y})$$

Recalling that $h(\mathbf{y}) > 0$, $t > 0$ and $\delta \in]0,1[$, we finally obtain $\langle \mathbf{n}(\mathbf{y}) \mid \mathbf{u}\rangle > 0$ and thus $\mathbf{u} = \mathbf{n}(\mathbf{y})$. To conclude, it remains to prove $H(\mathbf{y}) \geqslant 0$ and $K(\mathbf{y}) \geqslant 0$. Since $\mathbf{y}$ is a critical point for h, we can define the Hessian of h at $\mathbf{y}$ by:

$$\forall \mathbf{v} \in T_{\mathbf{y}}\Sigma, \qquad D^2_{\mathbf{y}} h(\mathbf{v}) := \frac{d^2}{dt^2}\bigg|_{t=0} (h\circ\alpha)(t) = -\langle \alpha''(0) \mid \mathbf{u}\rangle = \langle \mathbf{v} \mid D_{\mathbf{y}}\mathbf{n}(\mathbf{v})\rangle,$$

the last equality coming from the derivative at $t = 0$ of the relation $\langle \alpha'(t) \mid \mathbf{n}[\alpha(t)]\rangle = 0$, which holds true since $\alpha'(t) \in T_{\alpha(t)}\Sigma$. Finally, h attains its minimum at $\mathbf{y}$ so $D^2_{\mathbf{y}} h$ is semi-positive definite. We recall that the principal curvatures $\kappa_1(\mathbf{y})$ and $\kappa_2(\mathbf{y})$ are the eigenvalues of the self-adjoint operator $D_{\mathbf{y}}\mathbf{n}$. Therefore, if $\mathbf{v}_i$ refers to the principal direction (i.e. the unit eigenvector) associated with $\kappa_i(\mathbf{y})$, $i = 1, 2$, then we get:

$$\forall i \in \{1,2\}, \quad D^2_{\mathbf{y}} h(\mathbf{v}_i) = \langle \mathbf{v}_i \mid D_{\mathbf{y}}\mathbf{n}(\mathbf{v}_i)\rangle = \kappa_i(\mathbf{y})\langle \mathbf{v}_i \mid \mathbf{v}_i\rangle = \kappa_i(\mathbf{y}) \geqslant 0,$$

from which we deduce $H(\mathbf{y}) := \kappa_1(\mathbf{y}) + \kappa_2(\mathbf{y}) \geqslant 0$ and $K(\mathbf{y}) := \kappa_1(\mathbf{y})\kappa_2(\mathbf{y}) \geqslant 0$. □

Proposition 4.2 (Chern-Lashoff's Theorem). *For any compact C^2-surface $\Sigma \subset \mathbb{R}^3$, we have:*

$$\int_{\{\mathbf{x}\in\Sigma,\ H(\mathbf{x})\geqslant 0\}} \max(0,K)dA \geqslant 4\pi.$$

Proof. Let Σ be a compact C^2-surface of $\mathbb{R}^3$. We introduce the sets $A = \{\mathbf{x} \in \Sigma,\ H(\mathbf{x}) \geqslant 0\}$, $B = \{\mathbf{x} \in \Sigma,\ H(\mathbf{x}) \geqslant 0 \text{ and } K(\mathbf{x}) \geqslant 0\}$, and their respective characteristic functions $\mathbf{1}_A$ and $\mathbf{1}_B$. We recall that the Gaussian curvature $K(\mathbf{x})$ at any point $\mathbf{x} \in \Sigma$ corresponds to the determinant of $D_{\mathbf{x}}\mathbf{n}$. Therefore, we have $\mathbf{1}_A \max(0,K) = \mathbf{1}_B|\det(D\mathbf{n})|$ and from Proposition 4.1, the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^2$ restricted to B is surjective. Applying the area formula to Σ [73, Theorem 5.28], we finally obtain:

$$\int_A \max(0,K)dA = \int_\Sigma \mathbf{1}_B|\det(D\mathbf{n})|dA = \int_{\mathbb{S}^2}\left(\sum_{\mathbf{y}\in\mathbf{n}^{-1}(\mathbf{u})} \mathbf{1}_B(\mathbf{y})\right) dA(\mathbf{u}) \geqslant \int_{\mathbb{S}^2} 1 dA = 4\pi.$$

□

Proposition 4.3. *Let $H_0 < 0$. For any compact C^2-surface $\Sigma \subset \mathbb{R}^3$, we have:*

$$\frac{1}{4}\int_\Sigma (H - H_0)^2 dA > 4\pi,$$

and considering the sequence $(\mathbb{S}_a)_{a>0}$ of spheres, $\frac{1}{4}\int_{\mathbb{S}_a}(H-H_0)^2 dA \to 4\pi$ as their radius $a \to 0^+$.

Proof. Let $H_0 < 0$ and Σ be any compact C^2-surface of $\mathbb{R}^3$. Then, we have successively:

$$\frac{1}{4}\int_\Sigma (H-H_0)^2 dA \geqslant \frac{1}{4}\int_{H\geqslant 0}(H-H_0)^2 dA \quad > \quad \frac{1}{4}\int_{H\geqslant 0} H^2 dA \geqslant \int_{H\geqslant 0}\max(0,K)dA \geqslant 4\pi,$$

where the last inequality comes from Proposition 4.2. Finally, consider the sequence of spheres $\mathbb{S}_a$ with radius $a > 0$ and we have:

$$\frac{1}{4}\int_{\mathbb{S}_a}(H-H_0)^2 dA = 4\pi\left(1 - \frac{aH_0}{2}\right)^2 \underset{a\to 0^+}{\longrightarrow} 4\pi^+.$$

To conclude, among compact C^2-surface of $\mathbb{R}^3$, we have $\inf_\Sigma \frac{1}{4}\int_\Sigma (H-H_0)^2 = 4\pi$ and the infimum is not a minimum as mentionned in Table 4.1. □

4.2 The case of zero spontaneous curvature

An umbilical point of a smooth surface is a point at which the two principal curvatures coincide. The open subsets of planes and spheres are only open connected subset of surfaces in $\mathbb{R}^3$ which are totally umbilical i.e. on which all points are umbilical. As done in [73, Theorem 3.30], this assertion is usually proved by assuming at least C^3-regularity. Since this thesis is mainly concerned with the $C^{1,1}$-case, we first establish the regularity of such open connected subset of surfaces. This is then used to show that spheres are the only global minimizers of the Willmore energy (3.4).

Proposition 4.4. *Let U be a non-empty open connected subset of $\mathbb{R}^3$ and let Σ be a connected $C^{1,1}$-surface of $\mathbb{R}^3$. If $\Sigma \cap U$ is not empty and totally umbilical i.e. $\kappa_1 = \kappa_2$ at the points of $\Sigma \cap U$ where the principal curvatures κ_1 and κ_2 are defined, then $\Sigma \cap U$ has C^∞-regularity.*

Proof. Let Σ be any connected $C^{1,1}$-surface of $\mathbb{R}^3$. Hence, the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^2$ is Lipschitz continuous. Hence, the principal curvatures κ_1 and κ_2, i.e. the two eigenvalues of $D\mathbf{n}$, and their respective principal directions i.e. their unit eigenvector $\mathbf{e}_1$ and $\mathbf{e}_2$ exist for almost every $\mathbf{x} \in \Sigma$. In the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ of the tangent plane $T_{\mathbf{x}}\Sigma$, we thus have:

$$\forall \mathbf{v} \in T_{\mathbf{x}}\Sigma, \quad D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) = D_{\mathbf{x}}\mathbf{n}\left(\langle \mathbf{v} \mid \mathbf{e}_1\rangle \mathbf{e}_1 + \langle \mathbf{v} \mid \mathbf{e}_2\rangle \mathbf{e}_2\right) = \kappa_1(\mathbf{x})\langle \mathbf{v} \mid \mathbf{e}_1\rangle \mathbf{e}_1 + \kappa_2(\mathbf{x})\langle \mathbf{v} \mid \mathbf{e}_2\rangle \mathbf{e}_2.$$

Considering any non-empty open connected set $U \subseteq \mathbb{R}^3$ such that $\Sigma \cap U \neq \emptyset$, we assume that $\Sigma \cap U$ is totally umbilical and we set $\kappa := \kappa_1 = \kappa_2$ on $\Sigma \cap U$. From the foregoing, we deduce that $D_{\mathbf{x}}\mathbf{n}$ is proportional to the identity map on $T_{\mathbf{x}}(\Sigma \cap U) = T_{\mathbf{x}}\Sigma$:

$$\forall \mathbf{v} \in T_{\mathbf{x}}(\Sigma \cap U), \quad D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) = \kappa(\mathbf{x})\left(\langle \mathbf{v} \mid \mathbf{e}_1\rangle \mathbf{e}_1 + \langle \mathbf{v} \mid \mathbf{e}_2\rangle \mathbf{e}_2\right) = \kappa(\mathbf{x})\mathbf{v}. \tag{4.2}$$

Then, for any point $\mathbf{x}_0 \in \Sigma \cap U$, there exists a cylinder $C(\mathbf{x}_0) \subset \mathbb{R}^3$ in which $\Sigma \cap U$ is the graph of a $C^{1,1}$-map φ defined on a disk $D(\mathbf{x}_0) \subset \mathbb{R}^2$ (the base of the cylinder $C(\mathbf{x}_0)$). We can introduce the local $C^{1,1}$-parametrization $X : \mathbf{x}' \in D(\mathbf{x}_0) \mapsto (\mathbf{x}', \varphi(\mathbf{x}')) \in \Sigma \cap U \cap C(\mathbf{x}_0)$. In particular, X is an homeomorphism, its inverse is the restriction of the projection $(\mathbf{x}', x_n) \mapsto \mathbf{x}'$ and its differential $\mathbf{x}' \mapsto D_{\mathbf{x}'}X$ is injective. We deduce that $(\partial_1 X(\mathbf{x}'), \partial_2 X(\mathbf{x}'))$ forms a basis of the tangent plane $T_{X(\mathbf{x}')}\Sigma = D_{\mathbf{x}'}X(\mathbb{R}^2)$, not necessarily orthonormal. Consequently, for $i = 1, 2$, we get from (4.2):

$$\partial_i(\mathbf{n} \circ X) = D_{\mathbf{x}}\mathbf{n}(\partial_i X) = (\kappa \circ X)\partial_i X.$$

We now show that $\kappa \circ X$ is Lipschitz continuous. Note that if it is the case, the above relation implies that $\mathbf{n}\circ X$ has $C^{1,1}$-regularity. Since the outer unit normal is given in the parametrization by $\mathbf{n}\circ X = [1+(\partial_1\varphi)^2+(\partial_2\varphi)^2]^{-\frac{1}{2}}(-\partial_1\varphi, -\partial_2\varphi, 1)$ [46, (5.40)], we deduce that $\partial_i\varphi = -(\mathbf{n}\circ X)_i/(\mathbf{n}\circ X)_3$ has also $C^{1,1}$-regularity i.e. $\Sigma \cap U$ is a $C^{2,1}$-surface. Applying recursively the argument, we obtain that $\Sigma \cap U$ is a smooth surface. We first have:

$$\partial_1(\mathbf{n} \circ X) = (\kappa \circ X)\partial_1 X \Leftrightarrow \begin{cases} \partial_1\varphi\partial_2\varphi\partial_{12}\varphi - \left[1 + (\partial_2\varphi)^2\right]\partial_{11}\varphi = (\kappa \circ X)\left[1 + \|\nabla\varphi\|^2\right]^{\frac{3}{2}} \\[2ex] \partial_1\varphi\partial_2\varphi\partial_{11}\varphi - \left[1 + (\partial_1\varphi)^2\right]\partial_{12}\varphi = 0 \\[2ex] -\partial_1\varphi\partial_{11}\varphi - \partial_2\varphi\partial_{12}\varphi = (\kappa \circ X)\partial_1\varphi\left[1 + \|\nabla\varphi\|^2\right]^{\frac{3}{2}}, \end{cases} \tag{4.3}$$

and also

$$\partial_2(\mathbf{n} \circ X) = (\kappa \circ X)\partial_2 X \Leftrightarrow \begin{cases} \partial_1\varphi\partial_2\varphi\partial_{22}\varphi - \left[1 + (\partial_2\varphi)^2\right]\partial_{21}\varphi = 0 \\[2ex] \partial_1\varphi\partial_2\varphi\partial_{21}\varphi - \left[1 + (\partial_1\varphi)^2\right]\partial_{22}\varphi = (\kappa \circ X)\left[1 + \|\nabla\varphi\|^2\right]^{\frac{3}{2}} \\[2ex] -\partial_1\varphi\partial_{21}\varphi - \partial_2\varphi\partial_{22}\varphi = (\kappa \circ X)\partial_2\varphi\left[1 + \|\nabla\varphi\|^2\right]^{\frac{3}{2}}. \end{cases} \tag{4.4}$$

We can assume that $\nabla\varphi \neq (0,0)$, up to a rotation on a smaller cylinder than $C(\mathbf{x}_0)$. Using the first two relations in (4.3) and in (4.4), we obtain:

$$\begin{cases} \partial_{11}\varphi = -(\kappa \circ X)[1 + (\partial_1\varphi)^2]\sqrt{1 + \|\nabla\varphi\|^2} \\[2ex] \partial_{12}\varphi = \partial_{21}\varphi = -(\kappa \circ X)\partial_1\varphi\partial_2\varphi\sqrt{1 + \|\nabla\varphi\|^2} \\[2ex] \partial_{22}\varphi = -(\kappa \circ X)[1 + (\partial_2\varphi)^2]\sqrt{1 + \|\nabla\varphi\|^2}. \end{cases} \tag{4.5}$$

Furthermore, from the second relation in (4.3), we deduce that:

$$\frac{2\partial_1\varphi\partial_{11}\varphi}{1+(\partial_1\varphi)^2}-\frac{2\partial_{12}\varphi}{\partial_2\varphi}=0 \quad \text{i.e.} \quad \partial_1\left[\ln\left(\frac{1+(\partial_1\varphi)^2}{(\partial_2\varphi)^2}\right)\right]=0.$$

Consequently, there exists a map $f : x_2 \mapsto f(x_2)$ such that $1+[\partial_1\varphi(x_1,x_2)]^2 = e^{f(x_2)}[\partial_2\varphi(x_1,x_2)]^2$. In particular, this means that f is a Lipschitz continuous map. Similarly, we can use the first relation in (4.4) to get the existence of a Lipschitz continuous map $g : x_1 \mapsto g(x_1)$ satisfying $1+[\partial_2\varphi(x_1,x_2)]^2 = e^{g(x_1)}[\partial_1\varphi(x_1,x_2)]^2$. Note that the last two equalities imply that:

$$\left(1+e^{f(x_2)}\right)[\partial_2\varphi(x_1,x_2)]^2 = \left(1+e^{g(x_1)}\right)[\partial_1\varphi(x_1,x_2)]^2. \tag{4.6}$$

Moreover, we have $\nabla\varphi \neq (0,0)$ so the Lipschitz continuous maps $\partial_1\varphi$ and $\partial_2\varphi$ have constant sign on the simply connected disk $D(\mathbf{x}_0)$, let us say both positive. Hence, we get:

$$\forall (x_1,x_2)\in D(\mathbf{x}_0), \qquad \frac{\partial_1\varphi(x_1,x_2)}{\partial_2\varphi(x_1,x_2)} = \sqrt{\frac{1+e^{f(x_2)}}{1+e^{g(x_1)}}}. \tag{4.7}$$

Then, since f and g are Lipschitz continuous, they are differentiable almost everywhere and we can take successively the partial derivatives ∂_1 and ∂_2 in (4.6). Combined with (4.5) and (4.6), the two relations obtained become for almost every $(x_1,x_2)\in D(\mathbf{x}_0)$:

$$2(\kappa\circ X)(x_1,x_2)\sqrt{1+\|\nabla\varphi(x_1,x_2)\|^2} = \frac{\partial_1\varphi(x_1,x_2)g'(x_1)e^{g(x_1)}}{1+e^{g(x_1)}} = \frac{\partial_2\varphi(x_1,x_2)f'(x_2)e^{f(x_2)}}{1+e^{f(x_2)}}. \tag{4.8}$$

Finally, we deduce from the last equality of (4.8) and (4.7) that the following relation holds:

$$\forall \text{ a.e. } (x_1,x_2)\in D(\mathbf{x}_0), \qquad \frac{g'(x_1)e^{g(x_1)}}{\left[1+e^{g(x_1)}\right]\sqrt{1+e^{g(x_1)}}} = \frac{f'(x_2)e^{f(x_2)}}{\left[1+e^{f(x_2)}\right]\sqrt{1+e^{f(x_2)}}}.$$

Note that the left-hand side of the above equality only depends on x_1 while in the right-hand side, only x_2 appears. Consequently, there exists a constant $C>0$ such that :

$$\forall \text{ a.e. } (x_1,x_2)\in D(\mathbf{x}_0), \quad g'(x_1) = Ce^{-g(x_1)}\left[1+e^{g(x_1)}\right]^{\frac{3}{2}} \text{ and } f'(x_2) = Ce^{-f(x_2)}\left[1+e^{f(x_2)}\right]^{\frac{3}{2}}.$$

Hence, f and g are Lipschitz continuous maps whose derivatives are almost every equal to some continuous maps. Standard arguments [32, Lemma 4.7] show that the above relations hold for *any* $(x_1,x_2)\in D(\mathbf{x}_0)$, from which we deduce recursively that f and g have C^∞-regularity. Therefore, getting back to (4.8), the map $\kappa\circ X$ is Lipschitz continuous on $D(\mathbf{x}_0)$ for any $\mathbf{x}_0\in\Sigma\cap U$:

$$(\kappa\circ X)(x_1,x_2) = \frac{\partial_1\varphi(x_1,x_2)g'(x_1)e^{g(x_1)}}{2\left[1+e^{g(x_1)}\right]\sqrt{1+\|\nabla\varphi(x_1,x_2)\|^2}} = \frac{\partial_2\varphi(x_1,x_2)f'(x_2)e^{f(x_2)}}{2\left[1+e^{f(x_2)}\right]\sqrt{1+\|\nabla\varphi(x_1,x_2)\|^2}}.$$

To conclude, $\kappa\circ X$ is Lipschitz continuous so from (4.5), φ is of class $C^{2,1}$, and recursively, the local map $\varphi : D(\mathbf{x}_0)\to\mathbb{R}$ is smooth for any $\mathbf{x}_0\in\Sigma\cap U$ thus $\Sigma\cap U$ has C^∞-regularity. □

Proposition 4.5 (Willmore [93, Theorem 7.2.2]). *Let Σ be any compact C^2-surface of $\mathbb{R}^3$. Then, we have:*

$$\frac{1}{4}\int_\Sigma H^2 dA \geqslant 4\pi,$$

where the equality holds if and only if Σ is a sphere.

Proof. We essentially follow the original proof of Willmore [93, Theorem 7.2.2], which is more detailed in [73, Chapter 5, Exercise (15)]. Let Σ be any compact C^2-surface of $\mathbb{R}^3$. First, we have $\frac{1}{4}H^2 - K = \frac{1}{4}(\kappa_1-\kappa_2)^2 \geqslant 0$ so we get from Proposition 4.2:

$$\frac{1}{4}\int_\Sigma H^2 dA \geqslant \int_\Sigma \max(0,K)dA \geqslant \int_{H\geqslant 0}\max(0,K)dA \geqslant 4\pi.$$

Then, we consider any compact C^2-surface $\Sigma\subset\mathbb{R}^3$ satisfying $\frac{1}{4}\int_\Sigma H^2 dA = 4\pi$. From the foregoing, we deduce that $\frac{1}{4}H^2 = \max(K,0)$ so $\kappa_1=\kappa_2$ on the set of elliptic points $A=\{\mathbf{x}\in\Sigma,\ K(\mathbf{x})>0\}$, which is not empty since Σ is compact [73, Exercise 3.42] and open since $K : \Sigma \to \mathbb{R}$ is continuous [73, Proposition 3.43]. Finally, consider any $\mathbf{x} \in A$ and the (open) connected component C of Σ for which $\mathbf{x} \in C$. From Proposition 4.4, note that Σ has C^∞-regularity at the points of $C \cap A$. These are umbilical points so they are included in a sphere [73, Theorem 3.30]. Now assume by contradiction there exists a point $\mathbf{y} \in C \backslash A$. From connectedness, there exists a continuous curve $\alpha : [0,1] \to C$ such that $\alpha(0) = \mathbf{x}$ and $\alpha(1) = \mathbf{y}$. Consider the first time $t_0 \in (0,1]$ such that $\alpha(t_0) \notin A \cap C$ i.e. $K([\alpha(t_0)] \leqslant 0$ and $\alpha([0,t_0)) \subseteq C \cap A$. Hence, $\alpha([0,t_0))$ is included in a sphere and by continuity, $K[\alpha(t_0)] = \lim_{t \to t_0^-} K[\alpha(t)] > 0$, contradicting the definition of t_0. Therefore, the compact connected smooth surface C is totally umbilical thus it is a sphere [73, Corollary 3.31]. We get $\frac{1}{4}\int_C H^2 = 4\pi = \frac{1}{4}\int_\Sigma H^2$ so $\Sigma \equiv C$, otherwise $\Sigma \backslash C$ is a compact minimal C^2-surface, which is not possible [73, Exercise 3.42]. To conclude, spheres are the unique minimizers of (3.4). $\square$

4.3 The case of positive spontaneous curvature

The case of positive spontaneous curvature is an equivalent formulation of the Alexandrov Theorem [2] stating that spheres are the only compact connected smooth surfaces of constant mean curvature. The original proof of Alexandrov assumes the analyticity of the considered surfaces. Here, we treat the case of C^2-regularity and essentially follow [73, Section 6.4].

Proposition 4.6 (Heintze-Karcher's inequality [73, Theorem 6.16]). *Let Σ be any compact connected C^2-surface of $\mathbb{R}^3$, whose scalar mean curvature $H = \kappa_1 + \kappa_2$ is positive everywhere. Then, we have:*

$$3V(\Sigma) \leqslant \int_\Sigma \frac{2}{H} dA,$$

where $V(\Sigma)$ refers to the volume of the inner domain enclosed by Σ. Moreover, the equality holds if and only if Σ is a sphere.

Proof. Let Σ be any compact connected C^2-surface of $\mathbb{R}^3$. We order its principal curvatures i.e. $k_1 \leqslant \frac{1}{2}H \leqslant k_2$ and we assume that $H > 0$ everywhere on Σ. Hence, we have $k_2 > 0$ and we can introduce the set $A = \{(\mathbf{x}, t) \in \Sigma \times \mathbb{R},\ 0 \leqslant t \leqslant [k_2(\mathbf{x})]^{-1}\}$. If we denote by $\varepsilon > 0$ a real number strictly greater than the maximum of the continuous map $\frac{1}{k_2} : \Sigma \to]0, +\infty[$ on the compact set Σ, then A is a compact set contained in $\Sigma \times [0, \varepsilon)$. First, we show that $\Omega \subseteq F(A)$, where Ω is the inner domain enclosed by Σ and $F : (\mathbf{x}, t) \in \Sigma \times \mathbb{R} \to \mathbf{x} - t\mathbf{n}(\mathbf{x})$. Let $\mathbf{x} \in \Omega$. The continuous map $f : \mathbf{y} \in \Sigma \to \|\mathbf{y} - \mathbf{x}\|^2$ attains its minimum on the compact set Σ at a point $\mathbf{y}_0 \in \Sigma$. Moreover, f is differentiable on Σ so its differential at $\mathbf{y}_0$ is zero. Considering any $\mathbf{v} \in T_{\mathbf{y}_0}\Sigma$ i.e. a curve $\alpha : [-\delta, \delta] \to \Sigma$ such that $\alpha(0) = \mathbf{y}_0$ and $\alpha'(0) = \mathbf{v}$, we thus have:

$$\forall \mathbf{v} \in T_{\mathbf{y}_0}\Sigma, \quad D_{\mathbf{y}_0} f(\mathbf{v}) = \left. \frac{d}{ds} \right|_{s=0} (f \circ \alpha)(s) = 2\langle \mathbf{y}_0 - \mathbf{x} \mid \mathbf{v} \rangle = 0.$$

Hence, $\mathbf{y}_0 - \mathbf{x}$ is orthogonal to $T_{\mathbf{y}_0}\Sigma$ i.e. there exists $t \in \mathbb{R}$ such that $\mathbf{x} = F(\mathbf{y}_0, t)$. It remains to show that $0 \leqslant t \leqslant \frac{1}{k_2(\mathbf{y}_0)}$. Since $\mathbf{n}(\mathbf{y}_0)$ is the unit outer normal, for $s > 0$ small enough, we have $\mathbf{y}_0 + s\mathbf{n}(\mathbf{y}) \notin \overline{\Omega}$. Therefore, there exists $\eta \in]0,1[$ such that $\mathbf{y}_\eta := \eta\mathbf{x} + (1-\eta)[\mathbf{y}_0 + s\mathbf{n}(\mathbf{y})] \in \partial\Omega = \Sigma$, from which we deduce that $\|\mathbf{y}_\eta - \mathbf{x}\|^2 \geqslant \|\mathbf{y}_0 - \mathbf{x}\|^2 \Leftrightarrow t \geqslant -\frac{s}{2}$. By letting $s \to 0^+$, we obtain $t \geqslant 0$. It remains to prove $t \leqslant \frac{1}{k_2(\mathbf{y}_0)}$. Since $\mathbf{y}_0$ is a critical point, we can define the Hessian of f at $\mathbf{y}_0$ by:

$$\forall \mathbf{v} \in T_{\mathbf{y}_0}\Sigma, \quad D^2_{\mathbf{y}_0} f(\mathbf{v}) := \left. \frac{d^2}{ds^2} \right|_{s=0} (f \circ \alpha)(s) = 2\|\mathbf{v}\|^2 + 2t\langle \alpha''(0) \mid \mathbf{n}(\mathbf{y}_0)\rangle = 2\|\mathbf{v}\|^2 - 2t\langle \mathbf{v} \mid D_{\mathbf{y}}\mathbf{n}(\mathbf{v})\rangle,$$

the last equality coming from the derivative at $s = 0$ of the relation $\langle \alpha'(s) \mid \mathbf{n}[\alpha(s)]\rangle = 0$, which holds true since $\alpha'(s) \in T_{\alpha(s)}\Sigma$. Since $\mathbf{y}_0$ is a minimum for f, the Hessian of f at $\mathbf{y}_0$ is semi-definite positive. Considering the principal direction $\mathbf{v}_2$ associated with k_2, we get that $1 - tk_2(\mathbf{y}_0) \geqslant 0$ and $(\mathbf{y}_0, t) \in A$. Consequently, we have proved $\Omega \subseteq F(A)$. Then, we combine Fubini's Theorem [73, Theorem 5.18], the area formula [73, Theorem 5.27] and the fact that $\Omega \subseteq F(A)$ to get successively:

$$\begin{aligned}
\int_{\Sigma}\left(\int_0^{\frac{1}{k_2(\mathbf{y})}} |\det(D_{(\mathbf{y},t)}F)|dt\right) dA(\mathbf{y}) &= \int_{\Sigma\times(0,a)} \mathbf{1}_A(\mathbf{y},t)|\det(D_{(\mathbf{y},t)}F)|dV(\mathbf{y},t) \\
&= \int_{\mathbb{R}^3}\left(\sum_{(\mathbf{y},t)\in F^{-1}(\{x\})} \mathbf{1}_A(\mathbf{y},t)\right) dV(\mathbf{x}) \\
&\geqslant \int_{\mathbb{R}^3} \mathbf{1}_\Omega(\mathbf{x})dV(\mathbf{x}) := V(\Sigma).
\end{aligned}$$

Finally, we have to estimate the left-hand side of the above inequality. Using the orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{n})$, we get $|\det(D_{(\mathbf{y},t)}F)| = |\det[D_{\mathbf{y}}F_t(\mathbf{e}_1), D_{\mathbf{y}}F_t(\mathbf{e}_2), \mathbf{n}(\mathbf{y})]| = |(1-t\kappa_1(\mathbf{y}))(1-t\kappa_2(\mathbf{y}))|$. Moreover, for $0 \leqslant t \leqslant \frac{1}{k_2(\mathbf{y})}$, we have $\det(D_{(\mathbf{y},t)}F) \geqslant 0$ so we deduce that:

$$\forall(\mathbf{y},t)\in A,\ |\det(D_{(\mathbf{y},t)}F)| = \left[1-\frac{t}{2}H(\mathbf{y})\right]^2 - t^2\left[\frac{1}{4}H^2(\mathbf{y}) - K(\mathbf{y})\right] \leqslant \left[1-\frac{t}{2}H(\mathbf{y})\right]^2,$$

where equality holds in the last relation if and only if $\frac{1}{4}H^2 = K$ on Σ i.e. iff Σ is totally umbilical. Since $\frac{1}{k_2} \leqslant \frac{2}{H}$ and the integrand is positive, we obtain from the foregoing:

$$V(\Sigma) \leqslant \int_{\Sigma}\left(\int_0^{\frac{1}{k_2(\mathbf{y})}} |\det(D_{(\mathbf{y},t)}F)|dt\right) dA(\mathbf{y}) \leqslant \int_{\Sigma}\underbrace{\left(\int_0^{\frac{2}{H(\mathbf{y})}}\left[1-\frac{tH(\mathbf{y})}{2}\right]^2 dt\right)}_{=\frac{2}{3H(\mathbf{y})}} dA(\mathbf{y}).$$

Hence, we have $3V(\Sigma) \leqslant \int_\Sigma \frac{2}{H}dA$. Furthermore, if the equality holds then the compact surface Σ is totally umbilical, thus have C^∞-regularity from Proposition 4.4. Combined with connectedness, we get that Σ is a sphere [73, Corollary 3.31]. Conversely, any sphere satisfies $3V(\Sigma) = \int_\Sigma \frac{2}{H}dA$. $\square$

Proposition 4.7 (Alexandrov [2]). *Let $H_0 > 0$ and Σ be a compact C^2-surface of $\mathbb{R}^3$. Then, we have:*

$$\frac{1}{4}\int_\Sigma (H-H_0)^2 dA \geqslant 0,$$

where the equality holds if and only if Σ is the union of a finite number of pairwise disjoint copies of the sphere $\mathbb{S}_{H_0}$ with radius $2/H_0$.

Proof. Let $H_0 > 0$ and Σ be any compact C^2-surface of $\mathbb{R}^3$ satisfying $\frac{1}{4}\int_\Sigma(H-H_0)^2dA = 0$ i.e. such that Σ has constant scalar mean curvature $H = H_0 > 0$. Combining the divergence theorem for surfaces [73, Theorem 6.11] and the divergence theorem [73, Theorem 5.31], we have:

$$\begin{aligned}
\int_\Sigma \frac{2}{H}dA &= \frac{2A(\Sigma)}{H_0} = \frac{1}{H_0}\int_\Sigma \mathrm{div}_{\partial\Omega}(\mathbf{x})dA(\mathbf{x}) = \frac{1}{H_0}\int_\Sigma H(\mathbf{x})\langle \mathbf{x} \mid \mathbf{n}(\mathbf{x})\rangle dA(\mathbf{x}) \\
&= \int_\Sigma \langle \mathbf{x} \mid \mathbf{n}(\mathbf{x})\rangle dA(\mathbf{x}) = \int_\Omega \mathrm{div}(\mathbf{x})dV(\mathbf{x}) = 3V(\Omega),
\end{aligned}$$

where Ω is the inner domain enclosed by Σ. If Σ is connected, then we apply the equality case of Proposition 4.6 to get that Σ is the sphere $\mathbb{S}_{H_0}$ of radius $\frac{2}{H_0}$. Otherwise, from compactness, Σ has a finite number of connected components (cf. Lemma 25.13), each one being a copy of $\mathbb{S}_{H_0}$. $\square$

4.4 The case of the Canham-Helfrich energy

We study here the minimization of the Canham-Helfrich energy (3.2) among compact C^2-surfaces of $\mathbb{R}^3$. We just need to combine Theorem 3.1 with the previous results obtained for the Helfrich energy (3.3) in order to get similar statements.

Proposition 4.8. *Let $k_b > 0 > k_G$ and Σ be any compact C^2-surface of $\mathbb{R}^3$.*

(i) If $H_0 < 0$, then $\mathcal{E}(\Sigma) > 4\pi(2k_b + k_G)$ and the sequence of spheres $\mathbb{S}_a$ with radius $a > 0$ satisfies $\mathcal{E}(\mathbb{S}_a) \underset{a\to 0^+}{\longrightarrow} 4\pi(2k_b + k_G)$.

(ii) If $H_0 = 0$, then $\mathcal{E}(\Sigma) \geqslant 4\pi(2k_b + k_G)$ and the equality holds if and only if Σ is a sphere.

(iii) If $H_0 > 0$, then $\mathcal{E}(\Sigma) \geqslant 4\pi k_G$, where the equality holds if and only if Σ is the sphere $\mathbb{S}_{H_0}$ of radius $\frac{1}{H_0}$.

Proof. Let $k_b > 0 > k_G$ and Σ be any compact C^2-surface of $\mathbb{R}^3$. First, we treat the situation (i). If $H_0 < 0$, then we combine Proposition 4.3 and Theorem 3.1 to get:

$$2k_b\left(\frac{1}{4}\int_\Sigma (H - H_0)^2 dA\right) + k_G\int_\Sigma K dA > 8\pi k_b + 4\pi k_G[1 - g(\Sigma)] \geqslant 4\pi(2k_b + k_G).$$

Moreover, for any sphere $\mathbb{S}_a$ of radius $a > 0$, we have $\mathcal{E}(\mathbb{S}_a) = 8\pi k_b(1-\frac{aH_0}{2})^2+4\pi k_G$ and it converges to $4\pi(2k_b + k_G)$ as $a \to 0^+$. Similarly in (ii), if $H_0 = 0$, combine Proposition 4.5 and Theorem 3.1 to get $\mathcal{E}(\Sigma) \geqslant 4\pi(2k_b + k_G)$. Furthermore, if the equality holds, then $\int_\Sigma K dA = 4\pi = \frac{1}{4}\int_\Sigma H^2 dA$ and $\frac{1}{4}H^2 = K$ on Σ i.e Σ is a sphere. Conversely, any sphere satisfies the equality case. Finally, concerning (iii), if $H_0 > 0$, then from Proposition 4.7 and Theorem 3.1, we have $\mathcal{E}(\Sigma) \geqslant 4\pi k_G$. If the equality holds, we deduce $\int_\Sigma K dA = 4\pi$ and $H = H_0$ on Σ. Hence, from Theorem 3.1, Σ has the topology of spheres. In particular, Σ is connected and the equality case of Proposition 4.7 ensures Σ is the sphere $\mathbb{S}_{H_0}$ of radius $\frac{1}{H_0}$. To conclude, we conversely have $\mathcal{E}(\mathbb{S}_{H_0}) = 4\pi k_G$. □

4.5 The case of $C^{1,1}$-regularity

We minimize the Willmore energy (3.4) among compact simply connected $C^{1,1}$-surfaces. Indeed, we impose the topology of spheres because continuity arguments like those used in the proofs of Propositions 4.1, 4.5, and 4.6 does not hold in this case: the principal curvatures are only defined almost everywhere as $L^\infty(\Sigma)$-map. However, Theorem 3.1 and Proposition 4.4 still hold true.

Proposition 4.9. *Let Σ be any compact simply-connected $C^{1,1}$-surface of $\mathbb{R}^3$. Then, we have:*

$$\frac{1}{4}\int_\Sigma H^2 dA \geqslant 4\pi,$$

where the equality holds if and only if Σ is a sphere.

Proof. Let Σ be any compact simply-connected $C^{1,1}$-surface of $\mathbb{R}^3$. Since it has the topology of a sphere and $\frac{1}{4}H^2 \geqslant K$ almost everywhere on Σ, we have from Theorem 3.1:

$$\frac{1}{4}\int_\Sigma H^2 dA \geqslant \int_\Sigma K dA = 4\pi.$$

Moreover, if the equality holds, then $\frac{1}{4}H^2 = K$ almost everywhere on Σ. Apply Proposition 4.4 and Σ is a smooth surface which is totally umbilical. Then, we conclude with [73, Corollary 3.31]: Σ is a sphere and conversely, any sphere satisfies $\frac{1}{4}\int H^2 dA = 4\pi$. □

To conclude this chapter, we recall that in Chapter 3, we made a state of art on the modelling of vesicles and red blood cells. A simple model consists in minimizing the Helfrich energy (3.3) with prescribed genus, area, and enclosed volume. In Chapter 4, this problem is studied without constraint. Hence, our only contribution in this chapter was to get familiar with the basic tools of geometry in order to adapt standard proofs to solve Problem (4.1). Henceforth, the remaining part of this report details our work and the contributions of the thesis presented in the introduction.

Chapter 5

Minimizing the Helfrich energy under area constraint

In this chapter, we are interested in minimizing the Helfrich energy (3.3) among compact simply-connected $C^{1,1}$-surfaces of $\mathbb{R}^3$ with prescribed area $A_0 > 0$:

$$\inf_{A(\Sigma)=A_0} \frac{1}{4}\int_\Sigma (H - H_0)^2 dA. \tag{5.1}$$

We refer to Theorem 15.7, concerned with area and volume constraints, so as to get some existence results associated with (5.1). Indeed, apart from excluding the sphere from the class of admissible shapes, the volume constraint does not seem to play a specific role in the theoretical process used in calculus of variations i.e. neither in the compactness of the minimizing sequence, nor in the (lower semi-)continuity of the functional and constraints, nor in the regularity of minimizers.

Moreover, the sphere $\mathbb{S}_{A_0}$ of area A_0 seems a good candidate for being the minimizer of (5.1). In this chapter, we study in detail the global optimality of this sphere. The results are summarized in the first row of Table 5.1 below. They depend on a specific adimensional parameter:

$$c_0 := \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}, \tag{5.2}$$

and we prove the existence of two numbers $c_- \approx -0.575$ and $c_+ \approx 1.46$ which are threshold values.

Parameter $c_0 = \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}$	$-\infty$ c_- 0 1 c_+ c_{++} $+\infty$
Is the sphere a global minimizer ?	NO [? [YES] ?] NO
Is the sphere a local minimizer ?	YES] ?] NO
Is the sphere a critical point ?	YES

Table 5.1: Results obtained concerning the optimality in (5.1) of the sphere $\mathbb{S}_{A_0}$ with area A_0.

First, in Section 5.1, for any $c_0 \in [0, 1]$, we deduce from the Cauchy-Schwarz inequality that $\mathbb{S}_{A_0}$ is the unique global minimizer of (5.1). Then, in Section 5.2, for any $c_0 > c_+$, we establish that $\mathbb{S}_{A_0}$ is not a minimizer of (5.1) among cigars. In particular, in that case, we deduce that $\mathbb{S}_{A_0}$ is no longer a global minimizer, even in a smaller subclass of admissible shapes (convex, axisymmetric, uniform ball condition).

Finally, in Section 5.3, for any $c_0 < c_-$, we prove that a sequence of non-convex axisymmetric $C^{1,1}$-surfaces of constant area (converging to a double-sphere) has a strictly lower Helfrich energy (3.3) than $\mathbb{S}_{A_0}$, which is thus not a global minimizer of (5.1).

However, and this is the purpose of the second part in this report, if we restrict the class of admissible shapes to the ones enclosing a convex inner domain, or to the one bounding an *axiconvex* domain, i.e. an axisymmetric domain whose intersection with any plane orthogonal to the symmetry axis is either a disk or empty, then the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of (5.1) (cf. Inequality (7.1) and Theorem 7.1).

5.1 A case where the sphere is the unique global minimizer

Proposition 5.1. *Let $A_0 > 0$ and $H_0 \in \mathbb{R}$ such that $c_0 \in [0,1]$, where c_0 is defined by (5.2). Then, the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of (5.1).*

Proof. Let $H_0 \in \mathbb{R}$, $A_0 > 0$, and Σ be any compact simply connected $C^{1,1}$-surface of $\mathbb{R}^3$ with prescribed area $A_0 > 0$. First, we deduce from the Cauchy-Schwarz inequality:

$$\begin{aligned}\frac{1}{4}\int_\Sigma (H-H_0)^2 dA &= \frac{1}{4}\int_\Sigma H^2 dA - \frac{H_0}{2}\int_\Sigma H dA + \frac{H_0^2 A_0}{4}\\ &\geqslant \frac{1}{4}\int_\Sigma H^2 dA - |H_0|\sqrt{A_0}\sqrt{\frac{1}{4}\int_\Sigma H^2 dA} + \frac{H_0^2 A_0}{4}\\ &= 4\pi\left(\sqrt{\frac{\frac{1}{4}\int_\Sigma H^2 dA}{4\pi}} - \frac{|H_0|}{2}\sqrt{\frac{A_0}{4\pi}}\right)^2.\end{aligned}$$

Then, we set $c_0 := \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}$ and assume $c_0 \in [0,1]$. Combined with Proposition 4.9, this gives:

$$\sqrt{\frac{\frac{1}{4}\int_\Sigma H^2 dA}{4\pi}} - \frac{|H_0|}{2}\sqrt{\frac{A_0}{4\pi}} \geqslant 1 - |c_0| = 1 - c_0 \geqslant 0. \tag{5.3}$$

Finally, considering the sphere $\mathbb{S}_{A_0}$ of area A_0, we obtain from the foregoing:

$$\frac{1}{4}\int_\Sigma (H-H_0)^2 dA \geqslant 4\pi\left(\sqrt{\frac{\frac{1}{4}\int_\Sigma H^2 dA}{4\pi}} - \frac{|H_0|}{2}\sqrt{\frac{A_0}{4\pi}}\right)^2 \geqslant 4\pi(1-c_0)^2 = \frac{1}{4}\int_{\mathbb{S}_{A_0}} (H-H_0)^2 dA.$$

The last equality can be checked by direct calculation since the scalar mean curvature is constant over $\mathbb{S}_{A_0}$. Hence, $\mathbb{S}_{A_0}$ is a global minimizer of (5.1). To conclude, if Σ is a global minimizer of (5.1), then the equality holds in (5.3) i.e. $\frac{1}{4}\int_\Sigma H^2 dA = 4\pi$ and Proposition 4.9 ensures that $\Sigma \equiv \mathbb{S}_{A_0}$. $\square$

5.2 Minimizing among cigars of prescribed area

Definition 5.2. *We call cigar any cylinder of length $L \geqslant 0$ on which are glued two half spheres of radius $R > 0$, as illustrated in Figure 5.1.*

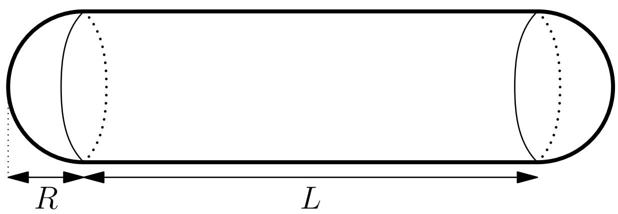

Figure 5.1: A cigar of length $L \geqslant 0$ and radius $R > 0$.

Hence, a cigar has the topology of spheres i.e. it is compact and simply connected. Moreover, it is axisymmetric, *axiconvex* (cf. Theorem 7.1), and it encloses a convex inner domain. Furthermore, it is a $C^{1,1}$-surface since it satisfies the R-ball condition (cf. Definition 15.1 and Theorem 16.6) but it is not C^2-regular since the scalar mean curvature is not continuous at the points of gluing.

Theorem 5.3. *Let $A_0 > 0$ and $H_0 \in \mathbb{R}$, c_0 as in (5.2) and $c_+ := \frac{1}{4}(1+\sqrt{2})^2 \sim 1.46$. If $c_0 < c_+$, then the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of (5.1) among the class of cigars. Moreover, if $c_0 > c_+$, then the minimizer is the cigar of area A_0 and radius:*

$$R_- := \sqrt{\frac{A_0}{3\pi}}\cos\left[\frac{1}{3}\arccos\left(\frac{-3}{H_0}\sqrt{\frac{3\pi}{A_0}}\right) + \frac{4\pi}{3}\right].$$

Finally, if $c_0 = c_+$, then $\mathbb{S}_{A_0}$ and the cigar of area A_0 and radius R_- are the only two global minimizers of (3.3) among cigars of prescribed area A_0.

The remaining part of Section 5.2 is devoted to the proof of Theorem 5.3. The optimization problem (5.1) is first formulated in a equivalent form more suitable to cigars. It is expressed as minimizing on an interval a real-valued function only depending on A_0, H_0 and the radius R of cigars. Then, we carefully study its variation, from which we have to distinguish different cases. In particular, we show that the global minimizer can only be the sphere and/or a specific cigar. Finally, a finer analysis is made to determine which one(s) is/are the minimizer(s).

5.2.1 A formulation of the problem in terms of radius

Admissible interval for the radius

First, the area of a cigar is $A_0 = 2\pi RL + 4\pi R^2 = 2\pi R(L + 2R)$. Hence, they are two limit cases. On the one hand, if $L \to 0^+$, then the cigar becomes a sphere of area A_0 with radius:

$$R_0 = \frac{1}{2}\sqrt{\frac{A_0}{\pi}}.$$

On the other hand, if $L \to +\infty$, then the cigar tends to become a line and thus $R \longrightarrow 0^+$. Finally, for any $R \in]0, R_0]$, by setting $L = \frac{A_0}{2\pi R} - 2R \geqslant 0$, we can build a cigar of given area $A_0 > 0$.

Expression of the functional

First, we express total mean curvature and the Willmore energy in terms of radius. We have:

$$\begin{cases} \dfrac{1}{2}\displaystyle\int H dA = \left(\frac{1}{R} + 0\right)\pi RL + \left(\frac{1}{R} + \frac{1}{R}\right)2\pi R^2 = \pi(L + 4R) = \pi\left(\frac{A_0}{2\pi R} + 2R\right) = \frac{A_0}{2R} + 2\pi R \\[2ex] \dfrac{1}{4}\displaystyle\int H^2 dA = \left(\frac{1}{2R}\right)^2 2\pi RL + \left(\frac{1}{R}\right)^2 4\pi R^2 = \frac{\pi}{2R}(L + 8R) = \frac{\pi}{2R}\left(\frac{A_0}{2\pi R} + 6R\right) = \frac{A_0}{4R^2} + 3\pi. \end{cases}$$

Therefore, we can write the Helfrich energy (3.3) in terms of $H_0 \in \mathbb{R}$, $A_0 > 0$, and $R \in]0, R_0]$:

$$\begin{aligned} \frac{1}{4}\int (H - H_0)^2 dA &= \frac{1}{4}\int H^2 dA - \frac{H_0}{2}\int H dA + \frac{H_0^2 A_0}{4} \\ &= \left(\frac{A_0}{4R^2} + 3\pi\right) - H_0\left(\frac{A_0}{2R} + 2\pi R\right) + \frac{H_0^2 A_0}{4} \\ &= \frac{A_0}{4R^2} - \frac{H_0 A_0}{2R} + \left(3\pi + \frac{H_0^2 A_0}{4}\right) - 2\pi H_0 R. \end{aligned}$$

The particular case $c_0 \leqslant 0$

If we assume $c_0 \leqslant 0$, then we get $H_0 \leqslant 0$ from (5.2). Combined with the identity $x^2 + y^2 \geqslant 2xy$ and the fact that $R \leqslant R_0$, we deduce that for any cigar Σ of given area A_0, we have:

$$\begin{aligned} \frac{1}{4}\int_\Sigma (H - H_0)^2 dA &= \left(\frac{A_0}{4R^2} + 3\pi\right) - H_0\left(\frac{A_0}{2R} + 2\pi R\right) + \frac{H_0^2 A_0}{4} \\ &\geqslant \left(\frac{A_0}{4R_0^2} + 3\pi\right) - 2H_0\sqrt{\frac{A_0}{2R}2\pi R} + \frac{H_0^2 A_0}{4} \\ &= \frac{A_0}{4}\left(\frac{1}{R_0} + \frac{1}{R_0} - H_0\right)^2 = \frac{1}{4}\int_{\mathbb{S}_{A_0}} (H - H_0)^2 dA. \end{aligned}$$

Hence, the sphere $\mathbb{S}_{A_0}$ of area A_0 is a global minimizer of (5.1) among cigars for any $c_0 \leqslant 0$. Moreover, it is unique since the equality holds in the above inequality if and only if $R = R_0$. Note that Theorem 7.1 in the second part will imply that these two last results still hold true in the wider class of *axiconvex* surfaces. Henceforth, we assume $c_0 > 0$ in the rest of Section 5.2.

Definition of the map f to minimize

Considering the expression previously obtained for the Helfrich energy (3.3), we introduce the following map:

$$\begin{array}{rcl} f: \ ]0, R_0] & \longrightarrow & [0, +\infty[\\ R & \longmapsto & f(R) = \dfrac{A_0}{4R^2} - \dfrac{H_0 A_0}{2R} + \left(3\pi + \dfrac{H_0^2 A_0}{4}\right) - 2\pi H_0 R, \end{array}$$

and we search for its minimum, which is the solution of Problem (5.1) among the class of cigars. For this purpose, we first study its variations. The function f is differentiable and we have:

$$\forall R \in]0, R_0], \quad f'(R) = \frac{-2A_0}{4R^3} + \frac{H_0 A_0}{2R^2} - 2\pi H_0 = -\frac{2\pi H_0}{R^3}\left(R^3 - \frac{A_0}{4\pi}R + \frac{A_0}{4\pi H_0}\right).$$

Since we assume $c_0 > 0$, we have $H_0 > 0$ from (5.2) and the sign of f' is thus the opposite of the following polynomial that we are going to study:

$$\begin{array}{rcl} P: \ ]0, R_0] & \longrightarrow & \mathbb{R} \\ R & \longmapsto & P(R) = R^3 - \dfrac{A_0}{4\pi}R + \dfrac{A_0}{4\pi H_0}. \end{array}$$

5.2.2 The variations of f

The variations of P

First, we have $P'(R) = 3R^2 - \frac{A_0}{4\pi}$ for any $R \in]0, R_0]$. This is a second-order polynomial whose sign on $]0, R_0]$ depends on the positive root of P' denoted by:

$$R_1 = \frac{1}{2}\sqrt{\frac{A_0}{3\pi}}.$$

Hence, we have $R_1 < R_0$ and we obtain the following table of variations for P.

R	0		R_1		R_0
$P'(R)$	$\vert$	$-$	0	$+$	
$P(R)$	$\vert$	$\searrow$	$P(R_1)$	$\nearrow$	

Finally, we obtain $\min_{R\in]0,R_0]} P(R) = P(R_1)$. Therefore, we distinguish two cases depending on the sign of $P(R_1)$.

The case where $P(R_1) \geqslant 0$

First, we search for an equivalent criteria. We have successively:

$$P(R_1) \geqslant 0 \iff R_1^3 - \frac{A_0}{4\pi}R_1 + \frac{A_0}{4\pi H_0} \geqslant 0 \iff \frac{A_0}{12\pi}\sqrt{\frac{A_0}{12\pi}} - \frac{A_0}{4\pi}\sqrt{\frac{A_0}{12\pi}} + \frac{A_0}{4\pi H_0} \geqslant 0$$

$$\iff \frac{A_0}{4\pi H_0} \geqslant \frac{A_0}{6\pi}\sqrt{\frac{A_0}{12\pi}} \iff 3\sqrt{3} \geqslant H_0\sqrt{\frac{A_0}{\pi}} \iff \frac{3\sqrt{3}}{4} \geqslant c_0.$$

Hence, if $0 < c_0 \leqslant \frac{3}{4}\sqrt{3}$, then the minimum of P on the interval $]0, R_0]$ is positive, from which we deduce f' is negative, i.e. f is decreasing. In this situation, $f(R) > f(R_0)$ for any $R \in]0, R_0[$ and the sphere is the unique global minimizer of (5.1) among cigars.

R	0		R_1		R_0
$P'(R)$	$\vert$	$-$	0	$+$	
$P(R)$	$\vert$	$\searrow$	$+$	$\nearrow$	
$f'(R)$	$\vert$		$-$		
$f(R)$	$+\infty$		$\searrow$		$f(R_0)$

The case where $\frac{3}{4}\sqrt{3} < c_0$

We now assume $\frac{3}{4}\sqrt{3} < c_0$ i.e. $P(R_1) < 0$. Since $P(0) > 0$, it implies that the third-order polynomial equation $P(R) = 0$ has three distinct real roots. Moreover, two of them are positive. Denoted by R_- and R_+, they satisfy $R_- \leqslant R_1 \leqslant R_+$. Furthermore, we get from $P(R_\pm) = 0$ on the one hand and on the other hand:

$$\begin{array}{rcl|rcl}
\frac{A_0}{4\pi}R_\pm & = & R_\pm^3 + \frac{A_0}{4\pi H_0} & \frac{A_0}{4\pi}R_\pm & = & R_\pm^3 + \frac{A_0}{4\pi H_0} \\
 & \Downarrow & & & \Downarrow & \\
\frac{A_0}{4\pi}R_\pm & > & R_\pm^3 & \frac{A_0}{4\pi}R_\pm & > & \frac{A_0}{4\pi H_0} \\
 & \Downarrow & & & \Downarrow & \\
\frac{A_0}{4\pi} & > & R_\pm^2. & R_\pm & > & \frac{1}{H_0}.
\end{array}$$

Hence, we have $\frac{1}{H_0} < R_\pm < R_0$. We deduce the following table of variations for P and f:

R	0		R_-		R_1		R_+		R_0
$P'(R)$	$\vert$		$-$		0		$+$		
$P(R)$	$\vert +$	$\searrow$	0	$\searrow$	$-$	$\nearrow$	0	$\nearrow$	$+$
$f'(R)$	$\vert$	$-$	0		$+$		0		$-$
$f(R)$	$+\infty$	$\searrow$	$f(R_-)$		$\nearrow$		$f(R_+)$	$\searrow$	$f(R_0)$

We have proved that if $c_0 > \frac{3}{4}\sqrt{3}$, then $\min_{R\in]0,R_0]} f(R) = \min\left(f(R_-), f(R_0)\right)$. In particular, the only global possible minimizers of (5.1) among cigars are the sphere and the cigar of radius R_-. It remains to determine which one it is.

5.2.3 The sign of $f(R_-) - f(R_0)$

In order to completely solve Problem (5.1) among cigars, we study the sign of $f(R_-) - f(R_0)$ in terms of c_0, assuming $c_0 > \frac{3}{4}\sqrt{3}$. If $f(R_-) - f(R_0) = 0$, then both are solutions, otherwise, if this quantity is positive, then it is the sphere, and if it is negative, then it is the cigar of radius R_-.

An explicit expression of R_-

We recall Cardano's method for the resolution of a polynomial equation of the form $x^3 + px + q = 0$ [89, Section 2.2.2]. If the discriminant $\Delta = q^2 + \frac{4}{27}p^3$ is negative, then the equation has three distinct real solutions denoted by x_0, x_1, and x_2 given by the following formula:

$$\forall k \in \{0,1,2\}, \quad x_k = 2\sqrt{\frac{-p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{-q}{2}\sqrt{\frac{27}{-p^3}}\right) + \frac{2k\pi}{3}\right].$$

In our case, we have $P(R) = R^3 - \frac{A_0}{4\pi}R + \frac{A_0}{4\pi H_0}$. It comes $p = \frac{-A_0}{4\pi}$ et $q = \frac{A_0}{4\pi H_0}$ and we deduce:

$$\Delta = \frac{A_0^2}{16\pi^2 H_0^2} - \frac{4}{27}\frac{A_0^3}{64\pi^3} = \frac{A_0^3}{27c_0^2 16\pi^3}\left[\left(\frac{3\sqrt{3}}{4}\right)^2 - c_0^2\right].$$

Hence, the hypothesis $c_0 > \frac{3}{4}\sqrt{3}$ is equivalent to $\Delta < 0$ so the three real roots of the equation $P(R) = 0$ are given by the following formulas:

$$\forall k \in \{0,1,2\}, \quad x_k = \sqrt{\frac{A_0}{3\pi}}\cos\left[\frac{1}{3}\arccos\left(\frac{-3}{H_0}\sqrt{\frac{3\pi}{A_0}}\right) + \frac{2k\pi}{3}\right].$$

Note that we can identify the three roots since we have successively:

$$c_0 > \frac{3\sqrt{3}}{4} \quad\Longleftrightarrow\quad -1 < \frac{-3}{H_0}\sqrt{\frac{3\pi}{A_0}} < 0 \quad\Longleftrightarrow\quad \frac{\pi}{6} < \frac{1}{3}\arccos\left(\frac{-3}{H_0}\sqrt{\frac{3\pi}{A_0}}\right) < \frac{\pi}{3}$$

Hence, we get $x_0 \in]R_1, R_0[$, $x_1 \in]-2R_1, -R_0[$ and $x_2 \in]0, R_1[$. We deduce that $x_2 = R_-$, $x_0 = R_+$ and x_1 corresponds to the negative root of $P(R) = 0$.

Evaluation of $f(R_-) - f(R_0)$ in terms of c_0

First, note that $P(R_-) = 0$ i.e. $R_-^3 + \frac{A_0}{4\pi H_0} = \frac{A_0}{4\pi} R_-$ is equivalent to $\frac{A_0}{4R_-^2} = \frac{H_0 A_0}{4R_-} - \pi H_0 R_-$, from which we deduce that:

$$f(R_-) = \frac{A_0}{4R_-^2} - \frac{H_0 A_0}{2R_-} + 3\pi + \frac{H_0^2 A_0}{4} - 2\pi H_0 R_- = \frac{H_0^2 A_0}{4} + 3\pi - 3\pi H_0 R_- - \frac{H_0 A_0}{4R_-}.$$

Then, we have $f(R_0) = 4\pi - 2H_0\sqrt{\pi A_0} + \frac{H_0^2 A_0}{4}$. Inserting this relation in the previous one gives:

$$f(R_-) - f(R_0) = 2H_0\sqrt{\pi A_0} - \pi - \frac{H_0 A_0}{4R_-} - 3\pi H_0 R_-.$$

Finally, the following map is well defined:

$$\begin{array}{rccl} g: &]\frac{3}{4}\sqrt{3}, +\infty[& \longrightarrow &]0, \frac{1}{2}[\\ & x & \longmapsto & \cos\left[\frac{1}{3}\arccos\left(\frac{-3\sqrt{3}}{4x}\right) + \frac{4\pi}{3}\right]. \end{array}$$

Moreover, we can compute $f(R_-) - f(R_0)$ as a function of c_0. Indeed, from the foregoing, we get $f(R_-) - f(R_0) = Q(c_0)$ where we set for any real $x > \frac{3}{4}\sqrt{3}$:

$$Q(x) = 8\pi x - \pi - \pi\sqrt{3}\frac{x}{g(x)} - 4\pi\sqrt{3}xg(x).$$

Studying the function g and the sign of Q

First, g is a decreasing continuous map. Indeed, it can be decomposed into:

$$]\tfrac{3\sqrt{3}}{4}, +\infty[\xrightarrow{\frac{-3\sqrt{3}}{4(\bullet)}}]-1, 0[\xrightarrow{\frac{\arccos(\bullet)+4\pi}{3}}]\tfrac{3\pi}{2}, \tfrac{5\pi}{3}[\xrightarrow{\cos(\bullet)}]0, \tfrac{1}{2}[.$$

Hence [13, Chapter IV, §2, Section 6, Theorem 5], the function g is an homeomorphism from $]\frac{3}{4}\sqrt{3}, +\infty[$ into $]0, \frac{1}{2}[$ whose inverse g^{-1} is decreasing. Then, we explicitly compute g^{-1}. Consider any $x > \frac{3}{4}\sqrt{3}$ and let us search for $y \in]0, \frac{1}{2}[$ such that:

$$y = g(x) \iff y = \cos\left[\frac{1}{3}\arccos\left(\frac{-3\sqrt{3}}{4x}\right) + \frac{4\pi}{3}\right] \iff 3\arccos(y) - 4\pi = \arccos\left(\frac{-3\sqrt{3}}{4x}\right)$$

$$\iff \cos\left[3\arccos(y)\right] = \frac{-3\sqrt{3}}{4x} \iff 3y - 4y^3 = \frac{3\sqrt{3}}{4x} \iff x = \frac{3\sqrt{3}}{4y(3-4y^2)}.$$

The above equivalences are justified because of the intervals in which x and y live, and also from the identity $\cos(3a) = 4\cos^3(a) - 3\cos(a)$. Consequently, we obtain an explicit expression for the inverse of g:

$$\begin{array}{rccl} g^{-1}: &]0, \frac{1}{2}[& \longrightarrow &]\frac{3}{4}\sqrt{3}, +\infty[\\ & y & \longmapsto & \dfrac{3\sqrt{3}}{4y(3-4y^2)} \end{array}$$

Finally, we can explicitly compute $Q \circ g^{-1}$. For any $y \in]0, \frac{1}{2}[$, we have:

$$\begin{aligned} Q \circ g^{-1}(y) &= 8\pi g^{-1}(y) - \pi - \pi\sqrt{3}\frac{g^{-1}(y)}{y} - 4\pi\sqrt{3}yg^{-1}(y) \\ &= \frac{4\pi}{y^2(3-4y^2)}\left(y^4 - 3y^2 + \frac{3\sqrt{3}}{2}y - \frac{9}{16}\right). \end{aligned}$$

We observe that the fourth-order polynomial appearing in the factorisation above has an explicit double root $\frac{\sqrt{3}}{2}$. An Euclidean division gives the following decomposition:

$$y^4 - 3y^2 + \frac{3\sqrt{3}}{2}y - \frac{9}{16} = \left(y - \frac{\sqrt{3}}{2}\right)^2\left(y^2 + \sqrt{3}y - \frac{3}{4}\right).$$

We now completely decompose the polynomial by calculating the discriminant $\Delta = 3+3 = 6$ so:

$$\begin{aligned} Q \circ g^{-1}(y) &= \frac{4\pi(y-\frac{\sqrt{3}}{2})^2}{y^2(3-4y^2)}\left(y+\frac{\sqrt{3}}{2}(\sqrt{2}+1)\right)\left(y-\frac{\sqrt{3}}{2}(\sqrt{2}-1)\right) \\ &= \frac{\pi}{y^2}\left(\frac{\frac{\sqrt{3}}{2}-y}{\frac{\sqrt{3}}{2}+y}\right)\left(y+\frac{\sqrt{3}}{2}(\sqrt{2}+1)\right)\left(y-\frac{\sqrt{3}}{2}(\sqrt{2}-1)\right). \end{aligned}$$

Therefore, the sign of $Q\circ g^{-1}$ on $]0,\frac{1}{2}[$ corresponds to the one of the monomial $y \mapsto y-y_0$ where we set $y_0 = \frac{1}{2}\sqrt{3}(\sqrt{2}-1)$. Since g^{-1} is decreasing, we deduce the sign of Q on the interval $]\frac{3}{4}\sqrt{3},+\infty[$ i.e. the sign of $f(R_-) - f(R_0)$ in terms of c_0.

c_0	$\frac{3\sqrt{3}}{4}$		$g^{-1}(y_0)$		$+\infty$
$y = g(c_0)$	$\frac{1}{2}$	$\searrow$	y_0	$\searrow$	0
$f(R_-) - f(R_0) = Q(c_0) = Q\circ g^{-1}(y)$		$+$	0	$-$	

To conclude, if $c_0 < g^{-1}(y_0)$, the sphere is the unique cigar of area A_0 which globally minimize the Helfrich energy $\frac{1}{4}\int(H-H_0)^2dA$, whereas if $c_0 > g^{-1}(y_0)$, then it is the cigar of radius R_-. Moreover, if $c_0 = g^{-1}(y_0)$ holds true, then Problem 5.1 has two global minimizer: the cigar of radius R_- and the sphere. It remains to calculate the critical value:

$$g^{-1}(y_0) = \frac{3\sqrt{3}}{4y_0(3-4y_0^2)} = \frac{3\sqrt{3}}{4}\frac{2}{\sqrt{3}(\sqrt{2}-1)}\frac{1}{6(\sqrt{2}-1)} = \frac{1}{4}(1+\sqrt{2})^2,$$

which concludes the proof of Theorem 5.3.

5.3 A sequence converging to a double sphere

Theorem 5.4. *Let $A_0 > 0$, $H_0 \in \mathbb{R}$, c_0 as in (2.5), and $c_- := \frac{1}{8\cos\theta} \approx -0.575$, where $\theta \approx 4.4934$ is the unique solution of $\tan x = x$ on the interval $]\pi, \frac{3\pi}{2}[$. Then, there exists a sequence $(\Sigma_r)_{r>0}$ of compact simply-connected non-convex axisymmetric $C^{1,1}$-surfaces of $\mathbb{R}^3$ such that:*

$$\frac{1}{4}\int_{\Sigma_r}(H-H_0)^2dA - \frac{1}{4}\int_{\mathbb{S}_{A_0}}(H-H_0)^2dA \underset{r\to 0^+}{\longrightarrow} 8\pi\left(c_0 - c_-\right).$$

Proof. We strongly advice the reader to read Chapter 8 before this proof in order to get familiar with the notation used to parametrize an axisymmetric surface. We first detail the construction of a sequence $(\Sigma_r)_{r>0}$ of axisymmetric $C^{1,1}$-surfaces with constant area $A_0 > 0$. Then, we show that its total mean curvature tends to zero as $r \to 0^+$ and we finally compute its Willmore energy.

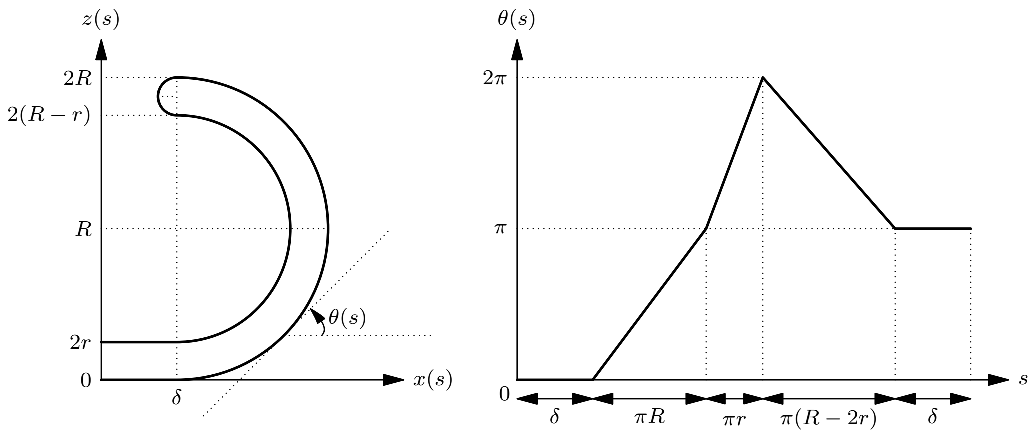

Figure 5.2: The construction of the sequence $(\Sigma_r)_{r>0}$ of axisymmetric $C^{1,1}$-surfaces.

Let us consider the sequence of surfaces $(\Sigma_r)_{r>0}$ described in Figure 5.2. They consist in two spheres of radius $R > 0$ and $R-2r > \delta > 0$ glued together at a distance $\delta > r > 0$ of the axis of revolution and such that the generating map $\theta : [0, L] \to \mathbb{R}$ is piecewise linear. More precisely, the generating map is given by:

$$\theta(s) = \begin{cases} 0 & \text{if } s \in [0, \delta] \\ \\ \dfrac{1}{R}(s-\delta) & \text{if } s \in [\delta, \delta + \pi R] \\ \\ \dfrac{1}{r}(s - \delta - \pi R) + \pi & \text{if } s \in [\delta + \pi R, \delta + \pi(R+r)] \\ \\ -\dfrac{1}{R-2r}(s - \delta - \pi R - \pi r) + 2\pi & \text{if } s \in [\delta + \pi(R+r), \delta + \pi(2R - r)] \\ \\ \pi & \text{if } s \in [\delta + \pi(2R-r), L], \end{cases}$$

where $L = 2\delta + \pi(2R - r) > 0$ is the total length of the generating curve. Then, a computation of $x(s) = \int_0^L \cos\theta(t)dt$ and $z(s) = \int_0^s \sin\theta(t)dt$ gives the following relations:

$$x(s) = \begin{cases} s & \text{if } s \in [0, \delta] \\ \delta + R\sin\theta(s) & \text{if } s \in [\delta, \delta + \pi R] \\ \delta + r\sin\theta(s) & \text{if } s \in [\delta + \pi R, \delta + \pi(R+r)] \\ \delta - (R-2r)\sin\theta(s) & \text{if } s \in [\delta + \pi(R+r), \delta + \pi(2R - r)] \\ L - s & \text{if } s \in [\delta + \pi(2R-r), L], \end{cases}$$

and also

$$z(s) = \begin{cases} 0 & \text{if } s \in [0, \delta] \\ R\,(1 - \cos\theta(s)) & \text{if } s \in [\delta, \delta + \pi R] \\ 2R - r(1 + \cos\theta(s)) & \text{if } s \in [\delta + \pi R, \delta + \pi(R+r)] \\ 2(R-r) - (R-2r)(1 - \cos\theta(s)) & \text{if } s \in [\delta + \pi(R+r), \delta + \pi(2R - r)] \\ 2r & \text{if } s \in [\delta + \pi(2R-r), L]. \end{cases}$$

Finally, we obtain the following expressions:

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_{\Sigma_r} H dA = \pi \int_0^L \left(\sin\theta(s) + \dot{\theta}(s)x(s)\right) ds = 4\pi r + \pi^2\delta \\ \\ A(\Sigma_r) = 2\pi \displaystyle\int_0^L x(s)ds = 2\pi\delta^2 + 2\pi^2\delta(2R - r) + 4\pi\left(R^2 - r^2 + (R-2r)^2\right). \end{cases}$$

Let us assume that δ is linear in r i.e. $\delta = kr$ with $k > 1$ fixed. The last relation is thus a second order polynomial in $R > 0$ and for each (sufficiently small) r, there exists a unique positive root R_r such that $A(\Sigma_r) = A_0$. Moreover, R_r converges to $R_0 = \sqrt{A_0/8\pi}$ as $r \to 0^+$. Then, we see that the total mean curvature converges to zero from above as r tends to 0^+.

The Willmore energy of the sequence

We recall that $\kappa_1(s) = \dot{\theta}(s)$, $\kappa_2(s) = \frac{\sin\theta(s)}{x(s)}$, and $dA(s) = 2\pi x(s)ds$, where $x(s) = \int_0^s \cos\theta(t)dt$. From Theorem 3.1, we have $\int_{\Sigma_r} KdA = 4\pi$ so we deduce:

$$\frac{1}{2\pi}\left(\int_{\Sigma_r} H^2 dA - 8\pi\right) = \frac{1}{2\pi}\int_{\Sigma_r}\left(\kappa_1^2 + \kappa_2^2\right) dA = \underbrace{\int_0^L \dot{\theta}^2(s)x(s)ds}_{:=A} + \underbrace{\int_0^L \frac{\sin^2\theta(s)}{x(s)}ds}_{:=B}. \tag{5.4}$$

In the right member of (5.4), we respectively denote by A and B the first and second term. On the one hand, concerning the term A, we have successively:

$$\begin{aligned}
A &= \int_{\delta}^{\delta+\pi R} \frac{\delta + R\sin\theta(s)}{R^2} ds + \int_{\delta+\pi R}^{\delta+\pi R+\pi r} \frac{\delta + r\sin\theta(s)}{r^2} ds + \int_{\delta+\pi(R+r)}^{\delta+\pi(2R-r)} \frac{\delta - (R-2r)\sin\theta(s)}{(R-2r)^2} ds \\
&= \frac{\delta\pi}{R} + \frac{\delta\pi}{r} + \frac{\delta\pi}{R-2r} - [\cos\theta(\bullet)]_{\delta}^{\delta+\pi R} - [\cos\theta(\bullet)]_{\delta+\pi R}^{\delta+\pi(R+r)} - [\cos\theta(\bullet)]_{\delta+\pi(R+r)}^{\delta+\pi(2R-r)} \\
&= \delta\pi \left(\frac{1}{R} + \frac{1}{r} + \frac{1}{2R-r} \right) \ + \ 2.
\end{aligned}$$

On the other hand, concerning the term B, we have:

$$B = \int_{\delta}^{\delta+\pi R} \frac{\sin^2\theta(s)}{\delta + R\sin\theta(s)} ds + \int_{\delta+\pi R}^{\delta+\pi R+\pi r} \frac{\sin^2\theta(s)}{\delta + r\sin\theta(s)} ds + \int_{\delta+\pi(R+r)}^{\delta+\pi(2R-r)} \frac{\sin^2\theta(s)}{\delta - (2R-r)\sin\theta(s)} ds$$

Then, since θ is piecelinear, we can make a change of variable in each integral above by setting $u = \theta(s)$. We obtain:

$$B = \int_0^{\pi} \frac{\sin^2 u}{\frac{\delta}{R} + \sin u} du + \int_0^{\pi} \frac{\sin^2 u}{\frac{\delta}{r} - \sin u} du + \int_0^{\pi} \frac{\sin^2 u}{\frac{\delta}{R-2r} + \sin u} du$$

Let us respectively denote by B_1, B_2 and B_3, the three integrals appearing in the above expression. We recall that $R > \delta + 2r > 3r > 0$. We first compute B_1. It comes:

$$\begin{aligned}
B_1 &= \int_0^{\pi} \frac{\sin^2 u}{\sin^2 u - \left(\frac{\delta}{R}\right)^2} \left(\sin u - \frac{\delta}{R} \right) du = \int_0^{\pi} \left(1 + \frac{\left(\frac{\delta}{R}\right)^2}{\sin^2 u - \left(\frac{\delta}{R}\right)^2} \right) \left(\sin u - \frac{\delta}{R} \right) du \\
&= [-\cos]_0^{\pi} - \frac{\delta\pi}{R} + \frac{\delta^2}{R^2} \int_0^{\pi} \frac{1}{\sin u + \frac{\delta}{R}} du \ = \ 2 - \frac{\delta\pi}{R} + \frac{\delta}{R} \int_0^{\pi} \frac{1}{1 + \frac{R}{\delta}\sin u} du
\end{aligned}$$

We make the change of variable $t = \tan\frac{u}{2}$. We find $du = \frac{2dt}{1+t^2}$ and $\sin u = \frac{2t}{1+t^2}$, which yields to:

$$\begin{aligned}
B_1 &= 2 - \frac{\delta\pi}{R} + \frac{2\delta}{R} \int_0^{+\infty} \frac{1}{1 + \frac{2R}{\delta}t + t^2} dt \\
&= 2 - \frac{\delta\pi}{R} + \frac{\delta^2}{R\sqrt{R^2-\delta^2}} \int_0^{+\infty} \left(\frac{1}{t + \frac{R-\sqrt{R^2-\delta^2}}{\delta}} - \frac{1}{t + \frac{R+\sqrt{R^2-\delta^2}}{\delta}} \right) dt \\
&= 2 - \frac{\delta\pi}{R} + \frac{\delta^2}{R\sqrt{R^2-\delta^2}} \left[\log \left(\frac{\bullet + \frac{R-\sqrt{R^2-\delta^2}}{\delta}}{\bullet + \frac{R+\sqrt{R^2-\delta^2}}{\delta}} \right) \right]_0^{+\infty} \\
&= 2 - \frac{\delta\pi}{R} + \frac{\delta^2}{R\sqrt{R^2-\delta^2}} \log \left(\frac{R + \sqrt{R^2-\delta^2}}{R - \sqrt{R^2-\delta^2}} \right).
\end{aligned}$$

Similarly, since $(R-2r)^2 - \delta^2 > 0$, we can compute B_2 and we obtain:

$$B_2 = 2 - \frac{\delta\pi}{R-2r} + \frac{\delta^2}{(R-2r)\sqrt{(R-2r)^2-\delta^2}} \log \left(\frac{(R-2r) + \sqrt{(R-2r)^2-\delta^2}}{(R-2r) - \sqrt{(R-2r)^2-\delta^2}} \right).$$

It remains to compute B_3. Following the same method, we have:

$$\begin{aligned}
B_3 &= \int_0^{\pi} \frac{\sin^2 u}{\left(\frac{\delta}{r}\right)^2 - \sin^2 u} \left(\sin u + \frac{\delta}{r} \right) du = \int_0^{\pi} \left(-1 + \frac{\left(\frac{\delta}{r}\right)^2}{\left(\frac{\delta}{r}\right)^2 - \sin^2 u} \right) \left(\sin u + \frac{\delta}{r} \right) du \\
&= [\cos]_0^{\pi} - \frac{\delta\pi}{r} + \frac{\delta^2}{r^2} \int_0^{\pi} \frac{1}{\frac{\delta}{r} - \sin u} du \ = \ -2 - \frac{\delta\pi}{r} + \frac{\delta}{r} \int_0^{\pi} \frac{1}{1 - \frac{r}{\delta}\sin u} du
\end{aligned}$$

As before, we make the change of variable $t = \tan\frac{u}{2}$. We find $du = \frac{2dt}{1+t^2}$ and $\sin u = \frac{2t}{1+t^2}$, which yields to:

$$\begin{aligned}
B_3 &= -2 - \frac{\delta\pi}{r} + \frac{2\delta}{r}\int_0^{+\infty} \frac{1}{1-\frac{2r}{\delta}t+t^2}dt \\
&= -2 - \frac{\delta\pi}{r} + \frac{2\delta^3}{r(\delta^2-r^2)}\int_0^{+\infty} \frac{1}{1+\frac{\delta^2}{\delta^2-r^2}(t-\frac{r}{\delta})^2}dt \\
&= -2 - \frac{\delta\pi}{r} + \frac{2\delta^2}{r\sqrt{\delta^2-r^2}}\left[\arctan\left(\frac{\delta}{\sqrt{\delta^2-r^2}}(\bullet - \tfrac{r}{\delta})\right)\right]_0^{+\infty} \\
&= -2 - \frac{\delta\pi}{r} + \frac{2\delta^2}{r\sqrt{\delta^2-r^2}}\left[\frac{\pi}{2} + \arctan\left(\frac{r}{\sqrt{\delta^2-r^2}}\right)\right].
\end{aligned}$$

Finally, inserting in (5.4) the expressions obtained for A and $B := B_1 + B_2 + B_3$, we get after simplifications:

$$\frac{1}{2\pi}\int_{\Sigma_r} H^2 dA = 8 + \frac{\log\left(\frac{R+\sqrt{R^2-\delta^2}}{R-\sqrt{R^2-\delta^2}}\right)}{\frac{R}{\delta}\sqrt{\left(\frac{R}{\delta}\right)^2-1}} + \frac{\log\left(\frac{(R-2r)+\sqrt{(R-2r)^2-\delta^2}}{(R-2r)-\sqrt{(R-2r)^2-\delta^2}}\right)}{\frac{R-2r}{\delta}\sqrt{\left(\frac{R-2r}{\delta}\right)^2-1}} + \frac{2\delta^2\left[\frac{\pi}{2}+\arctan\left(\frac{r}{\sqrt{\delta^2-r^2}}\right)\right]}{r\sqrt{\delta^2-r^2}}.$$

It remains to create a explicit dependence of δ in r. We set $\delta = kr^\alpha$. Since we need $\delta \to 0^+$ as $r \to 0^+$ in order to let $\int_{\Sigma_r} HdA \to 0^+$, we must have $\alpha \leqslant 1$. Moreover, the above expression is finite only if $\alpha = 1$. Hence, we set $\delta = kr$ with $k > 1$ and we get:

$$\lim_{r\to 0^+} \frac{1}{4}\int_{\Sigma_r} H^2 dA - \frac{1}{4}\int_{\mathbb{S}_{A_0}} H^2 dA = \frac{\pi k^2}{\sqrt{k^2-1}}\left[\frac{\pi}{2} + \arctan\left(\frac{1}{\sqrt{k^2-1}}\right)\right].$$

First, observe that the limit value of the Willmore deficit depends on $k > 1$ i.e. on the speed at which the sequence $(\Sigma_r)_{r>0}$ get closer from the axis of revolution. Then, we introduce the function:

$$\begin{aligned}
f : \ ]1,+\infty[&\longrightarrow [0,+\infty[\\
k &\longmapsto f(k) = \frac{\pi k^2}{\sqrt{k^2-1}}\left[\tfrac{\pi}{2} + \arctan\left(\frac{1}{\sqrt{k^2-1}}\right)\right],
\end{aligned}$$

and search for its minimum value. For this purpose, we slightly modify the expression of f by introducing the following map:

$$\begin{aligned}
g : \ ]1,+\infty[&\longrightarrow \ ]\pi, 2\pi[\\
k &\longmapsto g(k) = \pi + 2\arctan\left(\frac{1}{\sqrt{k^2-1}}\right),
\end{aligned}$$

It is differentiable and we have $g'(k) = \frac{-2}{k\sqrt{k^2-1}} < 0$ for any $k > 1$. Hence, g is a decreasing homeomorphism [13, Livre IV §2 Section 6 Théorème 5]. Furthermore, its inverse is explicitly given by the following explicit map:

$$\begin{aligned}
g^{-1} : \ ]\pi,2\pi[&\longrightarrow \ ]1,+\infty[\\
x &\longmapsto g^{-1}(x) = \frac{-1}{\cos\left(\frac{x}{2}\right)},
\end{aligned}$$

and $f \circ g^{-1}$ takes a nice form:

$$\forall x \in]\pi, 2\pi[, \quad (f\circ g^{-1})(x) = \frac{-\pi x}{\sin x}.$$

We can now evaluate $(f\circ g^{-1})'(x) = \frac{\pi}{\sin^2 x}(x\cos x - \sin x)$ and we deduce the following table of variations where x_0 is the unique solution in $]\pi, \frac{3\pi}{2}[$ of $\tan x = x$.

k	1		$g^{-1}(x_0)$		$+\infty$
$x = g(k)$	2π	$\searrow$	x_0	$\searrow$	π
$(f\circ g^{-1})'(x)$	$\vert$	$+$	0	$-$	$\vert$
$f'(k) = (f\circ g^{-1})'(x)g'(k)$	$\vert$	$-$	0	$+$	$\vert$
$f(k)$	$+\infty$	$\searrow$	$(f\circ g^{-1})(x_0)$	$\nearrow$	$+\infty$

To conclude, by setting $\delta = g^{-1}(x_0)r$, and recalling that x_0 is the unique solution of $\tan x_0 = x_0$ on the interval $]\pi, \frac{3\pi}{2}[$, we thus have proved:

$$\begin{aligned}
\lim_{r\to 0^+} \left(\frac{1}{4}\int_{\Sigma_r} (H-H_0)^2 dA - \frac{1}{4}\int_{\mathbb{S}_{A_0}} (H-H_0)^2 dA \right) &= (f\circ g^{-1})(x_0) + H_0\sqrt{4\pi A_0} \\
&= 8\pi\left(c_0 - \frac{x_0}{8\sin x_0}\right) \\
&= 8\pi\left(c_0 - \frac{\tan x_0}{8\sin x_0}\right) \\
&= 8\pi\left(c_0 - \frac{1}{8\cos x_0}\right)
\end{aligned}$$

Hence, if we set $c_- = \frac{1}{8\cos x_0} < 0$ and assume $c_0 < c_-$, then for r small enough, the surface Σ_r has a greater Helfrich energy than $\mathbb{S}_{A_0}$ and it is a non-convex axisymmetric compact simply-connected $C^{1,1}$-surface of $\mathbb{R}^3$ with same area A_0, which concludes the proof of Theorem 5.4. $\square$

Chapter 6

The sphere as a local minimizer of the Helfrich energy with given area

In this chapter, we look at the optimality of the sphere $\mathbb{S}_{A_0}$ of area A_0 as a local minimizer of (5.1). The results are summarized in the second row of Table 5.1. In the case $c_0 < 0$, using some results of the second part (Remark 11.1), $\mathbb{S}_{A_0}$ is a local minimizer of (5.1). Moreover, since $\mathbb{S}_{A_0}$ is a global minimizer of (5.1) for any $c_0 \in [0, 1]$, it is in particular a local minimizer of (5.1).

We now assume $c_0 > 1$, where c_0 is given by (5.2). In Section 6.1, we first study some local smooth axisymmetric perturbations of the sphere. We prove the existence of a threshold value above which the second-order shape derivative of the Helfrich energy (3.3) associated with these families of perturbations is negative. In Section 6.2, we prove that the final form of the critical value denoted by c_{++} is given by an optimization problem posed in a weighted Sobolev space.

Finally, in Section 6.3, we try to estimate from below the value of c_{++}. It becomes an eigenvalue problem associated with a non-linear fourth-order one-dimensional ordinary differential equation. In particular, an estimation from below of c_{++} is given by the lowest positive eigenvalue of this problem but we were not able to completely solve it. However, we obtain an explicit sequence of eigenvalues $(\lambda_{2i})_{i\geqslant 1} = \frac{1}{2}(2i+1)(2i+2)$. There is good numerical evidence to think that $\lambda_2 = 6$ is the lowest eigenvalue but we were not able to prove it.

