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par

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On stability and control of random jump linear systems

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dedicated to my family

Abstract

Dynamical systems with abrupt changes or several structures or interruptions to communications between subsystems can be effectively modelled by random jump linear systems. Random jump linear systems are a kind of stochastic hybrid systems, which consists of a set of linear systems, also called multiple modes, and the switching among them is governed by a random jump process. In particular, we consider that the underlying random jump process is time inhomogeneous. Roughly, a time inhomogeneous random jump process is one that has time varying transition probabilities (or rates) among its modes. It is a practical consideration, because, in general, the physical systems are time inhomogeneous across its different properties and transition probabilities (or rates) among its modes can also be viewed as a time inhomogeneous phenomenon.

Broadly, we categorize the thesis into two parts: stability and control subject to constraints.

In the first part, we investigate second moment stability and stabilization analysis of random jump linear systems. We first investigate second moment stability and stabilization of discrete-time inhomogeneous Markov jump linear systems. In particular, we consider the situation when the transition probability matrix of the inhomogeneous Markov Chain is varying in an interval. It is realistic to consider interval transition probability matrix, because, in general, the transition probabilities are estimated with errors, which can be represented as intervals. We obtain a sufficient condition for second moment stability of the proposed system by using results of interval analysis and graph theory in terms of a spectra radius of a suitably defined matrix. Alternatively, we obtain a convex hull representation of the interval transition probability matrix and give a sufficient condition for second moment stability in terms of linear matrix inequalities, using which we deal with stabilization. In continuous-time, we consider second moment stability and stabilization of a continuous-time state-dependent jump linear system, where the transition rates of the underlying random jump process depend on the state variable. It is a valid assumption to have state-dependent transition rates, because, transition rates between modes may depend on the present state of the system, which is true for systems prone to faults or a macroeconomic system with different economic situations etc. In particular, we assume the transition rates of the random jump process to have different values

across different sets to which the state of the system belong, and address the problem of stochastic stability and stabilization analysis. Using a stochastic version of Lyapunov's second method and the Dynkin's formula, we establish numerically solvable sufficient conditions for stochastic stability and state-feedback stabilization in terms of linear matrix inequalities. Based on these results, for the same system, we consider the presence of external disturbances with finite energy and synthesize a state-feedback $\mathcal{H}_\infty$ controller that guarantee disturbance rejection while ensuring stochastic stability and stabilization. We illustrate the obtained results on a macroeconomic system.

In the second part, we investigate control of random jump linear systems subject to constraints. We use model predictive control, also called receding horizon control, a widely used optimization algorithm to handle constraints in problems across disciplines. We first investigate a receding horizon control of discrete-time state-dependent jump linear systems subject to possibly unbounded random disturbances and probabilistic state constraints. Due to a nature of the dynamical system and the constraints, we consider a one-step receding horizon. We consider the receding horizon control law to have a linear state-feedback and an admissible offset term. We ensure mean square boundedness of the state variable via solving linear matrix inequalities off-line, and solve the receding horizon control problem on-line with control offset terms after obtaining its tractable representation. We illustrate the over all approach with a numerical example. For the same system, we consider the presence of noisy measurements and would like to apply the receding horizon approach afore mentioned. However, the unavailability of the state variable makes the formulation of state-dependent jump process intractable. Thus we confine ourselves to discrete-time inhomogeneous Markov jump linear systems, a simpler random jump linear systems, with process noise and noisy measurements and address the receding horizon control problem. Using conditional estimates and covariances of the state variable, we obtain a tractable representation of the receding horizon control problem and followed the similar approach afore mentioned. We illustrate the proposed approach on the dynamics of vertical take-off and landing vehicle.

Keywords: jump linear systems, second moment stability, linear state-feedback, stabilization, linear matrix inequalities, stochastic receding horizon control, probabilistic constraints

Résumé

Les systèmes dynamiques linéaires à sauts aléatoires représentent une classe de systèmes hybrides stochastiques. La commutation entre les différents modes de ce type de systèmes est régie par un processus stochastique à sauts. En particulier, nous considérons dans le cadre de cette thèse que le processus à sauts aléatoires sous-jacent est non-homogène en temps i.e. que les probabilités (ou taux) de transition entre les différents modes du système sont à temps variant. Il s'agit ici d'une modélisation motivée par des considérations d'ordre pratique. Cette considération d'ordre pratique sera motivée en détail dans la suite du document à travers divers exemples d'application.

Cette thèse se décompose en deux parties principales : une première partie est dédiée au problème de la stabilité/stabilisation des systèmes dynamiques linéaires à sauts aléatoires non-homogènes en temps et une seconde partie est dédiée au problème de contrôle sous contraintes de cette classe de systèmes.

Dans la première partie de cette thèse, nous nous intéressons d'une part à l'analyse de la stabilité en moyenne quadratique des systèmes linéaires à sauts Markoviens non-homogènes en temps, et ceci en temps discret. En particulier, nous considérons le cas où la matrice de probabilité de transition de la chaîne de Markov non-homogène varie dans un intervalle. L'approche basée intervalle se justifie ici par le fait qu'en pratique les probabilités de transition sont généralement estimées. Les erreurs d'estimation sont le plus souvent représentées sous forme d'intervalles. En nous appuyant sur des outils issus de l'analyse par intervalles ainsi que de la théorie des graphes, nous obtenons une condition suffisante de stabilité en moyenne quadratique de cette classe de systèmes. Ces conditions sont exprimées en terme du rayon spectrale d'une matrice définie de façon appropriée. D'autre part, nous considérons le problème de stabilité stochastique des systèmes linéaires à temps continu et à sauts aléatoires dépendant de l'état du système, i.e., où les taux de transition du processus de saut dépendent de la variable d'état. Ce type de modèle s'impose de façon naturelle dans le cas des systèmes sujets à des défauts/pannes, des systèmes macro-économiques etc. Ce point sera mis en exergue par la suite via quelques exemples d'application. En utilisant une version stochastique de la deuxième méthode de Lyapunov et la formule de Dynkin adaptée à cette classe de systèmes, nous établissons des conditions suffisantes de stabilité en terme d'inégalités matricielles linéaires (LMI pour Linear Matrix Inequalities). Sur la base de ces résultats, nous considérons par la suite le cas où le système est sujet à des perturbations exogènes à énergie finie et nous nous intéressons au problème de synthèse d'une loi de commande par retour d'état de type $\mathcal{H}_\infty$ stochastique. Ce type de commande vise à garantir le rejet des perturbations (dans un sens qui sera défini en temps voulu) tout en assurant la

stabilité stochastique du système.

Dans la seconde partie de cette thèse, nous étudions le problème de commande predictive des systèmes à sauts aléatoires sujets à des contraintes de type probabilistes. Nous étudions dans un premier temps un problème de commande à horizon glissant d'un système linéaire discret à sauts dépendant de l'état, soumis à des perturbations aléatoires éventuellement non bornées et à des contraintes de type probabilistes sur l'état du système. En raison de la nature du système dynamique et des contraintes considérées, nous considérons un horizon glissant à un seul pas. La loi de commande proposée se décompose en deux parties, un retour d'état linéaire, calculé hors-ligne, auquel vient se superposer un terme calculé pour sa part en ligne. Le retour d'état linéaire, obtenu par résolution d'un problème d'optimisation LMI, assure la pré-stabilisation du système dans un sens que l'on définira par la suite. Les contraintes probabilistes imposées à l'état du système sont quand à elles traitées via la résolution en ligne d'un problème de commande à horizon glissant et cela après transformation du problème original de nature stochastique en un problème de nature déterministe aisément résoluble par des outils de programmation linéaire. L'approche proposée est illustrée via un exemple macroéconomique. Dans un deuxième temps, nous relaxons l'hypothèse d'accessibilité de l'état du système et considérons le problème de commande à horizon glissant basée sur l'estimation de l'état d'un système linéaire discret à sauts Markoviens soumis à des perturbations aléatoires éventuellement non bornées et à des contraintes de type probabilistes sur l'état du système. En utilisant les estimées et les covariances conditionnelles de la variable d'état, nous arrivons à obtenir une formulation du problème de commande à horizon glissant pouvant être traitée en utilisant une approche similaire que celle décrite ci-dessus.

Mots-clés : Systèmes linéaires à sauts, stabilité en moyenne quadratique, retour d'état linéaire, LMI, contrôle à horizon glissant stochastique, contraintes probabilistes

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Contents

Abstract	iii
Résumé	v
Acknowledgements	vii
Contents	ix
1 Introduction	1
1.1 Random Jump Linear Systems (RJLSs): Description and Examples	1
1.2 Research Motivations	3
1.3 General Mathematical Model of RJLSs	5
1.4 Literature Survey	6
1.5 Our Approach and Contributions	8
1.6 Thesis Organization	10
1.7 Notations and Abbreviations	13
I Stability and Stabilization of RJLSs	15
2 Second Moment Stability and Stabilization of Discrete-time Inhomogeneous Markov Jump Linear Systems having Interval Transition Probability Matrix	17
2.1 Introduction	17
2.2 Model Description	18
2.2.1 A Related Background on Second Moment Stability	19
2.2.2 Characterization of TPM as an Interval Matrix	20

2.3	Mean Square Stability of Discrete-time Inhomogeneous MJLS with Interval TPM Using Interval Analysis	22
2.3.1	Brief Review on Interval Analysis	22
2.3.2	A Sufficient Condition for MSS Using Interval Analysis	25
2.3.3	State-feedback Stabilization	30
2.4	Stochastic Stability and Stabilization of Discrete-time Inhomogeneous MJLS with Interval TPM	30
2.4.1	Obtaining a Convex Combination of Stochastic Matrices of Interval TPM	31
2.4.2	A Sufficient Condition for SS	32
2.4.3	State-feedback Stabilization	34
2.4.4	An Illustrative Example of Vertical Take Off and Landing Vehicle	34
2.5	Conclusions	36
3	Stochastic Stability and Stabilization of a Class of Continuous-time State-dependent Jump Linear Systems	37
3.1	Introduction	37
3.2	Stochastic Stability and Stabilization of Continuous-time SDJLS	38
3.2.1	Model Description	38
3.2.2	A Related Background	40
3.2.3	An Equivalent Model of SDJLSs	41
3.2.4	Stochastic Stability	44
3.2.5	State-feedback Stabilization	47
3.2.6	An Illustrative Example of Macroeconomic System	49
3.3	$\mathcal{H}_\infty$ State-feedback Stabilization of Continuous-time SDJLS	51
3.3.1	Problem Formulation	51
3.3.2	An Equivalent Model of SDJLS	52
3.3.3	Stochastic Stability with $\mathcal{H}_\infty$ Performance	52
3.3.4	$\mathcal{H}_\infty$ State-feedback Stabilization	55
3.3.5	An Illustrative Example of Macroeconomic System	57
3.4	Proofs	58
3.5	Conclusions	60

II	Control of RJLSs Subject to Constraints	61
4	On Control of Discrete-time State-dependent Jump Linear Systems with Probabilistic Constraints: A Receding Horizon Approach	63
4.1	Introduction	63
4.2	Problem Setup	65
4.2.1	Control Policies	66
4.3	Pre-stabilization Results	67
4.4	A RHC with Probabilistic State Constraints	68
4.4.1	Probabilistic Constraints	69
4.4.2	A Tractable Representation of the RHC problem	70
4.5	A Numerical Example	72
4.6	Proofs	74
4.7	Conclusions	77
5	A Receding Horizon Control of Discrete-time Markov Jump Linear Systems with Probabilistic State Constraints by Incorporating State Estimation	79
5.1	Introduction	79
5.2	Problem Description	81
5.3	Pre-stabilization Results	84
5.3.1	Mean Boundedness of Conditional Covariance Process	84
5.3.2	Stabilization of the System via Mean Square Boundedness of the System States	86
5.4	A Tractable Representation of the RHC Problem	87
5.4.1	Converting Probabilistic Constraints to Deterministic Constraints	87
5.4.2	A Deterministic Representation of the Overall RHC Problem	88
5.5	An Illustrative Example of Vertical Take Off and Landing Vehicle	89
5.6	Proofs	92
5.7	Conclusions	94

6	Conclusions and Future Work	95
6.1	Summary of the Thesis	95
6.2	Directions for Future Research	96
A	Mathematical Preliminaries	99
B	A Non-exponential Sojourn Time Distribution for Change Detection of Exponentially Weighted Moving Average Algorithm	103
C	A Novel Mathematical Setup for Fault Tolerant Control Systems with State-dependent Failure Process	113

List of Figures

1.1	An illustration of RJLS	2
1.2	A flowchart of the thesis	12
2.1	Variation of first row $([e_1 \ e_2])$ of Π_k (2.7)	22
2.2	Illustration of theorem 2.18, where $\dot{\mathcal{B}} = \mathcal{A} $ if $\mathcal{A}$ is irreducible and $d(\mathcal{A}) \neq \mathbb{O}$	27
2.3	Directed Graph $G(\mathcal{A})$ of the interval matrix $\mathcal{A}$ in (2.17)	29
2.4	Plot of $\mathbb{E} [\ x_k\ ^2]$ obtained from Monte Carlo simulations	29
2.5	Plot of system states for a single path simulation	36
3.1	A schematic representation of the sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ in $\mathbb{R}^2$	39
3.2	A sample $\theta(t)$ and the corresponding stopping time	48
3.3	State trajectories of the system: a single sample path	48
3.4	Sample state trajectories of the system	50
3.5	A sample state-dependent $\theta(t)$ and the corresponding stopping times	50
3.6	Stabilized state trajectories of the system	58
3.7	A sample state-dependent $\theta(t)$	58
4.1	A sample evolution of modes θ_k and corresponding state values x_k	73
4.2	Illustration of probabilistic constraints with different values of ξ	74
5.1	Conditional covariances	91
5.2	Actual state values for different values of ξ	92
6.1	An illustration of distributed MPC of networked agent systems	97
B.1	PDF of fault detection delay $(f_{T_d}(t))$ and the time between false alarms $(f_{T_f}(t))$ and the normalized histograms	110

B.2	Effect of λ , $(\mu_1 - \mu_0)$ on the expectation of T_d	110
B.3	Effect of λ on the false alarms	111
B.4	Effect of λ on the variance of y_k	111
C.1	An illustration of the FTCS	114

Chapter 1

Introduction

Majority of the dynamical systems have been extensively modelled by linear time-invariant systems, and a large body of theory is available for design and analysis of such systems. In practice, it is not possible to use linear time-invariant system modelling of some dynamical systems. For example, systems with abrupt changes or breakdown of certain components or interruptions to communications between subsystems. These events can be modelled as random changes in the system structure that can lead to instability of the closed-loop system, which may result in a disastrous situation. Such a class of systems can be adequately modelled by *random lump linear systems*.

1.1 Random Jump Linear Systems (RJLSs): Description and Examples

RJLSs are a special class of stochastic hybrid systems, typically, a set of linear systems described by a set of classical differential (or difference) equations and a random jump process governing the jumps among them. An illustration of an RJLS with a set of N linear systems is schematically shown in Figure 1.1, which demonstrates the random switching among its linear systems by a random jump process. In the literature, RJLSs are also called as stochastic switching systems (Boukas, 2006) or piecewise deterministic stochastic systems (Davis, 1993) or jump linear systems (Mariton, 1990). RJLS modelling has been used in many engineering applications.

Manufacturing systems: Manufacturing systems produce several commodities depending on the demand rate; however, it may be subjected to sudden changes such as demand fluctuations, machine failures etc., at random instants of time, which affect its production rate. The manufacturing systems with multiple parts production, sudden failures of the components, and random production capacity can be modelled as Markov jump linear systems (MJLSs), which belong to the family of RJLSs, as in (Boukas and Liu, 2001),

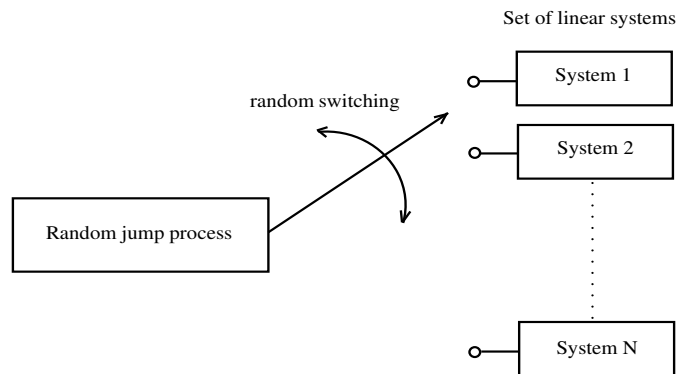


Figure 1.1: An illustration of RJLS

where the failure process of a component was modelled by a finite-state Markov process and the authors used stochastic optimization techniques to propose a control strategy. The problem of controlling the production rate of a failure prone manufacturing system so as to minimize the discounted inventory cost, under a constant demand rate for the commodity produced, was considered in (Akella and Kumar, 1986). The authors modelled the underlying dynamics as MJLS, and proposed an optimal inventory level corresponding to an infinite horizon discounted cost criterion. Also, a multiple-identical-machine manufacturing system with random breakdowns, repairs and preventive maintenance activities was considered in (Gharbi and Kenne, 2000), where the authors used MJLSs to model the dynamics of the inventory and obtained an optimal control that minimizes the total cost of the inventory. One can see (Gharbi and Kenne, 2000) for more references of RJLS modelling of the dynamics of manufacturing systems.

Vertical take-off and landing aircraft: The dynamics of vertical take-off and landing (VTOL) aircraft was modelled using RJLS in (De Farias et al., 2000; Aberkane et al., 2008). The authors modelled the dynamics of the horizontal velocity, the vertical velocity, the pitch rate and the pitch angle as MJLS, where the underlying Markov process states were categorized depending on the airspeed. Using the dynamics of VTOL aircraft that belong to different modes depending on the airspeed, a stability and optimal control problem was addressed.

Networked control systems: Networked control systems are a kind of distributed control systems, where the information of different control system components is exchanged via communication networks. Some of the network-induced constraints are network-induced time delays and packet dropouts, which can degrade the overall performance or even cause instability. MJLS modelling of networked control systems was given (Nilsson, 1998), where the underlying Markov process was considered to model the network-induced delays; similar modelling approach can be seen in (Huang and Nguang, 2008; Zhang et al., 2005) for example. In a similar context, MJLS modelling of networked control systems was addressed in (Wu and Chen, 2007), where the underlying Markov process represent the

pack dropouts; the similar modelling approach can be observed in (Xiong and Lam, 2007) and the references within.

Economics: In economics, a dynamical system based on Samuelson's multiplier accelerator model, which describes the relation between the government expenditure and the national income, was modelled as RJLS (Blair Jr, 1974; Blair Jr and Swarder, 1975) in addressing an optimal control problem. In this RJLS formulation, the economic situations are categorized as 'Normal', 'Boom', and 'Slump' and subsequently modelled the transitions between these situations as a Markov process.

Solar energy plant: The solar energy plant is basically composed of a set of adjustable mirrors, the heliostats, surrounding a tower which contains a boiler. By using the sunlight directed from the heliostats, the boiler gets heated. The objective in this situation is to control the feed-water flow rate to the boiler in order to maintain the outlet steam temperature within admissible limits. The stochastic nature of the problem arises because the system dynamics is heavily dependent on the instantaneous insolation. Cloud movements over the heliostats can cause sudden changes in insolation, and accordingly two modes are defined depending on the insolation: sunny, and cloudy. The transition between these two modes was modelled as Markov chain and the overall system as MJLS (Sworder and Rogers, 1983), where an optimal control for a solar energy was formulated.

Some additional interesting applications of RJLSs can be found in Uncrewed Air Vehicle (Stoica and Yaesh, 2002), papermills (Khanbaghi et al., 1998) etc.

In the next section, we present some of the research motivations for a class of systems we are going to investigate in this thesis.

1.2 Research Motivations

In this thesis, we consider RJLSs where the underlying random jump process is time inhomogeneous, which implies that the transition probabilities (or rates) among its subsystems are time varying. In this section, we give research motivations for specific RJLSs that we address in this thesis.

In RJLSs, if the random jump process is assumed to be a finite state *time-homogeneous* Markov process (or chain) with known transition rates (or probabilities), then this particular class of systems are widely known as MJLSs in the literature. It implies that the underlying transition probability matrix (or rate matrix) is time-invariant. The theory of stability, optimal and robust control, as well as important applications of such systems, can be found for instance in (Feng et al., 1992; Mariton, 1990; Kozin, 1969; Ji et al., 1991; Fang, 1994; Boukas, 2006; Dragan et al., 2010; Costa et al., 2013) and the references therein. In general, the studies of MJLSs assume that the underlying Markov process is

time-homogeneous, where the time-invariant property of the underlying transition probability matrix (or rate matrix) is quite a restrictive assumption.

For example, in MJLS modelling of manufacturing systems, if the underlying random process that represents failure process is modelled as a time-homogeneous Markov chain (or process), then it implies the constant failure probabilities (or rates). In reality, however, this assumption is often violated because the failure probabilities (or rates) of a component usually depend on many factors, for example, its age, the degree of usage, etc. It also means that the sojourn time of the failure process may not exponentially distributed¹. In economics and finance, the Colombian economic growth model was considered as a Markov switching regime model (Matha and Maria, 2006) with time-varying transition probabilities. Also, for instance, a study of banking systems fragility to a financial crisis was carried out in (Khallouli and Nabi, 2009), where the switchings between the tranquil state to the crisis state, and vice versa, were modelled as Markov chain with time-varying transition probabilities. As some of the motivations, this lead to studies of *time-inhomogeneous* MJLSs, in which transition probability matrix (or rate matrix) of the Markovian process is time variant.

In general, a reasonable course of direction is to analyze RJLSs when the transition probabilities (or rates) of the random jump process are time varying and also depend on some other factors such as state variable, input etc. For instance, in the analysis of random breakdown of components, the age, wear, and accumulated stress of a component affect its failure rate. Thus, it is reasonable to assume that the failure rate of a component is dependent on state of the component at age t (Bergman, 1978), where the state variable may be an amount of wear, stress etc. Also, for instance, in (Giorgio et al., 2010), a state-dependent Markov process was used to describe the random breakdown of cylinder lines in a heavy-duty marine diesel engine. As an another scenario, consider a case of stock market with two regimes: up and down. The transition rate between the regimes usually result from the state of economy, the general mood of the investors in the market etc., which can be regarded in general as the state of the market. Also, a state-dependent regime switching model was considered in (Kim et al., 2008) to model financial time series. One can find more examples or scenarios of this kind in the literature.

With regard to the research motivations pointed out above, in this thesis, we address the stability and control issues of two major classes of RJLSs:

- time-inhomogeneous MJLSs,

¹Also, for instance, in fault tolerant control literature, one usually considers the fault detection and identification (FDI) process as Markov process, where FDI process is modelled by employing memory-less statistical tests. However, in Appendix B, we obtain a non-exponential sojourn time distribution (Chitraganti et al., 2012) when the fault detection process is modelled by employing sequential change detection algorithms that consider the past observations also in deciding whether the fault has occurred or not.

- RJLSs, where the random jump process depends on the state variable.

1.3 General Mathematical Model of RJLSs

In this section, we provide general mathematical model of RJLSs that we deal within this thesis.

In this thesis, we consider a dynamical model of RJLSs

$$\begin{aligned} dx(t) &= A_{\theta(t)}x(t)dt + B_{\theta(t)}u(t)dt + E_{\theta(t)}dw(t), \\ dy(t) &= C_{\theta(t)}x(t)dt + F_{\theta(t)}dv(t), \end{aligned} \tag{1.1}$$

or

$$\begin{aligned} x_{k+1} &= A_{\theta_k}x_k + B_{\theta_k}u_k + E_{\theta_k}w_k, \\ y_k &= C_{\theta_k}x_k + F_{\theta_k}v_k, \end{aligned} \tag{1.2}$$

where $x(t)$ (or x_k) $\in \mathbb{R}^n$ is the state vector, $x(0)$ (or x_0) $\in \mathbb{R}^n$ is the initial state, $u(t)$ (or $u_k \in \mathbb{R}^k$) is the control, $y(t)$ (or $y_k \in \mathbb{R}^m$) is the measure variable, $w(t)$ and $v(t)$ are independent n -vector valued Wiener processes, $\{w_k\}$ and $\{v_k\}$ are sequences of independent n -vector random variables with zero mean and covariance matrix $\mathbb{I}_n$. Here $A_{\theta(t)}$ (or A_{θ_k}), $B_{\theta(t)}$ (or B_{θ_k}), $E_{\theta(t)}$ (or E_{θ_k}), $C_{\theta(t)}$ (or C_{θ_k}), $F_{\theta(t)}$ (or F_{θ_k}) be the system matrices of appropriate dimensions, which depend on $\theta(t)$ (or θ_k), and are assumed to be known. We usually assume the absence of noise, unless otherwise explicitly stated. Let $\{\theta(t), t \geq 0\}$ (or $\{\theta_k, k \in \mathbb{N}_{\geq 0}\}) \in S := \{1, 2, \dots, N\}$, describing the mode of the system at time t (or k), be a finite space random jump process.

Observe that the model (1.1) (or 1.2) has a multiple mode structure, which is evident by the dependence of all systems matrices on the mode $\theta(t)$ (or θ_k). The stochastic nature of $x(t)$ (or x_k) can be observed from its dependence on $\theta(t)$ (or θ_k), which is a stochastic process, and the noise terms.

Remark 1.1. If we notice the RJLS (1.1) (or 1.2), the actual state consists of two parts: the state $x(t)$ (or x_k) and the mode $\theta(t)$ (or θ_k). If the mode $\theta(t)$ (or θ_k) is fixed for all $t \geq 0$ (or $k \in \mathbb{N}_{\geq 0}$), then the RJLS (1.1) (or 1.2) is equivalent to the standard linear time-invariant system. Thus, the role of the mode $\theta(t)$ (or θ_k) is important in the RJLSs, and in general in any switching systems. With this background, we observe the mode dependence through out the thesis: in designing state-feedback gains, in handling constraints and in weighting matrices in the objective functions.

1.4 Literature Survey

We provide some important contributions in the field of RJLSs in the following.