6.1 The second-order shape derivative associated with some axisymmetric perturbations of the unit sphere

6.1.1 An equivalent formulation of the minimization problem

Instead of minimizing (5.1) whose formulation depends on two parameters, the prescribed area $A_0 > 0$ and the spontaneous curvature $H_0 \in \mathbb{R}$, we first express Problem (5.1) in an equivalent form depending only on the parameter c_0 defined in (5.2):

$$\inf_{A(\Sigma)=A_0} \frac{1}{4}\int_{\Sigma} (H - H_0)^2\, dA = \inf_{A(\widetilde{\Sigma})=4\pi} \frac{1}{4}\int_{\widetilde{\Sigma}} (H - 2c_0)^2\, dA. \tag{6.1}$$

Indeed, for any compact simply-connected $C^{1,1}$-surface $\Sigma \subset \mathbb{R}^3$ with area A_0, considering its rescaling $\widetilde{\Sigma} := \lambda\Sigma$ where $\lambda = \sqrt{4\pi/A_0}$, we have successively:

$$
\begin{aligned}
\frac{1}{4}\int_{\Sigma}(H-H_0)^2\,dA &= \frac{1}{4}\int_{\Sigma}H^2dA - \frac{H_0}{2}\int_{\Sigma}HdA + \frac{H_0^2A_0}{4}\\
&= \frac{1}{4}\int_{\lambda\Sigma}H^2dA - \frac{H_0}{2\lambda}\int_{\lambda\Sigma}HdA + \frac{H_0^2A(\lambda\Sigma)}{4\lambda^2}\\
&= \frac{1}{4}\int_{\widetilde{\Sigma}}H^2dA - \frac{H_0}{2}\sqrt{\frac{A_0}{4\pi}}\int_{\widetilde{\Sigma}}HdA + \frac{H_0^2A_0}{16\pi}A(\widetilde{\Sigma})\\
&= \frac{1}{4}\int_{\widetilde{\Sigma}}H^2dA - c_0\int_{\widetilde{\Sigma}}HdA + c_0^2A(\widetilde{\Sigma})\\
&= \frac{1}{4}\int_{\widetilde{\Sigma}}(H-2c_0)^2\,dA.
\end{aligned}
$$

Henceforth, we consider the new optimization problem given in the right member of (6.1). Note that the unit sphere $\mathbb{S}^2$ is admissible and we want to determine if it is always a local minimizer for any $c_0 > 1$. We strongly advice the reader to read Chapter 8 before going further in order to get familiar with the notation used to parametrize an axisymmetric surface.

6.1.2 An axisymmetric parametrization of the unit sphere

We consider an axisymmetric parametrization of the unit sphere $\mathbb{S}^2$ by arc length. In this case, all the geometric quantities are expressed as a function of the angle $\theta_0(s)$ made between the horizontal axis (orthogonal to the axis of revolution) and the tangent vector to the curve, the origin being taken to the south pole. We thus have $\theta_0(s) = s$ for any $s \in [0, L(0)]$ where $L(0) = \pi$. The planar coordinates are given by:

$$
\forall s \in [0,\pi], \quad \begin{cases} x_0(s) = \displaystyle\int_0^s \cos\theta_0(t)dt = \int_0^s \cos(t)dt = \sin s \\[2ex] z_0(s) = \displaystyle\int_0^s \sin\theta_0(t)dt = \int_0^s \sin(t)dt = 1-\cos s. \end{cases}
$$

Moreover, we can compute the infinitesimal area element $dA = 2\pi x_0(s)ds$ as well as the principal curvatures (see e.g. [18, Section 3.3, Example 4] or [86]):

$$
\forall s \in [0,\pi], \quad \begin{cases} \kappa_1(s) = \dfrac{\sin\theta_0(s)}{x_0(s)} = 1 \\[2ex] \kappa_2(s) = \dot{\theta}_0(s) = 1. \end{cases}
$$

In particular, the scalar mean curvature $H = \kappa_1 + \kappa_2$ is constant over the unit sphere.

6.1.3 Some axisymmetric admissible perturbations of the unit sphere

Let $\varepsilon > 0$. We consider an axisymmetric perturbation of the previous parametrization. It takes the form $\theta_\varepsilon : s \in [0, L(\varepsilon)] \mapsto \theta_\varepsilon(s) = s + \varepsilon\varphi(s)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-identically-zero C^∞-map with compact support, and where θ_ε is a parametrization of our perturbed sphere by arc length. We thus have:

$$
\forall s \in [0, L(\varepsilon)], \quad \begin{cases} x_\varepsilon(s) = \displaystyle\int_0^s \cos\theta_\varepsilon(t)dt = \int_0^s \cos[t+\varepsilon\varphi(t)]dt \\[2ex] z_\varepsilon(s) = \displaystyle\int_0^s \sin\theta_\varepsilon(t)dt = \int_0^s \sin[t+\varepsilon\varphi(t)]dt. \end{cases}
$$

Similarly, the infinitesimal area element is $dA = 2\pi x_\varepsilon(s)ds$. The principal curvatures are given by:

$$\forall s \in [0, L(\varepsilon)], \quad \begin{cases} \kappa_1(s) = \dfrac{\sin\theta_\varepsilon(s)}{x_\varepsilon(s)} = \dfrac{\sin[s+\varepsilon\varphi(s)]}{x_\varepsilon(s)} \\ \\ \kappa_2(s) = \dot{\theta}_\varepsilon(s) = 1 + \varepsilon\dot{\varphi}(s). \end{cases} \tag{6.2}$$

Furthermore, the perturbation $\theta_\varepsilon : s \in [0, L(\varepsilon)] \mapsto \theta_\varepsilon(s) = s + \varepsilon\varphi(s)$ must be admissible in the sense of Definition 8.1, which means θ_ε has to parametrize an axisymmetric surface Σ_ε without self-intersection or angular points. First, the boundary conditions (Point (i) in Definition 8.1) give:

$$\begin{cases} \theta_\varepsilon(0) = 0 \\ \\ \theta_\varepsilon[L(\varepsilon)] = \pi \end{cases} \Longrightarrow \begin{cases} \varphi(0) = 0 \\ \\ \dfrac{d\left(L(\varepsilon) + \varepsilon\varphi[L(\varepsilon)]\right)}{d\varepsilon} = 0 \end{cases} \Longrightarrow \begin{cases} \varphi(0) = 0 \\ \\ \varphi(\pi) = -\dot{L}(0). \end{cases}$$

The last relation is obtained by calculating the first-order derivative of $\theta_\varepsilon[L(\varepsilon)]$ at $\varepsilon = 0$. Then, we have some non-intersection conditions of the curve with itself and the axis of revolution (Point (iii) in Definition 8.1), which are $x_\varepsilon(s) > 0$ and $[x_\varepsilon(s) - x_\varepsilon(\tilde{s})]^2 + [z_\varepsilon(s) - z_\varepsilon(\tilde{s})]^2 > 0$ i.e:

$$\forall (s,\tilde{s}) \in]0, L(\varepsilon)[^2, \quad \begin{cases} \displaystyle\int_0^s \cos[t + \varepsilon\varphi(t)]dt > 0 \\ \\ \left(\displaystyle\int_{\tilde{s}}^s \cos[t+\varepsilon\varphi(t)]dt\right)^2 + \left(\displaystyle\int_{\tilde{s}}^s \sin[t+\varepsilon\varphi(t)]dt\right)^2 > 0. \end{cases}$$

We now prove this is automatically satisfied if ε is chosen sufficiently small. First, we show there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$, $x_\varepsilon(s) > 0$ for any $s \in]0, \frac{\pi}{2}]$. Let $s \in]0, \frac{\pi}{2}]$ and $t \in [0, s]$. Considering the second-order Taylor expansion of the map $f : \varepsilon \mapsto \cos[t+\varepsilon(t)]$, we have successively:

$$\begin{aligned} \cos[t+\varepsilon\varphi(t)] \quad &= \quad f(\varepsilon) \quad = \quad f(0) + \dot{f}(0)\varepsilon + \int_0^\varepsilon \ddot{f}(u)(\varepsilon - u)du \\ &= \quad \cos t - \varepsilon\varphi(t)\sin t - \varphi^2(t)\int_0^\varepsilon \cos[t + u\varphi(t)][\varepsilon - u]du \\ &\geqslant \quad \cos t - \varepsilon\|\varphi\|_\infty\left(1 + \frac{\varepsilon\|\varphi\|_\infty}{2}\right), \end{aligned}$$

where we set $\|\varphi\|_\infty = \sup_{s\in\mathbb{R}}|\varphi(s)|$. In integrating the last relation from $t = 0$ to $t = s$, it comes:

$$x_\varepsilon(s) = \int_0^s \cos[t+\varepsilon\varphi(t)]dt \geqslant \sin s - \varepsilon s\|\varphi\|_\infty\left(1 + \frac{\varepsilon\|\varphi\|_\infty}{2}\right) \geqslant s\left[\frac{2}{\pi} - \varepsilon\|\varphi\|_\infty\left(1 + \frac{\varepsilon\|\varphi\|_\infty}{2}\right)\right],$$

where the last inequality comes from the fact that $\sin s \geqslant \frac{2}{\pi}s$ for any $s \in [\delta, \frac{\pi}{2}]$. We deduce that:

$$x_\varepsilon(s) \geqslant \frac{-\|\varphi\|_\infty^2}{2}\left[\left(\varepsilon + \frac{1}{\|\varphi\|_\infty}\right)^2 - \frac{1}{\|\varphi\|_\infty^2}\left(1 + \frac{4}{\pi}\right)\right].$$

The right member of the above inequality is thus a second-order polynomial in ε which is positive if $\varepsilon \in]0, \varepsilon_0[$ where $\varepsilon_0 > 0$ refers to its positive root given by:

$$\varepsilon_0 = \frac{1}{\|\varphi\|_\infty}\left(\sqrt{1 + \frac{4}{\pi}} - 1\right).$$

Consequently, we have proved that for any $\varepsilon \in]0, \varepsilon_0[$, $x_\varepsilon(s) > 0$ for any $s \in]0, \frac{\pi}{2}]$. Similarly arguments can be used to show that $x_\varepsilon > 0$ in the neighbourhood of $L(\varepsilon)$ for small $\varepsilon > 0$ and we can treat in the same way $[x_\varepsilon(s) - x_\varepsilon(\tilde{s})]^2 + [z_\varepsilon(s) - z_\varepsilon(\tilde{s})]^2 > 0$ and also the second condition $z_\varepsilon[L(\varepsilon)] > 0$ appearing in Point (ii) of Definition 8.1.

Finally, in order to ensure θ_ε generates an admissible surface Σ_ε in the sense of Definition 8.1, it remains to impose the gluing condition $x_\varepsilon[L(\varepsilon)] = 0$ and also the area constraint $A(\Sigma) = 4\pi$. This two conditions gives new relations. Indeed, we have:

$$\begin{cases} A(\Sigma_\varepsilon) = 4\pi \\ x_\varepsilon[L(\varepsilon)] = 0 \end{cases} \Longleftrightarrow \begin{cases} \displaystyle\int_0^{L(\varepsilon)} \left(\int_0^s \cos[t + \varepsilon\varphi(t)]dt \right) ds = 2 \\ \displaystyle\int_0^{L(\varepsilon)} \cos[t + \varepsilon\varphi(t)]dt = 0. \end{cases}$$

The derivative at $\varepsilon = 0$ of the left member associated with the two relations above is thus zero. Using the formula $\frac{d}{d\varepsilon} \int_0^{L(\varepsilon)} f(\varepsilon, s)ds = \dot{L}(\varepsilon) f[L(\varepsilon), s] + \int_0^{L(\varepsilon)} \frac{\partial f}{\partial \varepsilon}(\varepsilon, s)ds$, we deduce that:

$$\dot{L}(\varepsilon) \underbrace{x_\varepsilon[L(\varepsilon)]}_{=0} - \int_0^{L(\varepsilon)} \left(\int_0^s \varphi(t) \sin[t + \varepsilon\varphi(t)]dt \right) ds = 0 \tag{6.3}$$

$$\dot{L}(\varepsilon) \underbrace{\cos\theta_\varepsilon[L(\varepsilon)]}_{=\cos\pi=-1} - \int_0^{L(\varepsilon)} \varphi(s) \sin[s + \varepsilon\varphi(s)]ds = 0. \tag{6.4}$$

Setting $\varepsilon = 0$, (6.4) gives $\dot{L}(0) + \int_0^\pi \varphi(t)\sin(t)dt = 0$ and (6.3) becomes $\int_0^\pi (\int_0^s \varphi(t)\sin(t)dt)ds = 0$, from which we deduce after an integration by parts $\pi\dot{L}(0) + \int_0^\pi s\varphi(s)\sin(s)ds = 0$. Finally, (6.3) is differentiated with respect to ε to obtain:

$$-\dot{L}(\varepsilon) \underbrace{\int_0^{L(\varepsilon)} \varphi(s)\sin[s + \varepsilon\varphi(s)]ds}_{=-\dot{L}(\varepsilon) \quad \text{from (6.4)}} - \int_0^{L(\varepsilon)} \left(\int_0^s \varphi(t)^2 \cos[t + \varepsilon\varphi(t)]dt \right) ds = 0.$$

We finally take $\varepsilon = 0$ in the above relation: $\dot{L}(0)^2 = \int_0^\pi (\int_0^s \varphi(t)^2 \cos(t)dt)ds$. To conclude with this subsection, we have proved the following.

Proposition 6.1. *Let us consider some well-defined maps of the form $\theta_\varepsilon : s \in [0, L(\varepsilon)] \mapsto s + \varepsilon\varphi(s)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-zero smooth map with compact support. We assume that each θ_ε is admissible in the sense of Definition 8.1, i.e. generates a (compact simply-connected) axisymmetric smooth surface $\Sigma_\varepsilon \subset \mathbb{R}^3$. Then, the map φ necessarily satisfies the following conditions:*

$$\varphi(0) = 0, \quad \varphi(\pi) = -\dot{L}(0) \tag{6.5}$$

$$\varphi(\pi) = \int_0^\pi \varphi(s)\sin(s)ds \tag{6.6}$$

$$\int_0^\pi \left(\int_0^s \varphi(t)\sin(t)dt \right) ds = 0 \quad \text{i.e.} \quad \int_0^\pi \varphi(s)\sin(s)ds = \frac{1}{\pi} \int_0^\pi s\varphi(s)\sin(s)ds \tag{6.7}$$

$$\varphi(\pi)^2 = \int_0^\pi \left(\int_0^s \varphi(t)^2 \cos(t)dt \right) ds \quad \text{i.e.} \quad \varphi(\pi)^2 = \int_0^\pi (\pi - s)\,\varphi(s)^2 \cos s ds. \tag{6.8}$$

6.1.4 Calculating the second-order shape derivative of Helfrich energy

The second-order derivative of total mean curvature

We introduce the functional $F_1 : \varepsilon \mapsto \frac{1}{2\pi} \int_{\Sigma_\varepsilon} HdA$, where Σ_ε is the surface of Proposition 6.1. Using (6.2), we thus have:

$$\begin{aligned} F_1(\varepsilon) &= \frac{1}{2\pi} \int_{\Sigma_\varepsilon} (\kappa_1 + \kappa_2)\, dA = \frac{1}{2\pi} \int_0^{L(\varepsilon)} \left(\frac{\sin\theta_\varepsilon(s)}{x_\varepsilon(s)} + \dot{\theta}_\varepsilon(s) \right) 2\pi x_\varepsilon(s)ds \\ &= \int_0^{L(\varepsilon)} \left(\sin[s + \varepsilon\varphi(s)] + [1 + \varepsilon\dot{\varphi}(s)] \int_0^s \cos[t + \varepsilon\varphi(t)]dt \right) ds. \end{aligned}$$

We first differentiate the above expression with respect to ε using the fact that $\frac{d}{d\varepsilon}\int_0^{L(\varepsilon)} f(\varepsilon,s)ds$ is equal to $\dot{L}(\varepsilon)f[L(\varepsilon),s]+\int_0^{L(\varepsilon)}\frac{\partial f}{\partial\varepsilon}(\varepsilon,s)ds$. We have successively:

$$
\begin{aligned}
\dot{F}_1(\varepsilon) &= \dot{L}(\varepsilon)\left(\underbrace{\sin\theta_\varepsilon[L(\varepsilon)]}_{=\sin\pi=0}+\dot{\theta}_\varepsilon[L(\varepsilon)]\underbrace{x_\varepsilon[L(\varepsilon)]}_{=0}\right)\\
&\quad+\int_0^{L(\varepsilon)}\Bigg(\varphi(s)\cos[s+\varepsilon\varphi(s)]+\dot{\varphi}(s)\int_0^s\cos[t+\varepsilon\varphi(t)]dt\\
&\qquad\qquad -[1+\varepsilon\dot{\varphi}(s)]\int_0^s\varphi(t)\sin[t+\varepsilon\varphi(t)]dt\Bigg)ds.
\end{aligned}
$$

If $\varepsilon=0$ in the above expression, then note that we find $F_1(\varepsilon)=0$ i.e. the unit sphere is a critical point of (6.1) for the family of perturbations (Σ_ε):

$$
\begin{aligned}
\dot{F}_1(0) &= \int_0^\pi\left(\varphi(s)\cos s+\dot{\varphi}(s)\underbrace{\int_0^s\cos(t)dt}_{=\sin s}-\int_0^s\varphi(t)\sin(t)dt\right)ds\\
&= [\varphi(s)\sin s]_0^\pi-\underbrace{\int_0^\pi\left(\int_0^s\varphi(t)\sin(t)dt\right)ds}_{=0\text{ from (6.7)}} \;=\; 0.
\end{aligned}
$$

In Theorem 11.3, we will prove this is also the case for any C^3-perturbation. Then, we differentiate with respect to ε the expression obtained for $\dot{F}_1(\varepsilon)$. It comes:

$$
\begin{aligned}
\ddot{F}_1(\varepsilon) &= \dot{L}(\varepsilon)\left(\varphi[L(\varepsilon)]\underbrace{\cos\theta_\varepsilon[L(\varepsilon)]}_{=\cos\pi=-1}+\dot{\varphi}[L(\varepsilon)]\underbrace{x_\varepsilon[L(\varepsilon)]}_{=0}-\dot{\theta}_\varepsilon[L(\varepsilon)]\int_0^{L(\varepsilon)}\varphi(s)\sin\theta_\varepsilon(s)ds\right)\\
&\quad-\int_0^{L(\varepsilon)}\Bigg(\varphi(s)^2\sin[s+\varepsilon\varphi(s)]+2\dot{\varphi}(s)\int_0^s\varphi(t)\sin[t+\varepsilon\varphi(t)]dt\\
&\qquad\qquad +[1+\varepsilon\dot{\varphi}(s)]\int_0^s\varphi(t)^2\cos[t+\varphi(t)]dt\Bigg)ds.
\end{aligned}
$$

If $\varepsilon=0$ in the last expression, we have:

$$
\begin{aligned}
\ddot{F}_1(0) &= \underbrace{\dot{L}(0)}_{=-\varphi(\pi)\text{ from (6.5)}}\left[-\varphi(\pi)-\underbrace{\int_0^\pi\varphi(s)\sin(s)ds}_{=\varphi(\pi)\text{ from (6.6)}}\right]-\int_0^\pi\varphi(s)^2\sin s ds\\
&\quad-2\underbrace{\int_0^\pi\left(\dot{\varphi}(s)\int_0^s\varphi(t)\sin(t)dt\right)ds}_{=\varphi(\pi)^2-\int_0^\pi\varphi(s)^2\sin(s)ds\text{ using (6.6)}}-\underbrace{\int_0^\pi\left(\int_0^s\varphi(t)^2\cos(t)dt\right)ds}_{=\varphi(\pi)^2\text{ from (6.8)}},
\end{aligned}
$$

from which we deduce after simplifications:

$$
\ddot{F}_1(0)=\int_0^\pi\varphi(s)^2\sin(s)ds-\varphi(\pi)^2. \tag{6.9}
$$

Let us check that $\ddot{F}_1(0)>0$ i.e. the unit sphere is a local minimizer of (6.1) for this family of perturbations $(\Sigma_\varepsilon)_{\varepsilon>0}$. In Remark 11.1, we will prove this is also the case for any C^2-perturbation. Combining (6.7) with the Cauchy-Schwarz inequality, we have:

$$
\pi^2\varphi(\pi)^2=\left(\int_0^\pi s\varphi(s)\sin(s)ds\right)^2\leqslant\left(\int_0^\pi\varphi(s)^2\sin(s)ds\right)\underbrace{\left(\int_0^\pi s^2\sin(s)ds\right)}_{=[2s\sin s+(2-s^2)\cos s]_0^\pi}.
$$

We thus obtain $\pi^2\varphi(\pi)^2 \leqslant (\ddot{F}_1(0)+\varphi(\pi)^2)(\pi^2-4)$ i.e. we have $\ddot{F}_1(0) \geqslant 4\varphi(\pi)^2 \geqslant 0$. But if $\ddot{F}_1(0) = 0$, then $\varphi(\pi) = 0$ and $\int_0^\pi \varphi(s)^2\sin(s)ds = 0$ so φ is identically zero, which is not the case. Contradiction. To conclude, we have shown $\dot{F}_1(0) = 0$ et $\ddot{F}_1(0) > 0$, where $\ddot{F}_1(0)$ is given by (6.9).

The second-order shape derivative of the Willmore energy

Let us now introduce the functional $F_2 : \varepsilon \mapsto \frac{1}{2\pi}\int_{\Sigma_\varepsilon} H^2 dA - 4$, where Σ_ε is the surface considered in Proposition 6.1. From Theorem 3.1, we have $\int_{\Sigma_\varepsilon} K dA = 4\pi$ which yields to:

$$\begin{aligned} F_2(\varepsilon) &= \frac{1}{2\pi}\int_{\Sigma_\varepsilon}\left(\kappa_1^2+\kappa_2^2\right)dA \quad = \quad \frac{1}{2\pi}\int_0^{L(\varepsilon)}\left(\frac{\sin^2\theta_\varepsilon(s)}{x_\varepsilon^2(s)}+\dot{\theta}_\varepsilon^2(s)\right)2\pi x_\varepsilon(s)ds \\ &= \int_0^{L(\varepsilon)}\left(\frac{\sin^2[s+\varepsilon\varphi(s)]}{\int_0^s\cos[t+\varepsilon\varphi(t)]dt}+[1+\varepsilon\dot{\varphi}(s)]^2\int_0^s\cos[t+\varepsilon\varphi(t)]dt\right)ds. \end{aligned}$$

First, we need to estimate the ratio $\frac{\sin\theta_\varepsilon(s)}{x_\varepsilon(s)}$ as $s \to L(\varepsilon)$. A first-order Taylor expansion gives:

$$\begin{cases} \sin\theta_\varepsilon(s) = \sin\theta_\varepsilon[L(\varepsilon)] + \dot{\theta}_\varepsilon[L(\varepsilon)]\cos\theta_\varepsilon[L(\varepsilon)][s-L(\varepsilon)] + o(s-L(\varepsilon)) \\ x_\varepsilon(s) = x_\varepsilon[L(\varepsilon)] + \cos\theta_\varepsilon[L(\varepsilon)][s-L(\varepsilon)] + o(s-L(\varepsilon)). \end{cases}$$

Using the boundary conditions $\theta(L(\varepsilon)) = \pi$ and the gluing condition $x[L(\varepsilon)] = 0$, we get:

$$\begin{cases} \sin\theta_\varepsilon(s) = (1+\varepsilon\dot{\varphi}[L(\varepsilon)])(L(\varepsilon)-s) + o(L(\varepsilon)-s) \\ x_\varepsilon(s) = L(\varepsilon)-s+o(L(\varepsilon)-s). \end{cases} \Longrightarrow \quad \frac{\sin\theta_\varepsilon(s)}{x_\varepsilon(s)} = 1+\varepsilon\dot{\varphi}[L(\varepsilon)] + \underset{s\to L(\varepsilon)}{o(1)}.$$

We now differentiate F_2 with respect to ε using $\frac{d}{d\varepsilon}\int_0^{L(\varepsilon)} f(\varepsilon,s)ds = \dot{L}(\varepsilon)f[L(\varepsilon),s] + \int_0^{L(\varepsilon)}\frac{\partial f}{\partial\varepsilon}(\varepsilon,s)ds$. We have:

$$\begin{aligned} \dot{F}_2(\varepsilon) &= \dot{L}(\varepsilon)\left(\underbrace{\sin\theta_\varepsilon[L(\varepsilon)]}_{=\sin\pi=0}\underbrace{\frac{\sin\theta_\varepsilon[L(\varepsilon)]}{x_\varepsilon[L(\varepsilon)]}}_{=1+\varepsilon\dot{\varphi}[L(\varepsilon)]} + \dot{\theta}_\varepsilon^2[L(\varepsilon)]\underbrace{x_\varepsilon[L(\varepsilon)]}_{=0}\right) \\ &\quad + \int_0^{L(\varepsilon)}\left(\frac{2\varphi(s)\sin\theta_\varepsilon(s)\cos\theta_\varepsilon(s)}{x_\varepsilon(s)} + \frac{\sin^2\theta_\varepsilon(s)}{x_\varepsilon(s)^2}\int_0^s\varphi(t)\sin\theta_\varepsilon(t)dt \right. \\ &\qquad\qquad \left. + 2\dot{\varphi}(s)\dot{\theta}_\varepsilon(s)x_\varepsilon(s) - \dot{\theta}_\varepsilon(s)^2\int_0^s\varphi(t)\sin\theta_\varepsilon(t)dt\right)ds. \end{aligned}$$

In particular, the ball is a critical point for F_2 as expected since spheres are the only global minimizer of the Willmore energy. Indeed, we have:

$$\dot{F}_2(0) = \int_0^\varepsilon\left(2\varphi(s)\cos s + 2\dot{\varphi}(s)\sin s\right)ds = [2\varphi(s)\sin s]_0^\pi = 0.$$

Then, we differentiate with respect to ε the expression obtained for $\dot{F}(\varepsilon)$ and it comes successively:

$$\ddot{F}_2(\varepsilon) = \dot{L}(\varepsilon)\left(2\varphi[L(\varepsilon)]\underbrace{\cos\theta_\varepsilon[L(\varepsilon)]}_{=\cos\pi=-1}\underbrace{\frac{\sin\theta_\varepsilon[L(\varepsilon)]}{x_\varepsilon[L(\varepsilon)]}}_{=1+\varepsilon\dot{\varphi}[L(\varepsilon)]} + 2\dot{\varphi}[L(\varepsilon)]\dot{\theta}_\varepsilon[L(\varepsilon)]\underbrace{x_\varepsilon[L(\varepsilon)]}_{=0}\right.$$

$$\left. + \underbrace{\frac{\sin^2\theta_\varepsilon[L(\varepsilon)]}{x_\varepsilon[L(\varepsilon)]^2}}_{=(1+\varepsilon\dot{\varphi}[L(\varepsilon)])^2}\int_0^{L(\varepsilon)}\varphi(s)\sin[s+\varepsilon\varphi(s)]ds - \dot{\theta}_\varepsilon^2[L(\varepsilon)]\int_0^{L(\varepsilon)}\varphi(s)\sin[s+\varepsilon\varphi(s)]ds\right)$$

$$+ \int_0^{L(\varepsilon)}\left(\frac{2\varphi(s)^2\cos^2\theta_\varepsilon(s)}{x_\varepsilon(s)} - \frac{2\varphi(s)^2\sin^2\theta_\varepsilon(s)}{x_\varepsilon(s)} + \frac{4\varphi(s)\cos\theta_\varepsilon(s)\sin\theta_\varepsilon(s)}{x_\varepsilon(s)^2}\int_0^s\varphi(t)\sin\theta_\varepsilon(t)dt\right.$$

$$+ \frac{\sin^2\theta_\varepsilon(s)}{x_\varepsilon(s)^2}\int_0^s\varphi(t)^2\cos\theta_\varepsilon(t)dt + \frac{2\sin^2\theta_\varepsilon(s)}{x_\varepsilon(s)^3}\left(\int_0^s\varphi(t)\sin\theta_\varepsilon(t)dt\right)^2 + 2\dot{\varphi}(s)^2\int_0^s\cos\theta_\varepsilon(t)dt$$

$$\left. -4\dot{\varphi}(s)\dot{\theta}_\varepsilon(s)\int_0^s\varphi(t)\sin\theta_\varepsilon(t)dt - \dot{\theta}_\varepsilon(s)^2\int_0^s\varphi^2(t)\cos\theta_\varepsilon(t)dt\right)ds$$

In the above expression, we set $\varepsilon = 0$ and we obtain:

$$\ddot{F}_2(0) = -2\varphi(\pi)\underbrace{\dot{L}(0)}_{\substack{=-\varphi(\pi)\\ \text{from (6.5)}}} + \int_0^\pi\left(\frac{2\varphi^2(s)\cos^2 s}{\sin s} - 2\varphi^2(s)\sin s + \frac{4\varphi(s)\cos s}{\sin s}\int_0^s\varphi(t)\sin t dt\right.$$

$$\left. + \frac{2}{\sin s}\left(\int_0^s\varphi(t)\sin t dt\right)^2 + 2\dot{\varphi}(s)^2\sin s - 4\dot{\varphi}(s)\int_0^s\varphi(t)\sin t dt\right)ds$$

$$= 2\varphi(\pi)^2 + \int_0^\pi\frac{2}{\sin s}\left(\varphi(s)\cos s + \int_0^s\varphi(t)\sin t dt\right)^2 ds + \int_0^\pi 2\sin s\left[\dot{\varphi}(s)^2 - \varphi(s)^2\right]ds$$

$$-4\varphi(\pi)\underbrace{\int_0^\pi\varphi(s)\sin s ds}_{=\varphi(\pi)\text{ from (6.6)}} + 4\int_0^\pi\varphi(s)^2\sin s ds$$

The last line is obtained from an integration by parts performed on $\int_0^\pi 4\dot{\varphi}(s)(\int_0^s\varphi(t)\sin t dt)ds$. Therefore, we have after simplifications:

$$\ddot{F}_2(0) = \int_0^\pi\frac{2}{\sin s}\left(\varphi(s)\cos s + \int_0^s\varphi(t)\sin t dt\right)^2 ds + \int_0^\pi 2\sin s\left[\dot{\varphi}(s)^2 + \varphi(s)^2\right]ds - 2\varphi(\pi)^2,$$

which is combined to (6.9) to finally obtain:

$$\frac{1}{2}\ddot{F}_2(0) = \ddot{F}_1(0) + \int_0^\pi\dot{\varphi}(s)^2\sin s ds + \int_0^\pi\frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s\varphi(t)\sin t dt\right)^2 ds. \tag{6.10}$$

Note that since we proved $\ddot{F}_1(0) > 0$, the above expression show that $\ddot{F}_2(0) > 0$. Hence, the sphere is a local minimizer as expected since spheres are the only global minimizer of the Willmore energy.

The second-order shape derivative of the Helfrich energy

We now get back to Problem (6.1). For any $c_0 > 0$, we define the following map:

$$F_{c_0} : \varepsilon \longmapsto F_{c_0}(\varepsilon) = \frac{1}{2}F_2(\varepsilon) + 2 - 2c_0F_1(\varepsilon) + 4c_0^2 = \frac{1}{4\pi}\int_{\Sigma_\varepsilon}(H - 2c_0)^2 dA. \tag{6.11}$$

Hence, $F_{c_0}(\varepsilon)$ (respectively $F_{c_0}(0)$) corresponds to the the Helfrich energy (3.2) of the axisymmetric perturbations Σ_ε (resp. of the unit sphere) considerered in Proposition 6.1. From the foregoing, we combine (6.9) with (6.10) to obtain its second-order derivative at $\varepsilon = 0$. We get that $\ddot{F}_{c_0}(0)$ is equal to:

$$(1-2c_0)\left(\int_0^\pi \varphi(s)^2 \sin s ds - \varphi(\pi)^2\right) + \int_0^\pi \left[\dot{\varphi}(s)^2 \sin s + \frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s \varphi(t)\sin t dt\right)^2\right] ds.$$

Since $\dot{F}_{c_0}(0) = \frac{1}{2}\dot{F}_2(0) - 2c_0\dot{F}_1(0) = 0$, we have $\ddot{F}_{c_0}(0) < 0$ if and only if for ε small enough, the Helfrich energy of Σ_ε given by $F_{c_0}(\varepsilon) = F_{c_0}(0) + \varepsilon^2 \ddot{F}_{c_0}(0) + o(\varepsilon^2)$ is strictly lower than the one of the unit sphere given by $F_{c_0}(0)$. Moreover, from the above relation we have:

$$\ddot{F}_{c_0}(0) < 0 \Longleftrightarrow c_0 \ > \ 1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \dot{\varphi}(s)^2 \sin s ds + \int_0^\pi \frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s \varphi(t)\sin t dt\right)^2 ds}{\displaystyle\int_0^\pi \varphi(s)^2 \sin s ds - \varphi(\pi)^2}. \quad (6.12)$$

Therefore, we define the following critical value:

$$c_{++} = \inf\left(1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \dot{\varphi}(s)^2 \sin s ds + \int_0^\pi \frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s \varphi(t)\sin t dt\right)^2 ds}{\displaystyle\int_0^\pi \varphi(s)^2 \sin s ds - \varphi(\pi)^2}\right), \quad (6.13)$$

where the infimum is taken among all non-zero map $\varphi : W^{1,1}_{\text{loc}}(0,\pi)$ satisfying with its weak derivative (in the sense of distribution) $\dot{\varphi}$ the following growth condition:

$$\int_0^\pi \dot{\varphi}(s)^2 \sin s ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \frac{\left(\int_0^s \varphi(t)\sin t dt\right)^2}{\sin s} ds < +\infty \quad (6.14)$$

and also the constraints (6.5)–(6.8) given in Proposition 6.1 which we recall for completeness:

$$\varphi(0) = 0,\ \varphi(\pi) = \int_0^\pi \varphi(s)\sin s ds = \frac{1}{\pi}\int_0^\pi s\varphi(s)\sin s ds,\ \varphi(\pi)^2 = \int_0^\pi \left(\int_0^s \varphi(t)^2 \cos t dt\right) ds.$$

We imposed the growth condition (6.14) to the map φ because we want the ratio appearing in (6.13) to be finite. Note that we *do not* have $\varphi \in H^1(0,\pi)$. However, (6.14) gives nevertheless severe restrictions for the behaviour of φ at 0 and π.

Lemma 6.2. *If $\varphi : [0,\pi] \to$ satisfies the growth condition* (6.14)*, then we have $\varphi(0) = \varphi(\pi) = 0$ and also:*

$$\int_0^\pi \varphi(s)\sin s ds = \lim_{s\to 0} \frac{1}{\sin s}\int_0^s \varphi(t)\sin t dt = \lim_{s\to\pi} \frac{1}{\sin s}\int_0^s \varphi(t)\sin t dt = 0.$$

Proof. Let $\varphi \in W^{1,1}_{\text{loc}}(0,\pi)$ satisfy (6.14). We first have from the Cauchy-Schwarz inequality:

$$\begin{aligned}\int_0^1 \left(\varphi(s)^2 + 2|\dot{\varphi}(s)\varphi(s)|\right) ds + \int_0^1 \frac{\varphi(s)^2}{s} ds \ &\leqslant \ \int_0^1 \frac{2\varphi(s)^2}{s} ds + 2\sqrt{\int_0^1 \dot{\varphi}(s)^2 \sin s ds \int_0^1 \frac{\varphi(s)^2}{\sin s} ds} \\ &\leqslant \ 3\int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \dot{\varphi}(s)^2 \sin s ds \ < \ +\infty\end{aligned}$$

Hence, $\varphi^2|_{[0,1]} \in W^{1,1}(0,1)$ and $(s \mapsto \varphi^2|_{[0,1]}(s)/s) \in L^1(0,1)$ so we can apply [15, Exercise 8.8.2] to get $\varphi(0) = 0$. Similarly, if we set $\tilde{\varphi}(s) = \varphi(\pi - s)$, then we get $\tilde{\varphi}|^2_{[0,1]} \in W^{1,1}(0,1)$ and $(s \mapsto \tilde{\varphi}^2|_{[0,1]}(s)/s) \in L^1(0,1)$ so [15, Exercise 8.8.2] gives $\tilde{\varphi}(0) = \varphi(\pi) = 0$. Let us now introduce the map $u(s) = \int_0^s \varphi(t)\sin t dt$. The growth condition (6.14) yield to:

$$\begin{aligned}\|u\|^2_{H^2(0,\pi)} \ &\leqslant \ \int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds \\ &\leqslant \ \int_0^\pi \frac{\left(\int_0^s \varphi(t)\sin t dt\right)^2}{\sin s} ds + 3\int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + 2\int_0^\pi \dot{\varphi}(s)^2 \sin s ds < +\infty\end{aligned}$$

Therefore, we obtain $u \in H^2(0,\pi)$. In particular, $u|_{[0,1]} \in H^2(0,1)$ and $u(0) = \dot{u}(0) = 0$ from which we deduce [15, Exercise 8.9.1] $(s \mapsto u|_{[0,1]}(s)/s^2) \in L^2(0,1)$ and $(s \mapsto \dot{u}|_{[0,1]}(s)/s) \in L^2(0,1)$. By setting $v(s) = u(s)/s$, we thus have $v|_{[0,1]} \in H^1(0,1)$ and $(s \mapsto v|_{[0,1]}(s)/s) \in L^2(0,1)$. We can apply again [15, Exercise 8.8.2] to obtain $v(0) = 0$ and thus:

$$\frac{1}{\sin s}\int_0^s \varphi(t)\sin t dt \quad = \quad \frac{s}{\sin s} v(s) \quad \underset{s\to 0}{\longrightarrow} \quad 0.$$

Finally, we introduce the map $\tilde{u}(s) = u(\pi - s)$. As we did for φ, we have:

$$\int_0^1 \left(\tilde{u}(s)^2 + 2|\tilde{u}'(s)\tilde{u}(s)|\right) ds + \int_0^1 \frac{\tilde{u}(s)^2}{s} ds \ \leqslant\ 3\int_0^\pi \frac{\tilde{u}(s)^2}{\sin s} ds + \int_0^\pi \tilde{u}'(s)^2 \sin s ds$$

Since $\sin(\pi - s) = \sin s$ on $[0,\pi]$, we make a change of variables in the two last integrals to obtain:

$$\int_0^1 \left(\tilde{u}(s)^2 + 2|\tilde{u}'(s)\tilde{u}(s)|\right) ds + \int_0^1 \frac{\tilde{u}(s)^2}{s} ds \leqslant 3\int_0^\pi \frac{\left(\int_0^s \varphi(t)\sin t dt\right)^2}{\sin s} ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds \ < \ +\infty.$$

Hence, $\tilde{u}|^2_{[0,1]} \in W^{1,1}(0,1)$ and $(s \mapsto \tilde{u}|^2_{[0,1]}(s)/s) \in L^1(0,1)$ so [15, Exercise 8.8.2] gives $\tilde{u}(0) = 0$. At last, we thus have $\tilde{u}(0) = \tilde{u}'(0) = 0$ and $\tilde{u}|_{[0,1]} \in H^2(0,1)$ so we use again [15, Exercise 8.9.1] to obtain $(s \mapsto \tilde{u}|_{[0,1]}(s)/s^2) \in L^2(0,1)$ and $(s \mapsto \dot{\tilde{u}}|_{[0,1]}(s)/s) \in L^2(0,1)$. By setting $\tilde{v}(s) = \tilde{u}(s)/s$, we have $\tilde{v}|_{[0,1]} \in H^1(0,1)$ and $(s \mapsto \tilde{v}|_{[0,1]}(s)/s) \in L^2(0,1)$. Applying again [15, Exercise 8.8.2], we obtain $\tilde{v}(0) = 0$, from which we deduce:

$$\lim_{s\to\pi} \frac{1}{\sin s}\int_0^s \varphi(t)\sin t dt = \lim_{s\to 0} \frac{1}{\sin(\pi - s)}\int_0^{\pi - s} \varphi(t)\sin t dt = \lim_{s\to 0}\frac{s}{\sin s}\tilde{v}(s) = 0.$$

To conclude, if $\varphi : [0,\pi] \to \mathbb{R}$ satisfies the growth condition (6.14), then we have $\varphi(0) = \varphi(\pi) = 0$ and also $u(0) = u(\pi) = v(0) = v(\pi) = 0$, where we set $u(s) = \int_0^s \varphi(t)\sin t dt$ and $v(s) = \frac{u(s)}{\sin s}$. $\square$

To conclude this subsection, we sum up the results we have obtained in the following statement.

Proposition 6.3. *Let c_0 be given by* (5.2) *and c_{++} as in* (6.13) *but where the infimum is taken among all non-zero maps satisfying* (6.14) *and only* (6.7)–(6.8). *If $c_0 < c_{++}$, then the unit sphere is a local minimizer of* (6.1) *among the class of perturbations Σ_ε given in Proposition 6.1. Conversely, if $c_0 > c_{++}$, provided we can build a perturbation Σ_ε from a map φ satisfying* (6.14) *and* (6.5)–(6.8), *then the sphere is not a local minimizer of* (6.1).

Proof. If $c_0 < c_{++}$, then $c_0 < R(\varphi)$ and (6.12) yields to $\ddot{F}_{c_0}(0) > 0$ for any $\varphi \in W^{1,1}_{\text{loc}}(0,\pi)$ satisfying (6.14) and (6.7)–(6.8). In particular, for ε small enough, any perturbation Σ_ε of the form given in Proposition 6.1 has a strictly greater Helfrich energy (6.11) than the one of the unit sphere, which is thus a local minimizer of (6.1) among this class of perturbations:

$$\begin{aligned}
\frac{1}{4\pi}\int_{\Sigma_\varepsilon}(H - 2c_0)^2 dA - \frac{1}{4\pi}\int_{\mathbb{S}^2}(H-2c_0)^2 dA \quad &= \quad F_{c_0}(\varepsilon) - F_{c_0}(0) = \dot{F}_{c_0}(0)\varepsilon + \ddot{F}_{c_0}(0)\frac{\varepsilon^2}{2} + o(\varepsilon^2) \\
&= \quad \frac{\varepsilon^2}{2}\left(\ddot{F}_{c_0}(0) + \frac{o(\varepsilon^2)}{\varepsilon^2}\right) \quad (\ > 0 \ \text{ for } \varepsilon \text{ small}).
\end{aligned}$$

Conversely, if $c_0 > c_{++}$, there exists $\varphi \in W^{1,1}_{\text{loc}}(0,\pi)$ satisfying the growth condition (6.14) and also, using Lemma 6.2, all the constraints (6.5)–(6.8), such that $c_0 > R(\varphi)$. We thus have $\ddot{F}_{c_0}(0) < 0$, provided we can build an extension $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of φ with compact support such that the family of maps $\theta_\varepsilon : s \in [0, L(\varepsilon)] \to s + \varepsilon\tilde{\varphi}(s)$ is well-defined around $\varepsilon = 0$ and admissible in the sense of Definition 8.1, i.e. generates some (compact simply-connected) axisymmetric surfaces $\Sigma_\varepsilon \subset \mathbb{R}^3$. If this is the case, for ε small enough, Σ_ε is a perturbation with strictly lower Helfrich energy (6.11) than the one of the unit sphere, which is thus not a local minimizer of (6.1). Moreover, with an appropriate rescaling, the sphere $\mathbb{S}_{A_0}$ of area A_0 is not a local minimizer of (5.1). $\square$

6.2 The critical value given by a new optimization problem

We are now in position to formulate the previous optimization problem (6.13) in a new form by considering $u(s) = \int_0^s \varphi(t)\sin t dt$. The result states as follows.

Theorem 6.4. *The value of c_{++} given by* (6.13) *is also determined by the following equivalent optimization problem:*

$$c_{++} = \inf \frac{1}{2} \frac{\displaystyle\int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds + 2 \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds}{\displaystyle\int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds}, \tag{6.15}$$

where the infimum is taken among all non-zero $H^2(0, \pi)$-maps satisfying the growth condition:

$$\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds + \int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds < +\infty, \tag{6.16}$$

and the following two constraints

$$\int_0^\pi u(s) ds = 0, \tag{6.17}$$

$$\int_0^\pi (\pi - s) \cos s \left(\frac{\dot{u}(s)}{\sin s} \right)^2 ds = 0. \tag{6.18}$$

Proof. First, we assume that φ is an admissible map of (6.13) i.e. (6.14) and (6.5)–(6.8) hold true. We set $u(s) = \int_0^s \varphi(t) \sin t dt$ and show that u is admissible for (6.15). From Lemma 6.2, $\varphi(\pi) = 0$ thus (6.5)–(6.6) gives (6.17)–(6.18). Concerning (6.16), $\dot{u}(s) = \varphi(s) \sin s$ and (6.14) holds for φ so we deduce:

$$\begin{aligned} \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds + \int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds &= \int_0^\pi \varphi(s)^2 \sin s ds + \int_0^\pi \frac{(\dot{\varphi}(s) \sin s + \varphi(s) \cos s)^2}{\sin s} ds \\ &\leqslant 2 \int_0^\pi \dot{\varphi}(s)^2 \sin s \, ds + 3 \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds < +\infty \end{aligned}$$

Moreover, we also have:

$$\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds = \int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{u(s)^2 \cos^2 s}{\sin^3 s} ds = \frac{1}{2} \int_0^\pi \frac{u(s)^2}{\sin s} ds - \frac{1}{2} \int_0^\pi \frac{d}{ds} \left(\frac{\cos s}{\sin^2 s} \right) u(s)^2 ds$$

In the last term above, we proceed to an integration by parts, and the boundary terms vanish by applying Lemma 6.2. We thus obtain from the Cauchy-Schwarz inequality:

$$\begin{aligned} \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds &= \frac{1}{2} \int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\dot{u}(s) u(s) \cos s}{\sin^2 s} ds \\ &\leqslant \frac{1}{2} \int_0^\pi \frac{u(s)^2}{\sin s} ds + \sqrt{\int_0^\pi \frac{\dot{u}(s)^2 \cos^2 s}{\sin^3 s} ds} \sqrt{\int_0^\pi \frac{u(s)^2}{\sin s} ds} \\ &\leqslant \int_0^\pi \frac{u(s)^2}{\sin s} ds + \frac{1}{2} \int_0^\pi \frac{\dot{u}(s)^2 \cos^2 s}{\sin^3 s} ds \\ &\leqslant \int_0^\pi \frac{\left(\int_0^s \varphi(t) \sin t dt \right)^2}{\sin s} ds + \frac{1}{2} \int_0^\pi \frac{\varphi^2(s)}{\sin s} ds < +\infty. \end{aligned}$$

Hence, u satisfies (6.16) and we proved that if φ is admissible for (6.13), then $u(s) = \int_0^s \varphi(t) \sin t dt$ is admissible for (6.15). Conversely, let us assume u is admissible for (6.15) then set $\varphi(s) = \dot{u}(s)/\sin s$. We have successively:

$$\begin{aligned} \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds + \int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds &= \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \dot{\varphi}(s)^2 \sin s ds \\ &\quad + \int_0^\pi 2 \varphi(s) \dot{\varphi}(s) \cos s ds \end{aligned}$$

Then, note that if we have $\varphi(0) = \varphi(\pi) = 0$, then we can perform two integration by parts on the last integral above such that the boundary terms are zero and we get:

$$\begin{aligned}
\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\dot u(s)^2}{\sin s} ds + \int_0^\pi \frac{\ddot u(s)^2}{\sin s} ds &= \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \dot\varphi(s)^2 \sin s ds \\
&\quad - \int_0^\pi u(s)\dot\varphi(s) ds \\
&\geqslant \int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \dot\varphi(s)^2 \sin s ds \\
&\quad - \int_0^\pi u(s)\dot\varphi(s) ds \\
&= \frac{1}{2}\int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \frac{1}{2}\int_0^\pi \dot\varphi(s)^2 \sin s ds \\
&\quad + \frac{1}{2}\int_0^\pi \left[\frac{u(s)}{\sqrt{\sin s}} - \dot\varphi(s)\sqrt{\sin s}\right]^2 ds \\
&\geqslant \frac{1}{2}\left(\int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\varphi(s)^2}{\sin s} ds + \int_0^\pi \dot\varphi(s)^2 \sin s ds\right),
\end{aligned}$$

from which we deduce that φ satisfies (6.14). Hence, we check that $\varphi(0) = \varphi(\pi) = 0$. First, from (6.16), we deduce that $\dot u|^2_{[0,1]} \in W^{1,1}(0,1)$ and $(s \mapsto \dot u(s)^2/\sin s) \in L^1(0,1)$ so [15, Exercise 8.8.2] gives $\dot u(0) = 0$. Then, we use successively the fact that the sine function is positive and increasing on $[0, \frac{\pi}{2}]$, the Cauchy-Schwarz inequality, and $\sin x \geqslant \frac{2}{\pi}x$ for any $x \in [0, \frac{\pi}{2}]$ in order to get $\varphi(0) = 0$ as follows:

$$\begin{aligned}
\forall s \in \left[0, \frac{\pi}{2}\right],\ |\varphi(s)| &= \frac{|\dot u(s)|}{\sin s} = \left| \int_0^s \frac{\ddot u(t)}{\sin s} dt \right| \leqslant \int_0^s \frac{|\ddot u(t)|}{\sin s} dt \leqslant \frac{1}{\sqrt{\sin s}} \int_0^s \frac{|\ddot u(t)|}{\sqrt{\sin t}} dt. \\
&\leqslant \sqrt{\frac{s}{\sin s}\int_0^s \frac{\ddot u(t)^2}{\sin t} dt} \leqslant \sqrt{\frac{\pi}{2}\int_0^s \frac{\ddot u(t)^2}{\sin t} dt} \underset{s\to 0}{\longrightarrow} 0.
\end{aligned} \tag{6.19}$$

Similarly, one can obtain that $\varphi(\pi) = 0$. Hence, we have proved that φ satisfies the growth condition (6.14). Applying Lemma 6.2 and using (6.17)–(6.18), we obtain that φ is admissible for (6.13). Finally, φ is admissible for (6.13) if and only if u is admissible for (6.15) so it remains to prove that the functional of the two problems are equal. Considering φ, we express (6.13) in terms of $u = \int_0^\bullet \varphi(t)\sin t dt$. We have:

$$\begin{aligned}
R(\varphi) &:= 1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \dot\varphi(s)^2 \sin s ds + \int_0^\pi \frac{1}{\sin s}\left(\varphi(s)\cos s + \int_0^s \varphi(t)\sin t dt\right)^2 ds}{\displaystyle\int_0^\pi \varphi(s)^2 \sin s ds - \varphi(\pi)^2} \\
&= 1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{1}{\sin^3 s}\left(\ddot u(s)\sin s - \dot u(s)\cos s\right)^2 ds + \int_0^\pi \frac{1}{\sin^3 s}\left(\dot u(s)\cos s + u(s)\sin s\right)^2 ds}{\displaystyle\int_0^\pi \frac{\dot u(s)^2}{\sin s} ds \quad - \quad 0} \\
&= 1 + \frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{2\dot u(s)^2\cos^2 s}{\sin^3 s} ds + \int_0^\pi \frac{u(s)^2 + \ddot u(s)^2}{\sin s} ds + \int_0^\pi 2\left(u(s)\dot u(s) - \dot u(s)\ddot u(s)\right)\frac{\cos s}{\sin^2 s} ds}{\displaystyle\int_0^\pi \frac{\dot u(s)^2}{\sin s} ds}
\end{aligned}$$

On the last term on the numerator, we proceed to an integration by parts where the boundary terms are zero by applying Lemma 6.2. We obtain that $R(\varphi)$ is equal to:

$$1+\frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{2\dot u(s)^2\cos^2 s}{\sin^3 s}ds+\int_0^\pi \frac{u(s)^2+\ddot u(s)^2}{\sin s}ds+\int_0^\pi \left(u(s)^2-\dot u(s)^2\right)\left(\frac{1}{\sin s}+\frac{2\cos^2 s}{\sin^3 s}\right)ds}{\displaystyle\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds}$$

$$\text{i.e. } R(\varphi)=1+\frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{\ddot u(s)^2}{\sin s}ds+2\int_0^\pi \frac{u(s)^2}{\sin s}ds-\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds}{\displaystyle\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds}=\frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{\ddot u(s)^2}{\sin s}ds+2\int_0^\pi \frac{u(s)^2}{\sin s}ds}{\displaystyle\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds}.$$

Conversely, considering u, we express (6.15) in terms of $\varphi(s)=\dot u(s)/\sin s$ and we obtain similarly after an integration by parts, that the boundary terms are zero from Lemma 6.2 and also:

$$\tilde R(u):=\frac{1}{2}\frac{\displaystyle\int_0^\pi \frac{\ddot u(s)^2}{\sin s}ds+2\int_0^\pi \frac{u(s)^2}{\sin s}ds}{\displaystyle\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds}=R(\varphi).$$

To conclude, we have proved that the minimization problems (6.13) and (6.15) are equivalent. □

6.3 An estimation of the threshold value

In this section, we try to evaluate the exact value of c_{++} by computing the critical points of (2.12). However, we did not manage to handle the non-linear constraint $\int_0^s(\pi-s)\cos s(\frac{\dot u(s)}{\sin s})^2ds=0$. Hence, we decided to compute the critical value of (2.12) without this constraint. It becomes an eigenvalue problem associated with a non-linear fourth-order one-dimensional differential equation. We first prove the existence of a minimizer to this problems.

6.3.1 The existence of a minimizer attaining c_{++}

Let us introduce the following weighted Sobolev space:

$$H^2_{\sin}(0,\pi)=\left\{u\in H^2(0,\pi),\quad \int_0^\pi \frac{u(s)^2}{\sin s}ds+\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds+\int_0^\pi \frac{\ddot u(s)^2}{\sin s}ds<+\infty\right\}, \tag{6.20}$$

which is an Hilbert space equipped with the norm $\|\bullet\|_{H^2_{\sin}(0,\pi)}=\langle\bullet\mid\bullet\rangle^{\frac{1}{2}}$ coming from the scalar product:

$$\langle u\mid v\rangle_{H^2_{\sin}(0,\pi)}:=\int_0^\pi \frac{u(s)v(s)}{\sin s}ds+\int_0^\pi \frac{\dot u(s)\dot v(s)}{\sin s}ds+\int_0^\pi \frac{\ddot u(s)\ddot v(s)}{\sin s}ds.$$

Note that in (6.20) the sine exponent appearing in first integral is different from the one given in (6.16) but we prove this leads to equivalent norms.

Lemma 6.5. *Let $u\in H^2_{\sin}(0,\pi)$. Then, we have:*

$$\|u\|^2_{H^2_{\sin}(0,\pi)}\leqslant\int_0^\pi \frac{u(s)^2}{\sin^3 s}ds+\int_0^\pi \frac{\dot u(s)^2}{\sin s}ds+\int_0^\pi \frac{\ddot u(s)^2}{\sin s}ds\leqslant 2\|u\|^2_{H^2_{\sin}(0,\pi)}.$$

Proof. Let $u\in H^2_{\sin}(0,\pi)$. Since $\sin s\in[0,1]$ for any $s\in[0,\pi]$, we have $\int_0^\pi \frac{u(s)^2}{\sin s}ds\leqslant\int_0^\pi \frac{u(s)^2}{\sin^3 s}ds$. Moreover, we also have:

$$\int_0^\pi \frac{u(s)^2}{\sin^3 s}ds=\int_0^\pi \frac{u(s)^2}{\sin s}ds+\int_0^\pi \frac{u(s)^2\cos^2}{\sin s}ds\ =\ \frac{1}{2}\int_0^\pi \frac{u(s)^2}{\sin s}ds+\int_0^\pi \frac{u(s)^2}{2}\frac{d}{ds}\left(\frac{-\cos s}{\sin^2 s}\right)ds$$

We can perform an integration by parts in the last integral above. As we did in the paragraph above (6.19) for the proof of Theorem 6.4, we can show that the boundary terms vanish and we get from the Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds &= \frac{1}{2}\int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{u(s)\dot{u}(s)\cos s}{\sin^2 s} ds \\
&\leqslant \frac{1}{2}\int_0^\pi \frac{u(s)^2}{\sin s} ds + \sqrt{\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds \int_0^\pi \frac{\dot{u}(s)^2 \cos^2 s}{\sin s} ds} \\
&\leqslant \frac{1}{2}\int_0^\pi \frac{u(s)^2}{\sin s} ds + \frac{1}{2}\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds + \frac{1}{2}\int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds.
\end{aligned}$$

After simplifications, we thus obtain:

$$\int_0^\pi \frac{u(s)^2}{\sin^3 s} ds \leqslant \int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds, \tag{6.21}$$

from which we conclude that the estimation of Lemma 6.5 holds true. □

Consequently, we deduce that the following functional is well-defined:

$$\forall u \in H^2_{\sin}(0,\pi), \quad G(u) := \frac{1}{2}\int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds + \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds < +\infty, \tag{6.22}$$

and we can prove the following.

Proposition 6.6. *There exists a minimizer to the optimization problem:*

$$\inf_{\substack{u \in H^2_{\sin}(0,\pi) \\ \int_0^\pi u(s)ds = 0 \\ \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds = 1}} G(u), \tag{6.23}$$

where G is given by (6.22)*. Moreover, there exists a minimizer to Problem* (6.13) *and* (6.15)*.*

Proof. We consider a minimizing sequence $(u_i)_{i\in\mathbb{N}}$ of (6.23). Combining Lemma 6.5, the constraints $\int_0^\pi \frac{\dot{u}_i(s)^2}{\sin s} ds = 1$ and the convergence of $G(u_i)$, we thus have $(u_i)_{i\in\mathbb{N}}$ bounded in $H^2_{\sin}(0,\pi)$. Since $H^2_{\sin}(0,\pi)$ is an Hilbert space, it is reflexive so we deduce that, up to a subsequence, $(u_i)_{i\in\mathbb{N}}$ weakly converges to $u \in H^2_{\sin}(0,\pi)$. Then, we use the fact that the space $H^2_{\sin}(0,\pi)$ is compactly embedded into the weighted Sobolev space:

$$H^1_{\sin}(0,\pi) = \left\{ v \in H^1(0,\pi), \quad \|u\|^2_{H^1_{\sin}(0,\pi)} := \int_0^\pi \frac{u(s)^2}{\sin s} ds + \int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds < +\infty \right\}$$

A proof of this fact can be found in [41, Theorem 2.3] where our one-dimensional setting fits with the hypothesis of the paper. Hence, up to a subsequence, $(u_i)_{i\in\mathbb{N}}$ weakly converges to u in $H^2_{\sin}(0,\pi)$ and strongly in $H^1_{\sin}(0,\pi)$. Combining (6.21) and the fact that the norm is lower semi-continuous with respect to the weak convergence, we deduce that:

$$\int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds \leqslant \liminf_{i\to+\infty} \int_0^\pi \frac{\ddot{u}_i(s)^2}{\sin s} ds \quad \text{and} \quad \lim_{i\to+\infty} \int_0^\pi \frac{u_i(s)^2}{\sin^3 s} ds = \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds.$$

Finally, the functional (6.22) is lower semi-continuous and it remains to prove the continuity of the constraints. From the strong convergence of $(u_i)_{i\in\mathbb{N}}$ in $H^1_{\sin}(0,\pi)$, we get $\int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds = 1$ and also:

$$\left| \int_0^\pi [u_i(s) - u(s)]\, ds \right| \leqslant \int_0^\pi |u_i(s) - u(s)| ds \leqslant \sqrt{\int_0^\pi \sin s ds \int_0^\pi \frac{[u_i(s) - u(s)]^2}{\sin s} ds} \leqslant \sqrt{2}\|u_i - u\|_{H^1_{\sin}}$$

Therefore, we have proved that u is a minimizer for (6.23). Finally, observe that if we add the constraint $\int_0^\pi (\pi - s)\cos s (\frac{\dot{u}(s)}{\sin s})^2 ds = 0$ to (6.23), then Problems (6.15) and (6.23) are equivalent.

We thus show that we can let $i \to +\infty$ in this constraint. We have successively using the Cauchy-Schwarz inequality and (6.21):

$$\begin{aligned}
\left| \int_0^\pi (\pi - s)\cos s \frac{\dot{u}_i(s)^2 - \dot{u}(s)^2}{\sin^2 s} ds \right| &\leqslant \int_0^\pi \frac{|u_i(s) - u(s)|}{\sqrt{\sin s}} \frac{|\dot{u}_i(s) + \dot{u}(s)|}{\sin^{\frac{3}{2}} s} ds \\
&\leqslant \sqrt{\int_0^\pi \frac{(u_i(s) - u(s))^2}{\sin s} ds \int_0^\pi \frac{(u_i(s) + u(s))^2}{\sin^3 s} ds} . \\
&\leqslant \sqrt{2} \|u_i - u\|_{H^1_{\sin}(0,\pi)} \, \|u_i + u\|_{H^1_{\sin}(0,\pi)} \underset{i \to +\infty}{\longrightarrow} 0
\end{aligned}$$

To conclude, Problem (6.15) has a minimizer and so does (6.13) by applying Theorem 6.4. □

6.3.2 Computing the critical points of the problem

We only study here the minimization problem (6.23), which is *not* equivalent to (6.13)–(6.15) since we drop the non-linear constraint $\int_0^\pi (\pi - s)\cos s (\frac{\dot{u}(s)}{\sin s})^2 ds = 0$. In particular, the minimum for (6.23) is an estimation of c_{++} from below since it is lower than (or equal to) c_{++}.

Moreover, although there exits a minimizer u to (6.23), we did not manage to prove a stronger regularity for u, which is often needed for the computation of critical points. We also have $H^2_{\sin}(0,\pi) \subseteq H^2_0(0,\pi)$. To see this last point, we can proceed by using [15, Exercise 8.8.2] as it is done in the proof of Lemma 6.2 or in the paragraph above (6.19).

We consider $t \in \mathbb{R}$, the minimizer $u \in H^2_{\sin}(0,\pi)$ of (6.23), any map $v \in C_c^\infty([0,\pi],\mathbb{R})$ and the Lagrangian associated with (6.23):

$$L(w) = \frac{1}{2}\int_0^\pi \frac{\ddot{w}(s)^2}{\sin s} ds + \int_0^\pi \frac{w(s)^2}{\sin^3 s} ds - \lambda \left(\int_0^\pi \frac{\dot{w}(s)^2}{\sin s} ds - 1 \right) - \mu \int_0^\pi u(s) ds,$$

where λ is an eigenvalue and μ the Lagrange multiplier associated with the constraint $\int_0^\pi u(s)ds = 0$. Then, we get after calculations:

$$\frac{L(u+tv) - L(u)}{t} \underset{t \to 0}{\longrightarrow} \int_0^\pi \frac{\ddot{u}(s)\ddot{v}(s)}{\sin s} ds + 2\int_0^\pi \frac{u(s)v(s)}{\sin s} ds - 2\lambda \int_0^\pi \frac{\dot{u}(s)\dot{v}(s)}{\sin s} ds - \mu \int_0^\pi v(s) ds.$$

Since u is a minimizer for (6.23), this limit must be zero for any $v \in C_c^\infty([0,\pi],\mathbb{R})$. Hence, the minimizer u satisfies the following non-linear fourth-order one-dimensional differential equation, whose solutions are the critical points of (6.23):

$$\forall s \in]0,\pi[, \quad \frac{d^2}{ds^2}\left(\frac{\ddot{u}(s)}{\sin s} \right) + 2\lambda \frac{d}{ds}\left(\frac{\dot{u}(s)}{\sin s} \right) + \frac{2u(s)}{\sin^3 s} = \mu, \quad u(0) = u(\pi) = \dot{u}(0) = \dot{u}(\pi) = 0, \tag{6.24}$$

More precisely, the equation given in (6.24) should be understood in the sense of distributions, since we do not have proved a stronger regularity for u. However, we now assume that u is smooth enough to consider equation (6.24) pointwise. Since $u \in H^2_{\sin}(0,\pi) \subset H^2_0(0,\pi)$, we impose the Dirichlet boundary conditions in (6.24) so as to make the problem well-posed.

We are interested in the case $\mu = 0$ and $\lambda > 0$ for which the solution to (6.24) is not identically zero. Indeed, the lowest positive value of such λ is the infimum in (6.23) and its associated solution u is the minimizer of (6.23). We did not manage to completely solve the problem but we were able to look some particular forms of solutions. We only describe here the results obtained and not the tedious calculus we made to get them.

First kind of solutions

First, we consider some maps u_n of the following form:

$$u_n(s) = \sin^3(s) \sum_{k=1}^n a_k \cos(2k-1)s. \tag{6.25}$$

Note that $u_n(0) = u_n(\pi) = \dot{u}_n(0) = \dot{u}_n(\pi)$. Moreover, we can see that any u_n of the form (6.25) satisfies the symmetry property:

$$\forall s \in \left[0, \frac{\pi}{2}\right], \quad u_n(\pi - x) = -u_n(x),$$

from which we deduce that such u_n automatically satisfy the constraint $\int_0^\pi u_n(s)ds = 0$. Then, we compute with u_n the left member of (6.24), we obtain after calculations:

$$\begin{aligned}
\frac{d^2}{ds^2}\left(\frac{\ddot{u}_n(s)}{\sin s}\right) + \lambda \frac{d}{ds}\left(\frac{2\dot{u}_n(s)}{\sin s}\right) + \frac{2u_n(s)}{\sin^3 s} &= \sum_{k=2}^{n-1} \cos(2k-1)s\,[k(2k-1)(\lambda - k(2k-1))\,a_{k-1} \\
&\qquad + \left[2 + (2k-1)^2\left(2k^2 - 2k - 1 - \lambda\right)\right] a_k \\
&\qquad + (k-1)(2k-1)\left[\lambda - (k-1)(2k-1)\right] a_{k+1}] \\
&\quad + \cos(2n-1)s\,[n(2n-1)\left[\lambda - n(2n-1)\right] a_{n-1} \\
&\qquad + \left(2 + (2n-1)^2\left(2n^2 - 2n - 1 - \lambda\right)\right) a_n] \\
&\quad + \left[\lambda - \frac{(2n+1)(2n+2)}{2}\right](n+1)(2n+1)\,a_n \cos\left[(2n+1)s\right].
\end{aligned}$$

In particular, for $n = 1$, we get that any u proportional to $s \mapsto \sin^3 s \cos s$ is a solution of (6.24) with $\lambda = 6$ and $\mu = 0$. For any $n \geqslant 2$, if we set $\lambda = \frac{1}{2}(2n+1)(2n+2)$ and $\mu = 0$, then we can find a solution u to (6.24) of the form (6.25) by deleting the terms appearing in front of each cosine in the above relation. The set of $(n-1)$ equations can be solved to get the coefficient $(a_k)_{2 \leqslant k \leqslant n}$ in terms of the coefficient a_1 which is the degree of freedom for the eigenvector space associated with the eigenvalue λ. We thus have proved the following.

Proposition 6.7. *The sequence of numbers $(\lambda_{2n})_{n \geqslant 1} = \frac{1}{2}(2n+1)(2n+2)$ are some eigenvalues of the following problem:*

$$\inf_{\substack{u \in H^2_{\sin}(0,\pi) \\ \int_0^\pi u(s)ds = 0}} \frac{\dfrac{1}{2}\displaystyle\int_0^\pi \frac{\ddot{u}(s)^2}{\sin s} ds + \int_0^\pi \frac{u(s)^2}{\sin^3 s} ds}{\displaystyle\int_0^\pi \frac{\dot{u}(s)^2}{\sin s} ds}. \tag{6.26}$$

Second kind of solutions

We now consider some maps v_n of the following form:

$$v_n(s) = \sin^3(s)\left(a_0 + \sum_{k=1}^{n} a_k \cos 2ks\right). \tag{6.27}$$

Again, we have $v_n(0) = v_n(\pi) = \dot{v}_n(0) = \dot{v}_n(\pi)$. This time, any v_n of the form (6.27) satisfies the symmetry property:

$$\forall s \in \left[0, \frac{\pi}{2}\right], \quad v_n(\pi - x) = v_n(x).$$

In this case, except for very specific coefficient a_k, there is little chance for such v_n to satisfy the constraint $\int_0^\pi v_n(s)ds = 0$. We proceed exactly in the same way we did for u_n.

We find that for any $n \in \mathbb{N}$, if we set $\lambda := \lambda_{2n+1} = \frac{1}{2}(2n+2)(2n+3)$ and $\mu = 2a_0$, then there exists some coefficients $(a_k)_{1 \leqslant k \leqslant n}$ such that v_n given by (6.27) is a solution of (6.24). However, if we set $a_0 = 0$, then we get $a_k = 0$ for any $k \geqslant 1$. Furthermore, for the small value of λ_{2n+1}, the corresponding solution does not satisfies the constraint $\int_0^\pi v_n(s)ds = 0$.

Moreover, we made some numerical analysis on (6.24). Assuming $\mu \neq 0$, we divide the equation by μ and we solve the equation of unknown $v = \frac{1}{\mu}u$:

$$\forall s \in]0,\pi[, \quad \frac{d^2}{ds^2}\left(\frac{\ddot{v}(s)}{\sin s}\right) + 2\lambda \frac{d}{ds}\left(\frac{\dot{v}(s)}{\sin s}\right) + \frac{2v(s)}{\sin^3 s} = 1, \quad u(0) = u(\pi) = \dot{u}(0) = \dot{u}(\pi) = 0, \tag{6.28}$$

with finite difference method. Computing the quantity $\int_0^\pi v(s)ds = \frac{1}{\mu}\int_0^\pi u(s)ds$ in terms of λ, we obtain the graph presented in Figure 6.1 below.

We assume that the solution u of (6.24) depends continuously on λ and μ. We observe that the graph has some picks. These are the values of λ for which $\int_0^\pi u(s)ds \neq 0$ and $\mu = 0$ i.e. $\int_0^\pi v(s) = +\infty$. Between each of these picks, the graph intersects one time the (x)-line. These are the values of λ for which $\mu \neq 0$ and $\int_0^\pi u(s)ds = 0$ with u non-identically zero. However, in this graph, it is not possible to show the value of λ for which we both have $\mu = 0$ and $\int_0^\pi u(s)ds = 0$ because the form is indeterminate.

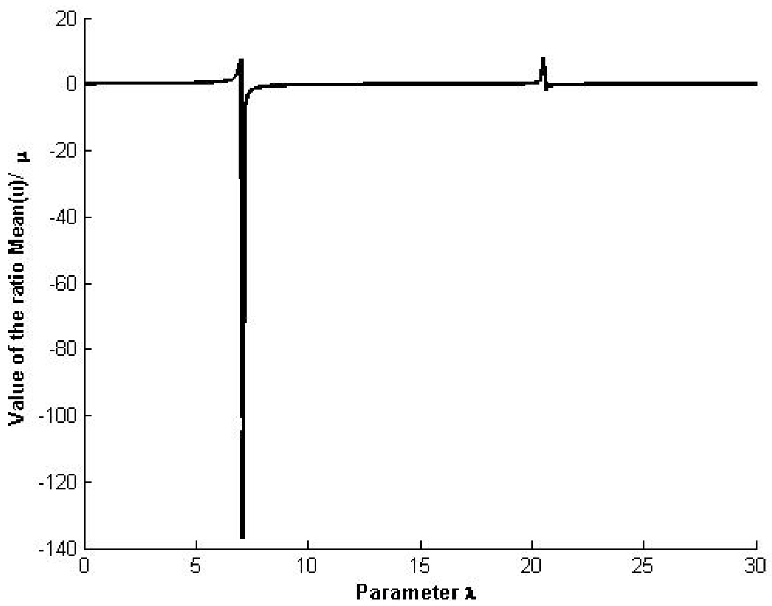

Figure 6.1: The quantity $\frac{1}{\mu}\int_0^\pi u(s)ds$ is plotted for different value of λ, where $u : [0,\pi] \to \mathbb{R}$ is the numerical solution of (6.28) for given λ, obtained by finite difference method.

Third kind of solution

Since λ_2 is a good candidate to be the infimum of (6.26), we wanted to prove they is no lower positive eigenvalue. Numerically, it seems that there could be only for $\lambda_1 = 3$ a non-zero solution u_1 to (6.24) with $\mu = 0$ but we find that $\int_0^\pi u_1(s)ds \neq 0$. Moreover, we were not able to solve theoretically (6.24) with $\lambda = 3$ and $\mu = 0$. However, we tried to compute (6.27) in (6.24) with $a_0 = 0$ but with an infinite sum:

$$v(s) = \sin^3(s)\left(\sum_{k=1}^{+\infty} a_k \cos 2ks\right). \tag{6.29}$$

Assuming $\lambda \neq \lambda_{2n}$ where λ_{2n} is given in Proposition 6.7, we obtain $a_2 = 4a_1$ and the following recurrence relation:

$$a_{k+1} = \frac{8k^2 a_k\left[2\lambda - \left(4k^4 - 3k^2 + \frac{1}{k^2}\right)\right] - 2k(2k+1)a_{k-1})\left[2\lambda - 2k(2k+1)\right]}{2k(2k-1)\left(2\lambda - 2k(2k-1)\right)},$$

for which we find an explicit solution: $a_k = k^2 a_1$ for any $k \geqslant 1$. However, although it satisfies formally (6.24), such a v does not represent a function. Indeed, the sequence $(a_k)_{k\geqslant 1}$ does not tends to zero so there is no chance for v to converge.

This ends to the first part of our work on the Helfrich energy. We now focus on the minimization of total mean curvature with prescribed area.

Part III

On the minimization of total mean curvature

Chapter 7

Introduction

This part is the reproduction of a submitted article entitled *on the minimization of total mean curvature* [23], done in collaboration with Simon Masnou, Antoine Henrot and Takéo Takahashi. We have added a more detailed exposition on the properties of non-decreasing rearrangements, and the proof of Minkowski's inequality (7.1) below with a complete treatment of the equality case.

In 1901, Minkowski proved that the following inequality holds for any non-empty bounded open convex subset $\Omega \subset \mathbb{R}^3$ whose boundary $\partial\Omega$ is a C^2-surface:

$$\frac{1}{2}\int_{\partial\Omega} H dA \geqslant \sqrt{4\pi A(\partial\Omega)}, \tag{7.1}$$

where the integration of the scalar mean curvature $H = \kappa_1 + \kappa_2$ is done with respect to the usual two-dimensional Hausdorff measure referred to as $A(\bullet)$.

Announced in [69], Inequality (7.1) is proved in [70, §7] assuming C^2-regularity. The proof can also be found in [73, Chapter 6, Exercise (10)] in the case of ovaloids, i.e. compact simply-connected smooth surfaces of $\mathbb{R}^3$ whose Gaussian curvature is positive everywhere.

The original proof of Minkowski is based on the isoperimetric inequality together with Steiner-Minkowski formulae. Therefore, Inequality (7.1) remains true if $\partial\Omega$ is only a surface of class $C^{1,1}$ (or equivalently, if $\partial\Omega$ has a positive reach, cf. Theorems 16.5–16.6). If we do not assume any regularity, the same inequality holds with the total mean curvature replaced by the mean width of the convex body.

Equality holds in (7.1) if and only if Ω is an open ball. This was stated by Minkowski in [70, §7] without proof. A proof due to Favard can be found in [31, Section 19] based on a Bonnesen-type inequality involving mixed volumes. In Chapter 13, we give a proof of inequality (7.1) with a complete treatment of the equality case, and also consider specifically the axisymmetric situation, inspired by Bonnesen [10, Section VI, §35 (74)].

Moreover, Inequality (7.1) is actually a consequence of a generalization due to Minkowski of the isoperimetric inequality. This generalization uses the notion of mixed volumes of convex bodies. We refer to [83, Theorem 6.2.1, Notes for Section 6.2] and [11, Sections 49,52,56] for a more detailed exposition on that question.

In this part, we are mainly interested in the validity of (7.1) under other various assumptions, and on the related problem of minimizing the total mean curvature with area constraint:

$$\inf_{\substack{\Sigma\in\mathfrak{C}\\ A(\Sigma)=A_0}} \frac{1}{2}\int_{\Sigma} H dA, \tag{7.2}$$

for a suitable class $\mathfrak{C}$ of surfaces in $\mathbb{R}^3$. We recall that the original motivation for Problem (7.2) is the study of Problem (5.1) in the particular case $H_0 < 0$. Indeed, one can wonder whether (5.1) can be solved by minimizing individually each term in the Helfrich energy (3.3):

$$\mathcal{H}(\Sigma) = \frac{1}{4}\int_{\Sigma} (H - H_0)^2 \, dA = \frac{1}{4}\int_{\Sigma} H^2 dA - \frac{H_0}{2}\int_{\Sigma} H dA + \frac{H_0^2 A(\Sigma)}{4}.$$

Since the Willmore energy (3.4) is invariant with respect to rescaling, and spheres are the only global minimizers of (3.4), this reduction makes sense only if the sphere $\mathbb{S}_{A_0}$ is also the only solutions to Problem (7.2). We prove in this part that this is true if the problem is tackled in a particular class of surfaces.

Let us first introduce two classes of embedded 2-surfaces in $\mathbb{R}^3$: the class $\mathcal{A}_{1,1}$ of all compact surfaces which are boundaries of axisymmetric domains (i.e. sets with rotational invariance around an axis), and the subclass $\mathcal{A}_{1,1}^+$ of *axiconvex* surfaces, i.e. surfaces bounding an axisymmetric domain whose intersection with any plane orthogonal to the symmetry axis is either a disk or empty. We first prove the following:

Theorem 7.1. *Consider the class $\mathcal{A}_{1,1}^+$ of axiconvex $C^{1,1}$-surfaces in $\mathbb{R}^3$. Then, we have:*

$$\forall \Sigma \in \mathcal{A}_{1,1}^+, \quad \frac{1}{2}\int_\Sigma H dA \geqslant \sqrt{4\pi A(\Sigma)},$$

where the equality holds if and only if Σ is a sphere. In particular, for any $A_0 > 0$, we have:

$$\frac{1}{2}\int_{\mathbb{S}_{A_0}} H dA = \min_{\substack{\Sigma \in \mathcal{A}_{1,1}^+ \\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma H dA = \sqrt{4\pi A_0},$$

and the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of this problem.

Then, we show this result cannot be extended to the general class of compact simply-connected $C^{1,1}$-surfaces in $\mathbb{R}^3$, and we even provide a negative clue for the extension to $\mathcal{A}_{1,1}$. More precisely:

Theorem 7.2. *Let $A_0 > 0$. There exists a sequence of $C^{1,1}$-surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ and a sequence of axisymmetric $C^{1,1}$-surfaces $(\widetilde{\Sigma}_n)_{n\in\mathbb{N}} \subset \mathcal{A}_{1,1}$ such that $A(\Sigma_i) = A(\widetilde{\Sigma}_i) = A_0$ for any $i \in \mathbb{N}$ with:*

$$\lim_{i\to+\infty} \frac{1}{2}\int_{\Sigma_i} H dA = -\infty \qquad \text{and} \qquad \lim_{i\to+\infty} \frac{1}{2}\int_{\widetilde{\Sigma}_i} H dA = 0^+.$$

It follows obviously that:

$$\inf_{\substack{\Sigma \in C^{1,1} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma H dA = -\infty \qquad \text{and} \qquad \inf_{\substack{\Sigma \in \mathcal{A}_{1,1} \\ A(\Sigma)=A_0}} \left|\frac{1}{2}\int_\Sigma H dA\right| = 0.$$

Therefore, Problem (7.2) has no solution in the class of (compact simply-connected) $C^{1,1}$-surfaces, and there is good reason to think that it might be the same within the class $\mathcal{A}_{1,1}$, but we were not able to prove it.

However, although Problem (7.2) has no global minimizer, it is easily seen that the sphere $\mathbb{S}_{A_0}$ of area A_0 is a local minimizer of (7.2) in the class of C^2-surfaces (Remark 11.1) and it can also be proved that $\mathbb{S}_{A_0}$ is the unique critical point of (7.2) in the class of C^3-surfaces (Theorem 11.3) by computing the first variation of total mean curvature and of area (Proposition 11.2).

Hence, this leads us naturally to consider another problem:

$$\inf_{\substack{\Sigma \in \mathcal{A}_{1,1} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma |H| dA, \tag{7.3}$$

for which we can prove the following:

Theorem 7.3. *Let $\mathcal{A}_{1,1}$ denotes the class of axisymmetric $C^{1,1}$-surfaces in $\mathbb{R}^3$. Then, we have:*

$$\forall \Sigma \in \mathcal{A}_{1,1}, \quad \frac{1}{2}\int_\Sigma |H| dA \geqslant \sqrt{4\pi A(\Sigma)},$$

where the equality holds if and only if Σ is a sphere. In particular, for any $A_0 > 0$, we have:

$$\frac{1}{2}\int_{\mathbb{S}_{A_0}} |H| dA = \min_{\substack{\Sigma \in \mathcal{A}_{1,1} \\ A(\Sigma)=A_0}} \frac{1}{2}\int_\Sigma |H| dA = \sqrt{4\pi A_0},$$

and the sphere $\mathbb{S}_{A_0}$ of area A_0 is the unique global minimizer of this problem.

Let us note that in 1973, Michael and Simon established in [68] a Sobolev-type inequality for m-dimensional C^2-submanifolds of $\mathbb{R}^n$, for which the case $m = 2$ and $n = 3$ with $f \equiv 1$ gives the following inequality:

$$\frac{1}{2}\int_\Sigma |H| dA \geqslant c_0 \sqrt{A(\Sigma)}.$$

More precisely, the constant appearing in the above inequality is $c_0 = \frac{1}{4^3}\sqrt{4\pi}$ [68, Theorem 2.1]. The better constant $c_0 = \frac{1}{2}\sqrt{2\pi}$ was obtained by Topping in [91, Lemma 2.1] and does not seem optimal. From Theorem 7.3, we think that an optimal constant should be $c_0 = \sqrt{4\pi}$.

We refer to the appendix of [91] for a concise proof of the above inequality using Simon's ideas. We also mention [19, Theorems 3.1, 3.2] for a weighted version of this inequality but less sharp as mentioned in the last paragraph of [19, Section 3.2].

Finally, we summarize in Table 7.1 several results and open questions related to Problems (7.2) and (7.3) (the term *inner-convex* refers to a closed surface which encloses a convex set). The paper is organized as follows. In Chapter 8, the notation and the basic definitions of surface, axisymmetry, and axiconvexity are recalled. In Chapter 9 and 10, we respectively give the proofs of Theorem 7.1 and 7.2. In Chapter 11, we study the optimality of the sphere for Problem (7.2) and Theorem 7.3 is proved in Chapter 12. Finally, Minkowski's inequality (7.1) is established in Chapter 13 and show some properties of non-decreasing rearrangements in Chapter 14.

Class of surfaces Σ	**Assertion**	**Proof**
$C^{1,1}$ compact inner-convex	$\frac{1}{2}\int_\Sigma H dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	[31, 70]
$C^{1,1}$ axisymmetric inner-convex	$\frac{1}{2}\int_\Sigma H dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	[10]
$C^{1,1}$ axiconvex	$\frac{1}{2}\int_\Sigma H dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	Thrm 7.1
$C^{1,1}$ axisymmetric	$\inf\limits_{A(\Sigma)=A_0} \left\lvert \frac{1}{2}\int_\Sigma H dA \right\rvert = 0$	Thrm 7.2
$C^{1,1}$ axisymmetric	$\frac{1}{2}\int_\Sigma H dA > 0$	**OPEN**
$C^{1,1}$ compact simply-connected	$\inf\limits_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma H dA = -\infty$	Thrm 7.2
C^2 compact simply-connected	$\mathbb{S}_{A_0}$ is a local minimizer of $\inf\limits_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma H dA$	Rmrk 11.1
C^3 compact simply-connected	$\mathbb{S}_{A_0}$ unique critical point of $\inf\limits_{A(\Sigma)=A_0} \frac{1}{2}\int_\Sigma H dA$	Thrm 11.3
$C^{1,1}$ axisymmetric	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	Thrm 7.3
C^2 compact simply-connected	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{\frac{\pi}{2} A(\Sigma)}$	[68, 91]
$C^{1,1}$ compact simply-connected	$\frac{1}{2}\int_\Sigma \lvert H\rvert dA \geqslant \sqrt{4\pi A(\Sigma)}$ (equality iff Σ sphere)	**OPEN**

Table 7.1: minimizing $\int H$ or $\int |H|$ with area constraint.

Chapter 8

Definitions and notation

We refer to Montiel and Ros [73, Definition 2.2] for the definition of $C^{k,\alpha}$-surfaces without boundary embedded in $\mathbb{R}^3$. We only consider here surfaces homeomorphic to spheres, i.e. compact and simply-connected.

In this part, we present several results on the particular class of $C^{1,1}$ axisymmetric surfaces. We focus on embedded axisymmetric surfaces which are obtained by rotating a planar open simple curve around the segment joining its ends, assuming that the segment meets the curve at no other point.