The class of RJLSs was introduced by (Krasovskii and Lidskii, 1961) in 1961, where the authors studied an optimal control problem. In 1969, the random jump linear quadratic control problem was considered in (Sworder, 1969) for the finite-horizon setting via maximum principle approach. After two years, in 1971, Wonham (Wonham, 1970) extended the previous work by including the Gaussian noise in the state equation, and studied the jump linear quadratic optimization problem via dynamic programming approach. While the studies of optimal control problem were going on, the stability studies lacked the proper impetus to move forward, partly due to the inadequate choice of the notion of stability. It posed a great challenge and after a decade or so, the researchers proceeded in defining the notions of stability in stochastic sense, and obtained some necessary and/or sufficient conditions. The relevant results were summarized in (Mariton, 1990), which is a seminal book on RJLSs. After that a lot contributions took place in the RJLSs, and we try to mention a significant sample of the literature.

We begin with stability and stabilization.

Stability and Stabilization: The investigation of stability and stabilization is broadly divided into two categories: second moment stability and almost sure stability. In reality, almost sure stability criteria is more preferred than second moment stability because we observe only the sample paths of a dynamical system than its moment paths. Using Lyapunov exponent techniques (Feng, 1990; Arnold et al., 1986; Arnold, 1984; Leizarowitz, 1990), some sufficient and necessary conditions have been established to verify the almost sure stability, but are very difficult in terms of computational tractability. Though restrictive, second moment stability has been able to attract many researchers because of feasibility of obtaining numerically tractable conditions. However, it was shown (Kozin, 1969, 1965) under some assumptions that second moment stability implies almost sure stability, but the converse is not true. In this thesis, we address only second moment stability along with control aspects. However, without being exhaustive, we review the existing literature in a more general fashion in the following.

In the framework of RJLSs, there are many results on second moment stability and stabilization that make use of stochastic Lyapunov's approach to give necessary and sufficient conditions in terms of LMIs. On the other hand there are few results in terms of spectral radii of some suitably defined matrices. However, the use of LMI framework allows one to design various control strategies, which is the reason for the vast literature related to necessary and sufficient conditions of second moment stability of RJLSs, or for any jump linear systems for that matter. There have been many results on second moment stability and stabilization of RJLSs in the literature. For instance, the second moment stability

and stabilization for time-homogeneous RJLSs was addressed by (Costa and Fragoso, 1993; Feng et al., 1992; Fang, 1994; Ji et al., 1991); for time-inhomogeneous RJLSs by (Aberkane, 2011; Costa et al., 1999a; El Ghaoui and Rami, 1996; Fang and Loparo, 2002; Krtolica et al., 1994; De Farias et al., 2000; Chitraganti et al., 2013a; Zhang and Lam, 2010; Bolzern et al., 2010); for RJLSs, when the random jump process is dependent on state variable and/or control variable, by (Boukas and Liu, 2001; Filar and Haurie, 2001; Sworder and Robinson, 1973; Chitraganti et al., 2013b, 2014c); for RJLSs in the presence of uncertainties in the system parameters by (de Souza, 2006; Xiong et al., 2005; Li et al., 2009; Gao et al., 2007; El Ghaoui and Rami, 1996), for RJLSs with delay by (Cao and Lam, 1999; Xiao et al., 2000; Gao et al., 2001; Huang and Shi, 2013a,b; Shi and Yu, 2009; Xu and Lam, 2008).

Now, we briefly review different control aspects of RJLSs.

Control Aspects: On the control front of RJLSs, optimal control problem with linear quadratic cost, finite/infinite time horizon, were systematically addressed by (Krasovskii and Lidskii, 1961; Sworder, 1969; Wonham, 1970; Ji and Chizeck, 1990a; Ji et al., 1991; Chizeck et al., 1986; Griffiths and Loparo, 1985; Costa and Do Val, 1998). A receding horizon control of RJLSs with state and/or input constraints was examined in (Costa and do Val, 2000; Lu et al., 2013; Vargas et al., 2006; Wen, 2013; Park and Kwon, 2002); for unconstrained state and input constraints, optimality of the receding horizon control was addressed via dynamic programming (Vargas et al., 2005), variational methods (Vargas et al., 2004), solving Riccati equations (Do Val and Başar, 1999); in the presence of stochastic constraints, the receding horizon control was addressed by (Vargas et al., 2013; Chitraganti et al., 2014d). In the presence of additive disturbances, $\mathcal{H}_2$, $\mathcal{H}_\infty$, robust control problems were investigated by (Aberkane, 2009; Costa and Do Val, 1996; Costa and Marques, 1998; De Farias et al., 2002; Sun et al., 2006; Dong and Yang, 2008). A control synthesis by game theoretic techniques was studied by (Başar, 1994; Zhu et al., 2014).

Structural Properties: For RJLSs, structural properties such as controllability, detectability, and observability have been addressed by (Costa and Do Val, 2001, 2002; Ji and Chizeck, 1990a; Liu et al., 2014; Shen et al., 2013).

Filtering: The filtering problem has been analyzed by (Allam et al., 2001; Blom and Bar-Shalom, 1988; Dufour and Elliott, 1999; Fragoso et al., 2005; Mahmoud and Shi, 2003; Shi et al., 1999), and the references therein.

Now, we give some of the widely referred books in the literature of RJLSs for an interested reader.

Significant Books: For detailed review of RJLSs, one can refer to the books (Costa et al., 2005, 2013; Mariton, 1990; Boukas, 2006; Dragan et al., 2010) and the references

therein, which give in-depth details of the different issues of RJLSs reviewed above.

1.5 Our Approach and Contributions

In this section, we present our approach to various issues of RJLSs that we address in this thesis.

As given in Section 1.2, the class of RJLSs that we address in this thesis are

- time-inhomogeneous MJLSs,
- RJLSs, where the random jump process depends on the state variable.

We first deal with the stability and stabilization issues that form Part I of the thesis. The important feature of stability of RJLSs is: the stability of the individual subsystems of RJLS is neither a sufficient nor a necessary condition for overall RJLS stability (Costa et al., 2005, 2013). Almost all the stability conditions for RJLS involve the characterization of the underlying jump process also. We begin with the time-inhomogeneous MJLS.

In **chapter 2**, we address the second moment stability and stabilization of time-inhomogeneous MJLSs in a discrete-time setting; here time-inhomogeneity is associated to the time-varying nature of the transition probability matrix (TPM) of the underlying random jump process, which is a discrete-time Markov chain. From now onwards, we use the term *inhomogeneity* for time-inhomogeneity for simplicity. We consider the time-varying TPM as an interval matrix, where each time-varying probability of TPM is assumed to vary in an interval with maximum interval being $[0, 1]$. Interval TPM is a more general method to represent TPMs of inhomogeneous Markov chains, because, in general, the transition probabilities are estimated with errors, thus it can be expressed as an interval with lower and upper bounds (Hartfield, 1998). It is a natural assumption for time varying TPM, using which we can also represent the well known convex hull representation of TPM with known vertices, as given in chapter 2, but not vice versa (Škulj, 2009). By using the tools of interval analysis, Kronecker products and graph theory, we provide a sufficient condition for the second moment stability in terms of spectral radius of a suitably defined matrix. As a special case, we show that our sufficient condition boils down to the sufficient condition of (Costa, 2004; Fang and Loparo, 2002; Ling and Deng, 2012) for time-homogeneous Markov chain. Since it is difficult to synthesize a controller for state-feedback stabilization with the obtained sufficient condition, alternatively, by using a convex hull representation of the interval TPM, we obtain a sufficient condition in terms of linear matrix inequalities (LMIs), which facilitate us in synthesizing a stabilizing controller.

In **chapter 3**, we address the second moment stability and stabilization of RJLSs in a continuous-time setting, when the random jump process depends on the state variable.

Such a class of systems are referred to as “state-dependent jump linear systems (SDJLSs)” in this thesis. In SDJLSs, the transition rates of the underlying random jump process depends on the state variable. In our approach, we consider the state-dependent transition rates explicitly as: the transition rates vary depending on which set the state of the system belongs to. This is a reasonable assumption because the state of the system at any time belongs to one of the predefined sets, and the transition rates can be considered to have different values across the predefined sets. Under the above assumption, using Dynkin’s formula and some formulations of stopping time, we obtain numerically tractable sufficient conditions for stochastic stability and stabilization in terms of LMIs. As a special case, we show that our sufficient conditions of stochastic stability and state-feedback stabilization are equivalent to the sufficient conditions of (Boukas et al., 1999; Feng et al., 1992) for time-homogeneous Markov process. In practice, dynamical systems are subjected to external disturbances, and one wants to design a controller that attenuates the effect of external disturbances. $\mathcal{H}_\infty$ control has been a widely useful tool in attenuating the effect of external disturbances with finite energy that act on system dynamics (Boyd, 1994; Zhou and Doyle, 1998). Subsequently, we consider a SDJLS subject to external disturbances with finite energy, and synthesize a state-feedback $\mathcal{H}_\infty$ controller that guarantees disturbance rejection while ensuring stochastic stability. We illustrate the obtained results on a macroeconomics problem.

After the stability and stabilization, we move to control the RJLSs subject to input and/or state constraints in Part II of the thesis. In this part, we address only the discrete-time versions of above systems for simplicity.

In **chapter 4**, the system under consideration is a discrete-time SDJLS, affected by a possibly unbounded stochastic disturbance, with perfect state availability and constraints on state variable. The constraints on state variable are regarded probabilistically, due to the presence of unbounded noise in the system dynamics. In the presence of constraints, model predictive control (MPC) is an effective control algorithm that has a great potential to address problems across different disciplines. MPC (Camacho and Bordons, 2004; Maciejowski and Huzmezan, 1997; Mayne et al., 2000; Rawlings and Muske, 1993) is a form of control in which the current input is obtained by solving on-line, at each sampling time, a finite horizon optimal control problem in the presence of constraints, using the current information and the predicted information over the finite horizon. Normally more than one input is computed at the current sampling time, however, only the first controller input will be implemented to the plant. At the next sampling time, these actions will be repeated. Due to a nature of the dynamical system and the constraints, we consider a one-step receding horizon. In our approach, we pre-stabilize the system by a state-feedback controller in a sense of bounding the second moment of the state variable; the probabilistic constraints are converted to suitable deterministic constraints via

inverse cumulative distribution function; MPC scheme with the converted deterministic constraints, to obtain an optimal control, is presented subsequently.

In the second part of the control problem, to extend the work of chapter 4, we would like to consider the SDJLSs with imperfect state availability. However, the unavailability of the state variable makes the formulation of state-dependent jump process intractable. Thus we confine ourselves to the inhomogeneous MJLSs with process noise and noisy measurements.

In **chapter 5**, we address a MPC of discrete-time MJLSs with process noise, noisy measurements and probabilistic state constraints. We assume that the state of the underlying inhomogeneous random jump process is available at each time, which facilitates us to estimate the state, via Kalman filter resembled recursions. Also, we consider the probabilistic constraints on state variable. Similar to chapter 4, we consider a one-step receding horizon and pre-stabilize the system by a state-feedback controller, using the estimated state, in a sense of bounding the second moment of the state variable. The probabilistic constraints are converted to deterministic constraints in terms of conditional estimates and covariances of state variable as proposed in (Yan and Bitmead, 2005) and the overall MPC problem is converted as a tractable deterministic receding horizon control problem.

1.6 Thesis Organization

This thesis is broadly organized into two parts: stability and stabilization of RJLS, control subject to constraints.

Part I: Stability and stabilization of RJLSs

chapter 2 is concerned with second moment stability and stabilization of discrete-time RJLSs. In Section 2.2, we describe in detail the type of RJLS under consideration, which is a discrete-time MJLS with interval transition probability matrix. In Section 2.3, we obtain a sufficient condition for second moment stability in terms of a spectral radius of a suitably defined matrix using interval analysis and alternatively sufficient conditions for second moment stability and stabilization in terms of LMIs in Section 2.4. We give concluding remarks in Section 2.5.

chapter 3 is concerned with second moment stability and stabilization of continuous-time RJLSs. In Section 3.2, we give a detailed description of the RJLS under consideration, which is continuous-time SDJLS. We give numerically solvable sufficient conditions for stochastic stability and stabilization in terms of LMIs in Section 3.2.4 and Section 3.2.5 respectively. In Section 3.3, we consider the SDJLS subject to external disturbances with finite energy, and obtain numerically solvable sufficient conditions for $\mathcal{H}_\infty$ state-feedback stabilization are established in terms of LMIs. We illustrate the obtained results

on a macroeconomics problem in Section 3.3.5. We give some proofs in Section 3.4 to facilitate readability of the chapter, followed by concluding remarks in Section 3.5.

Part II: Control of RJLSs subject to constraints

chapter 4 is concerned with receding horizon control of discrete-time RJLSs subject to constraints on state variable. In Section 4.2, we give the problem setup by describing the RJLS under consideration, which is a discrete-time SDJLS, subject to possibly unbounded random disturbance and probabilistic state constraints. In Section 4.3, we give pre-stabilization results by enforcing the second moment of the state variable via solving LMIs on-line. We obtain a tractable representation of the receding horizon control problem in Section 4.4. We illustrate our approach using a numerical example in Section 4.5. We give proofs in Section 4.6 and the concluding remarks in Section 4.7.

chapter 5 is concerned with receding horizon control of discrete-time MJLSs with process noise, measurement noise and is subjected to probabilistic constraints on state variable. In Section 5.2, by incorporating state estimation, we follow the similar approach of chapter 4 to address the one-step receding horizon control problem, where we give pre-stabilization results in Section 5.3 and convert the entire problem as a tractable deterministic receding horizon control problem in Section 5.4. We give proofs in Section 5.6 and the concluding remarks in Section 5.7.

We give a summary of the thesis highlighting the main contributions of the thesis in chapter 6, which also contain some of the future directions of research. Appendix A gives a brief description of some of the preliminaries required in this thesis. Appendix B contains a detailed description of an auxiliary result in chapter 1. Appendix C contains a brief description of a mathematical model indicated in chapter 6.

Flowchart of the Thesis

We give a flowchart of the thesis in Figure 1.2, which portrays a clear organization of the thesis. The main purpose of this flow chart is to indicate that the two parts of the thesis can be studied independently starting from any one of them. For a given part, it also gives the connections among the chapters.

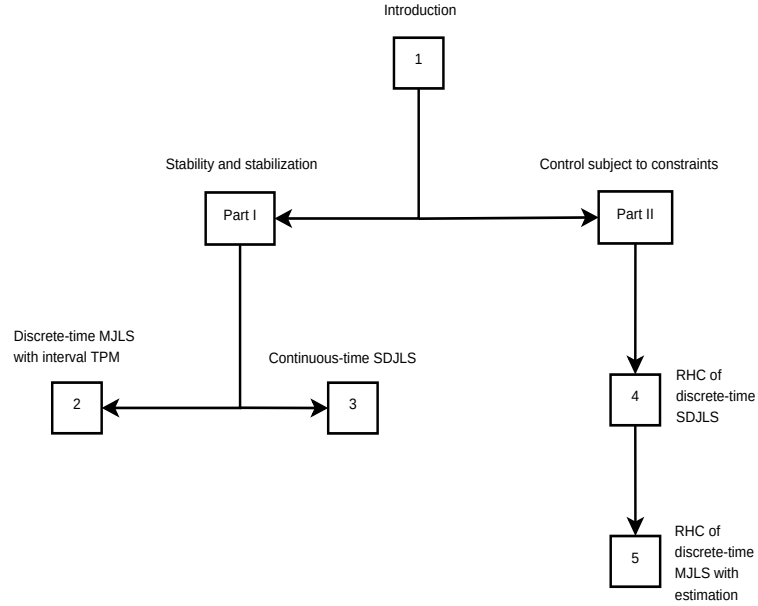


Figure 1.2: A flowchart of the thesis

1.7 Notations and Abbreviations

In this section, we give notations and abbreviations used in this thesis, which are fairly standard.

Notation	Meaning
$\mathbb{R}^n$	n-dimensional real Euclidean space
A^T	The transpose of a matrix A
$\lambda_{\min}(A)$ (or $\lambda_{\max}(A)$)	The minimum (or maximum) eigenvalue of the matrix A
$\rho(A)$	The spectral radius of the matrix A
$\text{tr}(A)$	The trace of the matrix A
$\text{He}\{A\}$	$A + A^T$
$L \succ M$ (or $L \prec M$)	The matrix $L - M$ is positive definite (or negative definite)
$L \succeq M$ (or $L \preceq M$)	The matrix $L - M$ is positive semi-definite (or negative semi-definite)
$L \geq_e M$ (or $L \leq_e M$)	The element-wise inequalities of matrices L, M . Also it is denoted that $L - M$ is non-negative (or negative) matrix.
$\ \cdot\ $	The standard vector norm in $\mathbb{R}^n$
$\ A\ $	An induced norm of the matrix A
$*$	To denote symmetric terms in block matrices
$AP\star$	APA^T
$x\star$	xx^T
$\mathbb{O}$	Zero matrix or a zero vector of appropriate dimension
$\mathbb{I}_n$	The identity matrix of dimension $n \times n$
ϕ	The empty set
$\text{diag}\{G(i)_N\}$ or $\text{diag}\{G(1), \dots, G(N)\}$	The diagonal matrix formed from arguments $G(1), \dots, G(N)$
$\mathbb{Q}$	The set of rational numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_{\geq 0}$	The set of natural numbers including 0
$\cup$ (or $\cap$)	Union (or intersection)
$A \setminus B$	$\{x \in A x \notin B\}$
$a \wedge b$	The minimum of two real numbers a, b

$I_A(x)$	An indicator function which has a value 1 if $x \in A$, otherwise has a value 0
I_A	An indicator function which has a value 1 if A is true otherwise has a value 0
$\mathbb{E}[g(X_t)]_{X_t=X}$	The expectation of the functional $g(X_t)$ at $X_t = X$
RJLS	Random jump linear system
MJLS	Markov jump linear system
SDJLS	State-dependent jump linear system
TPM	Transition probability matrix
SS	Stochastic stability
MSS	Mean square stability
LMI	Linear matrix inequality
MPC	Model predictive control
RHC	Receding horizon control
VTOL	Vertical take off and landing

Part I

Stability and Stabilization of RJLSs

Chapter 2

Second Moment Stability and Stabilization of Discrete-time Inhomogeneous Markov Jump Linear Systems having Interval Transition Probability Matrix

2.1 Introduction

MJLSs are an important class of RJLSs that consist of a set of linear systems and the switching among them are governed by a finite state Markov chain (or a finite state Markov process). In Section 1.1, we gave research motivations for the study of MJLSs with several examples, and in Section 1.2 for the study of time-inhomogeneous MJLSs. In the sequel, we present some of the related works on second moment stability of time-inhomogeneous MJLSs in the literature.

For time-inhomogeneous (or simply *inhomogeneous*) MJLSs, the investigation of exponential mean square stability using Lyapunov's second method was carried out in (Krtolica et al., 1994). A sufficient condition for the mean square stability using Kronecker products was obtained in (Fang and Loparo, 2002); in (Ling and Deng, 2012) it was also proved to be necessary. When the variation of underlying transition probability matrix (TPM) (or rate matrix) is confined to vary in a convex hull with known vertices, second moment stability was addressed in (Aberkane, 2011; Costa et al., 1999a; El Ghaoui and Rami, 1996). The other kind of description is in an element-wise description of TPM (or switching rate matrix). In this situation, the second moment stability was addressed with a given transition rate matrix along with its error bounds of each elements (Boukas et al., 1999; Xiong et al., 2005).

In this chapter, we consider second moment stability and stabilization of a discrete-time MJLS with a finite-state inhomogeneous Markov chain, which implies to contain a time variant TPM. Our contribution in this chapter is to consider the variation of TPM in an interval, which means the time-varying nature of TPM is captured as an interval TPM. Interval TPM is a more general method to represent TPMs of inhomogeneous Markov chains because the estimated transition probabilities with errors can be expressed as an interval with lower and upper bounds (Hartfield, 1998; Škulj, 2006). With a given interval TPM, we can also obtain the well known convex hull representation of TPM (Hartfield, 1998) with known vertices, as given in Section 2.2 and Section 2.4. However, given a convex hull representation, one may not always obtain an interval TPM (Škulj, 2009). Also, for instance, in the theory of Markov chains, a basic characteristic is the ability to calculate the probability of being in some state at some time given an initial probability. In this case, with interval TPM representation, the computational complexity is much lesser compared to convex hull representation (Hartfield, 1998), where the author used the term *Markov set-chains* to denote the sets of products of interval TPMs.

Thus, we consider an interval TPM and by using the results of interval analysis, we provide sufficient conditions for second moment stability, and also, by obtaining a convex hull representation of the interval TPM, we give sufficient conditions for stabilization of the system.

The rest of the chapter is organized as follows: Section 2.2 gives a description of a discrete-time MJLS with interval TPM. In Section 2.3, we obtain a sufficient condition for the MSS. In Section 2.4, we give sufficient conditions for the SS and stochastic stabilization, and Section 2.5 concludes the chapter.

2.2 Model Description

Consider a discrete-time MJLS:

$$x_{k+1} = A_{\theta_k} x_k, \quad k \in \mathbb{N}_{\geq 0}, \quad (2.1)$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $x_0 \in \mathbb{R}^n$ is the initial state, θ_k is the system mode, $A_{\theta_k} \in \mathbb{R}^{n \times n}$ be the system matrices that depend on θ_k , and are assumed to be known. The mode $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is described by a discrete-time Markov chain with state space $S = \{1, 2, \dots, N\}$ and TPM $\Pi_k \triangleq (\pi_{ij}(k))_{N \times N}$,

$$\Pr \{\theta_{k+1} = j | \theta_k = i, k\} = \pi_{ij}(k), \quad (2.2)$$

with $\pi_{ij}(k) \geq 0$ and $\sum_{j=1}^N \pi_{ij}(k) = 1, \forall k \geq 0$. Since Π_k is dependent on k , $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is called as an inhomogeneous Markov chain. Let $\pi_0 = (\pi_i(0))_{1 \times N}$ be the initial probability mass function of $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$, where $\pi_i(0) = \Pr\{\theta_0 = i\}$, $i = 1, \dots, N$.

2.2.1 A Related Background on Second Moment Stability

In this section, we briefly review the existing related results on second moment stability of the system (2.1) in general. Similar to the description in Section 1.5, for (2.1), the individual mode system's stability is neither sufficient nor necessary for the stability of the over all system (Fang, 1994). It implies that the system (2.1) can be stable even though the individual mode systems are not stable.

Before proceeding, we give definitions of various second moment stability concepts of discrete-time MJLS (2.1), which are also applicable to discrete-time RJLSs in general.

Definition 2.1. (Fang, 1994) For the system (2.1), the equilibrium point 0 is

- *stochastically stable* (SS), if for any initial state $x_0 \in \mathbb{R}^n$ and initial probability distribution of θ_k ,

$$\sum_{k=0}^{+\infty} \mathbb{E} [\|x_k\|^2] < +\infty, \quad (2.3)$$

where x_k is a sample solution of (2.1) initial from x_0 .

- *mean square stable* (MSS), if for any initial state $x_0 \in \mathbb{R}^n$ and initial probability distribution of θ_k ,

$$\lim_{k \rightarrow +\infty} \mathbb{E} [\|x_k\|^2] = 0, \quad (2.4)$$

- *exponentially mean square stable* (EMSS), if for any initial state $x_0 \in \mathbb{R}^n$ and initial probability distribution of θ_k , there exist constant $0 \leq \alpha \leq 1$ and $\beta > 0$ such that for all $k \geq 0$,

$$\mathbb{E} [\|x_k\|^2] \leq \beta \alpha^k \|x_0\|^2. \quad (2.5)$$

Consider the system (2.1) when $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is a finite-state time-homogeneous Markov chain with TPM $\Pi \triangleq (\pi_{ij})_{N \times N}$. For this case, in (Ji et al., 1991), the authors proved that all the second moment stabilities are equivalent. Also, a necessary and sufficient condition for second moment stability is given by the following theorem.

Theorem 2.2. (Ji et al., 1991) Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is a finite-state time-homogeneous Markov chain with TPM $\Pi \triangleq (\pi_{ij})_{N \times N}$, then the system (2.1) is second moment stable if and only if there exist positive definite matrices $P_1, \dots, P_N$ such that

$$A_i^T \left(\sum_{j=1}^N \pi_{ij} P_j \right) A_i - P_i \prec 0,$$

for all $i = 1, \dots, N$.

Observe that the condition in theorem 2.2 is computationally demanding as it involves solving a set of LMIs. A more computationally efficient condition is given by the following theorem.

Theorem 2.3. (Bhaurucha, 1961; Fang, 1994) Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is a finite-state time-homogeneous Markov chain with TPM $\Pi \triangleq (\pi_{ij})_{N \times N}$, then the system (2.1) is second moment stable if $\rho(\mathcal{A}) < 1$ where $\mathcal{A}$ is given by

$$\mathcal{A} = (\Pi^T \otimes \mathbb{I}_{n^2}) \text{diag} \{(A_i \otimes A_i)_N\}.$$

Here $\otimes$ denotes the Kronecker product¹. Observe that the above condition is computationally better than theorem 2.2. In (Fang and Loparo, 2002), the authors conjectured that the condition in theorem 2.3 is also necessary, which was subsequently proved in (Costa, 2004; Ling and Deng, 2012).

Now consider a case of discrete-time inhomogeneous MJLSs with the time varying TPM $\Pi_k \triangleq (\pi_{ij}(k))_{N \times N}$. In this case, a necessary and sufficient condition for second moment stability was obtained (Krtolica et al., 1994) in terms of LMIs similar to theorem 2.2, that is computationally intractable. Also, a sufficient condition similar to theorem 2.3 is given by

Theorem 2.4. (Fang and Loparo, 2002) Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is a finite-state inhomogeneous Markov chain with TPM $\Pi_k \triangleq (\pi_{ij}(k))_{N \times N}$ (2.2), then the system (2.1) is second moment stable if the deterministic product sequence $\{\mathcal{A}_1 \mathcal{A}_2 \cdots \mathcal{A}_k\}$ is exponentially convergent to the zero matrix, where $\mathcal{A}_k$ is given by

$$\mathcal{A}_k = (\Pi_{k-1}^T \otimes \mathbb{I}_{n^2}) \text{diag} \{(A_i \otimes A_i)_N\}.$$

Notice that the sufficient condition in theorem 2.4 is also computationally intractable as one has to verify the product sequence convergence to the zero matrix.