We choose the (xz)-plane as the curve plane and the z-line as the rotation axis. We denote by $L > 0$ the total length of the curve. We assume that the following parametrization holds for the curve (using the arc length s):

$$\begin{array}{rccl} \gamma: & [0, L] & \longrightarrow & \mathbb{R}^2 \\ & s & \longmapsto & \gamma(s) = \begin{pmatrix} x(s) \\ z(s) \end{pmatrix}, \end{array}$$

and we assume without loss of generality that $\gamma(0) = (0, 0)$. The axisymmetric surface Σ spanned by the rotation of γ is the surface Σ parametrized by:

$$\begin{array}{rccl} X: & [0, L] \times [0, 2\pi[& \longrightarrow & \mathbb{R}^3 \\ & (s,t) & \longmapsto & X(s,t) = \begin{pmatrix} x(s)\cos t \\ x(s)\sin t \\ z(s) \end{pmatrix}, \end{array} \tag{8.1}$$

where t refers to the rotation angle about the z-axis. It is well-known that all geometric quantities can be expressed with respect to the angle θ between the x-axis and the tangent line to the curve. This defines a Lipschitz continuous map $\theta : [0, L] \to \mathbb{R}$ such that:

$$\forall s \in [0, L], \quad \begin{pmatrix} \dot{x}(s) \\ \dot{z}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix},$$

therefore, recalling that $x(0) = z(0) = 0$,

$$\forall s \in [0, L], \quad x(s) = \int_0^s \cos\theta(t)dt \quad \text{and} \quad z(s) = \int_0^s \sin\theta(t)dt. \tag{8.2}$$

We also have $dA = 2\pi x(s)ds$, where dA is the infinitesimal area surface element. Moreover, applying Rademacher's Theorem, the principal curvatures κ_1 and κ_2, implicitly defined by the scalar mean curvature $H = \kappa_1 + \kappa_2$ and the Gaussian curvature $K = \kappa_1\kappa_2$, exist almost everywhere and are explicitly given by:

$$\text{for a.e. } s \in [0, L], \quad \kappa_1(s) = \frac{\sin\theta(s)}{x(s)} \quad \text{and} \quad \kappa_2(s) = \dot{\theta}(s)$$

Therefore total mean curvature $\frac{1}{2}\int_\Sigma H dA$ and area $A(\Sigma)$ are given by:

$$\int_\Sigma H dA = 2\pi \int_0^L \sin\theta(s) + \dot{\theta}(s)x(s)\, ds, \qquad A(\Sigma) = 2\pi \int_0^L x(s)\, ds. \tag{8.3}$$

All these expressions can be found for example in [18, Section 3.3, Example 4]. Note that the signs of κ_1 and κ_2 depend on the chosen orientation. Throughout the article, the Gauss map always represents the outer unit normal field to the surface. Hence, on the sphere of radius $R > 0$, one can check that $\theta(s) = \frac{s}{R}$ and $\kappa_1(s) = \kappa_2(s) = \frac{1}{R}$.

Definition 8.1. *We say that Σ is an axisymmetric $C^{1,1}$-surface and we write $\Sigma \in \mathcal{A}_{1,1}$ if it is generated as above by a Lipschitz continuous map $\theta : [0, L] \to \mathbb{R}$, which is admissible in the sense that the following three properties are fulfilled:*

(i) the map θ satisfies the boundary conditions $\theta(0) = 0$ and $\theta(L) = \pi$;

(ii) the map γ obtained from θ satisfies $x(0) = x(L) = 0$ and $z(L) > z(0) = 0$;

(iii) the map γ is one-to-one on $]0, L[$ and satisfies $x(s) > 0$ for any $s \in]0, L[$.

In particular, Σ has no boundary and no self-intersection.

Definition 8.2. *We say that Σ is an axiconvex $C^{1,1}$-surface and we write $\Sigma \in \mathcal{A}_{1,1}^{+}$ if $\Sigma \in \mathcal{A}_{1,1}$ and if the generating map θ is valued in $[0, \pi]$. In that case the intersection of the surface with any plane orthogonal to the axis of symmetry is either a circle or a point or the empty set.*

It is easy to check the strict inclusions: (convex and axisymmetric) $\subset$ axiconvex $\subset$ axisymmetric and to prove that an axisymmetric surface is axiconvex if and only if the ordinate function z is non-decreasing, also if and only if it is inner-convex in any direction orthogonal to the axis of revolution.

Chapter 9

The sphere is the unique minimizer of total mean curvature among axiconvex surfaces of given area

This chapter is devoted to the proof of Theorem 7.1. First, we note that any axiconvex $C^{1,1}$-surface Σ is generated by an admissible Lipschitz continuous map $\theta : [0,L] \to [0,\pi]$ as in Chapter 8 (and $L > 0$ refers to the total length of the generating curve) with the following conditions:

$$\theta(0) = 0, \quad \theta(L) = \pi, \tag{9.1}$$

$$\int_0^L \sin\theta(t)dt > 0, \quad \int_0^L \cos\theta(t)dt = 0, \tag{9.2}$$

$$\forall s \in]0,L[, \quad \int_0^s \cos\theta(t)dt > 0. \tag{9.3}$$

The first condition of (9.2) is verified if (9.1) holds and if $\theta([0,L]) \subset [0,\pi]$. The above conditions are also sufficient to obtain a $C^{1,1}$-axiconvex surface from $\theta : [0,L] \to [0,\pi]$. Indeed, the fact that the curve obtained from θ is simple can be deduced from this result.

Proposition 9.1. *Consider $L > 0$ and a continuous function $u : [0,L] \to [0,+\infty[$ generating a curve via the C^1-map $\gamma : s \in [0,L] \mapsto (\int_0^s \cos u(\tau)d\tau, \int_0^s \sin u(\tau)d\tau)$. If u is valued in $[0,\pi]$, then γ is a diffeomorphism. In particular, for every distinct $s,t \in]0,L[$:*

$$\left(\int_s^t \cos u(\tau)d\tau\right)^2 + \left(\int_s^t \sin u(\tau)d\tau\right)^2 > 0.$$

Proof. The map γ can be identified with the differentiable map $s \in [0,L] \mapsto \int_0^s e^{iu(\tau)}d\tau$. Obviously, $|\gamma'(s)| = 1$ for every $s \in [0,L]$. If u is valued in $[0,\pi]$, by the mean value theorem for vector-valued functions (see for instance [67]), γ is one-to-one, and therefore a diffeomorphism by the global inversion theorem. □

We also notice that the inner domain of Σ associated with $\theta : [0,L] \to [0,\pi]$ satisfying (9.1), (9.2), and (9.3) is a non-empty bounded open subset of $\mathbb{R}^3$ which is convex if and only if θ is non-decreasing. Indeed, in that case, the two principal curvatures are non-negative almost everywhere:

$$\kappa_1(s) = \frac{\sin\theta(s)}{x(s)} \geqslant 0 \qquad \text{and} \qquad \kappa_2(s) = \dot\theta(s) \geqslant 0 \qquad \text{a.e.}$$

We prove Theorem 7.1 by using a non-decreasing rearrangement of θ:

$$\forall s \in [0,L], \quad \theta^*(s) = \sup\left\{c \in [0,\pi], \quad s \in [L - |\{t \in [0,L], \quad \theta(t) \geqslant c\}|, L]\right\}, \tag{9.4}$$

where $|\bullet|$ refers here to the one-dimensional Lebesgue measure.

We split the proof into the following three steps:

1. We check that θ^* generates an axisymmetric inner-convex $C^{1,1}$-surface Σ^*.
2. We show that:
$$\frac{1}{2}\int_{\Sigma} HdA = \frac{1}{2}\int_{\Sigma^*} HdA \geqslant \sqrt{4\pi A(\Sigma^*)} \geqslant \sqrt{4\pi A(\Sigma)}.$$
3. We study the equality case.

It is convenient to first recall some well-known results about rearrangements.

Proposition 9.2. *Consider any Lipschitz continuous map $u : [0, L] \to [0, \infty[$ and its non-decreasing rearrangement u^* defined by:*

$$\forall s \in [0, L], \quad u^*(s) = \sup \{c \in [0, \infty[, \quad s \in [L - |\{t \in [0, L], \quad u(t) \geqslant c\}|, L]\}.$$

Then, the following properties hold true.

1. *The map u^* is non-decreasing.*
2. *The map u^* is Lipschitz continuous with the same Lipschitz modulus as u.*
3. *For any continuous map $F : [0, +\infty[\to \mathbb{R}$, we have the following equality:*
$$\int_0^L F(u(s))ds = \int_0^L F(u^*(s))ds.$$
4. *For any continuous increasing map $F : [0, +\infty[\to [0, +\infty[$, we have $(F(u))^* = F(u^*)$.*
5. *(Hardy–Littlewood inequality) If $v : [0, L] \to [0, +\infty[$ is another Lipschitz continuous map and v^* denotes its non-decreasing rearrangement, then:*
$$\int_0^L u(s)v(s)ds \leqslant \int_0^L u^*(s)v^*(s)ds.$$

Proof. The above results are quite classical. Chapter 14 is devoted to the proof of this proposition and we refer to [52, 54] for further references on the subject. □

Proof of Theorem 7.1. **Step 1:** the map θ^* defined by (9.4) generates an axisymmetric inner-convex $C^{1,1}$-surface Σ^*.

We only need to check (9.1), (9.2), and (9.3) for θ^*. Assertion (9.1) follows from the definition of θ^* given in (9.4). We define the functions:

$$\forall s \in [0, L], \quad x_*(s) = \int_0^s \cos\theta^*(t)dt \qquad \text{and} \qquad z_*(s) = \int_0^s \sin\theta^*(t)dt.$$

Note that x_*, z_* *are not* the rearrangements of x, z. From Property 3 in Proposition 9.2, we get $x_*(L) = x(L) = 0$ and $z_*(L) = z(L) > 0$ then the relations in (9.2) hold true for θ^*. Relation (9.3) is equivalent to $x_*(s) > 0$ for any $s \in]0, L[$. Since $\dot{x}_* = \cos\theta^*$, Property 1 in Proposition 9.2 combined with the fact that $\theta^*([0, L]) \subseteq [0, \pi]$ ensures x_* is a concave map, not identically zero. Hence, $x_* > 0$ in $]0, L[$.

Step 2: we compare the total mean curvature and the area of Σ with the ones of Σ^*.

First, observe that we can obtain from an integration by parts:

$$\int_\Sigma HdA = \int_0^L \left(\frac{\sin\theta(s)}{x(s)} + \dot{\theta}(s)\right) 2\pi x(s)ds = 2\pi \int_0^L F(\theta(s))ds,$$

where F is the continuous map $x \mapsto \sin x - x\cos x$. Using Property 3 in Proposition 9.2, we deduce that:

$$\int_\Sigma HdA = \int_{\Sigma^*} HdA. \tag{9.5}$$

Now, since Σ^* is an axisymmetric inner-convex $C^{1,1}$-surface, we can apply the Minkowski Theorem, see (7.1) or Corollary 13.6:

$$\frac{1}{2}\int_{\Sigma^*} HdA \geqslant \sqrt{4\pi A(\Sigma^*)}. \tag{9.6}$$

Then, we need to compare the areas of Σ and Σ^*. For that purpose, we are going to use the Hardy-Littlewood inequality combined with the following observation coming from an integration by parts:

$$A(\Sigma) = \int_\Sigma dA = \int_0^L 2\pi x(s)ds = -2\pi \int_0^L s\cos\theta(s)ds.$$

Set $u(s) = s$ and $v(s) = 1 - \cos\theta(s)$ for every $s \in [0, L]$. These two functions being non-negative and Lipschitz continuous, we get from Property 5 of Proposition 9.2:

$$\int_0^L u(s)v(s)ds \leqslant \int_0^L u^*(s)v^*(s)ds,$$

where the maps u^* and v^* are respectively the non-decreasing rearrangements of u and v. Since the continuous map $x \mapsto 1 - \cos x$ is non-negative and increasing on $[0, \pi]$, we use Property 4 in Proposition 9.2 in order to get $v^* = (1 - \cos(\theta))^* = 1 - \cos(\theta^*)$ but we have also $u^*(s) = u(s) = s$. Finally, we obtain that:

$$\frac{L^2}{2} + \frac{A(\Sigma)}{2\pi} = \int_0^L s(1 - \cos\theta(s))ds \leqslant \int_0^L s(1 - \cos\theta^*(s))ds = \frac{L^2}{2} + \frac{A(\Sigma^*)}{2\pi}. \tag{9.7}$$

Combining (9.5), (9.6), and (9.7), the inequality of Theorem 7.1 is therefore established:

$$\frac{1}{2}\int_\Sigma HdA = \frac{1}{2}\int_{\Sigma^*} HdA \geqslant \sqrt{4\pi A(\Sigma^*)} \geqslant \sqrt{4\pi A(\Sigma)}.$$

Step 3: the equality case.

Assume that there exists $\Sigma \in \mathcal{A}^+_{1,1}$ such that the equality holds in the previous inequalities. Then, we have:

$$\frac{1}{2}\int_\Sigma HdA = \frac{1}{2}\int_{\Sigma^*} HdA = \sqrt{4\pi A(\Sigma^*)} = \sqrt{4\pi A(\Sigma)}. \tag{9.8}$$

Therefore, since Σ^* is an inner-convex $C^{1,1}$-surface, using the Minkowski Theorem, we deduce that Σ^* must be a sphere (equality in (7.1), see Corollary 13.6). Now, we show that $\Sigma \equiv \Sigma^*$ i.e. $\theta = \theta^*$. From (9.7) and (9.8), we have the equality:

$$\int_0^L sv(s)ds = \int_0^L sv^*(s)ds,$$

where the map $v : s \mapsto v(s) = 1 - \cos\theta(s)$ has already been introduced. The above equality and an integration by parts yield to the following relation:

$$\int_0^L \left(\int_s^L v(c)dc\right) ds = \int_0^L \left(\int_s^L v^*(c)dc\right) ds. \tag{9.9}$$

Since $\mathbf{1}^*_{[s,L]} = \mathbf{1}_{[s,L]}$, the Hardy-Littlewood inequality implies that:

$$\forall s \in [0, L], \quad \int_s^L v(c)dc = \int_0^L \mathbf{1}_{[s,L]}(c)v(c)dc \leqslant \int_0^L \mathbf{1}^*_{[s,L]}(c)v^*(c)dc = \int_s^L v^*(c)dc.$$

Combining the above inequality and (9.9), we deduce that:

$$\forall s \in [0, L], \quad \int_s^L v(c)dc = \int_s^L v^*(c)dc,$$

thus $(1 - \cos[\theta^*]) = 1 - \cos[\theta]$ and $\theta = \theta^*$ on $[0, L]$. Hence, $\Sigma \equiv \Sigma^*$ and Σ must be a sphere. Conversely, any sphere Σ satisfies the equality $\frac{1}{2}\int_\Sigma HdA = \sqrt{4\pi A(\Sigma)}$, which concludes the proof of Theorem 7.1. □

Chapter 10

Two interesting sequences of axisymmetric surfaces

In this chapter, we give a proof of Theorem 7.2. We build two sequences of surfaces of constant area. The first one is not axisymmetric and its total mean curvature tends to $-\infty$ while the other one is axisymmetric and its total mean curvature tends to zero. Figures 10.1 et 10.2 describe their respective constructions.

10.1 Total mean curvature is not bounded from below

We first compute the total mean curvature of a sphere of radius $R > 0$ where a neighbourhood of the north pole has been removed, and replaced by an internal sphere of small radius $\varepsilon > 0$. The two parts are glued so that the resulting surface referred to as Σ_ε is an axisymmetric $C^{1,1}$-surface illustrated in Figure 10.1.

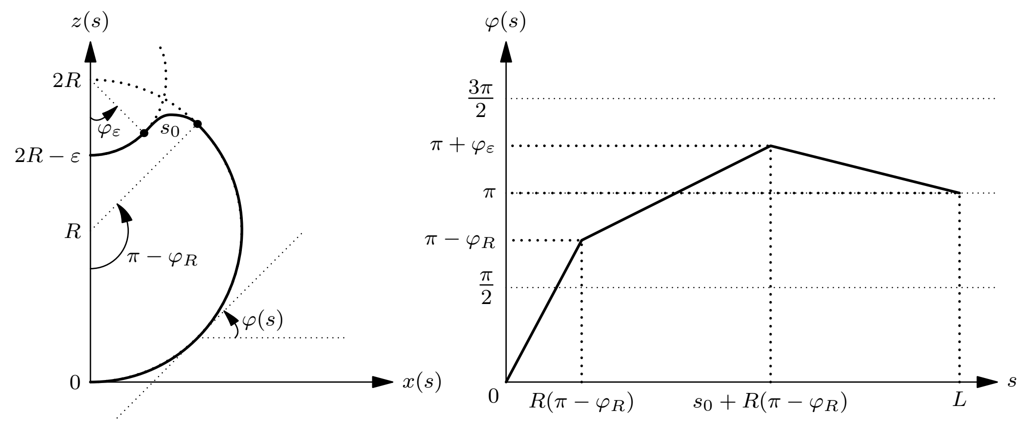

Figure 10.1: the construction of the sequence of axisymmetric surfaces $(\Sigma_\varepsilon)_{\varepsilon>0}$.

More precisely, let us fix $\varphi_\varepsilon = \frac{\pi}{2} - \varepsilon$ and let us consider the function $\varphi : [0, L] \to \mathbb{R}$ defined by:

$$\varphi(s) = \begin{cases} \dfrac{s}{R} & \text{if } s \in [0, R(\pi-\varphi_R)] \\[2ex] \dfrac{\varphi_R+\varphi_\varepsilon}{s_0}\,(s - R(\pi-\varphi_R)) + \pi - \varphi_R & \text{if } s \in [R(\pi-\varphi_R), s_0 + R(\pi-\varphi_R)] \\[2ex] -\dfrac{1}{\varepsilon}(s - s_0 - R(\pi-\varphi_R)) + \pi + \varphi_\varepsilon & \text{if } s \in [s_0 + R(\pi-\varphi_R), L], \end{cases}$$

with

$$\varphi_R,\ \varphi_\varepsilon = \frac{\pi}{2} - \varepsilon \in \left]0, \frac{\pi}{2}\right[, \quad s_0 > 0 \quad \text{and} \quad L = \varepsilon\varphi_\varepsilon + s_0 + R(\pi - \varphi_R).$$

In the above expression, there are three parameters φ_ε, φ_R and s_0, but actually we will have to impose two extra conditions (10.1) and (10.2) to express that $x(L) = 0$ and $z(L) = 2R - \varepsilon$. The map φ is continuous and piecewise linear, and satisfies (9.1), (9.2), (9.3). The surface Σ_ε is obtained through formulas (8.1), (8.2) when θ is replaced by φ. The first part of the definition of φ generates almost a sphere of radius $R > 0$ since φ_R will be chosen small. The third part generates almost an internal half-sphere of radius $\varepsilon > 0$. The second part corresponds to the gluing of the two spheres and has a length $s_0 > 0$. Let us note that $L > 0$ is the total length of the curve.

We compute $x(s) = \int_0^s \cos\varphi(t)dt$ and $z(s) = \int_0^s \sin\varphi(t)dt$ and taking into account that the expression for the last interval describes a part of the sphere of radius ε, we get:

$$x(s) = \begin{cases} R\sin\varphi(s) & \text{if } s \in [0, R(\pi - \varphi_R)] \\ \left(R - \dfrac{s_0}{\varphi_R + \varphi_\varepsilon}\right)\sin\varphi_R + \dfrac{s_0}{\varphi_R + \varphi_\varepsilon}\sin\varphi(s) & \text{if } s \in [R(\pi - \varphi_R), s_0 + R(\pi - \varphi_R)] \\ -\varepsilon\sin\varphi(s) & \text{if } s \in [s_0 + R(\pi - \varphi_R), L], \end{cases}$$

and also

$$z(s) = \begin{cases} R\,(1 - \cos\varphi(s)) & \text{if } s \in [0, R(\pi - \varphi_R)] \\ R + \left(R - \dfrac{s_0}{\varphi_R + \varphi_\varepsilon}\right)\cos\varphi_R - \dfrac{s_0}{\varphi_R + \varphi_\varepsilon}\cos\varphi(s) & \text{if } s \in [R(\pi - \varphi_R), s_0 + R(\pi - \varphi_R)] \\ 2R + \varepsilon\cos\varphi(s) & \text{if } s \in [s_0 + R(\pi - \varphi_R), L]. \end{cases}$$

We express now continuity of $x(s)$ and $z(s)$ at $s = s_0 + R(\pi - \varphi_R)$. The first relation gives s_0 explicitly in terms of φ_R and φ_ε. The second one gives an implicit relation between φ_R and φ_ε.

$$\left(R - \frac{s_0}{\varphi_R + \varphi_\varepsilon}\right)\sin\varphi_R - \frac{s_0}{\varphi_R + \varphi_\varepsilon}\sin\varphi_\varepsilon = \varepsilon\sin\varphi_\varepsilon \quad \text{i.e.} \quad s_0 = (\varphi_R + \varphi_\varepsilon)\frac{R\sin\varphi_R - \varepsilon\sin\varphi_\varepsilon}{\sin\varphi_R + \sin\varphi_\varepsilon}, \tag{10.1}$$

and

$$R + \left(R - \frac{s_0}{\varphi_R + \varphi_\varepsilon}\right)\cos\varphi_R + \frac{s_0}{\varphi_R + \varphi_\varepsilon}\cos\varphi_\varepsilon = 2R - \varepsilon\cos\varphi_\varepsilon. \tag{10.2}$$

The last relation can be rewritten, using the first relation, in the following form:

$$\frac{(R+\varepsilon)\cos\varphi_R - R}{\sin\varphi_R} + \frac{(R+\varepsilon)\cos\varphi_\varepsilon - R}{\sin\varphi_\varepsilon} = 0.$$

To see that this relation can be satisfied, we introduce the map $f : x \in]0, \frac{\pi}{2}[\mapsto \frac{(R+\varepsilon)\cos x - R}{\sin x}$, which is smooth, decreasing and surjective. Hence, it is an homeomorphism on its image and the previous relation become with this notation:

$$f(\varphi_R) + f(\varphi_\varepsilon) = 0 \Longleftrightarrow \varphi_R = f^{-1}(-f(\varphi_\varepsilon)).$$

We recall that $\varphi_\varepsilon = \frac{\pi}{2} - \varepsilon$ and we get by a straightforward computation:

$$f(\varphi_R) = R - R\epsilon + o(\epsilon).$$

Using the expression of f, we deduce that $\sin(\varphi_R) = \frac{\varepsilon}{R} + o(\varepsilon)$ and therefore, we obtain:

$$\varphi_R = \frac{\varepsilon}{R} + o(\varepsilon). \tag{10.3}$$

Now, we can compute the total mean curvature and the area of the surface. We obtain:

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{\Sigma_\varepsilon} HdA = \int_0^L (\sin\varphi(s) + \dot\varphi(s)x(s))\,ds = 4R - \left(2 - \frac{\pi}{2}\right)\varepsilon + o(\varepsilon) \\ \dfrac{A(\Sigma_\varepsilon)}{2\pi} = \displaystyle\int_0^L x(s)ds = 2R^2 + \frac{\varepsilon^2}{2} + o(\varepsilon^2). \end{cases} \tag{10.4}$$

We can notice in the above expressions a first term which is the contribution of the sphere of radius R and a second one due to the half-sphere of radius ε and the gluing. Note that the gluing has some first order impact on these relations, which is not obvious at first sight. We are now in position to prove the first part of Theorem 7.2.

Proof of Theorem 7.2. We decide to perform many perturbations of that kind all around the sphere. Notice that, for ε small enough, the perturbation we defined is contained in a ball of radius $\frac{3}{2}\varepsilon$ centred at the north pole. Thus it suffices to count how many such disjoint small balls we can put on the surface of the sphere of radius R. We will also use the fact that each perturbation makes a contribution for the total mean curvature and the area as $-\pi(2-\frac{\pi}{2})\varepsilon$ and $\pi\varepsilon^2$ (respectively) at first order, according to (10.4). We will denote by N_ε the number of perturbations. We first divide the surface of the sphere in slices S_k of latitude between $\frac{2\varepsilon}{R}(2k-1)$ and $\frac{2\varepsilon}{R}(2k+1)$, $k \in \{-K_\varepsilon \dots, K_\varepsilon\}$ with K_ε the integer part of $\frac{\pi R}{8\varepsilon} - \frac{1}{2}$. The (geodesic) width of each slice is 4ε. Now the slice S_k has a mean radius which is $R\cos(\frac{4k\varepsilon}{R})$, thus a perimeter which is $2\pi R\cos(\frac{4k\varepsilon}{R})$ and therefore, we can put on it $[2\pi R\cos(\frac{4k\varepsilon}{R})/4\varepsilon]$ patches of diameter close to 4ε, where $[.]$ refers to the integer part. On each patch, we can center a ball of radius $\frac{3\varepsilon}{2}$. Consequently, the total number of patches where we can put disjoint ball of diameter 3ε is given by:

$$N_\varepsilon = \sum_{k=-K_\varepsilon}^{K_\varepsilon - 1} \left[\frac{\pi R}{2\varepsilon}\cos\left(\frac{4k\varepsilon}{R}\right)\right]. \tag{10.5}$$

Using that K_ε satisfies

$$\frac{\pi R}{8\varepsilon} - \frac{3}{2} \leqslant K_\varepsilon \leqslant \frac{\pi R}{8\varepsilon} - \frac{1}{2},$$

we deduce from (10.5) that

$$N_\varepsilon = \frac{\pi R^2}{4\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right). \tag{10.6}$$

Then, the resulting $C^{1,1}$-surface obtained this way (written again Σ_ε) is compact simply-connected (and not axisymmetric). Moreover, we deduce from (10.6):

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_{\Sigma_\varepsilon} HdA = 4\pi R - \pi\left(2-\frac{\pi}{2}\right)N_\varepsilon\varepsilon + o(N_\varepsilon\varepsilon) = -\left(2-\frac{\pi}{2}\right)\dfrac{\pi^2R^2}{4\varepsilon} + o\left(\dfrac{1}{\varepsilon}\right), \\ A(\Sigma_\varepsilon) = 4\pi R^2 + \pi N_\varepsilon\varepsilon^2 + o(N_\varepsilon\varepsilon^2) = 4\pi R^2 + \dfrac{\pi^2R^2}{4} + o(1). \end{cases}$$

Finally, we make a rescaling of Σ_ε such that its area is exactly the required area A_0. First, we set $R > 0$ such that $4\pi R^2 = A_0$, i.e. the sphere of radius R has area A_0. Then we set:

$$t_\varepsilon = \sqrt{\frac{A_0}{A(\Sigma_\varepsilon)}} = \left(1 + \frac{\pi}{16} + o(1)\right)^{-1/2}.$$

Hence, the surface $t_\varepsilon\Sigma_\varepsilon$ has area A_0 and we have:

$$\frac{1}{2}\int_{t_\varepsilon\Sigma_\varepsilon} HdA = \frac{t_\varepsilon}{2}\left(\int_{\Sigma_\varepsilon} HdA\right) = -\left(1+\frac{\pi}{16}\right)^{-1/2}\left(2-\frac{\pi}{2}\right)\frac{\pi^2R^2}{4\varepsilon} + o\left(\frac{1}{\varepsilon}\right).$$

By letting ε tend to zero, we thus obtain the first part of Theorem 7.2. The total mean curvature, even constrained by area, is not bounded from below. $\square$

10.2 A sequence converging to a double sphere

We now detail the construction of a sequence $(\widetilde{\Sigma}_\varepsilon)_{\varepsilon>0}$ of axisymmetric $C^{1,1}$-surfaces of constant area whose total mean curvature tends to zero, which will end the proof of Theorem 7.2.

We consider the sequence of surfaces $(\widetilde{\Sigma}_\varepsilon)_{\varepsilon>0}$ described in Figure 10.2. They consist in two spheres of radius $R > 0$ and $R - 2r > 0$ glued together at a distance $\delta > r > 0$ of the axis of

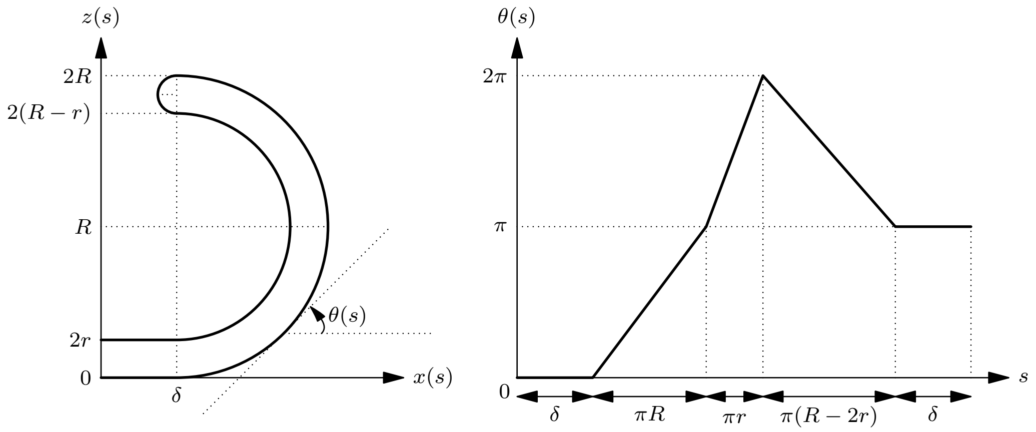

Figure 10.2: the construction of the sequence of axisymmetric surfaces $(\widetilde{\Sigma}_\varepsilon)_{\varepsilon>0}$.

revolution and such that the generating map $\theta : [0,L] \to \mathbb{R}$ is piecewise linear. More precisely, we have:

$$\theta(s) = \begin{cases} 0 & \text{if } s \in [0,\delta] \\ \dfrac{1}{R}(s-\delta) & \text{if } s \in [\delta, \delta+\pi R] \\ \dfrac{1}{r}(s-\delta-\pi R)+\pi & \text{if } s \in [\delta+\pi R, \delta+\pi(R+r)] \\ -\dfrac{1}{R-2r}(s-\delta-\pi R-\pi r)+2\pi & \text{if } s \in [\delta+\pi(R+r), \delta+\pi(2R-r)] \\ \pi & \text{if } s \in [\delta+\pi(2R-r), L], \end{cases}$$

where $L = 2\delta + \pi(2R-r) > 0$ is the total length of the generating curve. Then, a computation of $x(s) = \int_0^L \cos\theta(t)dt$ and $z(s) = \int_0^s \sin\theta(t)dt$ gives the following relations:

$$x(s) = \begin{cases} s & \text{if } s \in [0,\delta] \\ \delta + R\sin\theta(s) & \text{if } s \in [\delta, \delta+\pi R] \\ \delta + r\sin\theta(s) & \text{if } s \in [\delta+\pi R, \delta+\pi(R+r)] \\ \delta - (R-2r)\sin\theta(s) & \text{if } s \in [\delta+\pi(R+r), \delta+\pi(2R-r)] \\ L-s & \text{if } s \in [\delta+\pi(2R-r), L], \end{cases}$$

and also

$$z(s) = \begin{cases} 0 & \text{if } s \in [0,\delta] \\ R\,(1-\cos\theta(s)) & \text{if } s \in [\delta, \delta+\pi R] \\ 2R - r(1+\cos\theta(s)) & \text{if } s \in [\delta+\pi R, \delta+\pi(R+r)] \\ 2(R-r)-(R-2r)(1-\cos\theta(s)) & \text{if } s \in [\delta+\pi(R+r), \delta+\pi(2R-r)] \\ 2r & \text{if } s \in [\delta+\pi(2R-r), L]. \end{cases}$$

Finally, we obtain the following expressions:

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_{\widetilde{\Sigma}_\varepsilon} H dA = \pi \int_0^L \left(\sin\theta(s) + \dot{\theta}(s)x(s)\right) ds = 4\pi r + \pi^2\delta \\ A(\widetilde{\Sigma}_\varepsilon) = 2\pi \displaystyle\int_0^L x(s)ds = 2\pi\delta^2 + 2\pi^2\delta(2R-r) + 4\pi\left(R^2 - r^2 + (R-2r)^2\right). \end{cases}$$

Now, we impose that $\delta = 2r > r > 0$. The last relation is thus a second order polynomial in $R > 0$ and for each (small) r, there exists a unique positive root R_r such that $A(\widetilde{\Sigma}_\varepsilon) = A_0$. Moreover, R_r converges to $R_0 = \sqrt{A_0/8\pi}$ when $r \to 0$. Then, we see that the total mean curvature converges to zero from above as r tends to 0^+, which concludes the proof of Theorem 7.2.

Chapter 11

The sphere is the unique smooth critical point

According to Theorem 7.2, the sphere is not a global minimizer of (7.2) in the class of $C^{1,1}$-surfaces. However, in this chapter, we establish that the sphere is always a smooth local minimizer. Then, we compute the first variation of total mean curvature and area to obtain the Euler-Lagrange equation associated to (7.2). We deduce that the sphere is the unique smooth critical point of (7.2).

Remark 11.1. *Since the ball of radius R is a strictly convex set whose boundary has principal curvatures everywhere equal to $1/R$, any perturbation of class C^2 of the sphere yields a perturbation of class C^0 of its curvatures and then the perturbed domain remains convex. From* (7.1)*, the sphere is a global minimizer of* (7.2) *among compact inner-convex C^2-surfaces so the sphere is obviously a local minimizer of total mean curvature for small perturbations of class C^2.*

Proposition 11.2 (First variation of total mean curvature and area)**.** *Assume that Σ is a compact simply-connected C^2-surface. Consider a smooth vector field $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ and the family of maps $\phi_t : \mathbf{x} \in \Sigma \mapsto \mathbf{x} + t\mathbf{V}(\mathbf{x})$. Then, we have:*

$$\frac{d}{dt}\left(\int_{\phi_t(\Sigma)} 1 dA\right)_{t=0} = \int_\Sigma H\left(\mathbf{V}\cdot\mathbf{N}\right)dA,$$

where $\mathbf{N} : \Sigma \to \mathbb{S}^2$ refers to the Gauss map representing the outer unit normal field of Σ. Moreover, if Σ is a compact simply-connected C^3-surface, then we also get:

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\phi_t(\Sigma)} H dA\right)_{t=0} = \int_\Sigma K\left(\mathbf{V}\cdot\mathbf{N}\right)dA,$$

where $K = \kappa_1\kappa_2$ refers to the Gaussian curvature.

Proof. The first variation of area is classical, see for example [46, Corollary 5.4.16]. Concerning the first variation of total mean curvature, we refer to [26, Theorem 2.1] or [46, Theorem 5.4.17]. Using the notation of [26] i.e. $J(\Sigma) = \int_\Sigma H dA$, we get in the case where $\psi(x,\Sigma)$ represents any extension of the scalar mean curvature H, and $\psi'(\Omega;\mathbf{V})$ its shape derivative in the direction $\mathbf{V}$:

$$dJ(\Sigma;\mathbf{V}) = \int_\Sigma \psi'(\Omega;\mathbf{V})|_\Sigma dA + \int_\Sigma (\partial_\nu\psi + H\psi)V dA.$$

Now, Lemma 3.1 in [26] states $\psi'(\Sigma;\mathbf{V}) = -\Delta_\Sigma V$, where $V = \mathbf{V}\cdot\mathbf{N}$ and $\Delta_\Sigma = \mathrm{div}_\Sigma\nabla_\Sigma$ is the usual Laplace-Beltrami operator. Moreover, from [26, Lemma 3.2], and since Σ is C^3, we get $\partial_\nu H = -(\kappa_1^2+\kappa_2^2) = -H^2 + 2\kappa_1\kappa_2$. Therefore we deduce:

$$dJ(\Sigma;\mathbf{V}) = -\int_\Sigma \Delta_\Sigma V dA + \int_\Sigma (-H^2 + 2\kappa_1\kappa_2 + H^2)V dA = \int_\Sigma 2\kappa_1\kappa_2 V dA,$$

which gives the announced result and concludes the proof of Proposition 11.2. □

Theorem 11.3. *Within the class of compact simply-connected C^3-surfaces, if the area is constrained to be equal to a fixed positive number, then the corresponding sphere is the unique critical point of the total mean curvature.*

Proof. Consider any critical point Σ of (7.2) which is a compact simply-connected C^3-surface. From Proposition 11.2, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $2K = \lambda H$. Let us observe that $\lambda \neq 0$ otherwise $K = 0$ which is not possible (indeed, any compact surface has a point where $K > 0$ [73, Exercise 3.42]). Now assume that $\lambda < 0$. Then, from the relation $H^2 = (\kappa_1 + \kappa_2)^2 \geqslant 4\kappa_1\kappa_2 = 4K$, we get from the continuity of the scalar mean curvature and the connectedness of Σ that either $H \leqslant 2\lambda$ or $H \geqslant 0$. But this cannot happen since there is a point where $\lambda H = 2K > 0$ i.e. $H < 0$ and a point where $H \geqslant 0$. To see this last point, consider any plane far enough from the compact surface Σ and move it in a fixed direction. At the first point of contact between this plane and the surface Σ, it is locally convex i.e. $\kappa_1 \geqslant 0$ and $\kappa_2 \geqslant 0$. We deduce that at this point $H \geqslant 0$. Therefore, λ must be non-negative. In the same way, we prove that $H^2 \geqslant 4K = 2\lambda H$ impose that $H \geqslant 2\lambda$ everywhere and also that $K \geqslant \lambda^2 > 0$. Hence, Σ is an ovaloid, i.e. a compact simply-connected C^2-surface with $K > 0$, so its inner domain is a convex body [73, Theorem 6.1].

Integrating the relation $\lambda H = 2K$, we get $\lambda \int_\Sigma H dA = 2 \int_\Sigma K dA = 8\pi$, the last relation coming from the Gauss Bonnet Theorem [73, Theorem 8.38]. Now, multiply the relation $2K = \lambda H$ by the number $X \cdot \mathbf{N}(X)$, where X refer to the position of any point on the surface and $\mathbf{N}$ the outer unit normal field. Integrating over Σ and using [73, Theorem 6.11] give the following identity:

$$A(\Sigma) = \frac{1}{2}\int_\Sigma \mathrm{div}_\Sigma(\mathbf{x}) dA(\mathbf{x}) = \frac{1}{2}\int_\Sigma H X \cdot \mathbf{N}(X) dA = \frac{1}{\lambda}\int_\Sigma K X \cdot \mathbf{N}(X) dA = \frac{2}{\lambda}\int_\Sigma H dA = \frac{16\pi}{\lambda^2}.$$

Consequently, we obtain $\lambda = 2\sqrt{4\pi/A(\Sigma)}$ and $\frac{1}{2}\int_\Sigma H dA = \sqrt{4\pi A(\Sigma)}$. To conclude, we apply the equality case in Minkowski inequality (7.1): Σ has to be a sphere as required. □

Remark 11.4. *In the proof of Proposition 11.3, we show that when the Gaussian curvature K and the mean curvature H are proportional, the surface has to be a sphere. We only need C^2-regularity for this part. This result can be seen as a particular case of Alexandrov's uniqueness Theorem which deals with the similar question where a relation involving H and K holds. Usually, more regularity is required, see e.g. [73, Exercise 3.50] and [43, Appendix].*

Chapter 12

The sphere is the possible minimizer of absolute total mean curvature

This chapter is devoted to the proof of Theorem 7.3. We consider any axisymmetric $C^{1,1}$-surface $\Sigma \in \mathcal{A}_{1,1}$ generated by an admissible Lipschitz continuous map $\theta : [0, L] \to \mathbb{R}$, where $L > 0$ refers to the total length of the generating curve. We refer to Chapter 8 for precise definitions. The idea is to use again a certain rearrangement of θ:

$$\forall s \in [0, L], \quad \theta^{\bigstar}(s) = \begin{cases} \theta(s) - 2k\pi & \text{if} \quad \theta(s) \in [2k\pi, (2k+1)\pi[, \quad k \in \mathbb{Z} \\ \\ 2k\pi - \theta(s) & \text{if} \quad \theta(s) \in [(2k-1)\pi, 2k\pi[, \quad k \in \mathbb{Z}. \end{cases}$$

As shown in Figure 12.1, it consists in reflecting all parts of the range of θ which are outside the interval $[0, \pi]$ inside it. From a geometrical point of view, it is like unfolding the surface to make it inner-convex in any direction orthogonal to the axis of revolution.

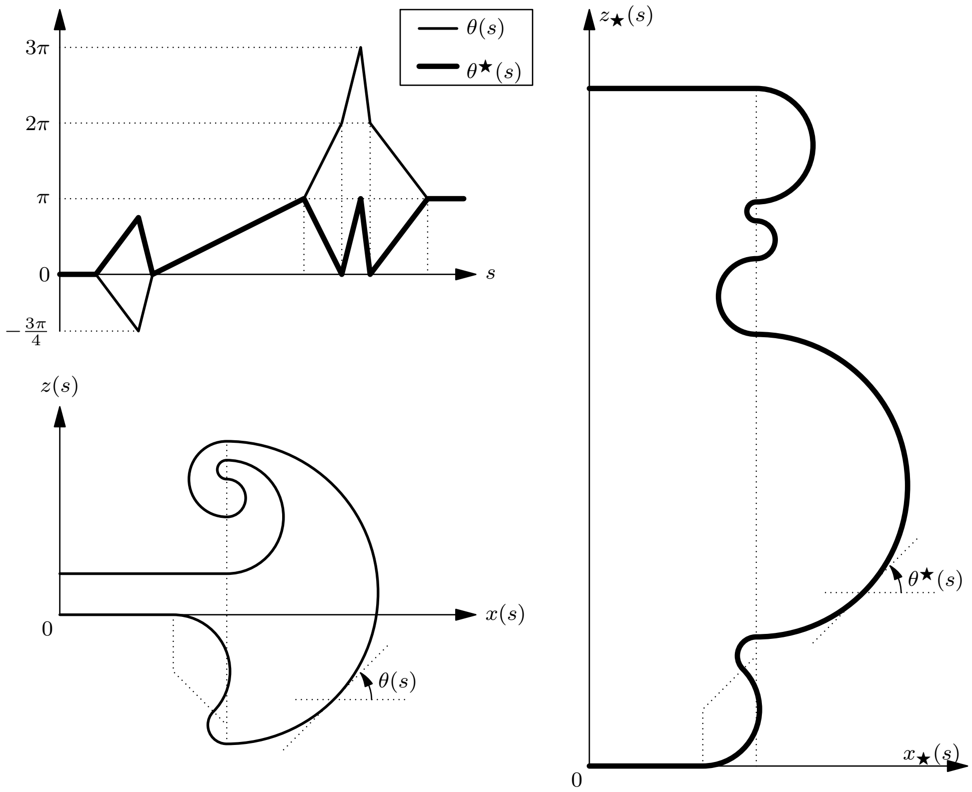

Figure 12.1: the rearrangement $\theta \mapsto \theta^{\bigstar}$ and the corresponding axisymmetric surfaces.

As in the proof of Theorem 7.1, this one is divided into three steps:

1. We show that $\theta^\star$ is generating an axiconvex $C^{1,1}$-surface $\Sigma^\star \in \mathcal{A}^+_{1,1}$.
2. We establish that:
$$\frac{1}{2}\int_\Sigma |H| dA \geqslant \frac{1}{2}\int_{\Sigma^\star} H dA \geqslant \sqrt{4\pi A(\Sigma^\star)} = \sqrt{4\pi A(\Sigma)}.$$
3. We study the equality case.

Proof of Theorem 7.3. **Step one:** $\Sigma^\star \in \mathcal{A}^+_{1,1}$.

The map $\theta^\star$ is Lipschitz continuous and valued in $[0,\pi]$ by construction. From Proposition 9.1, we have to check Relations (9.1), (9.2) and (9.3). The first one comes directly from the definition of $\theta^\star$. The second and third ones come from the odd and even parity of the cosine and sine functions. Indeed, observe that:

$$\forall s \in [0,L], \quad \begin{cases} x_\star(s) = \displaystyle\int_0^s \cos\theta^\star(t)dt = \int_0^s \cos\theta(t)dt = x(s) \\ z_\star(s) = \displaystyle\int_0^s \sin\theta^\star(t)dt = \int_0^s |\sin\theta(t)|dt \geqslant z(s). \end{cases}$$

Hence, we have $z_\star(L) \geqslant z(L) > 0$, $x_\star(L) = x(L) = 0$, and $x_\star(s) = x(s) > 0$ for any $s \in]0,L[$.

Step 2: comparing the total mean curvature and the area of Σ and $\Sigma^\star$.

Concerning the area, the equality is straightforward:

$$A(\Sigma^\star) = 2\pi\int_0^L x_\star(s)ds = 2\pi\int_0^L x(s)ds = A(\Sigma).$$

Then, we have:

$$\forall s \in [0,L], \quad \sin\theta^\star(s) + \dot{\theta}^\star(s)x_\star(s) = \begin{cases} \sin\theta(s) + \dot{\theta}(s)x(s) & \text{if } \theta(s) \in [2k\pi,(2k+1)\pi[, k \in \mathbb{Z} \\ -\sin\theta(s) - \dot{\theta}(s)x(s) & \text{if } \theta(s) \in [(2k-1)\pi, 2k\pi[, k \in \mathbb{Z}. \end{cases}$$

Consequently, we deduce that:

$$\begin{aligned} \frac{1}{2}\int_\Sigma |H|dA &= \pi\int_0^L |\sin\theta(s) + \dot{\theta}(s)x(s)|ds \geqslant \pi\int_0^L \left(\sin\theta^\star(s) + \dot{\theta}^\star(s)x_\star(s)\right)ds \\ &\geqslant \frac{1}{2}\int_{\Sigma_\star} H dA \geqslant \sqrt{4\pi A(\Sigma^\star)} = \sqrt{4\pi A(\Sigma)}, \end{aligned}$$

where the last inequality comes from Theorem 7.1 applied to the axiconvex $C^{1,1}$-surface $\Sigma^\star$.

Step 3: the equality case.

If we have equality in the above relation, it means that $\Sigma^\star$ is a sphere from the equality case of Theorem 7.1. Therefore, we have: $\theta^\star(s) = \frac{\pi}{L}s$. We prove by contradiction that θ is valued in $[0,\pi]$ which ensures from definition that $\theta = \theta^\star$ i.e. Σ is a sphere. Assume that there exists $s_0 \in]0,L[$ such that $\theta(s_0) < 0$. From the continuity of θ and the boundary conditions $\theta(0) = 0$, there exists $s_1 \in]0,L[$ such that $\theta(s_1) \in]-\pi,0[$. Then, from the definition of $\theta^\star$, $\theta(s_1) = -\theta^\star(s_1) = -\frac{\pi}{L}s_1$ and by the Lipschitz continuity of θ, we have:

$$\frac{\theta(L) - \theta(s_1)}{L - s_1} = \frac{\pi}{L}\frac{L+s_1}{L-s_1} \leqslant \|\dot{\theta}\|_{L^\infty(0,L)} = \|\dot{\theta}^\star\|_{L^\infty(0,L)} = \frac{\pi}{L},$$

Hence, the above inequality gives $L + s_1 \leqslant L - s_1$ which is not possible since $s_1 > 0$. Let us now assume that there exists $s_0 \in]0,L[$ such that $\theta(s_0) > \pi$. More precisely, since $\theta(0) = 0$, let us consider the first point $s_2 \in]0,L[$ such that $\theta(s_2) = \pi$. Since $0 \leq \theta(s) < \pi$ for any $s < s_2$; we have by definition $\theta(s) = \theta^\star(s) = \frac{\pi}{L}s$ for any $s < s_2$. But, passing to the limit $s \to s_2$, this leads to $\theta^\star(s_2) = \pi \Leftrightarrow s_2 = L$, which is not possible. To conclude, we proved that θ is valued in $[0,\pi]$. Hence, we have $\theta^\star = \theta$ so Σ must be a sphere. Conversely, any sphere satisfies the equality in (7.1), which concludes the proof of Theorem 7.3. □

Chapter 13

A proof of Minkowski's Theorem

In this chapter, a state of the art is made about inequality (7.1) and the equality case is also considered. In other words, we prove Minkowski's Theorem, which states as follows:

Theorem 13.1 (**Minkowski [69]**). *Consider the class $\mathfrak{C}$ of compact $C^{1,1}$-surfaces in $\mathbb{R}^3$ enclosing a convex inner domain. Then, we have the following inequality:*

$$\forall \Sigma \in \mathfrak{C}, \quad \frac{1}{2}\int_{\Sigma} H dA \geqslant \sqrt{4\pi A(\Sigma)},$$

where the equality holds if and only if Σ is a sphere.

Inequality (7.1) is announced in [69] assuming C^2-regularity and the proof can be found in [70, §7]. We also refer to [73, Chapter 6, Exercise (10)] for a proof considering ovaloids, i.e. compact simply-connected C^2-surfaces whose Gaussian curvature is positive everywhere.

First, the point of view of convex geometry is considered, without any regularity assumption, where the total mean curvature has to be replaced by the mean width of the convex body. The original proof of Minkowski is based on the isoperimetric inequality applied to the parallel sets for which Steiner-Minkowski formulas are available.

Then, we study the equality case of (7.1) which was stated by Minkowski in [70] without proof. We follow the ideas of Favard [31, Section 19] based on a Bonnesen-type inequality about mixed volume. Finally, Theorem 13.1 is proved and in the axisymmetric situation, we give a proof in our settings inspired by the one of Bonnesen [10, Section VI, §35 (74)].

For a proof involving more general cases, we refer to [83, Theorem 6.2.1 (6.2.3)] and the notes of [83, Section 6.2]. A more detailed exposition can also be found in [11, Section 49 (2')], [11, Section 52 (2')], and [11, Section 56 (6)].

13.1 Some results coming from convex geometry

Proposition 13.2 (**Minkowski [70]**). *Let K be a convex compact subset of $\mathbb{R}^3$ with some interior points. Then, we have the following inequality:*

$$\int_{\mathbb{S}^2} \sup_{\mathbf{y}\in K} \langle \mathbf{x} \mid \mathbf{y}\rangle dA(\mathbf{x}) \geqslant \sqrt{4\pi A(\partial K)}.$$

Proof. Consider any convex compact subset K of $\mathbb{R}^3$ with some interior points. We refer to [11] or [83] for the definitions and basic properties of convex subsets of $\mathbb{R}^3$ such as their volume $V(K)$, their area $A(K) := A(\partial K)$, and their mean width $M(K) = \int_{\mathbb{S}^2} \sup_{\mathbf{y}\in K}\langle \mathbf{x} \mid \mathbf{y}\rangle dA(\mathbf{x})$, related via the Steiner-Minkowski formulas:

$$\forall t \in [0,+\infty[, \quad \begin{cases} V(K+tB) = V(K) + A(K)t + M(K)t^2 + \frac{4\pi}{3}t^3 \\ \\ A(K+tB) = A(K) + 2M(K)t + 4\pi t^2, \end{cases}$$

where B refers to the unit closed ball of $\mathbb{R}^3$ and $K+tB$ to the set $\{\mathbf{x}+t\mathbf{y},\quad \mathbf{x}\in K \text{ and } \mathbf{y}\in B\}$. Set a real number $t\in]0,1[$. The Brunn-Minkowski inequality asserts that $\sqrt[3]{V}$ is a concave function (cf. [73, Theorem 6.22, Page 189] for a proof in $\mathbb{R}^3$) i.e. $\sqrt[3]{V(tK+(1-t)B)}\geqslant t\sqrt[3]{V}+(1-t)\sqrt[3]{B}$, so we get:

$$\frac{V(K+\frac{1-t}{t}B)-V(K)}{\frac{1-t}{t}}\geqslant 3V(K)^{\frac{2}{3}}V(B)^{\frac{1}{3}}+\frac{1-t}{t}3V(B)^{\frac{2}{3}}\left(V(K)^{\frac{1}{3}}+V(B)^{\frac{1}{3}}\right).$$

Using the Steiner-Minkowski formulas on the left member above, we obtain $A(K)\geqslant(6\sqrt{\pi}V(K))^{\frac{2}{3}}$ as $t\to 1^-$. This isoperimetric inequality is applied on $K+tB$, cubed, and multiplied by $s^6=\frac{1}{t^6}$, which gives, using again the Steiner-Minkowski formulas:

$$\left(A(K)s^2+2M(K)s+4\pi\right)^3-36\pi\left(V(K)s^3+A(K)s^2+M(K)s+\frac{4\pi}{3}\right)^2\geqslant 0.$$

Consequently, we have a positive polynomial $P(s)$ for every $s>0$. We find $P(0^+)=P'(0^+)=0$ so we must have $P''(0^+)=24\pi[M(K)^2-4\pi A(K)]\geqslant 0$. Hence, the inequality is established. □

Proposition 13.3 (Bonnesen [11]). *Consider a convex compact subset K of $\mathbb{R}^3$ with some interior points. Then, we have the following inequality:*

$$\lambda^2-\lambda\int_{\mathbb{S}^2}\sup_{\mathbf{y}\in K}\langle\mathbf{x}\mid\mathbf{y}\rangle dA(\mathbf{x})+\pi A(\partial K)\leqslant 0,$$

where λ refers to the total length (i.e. the one-dimensional Hausdorff measure) of the curve obtained by projecting orthogonally K on any plane of $\mathbb{R}^3$.

Proof. Let K and $\tilde{K}$ be two convex compact subsets of $\mathbb{R}^3$ with interior points. We still consider the notation introduced in the proof of Proposition 13.2. We refine the Brunn-Minkowski inequality, following [11, Section 50, Page 100]. Consider any plane P of $\mathbb{R}^3$ and the orthogonal projection K_P, $\tilde{K}_P$ of K, $\tilde{K}$ on P. If K_P and $\tilde{K}_P$ coincide, then K and $\tilde{K}$ are contained in a cylinder to which they are both tangent. Consider a line parallel to its axis of revolution (orthogonal to P). Any non-empty intersection with K, $\tilde{K}$, and $tK+(1-t)\tilde{K}$ are segments whose lengths satisfy: $L(tK+(1-t)\tilde{K})\geqslant tL(K)+(1-t)L(\tilde{K})$. Hence, using Cavalieri's principle, we deduce that the volume is also a concave function in this case: $V(tK+(1-t)\tilde{K})\geqslant tV(K)+(1-t)V(\tilde{K})$. Furthermore, if $A(K_P)=A(\tilde{K}_P)$, then we consider the Steiner symmetrization of K, $\tilde{K}$ with respect to P, followed by a Schwartz rearrangement K^*, $\tilde{K}^*$ around a line orthogonal to P. Since $K_P^*\equiv\tilde{K}_P^*$, we get:

$$\begin{aligned}V(tK+(1-t)\tilde{K}) &= V[(tK+(1-t)\tilde{K})^*] \geqslant V(tK^*+(1-t)\tilde{K}^*)\\ &\geqslant tV(K^*)+(1-t)V(\tilde{K}^*) = tV(K)+(1-t)V(\tilde{K}).\end{aligned}$$

Hence, the volume (and not only $\sqrt[3]{V}$) is a concave function in this case. Set a real number $t\in]0,1[$. We can apply the foregoing inequality to $A(K_P)^{-\frac{1}{2}}K$ and $A(\tilde{K}_P)^{-\frac{1}{2}}\tilde{K}$. Developing the left member with mixed volumes, one can obtain:

$$-\frac{(1+t)V(K)}{A(K_P)^{\frac{3}{2}}}+\frac{3tV(K,K,\tilde{K})}{A(K_P)A(\tilde{K}_P)^{\frac{1}{2}}}+\frac{3(1-t)V(K,\tilde{K},\tilde{K})}{A(\tilde{K}_P)A(K_P)^{\frac{1}{2}}}-\frac{(2-t)V(\tilde{K})}{A(\tilde{K}_P)^{\frac{3}{2}}}\geqslant 0,$$

thus $3A(\tilde{K}_P)^{-1}A(K_P)^{-\frac{1}{2}}V(K,\tilde{K},\tilde{K})-2A(\tilde{K}_P)^{-\frac{3}{2}}V(\tilde{K})-A(K_P)^{-\frac{3}{2}}V(K)\geqslant 0$ if we let $t\to 0^+$. Now apply this inequality to the sets K and $K+t\tilde{K}$, develop the expressions with mixed volumes, and expand the resulting relation in the neighbourhood of $t=0^+$. After calculation, we obtain:

$$-\frac{3A(K_P,\tilde{K}_P)^2V(K)}{A(K_P)^{\frac{7}{2}}}+\frac{6A(K_P,\tilde{K}_P)V(K,K,\tilde{K})}{A(K_P)^{\frac{5}{2}}}-\frac{3V(K,\tilde{K},\tilde{K})}{A(K_P)^{\frac{3}{2}}}+\frac{o(t^2)}{t^2}\geqslant 0.$$

Finally, if we choose the unit closed ball for K, then we have $3V(K)=4\pi$, $3V(K,K,\tilde{K})=M(\tilde{K})$, $3V(K,\tilde{K},\tilde{K})=A(\tilde{K})$, $A(K_P)=\pi$, and $2A(K_P,\tilde{K}_P)$ is the one-dimensional Hausdorff measure of ∂K_P referred as λ. Hence, as $t\to 0^+$, we get the required inequality: $\lambda^2-M(\tilde{K})\lambda+\pi A(\tilde{K})\leqslant 0$. □

Proposition 13.4 (Favard [31]). *If K is a convex compact subset of $\mathbb{R}^3$ with some interior points satisfying the equality $\int_{\mathbb{S}^2} \sup_{\mathbf{y}\in K}\langle \mathbf{x} \mid \mathbf{y}\rangle dA(\mathbf{x}) = \sqrt{4\pi A(\partial K)}$, then K must be a closed ball.*

Proof. We still use the notation introduced in the proof of Proposition 13.2. We follow the method described in [31, Section 19, Page 250]. Consider any convex compact subset K of $\mathbb{R}^3$ with some interior points satisfying $M(K) = \sqrt{4\pi A(K)}$. Set $\mathbf{u} \in \mathbb{S}^2$. Consider a plane $P_{\mathbf{u}}$ orthogonal to $\mathbf{u}$ and also the one-dimensional Hausdorff measure $L(\partial K_{P_{\mathbf{u}}})$ of the boundary $\partial K_{P_{\mathbf{u}}}$ associated to the orthogonal projection $K_{P_{\mathbf{u}}}$ of K on $P_{\mathbf{u}}$. Then, we get from Proposition 13.3: $L(\partial K_{P_{\mathbf{u}}})^2 = \pi A(\partial K)$. We integrate this relation over the unit sphere $\mathbb{S}^2$ and we apply Cauchy's Surface Area Formula $\pi A(\partial K) = \int_{\mathbb{S}^2} A(K_{P_{\mathbf{u}}})dA(\mathbf{u})$ (see e.g. [11, Section 32, Page 53]), in order to finally obtain:

$$\int_{\mathbb{S}^2} \left[L(\partial K_{P_{\mathbf{u}}})^2 - 4\pi A(K_{P_{\mathbf{u}}})\right] dA(\mathbf{u}) = 0.$$

From the two-dimensional isoperimetric inequality, we deduce $K_{P_{\mathbf{u}}}$ must be a disk for any $\mathbf{u} \in \mathbb{S}^2$. Hence, K is a closed ball as required. We refer to [10, Section III (43), Page 151] for a refined version of the two-dimensional isoperimetric inequality that allows a treatment of the equality case. □

Proof of Theorem 13.1. Combining Propositions 13.2 and 13.4, we only have to check that for any compact $C^{1,1}$-surface $\Sigma \subset \mathbb{R}^3$ enclosing a convex inner domain, we have the following relation:

$$\int_{\mathbb{S}^2} \sup_{\mathbf{y}\in K} \langle \mathbf{y} \mid \mathbf{x}\rangle dA(\mathbf{x}) = \frac{1}{2}\int_{\Sigma} HdA, \tag{13.1}$$

where $K = \Omega \cup \Sigma$ with Ω the inner domain of Σ. Since Σ is a compact $C^{1,1}$-surface, it has a positive reach (cf. Theorems 16.5–16.6). Hence, we can compare the Steiner-Minkwoski formulae of the convex body $K = \Omega \cup \Sigma$ with the one proved by Federer in [32], we get:

$$\begin{cases} A(K+tB) = A(\Sigma) + 2M(K)t + 4\pi t^2 \\ A(K+tB) = A(\Sigma) + t\displaystyle\int_{\Sigma} HdA + t^2 \int_{\Sigma} KdA \end{cases}$$

Since the compact surface Σ encloses a convex domain, it is simply-connected and from the Gauss-Bonnet Theorem valid for sets of positive reach [32, Theorem 5.19], we obtain $\int_{\Sigma} KdA = 4\pi$. To conclude, Relation (13.1) holds true, which ends the proof the Minkowski's Theorem.

□

13.2 The axisymmetric case

In this section we give a short proof, inspired by Bonnesen [10, Section 6,§35 (74)], of Minkowski's Theorem in the axisymmetric case. This result is used in particular in the proof of Theorem 7.1.

Proposition 13.5 (Bonnesen [10]). *Consider any axisymmetric $C^{1,1}$-surface Σ whose inner domain is assumed to be a convex subset of $\mathbb{R}^3$. Then, we have:*

$$4\pi\lambda^2 - \lambda\int_{\Sigma} HdA + A(\Sigma) \leqslant 0,$$

where $L = \pi\lambda$ refers to the total length of the generating curve.

Proof. Let $\Sigma \in \mathcal{A}_{1,1}$ and $\lambda \in \mathbb{R}$ be given. Using the notation of Chapter 8, we have in terms of generating map $\theta : [0, L] \to \mathbb{R}$:

$$\begin{aligned} 2\lambda^2 - \frac{\lambda}{2\pi}\int_{\Sigma} HdA + \frac{A(\Sigma)}{2\pi} &= \int_0^L \left[\lambda^2\dot{\theta}(s)\sin\theta(s)ds - \lambda\left(\sin\theta(s) + \dot{\theta}(s)x(s)\right) + x(s)\right] ds \\ &= \int_0^L (\lambda\sin\theta(s) - x(s))\left(\lambda\dot{\theta}(s) - 1\right) ds. \end{aligned}$$

We perform two integration by parts and we get:

$$\begin{aligned}
2\lambda^2 - \frac{\lambda}{2\pi}\int_\Sigma HdA + \frac{A(\Sigma)}{2\pi} &= -\int_0^L \cos\theta(s)\left(\lambda\theta(s)-s\right)\left(\lambda\dot{\theta}(s)-1\right)ds \\
&= \frac{1}{2}\left(\lambda\pi - L\right)^2 - \frac{1}{2}\int_0^L \left(\lambda\theta(s)-s\right)^2\dot{\theta}(s)\sin\theta(s)ds.
\end{aligned}$$

Now we set $\lambda = \frac{1}{\pi}L$ and we assume that Σ is inner-convex and axisymmetric. Therefore, the Gaussian curvature $K(s) = \kappa_1(s)\kappa_2(s) = \dot{\theta}(s)\frac{\sin\theta(s)}{x(s)}$ is non-negative on $[0, L]$. Hence, we obtain the required inequality:

$$4\pi\lambda^2 - \lambda\int_\Sigma HdA + A(\Sigma) = -\pi\int_0^L \left(\lambda\theta(s)-s\right)^2 K(s)x(s)ds \leqslant 0,$$

which concludes the proof of Proposition 13.5. $\square$

Corollary 13.6. *Consider any axisymmetric $C^{1,1}$-surface $\Sigma \subset \mathbb{R}^3$ which encloses a convex inner domain. Then, we have the following inequality:*

$$\frac{1}{2}\int_\Sigma HdA \geqslant \sqrt{4\pi A(\Sigma)},$$

where the equality holds if and only if Σ is a sphere.

Proof. From Proposition 13.5, the polynomial in λ has real roots, thus its discriminant must be non-negative, which gives the above inequality. Now if equality holds, $\lambda = L/\pi$ is a double root, that is:

$$\int_0^L (\lambda\theta(s)-s)^2\sin\theta(s)\dot{\theta}(s)ds = 0$$

Hence, the integrand must be zero almost everywhere, i.e. $\dot{\theta}$ is equal to zero or to $\frac{1}{\lambda}$ a.e. on $[0, L]$. But we have:

$$\int_0^L \dot{\theta}(s)ds = \theta(L) - \theta(0) = \pi = \frac{1}{\lambda}|\{s \in [0, L],\quad \dot{\theta}(s) \neq 0\}|$$

Since $\pi\lambda = L$, we get that $\dot{\theta}(s) \neq 0$ almost everywhere thus $\dot{\theta} = \frac{1}{\lambda}$ a.e. Hence, we get that θ is linear everywhere since the constant function $\frac{1}{\lambda}$ is continuous and Σ is a sphere as required. $\square$

Chapter 14

Some rearrangement properties

In this chapter, we give a proof of all the properties expressed in Proposition Parrangements.

Assumption 14.1. *Let $L > 0$ and $u : [0, L] \to [0, +\infty[$ be any continuous (non-negative) map satisfying $u(0) = 0$ and such that u is not identically zero.*

Lemma 14.2. *Let $L > 0$ and u be as in Assumption 14.1. We set $M = \max_{x\in[0,L]} u(x) > 0$. Then, the continuous map $u : [0, L] \to [0, M]$ is surjective.*

Proof. The image of the compact set $[0, L]$ through the continuous map u is a compact set in $\mathbb{R}$. Since u is non-negative and $u(0) = 0$, we get $f([0, L]) = [\min_{s\in[0,L]} f(s), \max_{x\in[0,L]} f(x)] = [0, M]$. □

Definition 14.3. *Considering Assumption 14.1 and the continuous surjective map $u : [0, L] \to [0, M]$ of Lemma 14.2, we introduce the following map:*

$$\begin{array}{rrcl} \rho : & [0, M] & \longrightarrow & [0, L] \\ & c & \longmapsto & \rho(c) = |\{x \in [0, L],\ u(x) \geqslant c\}|, \end{array}$$

where $|\bullet|$ refers to the usual one-dimensional Hausdorff measure.

Lemma 14.4. *Let $\rho : [0, M] \to [0, L]$ be the well-defined map of Definition 14.3. Then, ρ is decreasing and left-continuous.*

Proof. First, $\rho : [0, M] \to [0, L]$ is non-increasing. Indeed, for any $0 \leqslant c_1 \leqslant c_2 \leqslant M$, we have $\{u \geqslant c_2\} \subseteq \{u \geqslant c_1\}$ and thus $\rho(c_2) \leqslant \rho(c_1)$. Then, we assume $0 \leqslant c_1 < c_2 \leqslant M$ and we deduce:

$$\rho(c_1) - \rho(c_2) = |\{c_1 \leqslant u < c_2\}| = |u^{-1}\left([c_1, c_2[\right)| \geqslant |u^{-1}(]c_1, c_2[)|.$$

Since $]c_1, c_2[$ is a non-empty open subset of $[0, M]$, the continuity and the surjectivity of u ensures that $u^{-1}(]c_1, c_2[)$ is a non-empty open subset of $[0, L]$. In particular, it cannot be negligible so we obtain $\rho(c_1) - \rho(c_2) \geqslant |u^{-1}(]c_1, c_2[)| > 0$ and $\rho : [0, M] \to [0, L]$ is an increasing map. Finally, it remains to prove its left-continuity. Let $c \in]0, M]$ and $(c_i)_{i\in\mathbb{N}}$ be a sequence of converging to c such that $0 < c_i < c_{i+1} < c$ for any $i \in \mathbb{N}$. We set $A_i = \{u \geqslant c_i\}$. Hence, the sequence $(A_i)_{i\in\mathbb{N}}$ is decreasing and $\{u \geqslant c\} = \cap_{i\in\mathbb{N}} A_i \subseteq [0, L]$. We deduce that:

$$\rho(c) = |\{u \geqslant c\}| = \left| \bigcap_{i\in\mathbb{N}} A_i \right| = \lim_{i\to+\infty} |A_i| = \lim_{i\to+\infty} \rho(c_i).$$

Let $\varepsilon > 0$. From the foregoing, there exists $I \in \mathbb{N}$ such that $\rho(c) < \rho(c_I) < \rho(c) + \varepsilon$. We set $\delta = c - c_I > 0$ and choose any $x \in]c - \delta, c[$ i.e. $x \in]c_I, c[$. We get $0 < \rho(x) - \rho(c) < \rho(c_I) - \rho(c) < \varepsilon$ and ρ is left-continuous as required. □

Definition 14.5. *Considering Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. For any $A \subseteq [0, L]$, the non-decreasing rearrangement of A is defined by $A^* := [L - |A|, L]$. Similarly, the non-decreasing rearrangement of u denoted by $u^* : [0, L] \to [0, M]$ is defined as follows:*

$$\forall x \in [0, L], \quad u^*(x) = \sup\{c \in [0, M],\ x \in \{u \geqslant c\}^*\} = \sup\{c \in [0, M],\ x \in [L - \rho(c), L]\}.$$

Lemma 14.6. *Considering Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. Then, the map $u^* : [0, L] \to [0, M]$ of Definition 14.5 is non-decreasing.*

Proof. Let $\varepsilon > 0$ and choose any $x \in [0, L]$. From Definition 14.5, there exists $c_\varepsilon \in [0, M]$ such that $u^*(x)-\varepsilon < c_\varepsilon \leqslant u^*(x)$ and $L-\rho(c_\varepsilon) \leqslant x \leqslant L$. Considering any $y \in [x, L]$, we have $y \in [L-\rho(c_\varepsilon), L]$ so we get $u^*(y) \geqslant c_\varepsilon > u^*(x) - \varepsilon$. Then, we can let $\varepsilon \to 0^+$ to obtain $u^*(y) \geqslant u^*(x)$ for any $0 \leqslant x \leqslant y \leqslant L$. Hence, the map u^* is non-decreasing as required. □

Lemma 14.7. *Considering Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. In addition, we assume that u is L-Lipschitz continuous, $L > 0$. Then, we have:*

$$\forall (c_1, c_2) \in [0, M] \times [0, M], \quad c_1 < c_2 \quad \Longrightarrow \quad c_2 - c_1 \leqslant L\left[\rho\left(c_1\right) - \rho\left(c_2\right)\right],$$

where $\rho : [0, M] \to [0, L]$ is given in Definition 14.3.

Proof. Since u is L-Lipchitz continuous, we get from [30, Section 2.4.1 Theorem 1] that for any $A \subseteq [0, L]$ the inequality $|u(A)| \leqslant L|A|$ holds. Choose any $0 \leqslant c_1 < c_2 \leqslant M$ then set $A_1 = \{u \geqslant c_1\}$ and $A_2 = \{u \geqslant c_2\}$. Applying the previous estimation to $A = A_1 \backslash A_2$, we obtain:

$$L\left[\rho\left(c_1\right) - \rho\left(c_2\right)\right] = L|A_1 \backslash A_2| \geqslant |u(A_1 \backslash A_2)| = |u^{-1}\left\langle u\left([c_1, c_2[\right)\right\rangle| = c_2 - c_1.$$

In particular, note that the last equality holds because u is surjective as Lemma 14.2 shows. □

Lemma 14.8. *Considering Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. If in addition, the map u is L-Lipschitz continuous, $L > 0$, then the non-decreasing rearrangement u^* of Definition 14.5 is also an L-Lipschitz continuous map.*

Proof. Let $0 \leqslant x < y \leqslant L$. First, from Lemma 14.6, we have $u^*(x) \leqslant u^*(y)$. If equality holds, then we have obviously $|u^*(x) - u^*(y)| \leqslant L|x - y|$. We now assume $u^*(x) < u^*(y)$ then consider any $\varepsilon \in]0, \frac{1}{2}(u^*(y) - u^*(x))]$. From Definition 14.5, there exists $(c_x^\varepsilon, c_y^\varepsilon) \in [0, M] \times [0, M]$ such that $(x, y) \in [L - \rho(c_x^\varepsilon), L] \times [L - \rho(c_y^\varepsilon), L]$ with $u^*(x) - \varepsilon < c_x^\varepsilon \leqslant u^*(x)$ and $u^*(y) - \varepsilon < c_y^\varepsilon \leqslant u^*(y)$. Combining this previous relations with the bound on ε, we get $c_x^\varepsilon + 2\varepsilon \leqslant u^*(x) + 2\varepsilon \leqslant u^*(y) < c_y^\varepsilon + \varepsilon$. We deduce $0 \leqslant c_x^\varepsilon + \varepsilon < c_y^\varepsilon \leqslant M$ and apply Lemma 14.7 to get:

$$\begin{aligned} u^*(y) - u^*(x) - 2\varepsilon \quad &< \quad c_y^\varepsilon - (c_x^\varepsilon + \varepsilon) \\ &\leqslant \quad L\left(\rho\left(c_x^\varepsilon + \varepsilon\right) - \rho\left(c_y^\varepsilon\right)\right) \quad = \quad L\left(\left[L - \rho\left(c_y^\varepsilon\right)\right] - \left[L - \rho\left(c_x^\varepsilon + \varepsilon\right)\right]\right) \end{aligned}$$

We have $y \geqslant L - \rho(c_y^\varepsilon)$ and also $L - \rho(c_x^\varepsilon + \varepsilon) > x$, otherwise $u^*(x) \geqslant c_x^\varepsilon + \varepsilon$ which is not the case. We deduce $u^*(y) - u^*(x) - 2\varepsilon < L(y - x)$ and u^* is L-Lipschitz continuous by letting $\varepsilon \to 0^+$. □

Lemma 14.9. *Considering Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. Then, we have $\{u \geqslant c\}^* = \{u^* \geqslant c\}$ for any $c \in [0, M]$.*

Proof. Let $c \in [0, M]$. From Definition 14.5, we have $u^*(x) \geqslant c$ for any $x \in [L - \rho(c), L]$ i.e. $\{u \geqslant c\}^* \subseteq \{u^* \geqslant c\}$. Conversely, consider any $x \in [0, L]$ such that $u^*(x) \geqslant c$. Let $\varepsilon > 0$. There exists $c_x^\varepsilon \in]c - \varepsilon, u^*(x)]$ such that $x \in [L - \rho(c_\varepsilon), L]$. Since ρ is non-increasing, we deduce that $L - \rho(c + \varepsilon) \leqslant x \leqslant L$. Using the left-continuity of ρ proved in Lemma 14.4, we get $x \in [L - \rho(c), L]$ as $\varepsilon \to 0^+$. Hence, we have $\{u^* \geqslant c\} = \{u \geqslant c\}^* = [L - \rho(c), L]$. □

Lemma 14.10. *Considering Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. For any continuous map $F : [0, +\infty[\to \mathbb{R}$, we have:*

$$\int_0^L F\left[u\left(x\right)\right] dx = \int_0^L F\left[u^*\left(x\right)\right] dx.$$

Proof. Let $0 \leqslant a < b \leqslant L$. We have successively using Lemma 14.9:

$$\mathbf{1}_{[a,b[}(u^*) = \mathbf{1}_{(u^*)^{-1}([a,b[)} = \mathbf{1}_{\{u^* \geqslant a\} \backslash \{u^* \geqslant b\}} = \mathbf{1}_{\{u \geqslant a\}^* \backslash \{u \geqslant b\}^*}.$$

Integrating the previous equalities, we get:

$$\begin{aligned} \int_0^L \mathbf{1}_{[a,b[}(u^*) \quad &= \quad \int_0^L \mathbf{1}_{(\{u \geqslant a\} \backslash \{u \geqslant b\})^*} = |\{u \geqslant a\}^* \backslash \{u \geqslant b\}^*| = |[L - \rho(a), L - \rho(b)[| \\ &= \quad \rho(a) - \rho(b) = |u^{-1}([a, b[)| = \int_0^L \mathbf{1}_{u^{-1}([a,b[)} = \int_0^L \mathbf{1}_{[a,b[}(u). \end{aligned}$$

Therefore, by linearity, the same result holds for any step functions and thus for any regulated functions. In particular, this is the case for any continuous map $F : [0, +\infty[\to \mathbb{R}$. $\square$

Lemma 14.11. *Under Assumption 14.1, let $u : [0, L] \to [0, M]$ be the continuous surjective map of Lemma 14.2. For any continuous increasing map $F : [0, +\infty[\to [0, +\infty[$, we have $F(u^*) = [F(u)]^*$.*

Proof. First, note that any continuous increasing map is an homeomorphism on its image and its inverse is also increasing. Let $x \in [0, L]$ and $\varepsilon > 0$. From Definition 14.5, there exists $c_\varepsilon \in [0, M]$ such that $x \in [L - \rho(c_\varepsilon), L]$ and $u^*(x) - \varepsilon < c_\varepsilon \leqslant u^*(x)$. Since F is increasing, we get $F[u^*(x) - \varepsilon] < F(c_\varepsilon)$. Moreover, since $\rho(c_\varepsilon) = |\{u \geqslant c_\varepsilon\}| = |\{F(u) \geqslant F(c_\varepsilon)\}|$, we have $[F \circ u]^*(x) \geqslant F(c_\varepsilon) > F[u^*(x) - \varepsilon]$. Using the continuity of F, we get $F[u^*(x)] \geqslant [F \circ u^*](x)$ as $\varepsilon \to 0^+$. Similarly, there exists $\tilde{c}_\varepsilon \in [0, M]$ such that:

$$L - |\{F \circ u \geqslant \tilde{c}_\varepsilon\}| \leqslant x \leqslant L \qquad \text{and} \qquad [F \circ u]^*(x) - \varepsilon < \tilde{c}_\varepsilon \leqslant [F \circ u]^*(x).$$

Since $\rho[F^{-1}(\tilde{c}_\varepsilon)] = |\{F \circ u \geqslant \tilde{c}_\varepsilon\}|$, we deduce $F^{-1}(\tilde{c}_\varepsilon) \leqslant u^*(x)$ which implies $F[u^*(x)] \geqslant \tilde{c}_\varepsilon > [F \circ u]^*(x) - \varepsilon$. We obtain $F[u^*(x)] \geqslant [F \circ u]^*(x)$ as $\varepsilon \to 0^+$ so equality holds form the foregoing. $\square$

Lemma 14.12. *Let $L > 0$. For any measurable set $A \subseteq [0, L]$, we have $\mathbf{1}_{A^*} = (\mathbf{1}_A)^*$.*

Proof. Let $L > 0$ and A be any measurable subset of $[0, L]$. From Definition 14.5, we have:

$$\forall x \in [0, L], \quad (\mathbf{1}_A)^*(x) = \sup\{c \in [0, 1],\ x \in [L - |\{\mathbf{1}_A \geqslant c\}|, L]\}.$$

First, note that $|\{\mathbf{1}_A \geqslant c\}| = L$ if $c = 0$, otherwise $|\{\mathbf{1}_A \geqslant c\}| = |A|$ for any $c \in]0, 1]$. Therefore, if $x \in A^* = [L - |A|, L]$, then for any $c \in]0, 1]$, we have $x \in [L - |\{\mathbf{1}_A \geqslant c\}|, L]$ and thus $(\mathbf{1}_A)^*(x) = 1 = \mathbf{1}_{A^*}(x)$. Similarly, if $x \in [0, L] \backslash A_* = [0, L - |A|[$, then for any $c \in]0, 1]$, we have $x \notin [L - |\{\mathbf{1}_A \geqslant c\}|, L]$. Hence, we obtain $(\mathbf{1}_A)^*(x) = 0 = \mathbf{1}_{A^*}(x)$ in this case. To conclude, we proved $(\mathbf{1}_A)^*(x) = \mathbf{1}_{A^*}(x)$ for any $x \in [0, L]$. $\square$

Part IV

Uniform ball property and existence of optimal shapes for a wide class of geometric functionals

Chapter 15

Introduction

Using the shape optimization point of view, the aim of this part is to introduce a more reasonable class of surfaces, in which the existence of an enough regular minimizer is ensured for general functionals and constraints involving the first- and second-order geometric properties of surfaces. Inspired by what Chenais did in [20] when she considered the uniform cone property, we consider the (hyper-)surfaces that satisfy a uniform ball condition in the following sense.

Definition 15.1. *Let $\varepsilon > 0$ and $B \subseteq \mathbb{R}^n$ be open, $n \geqslant 2$. We say that an open set $\Omega \subseteq B$ satisfies the ε-ball condition and we write $\Omega \in \mathcal{O}_\varepsilon(B)$ if for any $\mathbf{x} \in \partial\Omega$, there exits a unit vector $\mathbf{d_x}$ of $\mathbb{R}^n$ such that:*

$$\begin{cases} B_\varepsilon(\mathbf{x} - \varepsilon\mathbf{d_x}) \subseteq \Omega \\ \\ B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d_x}) \subseteq B\backslash\overline{\Omega}, \end{cases}$$

where $B_r(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^n,\ \|\mathbf{y} - \mathbf{z}\| < r\}$ denotes the open ball of $\mathbb{R}^n$ centred at $\mathbf{z}$ and of radius r, where $\overline{\Omega}$ is the closure of Ω, and where $\partial\Omega = \overline{\Omega}\backslash\Omega$ refers to its boundary.

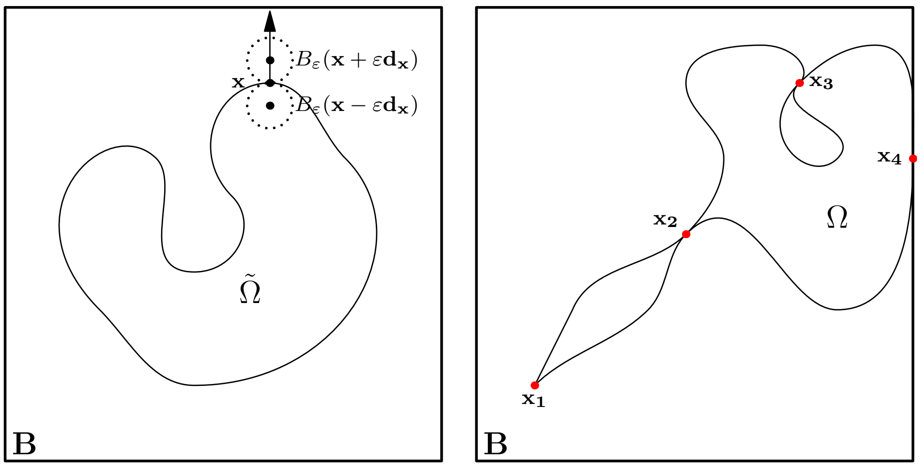

Figure 15.1: Example of an open set $\tilde{\Omega}$ in $\mathbb{R}^2$ satisfying the ε-ball condition, whereas Ω does not. Indeed, there is no circle passing through $\mathbf{x_1}$ or $\mathbf{x_2}$ (respectively $\mathbf{x_3}$ or $\mathbf{x_4}$) whose enclosed inner domain is included in Ω (respectively in $B\backslash\overline{\Omega}$).

The uniform (exterior/interior) ball condition was already considered by Poincaré in 1890 [78]. As illustrated in Figure 15.1, it avoids the formation of singularities such as corners, cuts, or self-intersections. In fact, it has been known to characterize the $C^{1,1}$-regularity of hypersurfaces for a long time by oral tradition, and also the positiveness of their reach, a notion introduced by Federer in [32]. An example is illustrated in Figure 15.2. We did not find any reference where these two characterizations were gathered. Hence, they are established in Chapter 16, reproducing an accepted proceeding entitled *some characterizations of a uniform ball property* [22]. We refer to Theorems 16.5 and 16.6 for precise statements.

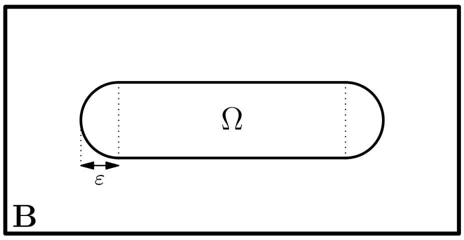

Figure 15.2: Example of a stadium $\Omega \in \mathcal{O}_{B,\varepsilon}$ is $C^{1,1}$ but not of class C^2.

Equipped with this class of admissible shapes, we can now state our main general existence result in the three-dimensional Euclidean space $\mathbb{R}^3$. We refer to Section 18.5 for its most general form in $\mathbb{R}^n$, but the following one is enough for the three physical applications we are presenting hereafter (further examples are also detailed in Section 18.5).

Theorem 15.2. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C, \widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, five continuous maps j_0, f_0, g_0, g_1, $g_2 : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$, and four maps j_1, j_2, f_1, f_2 : $\mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ which are continuous and convex in the last variable. Then, the following problem has at least one solution (see Notation 15.3):*

$$\inf \int_{\partial\Omega} j_0 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} j_1 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), H\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} j_2 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), K\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right),$$

where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying a finite number of constraints of the following form:

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} f_1 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), H\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} f_2 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right), K\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) \leqslant C \\ \displaystyle\int_{\partial\Omega} g_0 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} H\left(\mathbf{x}\right) g_1 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega} K\left(\mathbf{x}\right) g_2 \left[\mathbf{x}, \mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) = \widetilde{C}. \end{cases}$$

The proof of Theorem 15.2 only relies on basic tools of analysis and does not use the ones of geometric measure theory. We also mention that the particular case $j_0 \geqslant 0$ and $j_1 = j_2 = 0$ without constraints was obtained in parallel to our work in [40].