In the sequel, we present the time-varying TPM Π_k as an interval matrix and first obtain a sufficient condition of MSS in terms of a spectral radius of a deterministic matrix that looks similar in spirit to $\mathcal{A}_k$ in theorem 2.4. Then, in the subsequent section, we obtain an alternative sufficient condition of SS, which is used for stabilization in terms of LMIs that resemble LMIs given in theorem 2.2.

2.2.2 Characterization of TPM as an Interval Matrix

We represent the time-varying TPM Π_k as an interval matrix in the following.

Let $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$, a finite-state inhomogeneous Markov chain (2.2) with TPM Π_k , assumed to be varying in an interval. Broadly, it means that, at each time $k \in \mathbb{N}_{\geq 0}$, the transition probabilities of the TPM Π_k vary, but the variation is confined to a predefined interval.

Let $P \in \mathbb{R}^{N \times N}$ and $Q \in \mathbb{R}^{N \times N}$ be two non-negative matrices with $P \leq_e Q$. With a slight abuse of notation, we represent Π_k that is varying in an interval $[P, Q]$ as

$$\Pi_k = [P, Q] \in \mathbb{IR}^{N \times N}, \tag{2.6}$$

¹Please refer to Appendix A for description

where it implies that Π_k is $N \times N$ stochastic matrix at each $k \in \mathbb{N}_{\geq 0}$ and $P \leq_e \Pi_k \leq_e Q$ for all $k \in \mathbb{N}_{\geq 0}$. We see in the coming sections that the notation (2.6) will lead to a better handling of the mathematical intricacies. The notation $\mathbb{IR}^{NN}$ denotes the space of interval matrices with dimension $N \times N$, which is described in detail in Section 2.3.1. Assume that P and Q are such that $[P, Q] \neq \emptyset$. Here the matrices P and Q need not be stochastic matrices.

The important feature of the interval $[P, Q]$ is its tightness. If $[P, Q]$ is not tight, then Π_k may not be stochastic matrix for all $k \geq 0$ (Hartfield, 1998; Samuels, 2001). For the interval, $[P, Q]$ (2.6), its tightness can be verified by (Hartfield, 1998)

$$P_{ij} + \sum_{t \neq j} Q_{it} \geq 1 \text{ and } Q_{ij} + \sum_{t \neq j} P_{it} \leq 1,$$

for each $i \in S$,

where P_{ij} , Q_{ij} denote the elements of P , Q at position (i, j) respectively.

In this chapter, we assume that all the intervals $[P, Q]$ in (2.6) are tight. It is particularly useful, because, given a tight interval $[P, Q]$, one can always obtain its convex polytope representation. Also, even if the interval TPM is not tight, one can always tighten it; a more detailed exposition can be found in (Hartfield, 1998; Samuels, 2001).

Remark 2.5. As an illustration of time-varying nature of Π_k , given $\Pi_k = [P, Q]$, a convex polytope in which Π_k is varying can be obtained in the following examples.

Example 2.6. Let

$$\Pi_k = \begin{bmatrix} [0.2, 0.4] & [0.6, 0.8] \\ 0.5 & 0.5 \end{bmatrix} \in \mathbb{IR}^{22}, \quad (2.7)$$

then Π_k is varying in a convex polytope with vertices $\begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$ and $\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$ as partly observed in Figure 2.1, which just demonstrates the variation of the first row of Π_k . In this figure, the blue colored line gives the range of values the first row of Π_k can span in general, where $[e_1 \ e_2]$ denotes the first row elements of Π_k . Note that $e_1 + e_2 = 1$ (the blue colored line), since Π_k is a stochastic matrix. In the same figure, the intersection of the red colored box (that denotes the range of first row of (2.7)) with the blue colored line gives the effective range the first row of Π_k (2.7) can span.

Example 2.7. p.41 (Hartfield, 1998) Let

$$\Pi_k = \begin{bmatrix} [0.35, 0.45] & [0.55, 0.65] \\ [0.25, 0.35] & [0.65, 0.75] \end{bmatrix} \in \mathbb{IR}^{22},$$

then Π_k is varying in a convex polytope with vertices

$$\begin{bmatrix} 0.35 & 0.65 \\ 0.25 & 0.75 \end{bmatrix}, \begin{bmatrix} 0.35 & 0.65 \\ 0.35 & 0.65 \end{bmatrix}, \begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix} \text{ and } \begin{bmatrix} 0.45 & 0.55 \\ 0.35 & 0.65 \end{bmatrix}.$$

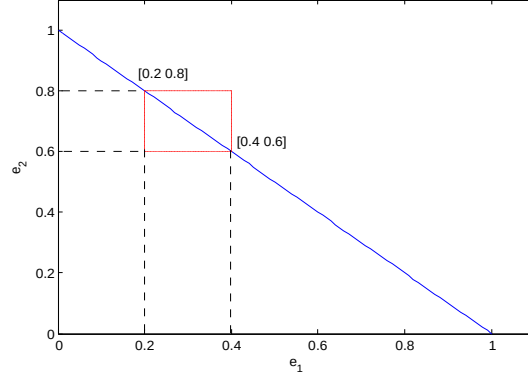


Figure 2.1: Variation of first row $([e_1 \ e_2])$ of Π_k (2.7)

Notice that the examples 2.6 and 2.7 deal only with small dimension of Π_k . One can infer that as the dimension of Π_k increases, the number of vertices of convex polytope also increases. Thus, in this scenario, obtaining vertices of convex polytope from a stochastic interval matrix is a cumbersome procedure. Interested reader can find a detailed investigation of obtaining vertices of polytopes of interval stochastic matrices in (Hartfield, 1998), which is also utilized in Section 2.4. In this section, we confine ourselves to dealing with interval matrices Π_k (2.6).

In the next section, we give a sufficient condition for the MSS of (2.1) using interval analysis.

2.3 Mean Square Stability of Discrete-time Inhomogeneous MJLS with Interval TPM Using Interval Analysis

In this section, we obtain a sufficient condition for the MSS of the system (2.1) in terms of spectral radius of deterministic matrix by using the interval analysis. Before proceeding, we briefly review some results on interval analysis that we use in this chapter.

2.3.1 Brief Review on Interval Analysis

We briefly provide a review of interval analysis adopted from (Alefeld and Herzberger, 1983), (Moore, 1966) that will be used in the subsequent sections. An interested reader can find a detailed exposition of interval analysis in the references listed.

We consider only real compact intervals, which are denoted by $[\underline{a}, \bar{a}]$, $\underline{a} \leq \bar{a}$, where $\underline{a}, \bar{a} \in \mathbb{R}$. We call $[\underline{a}, \bar{a}]$ a *non-degenerate interval* if $\underline{a} \neq \bar{a}$. Denote the set of all these intervals by $\mathbb{IR}$. Matrices with entries belonging to $\mathbb{IR}$ are denoted by $\mathcal{A}, \mathcal{B}, \dots$ or by $(\mathcal{A}_{ij}), (\mathcal{B}_{ij}), \dots$; $\mathcal{A}, \mathcal{B}, \dots$ are called interval matrices; the set of all $n \times m$ interval matrices are denoted by $\mathbb{IR}^{nm}$.

Denote $|\underline{a}, \bar{a}| \triangleq \max \{|\underline{a}|, |\bar{a}|\}$, and $|\mathcal{A}| \triangleq |\mathcal{A}_{ij}|$ for each i, j . Denote $d([\underline{a}, \bar{a}]) \triangleq \bar{a} - \underline{a}$, and $d(\mathcal{A}) \triangleq d(\mathcal{A}_{ij})$ for each i, j .

For $\mathcal{A} = (\mathcal{A}_{ij})$, $\mathcal{B} = (\mathcal{B}_{ij})$, the matrix operations $+$, $-$, $\cdot$ are defined as

$$\begin{aligned} \mathcal{A} \pm \mathcal{B} &: (\mathcal{A}_{ij} \pm \mathcal{B}_{ij}), \\ \mathcal{A} \cdot \mathcal{B} &: \left(\sum_s \mathcal{A}_{is} \mathcal{B}_{sj} \right). \end{aligned}$$

For $a \in \mathbb{R}$, $a = [a, a] \in \mathbb{IR}$; for $L \in \mathbb{R}^{n \times m}$, $L = [L, L] \in \mathbb{IR}^{nm}$. These particular intervals and matrices are called point intervals (or degenerate intervals) and point matrices respectively. Point matrices are denoted by $\dot{\mathcal{A}}$, $\dot{\mathcal{B}}$, ..., using a dot above the character. We denote the zero point matrix by $\mathbb{O}$. For a given interval matrix $\mathcal{A} \in \mathbb{IR}^{nn}$, we provide the following definitions.

Definition 2.8. Let $\mathcal{A} \in \mathbb{IR}^{nn}$, then the powers of $\mathcal{A}$ are defined by

$$\mathcal{A}^0 := \mathbb{I}_n; \quad \mathcal{A}^k := \mathcal{A}^{k-1} \cdot \mathcal{A}, \quad k = 1, 2, \dots$$

Definition 2.9. Let $\mathcal{A}^k = (\mathcal{A}_{ij}^{(k)}) = ([\underline{a}_{ij}^{(k)}, \bar{a}_{ij}^{(k)}])$, we say that $\mathcal{A}$ is convergent (to zero) iff the sequence $\{\mathcal{A}^k\}$ converges to $\mathbb{O}$, i.e; $\lim_{k \rightarrow \infty} \underline{a}_{ij}^{(k)} = 0$, $\lim_{k \rightarrow \infty} \bar{a}_{ij}^{(k)} = 0$, $i, j = 1, 2, \dots, n$. If $\mathcal{A}$ is not convergent, we call it divergent.

Definition 2.10. The directed graph $G(\mathcal{A})$ associated with $\mathcal{A} \in \mathbb{IR}^{nn}$ is the directed graph $G(|\mathcal{A}|)$ of a real matrix $|\mathcal{A}|$. Let a vertex set of $G(\mathcal{A})$ be $\{P_1, \dots, P_n\}$. Any two vertices P_i, P_j are connected by a direct path $\Gamma_{ij} \triangleq \overrightarrow{P_i P_j}$ iff the corresponding interval matrix entry $\mathcal{A}_{ij}$ is not zero.

Definition 2.11. The j^{th} column of $\mathcal{A}$ has $(*)$ property iff there exists a power $\mathcal{A}^m$, for some integer valued $m \geq 0$, which contains in the same j^{th} column of $\mathcal{A}^m$ at least one interval not degenerated to a point interval.

An example is presented here to illustrate the $(*)$ property among the columns of $\mathcal{A}$.

Example 2.12. p.206 (Mayer, 1984) Let

$$\mathcal{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & [0, 0.5] & 0 \end{bmatrix}, \quad (2.8)$$

then

$$\mathcal{A}^m = 2^{m-4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [-1, 1] & [-1, 1] & 0 \end{bmatrix}, \quad m \geq 3. \quad (2.9)$$

From (2.8), it is straightforward to say that the second column of $\mathcal{A}$ has the $(*)$ property because it contains a non-degenerate interval $[0, 0.5]$. But the other columns of $\mathcal{A}$ may

have the $(*)$ property. To verify the $(*)$ property for other columns, consider (2.9). From (2.9), we can conclude that first and second columns of $\mathcal{A}$ has the $(*)$ property because $\mathcal{A}^m, m \geq 3$ contains non-degenerate intervals $[-1, 1], [-1, 1]$. The procedure carried out in the above example is quite naive and cumbersome to find the $(*)$ property among the columns of an interval matrix $\mathcal{A}$. To know whether the j^{th} column of $\mathcal{A}$ has the $(*)$ property and its further relevance, the following lemmas are considered from (Mayer, 1984).

Lemma 2.13. *Each column of an irreducible interval matrix $\mathcal{A} \in \mathbb{IR}^{nn}$ has the $(*)$ property provided $d(\mathcal{A}) \neq \mathbb{O}$.*

Lemma 2.14. *The j^{th} column of $\mathcal{A} \in \mathbb{IR}^{nn}$ has property $(*)$ iff there exists a directed path $\Gamma_{ij} = \overrightarrow{P_i P_{i_1}}, \overrightarrow{P_{i_1} P_{i_2}}, \dots, \overrightarrow{P_{i_k} P_j}$, for some $1 \leq i \leq n$, in $G(\mathcal{A})$ which contains at least one non-degenerate interval.*

Lemma 2.15. *Let $\mathcal{A} \in \mathbb{IR}^{nn}$. Construct a point matrix $\dot{\mathcal{B}} = (b_{ij})$ by*

$$b_{ij} = \begin{cases} |\mathcal{A}_{ij}|, & \text{if the } j^{\text{th}} \text{ column of } \mathcal{A} \text{ has the } (*) \text{ property,} \\ \mathcal{A}_{ij}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}$ is convergent iff $\rho(\dot{\mathcal{B}}) < 1$.

The following lemma from graph theory will also be used in the sequel.

Lemma 2.16. (Seber, 2008) *A square matrix L is reducible if for all permutation matrices² P ,*

$$P^T L P = \begin{bmatrix} X & Y \\ \mathbb{O} & Z \end{bmatrix},$$

where X, Z are square matrices of order at least 1. A matrix is said to be irreducible if it is not reducible. Equivalently, a square matrix L is irreducible if and only if its directed graph $G(L)$ is strongly connected, which in other words, L is irreducible if and only if for each pair of indices (i, j) there is a sequence of entries in L such that $L_{ik_1} L_{k_1 k_2} \dots L_{k_t j} \neq 0$.

From lemma 2.16, we can say that the interval matrix $\mathcal{A}$ is irreducible iff either its directed graph $G(\mathcal{A})$ is strongly connected or $|\mathcal{A}|$ is irreducible.

In the next section, we present the main results based on the review of interval analysis given above.

²Please refer to Appendix A for description

2.3.2 A Sufficient Condition for MSS Using Interval Analysis

In this section, we obtain a sufficient condition for the MSS of system (2.1) in terms of a spectral radius of a deterministic matrix by using interval analysis. Before proceeding, we give some expressions to be used in the sequel.

From (2.6), $\Pi_k = [P, Q]$, and $\Pi_k^T = [P^T, Q^T]$. Thus

$$\Pi_k^T \otimes \mathbb{I}_{n^2} = [P^T \otimes \mathbb{I}_{n^2}, Q^T \otimes \mathbb{I}_{n^2}].$$

Let $G(i) = A_i \otimes A_i$ and consider

$$(\Pi_k^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\} = [\underline{M}, \overline{M}] \triangleq \mathcal{A}, \quad (2.10)$$

where

$$\begin{aligned} (\underline{M}_{ij}) &= \min \left\{ \sum_{l=1}^N \left((P^T \otimes \mathbb{I}_{n^2})_{il} \text{diag}\{G(i)_N\}_{lj} \right), \sum_{l=1}^N \left((Q^T \otimes \mathbb{I}_{n^2})_{il} \text{diag}\{G(i)_N\}_{lj} \right) \right\}, \\ (\overline{M}_{ij}) &= \max \left\{ \sum_{l=1}^N \left((P^T \otimes \mathbb{I}_{n^2})_{il} \text{diag}\{G(i)_N\}_{lj} \right), \sum_{l=1}^N \left((Q^T \otimes \mathbb{I}_{n^2})_{il} \text{diag}\{G(i)_N\}_{lj} \right) \right\}. \end{aligned}$$

The obtained interval matrix $\mathcal{A}$ in (2.10) is crucial in characterizing the results in this section.

Now consider the system (2.1), for which the solutions can be represented by

$$x_k = \Phi(k, 0)x_0 \triangleq \begin{cases} A_{\theta_{k-1}} \cdots A_{\theta_0} x_0 & \text{if } k \geq 1, \\ \mathbb{I}_n x_0 & \text{if } k = 0. \end{cases} \quad (2.11)$$

Thus

$$\begin{aligned} x_k x_k^T &= \Phi(k, 0) x_0 x_0^T \Phi^T(k, 0), \\ \text{vec}(x_k x_k^T) &= [\Phi(k, 0) \otimes \Phi(k, 0)] \text{vec}(x_0 x_0^T), \\ \mathbb{E}[\text{vec}(x_k x_k^T)] &= \mathbb{E}[\Phi(k, 0) \otimes \Phi(k, 0)] \text{vec}(x_0 x_0^T), \end{aligned} \quad (2.12)$$

where $\text{vec}(\cdot)$ denotes the vector formed from its arguments³. Denote

$$\Phi_i(k) \triangleq \Pr\{\theta_k = i\} \mathbb{E}[\Phi(k, 0) \otimes \Phi(k, 0) | \theta_k = i], \quad (2.13)$$

then

$$\begin{aligned} \Phi_i(0) &= \pi_i(0) \mathbb{I}_{n^2}, \\ \sum_{i=1}^N \Phi_i(k) &= \sum_{i=1}^N \Pr\{\theta_k = i\} \mathbb{E}[\Phi(k, 0) \otimes \Phi(k, 0) | \theta_k = i], \\ &= \mathbb{E}[\Phi(k, 0) \otimes \Phi(k, 0)]. \end{aligned} \quad (2.14)$$

The following lemma will be used in the sequel.

³Please refer to Appendix A for description

Lemma 2.17. Consider the interval matrix $\mathcal{A}$ (2.10). Then $V_\Phi(k) = \mathcal{A}^k V_\Phi(0)$ where $V_\Phi(k) \triangleq \begin{bmatrix} \Phi_1(k) & \cdots & \Phi_N(k) \end{bmatrix}^T$.

Proof. A sketch of the proof is as follows. First we obtain a recursive relation for $V_\Phi(k)$, then using the expression (2.10) we obtain the desired result.

From (2.2), we can denote $\Pi_k = (\pi_{ij}(k))_{N \times N}$. From (2.13),

$$\begin{aligned} \Phi_i(k) &= \Pr\{\theta_k = i\} \mathbb{E}[\Phi(k, 0) \otimes \Phi(k, 0) | \theta_k = i], \\ &= \Pr\{\theta_k = i\} \mathbb{E}\left[\left(A_{\theta_{k-1}} \otimes A_{\theta_{k-1}}\right) (\Phi(k-1, 0) \otimes \Phi(k-1, 0)) | \theta_k = i\right], \\ &= \Pr\{\theta_k = i\} \sum_{j=1}^N \Pr\{\theta_{k-1} = j | \theta_k = i\} (A_j \otimes A_j) \\ &\quad \mathbb{E}[(\Phi(k-1, 0) \otimes \Phi(k-1, 0)) | \theta_k = i, \theta_{k-1} = j], \\ &= \sum_{j=1}^N \pi_{ji}(k) (A_j \otimes A_j) \Phi_j(k-1). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} V_\Phi(k) &= \left(\Pi_{k-1}^T \otimes \mathbb{I}_{n^2}\right) \text{diag}\{(A_i \otimes A_i)_N\} V_\Phi(k-1), \\ &\triangleq \mathcal{A}_k V_\Phi(k-1), \end{aligned}$$

where $\mathcal{A}_k = \left(\Pi_{k-1}^T \otimes \mathbb{I}_{n^2}\right) \text{diag}\{G(i)_N\}$ and it is known that $\text{diag}\{G(i)_N\} = \text{diag}\{(A_i \otimes A_i)_N\}$. Recursively we obtain

$$V_\Phi(k) = \mathcal{A}_k \cdots \mathcal{A}_1 V_\Phi(0).$$

Since $\Pi_k = [P, Q]$, for each $k \in \mathbb{N}_{\geq 0}$, $\mathcal{A}_k = \mathcal{A}$ from (2.10). Thus, we obtain

$$V_\Phi(k) = \mathcal{A}^k V_\Phi(0),$$

which completes the proof. $\square$

To investigate further, let us consider two cases depending on the reducibility of the interval matrix $\mathcal{A}$ (2.10).

Case I – $\mathcal{A}$ is irreducible and $d(\mathcal{A}) \neq \mathbb{O}$:

We give a sufficient condition for the MSS of system (2.1) for case I.

Theorem 2.18. Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is given by (2.2) and (2.6), the interval matrix $\mathcal{A}$ by (2.10). Assume that $\mathcal{A}$ is irreducible and $d(\mathcal{A}) \neq \mathbb{O}$, then the system (2.1) is MSS if $\rho(|\mathcal{A}|) < 1$.

Proof. Since we assume that the interval matrix $\mathcal{A}$ given by (2.10) is irreducible and $d(\mathcal{A}) \neq \mathbb{O}$, every column of $\mathcal{A}$ has the (*) property from lemma 2.13. Thus from lemma 2.15, $\dot{\mathcal{B}} = |\mathcal{A}|$, and the convergence of $\{\mathcal{A}^k\}$ to $\mathbb{O}$ is guaranteed if and only if $\rho(|\mathcal{A}|) < 1$. To arrive at the desired result, we prove in the sequel that the convergence of $\{\mathcal{A}^k\}$ to

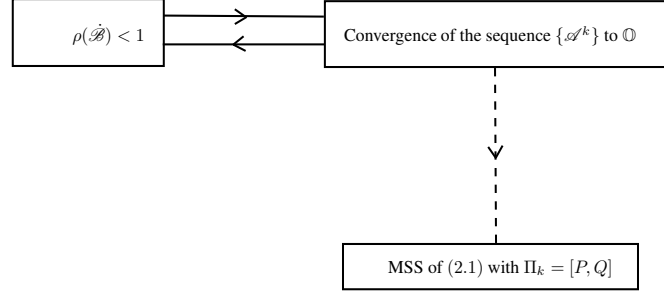


Figure 2.2: Illustration of theorem 2.18, where $\dot{\mathcal{B}} = |\mathcal{A}|$ if $\mathcal{A}$ is irreducible and $d(\mathcal{A}) \neq \mathbb{O}$

$\mathbb{O}$ implies the MSS of (2.1). The approach of the proof is shown in Figure 2.2, where we prove in the sequel the specific part with dashed lines. From (2.12), (2.14) and by lemma 2.17,

$$\begin{aligned}
 \mathbb{E} [\text{vec}(x_k x_k^T)] &= \mathbb{E} [\Phi(k, 0) \otimes \Phi(k, 0)] \text{vec} (x_0 x_0^T), \\
 &= \sum_{i=1}^N \Phi_i(k) \text{vec} (x_0 x_0^T), \\
 &= [\mathbb{I}_{n^2} \cdots \mathbb{I}_{n^2}] V_{\Phi}(k) \text{vec} (x_0 x_0^T), \\
 &= [\mathbb{I}_{n^2} \cdots \mathbb{I}_{n^2}] \mathcal{A}^k V_{\Phi}(0) \text{vec} (x_0 x_0^T).
 \end{aligned} \tag{2.15}$$

Also, it is apparent that

$$\mathbb{E} [\|x_k\|^2] = \mathbb{E} [x_k^T x_k] = \text{tr} (\mathbb{E} [x_k x_k^T]). \tag{2.16}$$

Thus from (2.15) and (2.16), the convergence of $\{\mathcal{A}^k\}$ to $\mathbb{O}$ implies that the system (2.1) is MSS. $\square$

Remark 2.19. If $\text{diag}\{G(i)_N\}$ is a non-negative matrix, then a simpler condition can be obtained for the MSS of (2.1). Thus, if $\text{diag}\{G(i)_N\} \geq_e 0$,

$$\begin{aligned}
 \mathcal{A} &= (\Pi_k^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\}, \\
 &= \left[(P^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\}, (Q^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\} \right].
 \end{aligned}$$

Hence, $|\mathcal{A}| = (Q^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\}$. In this case, theorem 2.18 can be simplified as follows.

Corollary 2.20. *Given the assumptions of theorem 1, and if $\text{diag}\{G(i)_N\} \geq_e 0$, then the system (2.1) is MSS if $\rho((Q^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\}) < 1$.*

Now consider the second case when $\mathcal{A}$ (2.10) is a reducible matrix.

Case II – $\mathcal{A}$ is reducible:

We give a sufficient condition for the MSS of (2.1) for case II in the following.

Theorem 2.21. Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is given by (2.2) and (2.6), the interval matrix $\mathcal{A}$ by (2.10). Assume that $\mathcal{A}$ is reducible, then the system (2.1) is MSS if $\rho(\dot{\mathcal{B}}) < 1$.

Proof. The proof is similar to theorem 2.18, hence omitted here. $\square$

Remark 2.22. Also in this case, if $\text{diag}\{G(i)_N\} \geq_e 0$, then from lemma 2.15,

$$\dot{\mathcal{B}} = |\mathcal{A}| = (Q^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\}.$$

Thus, we can obtain a result analogous to corollary 2.20.

We consider the mode process as a homogeneous Markov chain and show that the existing results on MSS of the system (2.1) in the literature can be obtained as a special case in the following remark.

Remark 2.23. When $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is a homogeneous Markov chain, i.e; $\Pi_k = \Pi$ for all k , then none of the columns of $\mathcal{A}$ has a non-degenerate interval. Hence, from lemma 2.15, $\dot{\mathcal{B}} = \mathcal{A}$, where $\mathcal{A} = (\Pi^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\}$. From theorem 2.21, the system (2.1) is MSS if $\rho(\mathcal{A}) < 1$, which is the sufficient condition described in (Costa, 2004; Fang and Loparo, 2002; Ling and Deng, 2012) for the MSS of discrete-time homogeneous MJLSs.

Now, we present a numerical example to illustrate the proposed result obtained in this section.