Notation 15.3. *We denote by $A(\bullet)$ (respectively $V(\bullet)$) the area (resp. the volume) i.e. the two (resp. three)-dimensional Hausdorff measure, and the integration on a surface is done with respect to A. The Gauss map $\mathbf{n} : \mathbf{x} \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^2$ always refers to the unit outer normal field of the surface, while $H = \kappa_1 + \kappa_2$ is the scalar mean curvature and $K = \kappa_1\kappa_2$ is the Gaussian curvature.*

Remark 15.4. *In the above theorem, the radius of B is large enough to avoid $\mathcal{O}_\varepsilon(B)$ being empty. Moreover, the assumptions on B can be relaxed by requiring B to be a non-empty bounded open set, smooth enough (Lipschitz for example) such that its boundary has zero three-dimensional Lebesgue measure, and large enough to contain at least an open ball of radius 3ε. Finally, for any set E, we recall that a well-defined map $j : E \times \mathbb{R} \to \mathbb{R}$ is said to be convex in its last variable if for any $(\mathbf{x}, t) \in E \times \mathbb{R}$ and any $\mu \in [0, 1]$, we have $j(\mathbf{x}, \mu t + (1-\mu)\tilde{t}) \leqslant \mu j(\mathbf{x}, t) + (1-\mu) j(\mathbf{x}, \tilde{t})$.*

15.1 First application: minimizing the Canham-Helfrich energy with area and volume constraints

We recall that the Canham-Helfrich energy is a simple model to characterize vesicles. Imposing the area of the bilayer and the volume of fluid it contains, their shape is a minimizer for the following free-bending energy (see Notation 15.3):

$$\mathcal{E}(\Sigma) = \frac{k_b}{2} \int_\Sigma (H - H_0)^2 \, dA + k_G \int_\Sigma K dA, \tag{15.1}$$

where the spontaneous curvature $H_0 \in \mathbb{R}$ measures the asymmetry between the two layers, and where $k_b > 0$, $k_G < 0$ are two other physical constants. Note that if $k_G > 0$, for any $k_b, H_0 \in \mathbb{R}$, the

Canham-Helfrich energy (15.1) with prescribed area A_0 and volume V_0 is not bounded from below. Indeed, in that case, from the Gauss-Bonnet Theorem, the second term $k_g \int K dA = 4\pi k_G(1-g)$ tends to $-\infty$ as the genus $g \to +\infty$, while the first term remains bounded by $4|k_b|(12\pi + \frac{1}{4}H_0^2 A_0)$. To see this last point, use [53, Remark 1.7 (iii) (1.5)], [84, Theorem 1.1], and [88, Inequality (0.2)]) in order to get successively:

$$\begin{aligned}
\inf_{\substack{A(\partial\Omega)=A_0 \\ V(\Omega)=V_0}} \mathcal{E}(\partial\Omega) &\leqslant 4|k_b| \left(\inf_{\substack{A(\partial\Omega)=A_0 \\ V(\Omega)=V_0 \\ \mathrm{genre}(\partial\Omega)=g}} \mathcal{W}(\partial\Omega) + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g) \\
&\leqslant 4|k_b| \left(\inf_{\mathrm{genre}(\partial\Omega)=g} \mathcal{W}(\partial\Omega) + \inf_{\substack{A(\partial\Omega)=A_0 \\ V(\Omega)=V_0 \\ \mathrm{genre}(\partial\Omega)=0}} \mathcal{W}(\partial\Omega) - 4\pi + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g) \\
&\leqslant 4|k_b| \left(8\pi + 8\pi - 4\pi + \frac{H_0^2 A_0}{4} \right) + 4\pi k_G(1-g).
\end{aligned}$$

The two-dimensional case of (15.1) is considered by Bellettini, Dal Maso, and Paolini in [5]. Some of their results is recovered by Delladio [24] in the framework of special generalized Gauss graphs from the theory of currents. Then, Choksi and Veneroni [21] solve the axisymmetric case of (15.1) assuming $-2k_b < k_G < 0$. In the general case, this hypothesis gives a fundamental coercivity property [21, Lemma 2.1]: the integrand of (15.1) is standard in the sense of [48, Definition 4.1.2]. Hence, we get a minimizer for (15.1) in the class of rectifiable integer oriented 2-varifold in $\mathbb{R}^3$ with L^2-bounded generalized second fundamental form [48, Theorem 5.3.2] [72, Section 2] [6, Appendix]. These compactness and lower semi-continuity properties were already noticed in [6, Section 9.3].

However, the regularity of minimizers remains an open problem and experiments show that singular behaviours can occur to vesicles such as the budding transition [85, 86]. As the temperature increases, an initially spherical vesicle becomes a prolate ellipsoid, then takes a pear shape with broken up/down symmetry, and finally the neck closes, resulting in two spherical compartments that are sitting on top of each other but still connected by a narrow constriction [85, Section 1.1, Figure 1]. This cannot happen to red blood cells because their skeleton prevents the membrane from bending too much locally [59, Section 2.1]. To take this aspect into account, the uniform ball condition of Definition 15.1 is also motivated by the modelization of the equilibrium shapes of red blood cells. We even have a clue for its physical order of magnitude [59, Section 2.1.5]. Our result states as follows.

Theorem 15.5. *Let $H_0, k_G \in \mathbb{R}$ and $\varepsilon, k_b, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Notation 15.3):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n) \\ A(\partial\Omega)=A_0 \\ V(\Omega)=V_0}} \frac{k_b}{2} \int_{\partial\Omega} (H-H_0)^2 dA + k_G \int_{\partial\Omega} K dA.$$

Remark 15.6. *From the isoperimetric inequality, if $A_0^3 < 36\pi V_0^2$, one cannot find any $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ satisfying the two constraints; and if equality holds, the only admissible shape is the ball of area A_0 and volume V_0. Moreover, in the above theorem, note that we did not assume the $\Omega \in \mathcal{O}_\varepsilon(B)$ as it is the case for Theorem 15.2 because a uniform bound on their diameter is already given by the functional and the area constraint [88, Lemma 1.1]. Finally, the result above also holds if H_0 is continuous function of the position and the normal.*

15.2 Second application: minimizing the Canham-Helfrich energy with prescribed genus, area, and volume

Since the Gauss-Bonnet Theorem is valid for sets of positive reach [32, Theorem 5.19], we get from Theorem 16.5 that $\int_\Sigma K dA = 4\pi(1-g)$ for any compact connected $C^{1,1}$-surface Σ (without boundary embedded in $\mathbb{R}^3$) of genus $g \in \mathbb{N}$. Hence, instead of minimizing (15.1), people usually fix the topology and search for a minimizer of the Helfrich energy (see Notation 15.3):

$$\mathcal{H}(\Sigma) = \int_{\Sigma} (H - H_0)^2 \, dA, \tag{15.2}$$

with prescribed area and enclosed volume. Like (15.1), such a functional depends on the surface but also on its orientation. However, in the case $H_0 \neq 0$, Energy (15.2) is not even lower semi-continuous with respect to the varifold convergence [6, Section 9.3]: the counterexample is due to Große-Brauckmann [38]. Using the framework of the ε-ball condition we prove the following.

Theorem 15.7. *Let $H_0 \in \mathbb{R}$, $g \in \mathbb{N}$, and $\varepsilon, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Notation 15.3 and Remark 15.6):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n) \\ \mathrm{genus}(\partial\Omega) = g \\ A(\partial\Omega) = A_0 \\ V(\Omega) = V_0}} \int_{\partial\Omega} (H - H_0)^2 dA,$$

where genus$(\partial\Omega) = g$ *has to be understood as $\partial\Omega$ is a compact connected $C^{1,1}$-surface of genus g.*

15.3 Third application: minimizing the Willmore functional with various constraints

The particular case $H_0 = 0$ in (15.2) is known as the Willmore functional (see Notation 15.3):

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_{\Sigma} H^2 dA. \tag{15.3}$$

It has been widely studied by geometers. Without constraint, Willmore [93, Theorem 7.2.2] proved that spheres are the only global minimizers of (15.3). Existence was established by Simon [88] for genus-one surfaces, Bauer and Kuwert [4] for higher genus. Recently, Marques and Neves [66] solved the so-called Willmore conjecture: the conformal transformations of the stereographic projection of the Clifford torus are the only global minimizers of (15.3) among smooth genus-one surfaces.

A main ingredient is the conformal invariance of (15.3), from which we can in particular deduce that minimizing (15.3) with prescribed isoperimetric ratio is equivalent to impose the area and the enclosed volume. In this direction, Schygulla [84] established the existence of a minimizer for (15.3) among analytic surfaces of zero genus and given isoperimetric ratio. For higher genus, Keller, Mondino, and Rivière [53] recently obtained similar results, using the point of view of immersions developed by Rivière [79] to characterize precisely the critical points of (15.3).

An existence result related to (15.3) is the particular case $H_0 = 0$ of Theorem 15.7. Again, the difficulty with these kind of functionals is not to obtain a minimizer (compactness and lower semi-continuity in the class of varifolds for example) but to show that it is regular in the usual sense (i.e. a smooth surface). Using again our result on the uniform ball condition, we now give a last application of Theorem 15.2 which comes from the modelling of vesicles. It is known as the bilayer-couple model [85, Section 2.5.3] and it states as follows.

Theorem 15.8. *Let $M_0 \in \mathbb{R}$ and $\varepsilon, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem has at least one solution (see Notation 15.3 and Remark 15.6):*

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n) \\ \mathrm{genus}(\partial\Omega) = g \\ A(\partial\Omega) = A_0 \\ V(\Omega) = V_0 \\ \int_{\partial\Omega} H dA = M_0}} \frac{1}{4} \int_{\partial\Omega} H^2 dA,$$

where genus$(\partial\Omega) = g$ *has to be understood as $\partial\Omega$ is a compact connected $C^{1,1}$-surface of genus g.*

To conclude this introduction, this part is organized as follows. In Chapter 16, we precisely state of the two characterizations associated with the uniform ball condition, namely Theorem 16.5 in terms of positive reach and Theorem 16.6 in terms of $C^{1,1}$-regularity. Then, we give the proofs of the theorem, as in [22].

Following the classical method from the calculus of variations, in Section 17.1, we first obtain the compactness of the class $\mathcal{O}_\varepsilon(B)$ for various modes of convergence. This essentially follows from the fact that the ε-ball condition implies a uniform cone property, for which we already have some compactness results.

Then, in the rest of Chapter 17, we prove the key ingredient of Theorem 15.2: we manage to parametrize in a fixed local frame simultaneously all the graphs associated with the boundaries of a converging sequence in $\mathcal{O}_\varepsilon(B)$. We then prove the C^1-strong and the $W^{2,\infty}$-weak-star convergence of these local graphs.

Finally, in Chapter 18, we show how to use this local result on a suitable partition of unity to get the global continuity of general geometric functionals. Merely speaking, the proof always consists in expressing the integral in the parametrization and show that the integrand is the product of a L^∞-weak-star converging term with an L^1-strong converging term.

We conclude by giving some existence results in Section 18.5. We prove Theorem 15.2, its generalization to $\mathbb{R}^n$, and detail many applications such as Theorems 15.5, 15.7, and 15.8, mainly coming from the modelling of vesicles and red blood cells.

Chapter 16

Two characterizations of the uniform ball property

In this chapter, we establish two characterizations of the ε-ball condition, namely Theorems 16.5 and 16.6. First, we show that this property is equivalent to the notion of positive reach introduced by Federer [32]. Then, we prove that it is equivalent to a uniform $C^{1,1}$-regularity of hypersurfaces. These are known facts. The proofs, already given in [22], are reproduced here for completeness.

Indeed, we did not find any reference where these two characterizations were gathered although many parts of Theorems 16.5 and 16.6 can be found in the literature as remarks [47, below Theorem 1.4] [71, (1.10)] [32, Remark 4.20], sometimes with proofs [39, Theorem 2.2] [62, §4 Theorem 1] [63, Proposition 1.4] [34, Section 2.1], or as consequences of more general results [35, Theorem 1.2] [3, Theorem 1.1 (1.2)].

16.1 Definitions, notation, and statements

Before stating the theorems, we recall some definitions and notation, used thereafter in the article. Consider any integer $n \geqslant 2$ henceforth set. The space $\mathbb{R}^n$ whose points are marked $\mathbf{x} = (x_1, \ldots, x_n)$ is naturally provided with its usual Euclidean structure, $\langle \mathbf{x} \mid \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_k$ and $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x} \mid \mathbf{x} \rangle}$, but also with a direct orthonormal frame whose choice will be specified later on. Inside this frame, every point $\mathbf{x}$ of $\mathbb{R}^n$ will be written into the form $(\mathbf{x}', x_n)$ such that $\mathbf{x}' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. In particular, the symbols $\mathbf{0}$ and $\mathbf{0}'$ respectively refer to the zero vector of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$.

First, some of the notation introduced in [32] by Federer are recalled. For every non-empty subset A of $\mathbb{R}^n$, the following map is well defined and 1-Lipschitz continuous:

$$d(.,A) : \begin{array}{ccl} \mathbb{R}^n & \longrightarrow & [0, +\infty[\\ \mathbf{x} & \longmapsto & d(\mathbf{x}, A) = \inf\limits_{\mathbf{a} \in A} \|\mathbf{x} - \mathbf{a}\|. \end{array}$$

Furthermore, we introduce:

$$\mathrm{Unp}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid \exists ! \mathbf{a} \in A, \quad \|\mathbf{x} - \mathbf{a}\| = d(\mathbf{x}, A)\}.$$

This is the set of points in $\mathbb{R}^n$ having a unique projection on A, that is the maximal domain on which the following map is well defined:

$$p_A : \begin{array}{ccl} \mathrm{Unp}(A) & \longrightarrow & A \\ \mathbf{x} & \longmapsto & p_A(\mathbf{x}), \end{array}$$

where $p_A(\mathbf{x})$ is the unique point of A such that $\|p_A(\mathbf{x}) - \mathbf{x}\| = d(\mathbf{x}, A)$. We can also notice that $A \subseteq \mathrm{Unp}(A)$ thus in particular $\mathrm{Unp}(A) \neq \emptyset$. We can now express what is a set of positive reach.

Definition 16.1. *Consider any non-empty subset A of $\mathbb{R}^n$. First, we set for any point $\mathbf{a} \in A$:*

$$\mathrm{Reach}(A, \mathbf{a}) = \sup \{r > 0, \quad B_r(\mathbf{a}) \subseteq \mathrm{Unp}(A)\},$$

with the convention $\sup \emptyset = 0$. *Then, we define the reach of* A *by the following quantity:*

$$\mathrm{Reach}(A) = \inf_{\mathbf{a}\in A} \mathrm{Reach}(A, \mathbf{a}).$$

Finally, we say that A *has a positive reach if we have* $\mathrm{Reach}(A) > 0$.

Remark 16.2. *From Definition 16.1, the reach of a subset of* $\mathbb{R}^n$ *is defined if it is not empty. Consequently, when considering the reach associated with the boundary of an open subset* Ω *of* $\mathbb{R}^n$*, we will have to ensure* $\partial\Omega \neq \emptyset$ *and to do so, we will assume* Ω *is not empty and different from* $\mathbb{R}^n$*. Indeed, if* $\partial\Omega = \emptyset$*, then* $\overline{\Omega} = \Omega \cup \partial\Omega = \Omega$ *thus* $\Omega = \emptyset$ *or* $\Omega = \mathbb{R}^n$ *because it is both open and closed.*

Then, we also recall the definition of a $C^{1,1}$-hypersurface in terms of local graph. Note that from the Jordan-Brouwer Separation Theorem, any compact topological hypersurface of $\mathbb{R}^n$ has a well-defined inner domain, and in particular a well-defined enclosed volume. If instead of being compact, it is connected and closed as a subset of $\mathbb{R}^n$, then it remains the boundary of an open set [73, Theorem 4.16] [25, Section 8.15], which is not unique and possibly unbounded in this case.

Definition 16.3. *Consider any subset* $\mathcal{S}$ *of* $\mathbb{R}^n$*. We say that* $\mathcal{S}$ *is a* $C^{1,1}$*-hypersurface if there exists an open subset* Ω *of* $\mathbb{R}^n$ *such that* $\partial\Omega = \mathcal{S}$*, and such that for any point* $\mathbf{x}_0 \in \partial\Omega$*, there exists a direct orthonormal frame centred at* $\mathbf{x}_0$ *such that in this local frame, there exists a map* $\varphi : D_r(\mathbf{0}') \to]-a, a[$ *continuously differentiable with* $a > 0$*, such that* φ *and its gradient* $\nabla\varphi$ *are L-Lipschitz continuous with* $L > 0$*, satisfying* $\varphi(\mathbf{0}') = 0$*,* $\nabla\varphi(\mathbf{0}') = \mathbf{0}'$*, and also:*

$$\begin{cases} \partial\Omega \cap (D_r(\mathbf{0}') \times]-a, a[) &= \{(\mathbf{x}', \varphi(\mathbf{x}')), \quad \mathbf{x}' \in D_r(\mathbf{0}')\} \\ \\ \Omega \cap (D_r(\mathbf{0}') \times]-a, a[) &= \{(\mathbf{x}', x_n), \quad \mathbf{x}' \in D_r(\mathbf{0}') \text{ and } -a < x_n < \varphi(\mathbf{x}')\}, \end{cases}$$

where $D_r(\mathbf{0}') = \{\mathbf{x}' \in \mathbb{R}^{n-1},\ \|\mathbf{x}'\| < r\}$ *denotes the open ball of* $\mathbb{R}^{n-1}$ *centred at the origin* $\mathbf{0}'$ *and of radius* $r > 0$.

Finally, we recall the definition of the uniform cone property introduced by Chenais in [20], illustrated in Figure 16.1, and from which the ε-ball condition is inspired. We also refer to [46, Definition 2.4.1].

Definition 16.4. *Let* $\alpha \in]0, \frac{\pi}{2}[$ *and* Ω *be an open subset of* $\mathbb{R}^n$*. We say that* Ω *satisfies the* α*-cone condition if for any point* $\mathbf{x} \in \partial\Omega$*, there exists a unit vector* $\xi_{\mathbf{x}}$ *of* $\mathbb{R}^n$ *such that:*

$$\forall \mathbf{y} \in B_\alpha(\mathbf{x}) \cap \Omega, \quad C_\alpha(\mathbf{y}, \xi_{\mathbf{x}}) \subseteq \Omega,$$

where $C_\alpha(\mathbf{y}, \xi_{\mathbf{x}}) = \{\mathbf{z} \in B_\alpha(\mathbf{y}),\ \|\mathbf{z} - \mathbf{y}\| \cos\alpha < \langle \mathbf{z} - \mathbf{y} \mid \xi_{\mathbf{x}} \rangle\}$ *refers to the open cone of corner* $\mathbf{y}$*, direction* $\xi_{\mathbf{x}}$*, and span* α.

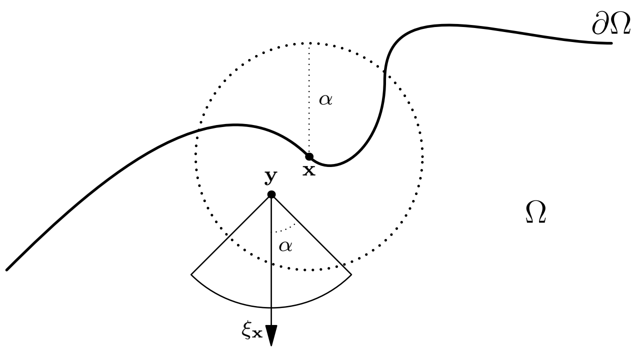

Figure 16.1: Illustration of the α-cone property.

We are now in position to precisely state the two main regularity results associated with the uniform ball condition.

Theorem 16.5 (**A characterization in terms of positive reach**). *Consider any non-empty open subset Ω of $\mathbb{R}^n$ different from $\mathbb{R}^n$. Then, the following implications are true:*

(i) if there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ as in Definition 15.1, then $\partial\Omega$ has a positive reach in the sense of Definition 16.1 and we have $\text{Reach}(\partial\Omega) \geqslant \varepsilon$;

(ii) if $\partial\Omega$ has a positive reach, then $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ for any $\varepsilon \in]0, \text{Reach}(\partial\Omega)[$, and moreover, if $\partial\Omega$ has a finite positive reach, then Ω also satisfies the $\text{Reach}(\partial\Omega)$*-ball condition.*

In other words, we have the following characterization:

$$\text{Reach}(\partial\Omega) = \sup\{\varepsilon > 0, \quad \Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)\},$$

with the convention $\sup\emptyset = 0$. *Moreover, this supremum becomes a maximum if it is not zero and finite. Finally, we get* $\text{Reach}(\partial\Omega) = +\infty$ *if and only if $\partial\Omega$ is an affine hyperplane of $\mathbb{R}^n$.*

Theorem 16.6 (**A characterization in terms of $C^{1,1}$-regularity**). *Let Ω be a non-empty open subset of $\mathbb{R}^n$ different from $\mathbb{R}^n$. If there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$, then its boundary $\partial\Omega$ is a $C^{1,1}$-hypersurface of $\mathbb{R}^n$ in the sense of Definition 16.3, where $a = \varepsilon$ and the constants L, r depend only on ε. Moreover, we have the following properties:*

(i) Ω satisfies the $f^{-1}(\varepsilon)$-cone property as in Definition 16.4 with $f : \alpha \in]0, \frac{\pi}{2}[\mapsto \frac{2\alpha}{\cos\alpha} \in]0, +\infty[$;

(ii) the vector $\mathbf{d}_\mathbf{x}$ of Definition 15.1 is the unit outer normal to the hypersurface at the point $\mathbf{x}$;

(iii) the Gauss map $\mathbf{d} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{d}_\mathbf{x} \in \mathbb{S}^{n-1}$ is well defined and $\frac{1}{\varepsilon}$-Lipschitz continuous.

Conversely, if $\mathcal{S}$ is a non-empty compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$ in the sense of Definition 16.3, then there exists $\varepsilon > 0$ such that its inner domain $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. In particular, it has a positive reach with $\text{Reach}(\mathcal{S}) = \max\{\varepsilon > 0,\ \Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)\}$.

Remark 16.7. *In the above assertion, note that a, L, and r only depend on ε for any point of the hypersurface. This uniform dependence of the $C^{1,1}$-regularity characterizes the class $\mathcal{O}_\varepsilon(\mathbb{R}^n)$. Indeed, the converse part of Theorem 16.6 also holds if instead of being compact, the non-empty $C^{1,1}$-hypersurface $\mathcal{S}$ satisfies: $\exists\varepsilon > 0, \forall\mathbf{x}_0 \in \mathcal{S}, \min(\frac{1}{L}, \frac{r}{3}, \frac{a}{3}) \geqslant \varepsilon$. In this case, we still have $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$ where Ω is the open set of Definition 16.3 such that $\partial\Omega = \mathcal{S}$.*

Remark 16.8. *From Point (iii) of Theorem 16.6, the Gauss map $\mathbf{d}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. Hence, it is differentiable almost everywhere and $\|D_\bullet\mathbf{d}\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$ [46, Section 5.2.2]. In particular, the principal curvatures (see Section 18.1 for definitions and* (18.19) *for details) satisfy $\|\kappa_l\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$.*

16.2 The sets of positive reach and the uniform ball condition

Throughout this section, Ω refers to any non-empty open subset of $\mathbb{R}^n$ different from $\mathbb{R}^n$. Hence, its boundary $\partial\Omega$ is not empty and $\text{Reach}(\partial\Omega)$ is well defined (cf. Remark 16.2). First, we establish some properties that were mentioned in Federer's paper [32], then we prove Theorem 16.5.

16.2.1 Positive reach implies uniform ball condition

Lemma 16.9. *For any $\mathbf{x} \in \partial\Omega$, we have:* $\text{Reach}(\partial\Omega, \mathbf{x}) = \min\left(\text{Reach}(\overline{\Omega}, \mathbf{x}), \text{Reach}(\mathbb{R}^n\backslash\Omega, \mathbf{x})\right)$.

Proof. We only sketch the proof. Observe $d(\mathbf{x}, \partial\Omega) = \max(d(\mathbf{x}, \overline{\Omega}), d(\mathbf{x}, \mathbb{R}^n\backslash\Omega))$ for any $\mathbf{x} \in \mathbb{R}^n$ to get $\text{Unp}(\partial\Omega) = \text{Unp}(\overline{\Omega}) \cap \text{Unp}(\mathbb{R}^n\backslash\Omega)$ and the equality of Lemma 16.9 follows from definitions. □

Proposition 16.10 (**Federer [32, Theorem 4.8]**). *Consider any non-empty closed subset A of $\mathbb{R}^n$, a point $\mathbf{x} \in A$, and a vector $\mathbf{v}$ of $\mathbb{R}^n$. If the set $\{t > 0,\ \mathbf{x} + t\mathbf{v} \in \text{Unp}(A)$ and $p_A(\mathbf{x} + t\mathbf{v}) = \mathbf{x}\}$ is not empty and bounded from above, then its supremum τ is well defined and $\mathbf{x} + \tau\mathbf{v}$ cannot belong to the interior of* $\text{Unp}(A)$.

Proof. We refer to [32] for a proof using Peano's Existence Theorem on differential equations. □

Corollary 16.11. *For any point* $\mathbf{x} \in \partial\Omega$ *satisfying* $\mathrm{Reach}(\partial\Omega, \mathbf{x}) > 0$*, there exists two different points* $\mathbf{y} \in \mathrm{Unp}(\overline{\Omega})\backslash\{\mathbf{x}\}$ *and* $\tilde{\mathbf{y}} \in \mathrm{Unp}(\mathbb{R}^n\backslash\Omega)\backslash\{\mathbf{x}\}$ *such that* $p_{\overline{\Omega}}(\mathbf{y}) = p_{\mathbb{R}^n\backslash\Omega}(\tilde{\mathbf{y}}) = \mathbf{x}$.

Proof. Consider $\mathbf{x} \in \partial\Omega$ satisfying $\mathrm{Reach}(\partial\Omega, \mathbf{x}) > 0$. From Lemma 16.9, there exists $r > 0$ such that $\overline{B_r}(\mathbf{x}) \subseteq \mathrm{Unp}(\overline{\Omega})$. Let $(\mathbf{x}_i)_{i\in\mathbb{N}}$ be a sequence of elements in $B_{\frac{r}{2}}(\mathbf{x})\backslash\overline{\Omega}$ converging to $\mathbf{x}$. We set:

$$\forall i \in \mathbb{N},\ \forall t \in \mathbb{R}, \quad \mathbf{z}_i(t) = p_{\overline{\Omega}}(\mathbf{x}_i) + t\frac{\mathbf{x}_i - p_{\overline{\Omega}}(\mathbf{x}_i)}{\|\mathbf{x}_i - p_{\overline{\Omega}}(\mathbf{x}_i)\|} \qquad \text{and} \qquad t_i = \frac{r}{2} + d(\mathbf{x}_i, \overline{\Omega}),$$

which is well defined since $\mathbf{x}_i \in \mathrm{Unp}(\overline{\Omega})$. First, $\mathbf{z}_i(t) \in \overline{B_{\frac{r}{2}}}(\mathbf{x}_i) \subseteq B_r(\mathbf{x}) \subseteq \mathrm{Unp}(\overline{\Omega})$ for any $t \in [0, t_i]$. Then, using Federer's result recalled in Proposition 16.10, one can prove by contradiction that:

$$\forall t \in [0, t_i], \quad p_{\overline{\Omega}}(\mathbf{z}_i(t)) = p_{\overline{\Omega}}(\mathbf{x}_i).$$

Finally, the sequence $\mathbf{y}_i = \mathbf{z}_i(t_i)$ satisfies $\|\mathbf{y}_i - \mathbf{x}_i\| = \frac{r}{2}$ and also $p_{\overline{\Omega}}(\mathbf{y}_i) = p_{\overline{\Omega}}(\mathbf{x}_i)$. Moreover, since it is bounded, $(\mathbf{y}_i)_{i\in\mathbb{N}}$ is converging, up to a subsequence, to a point denoted by $\mathbf{y} \in \overline{B_r}(\mathbf{x}) \subseteq \mathrm{Unp}(\overline{\Omega})$. Using the continuity of $p_{\overline{\Omega}}$ [32, Theorem 4.8 (4)], we get $\mathbf{y} \in \mathrm{Unp}(\overline{\Omega})\backslash\{\mathbf{x}\}$ and $p_{\overline{\Omega}}(\mathbf{y}) = p_{\overline{\Omega}}(\mathbf{x}) = \mathbf{x}$. To conclude, similar arguments work when replacing $\overline{\Omega}$ by the set $\mathbb{R}^n\backslash\Omega$ so Corollary 16.11 holds. □

Proof of Point (ii) in Theorem 16.5. Since $\Omega \notin \{\emptyset, \mathbb{R}^n\}$, $\partial\Omega \neq \emptyset$ thus its reach is well defined. We assume $\mathrm{Reach}(\partial\Omega) > 0$, choose $\varepsilon \in]0, \mathrm{Reach}(\partial\Omega)[$, and consider $\mathbf{x} \in \partial\Omega$. From Corollary 16.11, there exists $\mathbf{y} \in \mathrm{Unp}(\overline{\Omega})\backslash\{\mathbf{x}\}$ such that $p_{\overline{\Omega}}(\mathbf{y}) = \mathbf{x}$ so we can set $\mathbf{d}_\mathbf{x} = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$. From Lemma 16.9, we get $\mathbf{x} + [0, \varepsilon]\mathbf{d}_\mathbf{x} \subseteq \mathrm{Unp}(\overline{\Omega})$. Then, we use Proposition 16.10 again to prove by contradiction that:

$$\forall t \in [0, \varepsilon], \quad p_{\overline{\Omega}}(\mathbf{x} + t\mathbf{d}_\mathbf{x}) = \mathbf{x}.$$

In particular, we have $\|\mathbf{z} - (\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})\| > \varepsilon$ for any point $\mathbf{z} \in \overline{\Omega}\backslash\{\mathbf{x}\}$ from which we deduce that:

$$\overline{\Omega} \subseteq \{\mathbf{x}\} \cup \left(\mathbb{R}^n\backslash\overline{B_\varepsilon}(\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})\right) \quad \Longleftrightarrow \quad \overline{B_\varepsilon}(\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x})\backslash\{\mathbf{x}\} \subseteq \mathbb{R}^n\backslash\overline{\Omega}.$$

Similarly, there exists a unit vector $\xi_\mathbf{x}$ of $\mathbb{R}^n$ such that we get $\overline{B_\varepsilon}(\mathbf{x} + \varepsilon\xi_\mathbf{x})\backslash\{\mathbf{x}\} \subseteq \Omega$. Since we have $\overline{B_\varepsilon}(\mathbf{x} + \varepsilon\xi_\mathbf{x}) \cap \overline{B_\varepsilon}(\mathbf{x} + \varepsilon\mathbf{d}_\mathbf{x}) = \{\mathbf{x}\}$, we obtain $\mathbf{d}_\mathbf{x} = -\xi_\mathbf{x}$. To conclude, if $\mathrm{Reach}(\partial\Omega) < +\infty$, then observe that $B_{\mathrm{Reach}(\partial\Omega)}(\mathbf{x} \pm \mathrm{Reach}(\partial\Omega)\mathbf{d}_\mathbf{x}) = \bigcup_{0<\varepsilon<\mathrm{Reach}(\partial\Omega)} \overline{B_\varepsilon}(\mathbf{x} \pm \varepsilon\mathbf{d}_\mathbf{x})\backslash\{\mathbf{x}\}$ in order to check that Ω also satisfies the $\mathrm{Reach}(\partial\Omega)$-ball condition. □

16.2.2 Uniform ball condition implies positive reach

Proposition 16.12. *Assume there exists* $\varepsilon > 0$ *such that* $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$*. Then, we have:*

$$\forall(\mathbf{x}, \mathbf{y}) \in \partial\Omega \times \partial\Omega, \quad \|\mathbf{d}_\mathbf{x} - \mathbf{d}_\mathbf{y}\| \leqslant \frac{1}{\varepsilon}\|\mathbf{x} - \mathbf{y}\|. \tag{16.1}$$

In particular, if $\mathbf{x} = \mathbf{y}$*, then* $\mathbf{d}_\mathbf{x} = \mathbf{d}_\mathbf{y}$ *which ensures the unit vector* $\mathbf{d}_\mathbf{x}$ *of Definition 15.1 is unique. In other words, the map* $\mathbf{d} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{d}_\mathbf{x}$ *is well defined and* $\frac{1}{\varepsilon}$*-Lipschitz continuous.*

Proof. Let $\varepsilon > 0$ and $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Since $\Omega \notin \{\emptyset, \mathbb{R}^n\}$, $\partial\Omega$ is not empty so choose $(\mathbf{x}, \mathbf{y}) \in \partial\Omega^2 \times \partial\Omega$. First, from the ε-ball condition on $\mathbf{x}$ and $\mathbf{y}$, we have $B_\varepsilon(\mathbf{x} \pm \varepsilon\mathbf{d}_\mathbf{x}) \cap B_\varepsilon(\mathbf{y} \mp \varepsilon\mathbf{d}_\mathbf{y}) = \emptyset$, from which we deduce $\|\mathbf{x} - \mathbf{y} \pm \varepsilon(\mathbf{d}_\mathbf{x} + \mathbf{d}_\mathbf{y})\| \geqslant 2\varepsilon$. Then, squaring these two inequalities and summing them, one obtains the result (16.1) of the statement: $\|\mathbf{x} - \mathbf{y}\|^2 \geqslant 2\varepsilon^2 - 2\varepsilon^2\langle\mathbf{d}_\mathbf{x} \mid \mathbf{d}_\mathbf{y}\rangle = \varepsilon^2\|\mathbf{d}_\mathbf{x} - \mathbf{d}_\mathbf{y}\|^2$. □

Proof of Point (i) in Theorem 16.5. Let $\varepsilon > 0$ and assume that Ω satisfies the ε-ball condition. Since $\Omega \notin \{\emptyset, \mathbb{R}^n\}$, $\partial\Omega$ is not empty so choose any $\mathbf{x} \in \partial\Omega$ and let us prove $B_\varepsilon(\mathbf{x}) \subseteq \mathrm{Unp}(\partial\Omega)$. First, we assume $\mathbf{y} \in B_\varepsilon(\mathbf{x}) \cap \Omega$. Since $\partial\Omega$ is closed, there exists $\mathbf{z} \in \partial\Omega$ such that $d(\mathbf{y}, \partial\Omega) = \|\mathbf{z} - \mathbf{y}\|$. Moreover, we obtain from the ε-ball condition and $\mathbf{y} \in \Omega$:

$$\begin{cases} B_\varepsilon(\mathbf{z} + \varepsilon\mathbf{d}_\mathbf{z}) \subseteq \mathbb{R}^n\backslash\overline{\Omega} \\ \\ B_{d(\mathbf{y},\partial\Omega)}(\mathbf{y}) \subseteq \Omega \end{cases} \quad \Longrightarrow \quad B_\varepsilon(\mathbf{z} + \varepsilon\mathbf{d}_\mathbf{z}) \cap B_{d(\mathbf{y},\partial\Omega)}(\mathbf{y}) = \emptyset.$$

Therefore, we deduce that $\mathbf{y} = \mathbf{z} - d(\mathbf{y}, \partial\Omega)\mathbf{d}_\mathbf{z}$. Then, we show that such a $\mathbf{z}$ is unique. Considering another projection $\tilde{\mathbf{z}}$ of $\mathbf{y}$ on $\partial\Omega$, we get from the foregoing: $\mathbf{y} = \mathbf{z} - d(\mathbf{y}, \partial\Omega)\mathbf{d}_\mathbf{z} = \tilde{\mathbf{z}} - d(\mathbf{y}, \partial\Omega)\mathbf{d}_{\tilde{\mathbf{z}}}$. Using (16.1), we have:

$$\|\mathbf{d}_\mathbf{z} - \mathbf{d}_{\tilde{\mathbf{z}}}\| \leqslant \frac{1}{\varepsilon}\|\mathbf{z} - \tilde{\mathbf{z}}\| = \frac{d(\mathbf{y}, \partial\Omega)}{\varepsilon}\|\mathbf{d}_\mathbf{z} - \mathbf{d}_{\tilde{\mathbf{z}}}\|.$$

Since $d(\mathbf{y}, \partial\Omega) \leqslant \|\mathbf{x}-\mathbf{y}\| < \varepsilon$, the above inequality can only hold if $\|\mathbf{d_z}-\mathbf{d_{\tilde{z}}}\| = 0$ i.e. $\mathbf{z} = \tilde{\mathbf{z}}$. Hence, we obtain $B_\varepsilon(\mathbf{x}) \cap \Omega \subseteq \mathrm{Unp}(\partial\Omega)$ and similarly, one can prove that $B_\varepsilon(\mathbf{x}) \cap (\mathbb{R}^n\backslash\overline{\Omega}) \subseteq \mathrm{Unp}(\partial\Omega)$. Since $\partial\Omega \subseteq \mathrm{Unp}(\partial\Omega)$, we finally get $B_\varepsilon(\mathbf{x}) \subseteq \mathrm{Unp}(\partial\Omega)$. To conclude, we have $\mathrm{Reach}(\partial\Omega, \mathbf{x}) \geqslant \varepsilon$ for every $\mathbf{x} \in \partial\Omega$ i.e. $\mathrm{Reach}(\partial\Omega) \geqslant \varepsilon$ as required. ☐

Proposition 16.13. *Assume there exists $\varepsilon > 0$ such that $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Then, we have:*

$$\forall(\mathbf{a}, \mathbf{x}) \in \partial\Omega \times \partial\Omega, \quad |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a} \rangle| \leqslant \frac{1}{2\varepsilon}\|\mathbf{x} - \mathbf{a}\|^2. \tag{16.2}$$

Moreover, introducing the vector $(\mathbf{x} - \mathbf{a})' = (\mathbf{x} - \mathbf{a}) - \langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle\mathbf{d_a}$, if we assume $\|(\mathbf{x} - \mathbf{a})'\| < \varepsilon$ and $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle| < \varepsilon$, then the following local inequality holds:

$$\frac{1}{2\varepsilon}\|\mathbf{x} - \mathbf{a}\|^2 \leqslant \varepsilon - \sqrt{\varepsilon^2 - \|(\mathbf{x} - \mathbf{a})'\|^2}. \tag{16.3}$$

Proof. Let $\varepsilon > 0$ and $\Omega \in \mathcal{O}_\varepsilon(\mathbb{R}^n)$. Since $\Omega \notin \{\emptyset, \mathbb{R}^n\}$, $\partial\Omega$ is not empty so choose $(\mathbf{a}, \mathbf{x}) \in \partial\Omega \times \partial\Omega$. Observe that the point $\mathbf{x}$ cannot belong neither to $B_\varepsilon(\mathbf{a} - \varepsilon\mathbf{d_a}) \subseteq \Omega$ nor to $B_\varepsilon(\mathbf{a} + \varepsilon\mathbf{d_a}) \subseteq \mathbb{R}^n\backslash\overline{\Omega}$. Hence, we have $\|\mathbf{x} - \mathbf{a} \mp \varepsilon\mathbf{d_a}\| \geqslant \varepsilon$. Squaring these two inequalities, we obtain (16.2):

$$\|\mathbf{x} - \mathbf{a}\|^2 \geqslant 2\varepsilon|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle| \Longleftrightarrow |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle|^2 - 2\varepsilon|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle| + \|(\mathbf{x} - \mathbf{a})'\|^2 \geqslant 0.$$

It is a second-order polynomial inequality and we assume that its reduced discriminant is positive: $\Delta' = \varepsilon^2 - \|(\mathbf{x} - \mathbf{a})'\|^2 > 0$. Hence, the unknown cannot be located between the two roots: either $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle| \leqslant \varepsilon - \sqrt{\Delta'}$ or $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle| \geqslant \varepsilon + \sqrt{\Delta'}$. We assume $|\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle| < \varepsilon$ and the last case cannot hold. Squaring the remaining relation, we get the local inequality (16.3) of the statement: $\|\mathbf{x} - \mathbf{a}\|^2 = |\langle \mathbf{x} - \mathbf{a} \mid \mathbf{d_a}\rangle|^2 + \|(\mathbf{x} - \mathbf{a})'\|^2 \leqslant 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \|(\mathbf{x} - \mathbf{a})'\|^2}$. ☐

16.3 Uniform ball condition and compact $C^{1,1}$-hypersurfaces

In this section, Theorem 16.6 is proved. First, we show $\partial\Omega$ can be considered locally as the graph of a function whose $C^{1,1}$-regularity is then established. Finally, we demonstrate that the converse statement holds in the compact case. Hence, it is the optimal regularity we can expect from the uniform ball property. The proofs in Sections 16.2.2, 16.3.1, and 16.3.2 inspire those of Section 17.

16.3.1 A local parametrization of the boundary $\partial\Omega$

We now set $\varepsilon > 0$ and assume that the open set Ω satisfies the ε-ball condition. Since $\Omega \notin \{\emptyset, \mathbb{R}^n\}$, $\partial\Omega$ is not empty so we consider any point $\mathbf{x}_0 \in \partial\Omega$ and its unique vector $\mathbf{d}_{\mathbf{x}_0}$ from Proposition 16.12. We choose a basis $\mathcal{B}_{\mathbf{x}_0}$ of the hyperplane $\mathbf{d}_{\mathbf{x}_0}^\perp$ so that $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ is a direct orthonormal frame. Inside this frame, any point $\mathbf{x} \in \mathbb{R}^n$ is of the form $(\mathbf{x}', x_n)$ such that $\mathbf{x}' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. The zero vector $\mathbf{0}$ of $\mathbb{R}^n$ is now identified with $\mathbf{x}_0$ so we have $B_\varepsilon(\mathbf{0}', -\varepsilon) \subseteq \Omega$ and $B_\varepsilon(\mathbf{0}', \varepsilon) \subseteq \mathbb{R}^n\backslash\overline{\Omega}$.

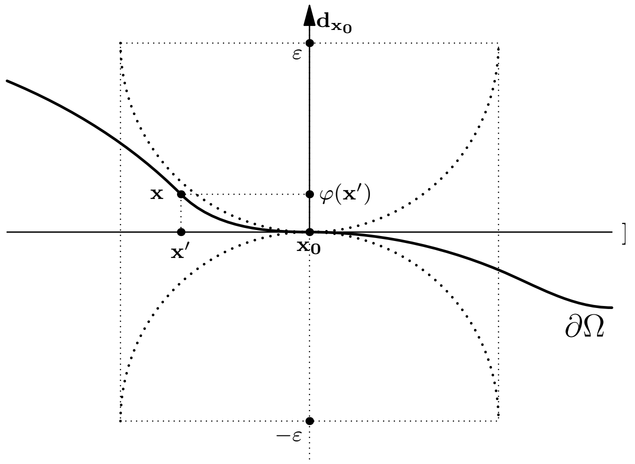

Figure 16.2: The orthonormal frame centred at $\mathbf{x_0}$ describing locally $\partial\Omega$ as the graph of a map φ.

Proposition 16.14. *The following maps $\varphi^{\pm}$ are well defined on $D_\varepsilon(\mathbf{0}') = \{\mathbf{x}' \in \mathbb{R}^{n-1},\ \|\mathbf{x}'\| < \varepsilon\}$:*

$$\begin{cases} \varphi^+ : & D_\varepsilon(\mathbf{0}') & \longrightarrow &]-\varepsilon,\varepsilon[\\ & \mathbf{x}' & \longmapsto & \sup\{x_n \in [-\varepsilon,\varepsilon], \quad (\mathbf{x}',x_n) \in \Omega\} \\ & & & \\ \varphi^- : & D_\varepsilon(\mathbf{0}') & \longrightarrow &]-\varepsilon,\varepsilon[\\ & \mathbf{x}' & \longmapsto & \inf\{x_n \in [-\varepsilon,\varepsilon], \quad (\mathbf{x}',x_n) \in \mathbb{R}^n\backslash\overline{\Omega}\}. \end{cases}$$

Moreover, for any $\mathbf{x}' \in D_\varepsilon(\mathbf{0}')$, introducing the points $\mathbf{x}^{\pm} = (\mathbf{x}', \varphi^{\pm}(\mathbf{x}'))$, we have $\mathbf{x}^{\pm} \in \partial\Omega$ and:

$$|\varphi^{\pm}(\mathbf{x}')| \leqslant \frac{1}{2\varepsilon}\|\mathbf{x}^{\pm} - \mathbf{x}_0\|^2 \leqslant \varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}. \tag{16.4}$$

Proof. Let $\mathbf{x}' \in D_\varepsilon(\mathbf{0}')$ and $g : t \in [-\varepsilon,\varepsilon] \mapsto (\mathbf{x}',t)$. Since $-\varepsilon \in g^{-1}(\Omega) \subseteq [-\varepsilon,\varepsilon]$, we can set $\varphi^+(\mathbf{x}') = \sup g^{-1}(\Omega)$. The map g is continuous so $g^{-1}(\Omega)$ is open and $\varphi^+(\mathbf{x}') \neq \varepsilon$ thus we get $\varphi(\mathbf{x}') \notin g^{-1}(\Omega)$ i.e. $\mathbf{x}^+ \in \overline{\Omega}\backslash\Omega$. Similarly, the map φ^- is well defined and $\mathbf{x}^- \in \partial\Omega$. Finally, we use (16.2) and (16.3) on the points $\mathbf{x}_0$ and $\mathbf{x} = \mathbf{x}^{\pm}$ in order to obtain (16.4). □

Lemma 16.15. *Let $r = \frac{\sqrt{3}}{2}\varepsilon$ and $\mathbf{x}' \in D_r(\mathbf{0}')$. We assume that there exists $x_n \in]-\varepsilon,\varepsilon[$ such that $\mathbf{x} = (\mathbf{x}', x_n) \in \partial\Omega$ and $\tilde{x}_n \in \mathbb{R}$ such that $|\tilde{x}_n| \leqslant \varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}$. Then, we introduce $\tilde{\mathbf{x}} = (\mathbf{x}', \tilde{x}_n)$ and the two following implications hold: $(\tilde{x}_n < x_n \Longrightarrow \tilde{\mathbf{x}} \in \Omega)$ and $(\tilde{x}_n > x_n \Longrightarrow \tilde{\mathbf{x}} \in \mathbb{R}^n\backslash\overline{\Omega})$.*

Proof. Let $\mathbf{x}' \in D_r(\mathbf{0}')$. Since $\tilde{\mathbf{x}} - \mathbf{x} = (\tilde{x}_n - x_n)\mathbf{d}_{\mathbf{x}_0}$, if we assume $\tilde{x}_n > x_n$, then we have:

$$\begin{aligned} \|\tilde{\mathbf{x}} - \mathbf{x} - \varepsilon\mathbf{d}_{\mathbf{x}}\|^2 - \varepsilon^2 &= |\tilde{x}_n - x_n|\left(|\tilde{x}_n - x_n| + \varepsilon\|\mathbf{d}_{\mathbf{x}} - \mathbf{d}_{\mathbf{x}_0}\|^2 - 2\varepsilon\right) \\ &\leqslant |\tilde{x}_n - x_n|\left(|\tilde{x}_n| + |x_n| + \tfrac{1}{\varepsilon}\|\mathbf{x} - \mathbf{x}_0\|^2 - 2\varepsilon\right) \\ &\leqslant |\tilde{x}_n - x_n|\left(2\varepsilon - 4\sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}\right) < |\tilde{x}_n - x_n|\left(2\varepsilon - 4\sqrt{\varepsilon^2 - r^2}\right) = 0. \end{aligned}$$

Indeed, we used (16.1) with $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} = \mathbf{x}_0$, (16.2) and (16.3) applied to $\mathbf{x} \in \partial\Omega$ and $\mathbf{a} = \mathbf{x}_0$, and also the hypothesis made on $\tilde{x}_n$. Hence, we proved that if $\tilde{x}_n > x_n$, then $\tilde{\mathbf{x}} \in B_\varepsilon(\mathbf{x} + \varepsilon\mathbf{d}_{\mathbf{x}}) \subseteq \mathbb{R}^n\backslash\overline{\Omega}$. Similarly, one can prove that if $\tilde{x}_n < x_n$, then we have $\tilde{\mathbf{x}} \in B_\varepsilon(\mathbf{x} - \varepsilon\mathbf{d}_{\mathbf{x}}) \subseteq \Omega$. □

Proposition 16.16. *Set $r = \frac{\sqrt{3}}{2}\varepsilon$. Then, the two maps $\varphi^{\pm}$ of Proposition 16.14 coincide on $D_r(\mathbf{0}')$. We denote by φ their common restriction. Moreover, we have $\varphi(\mathbf{0}') = 0$ and also:*

$$\begin{cases} \partial\Omega \cap (D_r(\mathbf{0}')\times]-\varepsilon,\varepsilon[) &= \{(\mathbf{x}', \varphi(\mathbf{x}')), \quad \mathbf{x}' \in D_r(\mathbf{0}')\} \\ & \\ \Omega \cap (D_r(\mathbf{0}')\times]-\varepsilon,\varepsilon[) &= \{(\mathbf{x}', x_n), \quad \mathbf{x}' \in D_r(\mathbf{0}') \text{ and } -\varepsilon < x_n < \varphi(\mathbf{x}')\}. \end{cases}$$

Proof. Assume by contradiction that there exists $\mathbf{x}' \in D_r(\mathbf{0}')$ such that $\varphi^-(\mathbf{x}') \neq \varphi^+(\mathbf{x}')$. We set $\mathbf{x} = (\mathbf{x}', \varphi^+(\mathbf{x}'))$ and $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi^-(\mathbf{x}'))$. By using (16.4), the hypothesis of Lemma 16.15 are satisfied for $\mathbf{x}$ and $\tilde{\mathbf{x}}$. Hence, either $(\varphi^-(\mathbf{x}') < \varphi^+(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}} \in \Omega)$ or $(\varphi^-(\mathbf{x}') > \varphi^+(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}} \in \mathbb{R}^n\backslash\overline{\Omega})$ whereas $\tilde{\mathbf{x}} \in \partial\Omega$. We deduce $\varphi^-(\mathbf{x}') = \varphi^+(\mathbf{x}')$ for any $\mathbf{x}' \in D_r(\mathbf{0}')$. Now consider $\mathbf{x}' \in D_r(\mathbf{0}')$ and $x_n \in]-\varepsilon,\varepsilon[$. We set $\mathbf{x} = (\mathbf{x}', \varphi(\mathbf{x}'))$ and $\tilde{\mathbf{x}} = (\mathbf{x}', x_n)$. If $x_n = \varphi(\mathbf{x}')$, then Proposition 16.14 ensures that $\mathbf{x} \in \partial\Omega$. Moreover, if $-\varepsilon < x_n < -\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}$, then $\tilde{\mathbf{x}} \in B_\varepsilon(\mathbf{0}', -\varepsilon) \subseteq \Omega$, and if $-\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2} \leqslant x_n < \varphi(\mathbf{x}')$, then apply Lemma 16.15 to get $\tilde{\mathbf{x}} \in \Omega$. Consequently, we proved $(-\varepsilon < x_n < \varphi(\mathbf{x}') \Longrightarrow (\mathbf{x}', x_n) \in \Omega)$ for any $\mathbf{x}' \in D_r(\mathbf{0}')$. Similar arguments hold when $\varepsilon > x_n > \varphi(\mathbf{x}')$ and imply $(\mathbf{x}', x_n) \in \mathbb{R}^n\backslash\overline{\Omega}$. To conclude, note that $\mathbf{x}_0 = \mathbf{0} = (\mathbf{0}', \varphi(\mathbf{0}'))$. □

16.3.2 The $C^{1,1}$-regularity of the local graph

Lemma 16.17. *The map $f : \alpha \in]0, \frac{\pi}{2}[\mapsto \frac{2\alpha}{\cos\alpha} \in]0, +\infty[$ is well defined, continuous, surjective and increasing. In particular, it is an homeomorphism and its inverse f^{-1} satisfies:*

$$\forall \varepsilon > 0, \quad f^{-1}(\varepsilon) < \frac{\varepsilon}{2}. \tag{16.5}$$

Proof. The proof is basic calculus. □

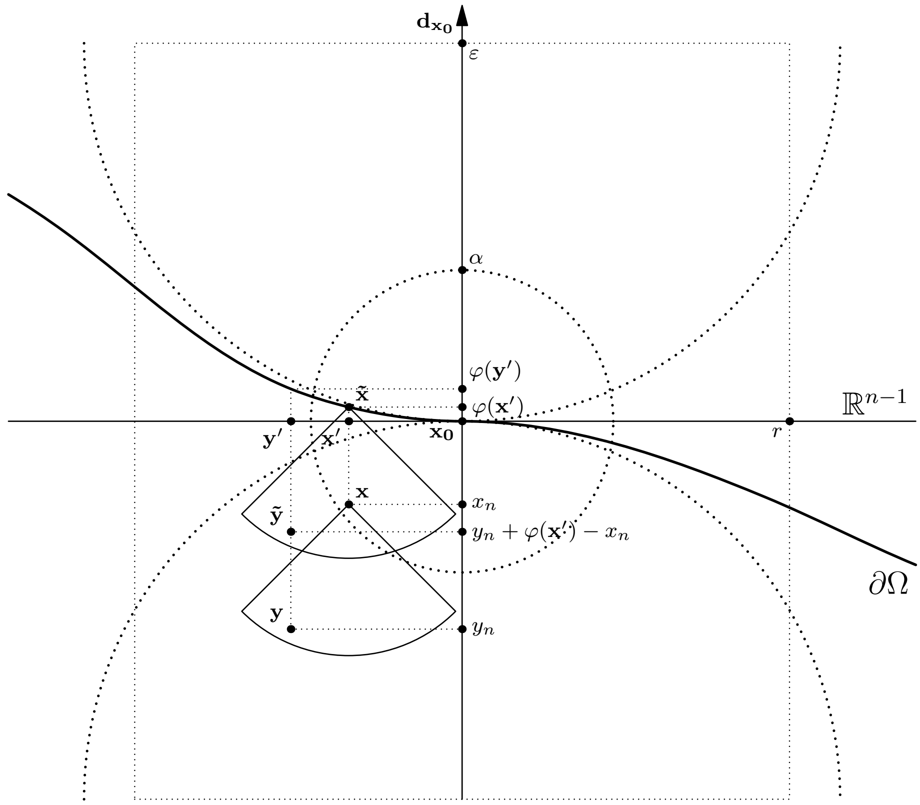

Figure 16.3: Scheme associated with the proof of Proposition 16.18.

Proposition 16.18 (**Point (i) of Theorem 16.6**). *Consider any $\alpha \in]0, f^{-1}(\varepsilon)]$ where f is defined in Lemma 16.17. Then, we have $C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega$ for any $\mathbf{x} \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega}$. In particular, the set Ω satisfies the $f^{-1}(\varepsilon)$-cone property in the sense of Definition 16.4.*

Proof. We set $r = \frac{\sqrt{3}}{2}\varepsilon$ and $\mathcal{C}_{r,\varepsilon} = D_r(\mathbf{0}')\times]-\varepsilon,\varepsilon[$. We choose any $\alpha \in]0, f^{-1}(\varepsilon)]$ then consider $\mathbf{x} = (\mathbf{x}', x_n) \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega}$ and $\mathbf{y} = (\mathbf{y}', y_n) \in C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0})$. The proof of the assertion $\mathbf{y} \in \Omega$ is divided into three steps illustrated in Figure 16.3:

- check that $\mathbf{x} \in \mathcal{C}_{r,\varepsilon}$ so as to introduce the point $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi(\mathbf{x}'))$ of $\partial\Omega$ satisfying $x_n \leqslant \varphi(\mathbf{x}')$;
- consider $\tilde{\mathbf{y}} = (\mathbf{y}', y_n + \varphi(\mathbf{x}') - x_n)$ and prove that $\tilde{\mathbf{y}} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}}) \subseteq \Omega$;
- show that $(\tilde{\mathbf{y}}, \mathbf{y}) \in \mathcal{C}_{r,\varepsilon} \times \mathcal{C}_{r,\varepsilon}$ in order to deduce $y_n + \varphi(\mathbf{x}') - x_n < \varphi(\mathbf{y}')$ and conclude $\mathbf{y} \in \Omega$.

First, from (16.5), we have: $\max(\|\mathbf{x}'\|, |x_n|) \leqslant \|\mathbf{x} - \mathbf{x}_0\| < \alpha \leqslant f^{-1}(\varepsilon) < \frac{\varepsilon}{2}$. Hence, we get $\mathbf{x} \in \overline{\Omega} \cap \mathcal{C}_{r,\varepsilon}$ and applying Proposition 16.16, it comes $x_n \leqslant \varphi(\mathbf{x}')$. We set $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi(\mathbf{x}')) \in \partial\Omega \cap \mathcal{C}_{r,\varepsilon}$. Note that $\tilde{\mathbf{x}} \in B_{\alpha\sqrt{2}}(\mathbf{x}_0)$ because Relation (16.4) applied to $\tilde{\mathbf{x}} = (\mathbf{x}', \varphi(\mathbf{x}'))$ gives:

$$\|\tilde{\mathbf{x}} - \mathbf{x}_0\|^2 \leqslant 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2} = \frac{4\varepsilon^2\|\mathbf{x}'\|^2}{2\varepsilon^2 + 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}} \leqslant 2\|\mathbf{x}'\|^2 \leqslant 2\|\mathbf{x} - \mathbf{x}_0\|^2 < 2\alpha^2.$$

Then, we prove $C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}})$ so consider any point $\mathbf{z} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0})$. Using the Cauchy-Schwarz inequality, (16.1) applied to $\tilde{\mathbf{x}} \in \partial\Omega$ and $\mathbf{y} = \mathbf{x}_0$, the fact that $\mathbf{z} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0})$, and the foregoing observation $\tilde{\mathbf{x}} \in B_{\alpha\sqrt{2}}(\mathbf{x}_0)$, we have successively:

$$\begin{array}{rcl} \|\mathbf{z} - \tilde{\mathbf{x}} + \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}}\|^2 - \varepsilon^2 & \leqslant & \|\mathbf{z} - \tilde{\mathbf{x}}\|^2 + 2\varepsilon\|\mathbf{z} - \tilde{\mathbf{x}}\|\|\mathbf{d}_{\tilde{\mathbf{x}}} - \mathbf{d}_{\mathbf{x}_0}\| + 2\varepsilon\left\langle \mathbf{z} - \tilde{\mathbf{x}} \mid \mathbf{d}_{\mathbf{x}_0}\right\rangle \\ & < & \|\mathbf{z} - \tilde{\mathbf{x}}\|^2 + 2\|\mathbf{z} - \tilde{\mathbf{x}}\|\|\tilde{\mathbf{x}} - \mathbf{x}_0\| - 2\varepsilon\|\mathbf{z} - \tilde{\mathbf{x}}\|\cos\alpha \\ & < & \|\mathbf{z} - \tilde{\mathbf{x}}\|\left[\left(1 + 2\sqrt{2}\right)\alpha - 2\varepsilon\cos\alpha\right] \;<\; 2\|\mathbf{z} - \tilde{\mathbf{x}}\|\cos\alpha\,(f(\alpha) - \varepsilon) \leqslant 0. \end{array}$$

Hence, we get $\mathbf{z} \in B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon \mathbf{d}_{\tilde{\mathbf{x}}})$ i.e. $C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}} - \varepsilon \mathbf{d}_{\tilde{\mathbf{x}}}) \subseteq \Omega$ using the ε-ball condition. Moreover, since $\tilde{\mathbf{y}} - \tilde{\mathbf{x}} = \mathbf{y} - \mathbf{x}$ and $\mathbf{y} \in C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0})$, we obtain $\tilde{\mathbf{y}} \in C_\alpha(\tilde{\mathbf{x}}, -\mathbf{d}_{\mathbf{x}_0})$ and thus $\tilde{\mathbf{y}} \in \Omega$. Finally, we show that $(\mathbf{y}, \tilde{\mathbf{y}}) \in \mathcal{C}_{r,\varepsilon} \times \mathcal{C}_{r,\varepsilon}$. We have successively:

$$\begin{cases} \|\mathbf{y}'\| \leqslant \|\mathbf{y}' - \mathbf{x}'\| + \|\mathbf{x}'\| < \sqrt{\alpha^2 - \alpha^2 \cos^2 \alpha} + \alpha = \dfrac{\alpha}{\cos \alpha} \left(\dfrac{1}{2} \sin 2\alpha + \cos \alpha \right) \leqslant \frac{3f(\alpha)}{4} \leqslant \frac{3\varepsilon}{4} < r \\ |y_n| \leqslant |y_n - x_n| + |x_n| \leqslant \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < 2\alpha < f(\alpha) \leqslant \varepsilon \\ |y_n + \varphi(\mathbf{x}') - x_n| \leqslant \|\mathbf{y} - \mathbf{x}\| + \varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2} < \alpha + \dfrac{\|\mathbf{x}'\|^2}{\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{x}'\|^2}} \leqslant \alpha + \frac{\alpha^2}{\varepsilon} < \frac{3}{2}\alpha \leqslant \varepsilon. \end{cases}$$

We used (16.4), (16.5), the fact that $\mathbf{y} \in C_\alpha(\mathbf{x}, -\mathbf{d}_{\mathbf{x}_0})$, and $\mathbf{x} \in B_\alpha(\mathbf{x}_0)$. To conclude, apply Proposition 16.16 to $\tilde{\mathbf{y}} \in \Omega \cap \mathcal{C}_{r,\varepsilon}$ in order to obtain $y_n + \varphi(\mathbf{x}') - x_n < \varphi(\mathbf{y}')$. Since we firstly proved $x_n \leqslant \varphi(\mathbf{x}')$, we deduce $y_n < \varphi(\mathbf{y}')$. Applying Proposition 16.16 to $\mathbf{y} \in \mathcal{C}_{r,\varepsilon}$, we get $\mathbf{y} \in \Omega$ as required. ☐

Corollary 16.19. *The map φ restricted to $D_{\frac{\sqrt{2}}{4} f^{-1}(\varepsilon)}(\mathbf{0}')$ is $\frac{1}{\tan[f^{-1}(\varepsilon)]}$-Lipschitz continuous.*

Proof. We set $\alpha = f^{-1}(\varepsilon)$, $r = \frac{\sqrt{3}}{2}\varepsilon$, and $\tilde{r} = \frac{\sqrt{2}}{4} f^{-1}(\varepsilon)$. We choose any $(\mathbf{x}'_+, \mathbf{x}'_-) \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. From (16.5), we get $\tilde{r} < r$ so we can consider $\mathbf{x}_\pm = (\mathbf{x}'_\pm, \varphi(\mathbf{x}'_\pm))$ and Proposition 16.14 gives:

$$\|\mathbf{x}_\pm - \mathbf{x}_0\|^2 \leqslant 2\varepsilon^2 - 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'_\pm\|^2} = \frac{4\varepsilon^2 \|\mathbf{x}'_\pm\|^2}{2\varepsilon^2 + 2\varepsilon\sqrt{\varepsilon^2 - \|\mathbf{x}'_\pm\|^2}} \leqslant 2\|\mathbf{x}'_\pm\|^2 < 2\tilde{r}^2 < \alpha^2.$$

Hence, we obtain $\mathbf{x}_\pm \in B_\alpha(\mathbf{x}_0) \cap \partial\Omega$. We also have: $\|\mathbf{x}_+ - \mathbf{x}_-\| \leqslant \|\mathbf{x}_+ - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}_-\| < 2\tilde{r}\sqrt{2} = \alpha$. Finally, applying Proposition 16.18, the points $\mathbf{x}_\pm$ cannot belong to the cones $C_\alpha(\mathbf{x}_\mp, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega$ thus we get: $|\langle \mathbf{x}_+ - \mathbf{x}_- \mid \mathbf{d}_{\mathbf{x}_0} \rangle| \leqslant \cos\alpha \|\mathbf{x}_+ - \mathbf{x}_-\| = \cos\alpha \sqrt{\|\mathbf{x}'_+ - \mathbf{x}'_-\|^2 + |\langle \mathbf{x}_+ - \mathbf{x}_- \mid \mathbf{d}_{\mathbf{x}_0} \rangle|^2}$. Consequently, one can re-arrange these terms in order to obtain the result of the statement: $|\varphi(\mathbf{x}'_+) - \varphi(\mathbf{x}'_-)| = |\langle \mathbf{x}_+ - \mathbf{x}_- \mid \mathbf{d}_{\mathbf{x}_0} \rangle| \leqslant \frac{1}{\tan\alpha} \|\mathbf{x}'_+ - \mathbf{x}'_-\|$. ☐

Proposition 16.20. *Set $\tilde{r} = \frac{\sqrt{2}}{4} f^{-1}(\varepsilon)$. The map φ of Proposition 16.16 restricted to $D_{\tilde{r}}(\mathbf{0}')$ is differentiable and its gradient $\nabla\varphi : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is L-Lipschitz continuous where $L > 0$ depends only on ε. Moreover, we have $\nabla\varphi(\mathbf{0}') = \mathbf{0}'$ and also:*

$$\forall \mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}'), \quad \nabla\varphi(\mathbf{a}') = \frac{-1}{\langle \mathbf{d}_\mathbf{a} \mid \mathbf{d}_{\mathbf{x}_0} \rangle} \mathbf{d}'_\mathbf{a}, \qquad \text{where} \quad \mathbf{a} = (\mathbf{a}', \varphi(\mathbf{a}')).$$

Proof. Let $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ and $\mathbf{x}' \in \overline{D_{\tilde{r} - \|\mathbf{a}'\|}}(\mathbf{a}')$. Consequently, we have $(\mathbf{a}', \mathbf{x}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$ and from (16.5), we get $\tilde{r} < \frac{\sqrt{3}}{2}\varepsilon$. Hence, using Proposition 16.16, we can introduce $\mathbf{x} := (\mathbf{x}', \varphi(\mathbf{x}'))$ and $\mathbf{a} := (\mathbf{a}', \varphi(\mathbf{a}'))$. Applying (16.2) to $(\mathbf{a}, \mathbf{x}) \in \partial\Omega \times \partial\Omega$ and using the Lipschitz continuity of φ on $D_{\tilde{r}}(\mathbf{0}')$ proved in Corollary 16.19, we deduce that:

$$|(\varphi(\mathbf{x}') - \varphi(\mathbf{a}'))\, \mathbf{d}_{\mathbf{a}n} + \langle \mathbf{d}'_\mathbf{a} \mid \mathbf{x}' - \mathbf{a}' \rangle| \leqslant \frac{1}{2\varepsilon} \|\mathbf{x} - \mathbf{a}\|^2 \leqslant \underbrace{\frac{1}{2\varepsilon} \left(1 + \frac{1}{\tan^2[f^{-1}(\varepsilon)]} \right)}_{:= C(\varepsilon) > 0} \|\mathbf{x}' - \mathbf{a}'\|^2,$$

where we set $\mathbf{d}_\mathbf{a} = (\mathbf{d}'_\mathbf{a}, \mathbf{d}_{\mathbf{a}n})$ with $\mathbf{d}_{\mathbf{a}n} = \langle \mathbf{d}_\mathbf{a} \mid \mathbf{d}_{\mathbf{x}_0} \rangle$. It represents a first-order Taylor expansion of the map φ if we can divide the above inequality by a uniform positive constant smaller than $\mathbf{d}_{\mathbf{a}n}$. Let us justify this assertion. Apply (16.1) to $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{x}_0$, then use (16.4) to get:

$$\mathbf{d}_{\mathbf{a}n} = 1 - \frac{1}{2} \|\mathbf{d}_\mathbf{a} - \mathbf{d}_{\mathbf{x}_0}\|^2 \geqslant 1 - \frac{1}{2\varepsilon^2} \|\mathbf{a} - \mathbf{x}_0\|^2 \geqslant 1 - \frac{\varepsilon - \sqrt{\varepsilon^2 - \|\mathbf{a}'\|^2}}{\varepsilon} = 1 - \frac{\|\mathbf{a}'\|^2}{\varepsilon(\varepsilon + \sqrt{\varepsilon^2 - \|\mathbf{a}'\|^2})}.$$

Hence, using (16.5), we obtain $\mathbf{d}_{\mathbf{a}n} > 1 - \frac{\tilde{r}^2}{\varepsilon^2} > \frac{31}{32} > 0$. Therefore, φ is a differentiable map at any point $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ and its gradient is the one given in the statement:

$$\forall \mathbf{x}' \in \overline{D_{\tilde{r} - \|\mathbf{a}'\|}}(\mathbf{a}'), \quad \left| \varphi(\mathbf{x}') - \varphi(\mathbf{a}') + \left\langle \frac{\mathbf{d}'_\mathbf{a}}{\mathbf{d}_{\mathbf{a}n}} \mid \mathbf{x}' - \mathbf{a}' \right\rangle \right| \leqslant \frac{32}{31} C(\varepsilon) \|\mathbf{x}' - \mathbf{a}'\|^2.$$

Moreover, for any $(\mathbf{a}',\mathbf{x}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$, we have successively:

$$\begin{aligned}\|\nabla\varphi(\mathbf{x}') - \nabla\varphi(\mathbf{a}')\| &\leqslant \left| \frac{1}{\mathbf{d_a}_n} - \frac{1}{\mathbf{d_x}_n} \right| \|\mathbf{d}'_\mathbf{x}\| + \left| \frac{1}{\mathbf{d_a}_n} \right| \|\mathbf{d}'_\mathbf{a} - \mathbf{d}'_\mathbf{x}\| \leqslant \left(\frac{32^2}{31^2} + \frac{32}{31}\right) \|\mathbf{d_a} - \mathbf{d_x}\| \\ &\leqslant \frac{32}{31\varepsilon}\left(1 + \frac{32}{31}\right)\|\mathbf{x}-\mathbf{a}\| \leqslant \frac{32}{31\varepsilon}\left(1+\frac{32}{31}\right)\sqrt{1 + \frac{1}{\tan^2[f^{-1}(\varepsilon)]}}\|\mathbf{x}'-\mathbf{a}'\|.\end{aligned}$$

We applied (16.1) to $\mathbf{x}$ and $\mathbf{y} = \mathbf{a}$, then used the Lipschitz continuity of φ proved in Corollary 16.19. Hence, $\nabla\varphi : \mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}') \mapsto \nabla\varphi(\mathbf{a}')$ is L-Lipschitz continuous with $L > 0$ depending only on ε. □

Corollary 16.21 (Points (ii) and (iii) of Theorem 16.6). *The unit vector $\mathbf{d}_{\mathbf{x}_0}$ of Definition 15.1 is the outer normal to $\partial\Omega$ at the point $\mathbf{x}_0$. In particular, the $\frac{1}{\varepsilon}$-Lipschitz continuous map $\mathbf{d} : \mathbf{x} \mapsto \mathbf{d_x}$ of Proposition 16.12 is the Gauss map of the $C^{1,1}$-hypersurface $\partial\Omega$.*

Proof. Consider the map $\varphi : D_{\tilde{r}}(\mathbf{0}') \to]-\varepsilon,\varepsilon[$ whose $C^{1,1}$-regularity comes from Proposition 16.20. We define the $C^{1,1}$-map $X : D_{\tilde{r}}(\mathbf{0}') \to \partial\Omega$ by $X(\mathbf{x}') = (\mathbf{x}',\varphi(\mathbf{x}'))$ then we consider $\mathbf{x}' \in D_{\tilde{r}}(\mathbf{0}')$. We denote by $(e_k)_{1\leqslant k\leqslant n-1}$ the first vectors of our local basis. The tangent plane of $\partial\Omega$ at $X(\mathbf{x}')$ is spanned by the vectors $\partial_k X(\mathbf{x}') = e_k + (\mathbf{0}', \partial_k\varphi(\mathbf{x}'))$. Since any normal vector $\mathbf{u} = (u_1,\ldots,u_n)$ to this hyperplane is orthogonal to this $(n-1)$ vectors, we have: $\langle \mathbf{u} \mid \partial_k X(\mathbf{x}')\rangle = 0 \Leftrightarrow u_k = \frac{u_n}{\mathbf{d_x}_n}\mathbf{d_x}_k$. Hence, we obtain $\mathbf{u} = \frac{u_n}{\mathbf{d_x}_n}\mathbf{d_x}$ so $\mathbf{u}$ is collinear to $\mathbf{d_x}$. Now, if we impose that $\mathbf{u}$ points outwards Ω and if we assume $\|\mathbf{u}\| = 1$, then we get $\mathbf{u} = \mathbf{d_x}$. □

16.3.3 The compact case: when $C^{1,1}$-regularity implies the uniform ball condition

Proof of Theorem 16.6. Combining Proposition 16.18 and Corollary 16.21, it remains to prove the converse part of Theorem 16.6. Consider any non-empty compact $C^{1,1}$-hypersurface $\mathcal{S}$ of $\mathbb{R}^n$ and its associated inner domain Ω. Choose any $\mathbf{x}_0 \in \partial\Omega$ and its local frame as in Definition 16.3. First, we have for any $(\mathbf{x}',\mathbf{y}') \in D_r(\mathbf{0}') \times D_r(\mathbf{0}')$ with $g : t \in [0,1] \mapsto \varphi(\mathbf{x}' + t(\mathbf{y}'-\mathbf{x}'))$:

$$\begin{aligned}|\varphi(\mathbf{y}') - \varphi(\mathbf{x}') - \langle\nabla\varphi(\mathbf{x}') \mid \mathbf{y}'-\mathbf{x}'\rangle| &= \left|\int_0^1 [g'(t) - g'(0)]\,dt\right| \leqslant \int_0^1 |g'(t)-g'(0)|dt \\ &\leqslant \int_0^1 \|\nabla\varphi(\mathbf{x}' + t(\mathbf{y}'-\mathbf{x}')) - \nabla\varphi(\mathbf{x}')\|\|\mathbf{y}'-\mathbf{x}'\|dt \\ &\leqslant \frac{L}{2}\|\mathbf{y}'-\mathbf{x}'\|^2.\end{aligned}$$

Then, we set $\varepsilon_0 = \min(\frac{1}{L},\frac{r}{3},\frac{a}{3})$ and consider any $\mathbf{x} \in B_{\varepsilon_0}(\mathbf{x}_0) \cap \partial\Omega$. Since $\varepsilon_0 \leqslant \min(r,a)$, there exists $\mathbf{x}' \in D_r(\mathbf{0}')$ such that $\mathbf{x} = (\mathbf{x}',\varphi(\mathbf{x}'))$. We introduce the notation $\mathbf{d_x}_n = (1 + \|\nabla\varphi(\mathbf{x}')\|^2)^{-\frac{1}{2}}$ and $\mathbf{d}'_\mathbf{x} = -\mathbf{d_x}_n\nabla\varphi(\mathbf{x}')$ so that $\mathbf{d_x} := (\mathbf{d}'_\mathbf{x},\mathbf{d_x}_n)$ is a unit vector. Now, let us show that Ω satisfy the ε_0-ball condition at the point $\mathbf{x}$ so choose any $\mathbf{y} \in B_{\varepsilon_0}(\mathbf{x}+\varepsilon_0\mathbf{d_x}) \subseteq B_{2\varepsilon_0}(\mathbf{x}) \subseteq B_{3\varepsilon_0}(\mathbf{x}_0)$. Since $3\varepsilon_0 \leqslant \min(r,a)$, there exists $\mathbf{y}' \in D_r(\mathbf{0}')$ and $y_n \in]-a,a[$ such that $\mathbf{y} = (\mathbf{y}',y_n)$. Moreover, we have $\mathbf{y} \in \mathbb{R}^n\backslash\overline{\Omega}$ iff $y_n > \varphi(\mathbf{y}')$. Observing that $\|\mathbf{y}-\mathbf{x}-\varepsilon_0\mathbf{d_x}\| < \varepsilon_0 \Leftrightarrow \frac{1}{2\varepsilon_0}\|\mathbf{y}-\mathbf{x}\|^2 < \langle \mathbf{y}-\mathbf{x} \mid \mathbf{d_x}\rangle$, we obtain successively:

$$\begin{aligned}y_n - \varphi(\mathbf{y}') &= \frac{1}{\mathbf{d_x}_n}\left[\mathbf{d_x}_n(y_n - \varphi(\mathbf{x}')) + \langle \mathbf{d}'_\mathbf{x} \mid \mathbf{y}'-\mathbf{x}'\rangle - \langle \mathbf{d}'_\mathbf{x} \mid \mathbf{y}'-\mathbf{x}'\rangle + \mathbf{d_x}_n(\varphi(\mathbf{x}')-\varphi(\mathbf{y}'))\right] \\ &= \frac{1}{\mathbf{d_x}_n}\langle\mathbf{y}-\mathbf{x} \mid \mathbf{d_x}\rangle - \varphi(\mathbf{y}') + \varphi(\mathbf{x}') + \langle\nabla\varphi(\mathbf{x}') \mid \mathbf{y}'-\mathbf{x}'\rangle \\ &> \frac{\|\mathbf{y}-\mathbf{x}\|^2}{2\varepsilon_0\mathbf{d_x}_n} - \frac{L}{2}\|\mathbf{y}'-\mathbf{x}'\|^2 > \frac{1}{2\mathbf{d_x}_n}\|\mathbf{y}'-\mathbf{x}'\|^2\left(\frac{1}{\varepsilon_0} - L\right) \geqslant 0.\end{aligned}$$

Consequently, we get $\mathbf{y} \notin \overline{\Omega}$ and we proved $B_{\varepsilon_0}(\mathbf{x}+\varepsilon_0\mathbf{d_x}) \subseteq \mathbb{R}^n\backslash\overline{\Omega}$. Similarly, we can obtain $B_{\varepsilon_0}(\mathbf{x}-\varepsilon_0\mathbf{d_x}) \subseteq \Omega$. Hence, for any $\mathbf{x}_0 \in \partial\Omega$, there exists $\varepsilon_0 > 0$ such that $\Omega \cap B_{\varepsilon_0}(\mathbf{x}_0)$ satisfies the ε_0-ball condition. Finally, as $\partial\Omega$ is compact, it is included in a finite reunion of such balls $B_{\varepsilon_0}(\mathbf{x}_0)$. Define $\varepsilon > 0$ as the minimum of this finite number of ε_0 and Ω will satisfy the ε-ball property. □

Chapter 17

Parametrization of a converging sequence from $\mathcal{O}_\varepsilon(B)$

In this chapter, we first recall a known compactness result about the uniform cone property [20]. Since we know from Point (i) of Theorem 16.6 that every set satisfying the ε-ball condition also satisfies the $f^{-1}(\varepsilon)$-cone property, we only have to check that $\mathcal{O}_\varepsilon(B)$ is closed under the Hausdorff convergence to get its compactness. Hence, we obtain the following result.

Proposition 17.1. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ a bounded open set, large enough to contain an open ball of radius 3ε, and smooth enough so that ∂B has zero n-dimensional Lebesgue measure. If $(\Omega_i)_{i\in\mathbb{N}}$ is a sequence of elements from $\mathcal{O}_\varepsilon(B)$, then there exists $\Omega \in \mathcal{O}_\varepsilon(B)$ such that a subsequence $(\Omega_{\psi(i)})_{i\in\mathbb{N}}$ converges to Ω in the following sense (see Definition 17.4 for the various modes of convergence):*

(i) $(\Omega_{\psi(i)})_{i\in\mathbb{N}}$ *converges to Ω in the Hausdorff sense;*

(ii) $(\partial\Omega_{\psi(i)})_{i\in\mathbb{N}}$ *converges to $\partial\Omega$ for the Hausdorff distance;*

(iii) $(\overline{\Omega_{\psi(i)}})_{i\in\mathbb{N}}$ *converges to $\overline{\Omega}$ for the Hausdorff distance;*

(iv) $(B\backslash\overline{\Omega_{\psi(i)}})_{i\in\mathbb{N}}$ *converges to $B\backslash\overline{\Omega}$ in the Hausdorff sense;*

(v) $(\Omega_{\psi(i)})_{i\in\mathbb{N}}$ *converges to Ω in the sense of compact sets;*

(vi) $(\Omega_{\psi(i)})_{i\in\mathbb{N}}$ *converges to Ω in the sense of characteristic functions.*

Then, in the rest of this chapter, we consider a sequence $(\Omega_i)_{i\in\mathbb{N}}$ of elements from $\mathcal{O}_\varepsilon(B)$ converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of Proposition 17.1, and we prove that locally the boundaries $\partial\Omega_i$ can be parametrized simultaneously by $C^{1,1}$-graphs in a fixed local frame associated with $\partial\Omega$. Finally, we get the C^1-strong and $W^{2,\infty}$-weak-star convergence of these local graphs as follows.

Theorem 17.2. *Let $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ converge to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of Proposition 17.1 (i)-(vi). Then, for any point $\mathbf{x}_0 \in \partial\Omega$, there exists a direct orthonormal frame centred at $\mathbf{x}_0$, and also $I \in \mathbb{N}$ depending only on $\mathbf{x}_0$, ε, Ω, and $(\Omega_i)_{i\in\mathbb{N}}$, such that inside this frame, for any integer $i \geqslant I$, there exists a continuously differentiable map $\varphi_i : \overline{D_{\tilde{r}}}(\mathbf{0}') \to]-\varepsilon,\varepsilon[$, whose gradient $\nabla\varphi_i$ and φ_i are L-Lipschitz continuous with $L > 0$ and $\tilde{r} > 0$ depending only on ε, and such that:*

$$\left\{\begin{array}{rcl} \partial\Omega_i \cap \left(\overline{D_{\tilde{r}}}(\mathbf{0}') \cap [-\varepsilon,\varepsilon]\right) & = & \left\{(\mathbf{x}',\varphi_i(\mathbf{x}')), \quad \mathbf{x}' \in \overline{D_{\tilde{r}}}(\mathbf{0}')\right\} \\ \Omega_i \cap \left(\overline{D_{\tilde{r}}}(\mathbf{0}') \cap [-\varepsilon,\varepsilon]\right) & = & \left\{(\mathbf{x}',x_n), \quad \mathbf{x}' \in \overline{D_{\tilde{r}}}(\mathbf{0}') \text{ and } -\varepsilon \leqslant x_n < \varphi_i(\mathbf{x}')\right\}. \end{array}\right.$$

Moreover, considering the map φ of Definition 16.3 associated with the point $\mathbf{x}_0$ of $\partial\Omega$, we have:

$$\varphi_i \to \varphi \quad \text{in } C^1\left(\overline{D_{\tilde{r}}}(\mathbf{0}')\right) \qquad \text{and} \qquad \varphi_i \rightharpoonup \varphi \quad \text{weak}-\text{star in } W^{2,\infty}\left(D_{\tilde{r}}(\mathbf{0}')\right). \tag{17.1}$$

Hence, the rest of this chapter is devoted to the proof of Theorem 17.2, which is done in the same spirit as Sections 16.2.2, 16.3.1, and 16.3.2. It is organized as follows.

- Some global and local geometric inequalities are established.

- The boundary $\partial\Omega_i$ is locally parametrized by a graph.
- We obtain the $C^{1,1}$-regularity of the local graph associated with $\partial\Omega_i$.
- We prove that (17.1) holds for the local graphs.

Remark 17.3. *Only Point (v) of Proposition 17.1 is needed to get the first part of Theorem 17.2. To obtain the second part, we also need to assume Point (ii) of Proposition 17.1. Indeed, this hypothesis ensures that the converging sequence of local graphs converges to the one associated with $\partial\Omega$.*

17.1 Compactness of the class $\mathcal{O}_\varepsilon(B)$

First, we quickly define the modes of convergence given in Proposition 17.1. Then, we state the compactness theorem associated with the uniform cone property. Finally, Proposition 17.1 is proved.

Definition 17.4. *The Hausdorff distance d_H between two compact sets $X, Y \subset \mathbb{R}^n$ is defined by $d_H(X,Y) = \max(\max_{\mathbf{x}\in X} d(\mathbf{x}, Y), \max_{\mathbf{y}\in Y} d(\mathbf{y}, X))$. We say that a sequence of compacts sets $(K_i)_{i\in\mathbb{N}}$ converges to a compact set K for the Hausdorff distance if $d_H(K_i, K) \to 0$. Let B be any non-empty bounded open subset of $\mathbb{R}^n$. A sequence of open sets $(\Omega_i)_{i\in\mathbb{N}} \subset B$ converges to $\Omega \subset B$:*

- *in the Hausdorff sense if $(\overline{B}\backslash\Omega_i)_{i\in\mathbb{N}}$ converges to $\overline{B}\backslash\Omega$ for the Hausdorff distance;*
- *in the sense of compact sets if for any compact sets K and L such that $K \subset \Omega$ and $L \subset B\backslash\overline{\Omega}$, there exists $I \in \mathbb{N}$ such that for any integer $i \geqslant I$, we have $K \subset \Omega_i$ and $L \subset B\backslash\overline{\Omega_i}$;*
- *in the sense of characteristic functions if we have $\int_{\overline{B}} |\mathbf{1}_{\Omega_i}(\mathbf{x}) - \mathbf{1}_\Omega(\mathbf{x})| d\mathbf{x} \to 0$, where $\mathbf{1}_X$ is the characteristic function of X, valued one for the points of X, otherwise zero.*

In [39, Theorem 2.8], Point (i) of Proposition 17.1 is proved. However, we can prove Proposition 17.1 by applying Theorem 16.6 (i) and the following result.