Example 2.24. Consider the system (2.1) with $\theta_k \in \{1, 2\}$. Let

$$A_1 = \begin{bmatrix} 2 & -1 \\ 0 & 0.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \Pi_k = \begin{bmatrix} [0, 0.2] & [0.8, 1] \\ 0.15 & 0.85 \end{bmatrix}.$$

Here if we write $\Pi_k = [P, Q]$, then

$$P = \begin{bmatrix} 0 & 0.8 \\ 0.15 & 0.85 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.2 & 1 \\ 0.15 & 0.85 \end{bmatrix}.$$

Using the expression (2.10), we obtain $\mathcal{A}$ as (2.17). To verify the irreducibility of the

$$\mathcal{A} = \begin{bmatrix} [0, 0.8] & [-0.4, 0] & [-0.4, 0] & [0, 0.2] & 0.0375 & 0.075 & 0.075 & 0.15 \\ 0 & [0, 0.2] & 0 & [-0.1, 0] & 0 & 0.075 & 0 & 0.15 \\ 0 & 0 & [0, 0.2] & [-0.1, 0] & 0 & 0 & 0.075 & 0.15 \\ 0 & 0 & 0 & [0, 0.05] & 0 & 0 & 0 & 0.15 \\ [3.2, 4] & [-2, -1.6] & [-2, -1.6] & [0.8, 1] & 0.2125 & 0.425 & 0.425 & 0.85 \\ 0 & [0.8, 1] & 0 & [-0.5, -0.4] & 0 & 0.425 & 0 & 0.85 \\ 0 & 0 & [0.8, 1] & [-0.5, -0.4] & 0 & 0 & 0.425 & 0.85 \\ 0 & 0 & 0 & [0.2, 0.25] & 0 & 0 & 0 & 0.85 \end{bmatrix}. \quad (2.17)$$

interval matrix $\mathcal{A}$, consider its directed graph in Figure 2.3. Its vertices are denoted by

the numbers $1, 2, \dots, 8$ for simplicity. If the interval matrix $\mathcal{A}$ contains a non-degenerate and non-zero interval, then the directed edge of graph of $\mathcal{A}$ is denoted by a dashed arrow. From Figure 2.3, it can be observed that the graph of $\mathcal{A}$ is not strongly connected, hence $\mathcal{A}$ is reducible from lemma 2.16. Thus, from case-II we consider $\dot{\mathcal{B}}$. From Figure 2.3,

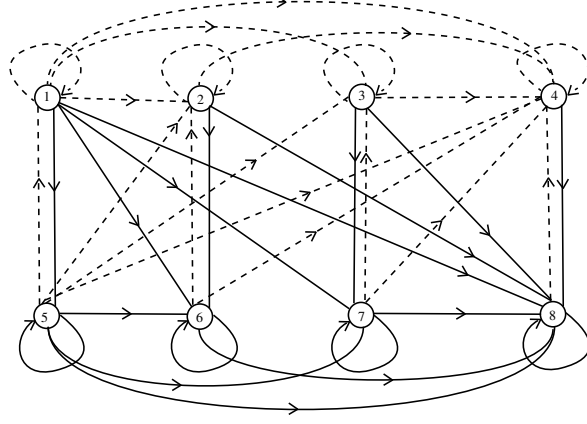


Figure 2.3: Directed Graph $G(\mathcal{A})$ of the interval matrix $\mathcal{A}$ in (2.17)

one can observe that for each vertex $i \in \{1, 2, \dots, 8\}$, there exists a directed path that ends at i and also contains a non-degenerate interval. Thus, from lemma 2.14, we conclude that every column of the interval matrix $\mathcal{A}$ has $(*)$ property. Subsequently, from lemma 2.15, we obtain $\dot{\mathcal{B}} = |\mathcal{A}|$. Here $\rho(|\mathcal{A}|) = 0.9923$, thus the system (2.1) is MSS according to theorem 2.21. Figure 2.4 shows the mean square value of the system state obtained from perform Monte Carlo simulations of 10,000 runs. One can observe that though one of the individual mode matrices A_1 and A_2 is unstable, the overall system is MSS in this case.

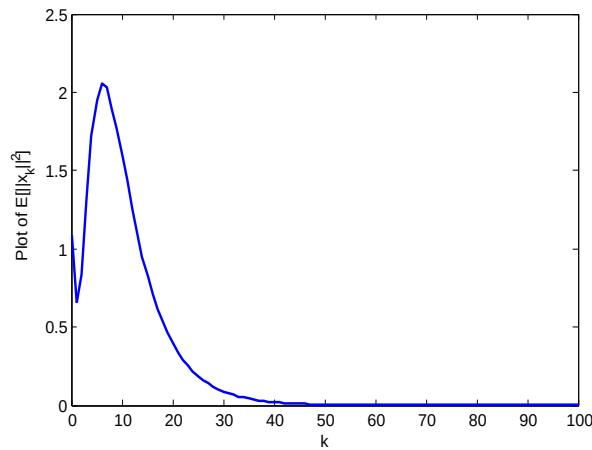


Figure 2.4: Plot of $\mathbb{E} [\|x_k\|^2]$ obtained from Monte Carlo simulations

So far, we have considered only stability of the system (2.1) in this chapter. In the next section, we consider a state-feedback stabilization problem.

2.3.3 State-feedback Stabilization

In this section, we address a mode dependent state-feedback stabilization approach for the proposed system.

Consider the system (2.1) with control input $u_k \in \mathbb{R}^m$

$$x_{k+1} = A_{\theta_k} x_k + B_{\theta_k} u_k, \quad (2.18)$$

where $B_{\theta_k} \in \mathbb{R}^{n \times m}$ and are assumed to be known for $\theta_k \in S$. We would like to utilize the results of this section to design a state-feedback stabilizing controller such that the system (2.18) is MSS. We assume that the system mode is available in real time, which lead to a state feedback control law of the form

$$u_k = K_{\theta_k} x_k, \quad K_{\theta_k} \in \mathbb{R}^{m \times n}. \quad (2.19)$$

Thus the overall system is given by

$$x_{k+1} = (A_{\theta_k} + B_{\theta_k} K_{\theta_k}) x_k. \quad (2.20)$$

If one observes carefully, with given state-feedback gains K_{θ_k} , we can verify the MSS of (2.20) using the results of this section, but it is not possible to synthesize the stabilizing state-feedback gains. To overcome this difficulty, in the next section we provide a method to synthesize state-feedback gains for second moment stabilization of (2.20) using a stochastic version of Lyapunov's second method (Costa et al., 2005), (Dragan et al., 2010). For convenience, we use SS as our second moment stability criterion. In the next section, we give sufficient conditions for SS and stochastic stabilization via solving a set of LMIs.

2.4 Stochastic Stability and Stabilization of Discrete-time Inhomogeneous MJLS with Interval TPM

In an ideal situation, we would like to synthesize a state-feedback controller for stabilization. As mentioned in the previous section, the results obtained in Section 2.3 can be used to verify the stability, but not for controller synthesis. Thus, we resort to representing the interval TPM as a convex combination of stochastic matrices, then one can readily perform the controller synthesis. In the literature, there are many existing results (Aberkane, 2011; Huang, 2013; Boukas et al., 1999) etc., which make use of convex combinations of stochastic matrices directly to obtain various results on stability and control. For completeness of this chapter, we show that one can represent the interval TPM as a convex combination of stochastic matrices and subsequently perform the controller synthesis for prestabilization.

In this section, we obtain sufficient conditions for SS of (2.1) and stochastic stabilization

of (2.20) and also subsequently synthesize state-feedback gains K_{θ_k} , for $\theta_k \in S$, by solving a set of LMIs. We begin with SS of (2.1).

The approach in this section involves representing the interval TPM as a convex combination of stochastic matrices, which will be useful in stabilization results.

2.4.1 Obtaining a Convex Combination of Stochastic Matrices of Interval TPM

In this section, given the interval TPM, we obtain a convex combination of stochastic matrices by a method presented in (Hartfield, 1998), which we give as the following algorithm.

Algorithm 1 Procedure to obtain a convex combination of stochastic matrices from a given interval TPM

Step 1: Consider the interval TPM $\Pi_k = [P, Q] \triangleq ([p_{ij}, q_{ij}])_{N \times N}$ as defined in Section 2.2.2;

Step 2: For each row of Π_k , choose a position in the row for a free variable;

- (a) For all other positions, put in the lowest or highest possible values of P and Q to obtain possible vectors with just one free variable;
- (b) From all the vectors, determine the value of the free variables that leads to stochastic vectors;
- (c) Change the position of the free variable, go to Step 2;

Step 3: For all the rows of Π_k , go to Step 2;

Step 4: Obtain stochastic matrices from the resulting combinations of set of stochastic vectors of each row; Denote them as $V^1, V^2, \dots, V^K$, for K stochastic matrices.

We give two examples in the sequel to illustrate algorithm 1.

Example 2.25. Consider Π_k from example 2.24, from which

$$P = \begin{bmatrix} 0 & 0.8 \\ 0.15 & 0.85 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.2 & 1 \\ 0.15 & 0.85 \end{bmatrix}.$$

Consider the first row, which we denote as $[x \ y]$ and choose the free variable as x . Then, for the first row, we apply the Step 2 of algorithm 1:

- (a) $[x \ 0.8]$. Set $x = 1 - 0.8 = 0.2$. Since $0 \leq x \leq 0.2$, $[0.2 \ 0.8]$ is a vertex.
- (b) $[x \ 1]$. Set $x = 1 - 1 = 0$. Since $0 \leq 0 \leq 0.2$, $[0 \ 1]$ is a vertex.

Thus $V^1 = \begin{bmatrix} 0.2 & 0.8 \\ 0.15 & 0.85 \end{bmatrix}$ and $V^2 = \begin{bmatrix} 0 & 1 \\ 0.15 & 0.85 \end{bmatrix}$ forms a convex combination of stochastic matrices for Π_k . Even if we choose the free variable as y , the result will be similar.

Example 2.26. Consider Π_k from example 2.7. By applying algorithm 1, we obtain the following vertices for the first row: $[0.35 \ 0.65]$, $[0.45 \ 0.55]$; second row: $[0.25 \ 0.75]$, $[0.35 \ 0.65]$. By different combinations, we can obtain the following vertices:

$$V^1 = \begin{bmatrix} 0.35 & 0.65 \\ 0.25 & 0.75 \end{bmatrix}, V^2 = \begin{bmatrix} 0.35 & 0.65 \\ 0.35 & 0.65 \end{bmatrix}, V^3 = \begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix} \text{ and } V^4 = \begin{bmatrix} 0.45 & 0.55 \\ 0.35 & 0.65 \end{bmatrix}.$$

By utilizing the vertices of interval TPM obtained from algorithm 1, we give a sufficient condition for SS in the following.

2.4.2 A Sufficient Condition for SS

In this section, we present a sufficient for SS of discrete-time inhomogeneous MJLS (2.1) with $\Pi_k = [P, Q] \triangleq ([p_{ij}, q_{ij}])_{N \times N}$ as defined in Section 2.2.2.

Remark 2.27. Since TPM $\Pi_k \triangleq (\pi_{ij}(k))_{N \times N}$ equals to $[P, Q]$, from algorithm 1, we can write Π_k as a convex combination $\Pi_k = \sum_{l=1}^K \varepsilon_l V^l$, where $\varepsilon_l \in [0, 1]$ for $1 \leq l \leq K$ and $\sum_{l=1}^K \varepsilon_l = 1$. Thus, we can write $\pi_{ij}(k)$ in (2.2) as a linear combination $\pi_{ij}(k) = \sum_{l=1}^K \varepsilon_l V_{ij}^l$, where V_{ij}^l denotes the element of V^l at a position (i, j) . By tuning ε_l , all possible $\pi_{ij}(k) \in [p_{ij}, q_{ij}]$ can be obtained. This approach is inspired from (Huang, 2013; Huang and Shi, 2013b), where the authors utilised the technique of parameter bounds and partitioning for continuous-time inhomogeneous MJLS to obtain numerically testable sufficient conditions for SS.

We give a sufficient condition for SS of (2.1) in the following theorem.

Theorem 2.28. Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is given by (2.2) and (2.6). Let $V^1, V^2, \dots, V^K$ be the vertices of the interval TPM Π_k (2.6) obtained from algorithm 1. Then the system (2.1) is SS if there exists $P_i \succ 0$ such that for all $i \in S$ and $1 \leq l \leq K$,

$$A_i^T \left(\sum_{j=1}^N V_{ij}^l P_j \right) A_i - P_i \triangleq -\Omega_{i,l} \prec 0. \quad (2.21)$$

Proof. Define a stochastic Lyapunov function $V(x_k, \theta_k)$ by

$$V(x_k, \theta_k) = x_k^T P_{\theta_k} x_k,$$

where $P_{\theta_k} \succ 0$ for $\theta_k \in S$. Consider

$$\begin{aligned} \mathbb{E}[V(x_{k+1}, \theta_{k+1}) | x_k, \theta_k] - V(x_k, \theta_k) &= x_k^T \left(A_{\theta_k}^T \sum_{j=1}^N \Pr\{\theta_{k+1} = j | \theta_k\} P_j A_{\theta_k} - P_{\theta_k} \right) x_k, \\ &= x_k^T \left(A_{\theta_k}^T \sum_{j=1}^N \pi_{\theta_k j}(k) P_j A_{\theta_k} - P_{\theta_k} \right) x_k. \end{aligned} \quad (2.22)$$

From remark 2.27, (2.22) is equivalent to

$$\begin{aligned}\mathbb{E}[V(x_{k+1}, \theta_{k+1}) | x_k, \theta_k] - V(x_k, \theta_k) &= x_k^T \left(A_{\theta_k}^T \sum_{j=1}^N \sum_{l=1}^K \varepsilon_l V_{\theta_{kj}}^l P_j A_{\theta_k} - P_{\theta_k} \right) x_k, \\ &= x_k^T \sum_{l=1}^K \varepsilon_l \left(A_{\theta_k}^T \sum_{j=1}^N V_{\theta_{kj}}^l P_j A_{\theta_k} - P_{\theta_k} \right) x_k, \\ &= -x_k^T \sum_{l=1}^K \varepsilon_l \Omega_{\theta_k, l} x_k.\end{aligned}\tag{2.23}$$

Let $\Omega_{\theta_k} \triangleq \sum_{l=1}^K \varepsilon_l \Omega_{\theta_k, l} \succ 0$, since $\Omega_{\theta_k, l} \succ 0$, for $1 \leq l \leq K$. Then by (2.21), (2.23) is equivalent to

$$\begin{aligned}\mathbb{E}[V(x_{k+1}, \theta_{k+1}) | x_k, \theta_k] - V(x_k, \theta_k) &= -x_k^T \Omega_{\theta_k} x_k, \\ &\leq -\rho_{\theta_k} x_k^T x_k, \\ &\leq -\rho x_k^T x_k,\end{aligned}\tag{2.24}$$

where $\rho_{\theta_k} = \lambda_{\min}\{\Omega_{\theta_k}\}$ and $\rho = \min_{\theta_k \in S} \rho_{\theta_k}$. Substituting $k = 0, k = 1$ in (2.24),

$$\mathbb{E}[V(x_1, \theta_1) | x_0, \theta_0] - V(x_0, \theta_0) \leq -\rho x_0^T x_0,\tag{2.25}$$

$$\mathbb{E}[V(x_2, \theta_2) | x_1, \theta_1] - V(x_1, \theta_1) \leq -\rho x_1^T x_1.\tag{2.26}$$

Taking the expectation $\mathbb{E}[\cdot | x_0, \theta_0]$ of (2.26), we obtain

$$\begin{aligned}\mathbb{E}[V(x_2, \theta_2) | x_0, \theta_0] &\leq V(x_0, \theta_0) - \rho \mathbb{E}[x_0^T x_0] - \rho \mathbb{E}[x_1^T x_1 | x_0, \theta_0], \\ \mathbb{E}\left[\sum_{k=0}^1 x_k^T x_k | x_0, \theta_0\right] &\leq \frac{1}{\rho} (V(x_0, \theta_0) - \mathbb{E}[V(x_2, \theta_2) | x_0, \theta_0]).\end{aligned}$$

Thus, for any $T \geq 1$,

$$\begin{aligned}\mathbb{E}\left[\sum_{k=0}^T \|x_k\|^2 | x_0, \theta_0\right] &\leq \frac{1}{\rho} (V(x_0, \theta_0) - \mathbb{E}[V(x_{T+1}, \theta_{T+1}) | x_0, \theta_0]), \\ &\leq \frac{1}{\rho} V(x_0, \theta_0).\end{aligned}$$

Therefore

$$\lim_{T \rightarrow \infty} \mathbb{E}\left[\sum_{k=0}^T \|x_k\|^2 | x_0, \theta_0\right] \leq \frac{1}{\rho} V(x_0, \theta_0) < \infty,$$

which implies that the system (2.1) is SS. □

Example 2.29. Consider the same parameters of example 2.24. We obtain its convex hull representation in example 2.25. From theorem 2.28, the LMIs (2.21) can be satisfied

$$P_1 = \begin{bmatrix} 1.3764 & -0.7494 \\ -0.7494 & 2.2009 \end{bmatrix} \succ 0 \text{ and } P_2 = \begin{bmatrix} 0.0791 & 0.1065 \\ 0.1065 & 5.2285 \end{bmatrix} \succ 0.$$

Thus the system (2.1) is SS.

2.4.3 State-feedback Stabilization

In this section, we synthesize mode dependent state-feedback control input (2.19) such that the controlled system (2.20) is SS, by utilizing theorem 2.28. The following theorem provides a sufficient condition for the existence of a stabilizing controller of the form (2.19).

Theorem 2.30. *Suppose that $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}}$ is given by (2.2) and (2.6). Let $V^1, V^2, \dots, V^K$ be obtained from algorithm 1. If there exist matrices $X_i \succ 0$, and Y_i , for all $i \in S$ and $1 \leq l \leq K$, such that*

$$\begin{bmatrix} -X_D & \mathcal{P}_{i,l}(A_i X_i + B_i Y_i) \\ * & -X_i \end{bmatrix} \prec 0, \quad (2.27)$$

where

$$\begin{aligned} \mathcal{P}_{i,l} &= \begin{bmatrix} \sqrt{V_{i1}^l} \mathbb{I}_n & \cdots & \sqrt{V_{iN}^l} \mathbb{I}_n \end{bmatrix}^T, \\ X_D &= \text{diag}\{X_1, \dots, X_N\}, \end{aligned}$$

then the system (2.20) is stochastically stabilized by (2.19), and the mode dependent controller is given by

$$K_i = Y_i X_i^{-1}. \quad (2.28)$$

Proof. It is apparent that the stochastic stabilizability of (2.20) by the state-feedback controller (2.19) is equivalent to the SS of (2.20). Hence, from theorem 2.28, by replacing A_i with $A_i + B_i K_i$ and after Schur complement, the LMIs (2.21) can be reformulated as

$$\begin{bmatrix} -P_1 & 0 & \cdots & 0 & \sqrt{V_{i1}^l} P_1 (A_i + B_i K_i) \\ * & -P_2 & \ddots & \vdots & \sqrt{V_{i2}^l} P_2 (A_i + B_i K_i) \\ * & * & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & -P_N & \sqrt{V_{iN}^l} P_N (A_i + B_i K_i) \\ * & * & \cdots & * & -P_i \end{bmatrix} \prec 0. \quad (2.29)$$

Let $X_i \triangleq P_i^{-1}$ and consider a diagonal matrix $\text{diag}\{X_1, \dots, X_N, X_i\}$, which is non-singular. Hence the congruent transformation of (2.29) by $\text{diag}\{X_1, \dots, X_N, X_i\}$, and applying a change of variable $Y_i \triangleq K_i X_i$ lead to (2.27). Also, the stabilizing controller is given by (2.28). This completes the proof. $\square$

2.4.4 An Illustrative Example of Vertical Take Off and Landing Vehicle

In this section, we illustrate the effectiveness of the proposed approach on vertical take-off and landing (VTOL) vehicle dynamical system. Let a discretized version of VTOL vehicle dynamics (Aberkane et al., 2008), with zero order hold and sampling time of 0.2 seconds,

be given by (2.18). Here $x_k \in \mathbb{R}^4$, $u_k \in \mathbb{R}^2$, $y_k \in \mathbb{R}^2$, $x_k(1)$, $x_k(2)$, $x_k(3)$, $x_k(4)$ denote horizontal velocity, vertical velocity, pitch rate, pitch angle respectively, and $u_k(1)$, $u_k(2)$ denote thrust along the aircraft axis, pitch moment respectively. Let $\theta_k \in \{1, 2, 3\}$ denote different air speeds, where the modes 1, 2, 3 denote the air speeds 135, 60, 170 knots respectively. The interval TPM Π_k along with the mode parameters are given as

$$\Pi_k = \begin{bmatrix} [0.1, 0.3] & [0.2, 0.4] & [0.4, 0.6] \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix}, \quad \begin{array}{|c|c|c|c|} \hline \theta_k(\text{knots}) & a_{32}(\theta_k) & a_{34}(\theta_k) & b_{21}(\theta_k) \\ \hline 135 & 0.0627 & 0.2398 & 0.6690 \\ \hline 60 & 0.0112 & 0.0167 & 0.2043 \\ \hline 170 & 0.0865 & 0.4452 & 0.9526 \\ \hline \end{array}.$$

The system matrices are as follows

$$A_{\theta_k} = \begin{bmatrix} 0.9927 & 0.0048 & -0.0053 & -0.0931 \\ 0.0082 & 0.8154 & -0.0717 & -0.7351 \\ 0.0191 & a_{32}(\theta_k) & 0.8924 & a_{34}(\theta_k) \\ 0.0019 & 0.0066 & 0.1882 & 1.0254 \end{bmatrix}, \quad B_{\theta_k} = \begin{bmatrix} 0.0912 & 0.0302 \\ b_{21}(\theta_k) & -1.3961 \\ -1.0145 & 0.7951 \\ -0.1042 & 0.0827 \end{bmatrix}.$$

By applying algorithm 1, we obtain the following vertices for Π_k :

$$V^1 = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix}, V^2 = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix}, V^3 = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix},$$

$$V^4 = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix}, V^5 = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix}, V^6 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix}.$$

By applying theorem 2.30 for stochastic stabilization, we obtain

$$X_1 = \begin{bmatrix} 0.7440 & -0.0255 & 0.0030 & 0.1235 \\ -0.0255 & 0.9976 & 0.0021 & 0.0124 \\ 0.0030 & 0.0021 & 1.0405 & -0.0689 \\ 0.1235 & 0.0124 & -0.0689 & 0.6428 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.0716 & 0.0366 & 0.1657 & 0.2656 \\ 0.0366 & 1.0068 & 0.0316 & 0.0110 \\ 0.1657 & 0.0316 & 1.3072 & -0.3201 \\ 0.2656 & 0.0110 & -0.3201 & 0.8069 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 1.0844 & 0.0190 & 0.1280 & 0.2439 \\ 0.0190 & 1.0041 & 0.0220 & 0.0406 \\ 0.1280 & 0.0220 & 1.2091 & -0.2615 \\ 0.2439 & 0.0406 & -0.2615 & 1.0850 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} -0.4032 & 0.7839 & 1.3290 & 0.1450 \\ -0.3320 & 0.9442 & 0.6074 & -0.2611 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -0.2203 & 0.5317 & 1.0455 & -0.1898 \\ -0.2324 & 0.6495 & 0.2434 & -0.4281 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -0.5394 & 1.0803 & 1.6667 & 0.4044 \\ -0.6010 & 1.2829 & 1.1878 & -0.2783 \end{bmatrix}.$$

Thus, the given system is SS and the stabilizing controller is given by

$$K_1 = \begin{bmatrix} -0.5985 & 0.7619 & 1.3083 & 0.4662 \\ -0.3675 & 0.9396 & 0.5634 & -0.2933 \end{bmatrix}, K_2 = \begin{bmatrix} -0.4265 & 0.5124 & 0.9044 & 0.2569 \\ -0.1351 & 0.6528 & 0.0735 & -0.4659 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.9234 & 1.0188 & 1.6616 & 0.9426 \\ -0.7276 & 1.2635 & 1.0614 & 0.1156 \end{bmatrix}.$$

With the given controller, we show a single sample path simulation in Figure 2.5, where one can observe the stabilizing state trajectories of the system.

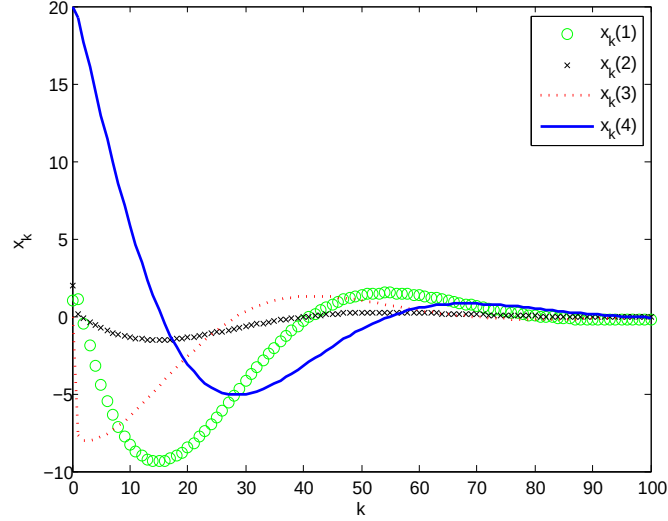


Figure 2.5: Plot of system states for a single path simulation

2.5 Conclusions

We considered a discrete-time MJLS with a finite-state time-inhomogeneous Markov chain whose TPM is varying in an interval. A sufficient condition for the MSS of the system was obtained in terms of the spectral radius of a deterministic matrix using the results of interval analysis and Kronecker products. Since, we could not synthesize a state-feedback controller using this condition, we obtained a convex hull representation of the interval TPM and alternatively provided sufficient conditions for the SS and stochastic stabilization via solving LMIs. Though, we obtained two alternative sufficient conditions for second moment stability, we could not relate them right now, which will be our future course of work. The sufficient conditions obtained in terms of LMIs are particularly interesting, because using the obtained LMIs, we can also address the robust control problem in the presence of uncertainties in system parameters, which will be our another direction of future work. As a final note, in continuous-time, MJLS with a transition rate matrix having interval transition rates was studied in detail in (Huang, 2013), thus we are not considering the continuous counterpart of our results.

Chapter 3

Stochastic Stability and Stabilization of a Class of Continuous-time State-dependent Jump Linear Systems

3.1 Introduction

In this chapter, we consider the analysis of continuous-time RJLSs when the underlying random jump process depends on the state variable. Such a class of systems are referred to as “state-dependent jump linear systems (SDJLSs)” in this thesis. In Section 1.2, we presented research motivations for SDJLS modelling of dynamical systems.

In SDJLSs, to the best of our knowledge, only a few works addressed the issues of stability and control. A study of hybrid switching diffusion processes, a kind of continuous state-dependent jump non-linear systems with diffusion, has been carried out by (Yin and Zhu, 2009; Basak et al., 1996) and references therein by treating existence, uniqueness, stability of the solutions etc. In (Sworder and Robinson, 1973), the authors considered that the transition rates of the random jump process depend on both the state variable and the control input, which affect the time scale of the random jump process, thus affecting its transition rate, and obtained a control policy for a given functional using stochastic maximum principle. A model for planning and maintenance in flexible manufacturing system was proposed in (Boukas and Haurie, 1990), where the failure rate of a machine was assumed to depend on the state variable, and computed an optimal control law using dynamic programming. In (Filar and Haurie, 2001), a two-time scale model of production plant was regarded as a jump diffusion model with state-dependent failure rates, and obtained an optimal control policy.