Theorem 17.5 (Chenais [46, Theorem 2.4.10]). *Let $\alpha \in]0, \frac{\pi}{2}[$ and B be as in Proposition 17.1. We set $\mathfrak{O}_\alpha(B)$ the class of non-empty open sets $\Omega \subseteq B$ that satisfy the α-cone property as in Definition 16.4. If $(\Omega_i)_{i\in\mathbb{N}}$ is a sequence of elements from $\mathfrak{O}_\alpha(B)$, then there exists $\Omega \in \mathfrak{O}_\alpha(B)$ such that a subsequence $(\Omega_{\psi(i)})_{i\in\mathbb{N}}$ converges to Ω in the sense of Proposition 17.1 (i)-(vi).*

Proof. We only sketch the proof and refer to [46, Theorem 2.4.10] for further details. First, consider any $(\Omega_i)_{i\in\mathbb{N}} \subset B$ and show that, up to a subsequence, it is converging to $\Omega \subset B$ in the Hausdorff sense. Then, use the uniform cone condition to get $\Omega \in \mathfrak{O}_\alpha(B)$ and $\lim_{i\to+\infty} d_H(\partial\Omega_i, \partial\Omega) = \lim_{i\to+\infty} d_H(\overline{\Omega_i}, \overline{\Omega}) = 0$. Next, deduce that $(B\backslash\overline{\Omega_i})_{i\in\mathbb{N}}$ converges to $B\backslash\overline{\Omega}$ in the Hausdorff sense, and $(\Omega_i)_{i\in\mathbb{N}}$ to Ω in the sense of compact sets. Finally, since $\Omega \in \mathfrak{O}_\alpha(B)$, $\partial\Omega$ is a finite reunion of Lipschitz graphs so it has zero n-dimensional Lebesgue measure [30, Section 2.4.2 Theorem 2] and so does ∂B by assumption. Combining this observation with the convergence in the sense of compacts, we obtain the convergence in the sense of characteristic functions. □

Proof of Proposition 17.1. Since $\mathcal{O}_\varepsilon(B) \subset \mathfrak{O}_{f^{-1}(\varepsilon)}(B)$ (Point (i) of Theorem 16.6), Theorem 17.5 holds and we only have to check $\Omega \in \mathcal{O}_\varepsilon(B)$. Consider any $\mathbf{x} \in \partial\Omega$. From [46, Proposition 2.2.14], there exists a sequence of points $\mathbf{x}_i \in \partial\Omega_i$ converging to $\mathbf{x}$. Then, we can apply the ε-ball condition on each point $\mathbf{x}_i$ so there exists a sequence of unit vector $\mathbf{d}_{\mathbf{x}_i}$ of $\mathbb{R}^n$ such that:

$$\forall i \in \mathbb{N}, \quad \begin{cases} B_\varepsilon(\mathbf{x}_i - \varepsilon \mathbf{d}_{\mathbf{x}_i}) \subseteq \Omega_i \\ \\ B_\varepsilon(\mathbf{x}_i + \varepsilon \mathbf{d}_{\mathbf{x}_i}) \subseteq B\backslash\overline{\Omega_i}. \end{cases}$$

Since $\|\mathbf{d}_{\mathbf{x}_i}\| = 1$, there exists a unit vector $\mathbf{d}_\mathbf{x}$ of $\mathbb{R}^n$ such that, up to a subsequence, $(\mathbf{d}_{\mathbf{x}_i})_{i\in\mathbb{N}}$ converges to $\mathbf{d}_\mathbf{x}$. Finally, the inclusion is stable under the Hausdorff convergence [46, (2.16)] and we get the ε-ball condition of Definition 15.1 by letting $i \to +\infty$ in the above inclusions. □

17.2 Some global and local geometric inequalities

In the rest of chapter 17, we consider a sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of Proposition 17.1 (i)-(vi) and we make the following hypothesis.

Assumption 17.6. *Let $\mathbf{x}_0 \in \partial\Omega$ henceforth set. From the ε-ball condition, a unit vector $\mathbf{d}_{\mathbf{x}_0}$ is associated with the point $\mathbf{x}_0$ (which is unique from Proposition 16.12). Moreover, we have:*

$$\begin{cases} B_\varepsilon(\mathbf{x}_0 - \varepsilon\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega \\ \\ B_\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{d}_{\mathbf{x}_0}) \subseteq B\backslash\overline{\Omega}. \end{cases}$$

Then, we also consider $\eta \in]0,\varepsilon[$. Since we assume Point (v) of Proposition 17.1, there exists $I \in \mathbb{N}$ depending on $(\Omega_i)_{i\in\mathbb{N}}$, Ω, $\mathbf{x}_0$, ε and η, such that for any integer $i \geqslant I$, we have:

$$\begin{cases} \overline{B_{\varepsilon-\eta}}(\mathbf{x}_0 - \varepsilon\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega_i \\ \\ \overline{B_{\varepsilon-\eta}}(\mathbf{x}_0 + \varepsilon\mathbf{d}_{\mathbf{x}_0}) \subseteq B\backslash\overline{\Omega_i}. \end{cases} \tag{17.2}$$

Finally, we consider any integer $i \geqslant I$.

Proposition 17.7. *Assume* (17.2)*. For any point $\mathbf{x}_i \in \partial\Omega_i$, we have the following inequality:*

$$\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2 \leqslant \frac{1}{\varepsilon^2}\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \frac{(2\varepsilon)^2 - (2\varepsilon-\eta)^2}{\varepsilon^2}. \tag{17.3}$$

Proof. Combine (17.2) with the ε-ball condition at $\mathbf{x}_i \in \partial\Omega_i$ to get $\overline{B_{\varepsilon-\eta}}(\mathbf{x}_0 \pm \varepsilon\mathbf{d}_{\mathbf{x}_0}) \cap B_\varepsilon(\mathbf{x}_i \mp \varepsilon\mathbf{d}_{\mathbf{x}_i}) = \emptyset$. We deduce $\|\mathbf{x}_i - \mathbf{x}_0 \mp \varepsilon(\mathbf{d}_{\mathbf{x}_i} + \mathbf{d}_{\mathbf{x}_0})\| \geqslant 2\varepsilon - \eta$. Squaring these two inequalities and summing them, we obtain the required one: $\|\mathbf{x}_i - \mathbf{x}_0\|^2 + 4\varepsilon^2 - (2\varepsilon-\eta)^2 \geqslant 2\varepsilon^2 - 2\varepsilon^2\langle \mathbf{d}_{\mathbf{x}_i} \mid \mathbf{d}_{\mathbf{x}_0}\rangle = \varepsilon^2\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2$. □

Proposition 17.8. *Under assumption 17.6, for any $\mathbf{x}_i \in \partial\Omega_i$, we have the following global inequality:*

$$|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \frac{1}{2\varepsilon}\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \frac{\varepsilon^2 - (\varepsilon-\eta)^2}{2\varepsilon}. \tag{17.4}$$

Moreover, if we introduce the vector $(\mathbf{x}_i - \mathbf{x}_0)' = (\mathbf{x}_i - \mathbf{x}_0) - \langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle\mathbf{d}_{\mathbf{x}_0}$ and if we assume that $\|(\mathbf{x}_i - \mathbf{x}_0)'\| \leqslant \varepsilon - \eta$ and $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \varepsilon$, then we have the following local inequality:

$$\frac{1}{2\varepsilon}\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \frac{\varepsilon^2 - (\varepsilon-\eta)^2}{2\varepsilon} < \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}. \tag{17.5}$$

Proof. From (17.2), any point $\mathbf{x}_i \in \partial\Omega_i$ cannot belong to the sets $\overline{B_{\varepsilon-\eta}}(\mathbf{x}_0 \pm \varepsilon\mathbf{d}_{\mathbf{x}_0})$. Hence, we have: $\|\mathbf{x}_i - \mathbf{x}_0 \mp \varepsilon\mathbf{d}_{\mathbf{x}_0}\| > \varepsilon - \eta$. Squaring these two inequalities, we get the first required relation (17.4): $\|\mathbf{x}_i - \mathbf{x}_0\|^2 + \varepsilon^2 - (\varepsilon-\eta)^2 > 2\varepsilon|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle|$. Then, by introducing the vector $(\mathbf{x}_i - \mathbf{x}_0)'$ of the statement, the previous inequality now takes the following form:

$$|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle|^2 - 2\varepsilon|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| + \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 + \varepsilon^2 - (\varepsilon-\eta)^2 > 0.$$

We assume that its left member is a second-order polynomial whose discriminant is non-negative: $\Delta' := (\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 \geqslant 0$. Hence, the unknown satisfies either $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < \varepsilon - \sqrt{\Delta'}$ or $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| > \varepsilon + \sqrt{\Delta'}$. We assume $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \varepsilon$ and the last case cannot hold. Squaring the remaining inequality, we get: $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle|^2 + \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 < \varepsilon^2 + (\varepsilon-\eta)^2 - 2\varepsilon\sqrt{\Delta'}$, which is the second required relation (17.5) since its left member is equal to $\|\mathbf{x}_i - \mathbf{x}_0\|^2$. □

Corollary 17.9. *With the same assumptions and notation as in Propositions 17.7 and 17.8, we have:*

$$\|\mathbf{x}_i - \mathbf{x}_0\| < 2\eta + 2\|(\mathbf{x}_i - \mathbf{x}_0)'\|, \tag{17.6}$$

$$\varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\| < 2\sqrt{2\varepsilon\eta} + \sqrt{2}\|(\mathbf{x}_i - \mathbf{x}_0)'\|. \tag{17.7}$$

Proof. Consider any $\mathbf{x}_i \in \partial\Omega_i$. We set $(\mathbf{x}_i - \mathbf{x}_0)' = (\mathbf{x}_i - \mathbf{x}_0) - \langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle \mathbf{d}_{\mathbf{x}_0}$. We assume $\|(\mathbf{x}_i - \mathbf{x}_0)'\| \leqslant \varepsilon - \eta$ and $|\langle \mathbf{x}_i - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \varepsilon$. The local estimation (17.5) of Proposition 17.8 gives:

$$\begin{aligned}
\|\mathbf{x}_i - \mathbf{x}_0\|^2 &< \varepsilon^2 + (\varepsilon-\eta)^2 - 2\varepsilon\sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2} \\
&= \frac{\left[\varepsilon^2 + (\varepsilon-\eta)^2\right]^2 - 4\varepsilon^2(\varepsilon-\eta)^2 + 4\varepsilon^2\|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}{\varepsilon^2 + (\varepsilon-\eta)^2 + 2\varepsilon\sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}} \\
&< \left[\frac{\varepsilon^2 - (\varepsilon-\eta)^2}{\varepsilon}\right]^2 + 4\|(\mathbf{x}_i - \mathbf{x}_0)'\|^2 \quad < \quad 4\eta^2 + 4\|(\mathbf{x}_i - \mathbf{x}_0)'\|^2.
\end{aligned}$$

Hence, we get: $\|\mathbf{x}_i - \mathbf{x}_0\| < 2\eta + 2\|(\mathbf{x}_i - \mathbf{x}_0)'\|$. Then, using (17.3), we also have:

$$\varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\| \leqslant \sqrt{4\varepsilon^2 - (2\varepsilon-\eta)^2 + \|\mathbf{x}_i - \mathbf{x}_0\|^2}.$$

Combining the above inequality with (17.5), we obtain:

$$\begin{aligned}
\varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\| &< \sqrt{4\varepsilon\eta - \eta^2 + \varepsilon^2 + (\varepsilon-\eta)^2 - 2\varepsilon\sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}} \\
&= \sqrt{2\varepsilon\frac{4\varepsilon\eta + \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}{\varepsilon + \eta + \sqrt{(\varepsilon-\eta)^2 - \|(\mathbf{x}_i - \mathbf{x}_0)'\|^2}}} \quad < \quad 2\sqrt{2\varepsilon\eta} + \sqrt{2}\|(\mathbf{x}_i - \mathbf{x}_0)'\|.
\end{aligned}$$

Consequently, the two required inequalities (17.6) and (17.7) are established so Corollary 17.9 holds. $\square$

17.3 A local parametrization of the boundary $\partial\Omega_i$

Henceforth, we consider a basis $\mathcal{B}_{\mathbf{x}_0}$ of the hyperplane $\mathbf{d}_{\mathbf{x}_0}^{\perp}$ such that $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ is a direct orthonormal frame. The position of any point is now determined in this local frame associated with $\mathbf{x}_0$. More precisely, for any point $\mathbf{x} \in \mathbb{R}^n$, we set $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ such that $\mathbf{x} = (\mathbf{x}', x_n)$. In particular, the symbols $\mathbf{0}$ and $\mathbf{0}'$ respectively refer to the zero vector of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. Moreover, since $\mathbf{x}_0$ is identified with $\mathbf{0}$ in this new frame, Relations (17.2) of Assumption 17.6 take new forms:

$$\left\{\begin{array}{l}
\overline{B_{\varepsilon-\eta}}(\mathbf{0}', -\varepsilon) \subseteq \Omega_i \\[1ex]
\overline{B_{\varepsilon-\eta}}(\mathbf{0}', \varepsilon) \subseteq B\backslash\overline{\Omega_i}.
\end{array}\right. \tag{17.8}$$

We introduce two functions defined on $\overline{D_{\varepsilon-\eta}}(\mathbf{0}') = \{\mathbf{x}' \in \mathbb{R}^{n-1},\ \|\mathbf{x}'\| \leqslant \varepsilon - \eta\}$. The first one determine around $\mathbf{x}_0$ the position of the boundary $\partial\Omega_i$ thanks to some exterior points, the other one with interior points. Then, we show these two maps coincide even if it means reducing η.

Proposition 17.10. *Under Assumption 17.6, the two following maps $\varphi_i^{\pm}$ are well defined:*

$$\left\{\begin{array}{rrcl}
\varphi_i^+ : & \overline{D_{\varepsilon-\eta}}(\mathbf{0}') & \longrightarrow &]-\varepsilon, \varepsilon[\\
& \mathbf{x}' & \longmapsto & \varphi_i^+(\mathbf{x}') = \sup\{x_n \in [-\varepsilon, \varepsilon],\quad (\mathbf{x}', x_n) \in \Omega_i\} \\[2ex]
\varphi_i^- : & \overline{D_{\varepsilon-\eta}}(\mathbf{0}') & \longrightarrow &]-\varepsilon, \varepsilon[\\
& \mathbf{x}' & \longmapsto & \varphi_i^-(\mathbf{x}') = \inf\{x_n \in [-\varepsilon, \varepsilon],\quad (\mathbf{x}', x_n) \in B\backslash\overline{\Omega_i}\},
\end{array}\right.$$

Moreover, for any $\mathbf{x}' \in \overline{D_{\varepsilon-\eta}}(\mathbf{0}')$, introducing the points $\mathbf{x}_i^{\pm} = (\mathbf{x}', \varphi_i^{\pm}(\mathbf{x}'))$, we have $\mathbf{x}_i^{\pm} \in \partial\Omega_i$ and also the following inequalities:

$$|\varphi_i^{\pm}(\mathbf{x}')| < \frac{1}{2\varepsilon}\|\mathbf{x}_i^{\pm} - \mathbf{x}_0\|^2 + \frac{\varepsilon^2 - (\varepsilon-\eta)^2}{2\varepsilon} < \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}. \tag{17.9}$$

Proof. Let $\mathbf{x}' \in \overline{D_{\varepsilon-\eta}}(\mathbf{0}')$ and $g : t \in [-\varepsilon, \varepsilon] \mapsto (\mathbf{x}', t)$. Since $-\varepsilon \in g^{-1}(\Omega_i) \subseteq [-\varepsilon, \varepsilon]$, we can set $\varphi_i^+(\mathbf{x}') = \sup g^{-1}(\Omega_i)$. The map g is continuous so $g^{-1}(\Omega_i)$ is open and $\varphi_i^+(\mathbf{x}') \neq \varepsilon$ thus we get $\varphi_i^+(\mathbf{x}') \notin g^{-1}(\Omega_i)$ i.e. $\mathbf{x}_i^+ \in \overline{\Omega_i}\backslash\Omega_i$. Similarly, the map φ_i^- is well defined and $\mathbf{x}_i^- \in \partial\Omega_i$. Finally, we use (17.4) and (17.5) on the points $\mathbf{x}_0$ and $\mathbf{x}_i = \mathbf{x}_i^{\pm}$ in order to obtain (17.9). $\square$

Lemma 17.11. *We make Assumption 17.6 and assume* $\eta < \frac{\varepsilon}{3}$. *We set* $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ *and* $\mathbf{x}' \in \overline{D_r}(\mathbf{0}')$. *Assume there exists* $x_n \in [-\varepsilon, \varepsilon]$ *such that* $\mathbf{x}_i := (\mathbf{x}', x_n)$ *belongs to* $\partial\Omega_i$. *We also consider* $\tilde{x}_n \in \mathbb{R}$ *satisfying the inequality* $|\tilde{x}_n| < \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}$. *Introducing* $\tilde{\mathbf{x}}_i = (\mathbf{x}', \tilde{x}_n)$, *then we have:* $(\tilde{x}_n < x_n \Longrightarrow \tilde{\mathbf{x}}_i \in \Omega_i)$ and $\left(\tilde{x}_n > x_n \Longrightarrow \tilde{\mathbf{x}}_i \in B\backslash\overline{\Omega_i}\right)$.

Proof. We assume $\eta < \frac{\varepsilon}{3}$ so we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$. Consider any $\mathbf{x}' \in \overline{D_r}(\mathbf{0}')$ and also $(x_n, \tilde{x}_n) \in [-\varepsilon, \varepsilon]^2$ such that $\mathbf{x}_i := (\mathbf{x}', x_n) \in \partial\Omega_i$ and $\tilde{\mathbf{x}}_i := (\mathbf{x}', \tilde{x}_n) \notin \overline{B_{\varepsilon-\eta}}(\mathbf{0}', \pm\varepsilon)$. We need to show that if $\tilde{x}_n \gtrless x_n$, then $\tilde{\mathbf{x}}_i \in B_\varepsilon(\mathbf{x}_i \pm \varepsilon\mathbf{d}_{\mathbf{x}_i})$. The ε-ball condition on Ω_i will give the result. Since $\mathbf{x}_i - \tilde{\mathbf{x}}_i = (x_n - \tilde{x}_n)\mathbf{d}_{\mathbf{x}_0}$, if we assume $\tilde{x}_n > x_n$, then we have:

$$\begin{aligned}
\|\tilde{\mathbf{x}}_i - \mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}\|^2 - \varepsilon^2 &= (\tilde{x}_n - x_n)^2 - 2\varepsilon(\tilde{x}_n - x_n)\langle \mathbf{d}_{\mathbf{x}_0} \mid \mathbf{d}_{\mathbf{x}_i}\rangle \\
&= |\tilde{x}_n - x_n|\left(|\tilde{x}_n - x_n| + \varepsilon\|\mathbf{d}_{\mathbf{x}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2 - 2\varepsilon\right) \\
&\leqslant |\tilde{x}_n - x_n|\left(|\tilde{x}_n| + |x_n| + \tfrac{\|\mathbf{x}_i - \mathbf{x}_0\|^2 + (2\varepsilon)^2 - (2\varepsilon-\eta)^2}{\varepsilon} - 2\varepsilon\right),
\end{aligned}$$

where the last inequality comes from Proposition 17.7 (17.3) applied to $\mathbf{x}_i \in \partial\Omega_i$. Finally, we use the inequality involving $\tilde{x}_n$ and the ones (17.4)-(17.5) of Proposition 17.8 applied to $\mathbf{x}_i \in \partial\Omega_i$ to obtain:

$$\|\tilde{\mathbf{x}}_i - \mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}\|^2 - \varepsilon^2 < 4|x_n - \tilde{x}_n| \underbrace{\left(\frac{\varepsilon+\eta}{2} - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}\right)}_{\leqslant\left(\frac{\varepsilon+\eta}{2} - \sqrt{(\varepsilon-\eta)^2 - r^2}\right) \;=\; 0}.$$

Hence, if $\tilde{x}_n > x_n$, then we get $\tilde{\mathbf{x}}_i \in B_\varepsilon(\mathbf{x}_i + \varepsilon\mathbf{d}_{\mathbf{x}_i}) \subseteq B\backslash\overline{\Omega_i}$. Similarly, one can prove that if $\tilde{x}_n < x_n$, then we have $\tilde{\mathbf{x}}_i \in B_\varepsilon(\mathbf{x}_i - \varepsilon\mathbf{d}_{\mathbf{x}_i}) \subseteq \Omega_i$. □

Proposition 17.12. *Let* η, r *be as in Lemma 17.11. Then, the two functions* $\varphi_i^{\pm}$ *of Proposition 17.10 coincide on* $\overline{D_r}(\mathbf{0}')$. *The map* φ_i *refers to their common restrictions and it satisfies:*

$$\begin{cases}
\partial\Omega_i \cap \left(\overline{D_r}(\mathbf{0}') \cap [-\varepsilon, \varepsilon]\right) &= \left\{(\mathbf{x}', \varphi_i(\mathbf{x}')), \quad \mathbf{x}' \in \overline{D_r}(\mathbf{0}')\right\} \\
\Omega_i \cap \left(\overline{D_r}(\mathbf{0}') \cap [-\varepsilon, \varepsilon]\right) &= \left\{(\mathbf{x}', x_n), \quad \mathbf{x}' \in \overline{D_r}(\mathbf{0}') \text{ and } -\varepsilon \leqslant x_n < \varphi_i(\mathbf{x}')\right\}.
\end{cases}$$

Proof. First, we assume by contradiction that there exists $\mathbf{x}' \in \overline{D_r}(\mathbf{0}')$ such that $\varphi_i^-(\mathbf{x}') \neq \varphi_i^+(\mathbf{x}')$. The hypothesis of Lemma 17.11 are satisfied for the points $\mathbf{x}_i := (\mathbf{x}', \varphi_i^+(\mathbf{x}'))$ and $\tilde{\mathbf{x}}_i := (\mathbf{x}', \varphi_i^-(\mathbf{x}'))$ by using (17.9). Hence, either $(\varphi_i^-(\mathbf{x}') < \varphi_i^+(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}}_i \in \Omega_i)$ or $(\varphi_i^-(\mathbf{x}') > \varphi_i^+(\mathbf{x}') \Rightarrow \tilde{\mathbf{x}}_i \in B\backslash\overline{\Omega_i})$ whereas $\tilde{\mathbf{x}}_i \in \partial\Omega_i$. We deduce that $\varphi_i^-(\mathbf{x}') = \varphi_i^+(\mathbf{x}')$ for any $\mathbf{x}' \in \overline{D_r}(\mathbf{0}')$. Then, we consider $\mathbf{x}' \in \overline{D_r}(\mathbf{0}')$ and $x_n \in [-\varepsilon, \varepsilon]$. We set $\mathbf{x}_i = (\mathbf{x}', \varphi_i(\mathbf{x}'))$ and $\tilde{\mathbf{x}}_i = (\mathbf{x}', x_n)$. Proposition 17.10 ensures that if $x_n = \varphi_i(\mathbf{x}')$, then $\mathbf{x}_i \in \partial\Omega_i$. Moreover, if $-\varepsilon \leqslant x_n \leqslant -\varepsilon + \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}$, then $\tilde{\mathbf{x}}_i \in \overline{B_{\varepsilon-\eta}}(\mathbf{0}', -\varepsilon) \subseteq \Omega_i$ and if $-\varepsilon + \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2} < x_n < \varphi(\mathbf{x}')$, then apply Lemma 17.11 in order to get $\tilde{\mathbf{x}}_i \in \Omega_i$. Consequently, we proved: $\forall \mathbf{x}' \in \overline{D_r}(\mathbf{0}'),\ -\varepsilon \leqslant x_n < \varphi_i(\mathbf{x}') \Longrightarrow (\mathbf{x}', x_n) \in \Omega_i$. To conclude, similar arguments hold when $\varepsilon \geqslant x_n > \varphi_i(\mathbf{x}')$ and imply $(\mathbf{x}', x_n) \in B\backslash\overline{\Omega_i}$. □

17.4 The $C^{1,1}$-regularity of the local graph φ_i

We previously showed that the boundary $\partial\Omega_i$ is locally described by the graph of a well-defined map $\varphi_i : \overline{D_r}(\mathbf{0}') \to]-\varepsilon, \varepsilon[$. Now we prove its $C^{1,1}$-regularity even if it means reducing η and r.

Lemma 17.13. *The following map is well defined, smooth, surjective and increasing:*

$$\begin{array}{rccc}
f_\eta : &]0, \frac{\pi}{2}[& \longrightarrow &]2\sqrt{2\varepsilon\eta}, +\infty[\\
& \alpha & \longmapsto & \dfrac{3\alpha + 2\sqrt{2\varepsilon\eta}}{\cos\alpha}.
\end{array}$$

In particular, it is an homeomorphism and its inverse f_η^{-1} *satisfies the following inequality:*

$$\forall \varepsilon > 0,\ \forall \eta \in \left]0, \frac{\varepsilon}{8}\right[, \quad f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{3}. \tag{17.10}$$

Proof. The proof is basic calculus. □

Proposition 17.14. *In Assumption 17.6, let $\eta < \frac{\varepsilon}{8}$ and consider $\alpha \in]0, f_\eta^{-1}(\varepsilon)]$, where f_η^{-1} has been introduced in Lemma 17.13. Then, we have:*

$$\forall \mathbf{x}_i \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega_i}, \quad C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega_i,$$

where $C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$ is defined in Definition 16.4.

Proof. Since we have $\eta < \frac{\varepsilon}{3}$, we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\mathcal{C}_{r,\varepsilon} = \overline{D_r}(\mathbf{0}') \times [-\varepsilon, \varepsilon]$. Moreover, we assume $\eta < \frac{\varepsilon}{8}$ i.e. $2\sqrt{2\varepsilon\eta} < \varepsilon$ so $f_\eta^{-1}(\varepsilon)$ is well defined. Choose $\alpha \in]0, f_\eta^{-1}(\varepsilon)]$ then consider $\mathbf{x}_i = (\mathbf{x}', x_n) \in B_\alpha(\mathbf{x}_0) \cap \overline{\Omega_i}$ and $\mathbf{y}_i = (\mathbf{y}', y_n) \in C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$. The proof of the assertion $\mathbf{y}_i \in \Omega_i$ is divided into the three following steps.

1. Check $\mathbf{x}_i \in \mathcal{C}_{r,\varepsilon}$ so as to introduce the point $\tilde{\mathbf{x}}_i = (\mathbf{x}', \varphi_i(\mathbf{x}'))$ of $\partial\Omega_i$ satisfying $x_n \leqslant \varphi_i(\mathbf{x}')$.
2. Consider $\tilde{\mathbf{y}}_i = (\mathbf{y}', y_n + \varphi_i(\mathbf{x}') - x_n)$ and prove $\tilde{\mathbf{y}}_i \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon \mathbf{d}_{\tilde{\mathbf{x}}_i}) \subseteq \Omega_i$.
3. Show $(\tilde{\mathbf{y}}_i, \mathbf{y}_i) \in \mathcal{C}_{r,\varepsilon} \times \mathcal{C}_{r,\varepsilon}$ in order to deduce $y_n + \varphi_i(\mathbf{x}') - x_n < \varphi_i(\mathbf{y}')$ and conclude $\mathbf{y}_i \in \Omega_i$.

First, from (17.10), we have: $\max(\|\mathbf{x}'\|, |x_n|) \leqslant \|\mathbf{x}_i - \mathbf{x}_0\| < \alpha \leqslant f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{3}$. Since $\eta < \frac{\varepsilon}{8}$, we get $r > \frac{1}{2}[4(\frac{7\varepsilon}{8})^2 - (\frac{9\varepsilon}{8})^2]^{\frac{1}{2}} > \frac{\varepsilon}{2}$ thus $\mathbf{x}_i \in \overline{\Omega_i} \cap \mathcal{C}_{r,\varepsilon}$. Hence, from Proposition 17.12, it comes $x_n \leqslant \varphi_i(\mathbf{x}')$. We set $\tilde{\mathbf{x}}_i = (\mathbf{x}', \varphi_i(\mathbf{x}')) \in \partial\Omega_i \cap \mathcal{C}_{r,\varepsilon}$. Then, we prove $C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i})$ so consider any $\mathbf{y} \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0})$. Combining the Cauchy-Schwarz inequality and $\mathbf{y} \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0})$, we get:

$$\begin{aligned}
\|\mathbf{y} - \tilde{\mathbf{x}}_i + \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i}\|^2 - \varepsilon^2 &\leqslant \|\mathbf{y} - \tilde{\mathbf{x}}_i\|^2 + 2\varepsilon\|\mathbf{y} - \tilde{\mathbf{x}}_i\|\|\mathbf{d}_{\tilde{\mathbf{x}}_i} - \mathbf{d}_{\mathbf{x}_0}\| - 2\varepsilon\|\mathbf{y} - \tilde{\mathbf{x}}_i\|\cos\alpha \\
&< 2\|\mathbf{y} - \tilde{\mathbf{x}}_i\|\left(\frac{\alpha}{2} + 2\sqrt{2\varepsilon\eta} + \sqrt{2}\|\mathbf{x}'\| - \varepsilon\cos\alpha\right) < 2\alpha\cos\alpha\underbrace{(f_\eta(\alpha) - \varepsilon)}_{\leqslant 0},
\end{aligned}$$

where we used (17.7) on $\tilde{\mathbf{x}}_i \in \partial\Omega_i \cap \mathcal{C}_{r,\varepsilon}$ and $\|\mathbf{x}'\| \leqslant \|\mathbf{x}_i - \mathbf{x}_0\| < \alpha$. Hence, $\mathbf{y} \in B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i})$ so $C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0}) \subseteq B_\varepsilon(\tilde{\mathbf{x}}_i - \varepsilon\mathbf{d}_{\tilde{\mathbf{x}}_i}) \subseteq \Omega_i$, using the ε-ball condition. Moreover, since $\tilde{\mathbf{y}}_i - \tilde{\mathbf{x}}_i = \mathbf{y}_i - \mathbf{x}_i$ and $\mathbf{y}_i \in C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$, we get $\tilde{\mathbf{y}}_i \in C_\alpha(\tilde{\mathbf{x}}_i, -\mathbf{d}_{\mathbf{x}_0})$, which ends the proof of $\tilde{\mathbf{y}}_i \in \Omega_i$. Finally, we check that $(\mathbf{y}_i, \tilde{\mathbf{y}}_i) \in \mathcal{C}_{r,\varepsilon} \times \mathcal{C}_{r,\varepsilon}$. We have successively:

$$\begin{cases}
\|\mathbf{y}'\| \leqslant \|\mathbf{y}' - \mathbf{x}'\| + \|\mathbf{x}'\| < \sqrt{\alpha^2 - \alpha^2\cos^2\alpha} + \alpha = \dfrac{\alpha}{\cos\alpha}\left(\dfrac{1}{2}\sin 2\alpha + \cos\alpha\right) < \dfrac{f_\eta(\alpha)}{2} \leqslant \dfrac{\varepsilon}{2} < r \\[2ex]
|y_n| \leqslant |y_n - x_n| + |x_n| \leqslant \|\mathbf{y}_i - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{x}_0\| < 2\alpha < f(\alpha) \leqslant \varepsilon \\[2ex]
|\tilde{y}_n| = |y_n + \varphi_i(\mathbf{x}') - x_n| \leqslant \|\mathbf{y}_i - \mathbf{x}_i\| + \varepsilon - \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2} < \alpha + \dfrac{\eta(2\varepsilon - \eta) + \|\mathbf{x}'\|^2}{\varepsilon + \sqrt{(\varepsilon-\eta)^2 - \|\mathbf{x}'\|^2}}.
\end{cases}$$

Here, we used Relation (17.9), the fact that $\mathbf{y}_i \in C_\alpha(\mathbf{x}_i, -\mathbf{d}_{\mathbf{x}_0})$ and $\mathbf{x}_i \in B_\alpha(\mathbf{x}_0)$. Hence, we obtain: $|\tilde{y}_n| < 2\alpha + 2\eta < 2f_\eta^{-1}(\varepsilon) + 2\frac{\varepsilon}{8} \leqslant \frac{2\varepsilon}{3} + \frac{\varepsilon}{4} < \varepsilon$. To conclude, apply Proposition 17.12 to $\tilde{\mathbf{y}}_i \in \Omega_i \cap \mathcal{C}_{r,\varepsilon}$ in order to get $y_n + \varphi_i(\mathbf{x}') - x_n < \varphi(\mathbf{y}')$. Since we firstly proved $x_n \leqslant \varphi_i(\mathbf{x}')$, we have $y_n < \varphi_i(\mathbf{y}')$. Applying Proposition 17.12 to $\mathbf{y}_i \in \mathcal{C}_{r,\varepsilon}$, we get $\mathbf{y}_i \in \Omega_i$ as required. □

Lemma 17.15. *The following map is well defined, smooth, surjective and increasing:*

$$\begin{array}{rccc}
g: &]0, \frac{\pi}{8}[& \longrightarrow &]0, +\infty[\\
& \eta & \longmapsto & \dfrac{32\eta}{\cos^2(4\eta)}.
\end{array}$$

In particular, it is an homeomorphism and its inverse g^{-1} satisfies the following relations:

$$\forall \varepsilon > 0, \quad g^{-1}(\varepsilon) < \frac{\varepsilon}{32} \qquad \text{and} \qquad g^{-1}(\varepsilon) < \frac{1}{4} f_{g^{-1}(\varepsilon)}^{-1}(\varepsilon), \tag{17.11}$$

where f_η^{-1} is defined in Lemma 17.13.

Proof. We only prove the inequality $g^{-1}(\varepsilon) < \frac{1}{4} f_{g^{-1}(\varepsilon)}^{-1}(\varepsilon)$. The remaining part is basic calculus. Consider any $\varepsilon > 0$. There exists a unique $\eta \in]0, \frac{\pi}{8}[$ such that $g(\eta) = \varepsilon$ or equivalently $\eta = g^{-1}(\varepsilon)$. Hence, we have $4\eta \in]0, \frac{\pi}{2}[$ so we can compute, using the first inequality $\eta < \frac{\varepsilon}{32}$:

$$f_\eta(4\eta) = \frac{2\sqrt{2\eta\varepsilon}}{\cos(4\eta)}\left(3\sqrt{\frac{2\eta}{\varepsilon}} + 1\right) < \frac{2\sqrt{2\eta\varepsilon}}{\cos(4\eta)}\left(3\sqrt{\frac{2}{32}} + 1\right) < \frac{4\sqrt{2\varepsilon\eta}}{\cos(4\eta)} = \sqrt{g(\eta)\varepsilon} = \varepsilon.$$

Since f_η is an increasing homeomorphism, so does f_η^{-1} and the inequality follows: $4\eta < f_\eta^{-1}(\varepsilon)$. □

Corollary 17.16. *In Assumption 17.6, we set $\eta = g^{-1}(\varepsilon)$, then consider $\alpha = f_\eta^{-1}(\varepsilon)$ and $\tilde{r} = \frac{1}{4}\alpha - \eta$. The restriction to $\overline{D_{\tilde{r}}}(\mathbf{0}')$ of the map φ_i defined in Proposition 17.12 is $\frac{1}{\tan\alpha}$-Lipschitz continuous.*

Proof. Let $\eta = g^{-1}(\varepsilon)$ and using (17.11), we have $\eta < \frac{\varepsilon}{32}$ so we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\alpha = f_\eta^{-1}(\varepsilon)$, but we also have $\tilde{r} := \frac{1}{4}\alpha - \eta > 0$. We consider any $(\mathbf{x}'_+, \mathbf{x}'_-) \in \overline{D_{\tilde{r}}}(\mathbf{0}') \times \overline{D_{\tilde{r}}}(\mathbf{0}')$. Using (17.10)-(17.11), we get $\tilde{r} < \frac{1}{4}f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{12} < \frac{1}{2}[4(\frac{31\varepsilon}{32})^2 - (\frac{33\varepsilon}{32})^2]^{\frac{1}{2}} < r$. From Proposition 17.12, we can define $\mathbf{x}_i^\pm := (\mathbf{x}'_\pm, \varphi_i(\mathbf{x}'_\pm)) \in \partial\Omega_i$. Then, we show that $\mathbf{x}_i^\pm \in \partial\Omega_i \cap B_\alpha(\mathbf{x}_0) \cap B_\alpha(\mathbf{x}_i^\mp)$. Relation (17.6) ensures that $\|\mathbf{x}_i^\pm - \mathbf{x}_0\| < 2\|\mathbf{x}'_\pm\| + 2\eta \leqslant 2\tilde{r} + 2\eta < \alpha$ and the triangle inequality gives $\|\mathbf{x}_i^+ - \mathbf{x}_i^-\| \leqslant \|\mathbf{x}_i^+ - \mathbf{x}_0\| + \|\mathbf{x}_0 - \mathbf{x}_i^-\| < 4\tilde{r} + 4\eta = \alpha$. Finally, we apply Proposition 17.14 to $\mathbf{x}_i^\pm \in \partial\Omega_i \cap B_\alpha(\mathbf{x}_0)$, which cannot belong to the cone $C_\alpha(\mathbf{x}_i^\mp, -\mathbf{d}_{\mathbf{x}_0}) \subseteq \Omega_i$. Hence, we obtain:

$$|\langle \mathbf{x}_i^+ - \mathbf{x}_i^- \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \cos\alpha \|\mathbf{x}_i^+ - \mathbf{x}_i^-\| = \cos\alpha\sqrt{\|\mathbf{x}'_+ - \mathbf{x}'_-\|^2 + |\langle \mathbf{x}_i^+ - \mathbf{x}_i^- \mid \mathbf{d}_{\mathbf{x}_0}\rangle|^2}.$$

Re-arranging the above inequality, we deduce that the map φ_i is L-Lipschitz continuous with $L > 0$ depending only on ε as required: $|\varphi_i(\mathbf{x}'_+) - \varphi_i(\mathbf{x}'_-)| = |\langle \mathbf{x}_i^+ - \mathbf{x}_i^- \mid \mathbf{d}_{\mathbf{x}_0}\rangle| \leqslant \frac{1}{\tan\alpha}\|\mathbf{x}'_+ - \mathbf{x}'_-\|$. □

Proposition 17.17. *We set $\tilde{r} = \frac{1}{4}f^{-1}_{g^{-1}(\varepsilon)}(\varepsilon) - g^{-1}(\varepsilon)$, where f and g are defined in Lemmas 17.13 and 17.15. Then, the restriction to $D_{\tilde{r}}(\mathbf{0}')$ of the map φ_i defined in Proposition 17.12 is differentiable:*

$$\forall \mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}'), \quad \nabla\varphi_i(\mathbf{a}') = \frac{-1}{\langle \mathbf{d}_{\mathbf{a}_i} \mid \mathbf{d}_{\mathbf{x}_0}\rangle}\mathbf{d}'_{\mathbf{a}_i} \qquad \text{where} \quad \mathbf{a}_i := (\mathbf{a}', \varphi_i(\mathbf{a}')).$$

Moreover, $\nabla\varphi_i : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is L-Lipschitz continuous with $L > 0$ depending only on ε.

Proof. Let $\eta = g^{-1}(\varepsilon)$ and using (17.11), we have $\eta < \frac{\varepsilon}{32}$ so we can set $r = \frac{1}{2}\sqrt{4(\varepsilon-\eta)^2 - (\varepsilon+\eta)^2}$ and $\alpha = f_\eta^{-1}(\varepsilon)$, but we also have $\tilde{r} := \frac{1}{4}\alpha - \eta > 0$. Let $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ and $\mathbf{x}' \in \overline{D_{\tilde{r}-\|\mathbf{a}'\|}}(\mathbf{a}')$. Hence, $(\mathbf{a}', \mathbf{x}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. Using (17.10)-(17.11), we get $\tilde{r} < \frac{1}{4}f_\eta^{-1}(\varepsilon) < \frac{\varepsilon}{12} < \frac{1}{2}[4(\frac{31\varepsilon}{32})^2 - (\frac{33\varepsilon}{32})^2]^{\frac{1}{2}} < r$. From Proposition 17.12, we can define $\mathbf{x}_i^\pm := (\mathbf{x}'_\pm, \varphi_i(\mathbf{x}'_\pm)) \in \partial\Omega_i$. Then, we apply (16.2) to Ω_i thus:

$$|\langle \mathbf{x}_i - \mathbf{a}_i \mid \mathbf{d}_{\mathbf{a}_i}\rangle| \leqslant \frac{1}{2\varepsilon}\|\mathbf{x}_i - \mathbf{a}_i\|^2 = \frac{1}{2\varepsilon}\left(\|\mathbf{x}' - \mathbf{a}'\|^2 + |\varphi_i(\mathbf{x}') - \varphi_i(\mathbf{a}')|^2\right) \leqslant \underbrace{\frac{1}{2\varepsilon}\left(1 + \frac{1}{\tan^2\alpha}\right)}_{:=C(\varepsilon)>0}\|\mathbf{x}' - \mathbf{a}'\|^2,$$

where we also used the Lipschitz continuity of φ_i on $\overline{D_{\tilde{r}}}(\mathbf{0}')$ established in Corollary 17.16. We note that $\mathbf{d}_{\mathbf{a}_i} = (\mathbf{d}'_{\mathbf{a}_i}, (\mathbf{d}_{\mathbf{a}_i})_n)$ where $(\mathbf{d}_{\mathbf{a}_i})_n = \langle \mathbf{d}_{\mathbf{a}_i} \mid \mathbf{d}_{\mathbf{x}_0}\rangle$. Hence, the above inequality takes the form:

$$|\left(\varphi_i(\mathbf{x}') - \varphi_i(\mathbf{a}')\right)(\mathbf{d}_{\mathbf{a}_i})_n + \langle \mathbf{d}'_{\mathbf{a}_i} \mid \mathbf{x}' - \mathbf{a}'\rangle| \leqslant C(\varepsilon)\|\mathbf{x}' - \mathbf{a}'\|^2.$$

This last inequality is a first-order Taylor expansion of φ_i if it can be divided by a uniform positive constant smaller than $(\mathbf{d}_{\mathbf{a}_i})_n$. Let us justify this last assertion. From (17.3) and (17.5), we deduce:

$$(\mathbf{d}_{\mathbf{a}_i})_n = 1 - \frac{1}{2}\|\mathbf{d}_{\mathbf{a}_i} - \mathbf{d}_{\mathbf{x}_0}\|^2 \geqslant 1 - \frac{1}{2\varepsilon^2}\|\mathbf{a}_i - \mathbf{x}_0\|^2 - \frac{4\varepsilon\eta - \eta^2}{2\varepsilon^2} > \frac{1}{\varepsilon}\sqrt{(\varepsilon-\eta)^2 - \|\mathbf{a}'\|^2} - \frac{\eta}{\varepsilon}.$$

Then, Inequality (17.11) gives $\frac{\eta}{\varepsilon} < \frac{1}{32}$ and from (17.10), it comes $\|\mathbf{a}'\| < \tilde{r} < \frac{\alpha}{4} < \frac{\varepsilon}{12}$. Consequently, we get $(\mathbf{d}_{\mathbf{a}_i})_n > [(\frac{31}{32})^2 - (\frac{1}{12})^2]^{\frac{1}{2}} - \frac{1}{32} > \frac{29}{32}$ and from the foregoing, we obtain:

$$\forall \mathbf{x}' \in \overline{D_{\tilde{r}-\|\mathbf{a}'\|}}(\mathbf{a}'), \quad \left|\varphi_i(\mathbf{x}') - \varphi_i(\mathbf{a}') + \left\langle \frac{\mathbf{d}'_{\mathbf{a}_i}}{(\mathbf{d}_{\mathbf{a}_i})_n} \mid \mathbf{x}' - \mathbf{a}'\right\rangle\right| \leqslant \frac{32C(\varepsilon)}{29}\|\mathbf{x}' - \mathbf{a}'\|^2.$$

Therefore, φ_i is differentiable at any point $\mathbf{a}' \in D_{\tilde{r}}(\mathbf{0}')$ with $\nabla\varphi_i(\mathbf{a}') = -\mathbf{d}'_{\mathbf{a}_i}/(\mathbf{d}_{\mathbf{a}_i})_n$. Finally, we show that $\nabla\varphi_i : D_{\tilde{r}}(\mathbf{0}') \to \mathbb{R}^{n-1}$ is Lipschitz continuous. Let $(\mathbf{x}', \mathbf{a}') \in D_{\tilde{r}}(\mathbf{0}') \times D_{\tilde{r}}(\mathbf{0}')$. We have:

$$\begin{aligned}
\|\nabla\varphi_i(\mathbf{x}') - \nabla\varphi_i(\mathbf{a}')\| &\leqslant \left|\tfrac{1}{(\mathbf{d}_{\mathbf{x}_i})_n} - \tfrac{1}{(\mathbf{d}_{\mathbf{a}_i})_n}\right|\|\mathbf{d}'_{\mathbf{x}_i}\| + \tfrac{1}{(d_{\mathbf{a}_i})_n}\|\mathbf{d}'_{\mathbf{a}_i} - \mathbf{d}'_{\mathbf{x}_i}\| \\
&\leqslant \frac{32}{29}\left(\frac{32}{29}|(\mathbf{d}_{\mathbf{a}_i})_n - (\mathbf{d}_{\mathbf{x}_i})_n| + \|\mathbf{d}_{\mathbf{a}_i} - \mathbf{d}_{\mathbf{x}_i}\|\right) \\
&\leqslant \frac{32}{29\varepsilon}\left(1 + \frac{32}{29}\right)\|\mathbf{x}_i - \mathbf{a}_i\| \;\leqslant\; \frac{32}{29\varepsilon}\left(1 + \frac{32}{29}\right)\sqrt{1 + \frac{1}{\tan^2\alpha}}\|\mathbf{x}' - \mathbf{a}'\|.
\end{aligned}$$

We used the fact that $(\mathbf{d}_{\mathbf{a}_i})_n < \frac{29}{32}$, the Lipschitz continuity of φ_i proved in Corollary 17.16 and the one of the map $\mathbf{x}_i \in \partial\Omega_i \mapsto \mathbf{d}_{\mathbf{x}_i}$ coming from Proposition 16.12 applied to $\Omega_i \in \mathcal{O}_\varepsilon(B)$. To conclude, $\nabla\varphi_i$ is an L-Lipschitz continuous map, where $L > 0$ depends only on ε. $\square$

Proof of Theorem 17.2. We set $K = \overline{D_{\frac{\tilde{r}}{2}}(\mathbf{0}')}$ where $\tilde{r} := \frac{1}{4} f^{-1}_{g^{-1}(\varepsilon)}(\varepsilon) - g^{-1}(\varepsilon) > 0$ from (17.11). From Propositions 17.12, 17.17 and Corollary 17.16, we proved that each Ω_i is parametrized by a local graph $\varphi_i : K \to]-\varepsilon, \varepsilon[$ as in Theorem 17.2. Hence, it remains to prove the convergence of these graphs. Since the sequence $(\varphi_i)_{i\in\mathbb{N}}$ is uniformly bounded and equi-Lipschitz continuous, from the Arzelà-Ascoli Theorem and up to a subsequence, it is converging to a continuous function $\tilde{\varphi} : K \to]-\varepsilon, \varepsilon[$. Considering the local map $\varphi : K \to]-\varepsilon, \varepsilon[$ associated with $\partial\Omega$, we now show that $\varphi \equiv \tilde{\varphi}$. Considering any $\mathbf{x}' \in K$, we set $\mathbf{x} = (\mathbf{x}', \tilde{\varphi}(\mathbf{x}'))$ and $\mathbf{x}_i = (\mathbf{x}', \varphi_i(\mathbf{x}'))$. There exists $\mathbf{y} \in \partial\Omega$ such that $d(\mathbf{x}_i, \partial\Omega) = \|\mathbf{x}_i - \mathbf{y}\|$. Then, we have:

$$\begin{aligned} d(\mathbf{x}, \partial\Omega) &\leqslant \|\mathbf{x} - \mathbf{y}\| \leqslant \|\mathbf{x} - \mathbf{x}_i\| + \|\mathbf{x}_i - \mathbf{y}\| = |\varphi_i(\mathbf{x}') - \tilde{\varphi}(\mathbf{x}')| + d(\mathbf{x}_\mathbf{i}, \partial\Omega) \\ &\leqslant \|\varphi_i - \tilde{\varphi}\|_{C^0(K)} + d_H(\partial\Omega_i, \partial\Omega). \end{aligned}$$

By letting $i \to +\infty$, we obtain $\mathbf{x} \in \partial\Omega \cap (K \times [-\varepsilon, \varepsilon])$. Hence, Proposition 17.12 gives $\mathbf{x} = (\mathbf{x}', \varphi(\mathbf{x}'))$ so $\varphi(\mathbf{x}') = \tilde{\varphi}(\mathbf{x}')$ for any $\mathbf{x}' \in K$. This also show that φ is the unique limit of any converging subsequence of $(\varphi_i)_{i\in\mathbb{N}}$. Hence, the whole sequence $(\varphi_i)_{i\in\mathbb{N}}$ is converging to φ uniformly on K. Similarly, $(\nabla\varphi_i)_{i\in\mathbb{N}}$ is uniformly bounded and equi-Lipschitz continuous, so it converges uniformly on K to a map, which must be $\nabla\varphi$ (use the convergence in the sense of distributions). To conclude, using [46, Section 5.2.2], each coefficient of the Hessian matrix of φ_i is uniformly bounded in $L^\infty(K)$. Hence [46, Lemma 2.2.27], each of them weakly-star converges in $L^\infty(K)$ to the one of φ. $\square$

Chapter 18

Continuity of some geometric functionals in the class $\mathcal{O}_\varepsilon(B)$

In this chapter, we prove that the convergence properties and the uniform $C^{1,1}$-regularity of the class $\mathcal{O}_\varepsilon(B)$ ensure the continuity of some geometric functionals. More precisely, with a suitable partition of unity, we show how to use the local convergence results of Theorem 17.2 to obtain the global continuity of linear integrals in the elementary symmetric polynomials of the principal curvatures. Throughout this chapter, we make the following hypothesis.

Assumption 18.1. *We assume that $(\Omega_i)_{i\in\mathbb{N}}$ is a sequence of elements from $\mathcal{O}_\varepsilon(B)$ converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of Proposition 17.1 (i)-(vi), where ε and B are as in Proposition 17.1.*

Remark 18.2. *Note that in this chapter, the proofs are based on the results of Theorem 17.2, so we only need to assume Points (ii) and (v) of Proposition 17.1 in the Assumption 18.1 (see Remark 17.3).*

Definition 18.3. *Let f, $(f_i)_{i\in\mathbb{N}} : E \to F$ be some continuous maps between two metric spaces. We say that $(f_i)_{i\in\mathbb{N}}$ diagonally converges to f if for any sequence $(t_i)_{i\in\mathbb{N}}$ converging to t in E, the sequence $(f_i(t_i))_{i\in\mathbb{N}}$ converges to $f(t)$ in F.*

Remark 18.4. *Note that the uniform convergence implies the diagonal convergence implying itself the pointwise convergence. Conversely, any sequence of equi-continuous maps converging pointwise is diagonally convergent. Moreover, from the Arzelà-Ascoli Theorem, it is uniformly convergent if in addition, it is uniformly bounded.*

The chapter is organized as follows. First, we recall the basic notions related to the geometry of hypersurfaces. Then, we study the continuity of functionals which depend on the position and the normal. Next, we consider linear functionals in the scalar mean curvature. Finally, we treat the case of the Gaussian curvature in $\mathbb{R}^3$ and we prove in $\mathbb{R}^n$ the following continuity result.

Theorem 18.5. *Let ε, B, Ω, $(\Omega_i)_{i\in\mathbb{N}}$ be as in Assumption 18.1. We consider some continuous maps $j^l, j^l_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that each sequence $(j^l_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j^l for any $l \in \{0, \dots, n-1\}$. Then, the following functional is continuous:*

$$J\left(\partial\Omega_i\right) := \sum_{l=0}^{n-1} \int_{\partial\Omega_i} \left[\sum_{1\leqslant n_1 < \ldots < n_l \leqslant n-1} \kappa_{n_1}^{\partial\Omega_i}\left(\mathbf{x}\right) \ldots \kappa_{n_l}^{\partial\Omega_i}\left(\mathbf{x}\right) \right] j^l_i\left[\mathbf{x}, \mathbf{n}^{\partial\Omega_i}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) \underset{i\to+\infty}{\longrightarrow} J(\partial\Omega),$$

where $\kappa_1, \ldots \kappa_{n-1}$ are the principal curvatures, $\mathbf{n}$ the unit outer normal field to the hypersurface, and where the integration is done with respect to the $(n-1)$-dimensional Hausdorff measure $A(\bullet)$.

Remark 18.6. *In the specific case of compact $C^{1,1}$-hypersurfaces, note that the above theorem is stronger than Federer's one on sets of positive reach [32, Theorem 5.9]. Indeed, in Theorem 18.5, taking $j^l_i(\mathbf{x}, \mathbf{n}(\mathbf{x})) = j^l(\mathbf{x})$ yields to the convergence of the curvature measures associated with $\partial\Omega_i$ to the ones of $\partial\Omega$ in the sense of Radon measures.*

18.1 On the geometry of hypersurfaces with $C^{1,1}$-regularity

Let us consider a non-empty compact $C^{1,1}$-hypersurface $\mathcal{S} \subset \mathbb{R}^n$. Merely speaking, for any point $\mathbf{x}_0 \in \mathcal{S}$, there exists $r_{\mathbf{x}_0} > 0$, $a_{\mathbf{x}_0} > 0$, and a unit vector $\mathbf{d}_{\mathbf{x}_0}$ such that in the cylinder defined by:

$$\mathcal{C}_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n, \quad |\langle \mathbf{x} - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < a_{\mathbf{x}_0} \text{ and } \|(\mathbf{x} - \mathbf{x}_0) - \langle \mathbf{x} - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle \mathbf{d}_{\mathbf{x}_0}\| < r_{\mathbf{x}_0}\}, \tag{18.1}$$

the hypersurface $\mathcal{S}$ is the graph of a $C^{1,1}$-map. Introducing the orthogonal projection on the affine hyperplane $\mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^{\perp}$:

$$\begin{array}{rccl} \Pi_{\mathbf{x}_0} : & \mathbb{R}^n & \longrightarrow & \mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^{\perp} \\ & \mathbf{x} & \longmapsto & \mathbf{x} - \langle \mathbf{x} - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle \mathbf{d}_{\mathbf{x}_0}, \end{array} \tag{18.2}$$

and considering the set $D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) = \Pi_{\mathbf{x}_0}(\mathcal{C}_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0))$, this means that there exists a continuously differentiable map $\varphi_{\mathbf{x}_0} : \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) \mapsto \varphi_{\mathbf{x}_0}(\mathbf{x}') \in]-a_{\mathbf{x}_0}, a_{\mathbf{x}_0}[$ such that its gradient $\nabla\varphi_{\mathbf{x}_0}$ and $\varphi_{\mathbf{x}_0}$ are $L_{\mathbf{x}_0}$-Lipschitz continuous maps, and such that:

$$\mathcal{S} \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) = \{\mathbf{x}' + \varphi_{\mathbf{x}_0}(\mathbf{x}')\mathbf{d}_{\mathbf{x}_0}, \quad \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{x}_0)\}.$$

Hence, we can introduce the local parametrization:

$$\begin{array}{rccl} X_{\mathbf{x}_0} : & D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) & \longrightarrow & \mathcal{S} \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) \\ & \mathbf{x}' & \longmapsto & \mathbf{x}' + \varphi_{\mathbf{x}_0}(\mathbf{x}')\mathbf{d}_{\mathbf{x}_0} \end{array}$$

and $\mathcal{S}$ is a $C^{1,1}$-hypersurface in the sense of [73, Definition 2.2]. Indeed, $X_{\mathbf{x}_0}$ is an homeomorphism, its inverse map is the restriction of $\Pi_{\mathbf{x}_0}$ to $C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0)$, and $X_{\mathbf{x}_0}$ is an immersion of class $C^{1,1}$.

We usually drop the dependence in $\mathbf{x}_0$ to lighten the notation, and consider a direct orthonormal frame $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ where $\mathcal{B}_{\mathbf{x}_0}$ is a basis of $\mathbf{d}_{\mathbf{x}_0}^{\perp}$. In this local frame, the point $\mathbf{x}_0$ is identified with the zero vector $\mathbf{0} \in \mathbb{R}^n$, the affine hyperplane $\mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^{\perp}$ with $\mathbb{R}^{n-1}$ and $\mathbf{x}_0 + \mathbb{R}\mathbf{d}_{\mathbf{x}_0}$ with $\mathbb{R}$. Hence, the cylinder $\mathcal{C}_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0)$ becomes $D_r(\mathbf{0}')\times]-a,a[$, $\varphi_{\mathbf{x}_0}$ is the $C^{1,1}$-map $\varphi : D_r(\mathbf{0}') \to]-a,a[$, the projection $\Pi_{\mathbf{x}_0}$ is $X^{-1} : (\mathbf{x}', x_n) \mapsto \mathbf{x}'$, and the parametrization $X_{\mathbf{x}_0}$ becomes the $C^{1,1}$-map $X : \mathbf{x}' \in D_r(\mathbf{0}') \mapsto (\mathbf{x}', \varphi(\mathbf{x}')) \in \mathcal{S} \cap (D_r(\mathbf{0}')\times]-a,a[)$. In this setting, $\mathcal{S}$ is a $C^{1,1}$-hypersurface in the sense of Definition 16.3.

Since $\mathbf{x}' \in D_r(\mathbf{0}') \mapsto D_{\mathbf{x}'}X$ is injective, the vectors $\partial_1 X, \ldots, \partial_{n-1}X$ are linearly independent. For any point $\mathbf{x} \in \mathcal{S}\cap(D_r(\mathbf{0}')\times]-a,a[)$, we define the tangent hyperplane $T_{\mathbf{x}}\mathcal{S}$ by $D_{X^{-1}(\mathbf{x})}X(\mathbb{R}^{n-1})$. It is an $(n-1)$-dimensional vector space so $(\partial_1 X, \ldots, \partial_{n-1}X)$ forms a basis of $T_{\mathbf{x}}\mathcal{S}$. However, this basis is not necessarily orthonormal. Consequently, the first fundamental form of $\mathcal{S}$ at $\mathbf{x}$ is defined as the restriction of the usual scalar product in $\mathbb{R}^n$ to the tangent hyperplane $T_{\mathbf{x}}\mathcal{S}$, i.e. as $\mathbf{I}(\mathbf{x}) : (\mathbf{v}, \mathbf{w}) \in T_{\mathbf{x}}\mathcal{S} \times T_{\mathbf{x}}\mathcal{S} \mapsto \langle \mathbf{v} \mid \mathbf{w}\rangle$. In the basis $(\partial_1 X, \ldots, \partial_{n-1}X)$, it is represented by a positive-definite symmetric matrix usually referred to as $(g_{ij})_{1\leqslant i,j\leqslant n-1}$ and its inverse denoted by $(g^{ij})_{1\leqslant i,j\leqslant n-1}$ is also explicitly given in this case:

$$g_{ij} = \langle \partial_i X \mid \partial_j X\rangle = \delta_{ij} + \partial_i\varphi\partial_j\varphi, \tag{18.3}$$

$$g^{ij} = \delta_{ij} - \frac{\partial_i\varphi\partial_j\varphi}{1 + \|\nabla\varphi\|^2}. \tag{18.4}$$

As a function of $\mathbf{x}'$, note that each coefficient of these two matrices is Lipschitz continuous so it is a $W^{1,\infty}$-map [30, Section 4.2.3], and from Rademacher's Theorem [30, Section 3.1.2], its differential exists almost everywhere. Moreover, any $\mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$ can be decomposed in the basis $(\partial_1 X, \ldots, \partial_{n-1}X)$. Denoting by V_i the component of $\partial_i X$ and $v_i = \langle \mathbf{v} \mid \partial_i X\rangle$, we have:

$$\mathbf{v} = \sum_{i=1}^{n-1} V_i\partial_i X \implies v_j = \sum_{i=1}^{n} V_i g_{ij} \implies V_i = \sum_{j=1}^{n-1} g^{ij}v_j \implies \mathbf{v} = \sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-1} g^{ij}v_j\right)\partial_i X. \tag{18.5}$$

In particular, we deduce $\mathbf{I}(\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{n-1} g^{ij}v_i w_j$. Then, the orthogonal of the tangent hyperplane is one dimensional. Hence, there exists a unique unit vector $\mathbf{n}$ orthogonal to the $(n-1)$ vectors $\partial_1 X, \ldots, \partial_{n-1}X$ and pointing outwards the inner domain of $\mathcal{S}$ i.e. $\det(\partial_1 X, \ldots, \partial_{n-1}X, \mathbf{n}) > 0$. It is called the unit outer normal to the hypersurface and we have its explicit expression:

$$\forall \mathbf{x}' \in D_r(\mathbf{0}'), \quad \mathbf{n} \circ X(\mathbf{x}') = \frac{1}{\sqrt{1 + \|\nabla\varphi(\mathbf{x}')\|^2}}\begin{pmatrix} -\nabla\varphi(\mathbf{x}') \\ 1 \end{pmatrix}. \tag{18.6}$$

It is a Lipschitz continuous map, like the coefficients of the first fundamental form. In particular, it is differentiable almost everywhere and introducing the Gauss map $\mathbf{n} : \mathbf{x} \in \mathcal{S} \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^{n-1}$, we can compute its differential almost everywhere called the Weingarten map:

$$\begin{array}{rrcl} D_{\mathbf{x}}\mathbf{n}: & T_{\mathbf{x}}\mathcal{S} = D_{X^{-1}(\mathbf{x})}X(\mathbb{R}^2) & \longrightarrow & T_{\mathbf{n}(\mathbf{x})}\mathbb{S}^{n-1} = D_{X^{-1}(\mathbf{x})}(\mathbf{n}\circ X)(\mathbb{R}^2) \\ & \mathbf{v} = D_{X^{-1}(\mathbf{x})}X(\mathbf{w}) & \longmapsto & D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) = D_{X^{-1}(\mathbf{x})}(\mathbf{n}\circ X)(\mathbf{w}). \end{array} \tag{18.7}$$

Note that $T_{\mathbf{n}(\mathbf{x})}\mathbb{S}^{n-1} = D_{X^{-1}(\mathbf{x})}(\mathbf{n}\circ X)(\mathbb{R}^2)$ because $\mathbf{n}\circ X$ is a Lipschitz parametrization of $\mathbb{S}^{n-1}$. Since $T_{\mathbf{n}(\mathbf{x})}\mathbb{S}^{n-1} \sim \mathbf{n}(\mathbf{x})^{\perp}$ can be identified with $T_{\mathbf{x}}\mathcal{S}$, the map $D_{\mathbf{x}}\mathbf{n}$ is an endomorphism of $T_{\mathbf{x}}\mathcal{S}$. Moreover, one can prove it is self-adjoint so it can be diagonalized to obtain $n-1$ eigenvalues denoted by $\kappa_1(\mathbf{x})$, …, $\kappa_{n-1}(\mathbf{x})$ and called the principal curvatures. Recall that the eigenvalues of an endomorphism do not depend on the chosen basis and thus are really properties of the operator. This assertion also holds for the coefficients of the characteristic polynomial associated with $D_{\mathbf{x}}\mathbf{n}$ so we can introduce them:

$$\forall l \in \{0, \ldots, n-1\}, \quad H^{(l)}(\mathbf{x}) = \sum_{1\leqslant n_1 < \ldots < n_l \leqslant n-1} \kappa_{n_1}(\mathbf{x}) \ldots \kappa_{n_l}(\mathbf{x}). \tag{18.8}$$

In particular, $H^{(0)} = 1$, $H^{(1)} = H$ is called the scalar mean curvature, and $H^{(n-1)} = K$ refers to the Gaussian curvature:

$$H(\mathbf{x}) = \kappa_1(\mathbf{x}) + \ldots + \kappa_{n-1}(\mathbf{x}) \qquad \text{and} \qquad K(\mathbf{x}) = \kappa_1(\mathbf{x})\kappa_2(\mathbf{x})\ldots\kappa_{n-1}(\mathbf{x}). \tag{18.9}$$

Moreover, introducing the symmetric matrix $(b_{ij})_{1\leqslant i,j\leqslant n-1}$ defined by:

$$b_{ij} = -\langle D\mathbf{n}(\partial_i X) \mid \partial_j X\rangle = -\langle \partial_i(\mathbf{n}\circ X) \mid \partial_j X\rangle = \frac{\operatorname{Hess}\varphi}{\sqrt{1+\|\nabla\varphi\|^2}} = \langle \mathbf{n}\circ X \mid \partial_{ij}X\rangle, \tag{18.10}$$

we get from (18.5) that the Weingarten map $D\mathbf{n}$ is represented in the local basis $(\partial_1 X, \ldots, \partial_{n-1}X)$ by the following symmetric matrix:

$$(h_{ij})_{1\leqslant i,j\leqslant n-1} = \left(-\sum_{k=1}^{n-1} g^{ik}b_{kj}\right) = \left(-\sum_{k=1}^{n-1}\left(\delta_{ik} - \frac{\partial_i\varphi\partial_j\varphi}{1+\|\nabla\varphi\|^2}\right)\frac{\partial_{kj}\varphi}{\sqrt{1+\|\nabla\varphi\|^2}}\right). \tag{18.11}$$

Finally, we introduce the symmetric bilinear form whose representation in the local basis is (b_{ij}). It is called the second fundamental form of the hypersurface and it is defined by:

$$\begin{array}{rrcl} \mathbf{II}(\mathbf{x}): & T_{\mathbf{x}}(\mathcal{S})\times T_{\mathbf{x}}(\mathcal{S}) & \longrightarrow & \mathbb{R} \\ & (\mathbf{v},\mathbf{w}) & \longmapsto & \langle -D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) \mid \mathbf{w}\rangle = \displaystyle\sum_{i,j,k,l=1}^{n-1} g^{ij}v_j g^{kl}w_l b_{il} = \sum_{i,j,k=1}^{n-1} g^{ij}v_j v_k h_{ki}. \end{array} \tag{18.12}$$

We can also decompose $\partial_{ij}X$ in the basis $(\partial_1 X, \ldots, \partial_{n-1}X, \mathbf{n})$ and its coefficients in the tangent space are the Christoffel symbols:

$$\partial_{ij}X = \sum_{k=1}^{n-1}\Gamma_{ij}^k \partial_k X + b_{ij}\mathbf{n}$$

Note that the Christoffel symbols are symmetric with respect to the lower indices: $\Gamma_{ij}^k = \Gamma_{ji}^k$. They can be expressed only in terms of coefficients of the first fundamental form:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{n-1} g^{kl}\left(\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}\right). \tag{18.13}$$

Like the first fundamental form, it is an intrinsic notion, which in particular do not depend on the orientation chosen for the hypersurface, while the Gauss map, the Weingarten map, and the second fundamental form does. Note that in local coordinates, the coefficients of the first fundamental form and the Gauss map are Lipschitz continuous functions i.e. $\mathbf{n}\circ X, g_{ij}, g^{ij} \in W^{1,\infty}(D_r(\mathbf{0}'))$. Hence, the Christoffel symbols, the Weingarten map and the coefficients of the second fundamental form exist almost everywhere and $\Gamma_{ij}^k, b_{ij}, h_{ij} \in L^{\infty}(D_r(\mathbf{0}'))$. Furthermore, one can prove that a

$C^{1,1}$-hypersurface satisfies the following relations in the sense of distributions, respectively called the Gauss and Codazzi-Mainardi equations:

$$\partial_l \Gamma_{ij}^k - \partial_j \Gamma_{il}^k + \sum_{m=1}^{n-1} \left(\Gamma_{ij}^m \Gamma_{ml}^k - \Gamma_{il}^m \Gamma_{mj}^k \right) = \sum_{m=1}^{n-1} g^{km} \left(b_{ij} b_{ml} - b_{il} b_{mj} \right) \tag{18.14}$$

$$\partial_k b_{ij} - \partial_j b_{ik} = \sum_{l=1}^{n-1} \left(\Gamma_{ik}^l b_{lj} - \Gamma_{ij}^l b_{lk} \right). \tag{18.15}$$

In fact, the converse statement is also true in $\mathbb{R}^3$: these equations characterize uniquely a surface and it is referred as the Fundamental Theorem of Surface Theory, valid with $C^{1,1}$-regularity [65]. Given a simply-connected open subset $\omega \subseteq \mathbb{R}^2$, a symmetric positive-definite matrix $(g_{ij})_{1\leqslant i,j \leqslant 2} \in W^{1,\infty}(\omega)$ and a symmetric matrix $(b_{ij})_{1\leqslant i,j\leqslant 2} \in L^\infty(\omega)$ satisfying (18.14) and (18.15) in the sense of distributions, then there exists an injective $C^{1,1}$-immersion $X : \omega \to \mathbb{R}^3$, unique up to proper isometries of $\mathbb{R}^3$, such that the surface $\mathcal{S} := X(\omega)$ has (g_{ij}) and (b_{ij}) as coefficients of the first and second fundamental forms. To conclude, we recall that $A(\bullet)$ (respectively $V(\bullet)$) refers to the $(n-1)$ (resp. n)-dimensional Hausdorff measure. The integration is always be done with respect to A and the infinitesimal area element is given by $(dA \circ X)(\mathbf{x}') = \sqrt{\det(g_{ij})} d\mathbf{x}' = \sqrt{1 + \|\nabla\varphi(\mathbf{x}')\|^2} d\mathbf{x}'$. We refer to [18, 73] for a more detailed exposition on all the notions quickly introduced here.

18.2 Geometric functionals involving the position and the normal

Proposition 18.7. *Under assumption 18.1, for any continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

In particular, the area and the volume are continuous: $A(\partial\Omega_i) \longrightarrow A(\partial\Omega)$ and $V(\Omega_i) \longrightarrow V(\Omega)$.

Remark 18.8. *Note that the above result states the convergence of $(\partial\Omega_i)_{i\in\mathbb{N}}$ to $\partial\Omega$ in the sense of oriented varifolds [6, Appendix B] [87]. Similar results were obtained in [40]. Moreover, the continuity of volume and the lower semi-continuity of area were already implied by the convergence in the sense of characteristic functions (Point (vi) in Proposition 17.1) [46, Proposition 2.3.6].*

Proof. Consider Assumption 18.1. Hence, from Theorem 17.2, the boundaries $(\partial\Omega_i)_{i\in\mathbb{N}}$ are locally parametrized by graphs of $C^{1,1}$-maps φ_i that converge strongly in C^1 and weakly-star in $W^{2,\infty}$ to the map φ associated with $\partial\Omega$. We now detail the procedure which allows to pass from this local result to the global one thanks to a suitable partition of unity. For any $\mathbf{x} \in \partial\Omega$, we introduce the cylinder $\mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x})$ defined by (18.1) and we assume that $\tilde{r} > 0$ is the one given in Theorem 17.2. In particular, it only depends on ε. Since $\partial\Omega$ is compact, there exists a finite number $K \geqslant 1$ of points written $\mathbf{x}_1, \ldots, \mathbf{x}_K$, such that $\partial\Omega \subseteq \bigcup_{k=1}^K \mathcal{C}_{\frac{\tilde{r}}{2},\frac{\varepsilon}{2}}(\mathbf{x}_k)$. We set $\delta = \min(\frac{\tilde{r}}{2}, \frac{\varepsilon}{2}) > 0$. From the triangle inequality, the tubular neighbourhood $\mathcal{V}_\delta(\partial\Omega) = \{\mathbf{y} \in \mathbb{R}^n, \quad d(\mathbf{y}, \partial\Omega) < \delta\}$ has its closure embedded in $\bigcup_{k=1}^K \mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)$. Then, we can introduce a partition of unity on this set. There exists K non-negative C^∞-maps ξ^k with compact support in $\mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)$ and such that $\sum_{k=1}^K \xi^k(\mathbf{x}) = 1$ for any point $\mathbf{x} \in \mathcal{V}_\delta(\partial\Omega)$. Now, we can apply Theorem 17.2 to the K points $\mathbf{x}_k$. There exists K integers $I_k \in \mathbb{N}$ and some maps $\varphi_i^k : \overline{D_{\tilde{r}}}(\mathbf{x}_k) \mapsto]-\varepsilon, \varepsilon[$, with $i \geqslant I_k$ and $K \geqslant k \geqslant 1$, such that:

$$\begin{cases} \partial\Omega_i \cap \overline{\mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)} &= \left\{(\mathbf{x}', \varphi_i^k(\mathbf{x}')), \quad \mathbf{x}' \in \overline{D_{\tilde{r}}}(\mathbf{x}_k)\right\} \\ \\ \Omega_i \cap \overline{\mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)} &= \left\{(\mathbf{x}', x_n), \quad \mathbf{x}' \in \overline{D_{\tilde{r}}}(\mathbf{x}_k) \quad \text{and} \quad -\varepsilon \leqslant x_n < \varphi_i^k(\mathbf{x}')\right\}. \end{cases}$$

Moreover, the K sequences of functions $(\varphi_i^k)_{i\geqslant I_k}$ and $(\nabla\varphi_i^k)_{i\geqslant I_k}$ converge uniformly on $\overline{D_{\tilde{r}}}(\mathbf{x}_k)$ respectively to the maps φ^k and $\nabla\varphi^k$ associated with $\partial\Omega$ at each point $\mathbf{x}_k$. From the Hausdorff convergence of the boundaries (Point (ii) in Proposition 17.1), there also exists $I_0 \in \mathbb{N}$ such that for any integer $i \geqslant I_0$, we have $\partial\Omega_i \in \mathcal{V}_\delta(\partial\Omega)$. Hence, we set $I = \max_{0\leqslant k\leqslant K} I_k$, which thus only depends on $(\Omega_i)_{i\in\mathbb{N}}$, Ω and ε. Then, we deduce that for any integer $i \geqslant I$, we have:

$$\begin{aligned}
J(\partial\Omega_i) \quad &:= \quad \int_{\partial\Omega_i} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \quad = \quad \int_{\partial\Omega_i \cap \mathcal{V}_\delta(\partial\Omega)} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\
&= \quad \int_{\partial\Omega_i} \left(\sum_{k=1}^{K} \xi^k(\mathbf{x})\right) j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \quad = \quad \sum_{k=1}^{K} \int_{\partial\Omega_i \cap \mathcal{C}_{r,\varepsilon}(\mathbf{x}_k)} \xi^k(\mathbf{x})\, j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\
&= \quad \sum_{k=1}^{K} \int_{D_{\tilde{r}}(\mathbf{x}_k)} \xi^k \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix} j\left[\begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix}, \begin{pmatrix} \frac{-\nabla\varphi_i^k(\mathbf{x}')}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \\ \frac{1}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \end{pmatrix}\right] \sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2} d\mathbf{x}'
\end{aligned}$$

The last equality comes from [73, Proposition 5.13] and Relation (18.6). The uniform convergence of the K sequences $(\varphi_i^k)_{i\geqslant I}$ and $(\nabla\varphi_i^k)_{i\geqslant I}$ on the compact set $\overline{D_{\tilde{r}}}(\mathbf{x}_k)$ combined with the continuity of j and $(\xi^k)_{1\leqslant k\leqslant K}$ allows one to let $i \to \infty$ in the above expression. Observing that the limit expression obtained is equal to $J(\partial\Omega)$, we proved that the functional J is continuous. Finally, for the area, take $j \equiv 1$ and for the volume, applying the Divergence Theorem, take $j[\mathbf{x}, \mathbf{n}(\mathbf{x})] = \frac{1}{n}\langle \mathbf{x} \mid \mathbf{n}(\mathbf{x})\rangle$. ☐

Proposition 18.9. *Consider Assumption 18.1 and some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j in the sense of Definition 18.3. Then, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} j_i\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

Proof. The proof is identical to the one of Proposition 18.7. Using the same partition of unity and the same notation, we get that $\int_{\partial\Omega_i} j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})] dA(\mathbf{x})$ is equal to:

$$\sum_{k=1}^{K} \int_{D_{\tilde{r}}(\mathbf{x}_k)} \xi^k \begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix} j_i\left[\begin{pmatrix} \mathbf{x}' \\ \varphi_i^k(\mathbf{x}') \end{pmatrix}, \begin{pmatrix} \frac{-\nabla\varphi_i^k(\mathbf{x}')}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \\ \frac{1}{\sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2}} \end{pmatrix}\right] \sqrt{1+\|\nabla\varphi_i^k(\mathbf{x}')\|^2} d\mathbf{x}'.$$

Then, instead of using the uniform convergence of each integrand on a compact set as it is the case in Proposition 18.7, we apply instead Lebesgue's Dominated Convergence Theorem. Indeed, the diagonal convergence ensures the pointwise convergence of each integrand, which are also, using the other hypothesis, uniformly bounded. Hence, we can let $i \to +\infty$ in the above expression. ☐

Definition 18.10. *Let $\mathcal{S}, \mathcal{S}_i$ be some non-empty compact C^1-hypersurfaces of $\mathbb{R}^n$ such that $(\mathcal{S}_i)_{i\in\mathbb{N}}$ converges to $\mathcal{S}$ for the Hausdorff distance: $d_H(\mathcal{S}_i, \mathcal{S}) \longrightarrow_{i\to+\infty} 0$. On each hypersurface $\mathcal{S}_i$, we also consider a continuous vector field $\mathbf{V}_i : \mathbf{x} \in \mathcal{S}_i \mapsto \mathbf{V}_i(\mathbf{x}) \in T_{\mathbf{x}}\mathcal{S}_i$. We say that $(\mathbf{V}_i)_{i\in\mathbb{N}}$ is diagonally converging to a vector field on $\mathcal{S}$ denoted by $\mathbf{V} : \mathbf{x} \in \mathcal{S} \mapsto \mathbf{V}(\mathbf{x}) \in T_{\mathbf{x}}\mathcal{S}$ if for for any point $\mathbf{x} \in \mathcal{S}$ and for any sequence of points $\mathbf{x}_i \in \mathcal{S}_i$ that converges to $\mathbf{x}$, we have $\|\mathbf{V}_i(\mathbf{x}_i) - \mathbf{V}(\mathbf{x})\| \longrightarrow_{i\to+\infty} 0$.*

Corollary 18.11. *Let $\varepsilon, B, \Omega, (\Omega_i)_{i\in\mathbb{N}}$ be as in Assumption 18.1, and consider some continuous vector fields $\mathbf{V}_i$ on $\partial\Omega_i$ converging to a continuous vector field $\mathbf{V}$ on $\partial\Omega$ as in Definition 18.10. We also assume that $(\mathbf{V}_i)_{i\in\mathbb{N}}$ is uniformly bounded. Considering a continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$, then we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \mathbf{V}_i(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \mathbf{V}(\mathbf{x})\right] dA(\mathbf{x}).$$

Of course, this continuity result can be extended to a finite number of vector fields.

Proof. We only have to check that the maps $j_i : (\mathbf{x}, \mathbf{u}) \in \partial\Omega_i \times \mathbb{S}^{n-1} \to j[\mathbf{x}, \mathbf{u}, \mathbf{V}_i(\mathbf{x})]$ can be extended to $\mathbb{R}^n \times \mathbb{S}^{n-1}$ such that their extension satisfy the hypothesis of Proposition 18.9. This is a standard procedure [46, Section 5.4.1]. Using the partition of unity given in Proposition 18.7 and introducing the $C^{1,1}$-diffeomorphisms $\Psi_i^k : (\mathbf{x}', x_n) \in \overline{\mathcal{C}_{r,\varepsilon}}(\mathbf{x_k}) \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}') - x_n)$, we can set:

$$\forall(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \quad j_i(\mathbf{x}, \mathbf{u}) = \sum_{k=1}^{K} \xi^k(\mathbf{x}) j\left[(\Psi_i^k)^{-1} \circ \Pi_{\mathbf{x}_k} \circ \Psi_i^k(\mathbf{x}), \mathbf{u}, \mathbf{V}_i \circ (\Psi_i^k)^{-1} \circ \Pi_{\mathbf{x}_k} \circ \Psi_i^k(\mathbf{x})\right].$$

We recall that $\Pi_{\mathbf{x}_k}$ is defined by (18.2). Finally, $(j_i)_{i\in\mathbb{N}}$ diagonally converges to the extension of $(\mathbf{x},\mathbf{u}) \mapsto j[\mathbf{x},\mathbf{u},\mathbf{V}(\mathbf{x})]$, since $(V_i)_{i\in\mathbb{N}}$ is diagonally converging to V. Moreover, $(\Omega_i)_{i\in\mathbb{N}} \subset B$, the Gauss map is always valued in $\mathbb{S}^{n-1}$, and $(V_i)_{i\in\mathbb{N}}$ is uniformly bounded. Hence, $(\mathbf{x},\mathbf{n}_{\partial\Omega_i}(\mathbf{x}),\mathbf{V}_i(\mathbf{x}))$ is valued in a compact set. Since j is continuous on this compact set, it is bounded and $(j_i)_{i\in\mathbb{N}}$ is thus uniformly bounded on $\overline{B}\times\mathbb{S}^{n-1}$. Finally, we can apply Proposition 18.9 to let $i\to+\infty$. □

18.3 Linear functions involving the second fundamental form

From Theorem 17.2, we only have the L^∞-weak-star convergence of the coefficients associated with the Hessian of the local maps φ_i^k so we consider here the case of functionals whose expressions in the parametrization are linear in $\partial_{pq}\varphi_i^k$. This is the case for the scalar mean curvature and the second fundamental form of two vector fields.

Proposition 18.12. *Consider Assumption 18.1 and a continuous map $j:\mathbb{R}^n\times\mathbb{S}^{n-1}\to\mathbb{R}$. Then, the following functional is continuous:*

$$\lim_{i\to+\infty}\int_{\partial\Omega_i} H(\mathbf{x})\, j\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} H(\mathbf{x})\, j\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

Proof. The proof is identical to the one of Proposition 18.7. Using the same notation and the same partition of unity, we have to check that in the parametrization $X_i^k : \mathbf{x}'\in\overline{D_{\tilde r}}(\mathbf{x}_k)\mapsto(\mathbf{x}',\varphi_i^k(\mathbf{x}'))$, the scalar mean curvature L^∞-weakly-star converges. It is the trace (18.9) of the Weingarten map defined by (18.7) so Relation (18.11) gives:

$$(H\circ X_i^k) = -\sum_{p,q=1}^{n-1} g^{pq}b_{qp} = -\sum_{p,q=1}^{n-1}\left(\delta_{pq} - \frac{\partial_p\varphi_i^k\partial_q\varphi_i^k}{1+\|\nabla\varphi_i^k\|^2}\right)\left(\frac{\partial_{pq}\varphi_i^k}{\sqrt{1+\|\nabla\varphi_i^k\|^2}}\right). \tag{18.16}$$

Using Theorem 17.2, the K sequences $(H\circ X_i^k)_{i\in\mathbb{N}}$ weakly-star converge in $L^\infty(D_{\tilde r}(\mathbf{x}_k))$ respectively to $H\circ X^k$. The remaining part of each integrand below uniformly converges to the one of $\partial\Omega$ so we can let $i\to+\infty$ inside:

$$\sum_{k=1}^{K}\int_{D_{\tilde r}(\mathbf{x}_k)}(H\circ X_i^k)(\mathbf{x}')(\xi^k\circ X_i^k)(\mathbf{x}')j[X_i^k(\mathbf{x}'),(\mathbf{n}\circ X_i^k)(\mathbf{x}')](dA\circ X_i^k)(\mathbf{x}'),$$

to get the limit asserted in Proposition 18.12. □

Corollary 18.13. *Consider Assumption 18.1 and a continuous map $j:\mathbb{R}^n\times\mathbb{S}^{n-1}\times\mathbb{R}\to\mathbb{R}$ which is convex in its last variable. Then, we have:*

$$\int_{\partial\Omega} j\left[\mathbf{x},\mathbf{n}(\mathbf{x}),H(\mathbf{x})\right] dA(\mathbf{x}) \leqslant \liminf_{i\to+\infty}\int_{\partial\Omega_i} j\left[\mathbf{x},\mathbf{n}(\mathbf{x}),H(\mathbf{x})\right] dA(\mathbf{x}).$$

Remark 18.14. *In particular, this result implies that the Helfrich* (15.2) *and Willmore functionals* (15.3) *are lower semi-continuous, and so does the p-th power norm of mean curvature $\int|H|^p dA$, $p\geqslant 1$. Note that we are able to treat the critical case $p=1$, while it is often excluded from many statements of geometric measure theory [24, Example 4.1] [72, Definition 2.2] [48, Definition 4.1.2].*

Proof. The arguments are standard [28, Theorem 2.2.1]. We only sketch the proof. First, assume that j is the maximum of finitely many affine functions according to its last variable:

$$\forall t\in\mathbb{R},\quad j(\mathbf{x},\mathbf{n}(\mathbf{x}),t) = \max_{0\leqslant l\leqslant L} j_l\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right]t + \tilde{j}_l\left[\mathbf{x},\mathbf{n}(\mathbf{x})\right]. \tag{18.17}$$

For simplicity, let us assume that j only depends on the position. Using a partition of unity as in Proposition 18.7, we introduce the local parametrizations $X^k:\mathbf{x}'\in D_{\tilde r}(\mathbf{x}_k)\mapsto(\mathbf{x}',\varphi^k(\mathbf{x}'))$ and we make a partition of the set $D_{\tilde r}(\mathbf{x}_k)$ into L disjoints sets. We define for any $l\in\{1,\dots L\}$:

$$D_l^k = \left\{\mathbf{x}'\in D_{\tilde r}(\mathbf{x}_k),\ j\left[X^k(\mathbf{x}'),\left(H\circ X^k\right)(\mathbf{x}')\right] = j_l\left[X^k(\mathbf{x}')\right]H\left[X^k(\mathbf{x}')\right] + \tilde{j}_l\left[X^k(\mathbf{x}')\right]\right\}.$$

Then, applying Proposition 18.12 and following [28, below (2.9)], we have successively:

$$
\begin{aligned}
\int_{\partial\Omega} j[\mathbf{x}, H(\mathbf{x})]dA(\mathbf{x}) &= \sum_{k=1}^{K} \int_{D_{\tilde{r}}(\mathbf{x}_k)} j[X^k, (H \circ X^k)](dA \circ X^k) \\
&= \sum_{k=1}^{K} \sum_{l=1}^{L} \int_{D_l^k} \left(j_l[X^k]H[X^k] + \tilde{j}_l[X^k]\right)(dA \circ X^k) \\
&= \sum_{k=1}^{K} \sum_{l=1}^{L} \lim_{i \to +\infty} \int_{D_l^k} \left(j_l[X_i^k]H[X_i^k] + \tilde{j}_l[X_i^k]\right)(dA \circ X_i^k) \\
&\leqslant \sum_{k=1}^{K} \sum_{l=1}^{L} \liminf_{i \to +\infty} \int_{D_l^k} j[X_i^k, (H \circ X_i^k)](dA \circ X_i^k) \\
&\leqslant \liminf_{i \to +\infty} \int_{\partial\Omega_i} j[\mathbf{x}, H(\mathbf{x})]dA(\mathbf{x}).
\end{aligned}
$$

The result holds for maps j that are maximum of finitely many planes. In the general case, we write $j = \lim_{L \to +\infty} j_L$ where j_L is defined by (18.17) and apply the Monotone Convergence Theorem. $\square$

Proposition 18.15. *Consider Assumption 18.1 and some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j in the sense of Definition 18.3. Then, the following functional is continuous:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} H(\mathbf{x})\, j_i\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} H(\mathbf{x})\, j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

Remark 18.16. *As in Corollary 18.11, we can consider here that j_i is a continuous map of the position, the normal, and a finite number of uniformly bounded vector fields diagonally converging in the sense of Definition 18.10.*

Proof. The proof is identical to the one of Proposition 18.12. Writing the functional in terms of local parametrizations, it remains to check that we can let $i \to +\infty$ in each integral. From (18.16), $(H \circ X_i^k)_{i\in\mathbb{N}}$ weakly-star converges in $L^\infty(\overline{D_{\tilde{r}}}(\mathbf{0}'))$ to $H \circ X^k$, while the remaining part of the integrand is strongly converging in $L^1(\overline{D_{\tilde{r}}}(\mathbf{0}'))$, since the hypothesis allows one to apply Lebesgue's Dominated Convergence Theorem. Hence, the functional is continuous. $\square$

Proposition 18.17. *Consider Assumption 18.1 and some uniformly bounded continuous vector fields $\mathbf{V}_i$ and $\mathbf{W}_i$ on $\partial\Omega_i$ that are diagonally converging to continuous vector fields $\mathbf{V}$ and $\mathbf{W}$ on $\partial\Omega$ in the sense of Definition 18.10. Let $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ be continuous maps such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j as in Definition 18.3. Then, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} \mathbf{II}(\mathbf{x})\left[\mathbf{V}_i(\mathbf{x}), \mathbf{W}_i(\mathbf{x})\right] j_i\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \int_{\partial\Omega} \mathbf{II}(\mathbf{x})\left[\mathbf{V}(\mathbf{x}), \mathbf{W}(\mathbf{x})\right] j\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

Remark 18.18. *Note that if $j_i = j$ for any $i \in \mathbb{N}$, then the above assertion states that a functional which is linear in the second fundamental form is continuous. Hence, using the same arguments than in Corollary 18.13, any functional whose integrand is a continuous map of the position, the normal, and the second fundamental form, convex in its last variable, is lower semi-continuous.*

Proof. We write the integral in terms of local parametrizations and check that we can let $i \to +\infty$. In the local basis $(\partial_1 X_i^k, \ldots, \partial_{n-1} X_i^k)$, using (18.12), the second fundamental form takes the form:

$$\left(\mathbf{II} \circ X_i^k\right)\left(\mathbf{V}_i \circ X_i^k, \mathbf{W}_i \circ X_i^k\right) = \sum_{p,q,r,s=1}^{n-1} \left\langle \mathbf{V}_i \circ X_i^k \mid \partial_p X_i^k \right\rangle g^{pq} b_{qr} g^{rs} \left\langle \mathbf{W}_i \circ X_i^k \mid \partial_s X_i^k \right\rangle.$$

Hence, each integrand is the product of $g^{pq}b_{qr}g^{rs}$ with a remaining term. Using the assumptions, the convergence results of Theorem 17.2, and Lebesgue's Dominated Convergence Theorem, we get that $g^{pq}b_{qr}g^{rs}$ weakly-star converges in L^∞, while the remaining term L^1-strongly converges. $\square$

18.4 Nonlinear functions involving the 2nd fundamental form

All the previous continuity results were obtained by expressing the integrals in the parametrizations associated with a suitable partition of unity, and by observing that each integrand is the product of b_{pq} converging L^∞-weakly-star with a remaining term converging L^1-strongly. We are wondering here if a non-linear function such as the determinant of the (b_{pq}) can also L^∞-weakly-star converge. Note that the convergence is in L^∞ and *not* in $W^{1,p}$ so we cannot use e.g. [29, Section 8.2.4.b].

However, the coefficients of the first and second fundamental forms are not random coefficients. They characterize the hypersurfaces through the Gauss-Codazzi-Mainardi equations (18.14) and (18.15). Hence, using the differential structure of these equations, we want to obtain the L^∞-weak-star convergence of non-linear functions of the b_{pq}. This is done by considering a generalization of the Div-Curl Lemma due to Tartar. We refer to [28, Section 5.5] for references and it states as follows.

Proposition 18.19 (Tartar 1979). *Let $n \geqslant 3$ and $U \subset \mathbb{R}^{n-1}$ be open and bounded with smooth boundary. Let us consider a sequence of maps $(u_i)_{i\in\mathbb{N}}$ weakly-star converging to u in $L^\infty(U,\mathbb{R}^M)$, $M \geqslant 1$, and a continuous functional $F : \mathbb{R}^M \to \mathbb{R}$ such that $(F(u_i))_{i\in\mathbb{N}}$ is weakly-star converging in $L^\infty(U,\mathbb{R})$. Let us suppose we are given P first-order constant coefficient differential operators $A^p v := \sum_{q=1}^{n-1}\sum_{m=1}^{M} a^p_{mq}\partial_q v_m$ so that the sequences $(A^p u_i)_{i\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(U)$. We also assume that $(u_i)_{i\in\mathbb{N}}$ is almost everywhere valued in K for some given compact set $K \subset \mathbb{R}^M$. We introduce the following wave cone:*

$$\Lambda = \left\{ \lambda \in \mathbb{R}^M \mid \exists \mu \in \mathbb{R}^{n-1}\backslash\{\mathbf{0}'\}, \forall p \in \{1,\dots P\}, \quad \sum_{q=1}^{n-1}\sum_{m=1}^{M} a^p_{mq}\lambda_m\mu_q = 0 \right\}.$$

If F is a quadratic form and $F = 0$ on Λ, then the weak-star limit of $(F(u_i))_{i\in\mathbb{N}}$ is $F(u)$.

We now treat the case of $\mathbb{R}^3$ to get familiar with the notation and observe how Proposition 18.19 can be used here to obtain the L^∞-weak-star convergence of the Gaussian curvature $K = \kappa_1\kappa_2$. Let $n = 3$, $U = D_{\tilde{r}}(\mathbf{x}_k)$, and $u_i : \mathbf{x}' \mapsto (b_{pq}) \in \mathbb{R}^{2^2}$ defined by (18.10) with $X_i^k : \mathbf{x}' \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}')) \in \partial\Omega_i$. First, we show that the assumptions of Proposition 18.19 are satisfied. From Theorem 17.2, $(u_i)_{i\in\mathbb{N}}$ $L^\infty(U)$-weakly-star converges to u and it is uniformly bounded so it is valued in a compact set. Moreover, in the case $n = 3$, there are only two Codazzi-Mainardi equations (18.15):

$$\begin{cases} \partial_1 b_{12} - \partial_2 b_{11} = \left(\Gamma^1_{11}b_{12} - \Gamma^1_{12}b_{11}\right) + \left(\Gamma^2_{11}b_{22} - \Gamma^2_{12}b_{21}\right) \\[2ex] \partial_1 b_{22} - \partial_2 b_{21} = \left(\Gamma^1_{21}b_{12} - \Gamma^1_{22}b_{11}\right) + \left(\Gamma^2_{21}b_{22} - \Gamma^2_{22}b_{21}\right). \end{cases}$$

Hence, the two differential operators $A^1 u_i := \partial_1 b_{12} - \partial_2 b_{11}$ and $A^2 u_i := \partial_1 b_{22} - \partial_2 b_{21}$ are valued and uniformly bounded in $L^\infty(U)$, which is compactly embedded in $H^{-1}(U)$ (Rellich-Kondrachov Embedding Theorem), so we deduce that up to a subsequence, $(A^1 u_i)_{i\in\mathbb{N}}$ and $(A^2 u_i)_{i\in\mathbb{N}}$ lies in a compact subset of $H^{-1}(U)$. Let us now have a look at the wave cone:

$$\Lambda = \left\{ \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in \mathbb{R}^{2^2} \mid \exists \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mu_1\lambda_{12} - \mu_2\lambda_{11} = 0 \text{ and } \mu_1\lambda_{22} - \mu_2\lambda_{21} = 0 \right\}.$$

Remark 18.20. *The wave cone Λ is the set of (2×2)-matrices with zero determinant.*

Consequently, if we want to apply Proposition 18.19 on a quadratic form in the b_{pq}, we get from Remark 18.20 that the determinant is one possibility. Indeed, if we set $F(u_i) = \det(u_i)$, then F is quadratic and $F(\lambda) = 0$ for any $\lambda \in \Lambda$. Since $(F(u_i))_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(U)$, up to a subsequence, it is converging and applying Proposition 18.19, the limit is $F(u)$. This also proves that $F(u)$ is the unique limit of any converging subsequence. Hence, the whole sequence is converging to $F(u)$ and we are now in position to prove the following result.

Proposition 18.21. *Consider Assumption 18.1 and some continuous maps $j, j_i : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^2$ and diagonally converges to j as in Definition 18.3. Then, we have (note that Remarks 18.16 and 18.18 also hold here):*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} K(\mathbf{x})\, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} K(\mathbf{x})\, j[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

In particular, the genus is continuous: $\mathrm{genus}(\partial\Omega_i) \longrightarrow_{i\to+\infty} \mathrm{genus}(\partial\Omega)$.

Proof. As in the proof of Proposition 18.7, we can express the functional in the parametrizations associated with the partition of unity. Then, we have to check we can let $i \to +\infty$ in each integral. Note that K is the determinant (18.9) of the Weingarten map (18.7) so we get from (18.11):

$$K \circ X_i^k = \det(h) = \det(-g^{-1}b) = -\frac{\det(b_{pq})}{\det(g_{rs})}.$$

From the foregoing and the uniform convergence of (g_{rs}), we get that the sequences $(K \circ X_i^k)_{i\in\mathbb{N}}$ converge L^∞-weakly-star respectively to $K \circ X^k$, whereas the remaining term in the integrand is L^1-strongly converging using the hypothesis and Lebesgue's Dominated Convergence Theorem. Hence, we can let $i \to +\infty$ and Proposition 18.21 holds. Finally, concerning the genus, we apply the Gauss-Bonnet Theorem $\int_{\partial\Omega_i} K dA = 4\pi(1-g_i) \longrightarrow_{i\to+\infty} \int_{\partial\Omega} K dA = 4\pi(1-g)$. □

We now establish the equivalent of Proposition 18.21 in $\mathbb{R}^n$. First, instead of working with the coefficients (b_{pq}) of the second fundamental form (18.10), we prefer to work with the ones (h_{pq}) representing the Weingarten map. We set $n > 3$, $U = D_{\tilde{r}}(\mathbf{x}_k)$, and $u_i : \mathbf{x}' \in U \mapsto (h_{pq}) \in \mathbb{R}^{(n-1)^2}$ defined by (18.11) in the local parametrizations $X_i^k : \mathbf{x}' \in U \mapsto (\mathbf{x}', \varphi_i^k(\mathbf{x}')) \in \partial\Omega_i$ introduced in the proof of Proposition 18.7. Then, we check that the hypothesis of Proposition 18.19 are satisfied. From Theorem 17.2, $(u_i)_{i\in\mathbb{N}}$ weakly-star converges to u in $L^\infty(U)$ and it is uniformly bounded so it is valued in a compact set. Using the Codazzi-Mainardi equations (18.15), the differential operators:

$$\partial_{q'} h_{pq} - \partial_q h_{pq'} = \sum_{m=1}^{n-1} ((\partial_{q'} g^{pm}) b_{mq} - (\partial_q g^{pm}) b_{mq'}) + \sum_{m=1}^{n-1} g^{pm} (\partial_{q'} b_{mq} - \partial_q b_{mq'}),$$

are valued and uniformly bounded in $L^\infty(U)$, which is compactly embedded in $H^{-1}(U)$ (Rellich-Kondrachov Embedding Theorem), so up to a subsequence, they lies in a compact set of $H^{-1}(U)$. Finally, we introduce the wave cone of Proposition 18.19:

$$\Lambda = \left\{ \lambda \in \mathbb{R}^{(n-1)^2} \mid \exists \mu \neq 0_{(n-1)\times 1}, \forall (p,q,m) \in \{1,\dots,n-1\}^3, \quad \mu_m \lambda_{pq} - \mu_q \lambda_{pm} = 0 \right\}.$$

Definition 18.22. *A pth-order minor of a square $(n-1)^2$-matrix M is the determinant of any $(p \times p)$-matrix $M[I,J]$ formed by the coefficients of M corresponding to rows with index in I and columns with index in J, where $I, J \subset \{1,\dots,n-1\}$ have p elements i.e. $\sharp I = \sharp J = p$.*

Remark 18.23. *The wave cone Λ is the set of square $(n-1)^2$-matrices of rank zero or one. In particular, any minor of order two is zero for such matrices.*

Consequently, Remark 18.23 combined with Proposition 18.19 tells us that continuous functionals are given by the ones whose expressions in the local parametrizations (cf. proof of Proposition 18.7) are linear in terms of the form $h_{pq}h_{p'q'} - h_{pq'}h_{p'q}$. However, such terms depend on the partition of unity and on the parametrizations i.e. on the chosen basis $(\partial_1 X_i^k, \dots, \partial_{n-1} X_i^k)$ whereas the integrand of the functional cannot. We now give three applications for which it is the case.

Proposition 18.24. *Consider Assumption 18.1 and some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ so that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j in the sense of Definition 18.3. Then, introducing $H^{(2)} = \sum_{1\leqslant p<q\leqslant n-1} \kappa_p\kappa_q$ defined in* (18.8)*, we have:*

$$\lim_{i\to+\infty} \int_{\partial\Omega_i} H^{(2)}(\mathbf{x})\, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}) = \int_{\partial\Omega} H^{(2)}(\mathbf{x})\, j[\mathbf{x}, \mathbf{n}(\mathbf{x})]\, dA(\mathbf{x}).$$

Note that Remarks 18.16 and 18.18 also hold for this functional.

Proof. First, using the notation of Definition 18.22, note that the characteristic polynomial of (h_{pq}), which is the matrix (18.11) representing the Weingarten map (18.7) in the basis $(\partial_1 X_i^k, \dots \partial_{n-1} X_i^k)$, can be expressed as:

$$P(t) = \det(h - tI_{n-1}) = (-1)^n t^n + \sum_{m=1}^{n-1} (-1)^{n-m} \left(\sum_{\sharp I = m} \det(h[I,I]) \right) t^{n-m},$$

but we can also represent the Weingarten map in the basis associated with the principal curvatures:

$$P(t) = \prod_{m=1}^{n-1} \left(\left(\kappa_m \circ X_i^k \right) - t \right) = \sum_{m=0}^{n-1} (-1)^{n-m} \left(H^{(m)} \circ X_i^k \right) t^{n-m}.$$

Since each coefficients of the characteristic polynomial do not depend on the chosen basis, we get:

$$\forall m \in \{0, \ldots, n-1\}, \quad H^{(m)} \circ X_i^k = \sum_{\sharp I = m} \det(h[I, I]). \tag{18.18}$$

If we set $F(\lambda) = \sum_{\sharp I=2} \det(\lambda[I, I])$, then F is quadratic and from Remark 18.23 we get $F(\lambda) = 0$ for any $\lambda \in \Lambda$. Since $(F(u_i))_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(U)$, up to a subsequence, it is converging and applying Proposition 18.19, the limit is $F(u)$, unique limit of any converging subsequence so the whole sequence is converging to $F(u)$. Using (18.18), we get that the sequences $(H^{(2)} \circ X_i^k)_{i\in\mathbb{N}}$ converge L^∞-weakly-star respectively to $H^{(2)} \circ X^k$, whereas the remaining term in the integrand is L^1-strongly converging using the hypothesis and Lebesgue's Dominated Convergence Theorem. Hence, we can let $i \to +\infty$ and the functional is continuous. □

Corollary 18.25. *Considering Assumption 18.1, a continuous map $j : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}$ convex in its last variable, and the (Frobenius) L^2-norm $\|D_\mathbf{x}\mathbf{n}\|_2 = \sqrt{\mathrm{trace}(D_\mathbf{x}\mathbf{n} \circ D_\mathbf{x}\mathbf{n}^T)} = (\sum_{m=1}^{n-1} \kappa_m^2)^{\frac{1}{2}}$ of the Weingarten map* (18.7)*, we have:*

$$\int_{\partial\Omega} j \left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \|D_\mathbf{x}\mathbf{n}\|_2^2 \right] dA(\mathbf{x}) \leqslant \liminf_{i\to+\infty} \int_{\partial\Omega_i} j \left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \|D_\mathbf{x}\mathbf{n}\|_2^2 \right] dA(\mathbf{x}).$$

In particular, the pth-power of the L^2-norm of the second fundamental form $\int \|\mathbf{II}\|_2^p dA$, $p \geqslant 2$ is lower semi-continuous.

Proof. First, assume that j is linear in its last argument. Note that the Frobenius norm $\|.\|_2$ does not depend on the chosen basis so we can consider the one associated with the principal curvatures, and we get $\|D\mathbf{n}\|_2^2 = \sum_{m=1}^{n-1} \kappa_m^2 = (\sum_{m=1}^{n-1} \kappa_m)^2 - \sum_{p\neq q} \kappa_p\kappa_q = H^2 - 2H^{(2)}$. Hence, there exists a continuous map $\tilde{j} : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $\int_{\partial\Omega_i} j[\mathbf{x}, \mathbf{n}(\mathbf{x}), \|D_\mathbf{x}\mathbf{n}\|_2^2] dA(\mathbf{x})$ is equal to:

$$\int_{\partial\Omega_i} H^2(\mathbf{x})\, \tilde{j}\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) - 2 \int_{\partial\Omega_i} H^{(2)}(\mathbf{x})\, \tilde{j}\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}).$$

In the left term, the integrand is convex in H so Corollary 18.13 furnishes its lower semi-continuity. Concerning the right one, apply Proposition 18.24 to get its continuity. Therefore, the functional is lower semi-continuous if j is linear in its last variable. Then, we can apply the standard procedure [28, Theorem 2.2.1] described in Corollary 18.13 to get the same result in the general case. Finally, $\|\mathbf{II}(\mathbf{x})\|_2^2 = \|D_\mathbf{x}\mathbf{n}\|_2^2$ and if $p \geqslant 2$, $t \mapsto t^{\frac{p}{2}}$ is convex thus $\int \|\mathbf{II}\|_2^p dA$ is lower semi-continuous. □

Proposition 18.26. *Consider Assumption 18.1, some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j as in Definition 18.3, and some vector fields $\mathbf{V}_i$ and $\mathbf{W}_i$ on $\partial\Omega_i$ uniformly bounded and diagonally converging to vector fields $\mathbf{V}$ and $\mathbf{W}$ on $\partial\Omega$ in the sense of Definition 18.10. Then, the following functional is continuous (note that Remarks 18.16 and 18.18 also hold here):*

$$J(\partial\Omega_i) = \int_{\partial\Omega_i} \langle D_\mathbf{x}\mathbf{n}\left[\mathbf{V}_i(\mathbf{x})\right] \mid D_\mathbf{x}\mathbf{n}\left[\mathbf{W}_i(\mathbf{x})\right] - H(\mathbf{x})\,\mathbf{W}_i(\mathbf{x})\rangle\, j_i\left[\mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J(\partial\Omega).$$

Proof. Again, the idea is to check that the expression of the functional in the parametrization is linear in a term of the form $b_{pq}b_{p'q'} - b_{pq'}b_{p'q'}$. First, the linear term can be expressed as:

$$\sum_{p,p',p"=1}^{n-1} \sum_{q,q',q"=1}^{n-1} \left\langle \mathbf{V}_i \circ X_i^k \mid \partial_q X_i^k \right\rangle g^{pq} g^{p'q'} \left(b_{q'p} b_{p"p'} - b_{q'p'} b_{pp"} \right) g^{p"q"} \left\langle \mathbf{W}_i \circ X_i^k \mid \partial_{q"} X_i^k \right\rangle$$

Note that until now, in Chapter 18, we never used the fact that (g_{pq}), (g^{pq}), (b_{pq}) or (h_{pq}) are symmetric matrices. Here, let us invert the two indices $b_{pp"} = b_{p"p}$ in the above expression. Then, $b_{q'p}b_{p"p'} - b_{q'p'}b_{p"p}$ is L^∞-weakly-star converging. Indeed, as we did for (h_{pq}), we can use the Codazzi-Mainardi equations (18.15) and Remark 18.23 to apply Proposition 18.19 on (b_{pq}). Finally, the hypothesis and the convergence results of Theorem 17.2 gives the L^1-strong convergence of the remaining term so we can let $i \to +\infty$ in each integral and the functional is continuous. □

Note that until now, in Chapter 18.4, we only used the Codazzi-Mainardi equations (18.15). We want here to use the Gauss equations (18.14) because from the foregoing, its right member is L^∞-weakly-star converging. For this purpose, we need to introduce some concepts of Riemannian geometry which are beyond the scope of the article. Hence, we refer to [93] for precise definitions. Merely speaking, the Riemann curvature tensor R of a Riemannian manifold measures the extend to which the first fundamental form is not locally isometric to a Euclidean space, i.e. the noncommutativity of the covariant derivative. In the basis $(\partial_1 X, \dots, \partial_{n-1} X)$, it has the following representation [93, Section 2.6]:

$$R^k_{jli} = \sum_{m=1}^{n-1} g^{km} R_{mjli} = \partial_l \Gamma^k_{ij} - \partial_j \Gamma^k_{il} + \sum_{m=1}^{n-1} \left(\Gamma^m_{ij} \Gamma^k_{ml} - \Gamma^m_{il} \Gamma^k_{mj} \right),$$

where the Christoffels symbols Γ^k_{ij} were defined in (18.13). Hence, the Gauss equations (18.14) state that in the local parametrization, the Riemann curvature tensor is given by:

$$R^k_{jli} = \sum_{m=1}^{n-1} g^{km} \left(b_{ij} b_{ml} - b_{il} b_{mj} \right),$$

which is thus L^∞-weakly-star converging, and so does the Ricci curvature tensor [93, Section 3.3] $Ric_{ij} = \sum_{k=1}^{n-1} R^k_{ikj}$ and the scalar curvature $\mathfrak{R} = \sum_{i,j=1}^{n-1} g^{ij} R_{ij}$. Hence, the following result holds.

Proposition 18.27. *Consider Assumption 18.1, some continuous maps $j, j_i : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ such that $(j_i)_{i\in\mathbb{N}}$ is uniformly bounded on $\overline{B} \times \mathbb{S}^{n-1}$ and diagonally converges to j as in Definition 18.3, and some vector fields $\mathbf{T}_i, \mathbf{U}_i, \mathbf{V}_i, \mathbf{W}_i$ on $\partial\Omega_i$ uniformly bounded and diagonally converging to vector fields $\mathbf{T}, \mathbf{U}, \mathbf{V}, \mathbf{W}$ on $\partial\Omega$ in the sense of Definition 18.10. Then, the three following functionals are continuous (note that Remarks 18.16 and 18.18 also hold here):*

$$\begin{cases} J(\partial\Omega_i) = \displaystyle\int_{\partial\Omega_i} \langle R_{\mathbf{x}} [\mathbf{T}_i(\mathbf{x}), \mathbf{U}_i(\mathbf{x})] \mathbf{V}_i(\mathbf{x}) \mid \mathbf{W}_i(\mathbf{x}) \rangle \, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})] \, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J(\partial\Omega) \\ J'(\partial\Omega_i) = \displaystyle\int_{\partial\Omega_i} Ric_{\mathbf{x}} [\mathbf{V}_i(\mathbf{x}), \mathbf{W}_i(\mathbf{x})] \, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})] \, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J'(\partial\Omega) \\ J"(\partial\Omega_i) = \displaystyle\int_{\partial\Omega_i} \mathfrak{R}(\mathbf{x}) \;\, j_i[\mathbf{x}, \mathbf{n}(\mathbf{x})] \, dA(\mathbf{x}) \underset{i\to+\infty}{\longrightarrow} J"(\partial\Omega) \end{cases}$$

Proof. The proof is same than previous ones. Write the functional in the local parametrizations, and observe that it is a finite sum of integrals whose integrand is the product of a L^∞-weakly-star converging term, while the other one is converging L^1-strongly so we can let $i \to +\infty$. □

Note that in the case $n = 3$, the scalar curvature $\mathfrak{R}$ is twice the Gaussian curvature $K = \kappa_1 \kappa_2$. Hence, the continuity of the last functional above is the generalization of Proposition 18.21 to $\mathbb{R}^n$, $n > 3$, which was the task of the section. We conclude by proving Theorem 18.5.

Proof of Theorem 18.5. Using Proposition 18.19 and (18.15), we showed how to get the L^∞-weakly-star convergence of any $h[pp', qq'] := h_{pq} h_{p'q'} - h_{pq'} h_{p'q}$ from the one of (h_{pq}) defined in (18.11). Now, we want to apply Proposition 18.19 to $(h[pp', qq'])$. For this purpose, we need to find differential operators which are valued and uniformly bounded in L^∞. Using (18.15), this is the case for:

$$\begin{aligned} \begin{vmatrix} \partial_q & h_{pq} & h_{p'q} \\ \partial_{q'} & h_{pq'} & h_{p'q'} \\ \partial_{q"} & h_{pq"} & h_{p'q"} \end{vmatrix} &= \partial_q h[pp', q'q"] - \partial_{q'} h[pp', qq"] + \partial_{q"} h[pp', qq'] \\ &= \left(\partial_q h_{pq'} - \partial_{q'} h_{pq} \right) h_{p'q"} + \left(\partial_{q'} h_{p'q} - \partial_q h_{p'q'} \right) h_{pq"} \\ &\quad + \left(\partial_q h_{p'q"} - \partial_{q"} h_{p'q} \right) h_{pq'} + \left(\partial_{q"} h_{p'q'} - \partial_{q'} h_{p'q"} \right) h_{pq} \\ &\quad + \left(\partial_{q"} h_{pq} - \partial_q h_{pq"} \right) h_{p'q'} + \left(\partial_{q'} h_{p'q"} - \partial_{q"} h_{pq'} \right) h_{p'q}. \end{aligned}$$

Then, the wave cone associated with these differential operators is thus given by:

$$\Lambda = \left\{ \lambda \in \mathbb{R}^{(n-1)^4} \mid \exists \mu \neq 0_{(n-1)\times 1}, \forall (p, p', q, q', q") \in \{0, \dots, n-1\}, \quad \begin{vmatrix} \mu_q & \lambda_{pq} & \lambda_{p'q} \\ \mu_{q'} & \lambda_{pq'} & \lambda_{p'q'} \\ \mu_{q"} & \lambda_{pq"} & \lambda_{p'q"} \end{vmatrix} = 0 \right\}.$$

As in Remark 18.20, one can check that the wave cone is given by all $(n-1)^2$-matrices for which any minor of order three are zero in the sense of Definition 18.22. Finally, combining (18.18) and Proposition 18.19, we get that functionals linear in $H^{(3)}$ are continuous. This procedure can be done recursively similarly to $H^{(l)}$ for any $l \geqslant 3$ so Theorem 18.5 holds. $\square$

18.5 Existence of a minimizer for some geometric functionals

We are now in position to establish general existence results in the class $\mathcal{O}_\varepsilon(B)$. More precisely, we can minimize any functional (and constraints) constructed from those given before in Chapter 18. Indeed, considering a minimizing sequence in $\mathcal{O}_\varepsilon(B)$, there exists a converging subsequence in the sense of Proposition 17.1 (i)-(vi). Then, applying the appropriate continuity results, we can pass to the limit in the functional and the constraints to get the existence of a minimizer.

In this section, we first give a proof of Theorem 15.2 and state/prove its generalization to $\mathbb{R}^n$. Then, we establish the existence for a very general model of vesicles. In particular, we prove that hold Theorems 15.5, 15.7, and 15.8. We refer to Sections 15.1, 15.2, and 15.3 of the introduction for a detailed exposition on these three models. Finally, we present two more applications that show how to use other continuity results to get the existence of a minimizer in the class $\mathcal{O}_\varepsilon(B)$.

Proof of Theorem 15.2. Consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$. From Proposition 17.1, up to a subsequence, it is converging to an open set $\Omega \in \mathcal{O}_\varepsilon(B)$. Since Assumption 18.1 holds, we can combine Propositions 18.7, 18.12, and 18.21 to let $i \to +\infty$ in the equalities of the form:

$$\int_{\partial\Omega_i} g_0\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega_i} H\left(\mathbf{x}\right) g_1\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \int_{\partial\Omega_i} K\left(\mathbf{x}\right) g_2\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) = \widetilde{C}.$$

Then, apply Proposition 18.7, Corollary 18.13 and Remark 18.18 on Proposition 18.21, to obtain the lower semi-continuity of the functional and that inequality contraints remain true as $i \to +\infty$. Therefore, Ω is a minimizer of the functional satisfying the constraints in the class $\mathcal{O}_\varepsilon(B)$. $\square$

Theorem 18.28. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ be a bounded open set containing a ball of radius 3ε such that ∂B has zero n-dimensional Lebesgue measure. Consider $(C,\widetilde{C}) \in \mathbb{R}\times\mathbb{R}$, some continuous maps $j_0, f_0, g_0, g_l : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$, and some maps $j_l, f_l : \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}$ which are continuous and convex in their last variable for any $l \in \{1,\ldots,n-1\}$. Then, the following problem has at least one solution (for the notation, we refer to Section 18.1):*

$$\inf \int_{\partial\Omega} j_0\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \sum_{l=1}^{n-1} \int_{\partial\Omega} j_l\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right), H^{(l)}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right),$$

where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying a finite number of constraints of the following form:

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \sum_{l=1}^{n-1} \int_{\partial\Omega} f_l\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right), H^{(l)}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) \leqslant C \\[2ex] \displaystyle\int_{\partial\Omega} g_0\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \sum_{l=1}^{n-1} \int_{\partial\Omega} H^{(l)}\left(\mathbf{x}\right) g_l\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) = \widetilde{C}. \end{cases}$$

Proof. Consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$. From Proposition 17.1, up to a subsequence, it is converging to an open set $\Omega \in \mathcal{O}_\varepsilon(B)$. Since Assumption 18.1 holds, we can apply Theorem 18.5 to let $i \to +\infty$ in the following equality:

$$\int_{\partial\Omega_i} g_0\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) + \sum_{l=1}^{n-1} \int_{\partial\Omega_i} H^{(l)}\left(\mathbf{x}\right) g_l\left[\mathbf{x},\mathbf{n}\left(\mathbf{x}\right)\right] dA\left(\mathbf{x}\right) = \widetilde{C}.$$

Then, we can use again Theorem 18.5 for any $l_0 \in \{1,\ldots,n-1\}$ by setting $j_{l_0} = g_{l_0}$ and $j_l = 0$ for any $l \neq l_0$ to obtain the continuity of any $\int H^{(l_0)}(.)g_{l_0}[.,\mathbf{n}(.)]$ and Remark 18.18 gives the lower semi-continuity of any $\int f_{l_0}[.,\mathbf{n}(.),H^{(l_0)}(.)]$ and $\int j_{l_0}[.,\mathbf{n}(.),H^{(l_0)}(.)]$. Hence, the functional is lower-semi-continuous and the inequality constraint remains true as $i \to +\infty$. Therefore, Ω is a minimizer of the functional satisfying the constraints. $\square$

Proposition 18.29. *Let $H_0, M_0, k_G, k_m \in \mathbb{R}$ and $\varepsilon, k_b, A_0, V_0 > 0$ such that $A_0^3 > 36\pi V_0^2$. Then, the following problem modelling the equilibrium shapes of vesicles [85, Section 2.5] has at least one solution (see Notation 15.3):*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(\mathbb{R}^n)\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + k_G\int_{\partial\Omega} K dA + k_m\left(\int_{\partial\Omega} H dA - M_0\right)^2.$$

Proof. Let us consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(\mathbb{R}^n)$ of the functional satisfying the area and volume constraints. First, we need to find an open ball B such that $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$. This can be done if we can bound the diameter thanks to the functional and the area constraint. The first step is to control the Willmore energy (15.3). Denoting by J the functional, we have:

$$\begin{aligned}
\frac{k_b}{4}\int_{\partial\Omega} H^2 dA &= \frac{k_b}{4}\int_{\partial\Omega}(H-H_0+H_0)^2 dA \leqslant \frac{k_b}{2}\int_{\partial\Omega}(H-H_0)^2 dA + \frac{k_b H_0^2}{2}A(\partial\Omega)\\
&\leqslant J(\partial\Omega) + \frac{k_b H_0^2}{2}A(\partial\Omega) + |k_G|\left|\int_{\partial\Omega} K dA\right| + |k_m|\left(\int_{\partial\Omega} H dA - M_0\right)^2\\
&\leqslant J(\partial\Omega) + \frac{k_b H_0^2}{2}A(\partial\Omega) + |k_G|\int_{\partial\Omega}|K| dA + 2|k_m|\left(\int_{\partial\Omega} H dA\right)^2 + 2|k_m|M_0^2.
\end{aligned}$$

The second step is to use Point (iii) in Theorem 16.6 and Remark 16.8. Considering a point $\mathbf{x} \in \partial\Omega$ in which the Gauss map $\mathbf{n}$ is differentiable, and a unit eigenvector $\mathbf{e}_l$ associated with the eigenvalue κ_l of the Weingarten map $D_{\mathbf{x}}\mathbf{n}$, we have:

$$|\kappa_l(\mathbf{x})| = \|\kappa_l(\mathbf{x})\mathbf{e}_l\| = \|D_{\mathbf{x}}\mathbf{n}(\mathbf{e}_l)\| \leqslant \|D_{\mathbf{x}}\mathbf{n}\|_{\mathcal{L}(T_{\mathbf{x}}\partial\Omega)}\|\mathbf{e}_l\| \leqslant \frac{1}{\varepsilon}, \tag{18.19}$$

from which we deduce that $\max_{1\leqslant l\leqslant n-1}\|\kappa_l\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\varepsilon}$. Hence, we obtain:

$$\frac{k_b}{4}\int_{\partial\Omega} H^2 dA \leqslant J(\partial\Omega) + \frac{k_b H_0^2}{2}A(\partial\Omega) + \frac{|k_G|}{\varepsilon^2}A(\partial\Omega) + \frac{8|k_m|}{\varepsilon^2}A(\partial\Omega)^2 + 2|k_m|M_0^2.$$

The final step is to apply [88, Lemma 1.1] to get four positive constants C_0, C_1, C_2, C_3 such that:

$$\mathrm{diam}(\Omega) \leqslant C_0 J(\partial\Omega)A(\partial\Omega) + C_1 A(\partial\Omega) + C_2 A(\partial\Omega)^2 + C_3 A(\partial\Omega)^3.$$

Hence, we can bound uniformly the diameter of the Ω_i and there exists a ball $B \subset \mathbb{R}^n$ sufficiently large such that $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$. From Proposition 17.1, up to a subsequence, it is converging to an $\Omega \in \mathcal{O}_\varepsilon(B)$. Then, we can apply:

- Corollary 18.13 with $j(x,y,z) = \frac{k_b}{2}(z-H_0)^2$ to get the lower semi-continuity of $\frac{k_b}{2}\int(H-H_0)^2$;
- Proposition 18.21 with $j_i \equiv 1$ to obtain the continuity of $\kappa_G \int K$;
- Proposition 18.12 with $j \equiv 1$ to have the continuity of $\int H dA$ thus the one of $k_m(\int H dA - M_0)^2$.

The functional is lower semi-continuous and from Proposition 18.7 with $j \equiv 1$ and $j(x,y) = \langle x \mid y\rangle$, the area and volume constraints are also continuous so let $i \to +\infty$ and Ω is a minimizer. □

Proof of Theorem 15.5. It is the particular case $k_m = 0$ in Proposition 18.29. This can be also deduced from Theorem 15.2, it suffices to follow the method described in the next proof. □

Proof of Theorem 15.7. First, as in the proof of Proposition 18.29 , one can show that minimizing in $\mathcal{O}_\varepsilon(\mathbb{R}^n)$ or in $\mathcal{O}_\varepsilon(B)$ is equivalent here. Then, apply Theorem 15.2 by setting $j_0 = j_2 \equiv 0$ and $j_1(x,y,z) = (z-H_0)^2$ which is continuous and convex in z. The area and volume constraints can be expressed as in Proposition 18.7 by setting $g_1 = g_2 \equiv 0$ and successively $g_0 \equiv 1$, $g_0(x,y) = \langle x \mid y\rangle$. Using the Gauss-Bonnet Theorem, the genus constraint is included in $\int K dA = 4\pi(1-g) := K_0$. Hence, Theorem 15.2 gives the existence of a minimizer satisfying the three constraints. Finally, we can apply [46, Proposition 2.2.17] to ensure that the compact minimizer is connected since it is the case for any minimizing sequence of compact sets. Hence, using again the Gauss-Bonnet Theorem, the minimizer has the right genus so Theorem 15.7 holds. □

Proof of Theorem 15.8. The proof is identical to the previous one. We just need to set $H_0 = 0$ and add a fourth equality constraint of the form $g_0 = g_2 \equiv 0$, $g_1 \equiv 1$. $\square$

Proposition 18.30. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^4$ be a bounded open set containing a ball of radius 3ε, and such that ∂B has zero 4-dimensional Lebesgue measure. We consider two bounded continuous vector fields of $\mathbb{R}^4$ denoted by $\mathbf{V}, \mathbf{W} : \mathbb{R}^4 \to \mathbb{R}^4$ and a continuous map $j : \mathbb{R}^4 \times \mathbb{S}^3 \times \mathbb{R}$, which is convex in its last variable. Then, the following problem has at least one solution (for the notation we refer to Section 18.1 and Proposition 18.27 above):*

$$\inf \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), Ric_{\mathbf{x}}\left(\mathbf{V}(\mathbf{x}) \wedge \mathbf{n}(\mathbf{x}), \mathbf{W}(\mathbf{x}) - \langle \mathbf{W}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \mathbf{n}(\mathbf{x})\right)\right] dA(\mathbf{x}),$$

where the infimum is taken among all $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the following constraint:

$$\int_{\partial\Omega} \mathfrak{R}(\mathbf{x}) \langle \mathbf{V}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \, dA(\mathbf{x}) = \int_{\partial\Omega} H^{(2)}(\mathbf{x}) \langle \mathbf{W}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \, dA(\mathbf{x}).$$

Proof. Consider a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ of the functional satisfying the constraint. From Proposition 17.1, up to a subsequence, it is converging to a set $\Omega \in \mathcal{O}_\varepsilon(B)$. We define $V_i := \mathbf{V} \wedge \mathbf{n}_{\partial\Omega_i}$ and $W_i := \mathbf{W} - \langle \mathbf{W} \mid \mathbf{n}_{\partial\Omega_i}\rangle \mathbf{n}_{\partial\Omega_i}$ which are two continuous vector fields on $\partial\Omega_i$, uniformly bounded since $\mathbf{V}$ and $\mathbf{W}$ are. We now check the diagonal convergence. Choose any sequence of points $\mathbf{x}_i \in \partial\Omega_i$ converging to $\mathbf{x} \in \partial\Omega$. Using the partition of unity introduced in Proposition 18.7, we get that $\mathbf{x} \in \partial\Omega \cap \mathcal{C}_{\tilde{r},\varepsilon}(\mathbf{x}_k)$ for some $k \in \{1, \ldots K\}$. Hence, there exists $\mathbf{x}' \in D_{\tilde{r}}(\mathbf{x}_k)$ such that $\mathbf{x} = (\mathbf{x}', \varphi^k(\mathbf{x}'))$. Since $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is converging to $\mathbf{x}$, for i sufficiently large, we can write $\mathbf{x}_i = (\mathbf{x}'_i, \varphi_i^k(\mathbf{x}'_i))$ with $\mathbf{x}'_i \in D_{\tilde{r}}(\mathbf{x}_k)$. Hence, $\mathbf{x}'_i \to \mathbf{x}'$ and $\varphi_i^k(\mathbf{x}'_i) \to \varphi^k(\mathbf{x}')$, but we also have from the triangle inequality:

$$\|\nabla\varphi_i^k(\mathbf{x}'_i) - \nabla\varphi^k(\mathbf{x}')\| \leqslant \|\nabla\varphi_i^k - \nabla\varphi^k\|_{C^0(\overline{D_{\tilde{r}}(\mathbf{x}_k)})} + \|\nabla\varphi^k(\mathbf{x}'_i) - \nabla\varphi^k(\mathbf{x}')\|.$$

From (17.1) and the continuity of $\nabla\varphi^k$, we can let $i \to +\infty$ and the diagonal convergence of $(\nabla\varphi_i^k)_{i\in\mathbb{N}}$ to $\nabla\varphi^k$ holds. Then, using (18.6), $n_{\partial\Omega_i}$ is also diagonally converging to $n_{\partial\Omega}$, and so does V_i and W_i. If j is linear in its last variable, we can apply Proposition 18.27 to obtain the continuity of the functional, otherwise we can use Remark 18.18 on the previous case to get the lower semi-continuity of the functional. Finally, apply Theorem 18.5 with $j_i^l \equiv 0$ if $l \neq 2$ and $j_i^2 = \langle \mathbf{V} \mid \mathbf{n}\rangle$ to have the continuity of the left member of the constraint. The continuity of the right one comes from Proposition 18.27 on J" with $j_i = \langle \mathbf{W} \mid \mathbf{n}\rangle$. Hence, we can let $i \to +\infty$ in the constraint.. $\square$

Proposition 18.31. *Let $\varepsilon, A_0, V_0 > 0$ be such that $A_0^3 > 36\pi V_0^2$, and let $B \subset \mathbb{R}^3$ be a ball of radius at least 3ε. We consider a bounded vector field in $\mathbb{R}^3$ denoted by $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ and a continuous map $j : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ which is convex in its last variable. Then, the following problem has at least one solution:*

$$\inf_{\substack{\Omega\in\mathcal{O}_\varepsilon(B)\\ A(\partial\Omega)=A_0\\ V(\Omega)=V_0}} \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}(\mathbf{x}), \kappa_{\mathbf{v}}(\mathbf{x})\right] dA(\mathbf{x}),$$

where $\kappa_{\mathbf{v}}$ is the normal curvature at $\mathbf{x}$ i.e. the curvature at $\mathbf{x}$ of the curve formed by the intersection of the surface $\partial\Omega$ with the plane spanned by $\mathbf{n}(\mathbf{x})$ and the vectore $\mathbf{v} := \mathbf{V}(\mathbf{x}) - \langle \mathbf{V}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle \mathbf{n}(\mathbf{x})$.

Proof. First, [73, Proposition 3.26, Remark 3.27] gives $\kappa_{\mathbf{v}} = \kappa_1 |\langle \mathbf{v}|\mathbf{e}_1\rangle|^2 + \kappa_2 |\langle \mathbf{v}|\mathbf{e}_2\rangle|^2 = \mathbf{II}(\mathbf{v}, \mathbf{v})$. Then, as in the previous proof, we can show that $\mathbf{v}_{\partial\Omega_i}$ is diagonally converging to $\mathbf{v}_{\partial\Omega}$. Finally, if j is linear in its last variable, we can apply Proposition 18.17 to get the continuity, otherwise use Remark 18.18 to get its lower semi-continuity. The area and volume constraints are continuous from Proposition 18.7. Hence, from Proposition 17.1, a minimizing sequence has a converging subsequence to an Ω and from the foregoing we can let $i \to +\infty$ in the functional and constraints so Ω is a minimizer. $\square$

Part V

Uniform ball property and existence of minimizers for functionals depending on the geometry and the solution of a state equation

Chapter 19

Introduction

This part is devoted to the extension of the existence results obtained in the previous part, in particular Theorems 15.2 and 18.28, for general geometric functionals also depending on the shape through the solutions of some second-order elliptic boundary value problems posed on the inner domain enclosed by the shape. Here, we present their three-dimensional version and the reader can easily adapt the proof to get their general form in $\mathbb{R}^n$.

A dependence through the solution of the Dirichlet Laplacian

For any domain $\Omega \in \mathcal{O}_\varepsilon(B)$, the associated boundary $\partial\Omega$ has $C^{1,1}$-regularity (cf. Theorem 16.6). Hence, we can consider the unique solution $u_\Omega \in H_0^1(\Omega,\mathbb{R}) \cap H^2(\Omega,\mathbb{R})$ of the Dirichlet Laplacian posed on a domain $\Omega \in \mathcal{O}_\varepsilon(B)$ with $f \in L^2(B,\mathbb{R})$ [37, Section 2.1 and Theorem 2.4.2.5]:

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega \\ \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{19.1}$$

Moreover, we say that the maps $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ and $g : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ have a quadratic growth in the first variable if there exists a constant $c > 0$ such that:

$$\forall(\mathbf{z},\mathbf{x},\mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \quad |f(\mathbf{z},\mathbf{x},\mathbf{y})| \leqslant c\left(1 + \|\mathbf{z}\|^2\right) \tag{19.2}$$

$$\forall(\mathbf{z},\mathbf{x},\mathbf{y},t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}, \quad |g(\mathbf{z},\mathbf{x},\mathbf{y},t)| \leqslant c\left(1 + \|\mathbf{z}\|^2\right). \tag{19.3}$$

Then, we prove the following extension of Theorem 15.2.

Theorem 19.1. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C,\widetilde{C}) \in \mathbb{R}^2$, five continuous maps $j_0,\ f_0,\ g_0,\ g_1,\ g_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ with quadratic growth* (19.2) *in the first variable, and four continuous maps $j_1,\ j_2,\ f_1,\ f_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth* (19.3) *in the first variable, and convex in the last variable. Then, the following problem has at least one solution:*

$$\begin{aligned} \inf \int_{\partial\Omega} j_0\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x}),H(\mathbf{x})\right] dA(\mathbf{x}) \\ + \int_{\partial\Omega} j_2\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x}),K(\mathbf{x})\right] dA(\mathbf{x}), \end{aligned}$$

where $u_\Omega \in H_0^1(\Omega,\mathbb{R}) \cap H^2(\Omega,\mathbb{R})$ is the unique solution of (19.1) *with $f \in L^2(B,\mathbb{R})$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:*

$$\begin{cases} \displaystyle\int_{\partial\Omega} f_0\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x}),H(\mathbf{x})\right] dA(\mathbf{x}) \\ \qquad\qquad \displaystyle + \int_{\partial\Omega} f_2\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x}),K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ \\ \displaystyle\int_{\partial\Omega} g_0\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ \qquad\qquad \displaystyle + \int_{\partial\Omega} K(\mathbf{x})\, g_2\left[\nabla u_\Omega(\mathbf{x}),\mathbf{x},\mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{cases}$$

In the above theorem, if we denote by $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growth (19.2)–(19.3) of the maps and from the continuity of the trace operator $H^2(\Omega, \mathbb{R}) \to H^1(\partial\Omega, \mathbb{R})$, we have:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant c \left[A(\partial\Omega) + \|\nabla u_\Omega\|^2_{L^2(\partial\Omega, \mathbb{R}^3)} \right] \leqslant \tilde{c} \left[A(\partial\Omega) + \|u\|^2_{H^2(\Omega, \mathbb{R})} \right] < +\infty.$$

We prove Theorem 19.1 with the same method used for Theorem 15.2. Considering a minimizing sequence $(\Omega_i)_{i \in \mathbb{N}}$, we first get from compactness a converging subsequence. Then, we parametrize simultaneously by local graphs of $C^{1,1}$-maps $(\varphi_i)_{i \in \mathbb{N}}$ the boundaries associated with the converging subsequence of domains. Moreover, from the previous part, $(\varphi_i)_{i \in \mathbb{N}}$ converges strongly in C^1 and weakly-star in $W^{2,\infty}$. Using a suitable partition of unity, we express the functional and constraints in this local parametrization. Therefore, it remains to show we can correctly let $i \to +\infty$.

Merely speaking, each integrand obtained is the product of a L^∞-weak-star converging term with a remaining term, on which we want to apply Lebesgue Domination Convergence Theorem to get its L^1-strong convergence. Hence, to let $i \to +\infty$, we need the almost-everywhere convergence and a uniform integrable bound for each integrand. Due to the continuity and the quadratic growth (19.2)–(19.3) hypothesis, this is the case if the local map $\mathbf{x}' \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ strongly converges in L^2 to the local map $\mathbf{x}' \mapsto \nabla u_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))$. We prove in Chapter 21 this assertion holds true.

A dependence through the solution of the Neumann/Robin Laplacian

The remaining work of this part is to extend the previous results to the Neumann/Robin boundary conditions. For any domain $\Omega \in \mathcal{O}_\varepsilon(B)$, since $\partial\Omega$ has $C^{1,1}$-regularity (Theorem 16.6), there exists a unique solution $v_\Omega \in H^2(\Omega, \mathbb{R})$ to the following problem [37, Section 2.1 and Theorem 2.4.2.7]:

$$\begin{cases} -\Delta v_\Omega + \lambda v_\Omega = f & \text{in } \Omega \\ \\ \partial_n(v_\Omega) = 0 & \text{on } \partial\Omega, \end{cases} \tag{19.4}$$

where $\lambda > 0$ and $f \in L^2(B, \mathbb{R})$. Moreover, there exists a unique solution $\tilde{v}_\Omega \in H^2(\Omega, \mathbb{R})$ to the following problem [37, Section 2.1 and Theorem 2.4.2.6]:

$$\begin{cases} -\Delta \tilde{v}_\Omega = f & \text{in } \Omega \\ \\ -\partial_n(\tilde{v}_\Omega) = \lambda \tilde{v}_\Omega & \text{on } \partial\Omega. \end{cases} \tag{19.5}$$

where $\lambda > 0$ and $f \in L^2(B, \mathbb{R})$. Furthermore, if the existence of a unique solution in $H^2(\Omega, \mathbb{R})$ is ensured, we are also able to treat in (19.5) some non-linear boundary conditions of the form $-\partial_n(\tilde{v}_\Omega) = \beta(\tilde{v}_\Omega)$, where $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map with $\beta(0) = 0$. Note that if $\beta(x) = \lambda x$, we get (19.5) and (19.4) is given by $\beta(x) = 0$.

Moreover, we say that the maps $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ have a quadratic growth in their two first variables if there exists a constant $c > 0$ such that:

$$\forall (s, \mathbf{z}, \mathbf{x}, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \quad |f(s, \mathbf{z}, \mathbf{x}, \mathbf{y})| \leqslant c \left(1 + s^2 + \|\mathbf{z}\|^2\right) \tag{19.6}$$

$$\forall (s, \mathbf{z}, \mathbf{x}, \mathbf{y}, t) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}, \quad |g(s, \mathbf{z}, \mathbf{x}, \mathbf{y}, t)| \leqslant c \left(1 + s^2 + \|\mathbf{z}\|^2\right). \tag{19.7}$$

Then, we prove the following.

Theorem 19.2. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C, \widetilde{C}) \in \mathbb{R}^2$, five continuous maps j_0, f_0, g_0, g_1, $g_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ with quadratic growth* (19.6) *in the two first variables, and four continuous maps j_1, j_2, f_1, $f_2 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth* (19.7) *in the two first variables, and convex in the last variable. Then, the following problem has at least one solution:*

$$\inf \int_{\partial\Omega} j_0 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}) \right] dA(\mathbf{x}) + \int_{\partial\Omega} j_1 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x}) \right] dA(\mathbf{x})$$
$$+ \int_{\partial\Omega} j_2 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x}) \right] dA(\mathbf{x}),$$

where $v_\Omega \in H^2(\Omega, \mathbb{R})$ is the unique solution of either (19.4) *or* (19.5) *with $f \in L^2(B, \mathbb{R})$ and $\lambda > 0$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:*

$$\left\{ \begin{aligned} &\int_{\partial\Omega} f_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) \\ &\qquad\qquad + \int_{\partial\Omega} f_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ &\int_{\partial\Omega} g_0\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} H(\mathbf{x})\, g_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ &\qquad\qquad + \int_{\partial\Omega} K(\mathbf{x})\, g_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}. \end{aligned} \right.$$

In the above theorem, if we denote by $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growth (19.6)–(19.7) of the maps and from the continuity of the trace operator $H^2(\Omega, \mathbb{R}) \to H^1(\partial\Omega, \mathbb{R})$, we have:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant c\left[A(\partial\Omega) + \|v_\Omega\|^2_{H^1(\partial\Omega,\mathbb{R})}\right] \leqslant \tilde{c}\left[A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

We prove Theorem 19.2 with the same method used for Theorem 2.16 and described in the previous section. The main task is to show that the local map $\mathbf{x}' \mapsto v_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ strongly converges in H^1 to the local map $\mathbf{x}' \mapsto v_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))$. It is the purpose of Chapter 22 to prove this holds true.

First application: some quadratic functionals on the domain involving the second-order derivatives of the Dirichlet/Neumann/Robin Laplacian

Until now, note that we only treat the case of functionals involving boundary integrals. Indeed, the case where the domain of integration corresponds to the one of (19.1) or (19.4)-(19.5) such as:

$$\int_\Omega j\left[\mathbf{x}, u_\Omega(\mathbf{x}), \nabla u_\Omega(\mathbf{x})\right] dV(\mathbf{x}),$$

is standard with the framework of the uniform cone property [46, Section 4.3]. Since the ε-ball condition implies an $\alpha(\varepsilon)$-cone property (cf. Point (i) in Theorem 16.6), we have not considered such functionals for the time being. However, the class $\mathcal{O}_\varepsilon(B)$ becomes interesting if some second-order partial derivatives of u_Ω appear in the above integrand. Our result states as follows. We say that a map $j : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2} \to \mathbb{R}$ has a quadratic growth in its three last variables if there exists a constant $c > 0$ such that:

$$\forall (\mathbf{x}, s, \mathbf{z}, Y) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2}, \quad |j(\mathbf{x}, s, \mathbf{z}, Y)| \leqslant c\left(1 + s^2 + \|\mathbf{z}\|^2 + \|Y\|^2\right), \tag{19.8}$$

where the Frobenius norm is considered on the set of (3×3)-matrices i.e. $\|Y\| = \sqrt{\mathrm{trace}([Y]^T Y)}$.

Theorem 19.3. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^3$ an open ball of radius large enough. Consider $(C, \widetilde{C}) \in \mathbb{R}^2$, three measurable maps $j_0, f_0, g_0 : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3^2} \to \mathbb{R}$ with quadratic growth* (19.8) *in their three last variables, and continuous in $(s, \mathbf{z}, Y)$ for almost every $\mathbf{x}$, five continuous maps j_1, f_1, g_1, g_2, $g_3 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ with quadratic growth* (19.6) *in the two first variables, and four continuous maps j_2, j_3, f_2, $f_3 : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ with quadratic growth* (19.7) *in the two first variables, and convex in the last variable. Then, the following problem has at least one solution:*

$$\inf \int_\Omega j_0\left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathrm{Hess}\ v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} j_1\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \quad + \\ \int_{\partial\Omega} j_2\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} j_3\left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}),$$

where $v_\Omega \in H^2(\Omega, \mathbb{R})$ is the unique solution of either (19.1) *or* (19.4) *or* (19.5) *with $f \in L^2(B, \mathbb{R})$*

and $\lambda > 0$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:

$$\begin{cases} C \geqslant \displaystyle\int_\Omega f_0 \left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \text{Hess } v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} f_1 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \\ \displaystyle\int_{\partial\Omega} f_2 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} f_3 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), K(\mathbf{x})\right] dA(\mathbf{x}) \\ \\ \widetilde{C} = \displaystyle\int_\Omega g_0 \left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \text{Hess } v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} g_1 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \\ \displaystyle\int_{\partial\Omega} H(\mathbf{x})\, g_2 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) + \int_{\partial\Omega} K(\mathbf{x}) g_3 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}). \end{cases}$$

Again, in the above theorem, if we denote by $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ the functional to minimize, note that it is well defined since from the quadratic growth (19.6)–(19.8) of the maps and from the continuity of the trace operator $H^2(\Omega, \mathbb{R}) \to H^1(\partial\Omega, \mathbb{R})$, we have:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant \tilde{c}\left[V(\Omega) + A(\partial\Omega) + \|u\|^2_{H^2(\Omega,\mathbb{R})}\right] < +\infty.$$

Also observe that the above statement treats the case where the integration is not done on the whole domain Ω but only on a measurable part $\widetilde{\Omega} \subseteq \Omega$. Indeed, it suffices to introduce the characteristic function $\mathbf{1}_{\widetilde{\Omega}}$ in the integrand j_0. This cannot be done for the boundary integrals but continuous cutoff functions can still be considered. Finally, the formulation adopted above allows constraints of the form $K \subset \Omega$ for a given compact set $K \subset B$, by setting $\widetilde{C} = V(K)$, $g_0 = \mathbf{1}_K$, and $g_1 = g_2 = g_3 = 0$.

Second application: boundary shape identification problems

Let $\varepsilon > 0$ and B be an open set as in Remark 2.11. We consider $\Omega_0 \in \mathcal{O}_\varepsilon(B)$, a subset $\Gamma_0 \subseteq \partial\Omega_0$, and $g_0 \in L^2(\Gamma_0, \mathbb{R})$. Imagine there is good reason to think that g_0 is the restriction to Γ_0 of the normal derivative associated with the solution u_Ω of the Dirichlet Laplacian (19.1) posed on an unknown domain $\Omega \in \mathcal{O}_\varepsilon(B)$ such that $\Gamma_0 \subseteq \partial\Omega$. In order to to find the *best* $\Omega \in \mathcal{O}_\varepsilon(B)$ such that $\partial_n(u_\Omega)|_{\partial\Omega_0} = g_0$, one possibility is to solve the following problem:

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(B) \\ \Gamma_0 \subseteq \partial\Omega}} \int_{\Gamma_0} \left[\partial_n(u_\Omega) - g_0\right]^2 dA. \tag{19.9}$$

Similarly, if we suspect that $f_0 \in L^2(\Gamma_0, \mathbb{R})$ is the restrictions to Γ_0 of the solution v_Ω to the Neumann/Robin Laplacian (19.4)-(19.5) posed on an unknown domain $\Omega \in \mathcal{O}_\varepsilon(B)$ such that $\Gamma_0 \subseteq \partial\Omega$, then we have to solve:

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(B) \\ \Gamma_0 \subseteq \partial\Omega}} \int_{\Gamma_0} (v_\Omega - f_0)^2\, dA. \tag{19.10}$$

Of course, we can build more complicated functionals but the main difficulty here is that the domain of integration is not the whole surface. We prove the following result.

Proposition 19.4. *Let $\Omega_0 \in \mathcal{O}_\varepsilon(B)$ and Γ_0 be a measurable subset of $\partial\Omega_0$. Then, Theorem 19.3 remains true if we add the constraint $\Gamma_0 \subseteq \partial\Omega$ and if the domain of integration $\partial\Omega$ in the functional and the constraints are restricted to Γ_0. In particular, Problems* (19.9)–(19.10) *have a minimizer.*

The identification of shape through its boundary like (19.9)–(19.10) often appear in inverse and optimal control problems. For example, let us try to detect a tumor in the brain. We put some electrods on the head Γ_0 of a patient, measure some electric activity g_0, and solve Problem (19.10). If no tumor exists, then the infimum is zero and the optimal shape is Γ_0, otherwise it is $\Gamma_0 \cup \Gamma_1$, where Γ_1 is the boundary of the tumor.

Third application: the MIT-bag model in relativistic quantum mechanics

During the conference MODE 2014 at the INSA-Rennes, Le Treust made a talk on his thesis [56]. He has studied some shape optimization problems coming from relativistic quantum mechanics. In particular, bag models are introduced to study the internal structure of hadrons. The energy of these particules is given by summing the energy of the quarks and anti-quarks living in the bag.

In the MIT-bag model, the wave functions of the quarks are the eigenvectors of the Dirac operator. Hence, the fundamental state problem corresponds to the minimization with prescribed volume of the first positive eigenvalue associated with this Dirac operator among non-empty open bounded subset of $\mathbb{R}^3$ with C^2-boundary. The existence of an optimal shape is actually open.

We did not study this problem but it seems that the framework of the uniform ball condition might be used again to approximate the fundamental state of the MIT-bag model:

$$\inf_{\substack{\Omega \in \mathcal{O}_\varepsilon(B) \\ V(\Omega)=V_0}} \quad \inf_{\substack{u \in H^1(\Omega,\mathbb{C}^2) \\ \int_\Omega u^2 = 1 \\ -\sigma \mathbf{n}_{\partial\Omega} u = u \text{ on } \partial\Omega}} \sqrt{m^2 + \int_\Omega \|\nabla u\|^2 + \int_{\partial\Omega} \left(m + \frac{H_{\partial\Omega}}{2} \right) |u|^2 dA},$$

where $m > 0$ is the mass and σ a given (3×3)-matrix. The difficult comes from the non-linear boundary condition of the associated eigenvalue problem.

To conclude this introduction, the last part is organized as follows. In Chapter 20, we establish H^2-*a priori* estimates for the solutions of (19.1) in the class $\mathcal{O}_\varepsilon(B)$, where the constant obtained depend only on ε, the diameter of B, and the dimension n of the space. We essentially follow the method suggested by Grisvard [37, Sections 3.1.1-3.1.2]. Then, in Chapters 21 and 22, we respectively treat the Dirichlet and the Neumann/Robin case. Finally, in Chapter 23, we give very general existence results in $\mathbb{R}^n$ and prove the theorems detailed in this introduction.

Chapter 20

Controlling uniformly the H^2-norm by the Laplacian

In this section, we want to control uniformly the constant appearing in *a priori* estimates related to the Dirichlet Laplacian. First, we recall some geometric definitions in the case of hypersurfaces with $C^{1,1}$-regularity. Then, we establish an identity for general functions, some Poincaré and trace inequalities, in order to finally prove Theorem 20.1. We follow essentially the method described in [37, Section 3.2] which treats the case of convex C^2-domains.

Theorem 20.1. *Let $\varepsilon > 0$, $n \geqslant 2$, and B be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We consider the class $\mathcal{O}_\varepsilon(B)$ formed by all the non-empty open subsets of B satisfying the ε-ball condition. We assume that the diameter D of B is large enough to ensure $\mathcal{O}_\varepsilon(B) \neq \emptyset$. Then, there exists a constant $C > 0$, depending only on ε, D, and n, such that:*

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \forall u \in H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R}), \quad \|u\|_{H^2(\Omega,\mathbb{R})} \leqslant C\,(\varepsilon, n, D)\, \|\Delta u\|_{L^2(\Omega,\mathbb{R})}.$$

20.1 On the geometry of hypersurfaces with $C^{1,1}$-regularity

Let us consider any non-empty bounded open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. We assume that its boundary $\partial\Omega$ is a $C^{1,1}$-hypersurface of $\mathbb{R}^n$ i.e. for any point $\mathbf{x}_0 \in \partial\Omega$, there exists $r_{\mathbf{x}_0} > 0$, $a_{\mathbf{x}_0} > 0$, and a unit vector $\mathbf{d}_{\mathbf{x}_0}$ of $\mathbb{R}^n$ such that in the cylinder defined by:

$$C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n,\ |\langle \mathbf{x}-\mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < a_{\mathbf{x}_0} \text{ and } \|(\mathbf{x}-\mathbf{x}_0) - \langle \mathbf{x}-\mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle \mathbf{d}_{\mathbf{x}_0}\| < r_{\mathbf{x}_0}\}, \tag{20.1}$$

the boundary $\partial\Omega$ is the graph of a $C^{1,1}$-map. To be more precise, introducing the orthogonal projection on the affine hyperplane $\mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^\perp$:

$$\begin{array}{rccl} \Pi_{\mathbf{x}_0}: & \mathbb{R}^n & \longrightarrow & \mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^\perp \\ & \mathbf{x} & \longmapsto & \mathbf{x} - \langle \mathbf{x}-\mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle \mathbf{d}_{\mathbf{x}_0}, \end{array}$$

and considering the set $D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) = \Pi_{\mathbf{x}_0}(C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0))$, this means that there exists a continuously differentiable map $\varphi_{\mathbf{x}_0} : \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) \mapsto \varphi_{\mathbf{x}_0}(\mathbf{x}') \in]-a_{\mathbf{x}_0}, a_{\mathbf{x}_0}[$ such that its gradient $\nabla\varphi_{\mathbf{x}_0}$ and $\varphi_{\mathbf{x}_0}$ are $L_{\mathbf{x}_0}$-Lipschitz continuous maps, $L_{\mathbf{x}_0} > 0$, and such that:

$$\left\{ \begin{array}{rcl} \partial\Omega \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) & = & \{\mathbf{x}' + \varphi_{\mathbf{x}_0}(\mathbf{x}')\mathbf{d}_{\mathbf{x}_0}, \quad \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{x}_0)\} \\ \\ \Omega \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) & = & \{\mathbf{x}' + x_n\mathbf{d}_{\mathbf{x}_0}, \quad \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) \text{ and } -a_{\mathbf{x}_0} < x_n < \varphi_{\mathbf{x}_0}(\mathbf{x}')\}. \end{array} \right.$$

Hence, we can introduce the local parametrization:

$$\begin{array}{rccl} X_{\mathbf{x}_0}: & D_{r_{\mathbf{x}_0}}(\mathbf{x}_0) & \longrightarrow & \partial\Omega \cap C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0) \\ & \mathbf{x}' & \longmapsto & \mathbf{x}' + \varphi_{\mathbf{x}_0}(\mathbf{x}')\mathbf{d}_{\mathbf{x}_0}, \end{array}$$

and $\partial\Omega$ is a $C^{1,1}$-hypersurface in the sense of [73, Definition 2.2]. Indeed, $X_{\mathbf{x}_0}$ is an homeomorphism, its inverse map is the restriction of $\Pi_{\mathbf{x}_0}$ to $C_{r_{\mathbf{x}_0},a_{\mathbf{x}_0}}(\mathbf{x}_0)$, and $X_{\mathbf{x}_0}$ is an immersion of class $C^{1,1}$.

We usually drop the dependence in $\mathbf{x}_0$ to lighten the notation, and consider a direct orthonormal frame $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ where $\mathcal{B}_{\mathbf{x}_0}$ is a basis of $\mathbf{d}_{\mathbf{x}_0}^{\perp}$. In this local frame, the point $\mathbf{x}_0$ is identified with the zero vector $\mathbf{0} \in \mathbb{R}^n$, the affine hyperplane $\mathbf{x}_0 + \mathbf{d}_{\mathbf{x}_0}^{\perp}$ with $\mathbb{R}^{n-1}$ and $\mathbf{x}_0 + \mathbb{R}\mathbf{d}_{\mathbf{x}_0}$ with $\mathbb{R}$. Hence, the cylinder $C_{r_{\mathbf{x}_0}, a_{\mathbf{x}_0}}(\mathbf{x}_0)$ becomes $D_r(\mathbf{0}')\times]-a,a[$, $\varphi_{\mathbf{x}_0}$ is the $C^{1,1}$-map $\varphi : D_r(\mathbf{0}') \to]-a,a[$, the projection $\Pi_{\mathbf{x}_0}$ is $X^{-1} : (\mathbf{x}', x_n) \mapsto \mathbf{x}'$, and the parametrization $X_{\mathbf{x}_0}$ becomes the $C^{1,1}$-map $X : \mathbf{x}' \in D_r(\mathbf{0}') \mapsto (\mathbf{x}', \varphi(\mathbf{x}')) \in \partial\Omega \cap (D_r(\mathbf{0}')\times]-a,a[)$.

Since $\mathbf{x}' \in D_r(\mathbf{0}') \mapsto D_{\mathbf{x}'}X$ is injective, the vectors $\partial_1 X, \ldots, \partial_{n-1}X$ are linearly independent. For any $\mathbf{x} \in \partial\Omega\cap(D_r(\mathbf{0}')\times]-a,a[)$, we define the tangent hyperplane $T_{\mathbf{x}}(\partial\Omega)$ by $D_{X^{-1}(\mathbf{x})}X(\mathbb{R}^{n-1})$. It is an $(n-1)$-dimensional vector space so $(\partial_1 X, \ldots, \partial_{n-1}X)$ forms a basis of $T_{\mathbf{x}}(\partial\Omega)$. However, this basis is not necessarily orthonormal. Consequently, the first fundamental form of $\partial\Omega$ at $\mathbf{x}$ is defined as the restriction of the usual scalar product in $\mathbb{R}^n$ to the tangent hyperplane $T_{\mathbf{x}}(\partial\Omega)$, i.e. as $\mathbf{I}(\mathbf{x}) : (\mathbf{v}, \mathbf{w}) \in T_{\mathbf{x}}(\partial\Omega) \times T_{\mathbf{x}}(\partial\Omega) \mapsto \langle \mathbf{v} \mid \mathbf{w}\rangle$. In the basis $(\partial_1 X, \ldots, \partial_{n-1}X)$, it is represented by a positive-definite symmetric matrix usually referred to as $(g_{ij})_{1\leqslant i,j\leqslant n-1}$ and its inverse denoted by $(g^{ij})_{1\leqslant i,j\leqslant n-1}$ is also explicitly given in this case:

$$g_{ij} = \langle \partial_i X \mid \partial_j X\rangle = \delta_{ij} + \partial_i\varphi\partial_j\varphi, \tag{20.2}$$

$$g^{ij} = \delta_{ij} - \frac{\partial_i\varphi\partial_j\varphi}{1 + \|\nabla\varphi\|^2}. \tag{20.3}$$

As a function of $\mathbf{x}'$, observe that each coefficient of these two matrices is Lipschitz continuous thus it is a $W^{1,\infty}$-map [30, Section 4.2.3], and from Rademacher's Theorem [30, Section 3.1.2], its differential exists almost everywhere. Moreover, any $\mathbf{v} \in T_{\mathbf{x}}(\partial\Omega)$ can be decomposed in the basis $(\partial_1 X, \ldots, \partial_{n-1}X)$. Denoting by V_i the component of $\partial_i X$ and $v_i = \langle \mathbf{v} \mid \partial_i X\rangle$, we have:

$$\mathbf{v} = \sum_{i=1}^{n-1} V_i \partial_i X \implies v_j = \sum_{i=1}^{n} V_i g_{ij} \implies V_i = \sum_{j=1}^{n-1} g^{ij} v_j \implies \mathbf{v} = \sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-1} g^{ij} v_j\right)\partial_i X. \tag{20.4}$$

In particular, we deduce $\mathbf{I}(\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{n-1} g^{ij} v_i w_j$. Then, the orthogonal of the tangent hyperplane is one dimensional. Hence, there exists a unique unit vector $\mathbf{n}$ orthogonal to the $(n-1)$ vectors $\partial_1 X, \ldots, \partial_{n-1}X$ and pointing outwards Ω i.e. $\det(\partial_1 X, \ldots, \partial_{n-1}X, \mathbf{n}) > 0$. It is called the unit outer normal to the hypersurface and we have its explicit expression in the parametrization:

$$\forall \mathbf{x}' \in D_r(\mathbf{0}'), \quad \mathbf{n}\circ X(\mathbf{x}') = \frac{1}{\sqrt{1 + \|\nabla\varphi(\mathbf{x}')\|^2}}\begin{pmatrix} -\nabla\varphi(\mathbf{x}') \\ 1 \end{pmatrix}. \tag{20.5}$$

It is a Lipschitz continuous map, like the coefficients of the first fundamental form. In particular, it is differentiable almost everywhere and introducing the Gauss map $\mathbf{n} : \mathbf{x} \in \partial\Omega \mapsto \mathbf{n}(\mathbf{x}) \in \mathbb{S}^{n-1}$, we can compute its differential almost everywhere called the Weingarten map:

$$\begin{array}{rrcl} D_{\mathbf{x}}\mathbf{n} : & T_{\mathbf{x}}(\partial\Omega) = D_{X^{-1}(\mathbf{x})}X(\mathbb{R}^2) & \longrightarrow & D_{X^{-1}(\mathbf{x})}(\mathbf{n}\circ X)(\mathbb{R}^2) \\ & \mathbf{v} = D_{X^{-1}(\mathbf{x})}X(\mathbf{w}) & \longmapsto & D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) = D_{X^{-1}(\mathbf{x})}(\mathbf{n}\circ X)(\mathbf{w}). \end{array}$$

Since $\|\mathbf{n}\circ X\|^2 = 1$, note that $D_{X^{-1}(\mathbf{x})}(\mathbf{n}\circ X)(\mathbb{R}^2) \subseteq \mathbf{n}(\mathbf{x})^{\perp} = T_{\mathbf{x}}(\partial\Omega)$ so the map $D_{\mathbf{x}}\mathbf{n}$ is an endomorphism of $T_{\mathbf{x}}(\partial\Omega)$. Moreover, one can prove it is self-adjoint so it can be diagonalized to obtain $n-1$ eigenvalues denoted by $\kappa_1(\mathbf{x}), \ldots, \kappa_{n-1}(\mathbf{x})$ and called the principal curvatures. Recall that the eigenvalues of an endomorphism do not depend on the chosen basis and thus are really properties of the operator. This also holds for the trace and the determinant of $D_{\mathbf{x}}\mathbf{n}$ so we can define the scalar mean curvature $H = \mathrm{Trace}(D_{\mathbf{x}}\mathbf{n})$ and the Gaussian curvature $K = \det(D_{\mathbf{x}}\mathbf{n})$:

$$H(\mathbf{x}) = \kappa_1(\mathbf{x}) + \ldots + \kappa_{n-1}(\mathbf{x}) \qquad \text{and} \qquad K(\mathbf{x}) = \kappa_1(\mathbf{x})\kappa_2(\mathbf{x})\ldots\kappa_{n-1}(\mathbf{x}). \tag{20.6}$$

Moreover, introducing the symmetric matrix $(b_{ij})_{1\leqslant i,j\leqslant n-1}$ defined by:

$$b_{ij} = -\langle D\mathbf{n}(\partial_i X) \mid \partial_j X\rangle = -\langle \partial_i(\mathbf{n}\circ X) \mid \partial_j X\rangle = \frac{\mathrm{Hess}\,\varphi}{\sqrt{1 + \|\nabla\varphi\|^2}} = \langle \mathbf{n}\circ X \mid \partial_{ij}X\rangle, \tag{20.7}$$

we get from (20.4) that the Weingarten map $D\mathbf{n}$ is represented in the local basis $(\partial_1 X, \ldots, \partial_{n-1}X)$ by the symmetric matrix $(-\sum_{k=1}^{n-1} g^{ik} b_{kj})_{1\leqslant i,j\leqslant n-1}$ and in particular, we have:

$$H\circ X = -\sum_{i,j=1}^{n-1} g^{ij} b_{ji} = -\sum_{i,j=1}^{n-1}\left(\delta_{ij} - \frac{\partial_i\varphi\partial_j\varphi}{1 + \|\nabla\varphi\|^2}\right)\frac{\partial_{ji}\varphi}{\sqrt{1 + \|\nabla\varphi\|^2}}. \tag{20.8}$$

Finally, we introduce the symmetric bilinear form whose representation in the local basis is (b_{ij}). It is called the second fundamental form of the hypersurface and it is defined by:

$$\begin{array}{rcl} \mathbf{II}(\mathbf{x}): \ T_{\mathbf{x}}(\partial\Omega)\times T_{\mathbf{x}}(\partial\Omega) & \longrightarrow & \mathbb{R} \\ (\mathbf{v},\mathbf{w}) & \longmapsto & \langle -D_{\mathbf{x}}\mathbf{n}(\mathbf{v}) \mid \mathbf{w}\rangle = \displaystyle\sum_{i,j,k,l=1}^{n-1} g^{ij}v_j g^{kl} w_l b_{il}. \end{array} \tag{20.9}$$

Note that in local coordinates, the coefficients of the first fundamental form and the Gauss map are Lipschitz continuous functions i.e. $\mathbf{n}\circ X, g_{ij}, g^{ij}\in W^{1,\infty}(D_r(\mathbf{0}'))$. Hence, the Weingarten map and the coefficients of the second fundamental form exist almost everywhere and $b_{ij}\in L^\infty(D_r(\mathbf{0}'))$. Henceforth, we do not indicate anymore the dependence on the point $\mathbf{x}$ or in the parameter $\mathbf{x}'$ such that $X(\mathbf{x}')=\mathbf{x}$. The same notation is now used to denote a map $f:\mathbf{x}\in\partial\Omega\cap C_{r,a}(\mathbf{x}_0)\mapsto f(\mathbf{x})$ and its parametrized version $\mathbf{x}'\in D_r(\mathbf{x}_0)\mapsto (f\circ X)(\mathbf{x}')$.

20.2 An identity based on two integrations by parts

In [37, Theorem 3.1.1.1], an identity based on two integration by parts is established in the case of domains with C^2-boundary. It is the main ingredient to get a uniform control on the constant appearing in *a priori* estimates associated with the Dirichlet/Neumann Laplacian. In this section, our only contribution is to show that Equality (20.10) remains true for domains with $C^{1,1}$-boundary.

Theorem 20.2 (Grisvard [37, Theorem 3.1.1.1]). *Let us consider any non-empty bounded open set $\Omega\subset\mathbb{R}^n$ such that its boundary is a $C^{1,1}$-hypersurface of $\mathbb{R}^n$, $n\geqslant 2$. Then, for any function $\mathbf{v}=(\mathbf{v}_1,\dots,\mathbf{v}_n)\in H^1(\Omega,\mathbb{R}^n)$, we have the following identity:*

$$\begin{aligned} \sum_{i,j=1}^n\int_\Omega \frac{\partial \mathbf{v}_i}{\partial x_j}\frac{\partial \mathbf{v}_j}{\partial x_i}-\int_\Omega |\mathrm{div}(\mathbf{v})|^2 \;&=\; 2\left\langle \nabla_{\partial\Omega}(v_{\mathbf{n}}) \mid \mathbf{v}_{\partial\Omega}\right\rangle_{H^{-\frac12}(\partial\Omega,\mathbb{R}^n),H^{\frac12}(\partial\Omega,\mathbb{R}^n)} \\ &\quad + \int_{\partial\Omega}\left[\mathbf{II}(\mathbf{v}_{\partial\Omega},\mathbf{v}_{\partial\Omega})-H(v_{\mathbf{n}})^2\right]dA, \end{aligned} \tag{20.10}$$

where $\mathbf{n}$ is the unit outer normal to the hypersurface as in (20.5)*, where $v_{\mathbf{n}}=\langle\mathbf{v}\mid\mathbf{n}\rangle$, $\mathbf{v}_{\partial\Omega}=\mathbf{v}-v_{\mathbf{n}}\mathbf{n}$, $\nabla_{\partial\Omega}(v_{\mathbf{n}})=\nabla(v_{\mathbf{n}})-\langle\nabla(v_{\mathbf{n}})\mid\mathbf{n}\rangle\mathbf{n}$, where H is the scalar mean curvature as in* (20.6) *and* **II** *refers to the second fundamental form defined in* (20.9).