In this chapter, we consider the state-dependent transition rates explicitly as: the transition rates vary depending on which set the state of the system belongs to. This is a reasonable assumption because the state of the system at any time belongs to one of the predefined sets, and the transition rates can be considered to have different values across the predefined sets. Under the above assumption, using Dynkin's formula, we obtain numerically tractable sufficient conditions for stochastic stability and state-feedback stabilization in terms of LMIs, though for a restricted class of SDJLSs described in Section 3.2.1.

The rest of the chapter is organized as follows: Section 3.2 gives the description of a mathematical model of SDJLS studied in this chapter and sufficient conditions for stochastic stability and stabilization. In Section 3.3, we address $\mathcal{H}_\infty$ state-feedback stabilization of SDJLS subject to external disturbances with finite energy and provide sufficient conditions. To facilitate the readability, we give some of the proofs in Section 3.4. Finally, we give concluding remarks in Section 3.5.

3.2 Stochastic Stability and Stabilization of Continuous-time SDJLS

In this section, we address the stochastic stability and stabilization of continuous-time SDJLS. We begin with the model description.

3.2.1 Model Description

Consider a SDJLS:

$$\begin{aligned}\dot{x}(t) &= A_{\theta(t)}x(t), \\ x(0) &= x_0,\end{aligned}\tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $x_0 \in \mathbb{R}^n$ is the initial state, $A_{\theta(t)} \in \mathbb{R}^{n \times n}$ be the system matrices which depend on $\theta(t)$, and are assumed to be known. Let $\theta(t) \in S := \{1, 2, \dots, N\}$ be a finite space continuous-time random jump process, describing the mode of the system at time t . Before proceeding, we make the following assumption related to the state-dependent jump process.

Assumption 3.1. Let $\mathcal{K} \triangleq \{1, 2, \dots, K\}$. we assume that the nonempty sets $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K \subseteq \mathbb{R}^n$ are Borel sets in $\mathbb{R}^n$. Roughly speaking, any subset of $\mathbb{R}^n$ that one can imagine are Borel sets, examples include open or closed area of the circle, cube, line etc. Almost all the subsets of $\mathbb{R}^n$ are Borel sets; however a set which is not Borel, its shape in $\mathbb{R}^n$ is complicated beyond our imagination (Royden, 1968) pp.63-64. We further assume that each of the sets $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K$ is a connected set, which broadly means that each set $\mathcal{C}_m$,

for $m \in \mathcal{K}$, is a onepiece set or equivalently it is not expressed as a union of separate nonempty pieces. Also, let $\cup_{l=1}^K \mathcal{C}_l = \mathbb{R}^n$ and $\mathcal{C}_p \cap \mathcal{C}_q = \emptyset$ for any $p, q \in \mathcal{K}$, $p \neq q$. At any time $t \geq 0$, we assume that $x(t)$ belongs to exactly one of the sets $\mathcal{C}_l$, $l \in \mathcal{K}$.

For example, as an illustration, we can consider $\mathbb{R}^2$ space with three sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, which can be defined as: $\mathcal{C}_1 := \{x(t) \in \mathbb{R}^2, x_1^2(t) + x_2^2(t) \leq \xi_1\}$, $\mathcal{C}_2 := \{x(t) \in \mathbb{R}^2, \xi_1 < x_1^2(t) + x_2^2(t) \leq \xi_2\}$ and $\mathcal{C}_3 := \{x(t) \in \mathbb{R}^2, x_1^2(t) + x_2^2(t) > \xi_2\}$ for some $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_1 < \xi_2$. The illustration is given in Figure 3.1, where the state variable $x(t)$ belong to exactly one of the sets at each time $t \geq 0$.

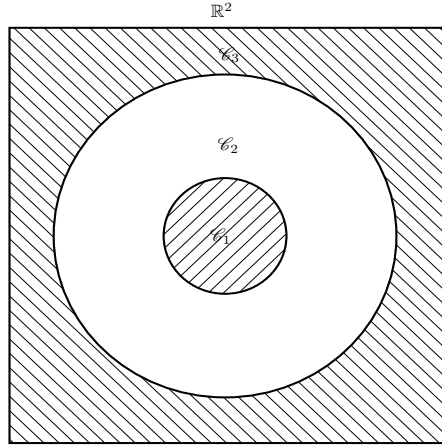


Figure 3.1: A schematic representation of the sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ in $\mathbb{R}^2$

Now the evolutions of the state-dependent jump process $\{\theta(t), t \geq 0\}$ for $i, j \in S$, $i \neq j$ are governed by:

$$\Pr\{\theta(t + \Delta) = j | \theta(t) = i, x(t)\} \triangleq \lambda_{ij}(x(t)) = \begin{cases} \lambda_{ij}^1 \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_1, \\ \lambda_{ij}^2 \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_2, \\ \vdots \\ \lambda_{ij}^K \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_K, \end{cases} \quad (3.2)$$

where $\Delta > 0$ and $\lim_{\Delta \rightarrow 0} \frac{o(\Delta)}{\Delta} = 0$. Here, for each $m \in \mathcal{K}$, λ_{ij}^m is the transition rate of $\theta(t)$ from mode i to mode j with $\lambda_{ij}^m \geq 0$ for $i \neq j \in S$, and $\lambda_{ii}^m = -\sum_{j=1, j \neq i}^N \lambda_{ij}^m$. We further assume that λ_{ij}^m is bounded for each $i, j \in S$ and $m \in \mathcal{K}$.

In words, λ_{ij}^m for $m \in \mathcal{K}$ represents the probability per time unit that $\theta(t)$ makes a transition from mode i to mode j when $x(t) \in \mathcal{C}_m$. A trivial remark here is that, for $m \in \mathcal{K}$, λ_{ij}^m in (3.2) is a notation followed in this chapter, but one should not get confused with the actual power of transition rates. Notice that the evolution of $\theta(t)$ depends on the state variable $x(t)$, hence $\theta(t)$ is called as a state-dependent jump process and the overall system (3.29) as SDJLS.

In the next section, we provide a related background on stability.

3.2.2 A Related Background

In this section, we give a brief background on MJLSs, the most widely known RJLSs, which are useful in understanding the stability results of SDJLSs that we obtain in this chapter. Before proceeding, we give definitions of various second moment stability concepts of SDJLS (3.1), which are also applicable to continuous-time RJLSs in general.

Definition 3.2. (Boukas, 2006) For the system (3.1), the equilibrium point 0 is

- *stochastically stable* (SS), if for any initial state $x_0 \in \mathbb{R}^n$ and initial probability distribution of $\theta(t)$,

$$\int_0^{+\infty} \mathbb{E} [\|x(t)\|^2] dt < \infty, \quad (3.3)$$

where $x(t)$ is a sample solution of (1.1) initial from x_0 ;

- *mean square stable* (MSS), if for any initial state $x_0 \in \mathbb{R}^n$ and initial probability distribution of $\theta(t)$,

$$\lim_{t \rightarrow +\infty} \mathbb{E} [\|x(t)\|^2] = 0, \quad (3.4)$$

- *exponentially mean square stable* (EMSS) if for any initial state $x_0 \in \mathbb{R}^n$ and initial probability distribution of $\theta(t)$, there exist constants $\alpha, \beta > 0$ such that

$$\mathbb{E} [\|x(t)\|^2] \leq \alpha \|x_0\|^2 e^{-\beta t}, \text{ for all } t \geq 0. \quad (3.5)$$

If the stochastic process $\theta(t)$ is time-homogeneous Markov, then the system (3.1) is called as MJLS. Let the transition rate of $\theta(t)$ be λ_{ij} . Consider the joint process $(x(t), \theta(t))$, and for any suitable Lyapunov function $V(x(t), \theta(t))$, the Dynkin's formula can be given by (Dynkin, 1965),

$$\mathbb{E}[V(x(t+\Delta), \theta(t+\Delta)) | x(t), \theta(t)] - V(x(t), \theta(t)) = \mathbb{E} \left[\int_t^{t+\Delta} \mathcal{L}V(x(s), \theta(s)) ds \right], \quad (3.6)$$

where $\mathcal{L}V(x(t), \theta(t))$ is the infinitesimal generator of $V(x(t), \theta(t))$, which is given by

$$\mathcal{L}V(x(t), \theta(t)) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \mathbb{E} \left[V(x(t+\Delta), \theta(t+\Delta)) | (x(t), \theta(t)) \right] - V(x(t), \theta(t)) \right\}.$$

In general, $\mathcal{L}V(x(t), \theta(t))$ can be understood as the average time rate of change of the function $V(x(t), \theta(t))$ given $(x(t), \theta(t))$ at time t . For system (3.1), consider the Lyapunov function $V(x(t), \theta(t)) \triangleq x^T(t) P_{\theta(t)} x(t)$, where $P_{\theta(t)} \succ 0$ for $\theta(t) \in S$. Then $\mathcal{L}V(x(t), \theta(t))$ is given by (Boukas, 2006; Costa et al., 2013),

$$\mathcal{L}V(x(t), \theta(t)) = x^T(t) \left(A_i^T P_i + P_i A_i + \sum_{j=1}^N \lambda_{ij} P_j \right) x(t). \quad (3.7)$$

By using the Dynkin's formula (3.6), if $\mathcal{LV}(x(t), \theta(t)) \prec 0$, then the given system is SS. This is equivalent to verifying the LMIs

$$\text{He}\{P_i A_i\} + \sum_{j=1}^N \lambda_{ij} P_j \prec 0, \quad (3.8)$$

for all $i \in S$.

The approach followed in this chapter to obtain conditions for stochastic stability will be similar to the approach presented in this section.

3.2.3 An Equivalent Model of SDJLSs

In this section, we slightly alter the notations of the SDJLS (3.1) and the mode $\theta(t)$ (3.2) such that handling of the state dependence becomes simpler.

For this purpose consider $\sigma_t \in \mathcal{K}$, which provides the information about the set the state variable $x(t)$ belongs at each time $t \geq 0$, as

$$\sigma_t = \begin{cases} 1, & \text{if } x(t) \in \mathcal{C}_1, \\ 2, & \text{if } x(t) \in \mathcal{C}_2, \\ \vdots & \\ K, & \text{if } x(t) \in \mathcal{C}_K. \end{cases} \quad (3.9)$$

Let $r(\sigma_t, t) \in S$ (which is equivalent to $\theta(t)$) be a finite space continuous-time jump process whose transitions depends on σ_t . Implicitly, $r(\sigma_t, t)$ depends on the state variable $x(t)$ as follows, for $i, j \in S$, $i \neq j$:

$$\Pr\{r(\sigma_{t+\Delta}, t+\Delta) = j | r(\sigma_t, t) = i, \sigma_t\} = \begin{cases} \lambda_{ij}^1 \Delta + o(\Delta), & \text{if } \sigma_t = 1, \\ \lambda_{ij}^2 \Delta + o(\Delta), & \text{if } \sigma_t = 2, \\ \vdots & \\ \lambda_{ij}^K \Delta + o(\Delta), & \text{if } \sigma_t = K, \end{cases} \quad (3.10)$$

where λ_{ij}^l , for $l \in \mathcal{K}$ are defined in (3.2).

Accordingly, the SDJLS (3.1) can be given by:

$$\begin{aligned} \dot{x}(t) &= A_{r(\sigma_t, t)} x(t), \\ x(0) &= x_0, \end{aligned} \quad (3.11)$$

where $A_{r(\sigma_t, t)} \in \mathbb{R}^{n \times n}$ be the system matrices (which are equivalent to $A_{\theta(t)}$) which depend on $r(\sigma_t, t)$ and are known from the description of (3.1). From now onwards, we analyse the system (3.11) with jump process (3.10), which is equivalent to analysing the system (3.1) with jump process (3.2). We adopt this equivalence, because the main results of this section will be based on the system description (3.11), which we use almost throughout

the chapter to facilitate readability.

Remark 3.3. Observe that system (3.11) can be viewed as a set of MJLSs with same mode matrices, but having different transition rate matrices depending on σ_t . Here σ_t is a stochastic function because its evolution depends on the state of the system that is stochastic. In this context, we would like to mention that a set of MJLSs with different mode matrices and transition rate matrices was considered in Bolzern et al. (2010) without uncertainties where the switching among different MJLSs is governed by a deterministic signal via a supervisor and addressed a problem of mean square stability.

Before we proceed, let $\mathcal{F}_t$ denote the natural filtration of $(x(t), r(\sigma_t, t), \sigma_t)$ on the interval $[0, t]$.

Remark 3.4. In (3.10), note that conditioning on $r(\sigma_t, t) = i \in S$, $r(\sigma_{t+\Delta}, t + \Delta)$ depends on σ_t for any $\Delta > 0$, and from (3.9), (3.11), which in turn depends on $r(\sigma_s, s)$, $s < t$. Hence $r(\sigma_t, t)$ is not a Markov process. Thus, intuitively, $(r(\sigma_t, t), \sigma_t)$ is a joint Markov process. Moreover, $(x(t), r(\sigma_t, t), \sigma_t)$ is also a joint Markov process. This point is stated and proved in the following lemma.

Lemma 3.5. *A vector valued random process $(x(t), r(\sigma_t, t), \sigma_t)$, which is a solution of (3.10) and (3.11), is a joint Markov process with respect to $\mathcal{F}_t$.*

Proof. Given in Section 3.4. □

In the sequel, we construct the solutions of (3.11) using first exit times. In that direction, we define first exit times from the sets $\mathcal{C}_l$, for $l \in \mathcal{K}$. We use a convention $\inf \phi \triangleq \infty$. Consider

- step 0: let $x(0) \in \mathcal{C}_{i_0}$, where $i_0 \in \mathcal{K}$. Let τ_0 denote the first exit time from $\mathcal{C}_{i_0}$ as follows:

$$\tau_0 = \inf \{t \geq 0 : \Phi_{i_0}(t, 0)x(0) \notin \mathcal{C}_{i_0}\},$$

- step 1: let $x(\tau_0) \in \mathcal{C}_{i_1}$, where $i_1 \neq i_0$, $i_1 \in \mathcal{K}$. Let τ_1 denote the first exit time from $\mathcal{C}_{i_1}$ after τ_0 as follows:

$$\tau_1 = \inf \{t \geq \tau_0 : \Phi_{i_1}(t, \tau_0)\Phi_{i_0}(\tau_0, 0)x(0) \notin \mathcal{C}_{i_1}\},$$

- step 2: let $x(\tau_1) \in \mathcal{C}_{i_2}$, where $i_2 \neq i_1$, $i_2 \in \mathcal{K}$. Let τ_2 denote the first exit time from $\mathcal{C}_{i_2}$ after τ_1 as follows:

$$\tau_2 = \inf \{t \geq \tau_1 : \Phi_{i_2}(t, \tau_1)\Phi_{i_1}(\tau_1, \tau_0)\Phi_{i_0}(\tau_0, 0)x(0) \notin \mathcal{C}_{i_2}\}.$$

In general, at any step $m \geq 1$, given τ_{m-1} , $i_{m-1} \in \mathcal{K}$ of the previous step $m - 1$ which is defined in a similar manner above,

- step m : let $x(\tau_{m-1}) \in \mathcal{C}_{i_m}$, where $i_m \neq i_{m-1}$, $i_m \in \mathcal{K}$. Let τ_m denote the first exit time from $\mathcal{C}_{i_m}$ after τ_{m-1} as follows:

$$\tau_m = \inf\{t \geq \tau_{m-1} : \Phi_{i_m}(t, \tau_{m-1})\Phi_{i_{m-1}}(\tau_{m-1}, \tau_{m-2}) \cdots \Phi_{i_0}(\tau_0, 0)x(0) \notin \mathcal{C}_{i_m}\}, \quad (3.12)$$

where the random flows $\Phi(., .)$ are described in the sequel.

We analyse, in general, one of the random flows $\Phi_{i_m}(t, \tau_{m-1})$ in (3.12), at step m , with $x(\tau_{m-1}) \in \mathcal{C}_{i_m}$, using which any random flow of (3.11) can be described in a similar fashion.

At step m , during the interval $[\tau_{m-1}, \tau_m)$, with $\tau_{-1} \triangleq 0$: let $n_m \in \mathbb{N}_0$ be the number of finite regime transitions of $r(\sigma_t, t)$; let $\{r_0^m, r_1^m, \dots, r_{n_m}^m\} \in S$ be the sequence of regimes visited by $r(\sigma_t, t)$; let $\{T_0^m, T_1^m, \dots, T_{n_m}^m\} \in [\tau_{m-1}, \tau_m)$ be the successive sojourn times of $r(\sigma_t, t)$, which are independent exponentially distributed random variables with parameters $\lambda_{r_p^m r_q^m}^{i_m}$, for $p, q \in \{0, 1, \dots, n_m\}$ and $p \neq q$. Let $S_{n_m}^m \triangleq \sum_{l=0}^{n_m-1} T_l^m$. Then $\Phi_{i_m}(t, \tau_{m-1})$ is given by

$$\Phi_{i_m}(t, \tau_{m-1}) = \begin{cases} e^{A_{r_{n_m}^m}(t-S_{n_m}^m-\tau_{m-1})} e^{A_{r_{n_m-1}^m} T_{n_m-1}^m} \cdots e^{A_{r_1^m} T_1^m} e^{A_{r_0^m} T_0^m}, & \text{if } n_m \geq 1, \\ e^{A_{r_{n_m}^m}(t-\tau_{m-1})}, & \text{if } n_m = 0. \end{cases}$$

Remark 3.6. From step 0, step 1, $\dots$, for $m \in \mathbb{N}_{\geq 0}$, in each of the intervals $[\tau_{m-1}, \tau_m)$, the state variable $x(t) \in \mathcal{C}_{i_m}$ that implies σ_t equals to m . Thus, in each $[\tau_{m-1}, \tau_m)$, system (3.11) can be viewed as MJLS with transition rates $\lambda_{ij}^{i_m}$, whose solutions exist and unique in almost sure sense, for instance from theorem 2.1 of Yin and Zhu (2009).

Remark 3.7. From (3.9) and from step 0, step 1, $\dots$ step $m, \dots$, σ_t can be described alternatively, for $t \geq 0$,

$$\sigma_t = \begin{cases} i_0, & \text{if } t \in [0, \tau_0), \\ i_1, & \text{if } t \in [\tau_0, \tau_1), \\ \vdots & \\ i_m, & \text{if } t \in [\tau_{m-1}, \tau_m), \\ \vdots & \end{cases}$$

where $\{i_0, i_1, \dots, i_m, \dots\} \in \mathcal{K}$. Lastly, we denote j^* the number of transitions made by σ_t .

Remark 3.8. From step 0, for any $i_0 \in \mathcal{K}$ and $x(0) \in \mathcal{C}_{i_0}$, if $\tau_0 = \infty$, then $x(t) \in \mathcal{C}_{i_0}$ for all $t \geq 0$. Thus, from remark 3.7, the overall system (3.11) is equivalent to time-homogeneous MJLS with random jump process being time-homogeneous Markov process with transition rates $\lambda_{ij}^{i_0}$.

In continuation to the discourse about the first exit times, we provide a definition of the stopping time in the following and prove that the first exit times $\tau_0, \tau_1, \tau_2, \dots$ are the stopping times.

Definition 3.9. (Gikhman and Skorokhod, 2004) Let $(\Omega, \mathcal{F}, \mathcal{G}_t, \Pr)$ be a filtered probability space, then a random variable $\tau : \Omega \rightarrow [0, \infty]$ (it may take the value ∞) is said to be an $\mathcal{G}_t$ –stopping time or a stopping time if $\{\tau \leq t\} \in \mathcal{G}_t$ for any $t \geq 0$, i.e; the event $\{\tau \leq t\}$ is $\mathcal{G}_t$ –measurable, which implies the event $\{\tau \leq t\}$ is completely determined by the knowledge of $\mathcal{G}_t$.

We give the following lemma, which shows that the first exit times given above are in fact stopping times.

Lemma 3.10. *The first exit times $\tau_0, \tau_1, \tau_2, \dots$ given in step 0, step 1, step 2, $\dots$ are $\mathcal{F}_t$ –stopping times.*

Proof. Given in Section 3.4. □

Based on lemma 3.5 and lemma 3.10, we provide a Dynkin’s formula that will be used in the next section.

Definition 3.11. Let a vector valued random process $(x(t), r(\sigma_t, t), \sigma_t)$, a solution of (3.10) and (3.11), be a Markov process and $\tau_0, \tau_1, \tau_2, \dots$, given in step 0, step 1, step 2 $\dots$ are stopping times. Let $\xi(t) \triangleq (x(t), r(\sigma_t, t), \sigma_t)$. For any suitable Lyapunov function $V(\xi(t))$, the Dynkin’s formula can be given by (Srichander and Walker, 1993; Wu et al., 2013),

$$\begin{aligned} \mathbb{E}[V(\xi(t))|\xi(0)] - V(\xi(0)) &= \mathbb{E} \left[\sum_{j=0}^{j^*} \int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(\xi(s)) ds | \xi(s) \right], \\ &= \sum_{j=0}^{j^*} \mathbb{E} \left[\int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(\xi(s)) ds | \xi(s) \right], \end{aligned} \quad (3.13)$$

where $\tau_{-1} = 0$, $j = 0, 1, \dots, j^*$, $j^* \leq \infty$, and $\tau_{j^*} \leq \infty$. Here $\mathcal{L}V(\xi(t))$ is the infinitesimal generator of $V(\xi(t))$ that is given by

$$\mathcal{L}V(\xi(t)) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \mathbb{E} \left[V(\xi(t + \Delta)) | \xi(t) \right] - V(\xi(t)) \right\}. \quad (3.14)$$

Also observe that, since $\xi(t)$ is Markov process, for any $t \geq 0$, the expectation terms in (3.13) are conditioned on $\xi(t)$, instead of the natural filtration of $\xi(t)$ on the interval $[0, t]$.

3.2.4 Stochastic Stability

We now give a sufficient condition for stochastic stability.

Theorem 3.12. *For each $\kappa \in \mathcal{K}$, if there exist positive definite matrices $P_i \succ 0$, for all $i \in S$, such that*

$$\text{He}\{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \triangleq -W_{\kappa,i} \prec 0, \quad (3.15)$$

then the system (3.11) is SS.

Proof. Consider a vector valued random process $(x(t), r(\sigma_t, t), \sigma_t)$, which is a solution of (3.10) and (3.11), and $P_{r(\sigma_t, t)} \succ 0$ for $r(\sigma_t, t) \in S$. Let $V(x(t), r(\sigma_t, t), \sigma_t) \triangleq x^T(t)P_{r(\sigma_t, t)}x(t)$, which is quadratic and positive in $x(t)$, hence a Lyapunov candidate function. From (3.14), the infinitesimal generator of $V(x(t), r(\sigma_t, t), \sigma_t)$, for any $i \in S$ and for any $\kappa \in \mathcal{K}$, is given by (3.7)

$$\begin{aligned} \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \mathbb{E} \left[V(x(t + \Delta), r(\sigma_{t+\Delta}, t + \Delta), \sigma_{t+\Delta}) \mid (x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \right] \right. \\ &\quad \left. - V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \right\}, \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \left[\sum_{j=1}^N \Pr\{r(\sigma_{t+\Delta}, t + \Delta) = j \mid r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa\} \right. \right. \\ &\quad \left. \left. x^T(t + \Delta)P_j x(t + \Delta) \right] - x^T(t)P_i x(t) \right\}, \end{aligned}$$

where $x(t + \Delta) = (\mathbb{I}_n + \Delta A_i)x(t) + o(\Delta^2)\mathbf{1}_n$ and $\mathbf{1}_n \in \mathbb{R}^n$ is a vector with all its elements as 1. From (3.10) and (3.11), it can be written as

$$\begin{aligned} \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \left[\sum_{j=1, j \neq i}^N \lambda_{ij}^\kappa \Delta x^T(t + \Delta)P_j x(t + \Delta) \right] \right. \\ &\quad \left. + \left[(1 + \lambda_{ii}^\kappa \Delta) x^T(t + \Delta)P_i x(t + \Delta) \right] - x^T(t)P_i x(t) \right\}, \end{aligned} \quad (3.16)$$

$$= x^T(t) \left[\text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \right] x(t). \quad (3.17)$$

Thus, by (3.15), (3.17),

$$\begin{aligned} \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) &= -x^T(t)W_{\kappa, i}x(t), \\ &\leq -\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa, i})\} x^T(t)x(t). \end{aligned} \quad (3.18)$$

From lemma 3.5 and lemma 3.10, $(x(t), r(\sigma_t, t), \sigma_t)$ is a Markov process, and $\tau_0, \tau_1, \tau_2, \dots$, given in step 0, step 1, step 2 $\dots$ are stopping times. Thus, for any $i_0 \in \mathcal{K}$, by Dynkin's formula (3.13), one obtains

$$\begin{aligned} \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) \mid (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ = \sum_{j=0}^{j^*} \mathbb{E} \left[\int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(x(s), r(\sigma_s, s), \sigma_s) ds \mid (x(s), r(\sigma_s, s), \sigma_s) \right], \end{aligned}$$

Let $\{i_0, i_1, i_2, \dots\} \in \mathcal{K}$ be the successive states visited by σ_t similar to remark 3.7. Then

$$\begin{aligned} & \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &= \mathbb{E} \left[\int_0^{\tau_0} \mathcal{L}V(x(s), r(\sigma_s = i_0, s), \sigma_s = i_0) ds | (x(s), r(\sigma_s = i_0, s), \sigma_s = i_0) \right] \\ &+ \mathbb{E} \left[\int_{\tau_0}^{\tau_1} \mathcal{L}V(x(s), r(\sigma_s = i_1, s), \sigma_s = i_1) ds | (x(s), r(\sigma_s = i_1, s), \sigma_s = i_1) \right] \\ &\vdots \\ &+ \mathbb{E} \left[\int_{t \wedge \tau_{j^*-1}}^{t \wedge \tau_{j^*}} \mathcal{L}V(x(s), r(\sigma_s = i_{j^*}, s), \sigma_s = i_{j^*}) ds | (x(s), r(\sigma_s = i_{j^*}, s), \sigma_s = i_{j^*}) \right]. \end{aligned}$$

By (3.18), one gets

$$\begin{aligned} & \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &\leq - \min_{\kappa \in \mathcal{K}, i \in S} \{ \lambda_{\min}(W_{\kappa, i}) \} \left(\mathbb{E} \left[\int_0^{\tau_0} x^T(s)x(s) ds \right] + \mathbb{E} \left[\int_{\tau_0}^{\tau_1} x^T(s)x(s) ds \right] + \right. \\ &\quad \left. \dots + \mathbb{E} \left[\int_{t \wedge \tau_{j^*-1}}^{t \wedge \tau_{j^*}} x^T(s)x(s) ds \right] \right). \end{aligned}$$

By noting $\sum_{j=0}^{j^*} \int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} = \int_0^t$, one obtains

$$\begin{aligned} & \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\ &\leq - \min_{\kappa \in \mathcal{K}, i \in S} \{ \lambda_{\min}(W_{\kappa, i}) \} \mathbb{E} \left[\int_0^t x^T(s)x(s) ds \right]. \end{aligned}$$

By rearranging the terms

$$\begin{aligned} & \min_{\kappa \in \mathcal{K}, i \in S} \{ \lambda_{\min}(W_{\kappa, i}) \} \mathbb{E} \left[\int_0^t x^T(s)x(s) ds \right] \\ &\leq V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) - \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right], \\ &\leq V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0). \end{aligned}$$

Thus

$$\mathbb{E} \left[\int_0^t x^T(s)x(s) ds \right] \leq \frac{V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)}{\min_{\kappa \in \mathcal{K}, i \in S} \{ \lambda_{\min}(W_{\kappa, i}) \}}.$$

By letting $t \rightarrow \infty$,

$$\mathbb{E} \left[\int_0^\infty x^T(s)x(s) ds \right] \leq \frac{V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)}{\min_{\kappa \in \mathcal{K}, i \in S} \{ \lambda_{\min}(W_{\kappa, i}) \}} < \infty.$$

Thus the system (3.11) is SS. □

Remark 3.13. Observe that, when remark 3.8 is true, LMIs (3.15) are equivalent to LMIs (3.8) with transition rates $\lambda_{ij}^{i_0}$.