Proof. Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$ whose boundary is a $C^{1,1}$-hypersurface. We consider $\mathbf{v}=(\mathbf{v}_1,\dots,\mathbf{v}_n)\in C^\infty(\overline{\Omega},\mathbb{R}^n)$ and we get from two integrations by parts:

$$\begin{aligned} \int_\Omega|\mathrm{div}(\mathbf{v})|^2 \;&=\; \sum_{i,j=1}^n\int_\Omega\frac{\partial\mathbf{v}_i}{\partial x_i}\frac{\partial\mathbf{v}_j}{\partial x_j}=\sum_{i,j=1}^n\int_{\partial\Omega}\frac{\partial\mathbf{v}_i}{\partial x_i}\mathbf{v}_j\mathbf{n}_j dA-\sum_{i,j=1}^n\int_\Omega\mathbf{v}_j\frac{\partial^2\mathbf{v}_i}{\partial x_j\partial x_i} \\ &=\; \sum_{i,j=1}^n\int_{\partial\Omega}\frac{\partial\mathbf{v}_i}{\partial x_i}\mathbf{v}_j\mathbf{n}_j dA-\sum_{i,j=1}^n\int_{\partial\Omega}\mathbf{v}_j\frac{\partial\mathbf{v}_i}{\partial x_j}\mathbf{n}_i dA+\sum_{i,j=1}^n\int_\Omega\frac{\partial\mathbf{v}_i}{\partial x_j}\frac{\partial\mathbf{v}_j}{\partial x_i}. \end{aligned}$$

Consequently, introducing the notation $v_{\mathbf{n}}=\langle\mathbf{v}\mid\mathbf{n}\rangle$, the above equality takes the following form:

$$\sum_{i,j=1}^n\int_\Omega\frac{\partial\mathbf{v}_i}{\partial x_j}\frac{\partial\mathbf{v}_j}{\partial x_i}-\int_\Omega|\mathrm{div}(\mathbf{v})|^2=\int_{\partial\Omega}\left[\langle\langle\mathbf{v}\mid\nabla\rangle(\mathbf{v})\mid\mathbf{n}\rangle-v_{\mathbf{n}}\mathrm{div}(\mathbf{v})\right]dA. \tag{20.11}$$

We now show that the right member of (20.11) is equal to the right one of (20.10) by expressing the right integrand of (20.11) in the local parametrization associated with $\partial\Omega$. More precisely, we set $\mathbf{x}_0\in\partial\Omega$. There exists a cylinder (20.1) simply denoted by $C(\mathbf{x}_0)$ in which $\partial\Omega$ is the graph of a $C^{1,1}$-map φ. Hence, we can introduce the local $C^{1,1}$-parametrization $X:\mathbf{x}'\to(\mathbf{x}',\varphi(\mathbf{x}'))\in\partial\Omega\cap C(\mathbf{x}_0)$ and we first assume that the smooth map $\mathbf{v}:\overline{\Omega}\to\mathbb{R}^n$ has compact support in $\overline{\Omega}\cap C(\mathbf{x}_0)$. We decompose it locally in the basis $(\partial_1X,\dots,\partial_{n-1}X,\mathbf{n})$ which is direct but not necessarily orthonormal. There is a tangential component denoted by $\mathbf{v}_{\partial\Omega}$ and a normal one. We set g_{ij}, g^{ij} as in (20.2)-(20.3), $v_i=\langle\mathbf{v}\mid\partial_iX\rangle$ for $i=1,\dots,n-1$, and $v_{\mathbf{n}}=\langle\mathbf{v}\mid\mathbf{n}\rangle$. We have from (20.4):

$$\mathbf{v}=\mathbf{v}_{\partial\Omega}+v_{\mathbf{n}}\mathbf{n}=\sum_i^{n-1}\left(\sum_{j=1}^{n-1}g^{ij}v_j\right)\partial_iX+v_{\mathbf{n}}\mathbf{n}. \tag{20.12}$$

Similarly, we can decompose the action of the gradient into tangential and normal components:

$$\nabla(\,.\,) = \nabla_{\partial\Omega}(\,.\,) + \partial_n(\,.\,)\mathbf{n} = \sum_{i}^{n-1}\left(\sum_{j=1}^{n-1} g^{ij}\partial_j(\,.\,)\right)\partial_i X + \partial_n(\,.\,)\mathbf{n},$$

where $\partial_n(\,.\,) = \langle \nabla(\,.\,)|\mathbf{n}\rangle$ and $\partial_j(\,.\,)$ are the partial derivatives in the parametrization. Observe that (20.12) shows that $\mathbf{v}$ is a Lipschitz continuous map in the parametrization, since it is a product and sum of such functions. Consequently, it is differentiable almost everywhere and we can compute:

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= \sum_{i,j=1}^{n-1} g^{ij}\langle \partial_j(\mathbf{v}) \mid \partial_i X\rangle \quad + \quad \langle \partial_n(\mathbf{v}) \mid \mathbf{n}\rangle \\
&= \sum_{i,j=1}^{n-1} g^{ij}\left\langle \partial_j\left(\mathbf{v}_{\partial\Omega} + v_{\mathbf{n}}\mathbf{n}\right) \mid \partial_i X\right\rangle + \left\langle \partial_n\left(\sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_{\mathbf{n}}\mathbf{n}\right) \mid \mathbf{n}\right\rangle \\
&= \sum_{i,j=1}^{n-1} g^{ij}\left(\left\langle \partial_j\left(\mathbf{v}_{\partial\Omega}\right) \mid \partial_i X\right\rangle + v_{\mathbf{n}}\left\langle \partial_j \mathbf{n} \mid \partial_i X\right\rangle + v_j\left\langle \partial_n(\partial_i X) \mid \mathbf{n}\right\rangle\right) + \langle \partial_n(v_{\mathbf{n}}\mathbf{n}) \mid \mathbf{n}\rangle.
\end{aligned}$$

To obtain the last expression, we used $\langle \partial_i X \mid \mathbf{n}\rangle = 0$. As we did for the gradient, we introduce the tangential component of the divergence operator $\operatorname{div}_{\partial\Omega}(\,.\,) = \sum_{i,j=1}^{n-1} g^{ij}\langle \partial_j(\,.\,) \mid \partial_i X\rangle$. Moreover, note that $H = \operatorname{div}_{\partial\Omega}(\mathbf{n}) = \sum_{i,j=1}^{n-1} g^{ij}\langle \partial_j \mathbf{n} \mid \partial_i X\rangle$ by using (20.7) and (20.8), so we can write:

$$\operatorname{div}(\mathbf{v}) = \operatorname{div}_{\partial\Omega}(\mathbf{v}_{\partial\Omega}) + H v_{\mathbf{n}} + \sum_{i,j=1}^{n-1} g^{ij} v_j \langle \partial_n(\partial_i X) \mid \mathbf{n}\rangle + \langle \partial_n(v_{\mathbf{n}}\mathbf{n}) \mid \mathbf{n}\rangle. \tag{20.13}$$

Similarly, we can express the operator $\langle \mathbf{v} \mid \nabla(\,.\,)\rangle$ in the basis and we obtain:

$$\langle \mathbf{v} \mid \nabla(\,.\,)\rangle = \left\langle \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_{\mathbf{n}}\mathbf{n} \mid \sum_{i,j=1}^{n-1} g^{ij}\partial_j(\,.\,)\partial_i X + \partial_n(\,.\,)\mathbf{n}\right\rangle = \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i(\,.\,) + v_{\mathbf{n}}\partial_n(\,.\,)$$

As we already noticed, $\mathbf{v}$ is Lipschitz continuous hence differentiable almost everywhere so we can compute $\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle$ and it is equal to:

$$\left\langle \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i\left(\sum_{i',j'=1}^{n-1} g^{i'j'} v_{j'}\partial_{i'} X + v_{\mathbf{n}}\mathbf{n}\right) \mid \mathbf{n}\right\rangle + v_{\mathbf{n}}\left\langle \partial_n\left(\sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X + v_{\mathbf{n}}\mathbf{n}\right) \mid \mathbf{n}\right\rangle.$$

After some simplifications using $\langle \partial_i X \mid \mathbf{n}\rangle = 0$ and $\langle \partial_i \mathbf{n} \mid \mathbf{n}\rangle = 0$ since $\|\mathbf{n}\|^2 = 1$, we get that $\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle$ is almost everywhere equal to:

$$\sum_{i,j=1}^{n-1} g^{ij}\left(\sum_{i',j'=1}^{n-1} g^{i'j'} v_j v_{j'}\left\langle \partial_{ii'} X \mid \mathbf{n}\right\rangle + v_j \partial_i(v_{\mathbf{n}}) + v_{\mathbf{n}} v_j\left\langle \partial_n(\partial_i X) \mid \mathbf{n}\right\rangle\right) + v_{\mathbf{n}}\langle \partial_n(v_{\mathbf{n}}\mathbf{n}) \mid \mathbf{n}\rangle.$$

Observing from (20.7) that we have $\langle \partial_{ii'} X \mid \mathbf{n}\rangle = b_{ii'}$, and recalling that the first fundamental form is defined as $\mathbf{I}(\mathbf{v}_{\partial\Omega}, \mathbf{w}_{\partial\Omega}) := \langle \mathbf{v}_{\partial\Omega} \mid \mathbf{w}_{\partial\Omega}\rangle = \sum_{i,j=1}^{n-1} g^{ij} v_i w_j$ and the second fundamental form in (20.9) by $\mathbf{II}(\mathbf{v}_{\partial\Omega}, \mathbf{w}_{\partial\Omega}) := \langle -D\mathbf{n}(\mathbf{v}_{\partial\Omega}) \mid \mathbf{w}_{\partial\Omega}\rangle = -\sum_{i,j,k,l=1}^{n-1} g^{ij} g^{kl} v_k v_j \langle \partial_i \mathbf{n} \mid \partial_l X\rangle$, then the above expression can be written as:

$$\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle = \mathbf{II}(\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}) + \mathbf{I}[\mathbf{v}_{\partial\Omega}, \nabla_{\partial\Omega}(v_{\mathbf{n}})] + v_{\mathbf{n}}\left(\sum_{i,j=1}^{n-1} g^{ij} v_j \langle \partial_n(\partial_i X) \mid \mathbf{n}\rangle + \langle \partial_n(v_{\mathbf{n}}\mathbf{n})|\mathbf{n}\rangle\right).$$

Finally, we combine the above relation with (20.13) to deduce the following identity:

$$\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle - v_{\mathbf{n}}\operatorname{div}(\mathbf{v}) = \mathbf{II}(\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}) + \mathbf{I}[\mathbf{v}_{\partial\Omega}, \nabla_{\partial\Omega}(v_{\mathbf{n}})] - H(v_{\mathbf{n}})^2 - v_{\mathbf{n}}\operatorname{div}_{\partial\Omega}(\mathbf{v}_{\partial\Omega}). \tag{20.14}$$

It remains to slightly modify the last term of right hand side in (20.14) by observing that:

$$\begin{aligned}
\mathrm{div}_{\partial\Omega}(v_{\mathbf{n}}\mathbf{v}_{\partial\Omega}) - v_{\mathbf{n}}\mathrm{div}_{\partial\Omega}(\mathbf{v}_{\partial\Omega}) &= \sum_{i,j=1}^{n-1} g^{ij}\partial_j(v_{\mathbf{n}}) \left\langle \underbrace{\sum_{i',j'=1}^{n-1} g^{i'j'} v_{j'}\partial_{i'}X}_{=\,\mathbf{v}_{\partial\Omega}} \mid \partial_i X \right\rangle \\
&= \sum_{i,j=1}^{n-1} g^{ij}\partial_j(v_{\mathbf{n}}) \sum_{j'=1}^{n-1} v_{j'} \underbrace{\left(\sum_{i'=1}^{n-1} g^{i'j'} g_{i'i}\right)}_{=\,\delta_{ij'}} = \sum_{i,j=1}^{n-1} g^{ij} v_i \partial_j(v_{\mathbf{n}}) \\
&= \mathbf{I}\left[\mathbf{v}_{\partial\Omega}, \nabla_{\partial\Omega}(v_{\mathbf{n}})\right].
\end{aligned}$$

Inserting this last relation in (20.14), we obtain:

$$\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle - v_{\mathbf{n}}\mathrm{div}(\mathbf{v}) = 2\mathbf{I}[\mathbf{v}_{\partial\Omega}, \nabla_{\partial\Omega}(v_{\mathbf{n}})] + \mathbf{II}(\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}) - H(v_{\mathbf{n}})^2 - \mathrm{div}_{\partial\Omega}(v_{\mathbf{n}}\mathbf{v}_{\partial\Omega}).$$

We can integrate over $\partial\Omega$ the above equality since $\mathbf{v}$ has compact support in $\overline{\Omega}\cap C(\mathbf{x}_0)$. We get:

$$\begin{aligned}
\int_{\partial\Omega} [\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle - v_{\mathbf{n}}\mathrm{div}(\mathbf{v})]\, dA = 2\int_{\partial\Omega} \langle \nabla_{\partial\Omega}(v_{\mathbf{n}}) \mid \mathbf{v}_{\partial\Omega}\rangle\, dA + \int_{\partial\Omega} \mathbf{II}(\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}) dA \\
- \int_{\partial\Omega} H(v_{\mathbf{n}})^2 dA - \int_{\partial\Omega} \mathrm{div}_{\partial\Omega}(v_{\mathbf{n}}\mathbf{v}_{\partial\Omega}) dA
\end{aligned} \tag{20.15}$$

Hence, Relation (20.15) holds for any smooth map $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^n$ with compact support in $\overline{\Omega}\cap C(\mathbf{x}_0)$ and for any point $\mathbf{x}_0 \in \partial\Omega$. We now extend the result globally thanks to a suitable partition of unity. Let $\mathbf{v} \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$. Since $\partial\Omega$ is compact, there exists a finite number $K \geqslant 1$ of points denoted by $\mathbf{x}_1, \ldots, \mathbf{x}_K$ such that $\partial\Omega \subset \cup_{k=1}^K C(\mathbf{x}_k)$. We build a partition of unity on this set. There exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $C(\mathbf{x}_k)$, and such that $\sum_{k=1}^K \xi_k = 1$ on $\partial\Omega$. Then, we have for $k = 1, \ldots, K$:

$$\left(\sqrt{\xi_k}\mathbf{v}\right)_{\mathbf{n}} \mathrm{div}\left(\sqrt{\xi_k}\mathbf{v}\right) = \sqrt{\xi_k} v_{\mathbf{n}} \left\langle \nabla\left(\sqrt{\xi_k}\right) \mid \mathbf{v}\right\rangle + \xi_k v_{\mathbf{n}}\mathrm{div}(\mathbf{v}) = \frac{1}{2}\left[v_{\mathbf{n}}\mathrm{div}\left(\xi_k\mathbf{v}\right) + \left(\xi_k\mathbf{v}_n\right)\mathrm{div}(\mathbf{v})\right].$$

Integrating the above relations on $\partial\Omega$ and summing them from $k = 1$ to K, we deduce that:

$$\sum_{k=1}^K \int_{\partial\Omega} \left(\sqrt{\xi_k}\mathbf{v}\right)_{\mathbf{n}} \mathrm{div}\left(\sqrt{\xi_k}\mathbf{v}\right) dA = \frac{1}{2}\sum_{k=1}^K \int_{\partial\Omega} \left[v_{\mathbf{n}}\mathrm{div}\left(\xi_k\mathbf{v}\right) + \left(\xi_k\mathbf{v}_n\right)\mathrm{div}(\mathbf{v})\right] dA = \int_{\partial\Omega} v_{\mathbf{n}}\mathrm{div}(\mathbf{v}) dA.$$

Similarly, one can prove that the following relation holds:

$$\sum_{k=1}^K \int_{\partial\Omega} \left\langle\left\langle \sqrt{\xi_k}\mathbf{v} \mid \nabla\right\rangle \left(\sqrt{\xi_k}\mathbf{v}\right) \mid \mathbf{n}\right\rangle dA = \int_{\partial\Omega} \langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle\, dA.$$

Combining the last two equalities and applying (20.15) since $\sqrt{\xi_k}\mathbf{v}$ has compact support in $\overline{\Omega}\cap C(\mathbf{x}_k)$, we obtain that $\int_{\partial\Omega}[\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle - v_{\mathbf{n}}\mathrm{div}(\mathbf{v})]dA$ is equal to:

$$\begin{aligned}
2\sum_{k=1}^K \int_{\partial\Omega} \left\langle \nabla_{\partial\Omega}\left[\left(\sqrt{\xi_k}\mathbf{v}\right)_{\mathbf{n}}\right] \mid \left(\sqrt{\xi_k}\mathbf{v}\right)_{\partial\Omega}\right\rangle dA + \sum_{k=1}^K \int_{\partial\Omega} \mathbf{II}\left[\left(\sqrt{\xi_k}\mathbf{v}\right)_{\partial\Omega}, \left(\sqrt{\xi_k}\mathbf{v}\right)_{\partial\Omega}\right] dA \\
-\sum_{k=1}^K \int_{\partial\Omega} H\left[\left(\sqrt{\xi_k}\mathbf{v}\right)_{\mathbf{n}}\right]^2 dA - \sum_{k=1}^K \int_{\partial\Omega} \mathrm{div}_{\partial\Omega}\left[\left(\sqrt{\xi_k}\mathbf{v}\right)_{\mathbf{n}}\left(\sqrt{\xi_k}\mathbf{v}\right)_{\partial\Omega}\right] dA,
\end{aligned}$$

from which we deduce that $\int_{\partial\Omega}[\langle\langle \mathbf{v} \mid \nabla\rangle(\mathbf{v}) \mid \mathbf{n}\rangle - v_{\mathbf{n}}\mathrm{div}(\mathbf{v})]dA$ is equal to:

$$\begin{aligned}
\sum_{k=1}^K \int_{\partial\Omega} \langle \nabla_{\partial\Omega}\left(\xi_k v_{\mathbf{n}}\right) \mid \mathbf{v}_{\partial\Omega}\rangle\, dA + \sum_{k=1}^K \int_{\partial\Omega} \langle \nabla_{\partial\Omega}\left(v_{\mathbf{n}}\right) \mid \xi_k\mathbf{v}_{\partial\Omega}\rangle\, dA + \sum_{k=1}^K \int_{\partial\Omega} \xi_k\mathbf{II}\left(\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}\right) dA \\
- \sum_{k=1}^K \int_{\partial\Omega} H\xi_k\left(v_{\mathbf{n}}\right)^2 dA - \sum_{k=1}^K \int_{\partial\Omega} \mathrm{div}_{\partial\Omega}\left(\xi_k v_{\mathbf{n}}\mathbf{v}_{\partial\Omega}\right) dA.
\end{aligned}$$

Since $\sum_{k=1}^{K} \xi_k = 1$ on $\partial\Omega$, we have proved that (20.15) holds for any map $\mathbf{v} \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$. Combining (20.11) and (20.15), then observing that $\int_{\partial\Omega} \mathrm{div}_{\partial\Omega}(v_{\mathbf{n}}\mathbf{v}_{\partial\Omega})dA = 0$ (we refer to the next result, namely Proposition 20.3, for a proof), we deduce that for any map $\mathbf{v} \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$, we have:

$$\sum_{i,j=1}^{n} \int_\Omega \frac{\partial \mathbf{v}_i}{\partial x_j}\frac{\partial \mathbf{v}_j}{\partial x_i} - \int_\Omega |\mathrm{div}(\mathbf{v})|^2 = \int_{\partial\Omega} \left[2\left\langle \nabla_{\partial\Omega}(v_{\mathbf{n}}) \mid \mathbf{v}_{\partial\Omega}\right\rangle \ + \ \mathbf{II}(\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}) \ - \ H(v_{\mathbf{n}})^2\right] dA \tag{20.16}$$

It remains to prove that (20.10) holds for $\mathbf{v} \in H^1(\Omega, \mathbb{R}^n)$ by a density argument. Let $\mathbf{v} \in H^1(\Omega, \mathbb{R}^n)$. Since $\partial\Omega$ has $C^{1,1}$-regularity, the domain Ω is Lipschitz and there exists a sequence of smooth maps $(\mathbf{v}^m)_{m\in\mathbb{N}} \subset C^\infty(\overline{\Omega}, \mathbb{R}^n)$ converging to $\mathbf{v}$ in $H^1(\Omega, \mathbb{R}^n)$. From the foregoing, (20.16) holds for any $\mathbf{v}^m$ and we now prove that we can let $m \to +\infty$. This is the case for the first term in left-hand side of (20.16) because we have from the Cauchy-Schwarz inequality for any $i, j, k, l = 1, \ldots, n$:

$$\begin{aligned}\left| \int_\Omega \frac{\partial \mathbf{v}_i^m}{\partial x_j}\frac{\partial \mathbf{v}_k^m}{\partial x_l} - \int_\Omega \frac{\partial \mathbf{v}_i}{\partial x_j}\frac{\partial \mathbf{v}_k}{\partial x_l} \right| \ \leqslant \ & \|\tfrac{\partial \mathbf{v}_i^m}{\partial x_j} - \tfrac{\partial \mathbf{v}_i}{\partial x_j}\|_{L^2(\Omega,\mathbb{R})} \left(\|\tfrac{\partial \mathbf{v}_k}{\partial x_l}\|_{L^2(\Omega,\mathbb{R})} + \|\tfrac{\partial \mathbf{v}_k^m}{\partial x_l} - \tfrac{\partial \mathbf{v}_k}{\partial x_l}\|_{L^2(\Omega,\mathbb{R})} \right) \\ & + \|\tfrac{\partial \mathbf{v}_i}{\partial x_j}\|_{L^2(\Omega,\mathbb{R})} \|\tfrac{\partial \mathbf{v}_k^m}{\partial x_l} - \tfrac{\partial \mathbf{v}_k}{\partial x_l}\|_{L^2(\Omega,\mathbb{R})},\end{aligned}$$

Similarly, the convergence holds for the second term in the left-hand side of (20.16) because we have $\int_\Omega |\mathrm{div}(\mathbf{v})|^2 = \sum_{i,j=1}^n \int_\Omega \frac{\partial \mathbf{v}_i}{\partial x_i}\frac{\partial \mathbf{v}_j}{\partial x_j}$. It remains to get the convergence in the right-hand side of (20.16). Firstly, we can combine the continuity of the two operators $(\ .\ )_{\partial\Omega} : H^1(\Omega, \mathbb{R}^n) \to H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^n)$ and $(\ .\ )_{\mathbf{n}} : H^1(\Omega, \mathbb{R}^n) \to H^{\frac{1}{2}}(\partial\Omega, \mathbb{R})$ with the one of $\nabla_{\partial\Omega} : H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}) \to H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^n)$ to deduce:

$$\int_{\partial\Omega} \left\langle \nabla_{\partial\Omega}\left[(\mathbf{v}^m)_{\mathbf{n}}\ \right] \mid (\mathbf{v}^m)_{\partial\Omega} \right\rangle dA \quad \underset{m\to+\infty}{\longrightarrow} \left\langle \nabla(v_{\mathbf{n}}) \mid \mathbf{v}_{\partial\Omega}\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^n), H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^n)}$$

Secondly, $\partial\Omega$ is a compact $C^{1,1}$-hypersurface hence there exists $\varepsilon > 0$ such that $\partial\Omega$ satisfies the ε-ball condition and in particular, the Gauss map $\mathbf{n} : \mathbf{x} \in \partial\Omega \to \mathbb{S}^{n-1}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. We deduce that the eigenvalues of its differential i.e. the principal curvatures $(\kappa_i)_{1\leqslant i\leqslant n-1}$ exists almost everywhere and belongs to $L^\infty(\partial\Omega, \mathbb{R})$. Considering the principal directions $(\mathbf{e}_i)_{1\leqslant i\leqslant n-1}$ associated with the principal curvatures, $(\mathbf{e}_1, \ldots, \mathbf{e}_{n-1})$ forms an orthonormal basis of the tangent hyperplane so we deduce that:

$$\int_{\partial\Omega} \mathbf{II}\left[(\mathbf{v}^m)_{\partial\Omega}, (\mathbf{v}^m)_{\partial\Omega}\right] dA = -\int_{\partial\Omega} \left\langle D_{\mathbf{x}}\mathbf{n}\left(\sum_{i=1}^{n-1} \langle (\mathbf{v}^m)_{\partial\Omega}(\mathbf{x}) | \mathbf{e}_i(\mathbf{x})\rangle \mathbf{e}_i(\mathbf{x})\right) | (\mathbf{v}^m)_{\partial\Omega}(\mathbf{x}) \right\rangle dA(\mathbf{x})$$

Since $D_{\mathbf{x}}\mathbf{n}[\mathbf{e}_i(\mathbf{x})] = \kappa_i(\mathbf{x})\mathbf{e}_i(\mathbf{x})$, we obtain from the linearity:

$$\int_{\partial\Omega} \mathbf{II}\left[(\mathbf{v}^m)_{\partial\Omega}, (\mathbf{v}^m)_{\partial\Omega}\right] dA = -\sum_{i=1}^{n-1} \int_{\partial\Omega} \kappa_i(\mathbf{x}) \left| \langle (\mathbf{v}^m)_{\partial\Omega} \ \mid \mathbf{e}_i(\mathbf{x})\rangle \right|^2 dA(\mathbf{x}),$$

from which we deduce with the Cauchy-Schwarz inequality:

$$\left| \int_{\partial\Omega} \left(\mathbf{II}\left[(\mathbf{v}^m)_{\partial\Omega}, (\mathbf{v}^m)_{\partial\Omega}\right] \ - \ \mathbf{II}\left[\mathbf{v}_{\partial\Omega}, \mathbf{v}_{\partial\Omega}\right]\right) dA \right| \leqslant \left(\sum_{i=1}^{n-1} \|\kappa_i\|_{L^\infty(\partial\Omega,\mathbb{R})}\right) \int_{\partial\Omega} \|(\mathbf{v}^m - \mathbf{v})_{\partial\Omega}\|^2 dA.$$

Using the continuity of $(\ .\ )_{\partial\Omega} : H^1(\Omega, \mathbb{R}^n) \to L^2(\partial\Omega, \mathbb{R}^n)$, we get the convergence of the second term in the right-hand side of (20.16). Concerning the third one, the arguments are similar because (20.6) gives $H = \kappa_1 + \ldots + \kappa_{n-1} \in L^\infty(\partial\Omega, \mathbb{R})$ and $(\ .\ )_{\mathbf{n}} : H^1(\Omega, \mathbb{R}^n) \to L^2(\partial\Omega, \mathbb{R})$ is continuous. To conclude, we can apply (20.16) on each $\mathbf{v}^m$ and let $m \to +\infty$ to obtain that (20.10) holds for any $\mathbf{v} \in H^1(\Omega, \mathbb{R})$ as required. $\square$

Proposition 20.3. *Let Σ be a compact $C^{1,1}$-hypersurface of $\mathbb{R}^n$. Then, for any $\mathbf{v} \in W^{1,1}(\Sigma, \mathbb{R}^n)$ such that $\langle \mathbf{v} \mid \mathbf{n}\rangle = 0$, we have:*

$$\int_\Sigma \mathrm{div}_\Sigma(\mathbf{v}) dA = 0.$$

Proof. Consider any compact $C^{1,1}$-hypersurface $\Sigma \subset \mathbb{R}^n$. Let $\mathbf{x}_0 \in \Sigma$. There exists a cylinder $C(\mathbf{x}_0)$ in which $\partial\Omega$ is the graph of a $C^{1,1}$-map φ. We thus introduce the local $C^{1,1}$-parametrization $X : \mathbf{x}' \in D(\mathbf{x}_0) \mapsto (\mathbf{x}', \varphi(\mathbf{x}')) \in \Sigma \cap C(\mathbf{x}_0)$ and we first assume that $\mathbf{v} : \Sigma \to \mathbb{R}^n$ is a smooth map with compact support in $\Sigma \cap C(\mathbf{x}_0)$. We use the same notation than in the proof of Theorem 20.2. Hence, we can decompose $\mathbf{v}$ in the basis $(\partial_1 X, \ldots, \partial_{n-1}X, \mathbf{n})$. Since $\langle \mathbf{v} \mid \mathbf{n}\rangle = 0$, we have:

$$\mathbf{v} = \sum_{i,j=1}^{n-1} g^{ij} \langle \mathbf{v} \mid \partial_j X\rangle \partial_i X + \langle \mathbf{v} \mid \mathbf{n}\rangle \mathbf{n} = \sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X.$$

In this decomposition, note that $\mathbf{v}$ is a Lipschitz continuous map so it is differentiable almost everywhere and we can compute:

$$\begin{aligned}
\mathrm{div}_\Sigma(\mathbf{v}) \quad = \quad & \sum_{k,l=1}^{n-1} g^{kl} \langle \partial_l(\mathbf{v}) \mid \partial_k X\rangle \quad = \quad \sum_{k,l=1}^{n-1} g^{kl} \left\langle \partial_l \left(\sum_{i,j=1}^{n-1} g^{ij} v_j \partial_i X \right) \mid \partial_k X \right\rangle
\end{aligned}$$

$$= \sum_{k,l=1}^{n-1} g^{kl} \left(\sum_{i,j=1}^{n-1} \partial_l \left(g^{ij} v_j \right) \langle \partial_i X \mid \partial_k X\rangle \right) + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \left(\sum_{k,l=1}^{n-1} g^{kl} \langle \partial_i(\partial_l X) \mid \partial_k X\rangle \right).$$

Since X is a $C^{1,1}$-map, it is twice-differentiable almost everywhere and at the point where it is the case, we have $\partial_l(\partial_i X) = \partial_i(\partial_l X)$. Moreover, the matrix (g^{kl}) is symmetric so we deduce that:

$$\begin{aligned}
\mathrm{div}_\Sigma(\mathbf{v}) \quad &= \quad \sum_{k,l=1}^{n-1} g^{kl} \left(\sum_{i,j=1}^{n-1} \partial_l \left(g^{ij} v_j \right) g_{ik} \right) + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \left(\sum_{k,l=1}^{n-1} g^{kl} \langle \partial_l(\partial_i X) \mid \partial_k X\rangle \right) \\
&= \quad \sum_{i,l=1}^{n-1} \partial_l \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \left(\sum_{k=1}^{n-1} g_{ik} g^{kl} \right) + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \left(\frac{1}{2} \sum_{k,l=1}^{n-1} g^{kl} \partial_i (g_{lk}) \right).
\end{aligned}$$

Then, we observe that the first term has a simplification since (g^{ij}) is the inverse matrix of (g_{ij}) and the second term is the differential of a determinant. Hence, we obtain:

$$\begin{aligned}
\mathrm{div}_\Sigma(\mathbf{v}) \quad &= \quad \sum_{i,l=1}^{n-1} \partial_l \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \delta_{il} + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \left(\frac{1}{2} \mathrm{Trace}\left(\partial_i(g) g^{-1} \right) \right) \\
&= \quad \sum_{i=1}^{n-1} \partial_i \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n-1} g^{ij} v_j \right) \frac{\partial_i(\det(g))}{2\det(g)} \\
&= \quad \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{n-1} \partial_i \left(\sqrt{\det(g)} \sum_{j=1}^{n-1} g^{ij} v_j \right).
\end{aligned}$$

Since $\mathbf{v}$ has compact support in $\Sigma \cap C(\mathbf{x}_0)$, so does $v_j = \langle \mathbf{v} \circ X \mid \partial_j X\rangle$ on $D(\mathbf{x}_0)$ and we get:

$$\int_\Sigma \mathrm{div}_\Sigma(\mathbf{v}) dA = \int_{D(\mathbf{x}_0)} \mathrm{div}_\Sigma(\mathbf{v} \circ X) \sqrt{\det(g)} = \sum_{i=1}^{n-1} \int_{D(\mathbf{x}_0)} \partial_i \left(\sqrt{\det(g)} \sum_{j=1}^{n-1} g^{ij} v_j \right) = 0$$

The result of Proposition 20.3 is thus established if $\mathbf{v} : \Sigma \to \mathbb{R}^n$ is a smooth map with compact support in $\Sigma \cap C(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \Sigma$. Then, we assume that $\mathbf{v} \in C^\infty(\Sigma, \mathbb{R}^n)$. Since Σ is compact, there exists a finite number $K \geqslant 1$ of points denoted by $\mathbf{x}_1, \ldots, \mathbf{x}_K$ such that $\Sigma \subset \cup_{k=1}^K C(\mathbf{x}_k)$. We can build a partition of unity on this set. There exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $C(\mathbf{x}_k)$, and such that $\sum_{k=1}^K \xi_k = 1$ on Σ. Hence, we have successively:

$$\int_\Sigma \mathrm{div}_\Sigma(\mathbf{v}) dA = \int_\Sigma \mathrm{div}_\Sigma \left(\sum_{k=1}^K \xi_k \mathbf{v} \right) dA = \sum_{k=1}^K \int_\Sigma \mathrm{div}_\Sigma(\xi_k \mathbf{v}) dA = 0,$$

where the last equality comes from the previous case because $\xi_k \mathbf{v}$ is a smooth map with compact support in $C(\mathbf{x}_k)$ for any $k = 1, \ldots, K$. The result of Proposition 20.3 holds for any $\mathbf{v} \in C^\infty(\Sigma, \mathbb{R}^n)$. Finally, we assume that $\mathbf{v} \in W^{1,1}(\Sigma, \mathbb{R}^n)$. By density, there exists a sequence of smooth maps $\mathbf{v}^m \in C^\infty(\Sigma, \mathbb{R}^n)$ such that $\mathbf{v}^m - \mathbf{v}$ tends to zero in $W^{1,1}(\Sigma, \mathbb{R}^n)$. We can apply the previous case on each $\mathbf{v}^m$ and we get:

$$\left| \int_\Sigma \mathrm{div}_\Sigma(\mathbf{v}) dA \right| = \left| \int_\Sigma \mathrm{div}_\Sigma(\mathbf{v} - \mathbf{v}^m) dA \right| \leqslant \sum_{i=1}^n \int_\Sigma \| \nabla_\Sigma (\mathbf{v}_i) - \nabla_\Sigma (\mathbf{v}_i^m) \| dA \underset{m \to +\infty}{\longrightarrow} 0.$$

To conclude, we proved $\int_\Sigma \mathrm{div}_\Sigma(\mathbf{v}) dA = 0$ for any map $\mathbf{v} \in W^{1,1}(\Sigma, \mathbb{R}^n)$ such that $\langle \mathbf{v} \mid \mathbf{n} \rangle = 0$. $\square$

20.3 Some Poincaré inequalities

We quickly recall here the well-known Poincaré inequality and deduce some of its consequences.

Proposition 20.4 (**Poincaré Inequality**). *Let Ω be any non-empty open subset of $\mathbb{R}^n$ which is bounded in a direction i.e. there exists a constant $D > 0$, a point $\mathbf{x}_0 \in \mathbb{R}^n$, and a unit vector $\mathbf{d}_{\mathbf{x}_0}$ of $\mathbb{R}^n$ such that $|\langle \mathbf{x} - \mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0} \rangle| \leqslant D$ for any point $\mathbf{x} \in \Omega$. Then, we have:*

$$\forall u \in H_0^1(\Omega, \mathbb{R}), \qquad \int_\Omega u^2 \leqslant 4D^2 \int_\Omega \|\nabla u\|^2.$$

Proof. We consider a basis $\mathcal{B}_{\mathbf{x}_0}$ of the orthogonal space $\mathbf{d}_{\mathbf{x}_0}^\perp$ such that $(\mathbf{x}_0, \mathcal{B}_{\mathbf{x}_0}, \mathbf{d}_{\mathbf{x}_0})$ is a direct orthonormal frame centred at $\mathbf{x}_0$. Henceforth, the position of any point is determined in this frame. In particular, any point $\mathbf{x} = (\mathbf{x}', x_n) \in \Omega$ must satisfy $|x_n| \leqslant D$. First, we assume $u \in C_c^\infty(\Omega, \mathbb{R})$. Then, the map u can be extended by zero to $\tilde{u} \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ and in particular, for any $\mathbf{x}' \in \mathbb{R}^{n-1}$, we have $\tilde{u}(\mathbf{x}', -D) = \lim_{x_n \to D^-} \tilde{u}(\mathbf{x}', x_n) = 0$ since $(\mathbf{x}', x_n) \notin \Omega$. Combining this observation with the Cauchy-Schwarz inequality, we have for any $(\mathbf{x}', x_n) \in \mathbb{R}^{n-1} \times [-D, D]$:

$$\tilde{u}(\mathbf{x}', x_n)^2 = \left(\int_{-D}^{x_n} \frac{\partial \tilde{u}}{\partial x_n}(\mathbf{x}', t) dt \right)^2 \leqslant (x_n + D) \int_{-D}^{x_n} \left(\frac{\partial \tilde{u}}{\partial x_n}(\mathbf{x}', t) \right)^2 dt \leqslant 2D \int_{-D}^{D} \left(\frac{\partial \tilde{u}}{\partial x_n}(\mathbf{x}', t) \right)^2 dt$$

Integrating this inequality in the x_n-variable on $[-D, D]$, and in the $\mathbf{x}'$-variable on $\mathbb{R}^{n-1}$, we obtain:

$$\int_{\mathbb{R}^{n-1}} \left(\int_{-D}^{D} \tilde{u}(\mathbf{x}', x_n)^2 dx_n \right) d\mathbf{x}' \leqslant 4D^2 \int_{\mathbb{R}^{n-1}} \left(\int_{-D}^{D} \left(\frac{\partial \tilde{u}}{\partial x_n}(\mathbf{x}', t) \right)^2 dt \right) d\mathbf{x}'$$

Then, we use again the observation $\tilde{u}(\mathbf{x}', x_n) = 0$ for any $\mathbf{x}' \in \mathbb{R}^{n-1}$ and $x_n \notin]-D, D[$. Thanks to the Fubini-Tonelli Theorem, we get:

$$\begin{aligned}
\int_\Omega u^2 &= \int_{\mathbb{R}^n} \tilde{u}^2 = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{+\infty} \tilde{u}^2 \right) = \int_{\mathbb{R}^{n-1}} \left(\int_{-D}^{D} \tilde{u}^2 \right) \leqslant 4D^2 \int_{\mathbb{R}^{n-1}} \left(\int_{-D}^{D} \left(\frac{\partial \tilde{u}}{\partial x_n} \right)^2 \right) \\
&\leqslant 4D^2 \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \left(\frac{\partial \tilde{u}}{\partial x_n} \right)^2 \right) = 4D^2 \int_{\mathbb{R}^n} \left(\frac{\partial \tilde{u}}{\partial x_n} \right)^2 = 4D^2 \int_\Omega \left(\frac{\partial u}{\partial x_n} \right)^2 \\
&\leqslant 4D^2 \sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i} \right)^2 = 4D^2 \int_\Omega \|\nabla u\|^2
\end{aligned}$$

Consequently, Proposition 20.4 is established for any $u \in C_c^\infty(\Omega, \mathbb{R})$. Finally, we assume that $u \in H_0^1(\Omega, \mathbb{R})$. There exists a sequence $(u_i)_{i \in \mathbb{N}} \subset C_c^\infty(\Omega, \mathbb{R})$ converging strongly to u in $H^1(\Omega, \mathbb{R})$. From the foregoing, we deduce that:

$$\begin{aligned}
\|u\|_{L^2(\Omega,\mathbb{R})} &\leqslant \|u - u_i\|_{L^2(\Omega,\mathbb{R})} + \|u_i\|_{L^2(\Omega,\mathbb{R})} \leqslant \|u - u_i\|_{L^2(\Omega,\mathbb{R})} + 2D\|\nabla u_i\|_{L^2(\Omega,\mathbb{R}^n)} \\
&\leqslant \|u - u_i\|_{L^2(\Omega,\mathbb{R})} + 2D\|\nabla u_i - \nabla u\|_{L^2(\Omega,\mathbb{R}^n)} + 2D\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)} \\
&\leqslant \max(1, 2D)\|u_i - u\|_{H^1(\Omega,\mathbb{R})} + 2D\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)}
\end{aligned}$$

To conclude, we let $i \to +\infty$ to obtain the required inequality : $\|u\|_{L^2(\Omega,\mathbb{R})} \leqslant 2D\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)}$. $\square$

Corollary 20.5. *Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$. If $D = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \|\mathbf{x}-\mathbf{y}\|$, then we have:*

$$\forall u \in H_0^1(\Omega,\mathbb{R}), \qquad \int_\Omega u^2 \leqslant 4D^2 \int_\Omega \|\nabla u\|^2.$$

Proof. Since Ω is bounded, $\overline{\Omega}$ is compact so the diameter D is finite and attained by two points $\mathbf{x}_0$ and $\mathbf{y}_0$. Moreover, it is positive because Ω is not empty and open. We get $\Omega \subseteq B_D(\mathbf{x_0})$ and applying Proposition 20.4 for the point $\mathbf{x}_0$ and the unit vector $\frac{1}{D}(\mathbf{y}_0 - \mathbf{x}_0)$, the inequality follows. □

Corollary 20.6. *Let Ω be a non-empty bounded open subset of $\mathbb{R}^n$. If $D = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \|\mathbf{x}-\mathbf{y}\|$, then we have:*

$$\forall u \in H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R}), \qquad \int_\Omega \|\nabla u\|^2 \leqslant 4D^2 \int_\Omega (\Delta u)^2.$$

Proof. Let any $u \in H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R})$. We get successively from an integration by parts, the inequality $xy \leqslant \frac{a}{2}x^2 + \frac{1}{2a}y^2$, and Corollary 20.5:

$$\int_\Omega \|\nabla u\|^2 = -\int_\Omega u\Delta u \leqslant \int_\Omega |u\Delta u| \leqslant 2D^2 \int_\Omega (\Delta u)^2 + \frac{1}{8D^2}\int_\Omega u^2 \leqslant 2D^2 \int_\Omega (\Delta u)^2 + \frac{1}{2}\int_\Omega \|\nabla u\|^2.$$

After simplification, we obtain the required inequality: $\|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)} \leqslant 2D\|\Delta u\|_{L^2(\Omega,\mathbb{R})}$. □

20.4 Some Trace inequalities

The constant appearing in the trace inequality depends on the C^1-norm of any partition of unity associated with an finite open covering of the hypersurface. Therefore, we need to build a partition of unity for which we can control uniformly the number of maps and the C^0-norm of their gradient.

Proposition 20.7. *Let $h > 0$, $n \geqslant 1$, and B be any non-empty open subset of $\mathbb{R}^n$ of diameter D, large enough to contain the origin. Then, there exists $N \in \mathbb{N}$ and a constant $C > 0$, both depending only on h, D and n, such that for any non-empty open set $\Omega \subseteq B$, there exists K distinct points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega$, $1 \leqslant K \leqslant N(h,D,n)$, such that the tubular neighbourhood $\mathcal{V}_{\frac{h}{4}}(\partial\Omega)$ has its closure embedded in $\cup_{k=1}^K B_h(\mathbf{x}_k)$, and there exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $B_h(\mathbf{x}_k)$, such that $\sum_{k=1}^K \xi_k = 1$ on $\overline{\mathcal{V}_{\frac{h}{4}}}(\partial\Omega)$ and $\sum_{k=1}^K \sum_{i=1}^n \|\frac{\partial \xi_k}{\partial x_i}\|_{C^0(\mathbb{R}^n,\mathbb{R})} \leqslant C(h,D,n)$.*

Proof. Let $h > 0$, $n \geqslant 1$, $B \subset \mathbb{R}^n$ be a non-empty open set of diameter D containing the origin $\mathbf{0}$, and Ω be a non-empty open subset of B. Since $\mathbf{0} \in B$, we have $\overline{\Omega} \subseteq \overline{B_D}(\mathbf{0})$ so it is included in the cube of length D centred at the origin. We set:

$$a := \frac{h}{2\sqrt{n}} \qquad \text{and} \qquad N(h,D,n) := \left(1 + \left[\frac{D}{a}\right]\right)^n,$$

where [.] denotes here the integer part. Hence, the larger cube of length $a(1+[\frac{D}{a}]) > D$ centred at the origin can be divided into $N(h,D,n)$ small cubes of length a. We denote by $(\mathbf{y}_k)_{1\leqslant k\leqslant N}$ the centres of these small cubes. Note that with our choice of a, their diameter is $\frac{h}{2}$ thus they are themselves contained in balls of radius $\frac{h}{4}$ centred at $\mathbf{y}_k$. In other words, $\overline{B_D}(\mathbf{0}) \subseteq \cup_{k=1}^N \overline{B_{\frac{h}{4}}}(\mathbf{y}_k)$. Then, we deduce that:

$$\partial\Omega \subseteq \bigcup_{\substack{1\leqslant k\leqslant N \\ \partial\Omega\cap\overline{B_{\frac{h}{4}}}(\mathbf{y}_k)\neq\emptyset}} \overline{B_{\frac{h}{4}}}(\mathbf{y}_k).$$

Therefore, we can relabel the points $(\mathbf{y}_k)_{1\leqslant k\leqslant N}$ such that there exists an integer $1 \leqslant K \leqslant N$ satisfying $\partial\Omega \subseteq \cup_{k=1}^K \overline{B_{\frac{h}{4}}}(\mathbf{x}_k)$ and $\partial\Omega \cap \overline{B_{\frac{h}{4}}}(\mathbf{y}_k) \neq \emptyset$ for $k = 1,\ldots,K$. In particular, $d(\mathbf{y}_k,\partial\Omega) \leqslant \frac{h}{4}$ so there exists K points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega$ such that $\|\mathbf{x}_k - \mathbf{y}_k\| \leqslant \frac{h}{4}$. From the triangle inequality, we successively deduce $\partial\Omega \subseteq \cup_{k=1}^K \overline{B_{\frac{h}{2}}}(\mathbf{x}_k)$ and $\overline{\mathcal{V}_{\frac{h}{4}}}(\partial\Omega) \subseteq \cup_{k=1}^K \overline{B_{\frac{3h}{4}}}(\mathbf{x}_k)$. Finally, it remains to build the partition of unity. This is a standard procedure. We introduce the following function:

$$\begin{array}{rccl} w: & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & \mathbf{x} & \longmapsto & w(x) = \begin{cases} e^{1-\frac{h^2}{h^2-(16\|\mathbf{x}\|)^2}} & \text{if } \|\mathbf{x}\| < \frac{h}{16} \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

One can check $w \in C^\infty(\mathbb{R}^n,[0,1])$ and its support is $\overline{B_{\frac{h}{16}}}(\mathbf{0})$. Then, we set $c(h,n) = 1/\int_{\mathbb{R}^n} w(\mathbf{x})d\mathbf{x}$, depending only on n and h. We consider the following maps:

$$\begin{array}{rcl} \Psi_k: \quad \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \mathbf{x} & \longmapsto & \Psi_k(\mathbf{x}) = c(h,n) \displaystyle\int_{\overline{B_{\frac{3h}{4}+\frac{h}{16}}(\mathbf{x}_k)}} w(\mathbf{x}-\mathbf{y})d\mathbf{y}. \end{array}$$

Similarly, one can show that $\Psi_k \in C^\infty(\mathbb{R}^n,[0,1])$, $\Psi_k = 1$ on $\overline{B_{\frac{3h}{4}}}(\mathbf{x}_k)$ and supp $\Psi_k \subseteq \overline{B_{h-\frac{h}{16}}}(\mathbf{x}_k)$ so it has compact support in $B_h(\mathbf{x}_k)$. Moreover, we have for $i = 1,\ldots,n$ and for $k = 1,\ldots,K$:

$$\begin{aligned} \left| \frac{\partial \Psi_k}{\partial x_i}(\mathbf{x}) \right| &= \left| c(h,n) \int_{\overline{B_{\frac{3h}{4}+\frac{h}{16}}(\mathbf{x}_k)}} \frac{\partial w}{\partial x_i}(\mathbf{x}-\mathbf{y})d\mathbf{y} \right| \\ &\leqslant c(h,n)\|\tfrac{\partial w}{\partial x_i}\|_{C^0(\mathbb{R}^n,\mathbb{R})} V\left(\overline{B_{\frac{3h}{4}+\frac{h}{16}}(\mathbf{x}_k)}\right) = \underbrace{c(h,n)\frac{2\exp(-1)}{h}\frac{4\pi}{3}\left(\frac{3h}{4}+\frac{h}{16}\right)^3}_{:=\tilde{c}(h,n)}. \end{aligned}$$

To conclude, we set $\xi_1 = \Psi_1$ and $\xi_k = \Psi_k \prod_{i=1}^{k-1}(1-\Psi_i)$ for any $2 \leqslant k \leqslant K$. We get that $\xi_k \in C^\infty(\mathbb{R}^n,[0,1])$ has compact support in $B_r(\mathbf{x}_k)$, and $\sum_{k=1}^K \xi_k = 1$ on $\cup_{k=1}^K \overline{B_{\frac{3h}{4}}}(\mathbf{x}_k)$ thus on the closure of $V_{\frac{h}{4}}(\partial\Omega)$. Furthermore, we have:

$$\sum_{k=1}^K \sum_{i=1}^n \|\tfrac{\partial \xi_k}{\partial x_i}\|_{C^0(\mathbb{R}^n,\mathbb{R})} \leqslant \sum_{k=1}^K \sum_{i=1}^n \sum_{j=1}^k \|\tfrac{\partial \Psi_k}{\partial x_i}\|_{C^0(\mathbb{R}^n,\mathbb{R})} \leqslant n\tilde{c}(h,n)N(h,D,n)^2,$$

and the constant $C(h,D,n) := n\tilde{c}(h,n)N(h,D,n)^2$ is the one required in the statement. □

Proposition 20.8. *Let $\alpha \in]0,\frac{\pi}{2}[$, $n \geqslant 2$ and B be a non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We consider the class $\mathfrak{O}_\alpha(B)$ formed by all the non-empty open subsets of B satisfying the α-cone property. We assume that the diameter D of B is large enough to ensure $\mathfrak{O}_\varepsilon(B) \neq \emptyset$. Then, there exists a constant $C > 0$, depending only on α, n, and D such that:*

$$\forall \Omega \in \mathfrak{O}_\alpha(B), \forall \eta \in]0,1[, \ \forall u \in H^1(\Omega,\mathbb{R}), \qquad \int_{\partial\Omega} u^2 dA \leqslant C(\alpha,D,n)\left(\eta \int_\Omega \|\nabla u\|^2 + \frac{1}{\eta}\int_\Omega u^2\right).$$

Proof. Let $\alpha \in]0,\frac{\pi}{2}[$, $n \geqslant 2$ and B be a non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. Introducing the class $\mathfrak{O}_\alpha(B)$ formed by all the non-empty open subsets of B satisfying the α-cone property, we consider $\Omega \in \mathfrak{O}_\alpha(B)$. Hence, from the uniform cone property, $\partial\Omega$ has a Lipschitz boundary i.e. for any point $\mathbf{x}_0 \in \partial\Omega$, there exits a cylinder $C_{r,a}(\mathbf{x}_0)$ as in (20.1) of direction a unit vector $\mathbf{d}_{\mathbf{x}_0}$ of $\mathbb{R}^n$ in which $\partial\Omega$ is the graph of a L-Lipschitz continuous map $\varphi_{\mathbf{x}_0}$, and in which Ω is the area below this graph. Moreover, the constants $r > 0$, $a > 0$, and $L > 0$ only depend on α. Consequently, Proposition 20.7 is applied to B with $h(\alpha) = \min(r,a)$ depending only on α. There exists K distinct points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega$, such that $\overline{\mathcal{V}_{\frac{h}{4}}}(\partial\Omega) \subset \cup_{k=1}^K C_{r,a}(\mathbf{x}_k)$, and there exists K smooth maps $\xi_k : \mathbb{R}^n \to [0,1]$ with compact support in $C_{r,a}(\mathbf{x}_k)$ such that $\sum_{k=1}^K \xi_k = 1$ on $\overline{\mathcal{V}_{\frac{h}{4}}}(\partial\Omega)$. Furthermore, we have $K \leqslant N(\alpha,D,n)$ and $\sum_{k=1}^K\sum_{i=1}^n \|\frac{\partial \xi_k}{\partial x_i}\|_{C^0(\mathbb{R}^n,\mathbb{R})} \leqslant C(\alpha,D,n)$, where $N \in \mathbb{N}$ and $C > 0$ depending only on α, D, and n. We set $\mathbf{m} = \sum_{k=1}^K \xi_k \mathbf{d}_{\mathbf{x}_k} \in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and we show that $\langle \mathbf{m} \mid \mathbf{n}\rangle \geqslant [1+L^2]^{-\frac{1}{2}}$ almost everywhere on $\partial\Omega$. Indeed, since $\varphi_{\mathbf{x}_k}$ is L-Lipschitz continuous, it is differentiable almost everywhere. Considering $\mathbf{x} \in \partial\Omega$ for which the normal exists, we have:

$$\langle \mathbf{m}(\mathbf{x}) \mid \mathbf{n}(\mathbf{x})\rangle = \sum_{k=1}^K \xi_k(\mathbf{x})\langle \mathbf{n}(\mathbf{x}) \mid \mathbf{d}_{\mathbf{x}_k}\rangle = \sum_{k=1}^K \frac{\xi_k(\mathbf{x}',\varphi_{\mathbf{x}_k}(\mathbf{x}'))}{\sqrt{1+\|\nabla\varphi_{\mathbf{x}_k}(\mathbf{x}')\|^2}} \geqslant \frac{\sum_{k=1}^K \xi_k(\mathbf{x})}{\sqrt{1+L^2}} = \frac{1}{\sqrt{1+L^2}}.$$

Let $u \in H^1(\Omega,\mathbb{R})$ and $\eta \in]0,1[$. We use successively the previous inequality, the Stokes Theorem, the Cauchy-Schwarz inequality, the one $2xy \leqslant \eta x^2 + \frac{1}{\eta}y_2$ and the fact that $\eta \in]0,1[$ to get:

$$\begin{aligned}
\int_{\partial\Omega} u^2 dA &\leqslant \sqrt{1+L^2}\int_{\partial\Omega} u^2\langle \mathbf{m} \mid \mathbf{n}\rangle dA \quad = \quad \sqrt{1+L^2}\int_{\Omega}\mathrm{div}(u^2\mathbf{m}) \\
&= \sqrt{1+L^2}\sum_{k=1}^{K}\int_{\Omega} 2\xi_k u\langle \nabla u \mid \mathbf{d}_{\mathbf{x}_k}\rangle \quad + \quad \sqrt{1+L^2}\sum_{k=1}^{K}\int_{\Omega} u^2\langle \nabla\xi_k \mid \mathbf{d}_{\mathbf{x}_k}\rangle \\
&\leqslant K\sqrt{1+L^2}\int_{\Omega} 2u\|\nabla u\| \quad + \quad \sqrt{1+L^2}\left(\sum_{k=1}^{K}\sum_{i=1}^{n}\|\frac{\partial \xi_k}{\partial x_i}\|_{C^0(\mathbb{R}^n,\mathbb{R})}\right)\int_{\Omega} u^2 \\
&\leqslant N\sqrt{1+L^2}\left(\eta\int_{\Omega}\|\nabla u\|^2 + \frac{1}{\eta}\int_{\Omega} u^2\right) \quad + \quad C\sqrt{1+L^2}\int_{\Omega} u^2 \\
&\leqslant (N+C)\sqrt{1+L^2}\left(\eta\int_{\Omega}\|\nabla u\|^2 + \frac{1}{\eta}\int_{\Omega} u^2\right)
\end{aligned}$$

To conclude, observe that the constant only depends on α, D and n as required. $\square$

Corollary 20.9. *Using the assumptions and notation of Proposition 20.8, we get for any $\Omega \in \mathfrak{O}_\alpha(B)$:*

$$\forall \eta \in]0,1[,\ \forall u \in H^2(\Omega,\mathbb{R}),\qquad \int_{\partial\Omega}\|\nabla u\|^2 dA \leqslant C(\alpha,D,n)\left(\eta\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 + \frac{1}{\eta}\int_{\Omega}\|\nabla u\|^2\right).$$

Proof. Apply Proposition 20.8 to each $\frac{\partial u}{\partial x_i} \in H^1(\Omega,\mathbb{R})$ and sum the n inequalities obtained. $\square$

Proof of Theorem 20.1. Let $\varepsilon > 0$, $n \geqslant 2$, and B be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. Introducing the class $\mathcal{O}_\varepsilon(B)$ formed by all the non-empty open subsets of B satisfying the ε-ball condition, we consider $\Omega \in \mathcal{O}_\varepsilon(B)$ and $\mathbf{u} \in H^2(\Omega,\mathbb{R}) \cap H_0^1(\Omega,\mathbb{R})$. First, since $u = 0$ on $\partial\Omega$, we deduce $\nabla u = \partial_n(u)\mathbf{n}$ i.e. $\nabla_{\partial\Omega}(u) = 0$. Applying Theorem 20.2 to $\mathbf{v} = \nabla u$, we get from (20.10):

$$\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 v_i}{\partial x_i\partial x_j}\right)^2 = \int_{\Omega}|\Delta u|^2 - \int_{\partial\Omega} H\|\nabla u\|^2 dA.$$

Then, recall that Ω satisfies the ε-ball condition so $\mathbf{n} : \mathbf{x} \in \partial\Omega \to \mathbb{S}^{n-1}$ is $\frac{1}{\varepsilon}$-Lipschitz continuous. We deduce that the eigenvalues of its differential i.e. the principal curvatures $(\kappa_i)_{1\leqslant i\leqslant n-1}$ exists almost everywhere and are essentially bounded by $\frac{1}{\varepsilon}$. Combining this observation with (20.6), we get $\|H\|_{L^\infty(\partial\Omega,\mathbb{R})} \leqslant \frac{n-1}{\varepsilon}$. Moreover, there exists $\alpha \in]0,\frac{\pi}{2}[$ depending only on ε such that Ω satisfies the $\alpha(\varepsilon)$-cone property so we deduce from Corollary 20.9 and the above equality:

$$\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 v_i}{\partial x_i\partial x_j}\right)^2 \leqslant \int_{\Omega}|\Delta u|^2 + \frac{(n-1)}{\varepsilon}C(\alpha,D,n)\left(\eta\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 + \frac{1}{\eta}\int_{\Omega}\|\nabla u\|^2\right).$$

Finally, we use Corollaries 20.5 and 20.6 to obtain:

$$\begin{cases}
\left(1 - \dfrac{\eta(n-1)C(\alpha,D,n)}{\varepsilon}\right)\displaystyle\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 v_i}{\partial x_i\partial x_j}\right)^2 \leqslant \left(1 + \dfrac{4D^2(n-1)C(\alpha,D,n)}{\varepsilon\eta}\right)\int_{\Omega}|\Delta u|^2. \\[2ex]
\displaystyle\int_{\Omega} u^2 + \int_{\Omega}\|\nabla u\|^2 \leqslant 4D^2(1+4D^2)\int_{\Omega}|\Delta u|^2.
\end{cases}$$

If we set $\eta(\varepsilon,\alpha,D,n) = \frac{1}{2}\min(1,\frac{\varepsilon}{(n-1)C(\alpha,D,n)})$, then we get the required estimation:

$$\|u\|^2_{H^2(\Omega,\mathbb{R})} \leqslant 2\left(1 + 2D^2(1+4D^2) + \frac{4D^2(n-1)C(\alpha(\varepsilon),D,n)}{\varepsilon\eta(\varepsilon,\alpha(\varepsilon),D,n)}\right)\|\Delta u\|^2_{L^2(\Omega,\mathbb{R})}.$$

To conclude, the constant appearing in the above inequality only depends on ε, D and n. $\square$

Chapter 21

Continuity of some geometric functionals based on PDE: the Dirichlet boundary condition

In this section, we want to extend the existence results obtained in $\mathcal{O}_\varepsilon(B)$ for general geometric functionals by allowing a dependence through the solutions of some partial differential equations. First, let us prove the sequential continuity in $\mathcal{O}_\varepsilon(B)$ of the following functional:

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad J(\Omega) := \int_{\partial\Omega} j\left[\mathbf{x}, \mathbf{n}_{\partial\Omega}(\mathbf{x}), \nabla u_\Omega(\mathbf{x})\right] dA(\mathbf{x}),$$

where $u_\Omega \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique solution of the Dirichlet Laplacian posed on a domain Ω with $C^{1,1}$-boundary [37, Section 2.1]:

$$\begin{cases} \Delta u_\Omega = f & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(B)$ and where $j : B \times \mathbb{S}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous functional satisfying an inequality of the form:

$$\exists C > 0, \quad \forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in B \times \mathbb{S}^{n-1} \times \mathbb{R}^{n-1}, \quad |j(\mathbf{x}, \mathbf{y}, \mathbf{z})| \leqslant C\left(1 + \|\mathbf{z}\|^2\right). \tag{21.1}$$

First, note that that the functional $J : \mathcal{O}_\varepsilon(B) \to \mathbb{R}$ is well defined. Indeed, we have from (21.1):

$$\forall \Omega \in \mathcal{O}_\varepsilon(B), \quad |J(\Omega)| \leqslant C\left(A(\partial\Omega) + \|\nabla u_\Omega\|^2_{L^2(\Omega)}\right) < +\infty.$$

Then, we recall that we managed to parametrize simultaneously by local graphs the boundaries associated with a converging sequence of domains in $\mathcal{O}_\varepsilon(B)$. More precisely, let $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ be a sequence converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ (in various senses: Hausdorff, characteristic functions, compact sets) whose boundaries $(\partial\Omega_i)_{i\in\mathbb{N}}$ also converges to $\partial\Omega$ for the Hausdorff distance.

For any $\mathbf{x} \in \partial\Omega$, this parametrization is made inside a cylinder $C_{r,\varepsilon}(\mathbf{x})$ whose base is a disk $D_{r(\varepsilon)}(\mathbf{x})$ of radius $r > 0$ depending only on ε, through some $C^{1,1}$-maps $\varphi^i_{\mathbf{x}} : D_r(\mathbf{x}) \to]-\varepsilon, \varepsilon[$. We consider the uniform partition of unity defined in Proposition 20.7 with $h(\varepsilon) = \min(r, \varepsilon) > 0$. Hence, there exists $K \geqslant 1$ distinct points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ of $\partial\Omega$ such that $\overline{\mathcal{V}_{\frac{h}{4}}}(\partial\Omega) \subseteq \cup_{k=1}^K C_{r,\varepsilon}(\mathbf{x}_k)$ and there exists K associated maps $\xi_k \in C_c^\infty(C_{r,\varepsilon}(\mathbf{x}_k), [0,1])$ such that $\sum_{k=1}^K \xi_k = 1$ on $\overline{\mathcal{V}_{\frac{h}{4}}}(\partial\Omega)$.

Considering the common parametrizations associated with $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$, there exists K integers $(I_k)_{1\leqslant k\leqslant K}$ such that for any $i \geqslant I_k$, there exists $C^{1,1}$-maps $\varphi^i_k : D_r(\mathbf{x}_k) \to]-\varepsilon, \varepsilon[$ such that:

$$\partial\Omega_i \cap C_{r,\varepsilon}(\mathbf{x}_k) = \{(\mathbf{x}', \varphi^i_k(\mathbf{x}')), \quad \mathbf{x}' \in D_r(\mathbf{x}_k)\}.$$

Moreover, φ^i_k converges in $C^1(\overline{D_r}(\mathbf{x}_k)) \cap W^{2,\infty}(D_r(\mathbf{x}_k))$ to the map $\varphi : D_r(\mathbf{x}_k) \to]-\varepsilon, \varepsilon[$ associated locally with the piece of boundary $\partial\Omega \cap C_{r,\varepsilon}(\mathbf{x}_k)$. Furthermore, there exists $I_0 \in \mathbb{N}$ such that for any integer $i \geqslant I_0$, we have $\partial\Omega_i \in \mathcal{V}_{\frac{h}{4}}(\partial\Omega)$.

We set $I = \max_{0\leqslant k\leqslant K} I_k$ and consider any integer $i \geqslant I$. We can now write the functional in terms of local graphs associated with the common partition of unity we built. We get that the functional $J(\Omega_i)$ can be written into the form:

$$\sum_{k=1}^{K}\int_{D_k}\xi_k\begin{pmatrix}\mathbf{x}'\\ \varphi_k^i(\mathbf{x}')\end{pmatrix}\ j\left[\begin{pmatrix}\mathbf{x}'\\ \varphi_k^i(\mathbf{x}')\end{pmatrix},\begin{pmatrix}\frac{-\nabla\varphi_k^i(\mathbf{x}')}{\sqrt{1+\|\nabla\varphi_k^i(\mathbf{x}')\|^2}}\\ \frac{1}{\sqrt{1+\|\nabla\varphi_k^i(\mathbf{x}')\|^2}}\end{pmatrix},\nabla u_{\Omega_i}\begin{pmatrix}\mathbf{x}'\\ \varphi_k^i(\mathbf{x}')\end{pmatrix}\right]\sqrt{1+\|\nabla\varphi_k^i(\mathbf{x}')\|^2}d\mathbf{x}',$$

where we set $D_k := D_r(\mathbf{x}_k)$. To let $i \to +\infty$, we want to apply Lebesgue Domination Convergence Theorem on each integral so we need the almost-everywhere convergence and a uniform bound of each integrand. Finally, due to the hypothesis (21.1) made on the j, note that this is case if the following proposition holds, which is the main task of this section.

Proposition 21.1. *The map $\mathbf{x}' \in D_k \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i^k(\mathbf{x}'))$ strongly converges in $L^2(D_k)$ to the map $\mathbf{x}' \in D_k \mapsto \nabla u_\Omega(\mathbf{x}', \varphi^k(\mathbf{x}'))$, where we set $D_k := D(\mathbf{x}_k)$.*

First, we show the sequence of maps is uniformly bounded in $L^2(D_k)$. Then, we show that the weak limit is the right one. Finally, we prove that the strong convergence holds. Note that this proposition can be used with similar arguments to extend the continuity result of the second part to functional depending on ∇u_Ω.

21.1 A uniform L^2-bound for the sequence

Proposition 21.2. *Let Ω be any non-empty bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then, we have for any $u \in L^1(\partial\Omega, \mathbb{R})$:*

$$\frac{1}{\sqrt{1+L^2}}\int_{\partial\Omega}|u(\mathbf{x})|dA(\mathbf{x}) \leqslant \sum_{k=1}^{K}\int_{\Pi_k(\partial\Omega\cap C_k)}|u(\mathbf{x}', \varphi_k(\mathbf{x}')|d\mathbf{x}' \leqslant n2^n(n!)(1+L)^n\int_{\partial\Omega}|u(\mathbf{x})|dA(\mathbf{x}),$$

where $L > 0$ is the maximum of the Lipschitz modulus of the maps $(\varphi_k)_{1\leqslant k\leqslant K}$ associated with any points $(\mathbf{x}_k)_{1\leqslant k\leqslant K}$ such that $\partial\Omega \subset \cup_{k=1}^K C_k$ with C_k a local cylinder centred at $\mathbf{x}_k$.

Proof. Since Ω is a non-empty bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, for any point $\mathbf{x}_0 \in \partial\Omega$, there exists a direct orthonormal frame centred at $\mathbf{x}_0$ such that in this local frame, there exists a L-Lispchitz continuous map $\varphi_{\mathbf{x}_0} : D_{r_{\mathbf{x}_0}}(\mathbf{0}') \to]-a_{\mathbf{x}_0}, a_{\mathbf{x}_0}[$ such that $\varphi_{\mathbf{x}_0}(\mathbf{0}') = 0$ and:

$$\begin{cases}\partial\Omega \cap \left(D_{r_{\mathbf{x}_0}}(\mathbf{0}')\times]-a_{\mathbf{x}_0}, a_{\mathbf{x}_0}[\right) = \{(\mathbf{x}', \varphi_{\mathbf{x}_0}(\mathbf{x}')),\quad \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{0}')\}\\[2ex] \Omega \cap \left(D_{r_{\mathbf{x}_0}}(\mathbf{0}')\times]-a_{\mathbf{x}_0}, a_{\mathbf{x}_0}[\right) = \{(\mathbf{x}', x_n),\quad \mathbf{x}' \in D_{r_{\mathbf{x}_0}}(\mathbf{0}') \text{ and } -a_{\mathbf{x}_0} < x_n < \varphi_{\mathbf{x}_0}(\mathbf{x}')\}.\end{cases}$$

We denote by $C_{r_{\mathbf{x}_0}, a_{\mathbf{x}_0}}(\mathbf{x}_0)$ the cylinder represented by $D_{r_{\mathbf{x}_0}}(\mathbf{0}')\times]-a_{\mathbf{x}_0}, a_{\mathbf{x}_0}[$ in the local frame and more generally we have:

$$C_{r_{\mathbf{x}_0}, a_{\mathbf{x}_0}}(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n,\quad |\langle \mathbf{x}-\mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle| < a_{\mathbf{x}_0} \text{ and } \|\mathbf{x}-\mathbf{x}_0 - \langle \mathbf{x}-\mathbf{x}_0 \mid \mathbf{d}_{\mathbf{x}_0}\rangle\mathbf{d}_{\mathbf{x}_0}\| < r_{\mathbf{x}_0}\},$$

where $\mathbf{d}_{\mathbf{x}_0}$ refers to the last vector of the basis associated with $\mathbf{x}_0$. Since $\partial\Omega$ is compact, we get from $\partial\Omega \subset \cup_{\mathbf{x}\in\partial\Omega} C_{r_{\mathbf{x}_0}, a_{\mathbf{x}_0}}(\mathbf{x}_0)$ the existence of a finite number $K \geqslant 1$ of points such that the inclusion $\partial\Omega \subset \cup_{k=1}^K C_{r_{\mathbf{x}_k}, a_{\mathbf{x}_k}}(\mathbf{x}_k)$ holds. Then, there exists K positive smooth maps $\xi_k : \mathbb{R}^n \to \mathbb{R}$ with compact support in $C_k := C_{r_{\mathbf{x}_k}, a_{\mathbf{x}_k}}(\mathbf{x}_k)$ and such that $\sum_{k=1}^K \xi_k(\mathbf{x}) = 1$ for any $\mathbf{x} \in \partial\Omega$. We set $L = \max_{1\leqslant kK} L_{\mathbf{x}_k}$ and introduce the Lipschitz continuous parametrization:

$$\begin{array}{rccc} X_k : & D_k & \longrightarrow & \partial\Omega \cap C_k \\ & \mathbf{x}' & \longmapsto & (\mathbf{x}', \varphi_k(\mathbf{x}'), \end{array}$$

whose inverse is the restriction of the projection $\Pi_k : \mathbf{x} \mapsto \mathbf{x} - \langle \mathbf{x}-\mathbf{x}_k \mid \mathbf{d}_{\mathbf{x}_k}\rangle\mathbf{d}_{\mathbf{x}_k}$ and where $D_k = \Pi_k(\partial\Omega \cap C_k)$. Let us choose $u \in L^1(\partial\Omega)$. We have:

$$\begin{array}{rcl}\displaystyle\int_{\partial\Omega}|u| & = & \displaystyle\int_{\partial\Omega}\left(\sum_{k=1}^K \xi_k\right)|u| \ = \ \sum_{k=1}^K\int_{\partial\Omega\cap C_k}\xi_k|u| \ = \ \sum_{k=1}^K\int_{D_k}[(\xi_k|u|)\circ X_k]\sqrt{1+|\nabla\varphi_k|^2} \\ & \leqslant & \displaystyle\sum_{k=1}^K\sqrt{1+L_{\mathbf{x}_k}^2}\int_{D_k}\left(\sum_{l=1}^K\xi_l|u|\right)\circ X_k \ \leqslant \ \sqrt{1+L^2}\sum_{k=1}^K\int_{D_k}|u\circ X_k|.\end{array}$$

It remains to prove the converse part of this inequality. Let $1 \leqslant k \leqslant K$ fixed and we have:

$$\int_{D_k} |u \circ X_k| = \int_{D_k} \left(\sum_{l=1}^{K} \xi_l |u| \right) \circ X_k = \sum_{l=1}^{L} \int_{\Pi_k(\partial\Omega \cap C_k)} (\xi_l |u|) \circ X_k.$$

Then, observe that $\xi_l(\mathbf{x}) = 0$ for any $\mathbf{x} \notin C_l$ so $\xi_l \circ X_k(\mathbf{x}') = 0$ for any $\mathbf{x}' \in \Pi_k(\partial\Omega \cap C_k \cap \mathbb{R}^n \backslash C_l)$. Hence, we deduce that:

$$\int_{D_k} |u \circ X_k| = \sum_{l=1}^{L} \int_{\Pi_k(\partial\Omega \cap C_k \cap C_l)} (\xi_l |u|) \circ X_k = \sum_{\substack{1 \leqslant l \leqslant K \\ \partial\Omega \cap C_k \cap C_l \neq \emptyset}} \int_{\Pi_k(\partial\Omega \cap C_k \cap C_l)} (\xi_l |u|) \circ X_k.$$

If $\partial\Omega \cap C_k \cap C_l \neq \emptyset$, we introduce the map $T_{kl} := \Pi_l \circ X_k : \Pi_k(\partial\Omega \cap C_k \cap C_l) \to \Pi_l(\partial\Omega \cap C_k \cap C_l)$ which is a bi-Lipschitz change of coordinates. Indeed, we have for any $(\mathbf{x}, \mathbf{y}) \in (\partial\Omega \cap C_l \cap C_k)^2$:

$$\begin{aligned}
\|\Pi_k(\mathbf{x}) - \Pi_k(\mathbf{y})\| &= \|\mathbf{x} - \mathbf{y} - \langle \mathbf{x} - \mathbf{y} \mid \mathbf{d}_{\mathbf{x}_k} \rangle \mathbf{d}_{\mathbf{x}_k}\| \quad \leqslant \quad 2\|\mathbf{x} - \mathbf{y}\| \\
&\leqslant 2\|\Pi_l(\mathbf{x}) - \Pi_l(\mathbf{y})\| + 2|\langle \mathbf{x} - \mathbf{x}_k \mid \mathbf{d}_{\mathbf{x}_k} \rangle - \langle \mathbf{y} - \mathbf{x}_k \mid \mathbf{d}_{\mathbf{x}_k} \rangle| \\
&= 2\|\Pi_l(\mathbf{x}) - \Pi_l(\mathbf{y})\| + 2|\varphi_l(\Pi_l(\mathbf{x})) - \varphi_l(\Pi_l(\mathbf{y}))| \\
&\leqslant 2(1 + L_{\mathbf{x}_l})\|\Pi_l(\mathbf{x}) - \Pi_l(\mathbf{y})\| \quad \leqslant \quad 2(1 + L)\|\Pi_l(\mathbf{x}) - \Pi_l(\mathbf{y})\|.
\end{aligned}$$

Moreover, the Jacobian of T_{kl} is L^∞-bounded. Indeed, we have for any $\mathbf{x} \in \partial\Omega \cap C_k \cap C_l$:

$$\begin{aligned}
J(T_{kl})(\Pi_k(\mathbf{x})) &= |\det D_{\Pi_k(\mathbf{x})} (\Pi_l \circ X_k)| \\
&\leqslant \sum_{\sigma \in \mathcal{S}_n} \prod_{m=1}^{n} |D_{\Pi_k(\mathbf{x})} (\Pi_l \circ X_k)_{m\sigma(m)}| \leqslant \sum_{\sigma \in \mathcal{S}_n} \prod_{m=1}^{n} \|D_{\Pi_k(\mathbf{x})} (\Pi_l \circ X_k)^T (\mathbf{e}_m)\| \\
&\leqslant \sum_{\sigma \in \mathcal{S}_n} \prod_{m=1}^{n} \|D_{\Pi_k(\mathbf{x})} (\Pi_l \circ X_k)^T\| \leqslant \sum_{\sigma \in \mathcal{S}_n} \prod_{m=1}^{n} \|D_{\Pi_k(\mathbf{x})} (\Pi_l \circ X_k)\| \\
&\leqslant \sum_{\sigma \in \mathcal{S}_n} \prod_{m=1}^{n} \operatorname*{sup\,ess}_{\mathbf{x} \in \partial\Omega \cap C_k \cap C_l} \|D_{\Pi_k(\mathbf{x})} (\Pi_l \circ X_k)\| \leqslant \sum_{\sigma \in \mathcal{S}_n} \prod_{m=1}^{n} 2(1 + L) \\
&\leqslant 2^n (n!)(1 + L)^n.
\end{aligned}$$

Consequently, we can make a change of variables and we obtain:

$$\begin{aligned}
\int_{D_k} |u \circ X_k| &= \sum_{\substack{1 \leqslant l \leqslant K \\ \partial\Omega \cap C_k \cap C_l \neq \emptyset}} \int_{\Pi_k(\partial\Omega \cap C_k \cap C_l)} (\xi_l |u|) \circ X_l \circ [\Pi_l \circ X_k]\, J(T_{kl}) J(T_{lk}) \\
&\leqslant 2^n (n!)(1 + L)^n \sum_{\substack{1 \leqslant k \leqslant K \\ \partial\Omega \cap C_k \cap C_l \neq \emptyset}} \int_{\Pi_k(\partial\Omega \cap C_k \cap C_l)} [(\xi_l |u|) \circ X_l \circ T_{kl}]\, J(T_{kl}) \\
&= 2^n (n!)(1 + L)^n \sum_{\substack{1 \leqslant l \leqslant K \\ \partial\Omega \cap C_k \cap C_l \neq \emptyset}} \int_{\Pi_l(\partial\Omega \cap C_k \cap C_l)} (\xi_l |u|) \circ X_l \\
&\leqslant 2^n (n!)(1 + L)^n \sum_{l=1}^{K} \int_{D_l} (\xi_l |u|) \circ X_l \quad = \quad 2^n (n!)(1 + L)^n \int_{\partial\Omega} |u|.
\end{aligned}$$

To conclude, we get the required inequality by summing the one above from $k = 1$ to K. □

Proposition 21.3. *Let $1 \leqslant k \leqslant K$. Considering the maps $v_k^i : \mathbf{x}' \mapsto \partial_n(u_{\Omega_i})(\mathbf{x}', \varphi_k^i(\mathbf{x}'))$, the sequence $(v_k^i)_{i \in \mathbb{N}}$ is uniformly bounded in $L^2(D_k)$.*

Proof. First, we apply Proposition 21.2 on $\partial\Omega_i$ to get:

$$\int_{D_k} (v_k^i)^2 \leqslant \sum_{k=1}^{K} \int_{D_k} \partial_n(u_{\Omega_i})^2 \circ X_k^i \leqslant n2^n n!(1+L)^n \int_{\partial\Omega_i} |\partial_n(u_{\Omega_i})|^2.$$

Then, $u_{\Omega_i} \in H_0^1(\Omega_i)$ and taking the partial derivatives in the relation $u_{\Omega_i} \circ X_k^i = 0$, we obtain that $\nabla u_{\Omega_i} = \partial_n(u_{\Omega_i})\mathbf{n}_{\partial\Omega_i}$ on $\partial\Omega_i$. Combined with Corollary 20.9, we obtain:

$$\int_{D_k} (v_k^i)^2 \leqslant n2^n n!(1+L)^n \int_{\partial\Omega_i} \|\nabla u_{\Omega_i}\|^2 \leqslant \tilde{C}(\varepsilon, n, D) \left(\|u_{\Omega_i}\|^2_{H^2(\Omega_i)} + \|f\|^2_{L^2(\Omega_i)} \right).$$

Finally, we can use the uniform bound proved in Theorem 20.1 to deduce the existence of a positive constant, which depends on D, ε, n, and ξ such that:

$$\int_{D_k} (v_k^i)^2 \leqslant C(\varepsilon, n, D) \int_B f^2.$$

□

21.2 The weak convergence in L^2-norm of the sequence

Proposition 21.4. *The sequence of maps $v_k^i : \mathbf{x}' \mapsto \partial_n(u_{\Omega_i})(\mathbf{x}', \varphi_k^i(\mathbf{x}'))$ converges weakly in $L^2(D_k)$ to the map $v_k : \mathbf{x}' \mapsto \partial_n(u_\Omega)(\mathbf{x}', \varphi_k(\mathbf{x}'))$, where $u_\Omega \in H_0^1(\Omega) \cap H^2(\Omega)$ is the unique solution the Dirichlet Laplacian on $\Omega \in \mathcal{O}_\varepsilon(B)$ and where Ω_i converges to Ω in the various sense of $\mathcal{O}_\varepsilon(B)$.*

Proof. Proposition 21.3 ensures we can bound uniformly the L^2-norm of $(v_k^i)_{i\in\mathbb{N}}$. Consequently, there exists $v_k^* \in L^2(D_k)$ such that, up to a subsequence, $(\mathbf{v}_k^i)_{i\in\mathbb{N}}$ weakly converges to v_k^* in $L^2(D_k)$. It remains to prove that for any weakly converging subsequence, the limit is unique i.e. $v_k^* = v_k$ in order to get the weak convergence of the full sequence to v_k. Let $w : B \to \mathbb{R}$ be any Lipschitz continuous map. From Rademacher's Theorem [30, Section 3.1.2], w is differentiable almost everywhere and $w \in W^{1,\infty}(B)$ [30, Section 4.2.3]. Then, we have:

$$\begin{aligned}
\int_{\partial\Omega_i} \partial_n(u_{\Omega_i})w &= \int_{\Omega_i} \operatorname{div}(w\nabla u_{\Omega_i}) = \int_{\Omega_i} \langle \nabla w \mid \nabla u_{\Omega_i}\rangle + \int_{\Omega_i} w\Delta u_{\Omega_i} \\
&= \int_B \langle \nabla w \mid \mathbf{1}_{\Omega_i}\nabla u_{\Omega_i}\rangle + \int_B \mathbf{1}_{\Omega_i} wf \\
&= \int_\Omega \langle \nabla w | \nabla u_\Omega\rangle + \int_\Omega wf + \int_B \langle \nabla w \mid \mathbf{1}_{\Omega_i}\nabla u_{\Omega_i} - \mathbf{1}_\Omega \nabla u_\Omega\rangle + \int_B (\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega)wf \\
&= \int_{\partial\Omega} \partial_n(u_\Omega)w + \int_B \langle \nabla w \mid \mathbf{1}_{\Omega_i}\nabla u_{\Omega_i} - \mathbf{1}_\Omega \nabla u_\Omega\rangle + \int_B (\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega)wf
\end{aligned}$$

The second term is bounded by $\|\nabla w\|_{L^\infty(B)}\sqrt{V(B)}\|\mathbf{1}_{\Omega_i}\nabla u_{\Omega_i} - \mathbf{1}_\Omega\nabla u_\Omega\|_{L^2(B)}$ while the third one is bounded by $\|w\|_{L^\infty(B)}\|f\|_{L^2(B)}\sqrt{\|\mathbf{1}_{\Omega_i} - \mathbf{1}_\Omega\|_{L^1(B)}}$. Using the convergence of $(\Omega_i)_{i\in\mathbb{N}}$ to Ω in the sense of characteristic functions and [46, Theorem 3.2.13], we can let $i \to +\infty$ in order to obtain:

$$\forall w \in W^{1,\infty}(B,\mathbb{R}), \quad \lim_{i\to+\infty} \int_{\partial\Omega_i} \partial_n(u_{\Omega_i})w = \int_{\partial\Omega} \partial_n(u_\Omega)w. \tag{21.2}$$

We now consider $w : B \to \mathbb{R}$ a Lipschitz continuous map with compact support in C_k. Then, we have:

$$\int_{\partial\Omega} \partial_n(u_{\Omega_i})w = \int_{D_k} [\partial_n(u_{\Omega_i})w] \circ X_k^i \sqrt{1 + \|\nabla\varphi_k^i\|^2} = \int_{D_k} v_k^i (w \circ X_k^i)\sqrt{1 + \|\nabla\varphi_k^i\|^2},$$

and we decompose the above expression into the following terms:

$$\begin{aligned}
\int_{\partial\Omega_i} \partial_n(u_{\Omega_i})w &= \int_{D_k} v_k^*(w \circ X_k)\sqrt{1+\|\nabla\varphi_k\|^2} \quad + \quad \int_{D_k} (v_k^i - v_k^*)(w \circ X_k)\sqrt{1+\|\nabla\varphi_k\|^2} \\
&+ \quad \int_{D_k} v_k^i \left[(w \circ X_k^i) - (w \circ X_k)\right] \|\nabla\varphi_k^i\|^2 \quad + \quad \int_{D_k} v_k^i w \circ X_k \frac{\|\nabla\varphi_k^i\|^2 - \|\nabla\varphi_k\|^2}{\sqrt{(1+\|\nabla\varphi_k^i\|^2)(1+\|\nabla\varphi_k\|^2)}}.
\end{aligned}$$

From the $L^2(D_k)$-weak convergence of $(v_k^i)_{i\in\mathbb{N}}$ to v_k^* , the second term tends to zero as $i \to +\infty$ since $(w \circ X_k)\sqrt{1+\|\nabla\varphi_k\|^2}$ is a Lipschitz continuous map and $A(D_k) = \pi r_{\mathbf{x}_k}^2$ thus it is an element of $L^2(D_k)$. The third term is bounded by $\|v_k^i\|_{L^2(D_k)}\sqrt{1+L^2}\|w\|_{W^{1,\infty}(B)}\|\varphi_k^i - \varphi_k\|_{L^\infty(D_k)}\sqrt{A(D_k)}$. We proved that $\|v_k^i\|_{L^2(D_k)} \leqslant C\|f\|_{L^2(B)}$ so the third term tends to zero as $i \to +\infty$. Concerning the fourth one, it is bounded by $\|v_k^i\|_{L^2(D_k)}\|w\|_{L^\infty}(B)2L\|\nabla\varphi_k^i - \nabla\varphi_k\|_{L^\infty(D_k)}\sqrt{A(D_k)}$ so the fourth terms converges to zero. Hence, we can let $i \to +\infty$ in the previous equality and we obtain:

$$\lim_{i\to+\infty}\int_{\partial\Omega_i}\partial_n(u_{\Omega_i})w = \int_{D_k} v_k^*(w\circ X_k)\sqrt{1+\|\nabla\varphi_k\|^2}.$$

But from (21.2), we also get:

$$\lim_{i\to+\infty}\int_{\partial\Omega_i}\partial_n(u_{\Omega_i})w = \int_{\partial\Omega}\partial_n(u_\Omega)w = \int_{D_k} v_k(w\circ X_k)\sqrt{1+\|\nabla\varphi_k\|^2}.$$

Consequently, we proved that for any Lipschitz continuous map $w : B \to \mathbb{R}$ with compact support in C_k, we have:

$$\int_{D_k}(v_k - v_k^*)(w\circ X_k)\sqrt{1+\|\nabla\varphi_k\|^2} = 0.$$

Let $\tilde{w} \in C_c^\infty(D_k,\mathbb{R})$ and we show that we can replace $w \circ X_k$ by $\tilde{w}$ in the above expression. For this purpose, we introduce the map:

$$\begin{array}{rccl} w: & B & \longrightarrow & \mathbb{R} \\ & (\mathbf{x}', x_n) & \longmapsto & \left\{ \begin{array}{ll} \tilde{w}(\mathbf{x}')\dfrac{a_{\mathbf{x}_k}^2 - x_n^2}{a_{\mathbf{x}_k}^2 - \varphi_k(\mathbf{x}')^2} & \text{if } (\mathbf{x}', x_n) \in C_k := D_k \times]-a_{\mathbf{x}_k}, a_{\mathbf{x}_k}[\\ 0 & \text{otherwise} \end{array}\right. \end{array}$$

One can check that w is Lipschitz continuous with compact support in D_k. Hence, we can insert w in the previous equality to get:

$$\forall \tilde{w} \in C_c^\infty(D_k,\mathbb{R}), \qquad \int_{D_k}(v_k - v_k^*)\tilde{w}\sqrt{1+\|\nabla\varphi_k\|^2} = 0$$

What has been done for v_i^k is also true for $v^k \in L^2(D_k)$. Moreover, we know that $v_k^* \in L^2(D_k)$ and $\sqrt{1+\|\nabla\varphi_k\|^2}$ is continuous. Hence, we deduce that $v_k = v_k^*$ for almost every $\mathbf{x}' \in D_k$ as required. To conclude, we proved that any weakly converging subsequence of $(v_k^i)_{i\in\mathbb{N}}$ converges to v_k so the results holds for the whole sequence. ☐

21.3 The strong convergence in L^2-norm of the sequence

First, we prove the result locally and then we establish the global strong convergence.