3.2.5 State-feedback Stabilization

In this section, we provide a sufficient condition for stochastic stabilization with a state-feedback controller.

Consider the system (3.11) with control input $u(t) \in \mathbb{R}^m$,

$$\dot{x}(t) = A_{r(\sigma_t, t)}x(t) + B_{r(\sigma_t, t)}u(t), \quad (3.19)$$

where $B_{r(\sigma_t, t)} \in \mathbb{R}^{n \times m}$ are assumed to be known for $r(\sigma_t, t) \in S$.

We make use of theorem 3.12 to design a state-feedback stabilizing controller such that the system (3.19) is SS. We assume that the system mode $r(\sigma_t, t)$ is available in real time, which lead to a state-feedback control law of the form

$$u(t) = K_{r(\sigma_t, t)}x(t), \quad K_{r(\sigma_t, t)} \in \mathbb{R}^{m \times n}, \quad (3.20)$$

and the overall system is given by

$$\dot{x}(t) = \tilde{A}_{r(\sigma_t, t)}x(t), \quad (3.21)$$

where $\tilde{A}_{r(\sigma_t, t)} = A_{r(\sigma_t, t)} + B_{r(\sigma_t, t)}K_{r(\sigma_t, t)}$. The following theorem gives a sufficient condition for the existence of a stabilizing controller of the form (3.20).

Theorem 3.14. *For each $\kappa \in \mathcal{K}$, if there exist matrices $X_i \succ 0$, and Y_i , for all $i \in S$, such that*

$$\begin{bmatrix} \text{He}\{A_i X_i + B_i Y_i\} + \lambda_{ii}^\kappa X_i & \mathcal{M}_{\kappa, i} \\ \star & -\mathcal{X}_i \end{bmatrix} \prec 0, \quad (3.22)$$

where

$$\mathcal{M}_{\kappa, i} = [\sqrt{\lambda_{i1}^\kappa} X_i \cdots \sqrt{\lambda_{ii-1}^\kappa} X_i \sqrt{\lambda_{ii+1}^\kappa} X_i \cdots \sqrt{\lambda_{iN}^\kappa} X_i], \quad (3.23)$$

$$\mathcal{X}_i = \text{diag}\{X_1 \cdots X_{i-1}, X_{i+1} \cdots X_N\}, \quad (3.24)$$

then the system (3.19) is stochastically stabilized by (3.20), and the stabilizing controller is given by

$$K_i = Y_i X_i^{-1}. \quad (3.25)$$

Proof. From theorem 3.12, the closed loop system (3.21) is SS if for each $\kappa \in \mathcal{K}$, there exist positive definite matrices $P_i \succ 0$, for all $i \in S$, such that

$$\text{He}\{P_i (A_i + B_i K_i)\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \prec 0, \quad (3.26)$$

Since (3.26) is nonlinear in P_i and K_i , we would like to transform (3.26) into an LMI. Let $X_i \triangleq P_i^{-1}$, and pre and post multiplying of (3.26) by X_i gives

$$\text{He}\{A_i X_i + B_i K_i X_i\} + X_i \sum_{j=1}^N \lambda_{ij}^\kappa X_j^{-1} X_i \prec 0. \quad (3.27)$$

Let $Y_i = K_i X_i$ and by (3.23), (3.24), the LMI (3.27) can be reformulated as

$$\text{He}\{A_i X_i + B_i Y_i\} + \lambda_{ii}^\kappa X_i + \mathcal{M}_{\kappa,i} \mathcal{X}_i^{-1} \mathcal{M}_{\kappa,i}^T \prec 0. \quad (3.28)$$

After Schur complement, the LMI (3.28) is equivalent to (3.22) and the state-feedback controller is given by (3.25). This completes the proof. $\square$

In the sequel, a numerical example is presented to illustrate the proposed result on stochastic stability.

Example 3.15. Consider the system (3.1) with $x(t) \triangleq [x_1(t), x_2(t)]^T \in \mathbb{R}^2$. Let $\theta(t) \in S := \{1, 2\}$, be a state-dependent jump process given by (3.2), with $\mathcal{C}_1 \triangleq \{x(t) \in \mathbb{R}^2 : x_1^2(t) + x_2^2(t) < 3\}$, $\mathcal{C}_2 \triangleq \{x(t) \in \mathbb{R}^2 : x_1^2(t) + x_2^2(t) \geq 3\}$, and for $\mathcal{K} := \{1, 2\}$ the transition rate matrices of $\theta(t)$ are given by

$$(\lambda_{ij}^1)_{2 \times 2} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \text{ and } (\lambda_{ij}^2)_{2 \times 2} = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}.$$

Let

$$A_1 = \begin{bmatrix} -1 & 5 \\ -0.5 & 0.9 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -4 & 2 \\ -2 & 0.1 \end{bmatrix}.$$

Then, from theorem 3.12, the LMIs (3.15), are satisfied with

$$P_1 = \begin{bmatrix} 0.3787 & -0.4069 \\ -0.4069 & 2.2977 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0.3891 & -0.6203 \\ -0.6203 & 1.9226 \end{bmatrix}.$$

Hence, the system (3.1) is SS. With $\theta(0) = 1$, $x(0) = [-1, 1]^T$, Figure 3.2 shows a sample evolution of $\theta(t)$, with the corresponding stopping times $\tau_0, \tau_1, \dots$ given in step 0, step 1, $\dots$. Here the dashed (green) line shows orchestration of the state of the system two sets $\mathcal{C}_1$ and $\mathcal{C}_2$. Figure 3.3 shows the corresponding sample stable state trajectories of the system.

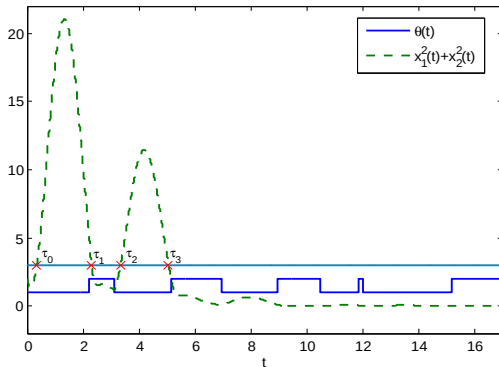


Figure 3.2: A sample $\theta(t)$ and the corresponding stopping time

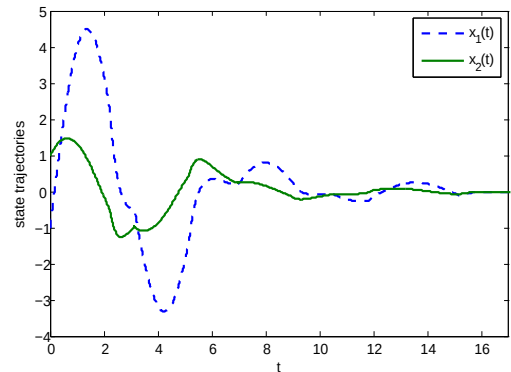


Figure 3.3: State trajectories of the system: a single sample path

In the next section, we illustrate our results on dynamics of a macroeconomic system.

3.2.6 An Illustrative Example of Macroeconomic System

In this section, we consider a macroeconomic system based on the Samuelson's multiplier accelerator model to illustrate our results. It has been considered by (Blair Jr, 1974; Blair Jr and Swarder, 1975; Chen et al., 2012; Costa et al., 1999b) etc., to illustrate the respective results on MJLSs. In these works, basically a dynamics of macroeconomic system was modelled as MJLS to describe the relation between the government expenditure (control input) and the national income (system state), where the different economic situations were modelled as modes and the switching among the modes was governed by a homogeneous Markov chain; one can refer to the above references for the detailed mathematical model. A more realistic scenario is to consider that the transitions among the economic situations (modes) depend on the state of the economic system, more explicitly, transition rates of the mode process depend on the state variable.

An economic system based on Samuelson's multiplier accelerator model with state-dependent economic situation can be given by the SDJLS (3.19), where $x(t) \in \mathbb{R}^2$. Here $x(t), u(t) \in \mathbb{R}$ refers to the government expenditure. Let $\theta(t) \in \{1, 2, 3\}$ denote modes of the economic system under three situations: $i = 1$ (normal), $i = 2$ (boom), $i = 3$ (slump). The continuous parameters of the mode matrices of (Chen et al., 2012; Costa et al., 1999b) can be given by:

$$A_1 = \begin{bmatrix} 9.1629 & -4.6522 \\ 4 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 14.5862 & -4.5519 \\ 8 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 26.5263 & 4.7368 \\ 8 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 4 & 0 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T, \quad B_3 = \begin{bmatrix} 2 & 0 \end{bmatrix}^T.$$

Consider the state-dependent jump process $\theta(t)$ (3.2) with $\mathcal{C}_1 \triangleq \{x(t) \in \mathbb{R}^2 : x_1^2(t) + x_2^2(t) < \delta\}$, $\mathcal{C}_2 \triangleq \{x(t) \in \mathbb{R}^2 : x_1^2(t) + x_2^2(t) \geq \delta\}$, where $\delta = 6$, and for $\mathcal{K} := \{1, 2\}$ let the transition rate matrices of $\theta(t)$ be given by

$$\left(\lambda_{ij}^1\right)_{3 \times 3} = \begin{bmatrix} -8 & 3 & 5 \\ 6 & -11 & 5 \\ 3 & 3 & -6 \end{bmatrix}, \left(\lambda_{ij}^2\right)_{3 \times 3} = \begin{bmatrix} -7 & 5 & 2 \\ 3 & -4 & 1 \\ 6 & 6 & -12 \end{bmatrix}.$$

Observe that when the economy (the state $x(t)$) belongs to $\mathcal{C}_2$ it represents a better economy, because in $\mathcal{C}_2$ the quantity of the economy is higher compared to $\mathcal{C}_1$; also in $\mathcal{C}_2$, the transitions among the modes follow $\left(\lambda_{ij}^2\right)_{2 \times 2}$ where the transition rates to boom are higher and the transition rates to slump are lower compared to $\left(\lambda_{ij}^1\right)_{2 \times 2}$ respectively. Thus, we observe the transitions among the economic situations depend on the state of the economy. From theorem 3.12, the LMIs (3.15), can be satisfied with

$$X_1 = \begin{bmatrix} 0.2606 & -0.0587 \\ -0.0587 & 0.2462 \end{bmatrix}, X_2 = \begin{bmatrix} 0.2597 & -0.0565 \\ -0.0565 & 0.2630 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0.2394 & -0.0495 \\ -0.0495 & 0.2477 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -0.8015 & 0.1692 \end{bmatrix}, Y_2 = \begin{bmatrix} -2.2811 & -0.0194 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -1.6295 & -0.4567 \end{bmatrix},$$

and the state-feedback gains are given by

$$K_1 = \begin{bmatrix} -3.0861 & -0.0488 \end{bmatrix}, K_2 = \begin{bmatrix} -9.2305 & -2.0560 \end{bmatrix}, K_3 = \begin{bmatrix} -7.4988 & -3.3428 \end{bmatrix}.$$

Thus, the system (3.19) is stochastically stabilizable. With $\theta(0) = 1$, $x(0) = [3, -2]^T$, Figure 3.4 shows sample trajectories of the system. We also show a sample evolution of

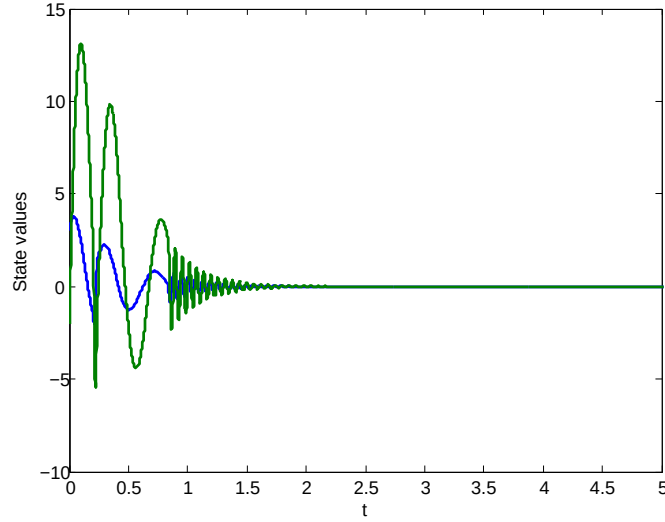


Figure 3.4: Sample state trajectories of the system

the state-dependent jump process $\theta(t)$ in figure Figure 3.5 along with stopping times $\tau_0, \tau_1, \dots$ given in step 0, step 1, $\dots$. In this sample simulation, during the initial period (the zoomed section of the figure), one can observe that the transition rate matrix of $\theta(t)$ is fluctuating between $(\lambda_{ij}^1)_{2 \times 2}$ and $(\lambda_{ij}^2)_{2 \times 2}$ depending on whether the state $x(t)$ belongs to $\mathcal{C}_1$ or $\mathcal{C}_2$ (also illustrated by the corresponding stopping times $\tau_0, \tau_1, \dots$ given in step 0, step 1, $\dots$) respectively.

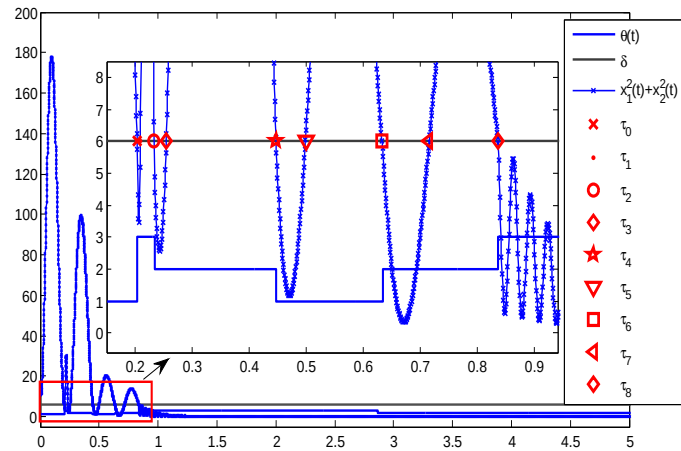


Figure 3.5: A sample state-dependent $\theta(t)$ and the corresponding stopping times

3.3 $\mathcal{H}_\infty$ State-feedback Stabilization of Continuous-time SDJLS

We have studied the stochastic stability and stabilization of continuous-time SDJLSs in Section 3.2. In practice, dynamical systems are subjected to external disturbances, and one wants to design a controller that attenuates the effect of external disturbances. $\mathcal{H}_\infty$ control has been a widely useful tool in attenuating the effect of external disturbances with finite energy that act on system dynamics (Boyd, 1994; Zhou and Doyle, 1998). The $\mathcal{H}_\infty$ controller design has been extended MJLSs, in (Boukas et al., 2001; Shi and Boukas, 1997; de Souza and Fragoso, 1993; Dragan et al., 1999; Todorov and Fragoso, 2008) and the references therein. To the best of our knowledge, the $\mathcal{H}_\infty$ control problem for SDJLSs has not been addressed in the literature.

In this section, we consider a SDJLS subject to external disturbances with finite energy, and synthesize a state-feedback $\mathcal{H}_\infty$ controller that guarantee disturbance rejection while ensuring stochastic stability (Chitraganti et al., 2014a).

3.3.1 Problem Formulation

Consider an extended version of the SDJLS (3.1)

$$\begin{aligned}\dot{x}(t) &= A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t), \\ z(t) &= C_{\theta(t)}x(t), \\ x(0) &= x_0,\end{aligned}\tag{3.29}$$

with the control input $u(t) \in \mathbb{R}^m$, the external disturbance $w(t) \in \mathbb{R}^r$ and the controlled output $z(t) \in \mathbb{R}^l$. The remaining parameters in (3.29) are described in the previous section, thus we are repeating them here for brevity. The disturbance is assumed to belong to $\mathcal{L}_2[0, \infty)$, i.e;

$$\|w\|_2 = \left(\int_0^\infty w^T(t)w(t)dt \right)^{\frac{1}{2}} < \infty,$$

which implies that the disturbance has finite energy.

Let $E_{\theta(t)} \in \mathbb{R}^{n \times r}$, $C_{\theta(t)} \in \mathbb{R}^{l \times n}$ be the system matrices that depend on $\theta(t)$, and are assumed to be known. Here $\theta(t) \in S := \{1, 2, \dots, N\}$ is described by (3.2).

The objective of this section is to design an $\mathcal{H}_\infty$ state-feedback controller such that the closed loop system is SS and guarantee the noise attenuation with a level $\gamma > 0$. More explicitly, we would like find a state-feedback controller $u(t)$ in (3.29) such that

$$\|z\|_{E_2} < \gamma \|w\|_2\tag{3.30}$$

under zero initial condition for any non-zero $w(t) \in \mathcal{L}_2[0, \infty)$, where $\|z\|_{E_2}$ is given by

$$\|z\|_{E_2} = \left(\mathbb{E} \left[\int_0^\infty z^T(t) z(t) dt \right] \right)^{\frac{1}{2}}.$$

3.3.2 An Equivalent Model of SDJLS

Similar to Section 3.2.3, we obtain an equivalent of the SDJLS (3.29) with $u(t) \equiv 0$ as

$$\begin{aligned} \dot{x}(t) &= A_{r(\sigma_t, t)} x(t) + E_{r(\sigma_t, t)} w(t), \\ z(t) &= C_{r(\sigma_t, t)} x(t), \\ x(0) &= x_0, \end{aligned} \tag{3.31}$$

where $r(\sigma_t, t)$ and σ_t are given by (3.10) and (3.9) respectively. Also one can prove that a vector valued random process $(x(t), r(\sigma_t, t), \sigma_t)$, which is a solution of a solution of (3.10) and (3.31), is a joint Markov process by following similar approach of lemma 3.5.

One can follow the similar approach of Section 3.2.3 to construct solution of the system and define the first exit times $\tau_0, \tau_1, \tau_2, \dots$. However due to the additive disturbance term $E_{r(\sigma_t, t)} w(t)$ in (3.11), it is difficult to derive exact expressions for the flows of the system.

Remark 3.16. Without loss of generality, a vector valued random process $(x(t), r(\sigma_t, t), \sigma_t)$, a solution of (3.10) and (3.31), is a Markov process and $\tau_0, \tau_1, \tau_2, \dots$, given in step 0, step 1, step 2 $\dots$ are stopping times, thus we can utilize the Dynkin's formula (3.13). One can follow the similar approach of the previous section to prove the above statement.

3.3.3 Stochastic Stability with $\mathcal{H}_\infty$ Performance

In this section, we present a sufficient condition for the stochastic stability of (3.31) while guaranteeing the $\mathcal{H}_\infty$ condition (3.30) in terms of LMIs. Using the sufficient condition, subsequently, we design a state-feedback controller via solving a set of LMIs.

Theorem 3.17. *For each $\kappa \in \mathcal{K}$, if there exist positive definite matrices $P_i \succ 0$, for all $i \in S$, such that*

$$\begin{bmatrix} \text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j + C_i^T C_i & P_i E_i \\ \star & -\gamma^2 \mathbb{I}_r \end{bmatrix} \prec 0 \tag{3.32}$$

then the system (3.31) is SS with γ -disturbance attenuation level given by (3.30).

Proof. Consider a vector valued random process $(x(t), r(\sigma_t, t), \sigma_t)$, which is a solution of (3.10) and (3.31), and $P_{r(\sigma_t, t)} \succ 0$ for $r(\sigma_t, t) \in S$. Let $V(x(t), r(\sigma_t, t), \sigma_t) \triangleq x^T(t) P_{r(\sigma_t, t)} x(t)$, which is quadratic and positive in $x(t)$, hence a Lyapunov candidate function. The infinitesimal generator of $V(x(t), r(\sigma_t, t), \sigma_t)$, for any $i \in S$ and for any $\kappa \in \mathcal{K}$, is given

by

$$\begin{aligned}
& \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \\
&= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \mathbb{E} \left[V(x(t + \Delta), r(\sigma_{t+\Delta}, t + \Delta), \sigma_{t+\Delta}) \mid (x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \right] \right. \\
&\quad \left. - V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \right\}, \\
&= x^T(t) \left[\text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \right] x(t) + 2x^T(t) P_i E_i w(t). \tag{3.33}
\end{aligned}$$

From (3.32),

$$\text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j + C_i^T C_i \prec 0,$$

which implies

$$\text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \triangleq -W_{\kappa,i} \prec 0. \tag{3.34}$$

From the inequality given in lemma A.2,

$$2x^T(t) P_i E_i w(t) \leq \delta_i^{-1} x^T(t) P_i E_i E_i^T P_i x(t) + \delta_i w^T(t) w(t), \tag{3.35}$$

for any $\delta_i > 0$. Thus, from (3.33), (3.34) and (3.35),

$$\begin{aligned}
& \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \\
&\leq -x^T(t) W_{\kappa,i} x(t) + \delta_i^{-1} x^T(t) P_i E_i E_i^T P_i x(t) + \delta_i w^T(t) w(t), \\
&\leq -\alpha x^T(t) x(t) + \delta_i w^T(t) w(t), \tag{3.36}
\end{aligned}$$

where $-\alpha = -\min_{\kappa \in \mathcal{K}, i \in S} \{\lambda_{\min}(W_{\kappa,i})\} + \delta_i^{-1} \max_{i \in S} \{\lambda_{\max}(P_i E_i E_i^T P_i)\}$, and $\alpha > 0$ for some $\delta_i > 0$. From remark 3.16, $(x(t), r(\sigma_t, t), \sigma_t)$ is a Markov process, and $\tau_0, \tau_1, \tau_2, \dots$, given in step 0, step 1, step 2 $\dots$ are stopping times. Thus, for any $i_0 \in \mathcal{K}$, by Dynkin's formula (3.13),

$$\begin{aligned}
& \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) \mid (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\
&= \sum_{j=0}^{j^*} \mathbb{E} \left[\int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} \mathcal{L}V(x(s), r(\sigma_s, s), \sigma_s) ds \mid (x(s), r(\sigma_s, s), \sigma_s) \right],
\end{aligned}$$

Let $\{i_0, i_1, i_2, \dots\} \in \mathcal{K}$ be the successive states visited by the process $\{\sigma_t, t \geq 0\}$ similar

to remark 3.7. Then

$$\begin{aligned}
& \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\
&= \mathbb{E} \left[\int_0^{\tau_0} \mathcal{L}V(x(s), r(\sigma_s = i_0, s), \sigma_s = i_0) ds | (x(s), r(\sigma_s = i_0, s), \sigma_s = i_0) \right] \\
&+ \mathbb{E} \left[\int_{\tau_0}^{\tau_1} \mathcal{L}V(x(s), r(\sigma_s = i_1, s), \sigma_s = i_1) ds | (x(s), r(\sigma_s = i_1, s), \sigma_s = i_1) \right] \\
&\vdots \\
&+ \mathbb{E} \left[\int_{t \wedge \tau_{j^*-1}}^{t \wedge \tau_{j^*}} \mathcal{L}V(x(s), r(\sigma_s = i_{j^*}, s), \sigma_s = i_{j^*}) ds | (x(s), r(\sigma_s = i_{j^*}, s), \sigma_s = i_{j^*}) \right].
\end{aligned}$$

By (3.36),

$$\begin{aligned}
& \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\
&\leq \mathbb{E} \left[\int_0^{\tau_0} [-\alpha x^T(s)x(s) + \delta_i w^T(s)w(s)] ds \right] + \mathbb{E} \left[\int_{\tau_0}^{\tau_1} [-\alpha x^T(s)x(s) + \delta_i w^T(s)w(s)] ds \right] \\
&+ \dots \mathbb{E} \left[\int_{t \wedge \tau_{j^*-1}}^{t \wedge \tau_{j^*}} [-\alpha x^T(s)x(s) + \delta_i w^T(s)w(s)] ds \right].
\end{aligned}$$

By noting $\sum_{j=0}^{j^*} \int_{t \wedge \tau_{j-1}}^{t \wedge \tau_j} = \int_0^t$, one obtains

$$\begin{aligned}
& \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right] - V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\
&\leq -\alpha \mathbb{E} \left[\int_0^t x^T(s)x(s) ds \right] + \delta_i \mathbb{E} \left[\int_0^t w^T(s)w(s) ds \right].
\end{aligned}$$

By rearranging the terms

$$\begin{aligned}
\alpha \mathbb{E} \left[\int_0^t x^T(s)x(s) ds \right] &\leq \delta_i \int_0^t w^T(s)w(s) ds + V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \\
&\quad - \mathbb{E} \left[V(x(t), r(\sigma_t, t), \sigma_t) | (x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0) \right], \\
&\leq \delta_i \int_0^t w^T(s)w(s) ds + V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0).
\end{aligned}$$

Thus

$$\mathbb{E} \left[\int_0^t x^T(s)x(s) ds \right] \leq \frac{\delta_i \int_0^t w^T(s)w(s) ds + V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)}{\alpha}.$$

By letting $t \rightarrow \infty$,

$$\mathbb{E} \left[\int_0^\infty x^T(s)x(s) ds \right] \leq \frac{\delta_i \|w\|_2 + V(x(0), r(\sigma_0 = i_0, 0), \sigma_0 = i_0)}{\alpha} < \infty.$$

Thus, the system (3.31) is SS.