Proposition 21.5. *Let $k \in \{1,\dots n\}$. For any Lipschitz continuous map $w : B \to \mathbb{R}$ with compact support in $C(\mathbf{x}_k)$, we have, up to a subsequence:*

$$\lim_{i\to+\infty}\int_{D_k}(w\circ X_k^i)\left(v_k^i - v_k\right)^2 = 0.$$

Proof. Let $w \in W^{1,\infty}(B,\mathbb{R})$ with compact support in $C(\mathbf{x}_k)$. We have:

$$\int_{D_k}(w\circ X_k^i)\left(v_k^i - v_k\right)^2 = \int_{D_k}(w\circ X_k^i)\left(v_k^i\right)^2 - 2\int_{D_k} v_k(w\circ X_k^i)v_k^i + \int_{D_k}(w\circ X_k^i)\left(v_k\right)^2. \quad (21.3)$$

First, considering the Lipschitz modulus $L > 0$ of w, the sequence $(w\circ X_i^k)_{i\in\mathbb{N}}$ uniformly converges to $w \circ X_k$. Indeed, we have:

$$\|(w\circ X_k^i) - (w\circ X_k)\|_{L^\infty(B,\mathbb{R})} \leqslant L\|X_k^i - X_k\|_{L^\infty(D_k)} = L\|\varphi_k^i - \varphi_k\|_{L^\infty(D_k)} \underset{i\to+\infty}{\longrightarrow} 0.$$

On the one hand, we deduce:

$$\int_{D_k}(w\circ X_k^i)\left(v_k\right)^2 = \int_{D_k}(w\circ X_k)\left(v_k\right)^2 + \int_{D_k}\left[(w\circ X_k^i) - (w\circ X_k)\right]\left(v_k\right)^2,$$

where the (absolute value of the) last term is bounded by $\|(w\circ X_k^i)-(w\circ X_k)\|_{L^\infty(B,\mathbb{R})}\|v_k\|_{L^2(D_k,\mathbb{R})}$ thus converges to zero as $i\to\infty$. On the other hand, we have:

$$\int_{D_k} v_k(w\circ X_k^i)v_k^i = \int_{D_k}(v_k)^2(w\circ X_k) + \int_{D_k} v_k(w\circ X_k)\left[v_k^i - v_k\right] + \int_{D_k} v_k v_k^i\left[(w\circ X_k^i)-(w\circ X_k)\right]$$

The last term is bounded by $\|(w\circ X_k^i)-(w\circ X_k)\|_{L^\infty(B,\mathbb{R})}\|v_k\|_{L^2(D_k,\mathbb{R})}\|v_k^i\|_{L^2(D_k,\mathbb{R})}$, and from Proposition 21.3, it is converging to zero as $i\to+\infty$. Moreover, the same holds for the second term according to Proposition 21.4. Therefore, to conclude the proof, it remains to show that the first term in the right-hand side of (21.3) $\int_{D_k}(w\circ X_k^i)(v_k^i)^2$ converges to $\int_{D_k}(w\circ X_k)(v_k)^2$ as $i\to+\infty$. Let us prove this last assertion. First, we get from $\nabla u_{\Omega_i}=\partial_n(u_{\Omega_i})\mathbf{n}_{\partial\Omega_i}$ and Stoke's Theorem:

$$\begin{aligned}\int_{\partial\Omega_i}\left[\partial_n\left(u_{\Omega_i}\right)\right]^2 w\langle\mathbf{d}_{\mathbf{x}_k}\mid\mathbf{n}_{\partial\Omega_i}\rangle dA &= \int_{\partial\Omega_i}\|\nabla u_{\Omega_i}\|^2 w\langle\mathbf{d}_{\mathbf{x}_k}\mid\mathbf{n}_{\partial\Omega_i}\rangle dA = \int_{\Omega_i}\operatorname{div}\left(\|\nabla u_{\Omega_i}\|^2 w\mathbf{d}_{\mathbf{x}_k}\right)\\ &= \int_{\Omega_i}2w\left\langle\left\langle\mathbf{d}_{\mathbf{x}_k}\mid\nabla\right\rangle\left(\nabla u_{\Omega_i}\right)\mid\nabla u_{\Omega_i}\right\rangle + \int_{\Omega_i}\|\nabla u_{\Omega_i}\|^2\langle\nabla w\mid\mathbf{d}_{\mathbf{x}_k}\rangle\end{aligned}$$

We denote by A_i and B_i respectively the first and second term in the right-hand side of the last equality above. We have:

$$\begin{aligned}\left|A_i-\int_\Omega\|\nabla u_\Omega\|^2\langle\nabla w\mid\mathbf{d}_{\mathbf{x}_k}\rangle\right| &\leqslant \|\nabla w\|_{L^\infty(B,\mathbb{R}^n)}\|\mathbf{1}_{\Omega_i}\nabla u_{\Omega_i}-\mathbf{1}_\Omega\nabla u_\Omega\|^2_{L^2(B,\mathbb{R}^n)}\\ &\quad+2\|\nabla w\|_{L^\infty(B,\mathbb{R}^n)}\|\mathbf{1}_{\Omega_i}\nabla u_{\Omega_i}-\mathbf{1}_\Omega\nabla u_\Omega\|_{L^2(B,\mathbb{R}^n)}\|\nabla u_\Omega\|_{L^2(\Omega,\mathbb{R}^n)}\end{aligned}$$

Since the ε-ball condition implies the $\alpha(\varepsilon)$-cone property, the sequence $(\mathbf{1}_{\Omega_i}\nabla u_{\Omega_i})_{\in\mathbb{N}}$ converges strongly in $L^2(B,\mathbb{R}^n)$ to the map $\mathbf{1}_\Omega\nabla u_\Omega$ [46, Theorem 2.3.13 and Proposition 3.2.4], from which we deduce that $(A_i)_{i\in\mathbb{N}}$ converges to $\int_\Omega\|\nabla u_\Omega\|^2\langle\nabla w\mid\mathbf{d}_{\mathbf{x}_k}\rangle$ as $i\to+\infty$. Concerning B_i, since $\Omega_i\in\mathcal{O}_\varepsilon(B)$ thus satisfies the $\alpha(\varepsilon)$-cone property, we get from [46, Proposition 3.7.2], a result due to Chenais [20], that $\nabla u_{\Omega_i}\in H^1(\Omega_i,\mathbb{R}^n)$ has a uniform extension $\mathbf{v}^i=(\mathbf{v}_1^i,\ldots,\mathbf{v}_n^i)\in H^1(B,\mathbb{R}^n)$ i.e. $\mathbf{v}^i|_{\Omega_i}=\nabla u_{\Omega_i}$ and $\|\mathbf{v}^i\|_{H^1(\Omega_i,\mathbb{R}^n)}\leqslant C(n,D,\varepsilon)\|\nabla u_{\Omega_i}\|_{H^1(\Omega_i,\mathbb{R}^n)}$, where $C(n,D,\varepsilon)>0$ is a constant depending only on D, n and ε. Applying Theorem 20.1, we get that $(\mathbf{v}^i)_{i\in\mathbb{N}}$ is uniformly bounded in $H^1(B,\mathbb{R}^n)$. Hence, up to a subsequence, it is converging to $\mathbf{v}\in H^1(B,\mathbb{R}^n)$, weakly in $H^1(B,\mathbb{R}^n)$ and strongly in $L^2(B,\mathbb{R}^n)$. We now show that $\mathbf{v}$ is an extension of Ω. Let $\varphi\in C_c^\infty(B,\mathbb{R})$ and $l\in\{1,\ldots,n\}$. We have successively:

$$\begin{aligned}\int_{\Omega_i}\mathbf{v}_l^i\varphi &= \int_{\Omega_i}\frac{\partial u_{\Omega_i}}{\partial x_l}\varphi = \int_{\partial\Omega_i}\underbrace{u_{\Omega_i}}_{=0}\varphi - \int_{\Omega_i}u_{\Omega_i}\frac{\partial\varphi}{\partial x_l} = -\int_{\Omega_i}u_{\Omega_i}\frac{\partial\varphi}{\partial x_l}\\ &= \int_\Omega u_\Omega\frac{\partial\varphi}{\partial x_l}+\int_{\Omega_i}u_{\Omega_i}\frac{\partial\varphi}{\partial x_l}-\int_\Omega u_\Omega\frac{\partial\varphi}{\partial x_l} = \int_\Omega\frac{\partial u_\Omega}{\partial x_l}\varphi+\int_B\underbrace{\left(\mathbf{1}_{\Omega_i}u_{\Omega_i}-\mathbf{1}_\Omega u_\Omega\right)}_{\|\bullet\|_{L^1(B,\mathbb{R})}\underset{i\to+\infty}{\longrightarrow}0}\frac{\partial\varphi}{\partial x_l}\\ &\underset{i\to+\infty}{\longrightarrow}\int_\Omega\frac{\partial u_\Omega}{\partial x_l}\varphi\end{aligned}$$

But we also have:

$$\int_{\Omega_i}\mathbf{v}_l^i\varphi=\int_\Omega\mathbf{v}_l\varphi+\int_B\mathbf{1}_{\Omega_i}\left(\mathbf{v}_l^i-\mathbf{v}_l\right)\varphi+\int_B\left(\mathbf{1}_{\Omega_i}-\mathbf{1}_\Omega\right)\mathbf{v}_l\varphi\quad\underset{i\to+\infty}{\longrightarrow}\int_\Omega\mathbf{v}_l\varphi.$$

Consequently, the uniqueness of the limit gives $\mathbf{v}_l=\frac{\partial u_\Omega}{\partial x_l}$ in the sense of distributions on Ω hence almost everywhere on Ω and thus $\mathbf{v}_l$ is an extension of $\partial\Omega$. In particular, we have the following property. Let $\varphi\in C_c^\infty(\Omega,\mathbb{R})$. From the convergence in the Hausdorff sense, for i large enough, we have $\varphi\in C_c^\infty(\Omega_i,\mathbb{R})$. We deduce that:

$$\int_{\Omega_i}\frac{\partial(\mathbf{v}_l^i)}{\partial x_m}\varphi=-\int_{\Omega_i}\mathbf{v}_l^i\frac{\partial\varphi}{\partial x_m}=\int_B\mathbf{1}_{\Omega_i}\frac{\partial u_{\Omega_i}}{\partial x_l}\frac{\partial\varphi}{\partial x_m}\underset{i\to+\infty}{\longrightarrow}\int_B\mathbf{1}_\Omega\frac{\partial u_\Omega}{\partial x_l}\frac{\partial\varphi}{\partial x_m}=\int_\Omega\frac{\partial^2u_\Omega}{\partial x_l\partial x_m}\varphi.$$

but we have from the convergence in the sense of characteristic functions and the weak convegrence of $\mathbf{v}^i$ in $H^1(B,\mathbb{R}^n)$ that the limit is also equal to $\int_B\mathbf{1}_\Omega\dfrac{\partial(\mathbf{v}_l)}{\partial x_m}\varphi$. Therefore, we obtain that $\dfrac{\partial(\mathbf{v}_l)}{\partial x_m}=$

$\dfrac{\partial^2 u_\Omega}{\partial x_l \partial x_m}$ in the sense of distribution in Ω thus almost everywhere on Ω. Finally, getting back to the convergence of B_i, we are going to use this property. We have:

$$\left| \int_{\Omega_i} \left(\frac{\partial(u_{\Omega_i})}{\partial x_m} \frac{\partial^2(u_{\Omega_i})}{\partial x_m \partial x_l} - \frac{\partial(u_\Omega)}{\partial x_m} \frac{\partial^2(u_\Omega)}{\partial x_m \partial x_l} \right) w(\mathbf{d}_{\mathbf{x}_k})_l \right| \leqslant \left| \int_B \mathbf{1}_\Omega \frac{\partial u_\Omega}{\partial x_m} w(\mathbf{d}_{\mathbf{x}_k})_l \left(\frac{\partial \mathbf{v}_l^i}{\partial x_m} - \frac{\partial \mathbf{v}_l}{\partial x_m} \right) \right|$$

$$+\|w\|_{L^\infty(B,\mathbb{R})} \|\tfrac{\partial \mathbf{v}_l^i}{\partial x_m}\|_{L^2(B,\mathbb{R})} \|\mathbf{1}_{\Omega_i} \tfrac{\partial u_{\Omega_i}}{\partial x_m} - \mathbf{1}_\Omega \tfrac{\partial u_\Omega}{\partial x_m}\|_{L^2(B,\mathbb{R})}$$

Since the right-hand side of the above inequality converges to zero as $i \to +\infty$, so does the left-hand side. Summing from $m, l = 1$ to n gives:

$$B_i := \int_{\Omega_i} 2w \left\langle \langle \mathbf{d}_{\mathbf{x}_k} \mid \nabla \rangle (\nabla u_{\Omega_i}) \mid \nabla u_{\Omega_i} \right\rangle \underset{i \to +\infty}{\longrightarrow} \int_\Omega 2w \left\langle \langle \mathbf{d}_{\mathbf{x}_k} \mid \nabla \rangle (\nabla u_\Omega) \mid \nabla u_\Omega \right\rangle$$

Combining the convergence result of A_i and B_i, we deduce that:

$$\int_{\partial\Omega_i} [\partial_n(u_{\Omega_i})]^2 w \langle \mathbf{d}_{\mathbf{x}_k} \mid n_{\partial\Omega_i} \rangle dA \underset{i \to +\infty}{\longrightarrow} \int_{\partial\Omega} [\partial_n(u_\Omega)]^2 w \langle \mathbf{d}_{\mathbf{x}_k} \mid n_{\partial\Omega} \rangle dA$$

Since w has compact support in $C(\mathbf{x}_k)$, it remains to look at the local expression of the integrals to obtain the required result:

$$\int_{\partial\Omega_i} [\partial_n(u_{\Omega_i}]^2 w \langle \mathbf{d}_{\mathbf{x}_k} \mid \mathbf{n}_{\partial\Omega_i} \rangle dA = \int_{D_k} (w \circ X_k^i)(v_k^i)^2 \underset{i \to +\infty}{\longrightarrow} \int_{D_k} (w \circ X_k)(v_k)^2.$$

To conclude, we have proved that the right-hand side of (21.3) converges to zero as $i \to +\infty$. □

Proof of Proposition 21.1. Considering Proposition 21.5, it remains to delete the local map w. This is done in a similar way than in the proof of Proposition 21.2 and the same notation are used. We have:

$$\int_{D_k} \left(v_k^i - v_k\right)^2 = \int_{D_k} \left(\sum_{l=1}^K (\xi_l \circ X_k^i) \right) \left(v_k^i - v_k\right)^2 = \sum_{l=1}^L \int_{\Pi_k(\partial\Omega \cap C_k)} \left(\xi_l \circ X_k^i\right) \left(v_k^i - v_k\right)^2.$$

Then, observe that $\xi_l(\mathbf{x}) = 0$ for any $\mathbf{x} \notin C_l$ so $\xi_l \circ X_k(\mathbf{x}') = 0$ for any $\mathbf{x}' \in \Pi_k(\partial\Omega \cap C_k \cap \mathbb{R}^n \backslash C_l)$. Hence, we deduce that:

$$\int_{D_k} \left(v_k^i - v_k\right)^2 = \sum_{\substack{1 \leqslant l \leqslant K \\ \partial\Omega \cap C_k \cap C_l \neq \emptyset}} \int_{\Pi_k(\partial\Omega \cap C_k \cap C_l)} \left(\xi_l \circ X_k^i\right) \left[\left(\partial_n(u_{\Omega_i}) - \partial_n(u_\Omega)\right)^2 \circ X_k^i \right].$$

If $\partial\Omega \cap C_k \cap C_l \neq \emptyset$, we introduce the map $T_{kl}^i := \Pi_l \circ X_k^i : \Pi_k(\partial\Omega_i \cap C_k \cap C_l) \to \Pi_l(\partial\Omega_i \cap C_k \cap C_l)$ which is a uniform bi-Lipschitz change of coordinates. We make a change of variable and we obtain:

$$\int_{D_k} \left(v_k^i - v_k\right)^2 = \sum_{\substack{1 \leqslant l \leqslant K \\ \partial\Omega \cap C_k \cap C_l \neq \emptyset}} \int_{\Pi_l(\partial\Omega \cap C_k \cap C_l)} \left(\xi_l \circ X_l^i\right) \left[\left(\partial_n(u_{\Omega_i}) - \partial_n(u_\Omega)\right]^2 \circ X_l^i \right] |\det(D_\bullet T_{lk}^i)|.$$

Then, ξ_l has compact support in $D_l = \Pi_l(\partial\Omega_i \cap C_l)$. Applying Proposition 21.5, up to a subsequence, the quantity $\left(\xi_l \circ X_l^i\right) \left[\left(\partial_n(u_{\Omega_i}) - \partial_n(u_\Omega)\right]^2 \circ X_l^i$ strongly converges to zero in $L^1(\Omega)$. Hence, [51, Chapter 1, Proposition 4.11], up to a subsequence, this quantity is uniformly bounded by $L^1(D_k)$ function and converges almost everywhere to zero. Similarly, in the proof of Proposition 21.2, we proved that the Jacobian of T_{lk}^i is uniformly bounded and from the continuity of Π_l and the determinant, since X_l^i converges uniformly to X_l, we get that the Jacobian of Π_{lk}^i converges almost everywhere. Applying Lebesgue Dominated Convergence Theorem, we deduce that we can remove the i in each term of the sum of the above relation. We deduce that, up to a subsequence, we $(v_k^i)_{i \in \mathbb{N}}$ strongly converge to v_k in $L^2(D_k)$. Since the limit is unique, we deduce that the convergence of the whole sequence, which concludes the proof. □

Chapter 22

Continuity of some geometric functionals based on PDE: the Neumann/Robin boundary condition

In this section, we assume that there exists a unique solution $u_\Omega \in H^2(\Omega)$ associated with the $C^{1,1}$-domain Ω and satisfying:

$$\begin{cases} -\Delta u_\Omega + \lambda u_\Omega = f & \text{in } \Omega \\ \\ -\partial_n(u) = \beta(u) & \text{on } \partial\Omega, \end{cases} \tag{22.1}$$

where $\lambda > 0$, $f \in L^2(\Omega)$, and $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map satisfying $\beta(0) = 0$. Note that if β is identically zero, then the above problem is the Laplacian with Neumann boundary condition, and if β is linear, then it is the Robin boundary condition. At least for these two cases, we know there exists a unique solution $u_\Omega \in H^2(\Omega)$ [37, Theorems 2.4.2.6 and 2.4.2.7]. We now establish an *a priori* H^2-estimate for thus problem, where the constant is controlled. We essentially follow [37, Theorem 3.1.2.3] which treat the case of convex domain with C^2-boundary. Our only contribution is to treat the $C^{1,1}$-case with the ε-ball condition

22.1 A uniform *a priori* H^2-estimate for the Neuman/Robin Laplacian

Theorem 22.1. *Let $\varepsilon > 0$, $n \geqslant 2$, and B be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We consider the class $\mathcal{O}_\varepsilon(B)$ formed by all the non-empty open subsets of B satisfying the ε-ball condition. We assume that the diameter D of B is large enough to ensure $\mathcal{O}_\varepsilon(B) \neq \emptyset$. Then, there exists a constant $C > 0$, depending only on ε, D, and n, such that for any $\Omega \in \mathcal{O}_\varepsilon(B)$, we have:*

$$\forall u \in \{v \in H^2(\Omega, \mathbb{R}), -\partial_n(v) = \beta(v) \text{ on } \partial\Omega\}, \quad \|u\|_{H^2(\Omega,\mathbb{R})} \leqslant C\,(\lambda, \varepsilon, n, D)\, \| - \Delta u + \lambda u\|_{L^2(\Omega,\mathbb{R})},$$

where $\partial_n(u) := \langle \nabla u \mid \mathbf{n}_{\partial\Omega}\rangle$, $\lambda > 0$, and $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map satisfying $\beta(0) = 0$.

Proof. We apply (20.10) with $\mathbf{v} = \nabla u$. We obtain:

$$\sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 - \int_\Omega |\Delta u|^2 = \int_{\partial\Omega} \left[-2\beta'(u)\|\nabla_{\partial\Omega} u\|^2 + \mathbf{II}\left(\nabla_{\partial\Omega}(u), \nabla_{\partial\Omega}(u)\right) - H\partial_n(u)^2\right] dA$$

Note that in (20.10), we can rewrite the bracket as an integral because $v_{\mathbf{n}} = \partial_n(u) = -\beta(u)$. Indeed, since $u \in H^2(\Omega)$, we have $u \in H^1(\partial\Omega)$ and since β is Lipchitz continuous, we get $\beta(u) \in H^1(\partial\Omega)$ so $\nabla_{\partial\Omega}(v_{\mathbf{n}}) = \beta'(u)\nabla_{\partial\Omega}(u) \in L^2(\partial\Omega)$. Then, observe that the first term in the expression above is non-negative since β is non-decreasing. We deduce that:

$$\sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 \leqslant \int_\Omega |\Delta u|^2 + \int_{\partial\Omega} \left[\mathbf{II}\left(\nabla_{\partial\Omega}(u), \nabla_{\partial\Omega}(u)\right) - H\partial_n(u)^2\right] dA \tag{22.2}$$

Next, we can rewrite the last term of the above expression. Considering the orthonormal basis of $\mathbb{R}^n$ denoted $(\mathbf{e}_1, \mathbf{e}_{n-1}, \mathbf{n}_{\partial\Omega})$ associated with the principal curvature $(\kappa_i)_{1\leqslant i\leqslant n}$, we have:

$$\begin{aligned}
\mathbf{II}\left(\nabla_{\partial\Omega}(u), \nabla_{\partial\Omega}(u)\right) &:= \langle -D\mathbf{n}_{\partial\Omega}\left(\nabla_{\partial\Omega}(u)\right) \mid \nabla_{\partial\Omega}(u)\rangle \\
&= \left\langle -D\mathbf{n}_{\partial\Omega}\left(\sum_{i=1}^{n-1}\langle\nabla_{\partial\Omega}u \mid \mathbf{e}_i\rangle\mathbf{e}_i\right) \mid \nabla_{\partial\Omega}(u)\right\rangle \\
&= -\sum_{i=1}^{n-1}\langle\nabla_{\partial\Omega}u \mid \mathbf{e}_i\rangle\left\langle \underbrace{D\mathbf{n}_{\partial\Omega}(\mathbf{e}_i)}_{:=\kappa_i\mathbf{e}_i} \mid \nabla_{\partial\Omega}(u)\right\rangle \\
&= -\sum_{i=1}^{n-1}\kappa_i|\langle\nabla_{\partial\Omega}(u) \mid \mathbf{e}_i\rangle|^2
\end{aligned}$$

Recalling that $H = \sum_{i=1}^{n-1}\kappa_i$ and inserting the above relation in the right member of (22.2), it comes:

$$\begin{aligned}
\sum_{i,j=1}^{n}\int_\Omega\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 &\leqslant \int_\Omega|\Delta u|^2 - \sum_{i=1}^{n-1}\int_{\partial\Omega}\kappa_i\|\nabla u\|^2 dA \\
&\leqslant 2\int_\Omega|-\Delta u+\lambda u|^2 + 2\lambda^2\int_\Omega u^2 + \frac{n-1}{\varepsilon}\int_{\partial\Omega}\|\nabla u\|^2 dA.
\end{aligned}$$

In the last inequality, we use the fact that $\Omega \in \mathcal{O}_\varepsilon(B)$ hence its Gauss map $\mathbf{n}_{\partial\Omega} : \partial\Omega \to \mathbb{S}^2$ is $\frac{1}{\varepsilon}$-Lipschitz continuous (cf. Point (ii) Theorem 16.6) so it is differentiable almost everywhere and its principal curvature are essentially bounded on $\partial\Omega$ by $\frac{1}{\varepsilon}$ (cf. Remark 16.8). Finally, we get from Point (i) in Theorem 16.6 that Ω satisfies the $\alpha(\varepsilon)$-cone condition so we can apply Corollary 20.9 to deduce:

$$\begin{aligned}
\left(1-\frac{\eta(n-1)C(\alpha,D,n)}{\varepsilon}\right)\sum_{i,j=1}^{n}\int_\Omega\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 &\leqslant 2\int_\Omega|-\Delta u+\lambda u|^2 \quad + \quad 2\lambda^2\int_\Omega u^2 \\
&\quad + \quad \frac{(n-1)C(\alpha,D,n)}{\varepsilon\eta}\int_\Omega\|\nabla u\|^2.
\end{aligned}$$

It remains to obtain an *a priori* estimate for the H^1-norm. We have:

$$\int_\Omega(-\Delta u+\lambda u)\,u = -\int_{\partial\Omega}u\partial_n(u)dA + \int_\Omega\|\nabla u\|^2 + \lambda\int_\Omega u^2 = \int_{\partial\Omega}u\beta(u)dA + \int_\Omega\|\nabla u\|^2 + \lambda\int_\Omega u^2.$$

Since $\beta(0) = 0$ and β is non-decreasing, we deduce that $\beta(u)u \geqslant 0$. Combining this observation with the Cauchy-Schwarz inequality, we get:

$$\lambda\int_\Omega u^2 + \int_\Omega\|\nabla u\|^2 \leqslant \int_\Omega(-\Delta u+\lambda u)\,u \leqslant \|-\Delta u+\lambda u\|_{L^2(\Omega,\mathbb{R})}\|u\|_{L^2(\Omega,\mathbb{R})}.$$

We deduce that $\|u\|_{L^2(\Omega,\mathbb{R})} \leqslant \frac{1}{\lambda}\|-\Delta u+\lambda u\|_{L^2(\Omega,\mathbb{R})}$ and $\|\nabla u\|^2_{L^2(\Omega,\mathbb{R}^n)} \leqslant \frac{1}{\lambda}\|-\Delta u+\lambda u\|^2_{L^2(\Omega,\mathbb{R})}$, which yields to:

$$\left\{\begin{aligned}
&\left(1-\frac{\eta(n-1)C(\alpha,D,n)}{\varepsilon}\right)\sum_{i,j=1}^{n}\int_\Omega\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 \leqslant \left(4+\frac{(n-1)C(\alpha,D,n)}{\varepsilon\eta\lambda}\right)\int_\Omega|-\Delta u+\lambda u|^2 \\
&\int_\Omega u^2 + \int_\Omega\|\nabla u\|^2 \leqslant \left(\frac{1}{\lambda^2}+\frac{1}{\lambda}\right)\int_\Omega|-\Delta u+\lambda u|^2
\end{aligned}\right.$$

Finally, we set $\eta = \dfrac{1}{2}\min(1, \frac{\varepsilon}{(n-1)C(\alpha,D,n)})$, which depends only on ε, D, and n, in order to obtain the required result:

$$\|u\|^2_{H^2(\Omega,\mathbb{R})} \leqslant \left(8+\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{2(n-1)C(\alpha,D,n)}{\varepsilon\eta\lambda}\right)\|-\Delta u+\lambda u\|^2_{L^2(u,\mathbb{R})}.$$

To conclude, observe that the constant above only depends on ε, D, λ and n. $\square$

22.2 Extending the continuity result to the Neuman/Robin case

We only sketch the procedure to obtain similar results in this case. Indeed, note that all the results and arguments used in Chapter 21 are only based on the H^2-estimation, which also holds for the solution u_Ω of the Neumann/Robin boundary condition. Therefore, we can proceed exactly in the same way than we did for the Dirichlet boundary condition. Considering a minimizing sequence of domains $(\Omega_i)_{i\in\mathbb{N}}$, this uniform bound ensures the the local maps $\mathbf{x}' \mapsto u_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ is uniformly bounded in H^1. Considering a weakly converging subsequence, we can prove it is converging in H^1 to the map $\mathbf{x}' \mapsto u_{\Omega_i}(\mathbf{x}', \varphi(\mathbf{x}'))$. We obtain

Proposition 22.2. *The map $\mathbf{x}' \in D_k \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i^k(\mathbf{x}'))$ strongly converges in $H^1(D_k)$ to the map $\mathbf{x}' \in D_k \mapsto \nabla u_\Omega(\mathbf{x}', \varphi^k(\mathbf{x}'))$, where we set $D_k := D(\mathbf{x}_k)$.*

Therefore, all the continuity results of the previous part can be extended to the Robin/ Neuman case. In the three-dimensional case, there is a simpler way to get Proposition 22.2 for functional depending only on u_Ω and not on ∇u_Ω. Indeed, we can combine the uniform H^2-bound we establish in the previous section with the Morrey embedding.

Proposition 22.3. *Let $n = 3$, $\varepsilon > 0$, and B be any non-empty open bounded subset of $\mathbb{R}^n$ containing the origin. We assume that for any $\Omega \in \mathcal{O}_\varepsilon(B)$, there exists a unique solution $u_\Omega \in H^2(\Omega, \mathbb{R})$ to (22.1). Then, for any $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ converging to $\Omega \in \mathcal{O}_\varepsilon(B)$ in the sense of Proposition 17.1 (i)-(vi), the sequence of maps $u_i : \mathbf{x}' \in \overline{D_r}(\mathbf{x}_0) \mapsto u_{\Omega_i}(\mathbf{x}', \varphi^i_{\mathbf{x}_0}(\mathbf{x}'))$ converges uniformly on $\overline{D_r}(\mathbf{x}_0)$ to the map $u : \mathbf{x}' \in \overline{D_r}(\mathbf{x}_0) \mapsto u_\Omega(\mathbf{x}', \varphi_{\mathbf{x}_0}(\mathbf{x}'))$, where $D_r(\mathbf{x}_0)$ is the disk of Theorem 17.2 associated with any $\mathbf{x}_0 \in \partial\Omega$.*

Proof. Let $\Omega \in \mathcal{O}_\varepsilon(B)$ and $u \in H^2(\Omega, \mathbb{R})$. First, from Point (i) in Theorem 16.6, any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfies the $\alpha(\varepsilon)$-cone property in the sense of Chenais [20]. Hence, we can apply [20, Theorem II.1]: there exists a map $\tilde{u} \in H^2(\mathbb{R}^3, \mathbb{R})$ such that $\|\tilde{u}\|_{H^2(\mathbb{R}^3,\mathbb{R})} \leqslant c(\varepsilon)\|u\|_{H^2(\Omega,\mathbb{R})}$, where the constant $c > 0$ only depends on ε (maybe also on D and n). Then, we want to use Morrey's embeddings but we have to be careful with the constants. First, since $\tilde{u} \in H^2(\mathbb{R}^n, \mathbb{R})$, we deduce from the Gargliardo-Nirenberg-Sobolev inequality [30, Section 4.5.1 Theorem 1] that there exists a constant $c_1(p, n) > 0$ depending only on $p = 2$ and $n = 3$ such that $\|\tilde{u}\|_{W^{1,6}(\mathbb{R}^3,\mathbb{R})} \leqslant c_1\|u\|_{H^2(\mathbb{R}^3,\mathbb{R}}$. Next, we use Morrey's inequality [29, Section 5.6.2 Theorem 4]: there exists a constant $c_2(p, n) > 0$ depending only on $p = 2$ and $n = 3$ such that $\|\tilde{u}\|_{C^{0,\frac{1}{2}}(\mathbb{R}^3,\mathbb{R})} \leqslant c_2\|\tilde{u}\|_{W^{1,6}(\mathbb{R}^3,\mathbb{R})}$. Combining all these estimations, we have successively:

$$\|\tilde{u}\|_{C^{0,\frac{1}{2}}(\mathbb{R}^3,\mathbb{R})} \leqslant c_2\|\tilde{u}\|_{W^{1,6}(\mathbb{R}^3,\mathbb{R})} \leqslant c_2c_1\|u\|_{H^2(\mathbb{R}^3,\mathbb{R})} \leqslant c_2c_1c(\varepsilon)\|u\|_{H^2(\Omega,\mathbb{R})}.$$

Finally, we assume that u is the unique solution $u_\Omega \in H^2(\Omega, \mathbb{R})$ to (22.1). Applying Theorem 22.1, there exists a constant $C(\varepsilon, D, n)$ depending only on ε, D and $n = 3$ such that:

$$\|\tilde{u}\|_{C^{0,\frac{1}{2}}(\mathbb{R}^3,\mathbb{R})} \leqslant C(\varepsilon, D)\|f\|_{L^2(B,\mathbb{R})}.$$

In particular, if we consider the maps $(u_i)_{i\in\mathbb{N}}$ and u of the statement, we obtain:

$$\begin{aligned}
|u_i(\mathbf{x}') - u(\mathbf{x}')| &= |u_{\Omega_i}(\mathbf{x}', \varphi^i(\mathbf{x}')) - u_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))| = |\tilde{u}_{\Omega_i}(\mathbf{x}', \varphi^i(\mathbf{x}')) - \tilde{u}_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))| \\
&\leqslant ||\tilde{u}_{\Omega_i}(\mathbf{x}', \varphi^i(\mathbf{x}')) - \tilde{u}_{\Omega_i}(\mathbf{x}', \varphi(\mathbf{x}'))| + |\tilde{u}_{\Omega_i}(\mathbf{x}', \varphi(\mathbf{x}')) - \tilde{u}_\Omega(\mathbf{x}', \varphi(\mathbf{x}'))| \\
&\leqslant \|\tilde{u}_{\Omega_i}\|_{C^{0,\frac{1}{2}}(\mathbb{R}^3,\mathbb{R})}|\varphi^i(\mathbf{x}') - \varphi(\mathbf{x}')| + \|\tilde{u}_{\Omega_i} - \tilde{u}_\Omega\|_{C^0(\mathbb{R}^3,\mathbb{R})} \\
&\leqslant C(\varepsilon, D)\|f\|_{L^2(B,\mathbb{R})}\|\varphi^i - \varphi\|_{C^0(\overline{D_r}(\mathbf{x}_0))} + c_0(\varepsilon, D)\|\tilde{u}_{\Omega_i} - \tilde{u}_\Omega\|_{H^2(B,\mathbb{R})}
\end{aligned}$$

To conclude, we can let $i \to +\infty$ only if $\tilde{u}_i$ converge strongly to $\tilde{u}$. Using relation (20.10) with $\mathbf{v} = \nabla u_{\Omega_i}$, we can express the L^2-norm of the second derivative of $\tilde{u}$ as boundary term and show these terms tends to zero as we did in the previous section. $\square$

Chapter 23

A general existence result

In this short chapter, we detail the procedure to prove the theorems expressed in the introduction. In $\mathbb{R}^n$, we have the following version of Theorem 18.28.

Theorem 23.1. *Let $\varepsilon > 0$ and $B \subset \mathbb{R}^n$ be a bounded open set containing the origin and a ball of radius 3ε such that ∂B has zero n-dimensional Lebesgue measure. Consider $(C, \widetilde{C}) \in \mathbb{R} \times \mathbb{R}$, three measurable maps $j_n, f_n, g_n : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ with quadratic growth* (19.8) *in their three last variables, and continuous in $(s, \mathbf{z}, Y)$ for almost every $\mathbf{x}$, some continuous maps j_0, f_0, g_0, $g_l : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ with quadratic growth* (19.6) *in the two first variables, and some continuous maps j_l, $f_l : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}$ with quadratic growth* (19.7) *in the two first variables, and convex in the last variable. Then, the following problem has at least one solution:*

$$\begin{aligned}\inf \int_{\Omega} j_n \left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \text{Hess } v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} j_0 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ + \sum_{l=1}^{n-1} \int_{\partial\Omega} j_l \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H^{(l)}(\mathbf{x})\right] dA(\mathbf{x}),\end{aligned}$$

where $v_\Omega \in H^2(\Omega, \mathbb{R})$ is the unique solution of either (19.1) *or* (19.4) *or* (19.5) *with $f \in L^2(B, \mathbb{R})$ and $\lambda > 0$, and where the infimum is taken among any $\Omega \in \mathcal{O}_\varepsilon(B)$ satisfying the constraints:*

$$\left\{\begin{aligned}&\int_{\Omega} f_n \left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \text{Hess } v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} f_0 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ &\qquad + \sum_{l=1}^{n-1} \int_{\partial\Omega} f_l \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x}), H^{(l)}(\mathbf{x})\right] dA(\mathbf{x}) \leqslant C \\ &\int_{\Omega} g_n \left[\mathbf{x}, v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \text{Hess } v_\Omega(\mathbf{x})\right] dV(\mathbf{x}) + \int_{\partial\Omega} g_0 \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) \\ &\qquad + \sum_{l=1}^{n-1} \int_{\partial\Omega} H^{(l)}(\mathbf{x})\, g_l \left[v_\Omega(\mathbf{x}), \nabla v_\Omega(\mathbf{x}), \mathbf{x}, \mathbf{n}(\mathbf{x})\right] dA(\mathbf{x}) = \widetilde{C}.\end{aligned}\right.$$

Moreover, we can add a constraint of the form $\Gamma_0 \subseteq \partial\Omega$ where Γ_0 is a measurable subset of $\partial\Omega_0$ with $\Omega_0 \in \mathcal{O}_\varepsilon(B)$. In this case, we can also replace $\partial\Omega$ by Γ_0 in the domain of integration associated with the functional and the constraints.

Proof of Theorems 19.1–19.2. We prove Theorems 19.1–19.2 by following the same method than the one we use to prove Theorem 15.2. Considering a minimizing sequence $(\Omega_i)_{i\in\mathbb{N}}$, we first get from compactness a converging subsequence. Then, we parametrize simultaneously by local graphs of $C^{1,1}$-maps $(\varphi_i)_{i\in\mathbb{N}}$ the boundaries associated with the converging subsequence of the domains. Moreover, from the previous part, $(\varphi_i)_{i\in\mathbb{N}}$ converges strongly in C^1 and weakly in $W^{2,\infty}$. Using a suitable partition of unity, we express the functional and the constraints in this local parametrization. Therefore, it remains to show that we can correctly let $i \to +\infty$.

Then, each integrand obtained is the product of a L^∞-weak-star converging term with a remaining term, on which we want to apply Lebesgue Domination Convergence Theorem to get its L^1-strong convergence. Hence, to let $i \to +\infty$, we need the almost-everywhere convergence and a uniform integrable bound for each integrand. Due to the continuity and the quadratic growth (19.2)–(19.3) hypothesis, this is the case from Propositions 21.1–22.2: the local map $\mathbf{x}' \mapsto \nabla u_{\Omega_i}(\mathbf{x}', \varphi_i(\mathbf{x}'))$ strongly converges in H^1 to the local map $\mathbf{x}' \mapsto \nabla u_{\Omega}(\mathbf{x}', \varphi(\mathbf{x}'))$. Hence, we can pass to the limit in the functional and constraint. The existence of a minimizer is thus ensured. Similarly, this results hold true in $\mathbb{R}^n$ where the functional and constraints are those given in Theorem 18.28 but with a dependence in $u_\Omega, \nabla u_\Omega$ and quadratic growth assumptions on the integrands. □

Proof of Theorem 19.3. From the foregoing, we only need to prove that for any converging sequence of domains $(\Omega_i)_{i\in\mathbb{N}} \subset \mathcal{O}_\varepsilon(B)$ we have that $\mathbf{1}_{\Omega_i} u_{\Omega_i}$ converges to $\mathbf{1}_\Omega u_\Omega$ in $H^2(B, \mathbb{R})$, where $\Omega \in \mathcal{O}_\varepsilon(B)$ is the limit domain. The $H^1(B, \mathbb{R})$ convergence is standard in the framework of the uniform cone property. Since the uniform ball condition implies a uniform cone property, to prove the assertion, we only have to express the second-order terms as boundary terms and apply the previous results. This can be done using the estimation we proved in Theorem 20.2 with $\mathbf{v} = \nabla u_{\Omega_i}$. □

Proof of Proposition 19.4. The local parametrization we use is made on the limit boundary $\partial\Omega$. Hence, since the Hausdorff convergence is stable for the inclusion. The constraint $\Gamma_0 \subseteq \partial\Omega_i$ pass to the limit and we have $\Gamma_0 \subseteq \partial\Omega$ Then, we can proceed as before with a partition of unity only made on Γ_0 and the result follows. □

Chapter 24

Some perspectives

Minimizing the Helfrich energy a volume constraint

Apart from the case $H_0 = 0$ for which the sphere is the unique global minimizer, very few is known about this problem. The main difficulty comes from the lack of compactness due to the poor control on the area of a minimizing sequence. Even in the two-dimensional case, it seems open the existence of a smooth Jordan curve minimizing $\int \kappa^2(s)ds$ with prescribed enclosed area $A_0 > 0$.

However, in the case of negative spontaneous curvature $H_0 < 0$, thanks to the isoperimetric inequality and the results of the second part, the sphere $\mathbb{S}_{V_0}$ of volume V_0 is the unique minimizer of the Helfrich energy (2.3) with prescribed enclosed volume $V_0 > 0$ among compact simply-connected $C^{1,1}$-surfaces of $\mathbb{R}^3$ enclosing a convex inner domain, or those bounding an *axiconvex* domain, i.e. an axisymmetric domain whose intersection with any plane orthogonal to the symmetry axis is either a disk or empty (cf. Inequality (2.14) and Theorem 2.5).

Existence for small spontaneous curvature

We give some clues concerning the minimization of (2.3) with both area and volume constraints. In the case $H_0 = 0$, due to the conformal invariance of the Willmore functional (2.2), the problem is equivalent to minimize (2.2) with prescribed isoperimetric ratio. Two methods have been developed to tackle with the regularity issue of this problem:

- Simon's *cut-and-paste* procedure [88] adapted by Schygulla to solve the zero-genus case [84];
- Rivière's *immersions* approach [79] used by Keller, Mondino and Rivière [53] for higher genus.

Both of them are strongly based on the fact that the Willmore energy (2.2) of a minimizing sequence is bounded by $8\pi - \delta$ for some fixed $\delta > 0$. In Simon's approach, this is combined with the monotonicity formula to ensure that the limit integral varifold has multiplicity one. In Rivière's approach, Li-Yau 8π-estimate [58] ensures that the immersion is in fact an embedding.

Therefore, in the case of small spontaneous curvature, we prove this estimation holds for the Willmore energy (2.2) of a minimizing sequence associated with the minimization of the Helfrich energy (2.3) under area and volume constraints.

Proposition 24.1. *Let $A_0 > 0$ and $V_0 > 0$ satisfy the isoperimetric inequality: $A_0^3 > 36\pi V_0^2$. Consider the family $\mathcal{S}_{V_0}^{A_0}$ of embedded spherical surfaces $\vec{\Phi} : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ with area A_0 and enclosed volume V_0. Then, there exists $H_0^*(A_0, V_0) > 0$, depending continuously on A_0 and V_0, such that for any $H_0 \in]-H_0^*, H_0^*[$, the following holds true:*

$$\text{any minimizing sequence } (\vec{\Phi}_i)_{i\in\mathbb{N}} \text{ of } \inf_{\vec{\Phi}\in\mathcal{S}_{V_0}^{A_0}} \frac{1}{4}\int_{\vec{\Phi}} (H - H_0)^2 dA \text{ satisfies } \limsup_{i\to+\infty} \frac{1}{4}\int_{\vec{\Phi}_i} H^2 dA < 8\pi.$$

Hence, considering Rivière's approach, there is good evidence to think that for H_0 small enough, we can prove the existence of a smooth minimizer of the Helfrich energy (2.3) among compact simply-connected smooth surfaces of $\mathbb{R}^3$ with prescribed area and volume.

Proof. From the Cauchy-Schwarz inequality, we have $|\int_{\vec{\Phi}} H dA| \leqslant \sqrt{\int_{\vec{\Phi}} H^2}\sqrt{A_0}$ for any $\vec{\Phi} \in \mathcal{S}_{A_0,V_0}$ so we get:

$$\left(\sqrt{\int_{\vec{\Phi}} H^2 dA} - |H_0|\sqrt{A_0}\right)^2 \leqslant \int_{\vec{\Phi}} (H - H_0)^2 dA \leqslant \left(\sqrt{\int_{\vec{\Phi}} H^2 dA} + |H_0|\sqrt{A_0}\right)^2$$

If we assume that $|H_0|\sqrt{A_0} \leqslant \sqrt{4\pi}$, then we obtain using [93, Theorem 7.2.2]:

$$\left(0 \leqslant \sqrt{4\pi} - |H_0|\sqrt{A_0} \leqslant\right) \sqrt{\int_{\vec{\Phi}} H^2 dA} - |H_0|\sqrt{A_0} \leqslant \sqrt{\int_{\vec{\Phi}} (H - H_0)^2 dA} \leqslant \sqrt{\int_{\vec{\Phi}} H^2 dA} + |H_0|\sqrt{A_0}$$

Consequently, we deduce that:

$$\forall \vec{\Phi} \in \mathcal{S}_{A_0,V_0}, \quad \left| \sqrt{\int_{\vec{\Phi}} H^2 dA} - \sqrt{\int_{\vec{\Phi}} (H - H_0)^2 dA} \right| \leqslant |H_0|\sqrt{A_0}, \tag{24.1}$$

and also

$$\left| \inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \sqrt{\int_{\vec{\Phi}} H^2 dA} - \inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \sqrt{\int_{\vec{\Phi}} (H - H_0)^2 dA} \right| \leqslant |H_0|\sqrt{A_0}. \tag{24.2}$$

Now, we consider a minimizing sequence $(\vec{\Phi}_k)_{k\in\mathbb{N}}$ of $\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}} (H - H_0)^2 dA$. Since we know from [84, Lemma 2.1] that $\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}} H^2 dA < 8\pi$ (using here the conformal invariance of the Willmore energy which ensures the equivalence between an isoperimetric-ratio constraint and the volume+area constraints), we can assume that:

$$|H_0| < c_0(A_0, V_0) := \frac{\sqrt{8\pi} - \sqrt{\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}} H^2 dA}}{2\sqrt{A_0}} \quad \left(< \sqrt{\frac{4\pi}{A_0}} \right).$$

Hence, we can find $\varepsilon > 0$ such that we have:

$$\sqrt{\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}} H^2 dA + \varepsilon} + 2|H_0|\sqrt{A_0} < \sqrt{8\pi}. \tag{24.3}$$

There exists $K \in \mathbb{N}$ such that for any integer $k \geqslant K$, we have:

$$\sqrt{\int_{\vec{\Phi}_k} (H - H_0)^2 dA} \leqslant \sqrt{\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}_k} (H - H_0)^2 dA + \varepsilon} \tag{24.4}$$

Combining successively (24.1), (24.4), (24.2), and (24.3), we finally get:

$$\begin{aligned}
\sqrt{\int_{\vec{\Phi}_k} H^2 dA} &\leqslant \sqrt{\int_{\vec{\Phi}_k} (H - H_0)^2 dA} + |H_0|\sqrt{A_0} \\
&\leqslant \sqrt{\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}_k} (H - H_0)^2 dA + \varepsilon} + |H_0|\sqrt{A_0} \\
&\leqslant \sqrt{\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}_k} H^2 dA + \varepsilon} + 2|H_0|\sqrt{A_0} \\
&< \sqrt{8\pi}
\end{aligned}$$

To conclude, $c_0(A_0, V_0)$ depends continuously on A_0 and V_0. Indeed, from [84, Theorem 1.1], $\inf_{\vec{\Phi} \in \mathcal{S}_{A_0,V_0}} \int_{\vec{\Phi}} H^2 dA$ is a continuous function of the isoperimetric ratio. $\square$

Existence for more general operators

It seems that the previous results can be extended to general strongly elliptic operator. The proof might be quite technical but the arguments used for the Dirichlet/Neumann/Robin Laplacian should also work in this situation.

Moreover, the H^2- a priori estimation of Theorem 22.1 work for boundary conditions of the form $-\partial_n(u) = \beta(u)$ where $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous map satisfying $\beta(0) = 0$. Hence, Theorem 19.2 also holds in this case provided that for any $\Omega \in \mathcal{O}$, there exists a unique solution $u_\Omega \in H^2(\Omega)$.

However, if we want to extend Theorem 19.3 to this case, we must assume β to have C^1-regularity. Indeed, in the proof of the Theorem 19.3, we express the L^2-norm of the second order derivatives of u as boundary integrals using (20.10) with $\mathbf{v} = \nabla u_{\Omega_i}$. Although the bracket can be rewritten as a boundary integral if β is only Lipschitz, we need the continuity of $\dot{\beta}$ to correctly let $i \to +\infty$ in this term.

Finally, it remains to study some more physical functional, where the dependence appears through the solution of more general equations, such as a Stokes system. This could allow to model some interactions between a vesicle and a fluid.

Unfortunately, in this thesis, we did not really consider the numerical aspects of the Helfrich energy. Indeed, we were missing some time. However, phase-field methods and the accurate way to compute total mean curvature and also a conservative scheme for the area constraint are very interesting problems. Moreover, an efficient algorithm coupling the Helfrich energy with some Navier-Stokes equations is a big issue.

Part VI

Annexe

Chapter 25

Some results coming from algebraic topology

In many textbooks and papers concerning geometry such as [18] or [73], the surfaces considered are usually assumed to have C^∞-regularity. This is justified in a introductory course where the author(s) do(es) not want to loose their reader into details. This is also the case for example, when geometers are working on minimal surfaces for which we already have regularity results.

However, results like the Jordan-Brouwer Separation Theorem come from algebraic topology. Even though the proof can be sometimes greatly simplified if a certain amount of regularity is available, these kind of properties remain of topological nature. Hence, many people from this field usually work without any regularity assumption.

For those who are dealing with partial differential equations, some regularity assumptions are often needed on the boundary of the domain. Usually, we do not have the C^∞-regularity, think about polygons, but it is often better than just a topological surface. In this in-between situation, we have to distinguish what properties hold or not, depending on the available regularity.

In this chapter, we establish a naive property we often use throughout the text: the existence of an inner domain for any compact hypersurface. In the connected case, this is known as the Jordan-Brouwer Separation Theorem. We extend here the notion to the non-connected case. We also recall the classification of surfaces according to their genus.

25.1 The notion of topological n-manifold, $(n-1)$-submanifold and C^0-hypersurface

First, we quickly recall some basic definitions of topology. A set X is called a topological space if we specify among its subsets the family of its open sets. It must contain X and the empty set $\emptyset$ but has also to be stable under union and finite intersection. A neighbourhood of a point $x \in X$ is a subset of X that contains an open set containing x.

A topological space X is connected if it cannot be represented as the union of two disjoint non-empty open sets. It is compact if every open covering of X contains a finite subcovering. It is second-countable if there exists a countable family $\mathcal{F}$ of open sets such that every open set can be written as an union of elements of $\mathcal{F}$.

Considering two topological spaces X and Y, a well-defined map $f : X \to Y$ is continuous if the inverse image of any open set in Y is an open subset of X. Moreover, f is called an homeomorphism if it is a continuous bijective map whose inverse f^{-1} is also continuous. A topological space X is pathwise connected if any pair of points can be joined by a continuous path $\gamma : [0,1] \to X$.

Then, we define the notion of topological n-manifold. We also emphasize the distinction made between a topological $(n-1)$-submanifold and a C^0-hypersurface.

Definition 25.1. *Let $n \geqslant 1$. A topological n-manifold (without boundary) is a non-empty Hausdorff space which is locally Euclidean, i.e. a non-empty topological space where distinct points have disjoint neighbourhoods, and such that any point has an open neighbourhood homeomorphic to an open subset of the usual n-dimensional Euclidean space $\mathbb{R}^n$.*

Definition 25.2. *Let $n \geqslant 2$ and M be a topological n-manifold. We say that Σ is a topological $(n-1)$-submanifold of M if it is a topological $(n-1)$-manifold embedded in M i.e. for which there exists a map $i : \Sigma \to M$ which is an homeomorphism on its image $i(\Sigma)$.*

Observe that $i(\Sigma)$ is a topological $(n-1)$-manifold for the induced topology of M. Henceforth, any topological $(n-1)$-submanifold Σ is identified with its image $i(\Sigma)$. In particular, Σ is seen as a subset of M through the inclusion map $i : \Sigma \to M$. Moreover, if M is second-countable, so does Σ, because this property is invariant under homeomorphism.

Definition 25.3. *Let $n \geqslant 2$ and M be a topological n-manifold. We say that Σ is a C^0-hypersurface of M if it is a topological $(n-1)$-submanifold $\Sigma \subset M$ which is locally flat, i.e. for any point $x \in \Sigma$, there exists an open set $U_x \subset M$ containing x, and a map $\Psi_x : U_x \to \mathbb{R}^n$ which an homeomorphism on its image, such that $\Psi_x(U_x \cap \Sigma) = \Psi_x(U_x) \cap (\mathbb{R}^{n-1} \times \{0\})$.*

25.2 The separation of topological $(n-1)$-submanifolds

We refer to [25, Chapter VIII §2] for a definition of algebraic orientability. We set an integer $n \geqslant 2$. In this section, we prove a general version of the Jordan-Brouwer Separation Theorem.

Theorem 25.4. *Let Σ be a pathwise- and simply-connected topological n-manifold. For any compact connected topological $(n-1)$-submanifold $K \subset \Sigma$, the set $\Sigma \backslash K$ has exactly two non-empty connected components. Moreover, K is orientable in the algebraic sense [25, Chapter VIII §2].*

In order to prove the above assertion, we need to establish the following three propositions. The first one deals with some homology groups, the second one concerns a duality property, and the third one is about orientability.

Proposition 25.5. *Let X be a non-empty pathwise-connected topological space. If we denote by H_* the singular homology and $\widetilde{H}_*$ the reduced singular homology, then we get:*

$$\widetilde{H}_0(X;\mathbb{Z}) = \widetilde{H}_0(X;\mathbb{Z}/2\mathbb{Z}) = \{0\}.$$

If in addition, we assume that X is simply connected i.e. his first homotopy group $\pi_1(X)$ is trivial, then we also have:

$$\widetilde{H}_1(X;\mathbb{Z}) = \widetilde{H}_1(X;\mathbb{Z}/2\mathbb{Z}) = \{0\},$$

and considering a set $Y \subset X$ such that $Y \neq X$, we obtain:

$$\widetilde{H}_0(X\backslash Y;\mathbb{Z}) \simeq H_1(X, X\backslash Y;\mathbb{Z}) \qquad \text{and} \qquad \widetilde{H}_0(X\backslash Y;\mathbb{Z}/2\mathbb{Z}) \simeq H_1(X, X\backslash Y;\mathbb{Z}/2\mathbb{Z}).$$

Beware of the distinction between the difference operator $\backslash$ and the quotient one $/$.

Proof. Let X be a non-empty pathwise-connected topological space. From [25, Chapter III §4.11], we get $H_0(X;\mathbb{Z}) \simeq \mathbb{Z}$ and $H_0(\{x\};\mathbb{Z}) \simeq \mathbb{Z}$ for any $x \in X$. According to [25, Chapter III §4.3], we set $\widetilde{H}_0(X;\mathbb{Z}) := \mathrm{Ker}(\tilde{f}_0)$ where the morphism $\tilde{f}_0 : H_0(X;\mathbb{Z}) \to H_0(\{x\};\mathbb{Z})$ is the one induced by the constant function $f_0 : X \to \{x\}$. From the foregoing, $\tilde{f}_0$ is an endomorphism of $\mathbb{Z}$ so it has the form $t \mapsto ct$ where $c \in \mathbb{Z}$. Moreover, it is in fact an isomorphism, because $\tilde{f}_0 \circ \tilde{g}_0 = Id$ where $\tilde{g}_0 : H_0(\{x\};\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is the morphism induced by the constant map $g_0 : \{x\} \to X$. Hence, $\tilde{f}_0$ is a surjective map from which we deduce $c \neq 0$, thus it is also an injective map and $\widetilde{H}_0(X;\mathbb{Z}) = \mathrm{Ker}(\tilde{f}_0) = \{0\}$. Then, we show that $\mathbb{Z}$ can be replaced by $\mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/2\mathbb{Z}$ is a field, it is a free $\mathbb{Z}/2\mathbb{Z}$-module [25, Chapter VI §1.10] so [25, Chapter VI §7.22.4 and §5.12 (5.14)] gives:

$$H_0(X;\mathbb{Z}/2\mathbb{Z}) \simeq H_0(X;\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$$

The same result holds if X is replaced by $\{x\}$. Since [25, Chapter VI §7.7] defines $\widetilde{H}_0(X;\mathbb{Z}/2\mathbb{Z})$ as $\mathrm{Ker}(\tilde{f}_0 : H_0(X;\mathbb{Z}/2\mathbb{Z}) \to H_0(\{x\};\mathbb{Z}/2\mathbb{Z}))$, we can proceed as we did for $\widetilde{H}_0(X;\mathbb{Z})$ and show that $\tilde{f}_0$ is an isomorphism of $\mathbb{Z}$ so $\widetilde{H}_0(X;\mathbb{Z}/2\mathbb{Z}) = \{0\}$. Finally, we assume that X is simply connected i.e.

$\pi_1(X) = \{0\}$. Applying the Hurewicz Theorem (a reference is given in [25, Chapter VIII §2.12]), we immediatly get $H_1(X;\mathbb{Z}) = \pi_1(X)^{ab} = \{0\}$. We also have $H_1(\{x\};\mathbb{Z}) = \{0\}$ from which we deduce $\widetilde{H}_1(X;\mathbb{Z}) = \mathrm{Ker}(\tilde{f}_1 : H_1(X,\mathbb{Z}) \to H_1(\{x\}) = \mathrm{Ker}(\tilde{f}_1 : \{0\} \to \{0\}) = \{0\}$. Similarly, we get $\widetilde{H}_1(X,\mathbb{Z}/2\mathbb{Z}) = \{0\}$. To conclude, let $Y \subset X$ with $X\backslash Y \neq \emptyset$. We obtain from [25, Chapter I §4.4 and Chapter VI §7.7] that the following two sequences are exact:

$$\begin{cases} \{0\} = \widetilde{H}_1(X;\mathbb{Z}) \longrightarrow H_1(X, X\backslash Y;\mathbb{Z}) \longrightarrow \widetilde{H}_0(X\backslash Y;\mathbb{Z}) \longrightarrow \widetilde{H}_0(X;\mathbb{Z}) = \{0\} \\ \\ \{0\} = \widetilde{H}_1(X;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(X, X\backslash Y;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \widetilde{H}_0(X\backslash Y;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \widetilde{H}_0(X;\mathbb{Z}/2\mathbb{Z}) = \{0\}. \end{cases}$$

Hence, we have $H_1(X, X\backslash Y;\mathbb{Z}) \simeq \widetilde{H}_0(X\backslash Y;\mathbb{Z})$ and $H_1(X, X\backslash Y;\mathbb{Z}/2\mathbb{Z}) \simeq \widetilde{H}_0(X\backslash Y;\mathbb{Z}/2\mathbb{Z})$. □

Proposition 25.6. *Let Σ be a topological n-manifold and we denote by $\check{H}^*$ the Čech cohomology. Then, for any compact set $K \subset \Sigma$, we have $\check{H}^{n-1}(K;\mathbb{Z}/2\mathbb{Z}) \simeq H_1(\Sigma, \Sigma\backslash K;\mathbb{Z}/2\mathbb{Z})$. If in addition, we assume that Σ is orientable, then we also have: $\check{H}^{n-1}(K;\mathbb{Z}) \simeq H_1(\Sigma, \Sigma\backslash K;\mathbb{Z})$.*

Proof. This is a particular case of the Pointcaré-Lefschetz Duality Theorem [25, Chapter VIII §7.2] with $M = \Sigma$, $L = \emptyset$, and $i = 1$. Note that this duality theorem uses the Čech cohomology, which only differs from the usual one if the manifold is not second countable. □

Proposition 25.7. *Let Σ be a connected topological n-manifold and we denote by $\check{H}^*_c$ the Čech cohomology with compact support. Then, we have $\check{H}^n_c(\Sigma;\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and Σ is orientable if and only if $\check{H}^n_c(\Sigma;\mathbb{Z}) = \mathbb{Z}$. Moreover, if Σ is pathwise and simply connected, then it is orientable.*

Proof. The first two assertions come from [25, Chapter VIII §6.25] with $Y = \emptyset$. Concerning the last one, it suffices to apply [25, Chapter VIII Proposition 2.12] to Σ for which $\pi_1(\Sigma) = \{0\}$. □

Proof of Theorem 25.4. Let Σ be a pathwise- and simply-connected topological n-manifold, and let $K \subset \Sigma$ be a compact connected topological $(n-1)$-submanifold . First, we combine Propositions 25.5, 25.6, and 25.7 in order to get successively:

$$\widetilde{H}_0(\Sigma\backslash K;\mathbb{Z}/2\mathbb{Z}) \simeq H_1(\Sigma, \Sigma\backslash K;\mathbb{Z}/2\mathbb{Z}) \simeq \check{H}^{n-1}(K;\mathbb{Z}/2\mathbb{Z}) \simeq \check{H}^{n-1}_c(K;\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Note that all the hypothesis made on Σ and K are needed. The penultimate equality holds because K is compact [25, Chapter VIII §6.22]. Hence, the rank of $H_0(X\backslash K;\mathbb{Z}/2\mathbb{Z})$ is two so [25, Chapter VIII §6.22] $\Sigma\backslash K$ has exactly two non-empty pathwise-connected components, which are the connected components since Σ is pathwise connected. Then, applying again Propositions 25.5, 25.6, and 25.7, we can get back with integer coefficients:

$$\mathbb{Z} \simeq \tilde{H}_0(\Sigma\backslash K;\mathbb{Z}) \simeq H_1(\Sigma, \Sigma\backslash K;\mathbb{Z}) \simeq \check{H}^{n-1}(K;\mathbb{Z}) = \check{H}^{n-1}_c(K;\mathbb{Z}).$$

Note that the two last equalities use the fact that X is orientable since it is pathwise and simply connected. To conclude, K is orientable from Proposition 25.7. □

Corollary 25.8 (Jordan-Brouwer Separation Theorem). *Let K be any compact connected topological $(n-1)$-submanifold of $\mathbb{R}^n$. Then, $\mathbb{R}^n\backslash K$ has exactly two non-empty connected components. Both have K as boundary and only one of them is bounded.*

Proof. It suffices to apply Theorem 25.4 to $\Sigma = \mathbb{R}^n$. Hence, there exists two non-empty disjoint connected sets C_1 and C_2 such that $\mathbb{R}^n\backslash K = C_1 \sqcup C_2$. Since K is compact, there also exists a ball B such that $K \subset B$, which is connected, so $\mathbb{R}^n\backslash B$ is unbounded and belongs to one of the two connected components, let us say $\mathbb{R}^n\backslash B \subseteq C_1$. Therefore, C_1 is unbounded and we have $C_2 \sqcup K = \mathbb{R}^n\backslash C_1 \subseteq B$ thus C_2 is bounded. Finally, it remains to prove that $\partial C_1 = \partial C_2 = K$. Note that C_2 is open because $\mathbb{R}^n\backslash K$ is locally connected. We deduce that $C_1 \sqcup K = \mathbb{R}^n\backslash C_2$ is closed so $\overline{C_1} \subset C_1 \sqcup K$ thus $\partial C_1 \subseteq K$. Let us now prove $\partial C_1 = K$. Assume by contradiction that there exists a point $x \in K$ such that $x \notin \partial C_1$. Since $x \notin C_1$, we deduce $x \in \mathbb{R}^n\backslash\overline{C_1}$ which is open so there exists an open ball B_x containing x such that $B_x \subseteq \mathbb{R}^n\backslash\overline{C_1}$. Then, apply Theorem 25.4 to $\Sigma = B_x$ and $K \cap B_x$ in order to obtain that $B_x\backslash(B_x \cap K)$ has exactly two non-empty connected components. Hence, there exists two disjoint non-empty connected sets A_1 and A_2 such that:

$$A_1 \sqcup A_2 = B_x\backslash(B_x \cap K) \subseteq \mathbb{R}^n\backslash(\overline{C_1} \cup K) \subseteq \mathbb{R}^n\backslash(C_1 \sqcup K) = C_2.$$

This contradicts the connectedness of C_2 and such a point x cannot exist. To conclude, we obtain $\partial C_1 = K$ and the same arguments hold for $\partial C_2 = K$. □

Corollary 25.9 (**Brouwer-Samelson Theorem**). *Let K be any compact connected topological $(n-1)$-submanifold of $\mathbb{R}^n$. Then, K is orientable in the algebraic sense [25, Chapter VIII §2].*

Proof. This is also a direct consequence of Theorem 25.4 applied to $\Sigma = \mathbb{R}^n$. $\square$

Remark 25.10. *Corollaries 25.8 and 25.9 remain true if K is only a connected topological $(n-1)$-submanifold which is closed as a subset of $\mathbb{R}^n$. The proof is similar to the one of Theorem 25.4. It uses Alexander duality on the compactification $\mathbb{R}^n \cup \{\infty\} \simeq \mathbb{S}^n$ [25, Chapter VIII §8.15]. We also refer to [60] and [82] for a proof assuming C^2-regularity.*

25.3 The inner domain associated with a compact topological $(n-1)$-submanifold

Let $n \geqslant 2$. From Corollary 25.8, any compact connected topological $(n-1)$-submanifold $K \subset \mathbb{R}^n$ has a well-defined inner domain Ω i.e. a unique open bounded (connected) set $\Omega \subset \mathbb{R}^n$ such that we have $\partial\Omega = K$. In this section, we prove that we can still define an inner domain if we drop the connectedness hypothesis. The result states as follows.

Theorem 25.11. *Let K be a compact topological $(n-1)$-submanifold of $\mathbb{R}^n$. Then, there exists a unique open bounded set $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega = K$. Moreover, Ω is connected iff K is connected.*

Definition 25.12. *For any compact topological $(n-1)$-submanifold $K \subset \mathbb{R}^n$, the inner domain of K is the unique open bounded set $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega = K$ whose existence is guaranteed by Theorem 25.11. The enclosed volume $V(K)$ (respectively the area $A(K)$) of K is defined as the n(resp. $n-1$)-dimensional Hausdorff measure of Ω (resp. of K).*

Lemma 25.13. *Let K be a compact $(n-1)$-dimensional topological submanifold of $\mathbb{R}^n$. Then, K has a finite number of non-empty connected components.*

Proof. Let us assume by contradiction that K has an infinite number of non-empty connected components. First, there exists a sequence of pairwise distinct points x_i belonging to each connected components of K. Since K is compact, up to a subsequence, $(x_i)_{i\in\mathbb{N}}$ is converging to a point $x \in K$. Then, there exists an open set U_x of $\mathbb{R}^n$ containing x such that $U_x \cap K$ is homeomorphic to a ball of $\mathbb{R}^{n-1}$. Finally, $U_x \cap K$ has an infinite number of components so it cannot be homeomorphic to this ball. Contradiction. Hence, K has a finite number of non-empty connected components. $\square$

Proposition 25.14. *Let C_1 and C_2 be two distinct non-empty connected components of a compact topological $(n-1)$-submanifold of $\mathbb{R}^n$. Then, there exists two unique non-empty open bounded connected sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ such that $\partial\Omega_1 = C_1$ and $\partial\Omega_2 = C_2$. Moreover, only one of these three disjoint possibilities can hold: (i) $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$; (ii) $\overline{\Omega_1} \subseteq \Omega_2$; (iii) $\overline{\Omega_2} \subseteq \Omega_1$.*

Proof. The first part of the statement comes from Corollary 25.8 applied to C_1 and C_2. Since we have $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, we can write $\partial\Omega_1 = (\partial\Omega_1 \cap \Omega_2) \sqcup (\partial\Omega_1 \cap (\mathbb{R}^n \backslash \overline{\Omega_2}))$, which is a partition of $\partial\Omega_1$ with two disjoint open sets. Hence, the connectedness of $\partial\Omega_1$ imposes that either $\partial\Omega_1 \cap \Omega_2 = \emptyset$ or $\partial\Omega_1 \cap (\mathbb{R}^n \backslash \overline{\Omega_2}) = \emptyset$. Let us respectively denote these two cases by (1) and (2). If (1) occurs, then $\Omega_2 \subseteq \Omega_1 \sqcup (\mathbb{R}^n \backslash \overline{\Omega_1})$ but Ω_2 is connected so either $\Omega_2 \subseteq \Omega_1$ or $\Omega_1 \cap \overline{\Omega_2} = \emptyset$. Using again $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, we obtain that (1) leads either to (iii) or to (i). Finally, if (2) occurs, then $\mathbb{R}^n \backslash \overline{\Omega_2} \subseteq \Omega_1 \sqcup (\mathbb{R}^n \backslash \overline{\Omega_1})$ but $\mathbb{R}^n \backslash \overline{\Omega_2}$ is connected and unbounded so we must have $\mathbb{R}^n \backslash \overline{\Omega_2} \subseteq \mathbb{R}^n \backslash \overline{\Omega_1}$ i.e. $\Omega_1 \subseteq \Omega_2$. But (2) also mean $\partial\Omega_2 \subseteq \overline{\Omega_1}$ i.e. $\partial\Omega_2 \subseteq \Omega_1$ since $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, so we get (iii). $\square$

Proof of Theorem 25.11. Let K be a compact topological $(n-1)$-submanifold of $\mathbb{R}^n$. We define the inner domain of K thanks to a graph. From Lemma 25.13, K has a finite number of connected components. We represent them as the vertices of a planar graph. From Proposition 25.14, we can pairwise compare the vertices: if (ii) or (iii) occur, then an edge is created between the two vertices with the orientation given by the inclusion, otherwise (i) occurs and no edge is added. We finish our construction by adding a vertex denoted by ∞. We add an edge between ∞ and any vertex having no departure edge, with an orientation pointing towards ∞. Hence, we obtain a simple planar connected oriented graph with no cycle i.e. a tree referred to as T.

Then, we define a map $f : \mathbb{R}^n \backslash K \to T$. Consider any point $x \in \mathbb{R}^n \backslash K$. First, x is placed at the vertex ∞ and moved in the graph according to the following procedure. Assume x is located at a vertex v. Consider the vertices connected to v by an edge pointing towards v. We can apply Corollary 25.8 on each of them: x belongs or not to their inner domain. If it the case, then the point x is moved to the corresponding vertex, otherwise it remains at v. Only two cases can occur: either x is in the outer domain of all the vertices connected to v by an edge pointing towards v and we stop the procedure leaving x on v, otherwise x can only belong to one inner domain. Indeed, from our construction, there is no cycle and two vertices with no edge have disjoint inner domains and thus cannot contain at the same time a point x. Doing this operation recursively leads to locate uniquely the point x in the graph. Of course, if a vertex has no arrival edge, x is left on it. Consequently, we denoted by $f(x)$ the final vertex on which is any point $x \in \mathbb{R}^n \backslash K$ and the map $f : \mathbb{R}^n \backslash K \to T$ is well-defined i.e. $f(x)$ is uniquely determined.

Finally, for any point $x \in \mathbb{R}^n \backslash K$, there exists an oriented path between $f(x)$ and ∞, which is unique otherwise there exists a cycle in the graph, i.e. the inner and outer domain of a vertex have a non-empty intersection, which cannot be the case. We denote by $n(x)$ the number of distinct vertices in this path (counting ∞ and the departure point if it is not ∞). Hence, the map $n : x \in \mathbb{R}^n \backslash K \mapsto n(x) \in \mathbb{N}^*$ is well-defined and we set:

$$\Omega = \{x \in \mathbb{R}^n \backslash K, \quad n(x) \text{ is even}\} .$$

It remains to show that Ω is an open bounded set satisfying $\partial\Omega = K$. Consider any $x \in \Omega$. From our construction, x belongs to the intersection of the inner domain of $f(x)$ with the outer domains of every vertices connected to $f(x)$ by an edge pointing towards $f(x)$. This is a finite intersection of open sets so there exists a neighbourhood U_x of x included in this intersection. Therefore, we have $f(U_x) = \{f(x)\}$ thus $n(U_x) = \{n(x)\}$ i.e. $U_x \subseteq \Omega$ and Ω is open. Similarly, one can prove that $\mathbb{R}^n \backslash (K \cup \Omega)$ is open. We now show that $K = \partial\Omega$. Let $y \in K$. It is a vertex of our graph and from Corollary 25.8, it is the boundary of the inner (respectively outer) domain of y. Hence, there exists a sequence of points $(y_i)_{i\in\mathbb{N}}$ (resp. $(x_i)_{i\in\mathbb{N}}$) from the inner (resp. outer) domain converging to y. From Proposition 25.14 (ii)-(iii), y belongs to the outer (resp. inner) domain of any (resp. the unique) vertex connected to y by an edge pointing towards y (resp. this vertex). Hence, we can assume that $(y_i)_{i\in\mathbb{N}}$ (resp. $(x_i)_{i\in\mathbb{N}}$) belongs to the intersection of the inner (resp. outer) domain of y with the outer domains (resp. the inner domain), since it is a finite reunion of open sets. Therefore, we get $f(x_i) = y$ and $f(y_i)$ is the unique vertex connected to y and pointing towards $f(y_i)$. Moreover, $n(x_i)$ and $n(y_i)$ are constant with $n(x_i) = n(y_i) + 1$. We deduce that of the two sequences, let us say $(y_i)_{i\in\mathbb{N}} \subset \Omega$ while $(x_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n \backslash (K \cup \Omega)$. Hence, $y \in \partial\Omega$ for any point $y \in K$ so we proved $K \subseteq \partial\Omega$. But if there exists a point $x \in \partial\Omega$ such that $x \notin K$, then $x \in \mathbb{R}^n \backslash (K \cup \Omega)$ which is open so there exists a neighbourhood of x written $U_x \subseteq \mathbb{R}^n \backslash (K \cup \Omega)$. We get $U_x \cap \Omega = \emptyset$ contradicting the fact that $x \in \partial\Omega$. Consequently, $K = \partial\Omega$. To conclude, K is compact so it is included in a closed ball B. We have $f(\mathbb{R}^n \backslash B) = \{\infty\}$ and $n(\mathbb{R}^n \backslash B) = \{1\}$ so $\mathbb{R}^n \backslash B \subseteq \mathbb{R}^n \backslash (K \cup \Omega) = \mathbb{R}^n \backslash \overline{\Omega}$. We obtain $\overline{\Omega} \subseteq B$ and Ω is bounded as required. □

25.4 The class of topological surfaces

In this section, we recall the results obtained in the particular case $n = 3$. First, we present the classification of topological 2-manifolds. For this purpose, we need to define their orientation in a more geometrical way than [25, Chapter VIII §2]. This can be done thanks to the following result.

Proposition 25.15 (Jordan Curve Theorem). *If $\gamma : [0, 1[\to \mathbb{R}^2$ is a Jordan curve i.e. a continuous injective map with $\gamma(0) = \gamma(1^-)$, then the complement of the image of γ in $\mathbb{R}^2$ is the union of two disjoint non-empty open connected sets. Moreover, both have $\gamma([0, 1[)$ as boundary and only one of them is bounded.*

Proof. We refer to [42] for a reference describing the original proof of Jordan. □

From Proposition 25.15, the bounded connected component defines well an inner domain for γ. A Jordan curve is positively oriented if its inner domain always lies on the left when travelling on it. An homeomorphism $f : U \to V$ is sense-preserving if for any positively oriented Jordan curve $\gamma : [0, 1[\to U$, the map $f \circ \gamma$ is a positively oriented Jordan curve.

Definition 25.16. *A topological 2-manifold Σ is orientable if one can build an open covering of Σ by a family of open sets U_i homeomorphic to a disk D via the maps $f_i : U_i \to D$ such that each map $f_i \circ f_j^{-1} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ is a sense-preserving homeomorphism if $U_i \cap U_j \neq \emptyset$.*

Proposition 25.17. *A compact connected second-countable 2-manifold Σ is orientable in the sense of Definition 25.16 if and only if it is orientable in the algebraic sense [25, Chapter VIII §2].*

Proof. Let Σ be any compact connected second-countable 2-manifold. Assume that Σ is orientable in the sense of Definition 25.16. First, note that this definition is equivalent to the one given in [1, Chapter I §11-12]. Since a second-countable connected two-dimensional manifold is triangulable [1, Chapter I §7], there exists a compact connected triangulation Σ_g of Σ and we get from [1, Chapter I §25C-25D]: if Σ_g is orientable, then $H_2(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}$, otherwise $H_2(\Sigma_g;\mathbb{Z}) = \{0\}$. Hence, we obtain $H_2(\Sigma;\mathbb{Z}) \simeq H_2(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}$. We conclude by using [25, Chapter VIII §3.4] with $M = \Sigma$ and $C = \emptyset$: Σ is orientable in the algebraic sense [25, Chapter VIII §2]. Conversely, if Σ is not orientable as in Definition 25.16, then $H_2(\Sigma;\mathbb{Z}) = H_2(\Sigma_g;\mathbb{Z}) = \{0\}$ and [25, Chapter VIII §3.4] ensures that Σ is not orientable in the algebraic sense [25, Chapter VIII §2]. $\square$

Theorem 25.18 (Topological Classification Theorem). *If Σ is a two-dimensional topological manifold which is connected, compact, second-countable, and orientable, then it is homeomorphic either to a sphere ($g = 0$) or to a sphere with g handles ($g \geqslant 1$). In other words, we have a topological characterization of such Σ according to their genus $g \in \mathbb{N}$.*

Proof. We refer to [90] for a self-contained proof using graph theory. The Jordan Curve Theorem (Proposition 25.15) and its refinement the Jordan-Schönflies Curve Theorem (Proposition 25.23) are first established, then used to prove that such Σ are triangulable. Finally, all triangulated surfaces are classified. We also mention [1, Chapter I §7-8] for a proof based on homology and homotopy theory, and [36, Chapter V] if a C^∞-differential structure is added to use the Morse theory. $\square$

Then, we recall Corollaries 25.8 and 25.8 for two-dimensional topological submanifolds of $\mathbb{R}^3$.

Proposition 25.19 (Jordan-Brouwer Separation Theorem). *If Σ is a compact connected topological 2-submanifold of $\mathbb{R}^3$, then $\mathbb{R}^3\backslash\Sigma$ is the union of two disjoint non-empty open connected sets. Moreover, both have Σ as boundary and only one of them is bounded.*

Proof. It suffices to apply Corollary 25.8 for $n = 3$. The proof is much easier if Σ is assumed to be a C^1-surface [73, Sections 4.2-4.4]. Although the authors assume C^∞-regularity in [73, Definition 2.2], the proof is valid with C^1-regularity only. Indeed, it is based on topological considerations combined with the Inverse Function Theorem and Sard's Theorem. We also mention [36, Chapter VII] for a proof assuming the C^∞-regularity of surfaces and the Morse theory. $\square$

Proposition 25.20 (Brouwer-Samelson Theorem). *If Σ is a compact connected topological 2-submanifold of $\mathbb{R}^3$, then Σ is an orientable.*

Remark 25.21. *The above assertion remains valid if Σ is a connected two-dimensional topological submanifold which is closed as a subset of $\mathbb{R}^3$ [25, Chapter VIII §8.15] [73, Chapter 4 Exercise (6)]. We also refer to [73, Chapter 2 Exercise (2)] for an example of non-closed surface.*

Corollary 25.22. *If Σ is a compact 2-submanifold, then Σ is countable, orientable and $i(\Sigma)$ has a well-defined inner domain $\mathrm{Int}(i(\Sigma))$. Henceforth, Σ is identified with $i(\Sigma)$: its area $A(\Sigma)$ and volume $V(\Sigma)$ are respectively defined by $\mathcal{H}_2(i(\Sigma))$ and $\mathcal{H}_3(\mathrm{Int}(i(\Sigma)))$, where $\mathcal{H}_n$ is the ordinary n-dimensional Hausdorff measure. Moreover, if Σ is connected, its topology is only characterized by its genus $g \in \mathbb{N}$.*

Proof First, consider a 2-submanifold Σ. As $\mathbb{R}^3$ is countable, so is $i(\Sigma)$, thus so is also Σ because countability is a property invariant under homeomorphism. Then, assume that Σ is compact and observe it has a finite number of connected components (cf. [36] V §1 Th. 1) that cannot intersect. Hence, one can check that there is only one way to extend to Σ the definition of orientability and inner domain ensured by Proposition 25.20 for compact connected 2-submanifolds. Finally, if Σ is compact and connected, then apply Proposition 25.18 to Σ $\boxdot$

Consequently, any non-orientable 2-manifold such as the projective plane or the Klein bottle cannot be embedded in $\mathbb{R}^3$. Hence, although a 2-submanifold is always a 2-manifold, the converse is not true in general. We now investigate the possibility of extending an embedding to an homeomorphism on a whole space.

Proposition 25.23 (Jordan-Schönflies Curve Theorem). *If $\gamma : [0, 1[\to \mathbb{R}^2$ is a Jordan curve, then there exists an homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the image of $\gamma([0, 1[)$ through f is the unit circle of the plane centred at the origin.*

However, Proposition 25.23 (cf. [90] §3 for proof) does not hold for 2-submanifolds. The Alexander's horned sphere is an embedding of a sphere in $\mathbb{R}^3$ and thus separates space into two regions (cf. Proposition 25.20), but those two are so widely knotted that the outer domain is not homeomorphic to the outside of a sphere. This motivates the next definition, requiring a local existence of such extension.

Definition 25.24. *A 2-submanifold Σ is called a C^0-surface if it is locally flat, i.e. for every point $\mathbf{x} \in \Sigma$, there exists an open neighbourhood U of point $i(\mathbf{x})$ in $\mathbb{R}^3$, an open neighbourhood V of the origin and an homeomorphism $\Psi : U \to V$ such that $\Psi(U \cap i(\Sigma)) = V \cap (\mathbb{R}^2 \times \{0\})$. Moreover, we define $\mathcal{C}_0$ as the family of all compact connected C^0-surfaces of zero genus. Hence, according to Corollary 25.22, $\mathcal{C}_0$ exactly contains the C^0-surfaces whose topology is the one of a sphere.*

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Abstract

En biologie, lorsqu'une quantité importante de phospholipides est insérée dans un milieu aqueux, ceux-ci s'assemblent alors par paires pour former une bicouche, plus communément appelée vésicule. En 1973, Helfrich a proposé un modèle simple pour décrire la forme prise par une vésicule. Imposant la surface de la bicouche et le volume de fluide qu'elle contient, leur forme minimise une énergie élastique faisant intervenir des quantités géométriques comme la courbure, ainsi qu'une courbure spontanée mesurant l'asymétrie entre les deux couches. Les globules rouges sont des exemples de vésicules sur lesquels sont fixés un réseau de protéines jouant le rôle de squelette au sein de la membrane. Un des principaux travaux de la thèse fut d'introduire et étudier une condition de boule uniforme, notamment pour modéliser l'effet du squelette.

Dans un premier temps, on cherche à minimiser l'énergie de Helfrich sans contrainte puis sous contrainte d'aire. Le cas d'une courbure spontanée nulle est connu sous le nom d'énergie de Willmore. Comme la sphère est un minimiseur global de l'énergie de Willmore, c'est un bon candidat pour être un minimiseur de l'énergie de Helfrich parmi les surfaces d'aire fixée. Notre première contribution dans cette thèse a été d'étudier son optimalité. On montre qu'en dehors d'un certain intervalle de paramètres, la sphère n'est plus un minimum global, ni même un minimum local. Par contre, elle est toujours un point critique.

Ensuite, dans le cas de membranes à courbure spontanée négative, on se demande si la minimisation de l'énergie de Helfrich sous contrainte d'aire peut être effectuée en minimisant individuellement chaque terme. Cela nous conduit à minimiser la courbure moyenne totale sous contrainte d'aire et à déterminer si la sphère est la solution de ce problème. On montre que c'est le cas dans la classe des surfaces axisymétriques axiconvexes mais que ce n'est pas vrai en général.

Enfin, lorsqu'une contrainte d'aire et de volume sont considérées simultanément, le minimiseur ne peut pas être une sphère qui n'est alors plus admissible. En utilisant le point de vue de l'optimisation de formes, la troisième et plus importante contribution de cette thèse est d'introduire une classe plus raisonnable de surfaces, pour laquelle l'existence d'un minimiseur suffisamment régulier est assurée pour des fonctionnelles et des contraintes générales faisant intervenir les propriétés d'ordre un et deux des surfaces. En s'inspirant de ce que fit Chenais en 1975 quand elle a considéré la propriété de cône uniforme, on considère les surfaces satisfaisant une condition de boule uniforme. On étudie d'abord des fonctionnelles purement géométriques puis nous autorisons la dépendance à travers la solution de problèmes aux limites elliptiques d'ordre deux posés sur le domaine intérieur à la surface.

In biology, when a large amount of phospholipids is inserted in aqueous media, they immediatly gather in pairs to form bilayers also called vesicles. In 1973, Helfrich suggested a simple model to characterize the shapes of vesicles. Imposing the area of the bilayer and the volume of fluid it contains, their shape is minimizing a free-bending energy involving geometric quantities like curvature, and also a spontanuous curvature measuring the asymmetry between the two layers. Red blood cells are typical examples of vesicles on which is fixed a network of proteins playing the role of a skeleton inside the membrane. One of the main work of this thesis is to introduce and study a uniform ball condition, in particular to model the effects of the skeleton.

First, we minimize the Helfrich energy without constraint then with an area constraint. The case of zero spontaneous curvature is known as the Willmore energy. Since the sphere is the global minimizer of the Willmore energy, it is a good candidate to be a minimizer of the Helfrich energy among surfaces of prescribed area. Our first main contribution in this thesis was to study its optimality. We show that apart from a specific interval of parameters, the sphere is no more a global minimizer, neither a local minimizer. However, it is always a critical point.

Then, in the specific case of membranes with negative spontaneous curvature, one can wonder whether the minimization of the Helfrich energy with an area constraint can be done by minimizing individually each term. This leads us to minimize total mean curvature with prescribed area and to determine if the sphere is a solution to this problem. We show that it is the case in the class of axisymmetric axiconvex surfaces but that it does not hold true in the general case.

Finally, considering both area and volume constraints, the minimizer cannot be the sphere, which is no more admissible. Using the shape optimization point of view, the third main and most important contribution of this thesis is to introduce a more reasonable class of surfaces, in which the existence of an enough regular minimizer is ensured for general functionals and constraints involving the first- and second-order geometric properties of surfaces. Inspired by what Chenais did in 1975 when she considered the uniform cone property, we consider surfaces satisfying a uniform ball condition. We first study purely geometric functionals then we allow a dependence through the solution of some second-order elliptic boundary value problems posed on the inner domain enclosed by the shape.