Now, we prove that the $\mathcal{H}_\infty$ condition (3.30) is satisfied. To proceed, let us consider

$$J(T) = \int_0^T [z^T(t)z(t) - \gamma^2 w^T(t)w(t)] dt.$$

We prove in the sequel that $\mathbb{E}[J(T)] < 0$ as $T \rightarrow \infty$, which is equivalent to proving (3.30).

Consider (3.33), which can also be written as

$$\begin{aligned} \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) \\ = x^T(t) \left[\text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j \right] x(t) + x^T(t) P_i E_i w(t) + w^T(t) E_i^T P_i x(t). \end{aligned} \quad (3.37)$$

Also

$$z^T(t)z(t) - \gamma^2 w^T(t)w(t) = x^T(t) C_i^T C_i x(t) - \gamma^2 w^T(t)w(t). \quad (3.38)$$

Consider

$$\begin{aligned} \mathbb{E}[J(T)] &= \mathbb{E} \left[\int_0^T [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa)] dt \right] \\ &\quad - \mathbb{E} \left[\int_0^T \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa) dt \right], \\ &\leq \mathbb{E} \left[\int_0^T [z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \mathcal{L}V(x(t), r(\sigma_t = \kappa, t) = i, \sigma_t = \kappa)] dt \right]. \end{aligned}$$

From (3.37) and (3.38), it implies to

$$\mathbb{E}[J(T)] \leq \mathbb{E} \left[\int_0^T \tilde{x}^T(t) \Gamma_{\kappa, i} \tilde{x}(t) dt \right]$$

where $\tilde{x}(t) \triangleq \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix}^T$ and

$$\Gamma_{\kappa, i} \triangleq \begin{bmatrix} \text{He} \{P_i A_i\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j + C_i^T C_i & P_i E_i \\ \star & -\gamma^2 \mathbb{I}_r \end{bmatrix}.$$

Since $\Gamma_{\kappa, i} \prec 0$ from (3.32), $\mathbb{E}[J(T)] < 0$, which satisfies the $\mathcal{H}_\infty$ condition (3.30). This completes the proof. $\square$

In the next section, we give an $\mathcal{H}_\infty$ state-feedback stabilization result.

3.3.4 $\mathcal{H}_\infty$ State-feedback Stabilization

In this section, we provide a sufficient condition for stochastic stabilization with a state-feedback controller while guaranteeing the $\mathcal{H}_\infty$ condition (3.30).

We assume that the system mode $r(\sigma_t, t)$ is available in real time, and consider a state-

feedback control law of the form

$$u(t) = K_{r(\sigma_t, t)}x(t), \quad K_{r(\sigma_t, t)} \in \mathbb{R}^{m \times n}. \quad (3.39)$$

Thus, the overall system (3.31) is given by

$$\begin{aligned} \dot{x}(t) &= \tilde{A}_{r(\sigma_t, t)}x(t) + E_{r(\sigma_t, t)}w(t), \\ z(t) &= C_{r(\sigma_t, t)}x(t), \\ x(0) &= x_0, \end{aligned} \quad (3.40)$$

where $\tilde{A}_{r(\sigma_t, t)} = A_{r(\sigma_t, t)} + B_{r(\sigma_t, t)}K_{r(\sigma_t, t)}$.

The following theorem gives a sufficient condition for the existence of a stabilizing controller of the form (3.39) while guaranteeing the $\mathcal{H}_\infty$ condition (3.30).

Theorem 3.18. *For each $\kappa \in \mathcal{K}$, if there exist positive definite matrices $\hat{P}_i \succ 0$, and U_i for all $i \in S$, such that*

$$\begin{bmatrix} \text{He} \{A_i \hat{P}_i + B_i U_i\} + \lambda_{ii}^\kappa \hat{P}_i & E_i & \hat{P}_i C_i^T & \Omega_{i,k} \hat{P}_i \\ \star & -\gamma^2 \mathbb{I}_r & 0_{r \times l} & 0_{r \times (N-1)n} \\ \star & \star & -\mathbb{I}_l & 0_{l \times (N-1)n} \\ \star & \star & \star & \Upsilon_i \end{bmatrix} \prec 0 \quad (3.41)$$

where

$$\begin{aligned} \Upsilon_i &= -\text{diag} \{ \hat{P}_1, \dots, \hat{P}_{i-1}, \hat{P}_{i+1}, \dots, \hat{P}_N \}, \\ \Omega_{i,k} &= \left[\sqrt{\lambda_{i1}^\kappa} \hat{P}_i, \dots, \sqrt{\lambda_{ii-1}^\kappa} \hat{P}_i, \sqrt{\lambda_{ii+1}^\kappa} \hat{P}_i, \dots, \sqrt{\lambda_{iN}^\kappa} \hat{P}_i \right]. \end{aligned}$$

Then, the system (3.40) is SS with γ -disturbance attenuation level given by (3.30). The stabilizing controller is given by

$$K_i = U_i \hat{P}_i^{-1}. \quad (3.42)$$

Proof. By following theorem 3.17, the system (3.40) is SS while guaranteeing the $\mathcal{H}_\infty$ condition (3.30) if

$$\begin{bmatrix} \text{He} \{P_i (A_i + B_i K_i)\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j + C_i^T C_i & P_i E_i \\ \star & -\gamma^2 \mathbb{I}_r \end{bmatrix} \prec 0. \quad (3.43)$$

By Schur complement of (3.43), we obtain

$$\begin{bmatrix} \text{He} \{P_i (A_i + B_i K_i)\} + \sum_{j=1}^N \lambda_{ij}^\kappa P_j & P_i E_i & C_i^T \\ \star & -\gamma^2 \mathbb{I}_r & 0_{r \times l} \\ \star & \star & -\mathbb{I}_l \end{bmatrix} \prec 0. \quad (3.44)$$

Since $P_i \succ 0$, let $\hat{P}_i = P_i^{-1}$ and the congruent transformation of (3.44) by $\text{diag} \{ \hat{P}_i, \mathbb{I}_r, \mathbb{I}_l \}$ lead to

$$\begin{bmatrix} \text{He} \left\{ (A_i + B_i K_i) \hat{P}_i \right\} + \sum_{j=1}^N \lambda_{ij}^\kappa \hat{P}_i \hat{P}_j^{-1} \hat{P}_i & E_i & \hat{P}_i C_i^T \\ \star & -\gamma^2 \mathbb{I}_r & 0_{r \times l} \\ \star & \star & -\mathbb{I}_l \end{bmatrix} \prec 0. \quad (3.45)$$

Again, by applying Schur complement on (3.45), we obtain

$$\begin{bmatrix} \text{He} \left\{ (A_i + B_i K_i) \hat{P}_i \right\} + \lambda_{ii}^\kappa \hat{P}_i & E_i & \hat{P}_i C_i^T & \Omega_{i,k} \hat{P}_i \\ \star & -\gamma^2 \mathbb{I}_r & 0_{r \times l} & 0_{r \times (N-1)n} \\ \star & \star & -\mathbb{I}_l & 0_{l \times (N-1)n} \\ \star & \star & \star & \Upsilon_i \end{bmatrix} \prec 0. \quad (3.46)$$

By denoting $U_i = K_i \hat{P}_i$ will result in (3.41), which completes the theorem. $\square$

3.3.5 An Illustrative Example of Macroeconomic System

In this section, we consider a macroeconomic system considered in Section 3.2.6 to illustrate our results. An economic system based on Samuelson's multiplier accelerator model with state-dependent economic situation can be given by the SDJLS (3.40), where $x(t) \in \mathbb{R}^2$. Here $x(t)$ denotes the national income, $u(t) \in \mathbb{R}$ refers to the government expenditure. We assume that the economy is subject to fluctuations, and let $w(t) \in \mathcal{L}_2[0, \infty)$ represents the fluctuations in the national income. Let $\theta(t) \in \{1, 2, 3\}$ denote modes of the economic system under three situations: $i = 1$ (normal), $i = 2$ (boom), $i = 3$ (slump). Let $E_{\theta(t)} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $C_{\theta(t)} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Consider the state-dependent jump process $\theta(t)$ (3.2) where $\delta = 0.01$. We would like to observe what happens if the origin is on the border (or almost on the border) of the partitioned sets.

In this example, the controlled output $z(t)$ is $x_2(t)$. Let the $\mathcal{H}_\infty$ attenuation level γ be 1, then, from theorem 3.18, the LMIs (3.41), can be satisfied with

$$\begin{aligned} \hat{P}_1 &= \begin{bmatrix} 0.5306 & -0.1877 \\ -0.1877 & 0.4202 \end{bmatrix}, \hat{P}_2 = \begin{bmatrix} 0.5252 & -0.1807 \\ -0.1807 & 0.4802 \end{bmatrix}, \hat{P}_3 = \begin{bmatrix} 0.4849 & -0.1646 \\ -0.1646 & 0.4541 \end{bmatrix}, \\ U_1 &= \begin{bmatrix} -1.6928 & 0.4292 \end{bmatrix}, U_2 = \begin{bmatrix} -4.7233 & 0.3661 \end{bmatrix}, U_3 = \begin{bmatrix} -3.2042 & -0.4351 \end{bmatrix}, \end{aligned}$$

and the state-feedback gains are given by

$$K_1 = \begin{bmatrix} -3.3600 & -0.4794 \end{bmatrix}, K_2 = \begin{bmatrix} -10.0288 & -3.0119 \end{bmatrix}, K_3 = \begin{bmatrix} -7.9063 & -3.8251 \end{bmatrix}.$$

Thus, the system (3.29) is stochastically stabilizable with state-feedback controller while satisfying the $\mathcal{H}_\infty$ attenuation. Let us consider $w(t)$ be $e^{-0.2t} \sin(t)$. With $\theta(0) = 1$, $x(0) = [-3, 2]^T$, Figure 3.6 shows the stabilized state trajectories of the system for Monte Carlo simulations of 50 runs. From simulation results, $\|z(t)\|_{E_2} / \|w(t)\|_2 = 0.141 < \gamma$. We also show a sample evolution of the state-dependent jump process $\theta(t)$ in figure Figure 3.7. Because the origin is close to the boundary of the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and the presence of norm

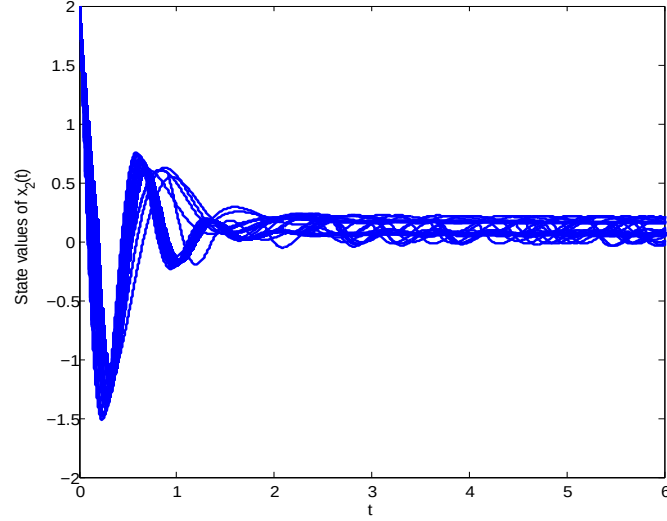


Figure 3.6: Stabilized state trajectories of the system

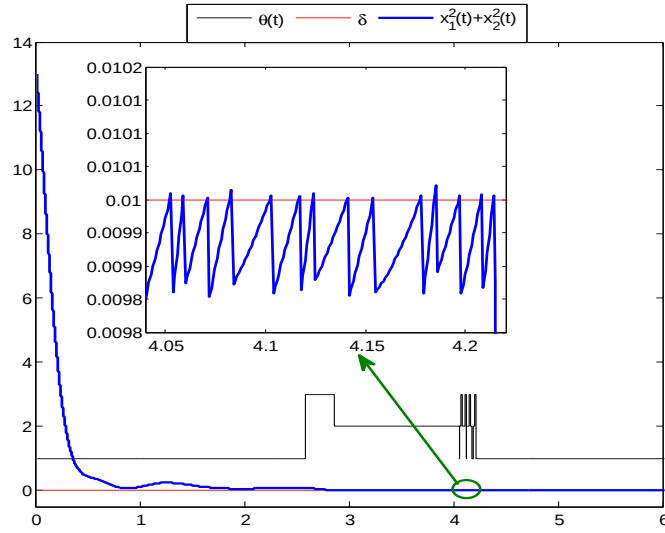


Figure 3.7: A sample state-dependent $\theta(t)$

bounded disturbances cause the fast transitions of the state variable between the sets during the interval that is shown in this sample simulation in the zoomed section. This caused fast switching among the modes during that particular interval. However, we observe that the stabilizing state trajectories in Figure 3.6. Of course fast switching is an undesired effect for the actuators. We would like to study this phenomenon in our future course of work.

3.4 Proofs

Proof of Lemma 3.5:

We follow the approach used in (Krystul and Blom, 2005) and (Mao and Yuan, 2006) to proceed. We prove in the sequel that $(x(t), r(\sigma_t, t), \sigma_t)$ is a joint Markov process with

respect to $\mathcal{F}_t$. Let $\mathcal{F}^t$ denote the natural filtrations of $(x(t), r(\sigma_t, t), \sigma_t)$ on the interval $[t, \infty)$. Let $x(t)$, $r(\sigma_t, t)$ and σ_t from (3.11), (3.10) and (3.9) be denoted, starting from $t_0 \geq 0$, as

$$x(t) \triangleq x_t^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}, r(\sigma_t, t) \triangleq r_t^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}, \text{ and} \\ \sigma_t \triangleq \sigma_t^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}.$$

Denote the joint process $\xi_t^{t_0, \eta(t_0)}$ starting from t_0 as

$$\xi_t^{t_0, \eta(t_0)} \triangleq \left(x_t^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}, r_t^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}, \sigma_t^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}} \right),$$

with $\eta(t_0) = \left(x_{t_0}^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}, r_{t_0}^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}}, \sigma_{t_0}^{t_0, x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0}} \right)$, which is simply

$(x(t_0), r(\sigma_{t_0}, t_0), \sigma_{t_0})$. For $0 < s < t$, $\xi_t^{s, \eta(s)}$ is a random process on $[s, \infty)$ with $\eta(s) = \left(x_s^{s, x(s), r(\sigma_s, s), \sigma_s}, r_s^{s, x(s), r(\sigma_s, s), \sigma_s}, \sigma_s^{s, x(s), r(\sigma_s, s), \sigma_s} \right)$, thus it is $\mathcal{F}^s$ -measurable. Let $\eta(s)$ be an arbitrary $\mathcal{F}_s$ -measurable random variable. For $0 < \tau < s < t$, $\xi_t^{\tau, \eta(\tau)}$ is a $\mathcal{F}^s$ -measurable random process on $[s, \infty)$ with initial condition $\xi_s^{\tau, \eta(\tau)}$. Thus we can write

$$\xi_t^{\tau, \eta(\tau)} = \xi_t^{s, \xi_s^{\tau, \eta(\tau)}}, \quad \text{for } 0 < \tau < s < t. \quad (3.47)$$

Let B be a set in the σ -algebra of Borel sets on $\mathbb{R}^n \times S \times \mathcal{K}$. Then

$$\begin{aligned} \Pr\{\xi_t^{\tau, \eta(\tau)} \in B | \mathcal{F}_s\} &= \mathbb{E} \left[I_B \left(\xi_t^{\tau, \eta(\tau)} \right) | \mathcal{F}_s \right] = \mathbb{E} \left[I_B \left(\xi_t^{s, \xi_s^{\tau, \eta(\tau)}} \right) | \mathcal{F}_s \right], \\ &= \mathbb{E} \left[I_B \left(\xi_t^{s, \eta(s)} \right) \right]_{\eta(s) = \xi_s^{\tau, \eta(\tau)}}, \\ &= \Pr\{\xi_t^{s, \eta(s)} \in B\}, \end{aligned}$$

which completes the proof. $\square$

Proof of Lemma 3.10:

Consider

$$\begin{aligned} \{\tau_0 \leq t\} &= \bigcup_{s \in \mathbb{Q} \cap [0, t]} \{\Phi_{i_0}(s, 0)x(0) \notin \mathcal{C}_{i_0}\}, \\ &= \Omega \setminus \bigcap_{s \in \mathbb{Q} \cap [0, t]} \{\Phi_{i_0}(s, 0)x(0) \in \mathcal{C}_{i_0}\}. \end{aligned}$$

From the above argument, observe that, each event $\{\Phi_{i_0}(s, 0)x(0) \in \mathcal{C}_{i_0}\}$ is $\mathcal{F}_t$ -measurable for all $s \in \mathbb{Q} \cap [0, t]$. Consequently the event $\{\tau_0 \leq t\}$ is also $\mathcal{F}_t$ -measurable, as the complement of the intersection of $\mathcal{F}_t$ -measurable events are also $\mathcal{F}_t$ -measurable. Thus τ_0 is a stopping time. The similar arguments are applied to $\tau_1, \tau_2, \dots$. $\square$

3.5 Conclusions

In this chapter we studied the stochastic stability and stabilization of continuous-time SDJLSs. For the underlying state-dependent random jump process, we explicitly considered different transition rates depending on the set to which the state of the system belong. By Dynkin's formula, we obtained a sufficient condition for stochastic stability in terms of LMIs. Using the sufficient condition, we synthesized a state-feedback controller that stochastically stabilizes the system. Subsequently, we considered SDJLS with additive bounded disturbance and addressed $\mathcal{H}_\infty$ state-feedback stabilization problem. For the unforced system, by using Dynkin's formula, we provided a sufficient condition for stochastic stability while guaranteeing the $\mathcal{H}_\infty$ condition in terms of LMIs. Using the sufficient condition, we synthesized a state-feedback stabilizing controller while guaranteeing the $\mathcal{H}_\infty$ condition. We illustrated the obtained results on a macroeconomic system. As one of the future directions, we would like to consider bounded uncertainties in different parameters of the system and address robust stabilization problem in addition to the $\mathcal{H}_\infty$ state-feedback stabilization dealt in this chapter. This problem will be studied in depth in future. Also, we can consider a discrete-time counter part of our results, which may not be relatively difficult compared to the approach in this chapter. In the next chapter, we consider a discrete-time SDJLS to address the issues of control in the presence of constraints, where we can observe the relative comfort in obtaining stabilization results.

Part II

Control of RJLSs Subject to Constraints

Chapter 4

On Control of Discrete-time State-dependent Jump Linear Systems with Probabilistic Constraints: A Receding Horizon Approach

4.1 Introduction

In this chapter, the system under consideration is a discrete-time SDJLS. In Section 3.1, we gave research motivations to study SDJLSs and also reviewed the existing results related to stability. Now, we address the receding horizon control of discrete-time SDJLSs subject to constraints.

We consider that the SDJLS is affected by possibly unbounded stochastic disturbances, and the perfect state information is assumed. We also deal with constraints on different variables of the system that is inherent in all practical systems. Model predictive control (MPC), also called receding horizon control (RHC), is an effective control algorithm that has a great potential to handle input and/or state constraints, for problems across different disciplines. MPC is a form of control in which the current input is obtained at each sampling time by solving on-line a finite horizon optimal control problem in the presence of constraints, using the current information and the predicted information over the finite horizon. Normally more than one input is obtained at the current sampling time, however, only the first controller input will be implemented to the plant. At the next sampling time, these actions will be repeated, that is why the MPC is also called the RHC. One can refer to (Camacho and Bordons, 2004; Maciejowski, 2002; Mayne et al., 2000; Rawlings

and Muske, 1993) etc., for classic contributions in the MPC literature. In the context of RJLSs, of late, the RHC scheme has been extended to discrete-time MJLSs. For discrete-time MJLSs, the RHC with hard symmetric constraints and bounded uncertainties in system parameters was dealt by (Lu et al., 2013; Wen, 2013), where the constraints and the objective function were posed as sufficient conditions in terms of LMIs to be solved at each sampling time; for the similar case without constraints, the RHC was addressed by (Park and Kwon, 2002) following the similar approach. Also, for unconstrained state and control input, optimality of the RHC was addressed via dynamic programming (Vargas et al., 2005), variational methods (Vargas et al., 2004), solving Riccati equations (Do Val and Başar, 1999), etc.

One major issue in the presence of unbounded disturbances is that the RHC cannot necessarily guarantee the satisfaction of constraints. For instance, an additive unbounded disturbance eventually drives the state variable outside any bounded limits, no matter how arbitrarily large they may be. Also, due to the former reason, it is impossible to ensure bounds on the control input with linear state-feedback policy (Hokayem et al., 2012). A possible alternative is to consider the satisfaction of constraints in stochastic manner, which allow occasional constraint violations. In this direction, recent approaches (Primbs and Sung, 2009; Cannon et al., 2009; Korda et al., 2011), and the references therein treat the RHC of discrete-time linear systems with stochastic constraints. For discrete-time MJLSs, in case of perfect state availability, a linear quadratic regulator problem with second moment constraints was considered in (Vargas et al., 2013), where the entire problem was converted to a set of LMIs. However, to the best of our knowledge, the RHC of discrete-time SDJLSs with probabilistic constraints has not been examined.

In this chapter, we address a one-step RHC of discrete-time SDJLSs with additive Gaussian process noise (its distribution has an unbounded support), and probabilistic state constraints under perfect state availability. We would like to highlight several challenges in our problem set-up. First, in the presence of additive process noise with unbounded support, it is impossible to guarantee hard bounds on the state, and also on the linear state-feedback control. Second, one needs to pre-stabilize the system before addressing the RHC problem. Third, one needs to obtain a tractable representation of the RHC problem in the presence of probabilistic constraints.

Our approach along with main contributions in this chapter can be listed as follows. In our problem set-up, we consider the control to have a linear state-feedback and an offset term (Bemporad and Morari, 1999; Mayne et al., 2005), where linear state-feedback gains are computed off-line for pre-stabilization and admissible offset terms are computed on-line to solve the RHC problem. In the presence of unbounded process noise, it is not possible to ensure hard bounds on the state and the control variables that follow state-feedback law, thus we consider the state variable to be probabilistically constrained and the control to be

unconstrained, except for the offset term. Using inverse cumulative distribution function, we convert the probabilistic state constraints to deterministic constraints and the overall stochastic RHC problem is replaced by a tractable deterministic RHC problem.

To summarize, for SDJLS subject to possibly unbounded random disturbances and probabilistic state constraints, our contributions in this chapter are:

- pre-stabilizing the system state by a state-feedback controller in mean square sense,
- implementing a one-step RHC scheme on-line with probabilistic constraints on the state variable, which are converted to deterministic constraints.

The chapter is organized as follows. Section 4.2 presents the problem setup. We present the pre-stabilization of the system by a state-feedback controller in Section 4.3. We give a one-step RHC scheme with probabilistic constraints in Section 4.4, and obtain its tractable representation. Section 4.5 presents an illustrative example. We give majority of the proofs in Section 4.6 to improve readability followed by conclusions in Section 4.7.

4.2 Problem Setup

Consider a discrete-time SDJLS:

$$x_{k+1} = A_{\theta_k} x_k + B_{\theta_k} u_k + E_{\theta_k} w_k, \quad k \in \mathbb{N}_{\geq 0}, \quad (4.1)$$

with the state $x_k \in \mathbb{R}^n$, the control input $u_k \in \mathbb{R}^m$, the system mode θ_k . Let A_{θ_k} , B_{θ_k} be the system matrices of appropriate dimensions that are assumed to be known, and without loss of generality we assume that E_{θ_k} are identity matrices. Perfect state information and the mode availability is assumed at each time step $k \in \mathbb{N}_{\geq 0}$. We set $\mathcal{F}_k$ the σ -algebra generated by $\{(\theta_t, x_t); 0 \leq t \leq k\}$. Regarding the different stochastic processes, the following assumptions are made:

Assumption 4.1. Let $w_k \in \mathbb{R}^n$ be an n -dimensional standard Gaussian distributed random vector i.e. $w_k \sim \mathcal{N}(0, \mathbb{I}_n)$, independent of $\mathcal{F}_k$.

Assumption 4.2. Let the mode $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}} \in S = \{1, 2, \dots, N\}$ be a finite-state random jump process described by

$$\Pr\{\theta_{k+1} = j | \mathcal{F}_k\} = \Pr\{\theta_{k+1} = j | (\theta_k, x_k)\} = \pi_{\theta_k j}(x_k) = \begin{cases} \lambda_{\theta_k j}, & \text{if } x_k \in \mathcal{C}_1, \\ \mu_{\theta_k j}, & \text{if } x_k \in \mathcal{C}_2, \end{cases} \quad (4.2)$$

where $\lambda_{ij} \geq 0$ ($\mu_{ij} \geq 0$) is the transition probability from a state i to a state j , and $\sum_{j=1}^N \lambda_{ij} = 1$ ($\sum_{j=1}^N \mu_{ij} = 1$). Here we further assume that $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathbb{R}^n$, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

Remark 4.3. In chapter 3, we dealt with the continuous SDJLS, where we considered the sets $\mathcal{C}_1, \dots, \mathcal{C}_K$ in (3.2). For simplicity of the exposition, we just considered two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ (4.2) in this chapter.

The sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in (4.2) are described explicitly as follows. We consider the set $\mathcal{C}_1$ as a polyhedral intersection of finite constraints on the state variable and $\mathcal{C}_2$ as $\mathbb{R}^n - \{\mathcal{C}_1\}$. So, in this chapter, the states of the system that belong to $\mathcal{C}_1$ are specifically denoted by

$$x_k \in \mathcal{C}_1 \iff Gx_k \leq H, \quad (4.3)$$

where $G \in \mathbb{R}^{r \times n}$ and $H \in \mathbb{R}^r$. We can regard $\mathcal{C}_1$ as a desirable set, and ideally we want the system state to be constrained in $\mathcal{C}_1$. However, the state variable being affected by Gaussian noise that is unbounded in distribution, it is impossible to constrain the state to the set $\mathcal{C}_1$ at all times. Thus, we consider probabilistic state constraints, and in Section 4.4.1, we describe in detail the type of probabilistic state constraints that we deal within this chapter.

In our RHC problem, we address the minimization of the following one-step objective function

$$J_k = \mathbb{E} \left[x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + x_{k+1}^T \Psi_{\theta_{k+1}} x_{k+1} | \mathcal{F}_k \right] \quad (4.4)$$

subject to probabilistic state constraints and the control input that follows an affine state-feedback law. Here $Q_i \succ 0$, $R_i \succ 0$, $\Psi_i \succ 0$ are assumed to be known for $i \in S$. In this formulation, the terminal cost depend on the matrices $\Psi_{(\cdot)}$ and the state of the system at time $k+1$. The choice of one-step prediction in (4.4) is explained in detail in Section 4.4 in perspective of the system and the constraints.

4.2.1 Control Policies

We parametrize the control input as (inspired by (Bemporad and Morari, 1999; Mayne et al., 2005))

$$u_k = K_{\theta_k} x_k + \nu_k, \quad (4.5)$$

where feedback gains K_{θ_k} , for $\theta_k \in S$, will be designed to pre-stabilize the system in mean square sense that will be given in the next section. An on-line minimization of J_k in (4.4) subject to constraints will be carried out by an appropriate choice of ν_k .

Remark 4.4. Notice that the state-feedback control input (4.5) is not bounded a priori. The reason is as follows. From (4.1), the state x_k in (4.5) is a function of Gaussian noise. Since Gaussian distribution has an unbounded support, we cannot assume the a priori boundedness of u_k (4.5). Thus, we just consider ν_k to be bounded to obtain a tractable solution of minimization of J_k in (4.4) subject to constraints.

Thus, we give an explicit assumption on ν_k in the following.

Assumption 4.5. We assume that ν_k belong to a compact set $\mathbb{U} \subset \mathbb{R}^m$.

Remark 4.6. Note that ν_k is obtained by solving $\mathcal{P}_3$ on-line that will be introduced in Section 4.4. It will be observed that the computation of ν_k relies on the available infor-

mation (θ_k, x_k) , for each $k \in \mathbb{N}_{\geq 0}$. Thus ν_k can be regarded as $\mathcal{F}_k$ -measurable random variable taking values in $\mathbb{U}$.

Substituting (4.5) into (4.1), we obtain the closed loop system as

$$x_{k+1} = \tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k + w_k, \quad (4.6)$$

with $\tilde{A}_{\theta_k} = A_{\theta_k} + B_{\theta_k} K_{\theta_k}$.

Remark 4.7. In this chapter, we consider the linear state-feedback control input (4.5) for pre-stabilization, where we could not ensure hard bounds on the control because of the stochastic unbounded disturbance. To incorporate bounded controls in the class of causal feedback policies, one alternative could be employing an affine disturbance feedback control with non-linear saturation of noise terms (Hokayem et al., 2010, 2012). In this approach, the disturbance implicitly depends on the output, hence on the state in a causal manner. However, in this case the system under consideration needs to be internally stable.

In the next section, we synthesize the state-feedback gains K_{θ_k} in (4.5), which pre-stabilize (4.6) in bounding the state-trajectories in mean square sense.

4.3 Pre-stabilization Results

In this section, we pre-stabilize the system (4.6) with state-feedback gains K_{θ_k} by solving sufficient LMI conditions off-line.

In general, in the presence of unbounded disturbances, it is difficult to guarantee or establish stability. It is impossible to expect asymptotic stability to origin. However, we synthesize a state-feedback controller that bound the second moment of the state-trajectories. The approach of this section rely on a stochastic version of Lyapunov's second method (Costa et al., 2005; Dragan et al., 2010) by the choice of mode-dependent Lyapunov function $V(x_k, \theta_k) = x_k^T P_{\theta_k} x_k$ with $P_{\theta_k} \succ 0, \forall \theta_k \in S$. We begin with the following notations.

Let $\mathcal{T} \subset \mathbb{R}^{m \times n}$ be a linear subspace of real matrices. Set $\mathcal{T}^N = \mathcal{T} \times \mathcal{T} \times \cdots \times \mathcal{T}$. Also, let $\mathcal{S}_n \subset \mathbb{R}^{n \times n}$ be a linear subspace of real symmetric matrices. Set $\mathcal{S}_n^N = \mathcal{S}_n \times \mathcal{S}_n \times \cdots \times \mathcal{S}_n$. We give a proposition to pre-stabilize the system (4.6) in the following.

Proposition 4.8. *If there exist matrices $K = (K_1 \ \cdots \ K_N) \in \mathcal{T}^N$ and $P = (P_1 \ \cdots \ P_N) \in \mathcal{S}_n^N$, with $P_i \succ 0, 1 \leq i \leq N$ such that*

$$\tilde{A}_i^T \left(\sum_{j=1}^N \lambda_{ij} P_j \right) \tilde{A}_i - P_i = \mathcal{E}_i^1 \prec 0, \quad (4.7)$$

and

$$\tilde{A}_i^T \left(\sum_{j=1}^N \mu_{ij} P_j \right) \tilde{A}_i - P_i = \mathcal{E}_i^2 \prec 0. \quad (4.8)$$

then the trajectories of (4.6) satisfy the constraints:

$$\mathbb{E} [\|x_k\|^2] \leq \beta_1 \|x_0\|^2 q^k + \beta_2, \forall k \in \mathbb{N}_{\geq 0}, \quad (4.9)$$

for some $\beta_1 \geq 1$, $\beta_2 > 0$ and $q \in (0, 1)$.

Proof. Given in Section 4.6. □

In the sequel, we provide a proposition to synthesize gains K_{θ_k} , for $\theta_k \in S$.

Proposition 4.9. *If there exist matrices $X_i \succ 0$ and Y_i , $1 \leq i \leq N$, such that*

$$\begin{bmatrix} -X_D & \Lambda_i (A_i X_i + B_i Y_i) \\ * & -X_i \end{bmatrix} \prec 0, \quad (4.10)$$

and

$$\begin{bmatrix} -X_D & \Gamma_i (A_i X_i + B_i Y_i) \\ * & -X_i \end{bmatrix} \prec 0, \quad (4.11)$$

where

$$\Lambda_i = [\sqrt{\lambda_{i1}} \mathbb{I}_n \quad \cdots \quad \sqrt{\lambda_{iN}} \mathbb{I}_n]^T, \Gamma_i = [\sqrt{\mu_{i1}} \mathbb{I}_n \quad \cdots \quad \sqrt{\mu_{iN}} \mathbb{I}_n]^T, X_D = \text{diag}\{X_1, \dots, X_N\},$$

then the trajectories of the system (4.6) satisfy (4.9), and K_i is given by

$$K_i = Y_i X_i^{-1}. \quad (4.12)$$

Proof. Briefly, we explain the steps to arrive at the result. Consider the Schur complement of (4.7), (4.8), and let $X_i \triangleq P_i^{-1}$. By the congruent transformation of Schur complement of (4.7), (4.8) by $\text{diag}\{X_1, \dots, X_N, X_i\}$, and applying a change of variable $Y_i \triangleq K_i X_i$ one obtains (4.10), (4.11). Hence, if (4.10), (4.11) are feasible and using proposition 4.8, it follows that the trajectories of (4.6) satisfy the estimates (4.9). Hence the proof is complete. □

Remark 4.10. Though the conditions to obtain the state-feedback gains K_{θ_k} , for $\theta_k \in S$, in proposition 4.9 seems computationally heavy, we perform these computations off-line, and only once. Once we obtain the state-feedback gains K_{θ_k} , we use them as a lookup table, where depending on the mode θ_k at each time step $k \in \mathbb{N}_{\geq 0}$, the corresponding value of the state-feedback gain is used in updating the state in (4.6).

4.4 A RHC with Probabilistic State Constraints

In this section, we provide a RHC problem with the objective function (4.4) subject to the probabilistic constraints and the state-feedback control input (4.5). We begin with probabilistic constraints.

4.4.1 Probabilistic Constraints

In this section, we consider the probabilistic state constraints that we address in this chapter. In general, the inclusion of probabilistic constraints in a RHC problem lead to intractability. Thus to make the RHC problem tractable, we replace the probabilistic constraints by deterministic constraints.

We mentioned in Section 4.2, that it is impossible to constrain the state variable to the set $\mathcal{C}_1$, which is polyhedron $Gx_k \leq H$, where $G \in \mathbb{R}^{r \times n}$ and $H \in \mathbb{R}^r$. Rather, let us consider probabilistic constraint

$$\Pr\{Gx_k \leq H\} \geq \xi, \quad (4.13)$$

which portrays the probabilistic scenario of staying in the polyhedron. Here ξ represents the level of constraint satisfaction, which needs to be high enough ($\xi \gg 0$). However, without loss of generality, we assume that ξ may take any values in the interval $(0, 1)$. In the sequel, we try to obtain deterministic constraints that imply the probabilistic constraint (4.13). We say it as converting probabilistic constraint to deterministic ones. A sufficient condition to satisfy (4.13) is given by (Korda et al., 2011),

$$\Pr\{Gx_{k+1} \leq H | \mathcal{F}_k\} \geq \xi. \quad (4.14)$$

Now, we present a sufficient deterministic condition to satisfy (4.14) via approximation of inscribed ellipsoidal constraints (Van Hessem et al., 2001; Van Hessem and Bosgra, 2002).

Lemma 4.11. *Let $\delta = \sqrt{F_{n,Chi}^{-1}(\xi)}$, where $F_{n,Chi}^{-1}(\xi)$ denotes Chi-square inverse cumulative distribution function with a given probability ξ and n degrees of freedom. If*

$$G_j(\tilde{A}_{\theta_k}x_k + B_{\theta_k}\nu_k) \leq H_j - \|G_j\|_2\delta, \text{ for } 1 \leq j \leq r, \quad (4.15)$$

then $\Pr\{Gx_{k+1} \leq H | \mathcal{F}_k\} \geq \xi$, where G_j and H_j are j^{th} rows of G and H respectively.

Proof. The proof is given on the similar lines of (Van Hessem et al., 2001; Van Hessem and Bosgra, 2002) and is outlined in Section 4.6. $\square$

Note that (4.15) is an over conservative condition, because the deterministic condition (4.15) implies (4.14), which finally implies (4.13). Even for this special case, we could not obtain an equivalent representation of (4.14), which is non-convex in general, where it is hard to find its feasible region (Prékopa, 1995). So, alternatively, we propose the individual probabilistic constraints of type

$$\Pr\{G_jx_k \leq H_j\} \geq \xi, \quad 1 \leq j \leq r, \quad (4.16)$$

where G_j and H_j represent the j^{th} row of G and H respectively. With given ξ , the constraints (4.16) offer satisfaction of each individual constraint of the polyhedron probabilistically, but with more constraint violations (Arnold et al., 2013) than (4.13) that

will also be observed in Section 4.5. However, we consider the individual probabilistic constraints (4.16), because they are simpler to handle and in general convex (Prékopa, 1995; Arnold et al., 2013). Similar to the above treatment, a sufficient condition to satisfy (4.16) is given by

$$\Pr\{G_j x_{k+1} \leq H_j | \mathcal{F}_k\} \geq \xi, \quad 1 \leq j \leq r. \quad (4.17)$$

Similar to the approach of (Yan and Bitmead, 2005; Korda et al., 2011; Schwarm and Nikolaou, 1999), one obtains,

$$\Pr\{G_j(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k + w_k) \leq H_j\} \geq \xi \quad (4.18)$$

$$\begin{aligned} &\iff \Pr\{G_j w_k \leq H_j - G_j(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k)\} \geq \xi \\ &\iff F_{G_j w_k}(H_j - G_j(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k)) \geq \xi \\ &\iff G_j(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k) \leq H_j - F_{G_j w_k}^{-1}(\xi), \end{aligned} \quad (4.19)$$

where $F_{G_j w_k}(\cdot)$ and $F_{G_j w_k}^{-1}(\cdot)$ are the cumulative distribution and inverse cumulative distribution of the random variable $G_j w_k$ respectively.

Remark 4.12. The probabilistic constraints (4.16) result in handling with uni-variate Gaussian random variables $G_j w_k$ when converting to deterministic constraints (4.19), which is straightforward. In this case, given ξ , $F_{G_j w_k}^{-1}(\xi)$ can easily be obtained.

Observe that the conditions (4.17) and (4.19) are equivalent, which imply (4.16).

Remark 4.13. The probabilistic constraints (4.13) and (4.16) are two different ways of treating the constraints $Gx_k \leq H$ in a probabilistic fashion. We consider the probabilistic constraints (4.17) because of the simplicity and low conservatism involved.

4.4.2 A Tractable Representation of the RHC problem

In this section, we obtain a tractable representation of the RHC problem. At each $k \in \mathbb{N}_{\geq 0}$, we consider the following one-step RHC problem

$$\begin{aligned} \mathcal{P}_1 : \min_{\nu_k} J_k &= \mathbb{E} \left[x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + x_{k+1}^T \Psi_{\theta_{k+1}} x_{k+1} | \mathcal{F}_k \right] \\ \text{s.t. } &(4.1), (4.2), \\ &\Pr\{G_j x_{k+1} \leq H_j | \mathcal{F}_k\} \geq \xi, \quad 1 \leq j \leq r, \quad \xi \in (0, 1), \end{aligned} \quad (4.20)$$

$$u_k = K_{\theta_k} x_k + \nu_k, \quad \nu_k \in \mathbb{U}. \quad (4.21)$$

Remark 4.14. In $\mathcal{P}_1$, we choose the prediction horizon to be one because of the probabilistic constraints (4.20) and the system (4.6). In Section 4.4.1, we obtained a deterministic equivalence of the constraints (4.20) in terms of the state variable and the mode at time k that are known. In general, the larger prediction horizon result in better performance and more computational burden depending on the system (Maciejowski, 2002). The choice of the prediction horizon depends on the performance requirements and computational

capability. Suppose, if we consider a multi-step prediction, the probabilistic constraints in $\mathcal{P}_1$ look like $\Pr\{G_j x_{k+N} \leq H_j | \mathcal{F}_k\} \geq \xi$, for $N \geq 2$. By proceeding with the similar approach of Section 4.4.1, we can obtain an equality similar to (4.18) that contain additional unknown random variables θ_{k+m} , $1 \leq m \leq N-1$, where it is not possible to obtain its deterministic equivalence. Thus, we choose a one-step prediction horizon to obtain a deterministic equivalence of the probabilistic constraints (4.20) for tractability of $\mathcal{P}_1$. At this point, we give a brief review of some works in the literature of RJLSs with multi-step prediction. In case of perfect state availability, without process noise, a multi-step prediction horizon was considered for discrete-time MJLSs with hard symmetric input and state constraints (Lu et al., 2013; Wen, 2013), where an upper bound of the objective function was minimized and the overall problem with the constraints was converted to sufficient LMIs. The approach was based on (Costa et al., 1999b), where the state variable was restricted to an invariant ellipsoid in the constrained space and the techniques of LMIs and Lyapunov method were utilized. However, it brings a computational burden because of the additional LMIs, and more importantly it reduces the feasibility region drastically depending on the invariant ellipsoid. To avoid the sufficiency, the authors in (Vargas et al., 2013) directly considered ellipsoidal constraints (in terms of second moment constraints) for discrete-time MJLSs with multi-step prediction, where the constraints were replaced by a set of LMIs.

In RHC problems in general, it is important to guarantee the feasibility of constraints before starting the optimization problem. Notice that it can be possible that the problem $\mathcal{P}_1$ becomes infeasible when the constraints (4.20) are tight with a given admissible input. To resolve this issue, the constraints can be relaxed by an additional slack variable $\rho_k \geq 0$ as (Schwarm and Nikolaou, 1999),

$$\Pr\{G_j x_{k+1} \leq H_j + \rho_k | \mathcal{F}_k\} \geq \xi. \quad (4.22)$$

Thus from (4.19),

$$G_j (\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k) \leq H_j - F_{G_j w_k}^{-1}(\xi) + \rho_k. \quad (4.23)$$

The addition of a variable ρ_k can be compensated by adding a variable $\alpha \rho_k$ to the objective function in $\mathcal{P}_1$. In particular, $\alpha > 0$ should be chosen as a very high value, which act as a penalty to discourage the use of slack variable (Chinneck, 2007). Thus $\mathcal{P}_1$ will be

$$\mathcal{P}_2 : \min_{\nu_k, \rho_k} J_k = \mathbb{E} \left[x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + x_{k+1}^T \Psi_{\theta_{k+1}} x_{k+1} | \mathcal{F}_k \right] + \alpha \rho_k$$

$$\text{s.t. (4.1), (4.2),}$$

$$G_j (\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k) \leq H_j - F_{G_j w_k}^{-1}(\xi) + \rho_k, \quad 1 \leq j \leq r, \quad \xi \in (0, 1), \quad (4.24)$$

$$u_k = K_{\theta_k} x_k + \nu_k, \quad \nu_k \in \mathbb{U}, \quad (4.25)$$

$$\rho_k \geq 0. \quad (4.26)$$

Notice that due to (4.24), (4.26), $\mathcal{P}_2$ is always feasible. In order to write the $\mathcal{P}_2$ in terms of tractable terms, consider the term $\mathbb{E} \left[x_{k+1}^T \Psi_{\theta_{k+1}} x_{k+1} | \mathcal{F}_k \right]$. From (4.6), x_{k+1} is a random vector with

$$\mathbb{E} [x_{k+1} | \mathcal{F}_k] = \tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k,$$

and

$$\mathbb{E} \left[\left\{ x_{k+1} - \left(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k \right) \right\} \star | \mathcal{F}_k \right] = \mathbb{E} [w_k w_k^T] = \mathbb{I}_n.$$

And now, let us consider the following lemma which is a classic result,

Lemma 4.15. *Let x be a random vector with mean m and covariance matrix R , and let K be a given matrix of suitable dimensions. Then*

$$\mathbb{E} [x^T K x] = m^T K m + \text{tr}(K R). \quad (4.27)$$

The result (4.27) and the law of total probability gives that

$$\mathbb{E} [x_{k+1}^T \Psi_{\theta_{k+1}} x_{k+1} | \mathcal{F}_k] = \left(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k \right)^T \Pi_{\theta_k} \star + \text{tr}(\Pi_{\theta_k}), \quad (4.28)$$

where $\Pi_{\theta_k} \triangleq \sum_{j=1}^N \pi_{\theta_k j}(x_k) \Psi_j$. By (4.28), the objective function in $\mathcal{P}_2$ can be expressed as

$$J_k = x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + \left(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k \right)^T \Pi_{\theta_k} \star + \text{tr}(\Pi_{\theta_k}) + \alpha \rho_k.$$

Thus, the problem $\mathcal{P}_2$ is equivalent to

$$\begin{aligned} \mathcal{P}_3 : \min_{\nu_k, \rho_k} J_k &= x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + \left(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k \right)^T \Pi_{\theta_k} \star + \text{tr}(\Pi_{\theta_k}) + \alpha \rho_k \\ \text{s.t. } &(4.1), (4.2), \\ &G_j(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k) \leq H_j - F_{G_j w_k}^{-1}(\xi) + \rho_k, \quad 1 \leq j \leq r, \quad \xi \in (0, 1), \quad (4.29) \\ &u_k = K_{\theta_k} x_k + \nu_k, \quad \nu_k \in \mathbb{U}, \quad (4.30) \\ &\rho_k \geq 0. \quad (4.31) \end{aligned}$$

We denote J_k^* as an optimal solution of $\mathcal{P}_3$. To solve $\mathcal{P}_3$, which is a Quadratic Programming (QP) problem, we use the default QP solver in MATLAB. When we solve $\mathcal{P}_3$, if ρ_k is zero, for all $k \geq 0$, then we have solved original $\mathcal{P}_1$, and the solution of $\mathcal{P}_3$ would be equivalent to the solution of $\mathcal{P}_1$. If ρ_k is positive, then we conclude that $\mathcal{P}_1$ was infeasible, and the solution of $\mathcal{P}_3$ would approximate the solution of $\mathcal{P}_1$. This approximation depends on the magnitude of ρ_k , which needs to be lower for better approximation.

4.5 A Numerical Example

In this section, we illustrate the proposed approach on a numerical example. Consider SDJLS (4.1) with the $x_k \in \mathbb{R}^2$, $x_0 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$, $\nu_k \in [-10, 10] \times [-10, 10] \subset \mathbb{R}^2$, $\theta_k = \{1, 2\}$,

$w_k \sim \mathcal{N}(0, \mathbb{I}_2)$, and let the mode matrices be:

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.5 & 2 \\ 3 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The transition probabilities for θ_k are given by (4.2), where

$$\lambda = \begin{bmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{bmatrix}, \mu = \begin{bmatrix} 0.1 & 0.9 \\ 0.25 & 0.75 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}^T, H = \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}^T.$$

For pre-stabilization, we consider proposition 4.9 that can be satisfied with

$$X_1 = \begin{bmatrix} 1.1231 & -0.2880 \\ -0.2880 & 2.1246 \end{bmatrix}, X_2 = \begin{bmatrix} 0.1463 & -0.2828 \\ -0.2828 & 0.6060 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} -0.2151 & -0.5485 \\ -0.5485 & 1.5483 \end{bmatrix}, Y_2 = \begin{bmatrix} -0.0479 & 0.1057 \\ 0.1057 & -0.2570 \end{bmatrix}.$$

Thus the system is pre-stabilized in the sense of (4.9), and the state-feedback gains are given by

$$K_1 = \begin{bmatrix} -0.2670 & -0.2944 \\ -0.3124 & 0.6864 \end{bmatrix}, K_2 = \begin{bmatrix} 0.0999 & 0.2210 \\ -0.9956 & -0.8888 \end{bmatrix}.$$

Let us consider the soft constraint (4.17) with $\xi=0.85$. Let $\Psi_1 = 2\mathbb{I}_2$, $\Psi_2 = \mathbb{I}_2$, $Q_1 = \mathbb{I}_2$, $Q_2 = 2\mathbb{I}_2$, $R_1 = 1$, and $R_2 = 2$, $\alpha = 1000$. Using the parameters, we solve $\mathcal{P}_3$ at each time k . A sample evolution of modes θ_k and corresponding state values x_k are portrayed in Figure 4.1.

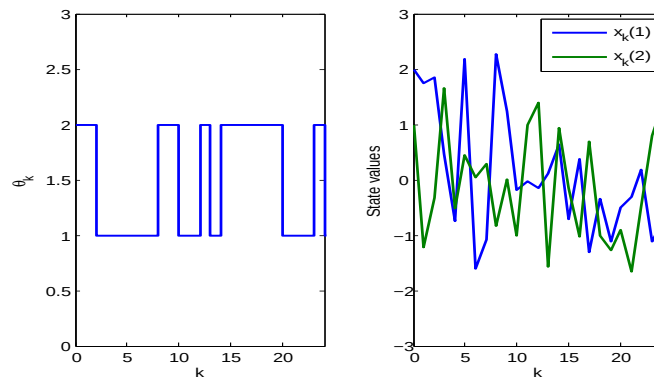


Figure 4.1: A sample evolution of modes θ_k and corresponding state values x_k

To compare the probabilistic state constraints (4.17) with different values of ξ qualitatively, we perform Monte Carlo simulations of 100 runs for values of ξ (0.95, 0.5, 0.3) separately by keeping the remaining parameters same as above, and the phase plots are shown in Figure 4.2. Observe that the larger the level of constraint satisfaction ξ , the

better the states meet the required target. For this experiment, we obtained ρ_k as zero, for all k , when ξ is 0.5 and 0.3; $\rho_k > 0$, for some k , when ξ is 0.95. It implies that the higher the level of constraint satisfaction ξ , the more the chance of becoming the original problem $\mathcal{P}_1$ infeasible. This can also be observed from (4.19), where the larger values of ξ makes the right hand side of (4.19) smaller, thus reducing its feasibility region.

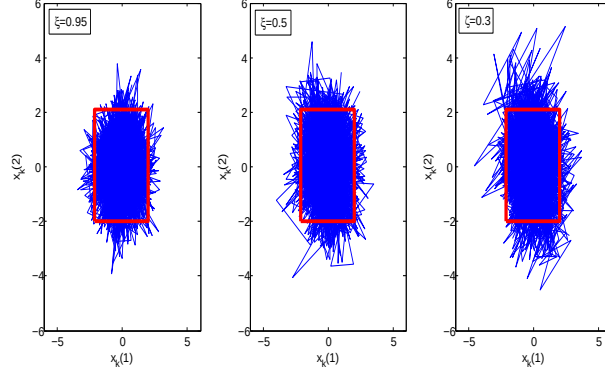


Figure 4.2: Illustration of probabilistic constraints with different values of ξ

4.6 Proofs

Proof of Proposition 4.8:

Consider the given system

$$x_{k+1} = \tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k + w_k, \quad (4.32)$$

where θ_k is defined in (4.2) and $\tilde{A}_{\theta_k} = A_{\theta_k} + B_{\theta_k} K_{\theta_k}$. From remark 4.6, $\{\nu_k\}_{k \in \mathbb{N}_{\geq 0}}$ is a stochastic process with the property that for each $k \in \mathbb{N}_{\geq 0}$, the vector ν_k is $\mathcal{F}_k$ -measurable and $\mathbb{E}[\|\nu_k\|^2] \leq \delta < \infty$.

One knows that there exist matrices $K_i \in \mathcal{T}^N$ and $P_i \in \mathcal{S}_n^N$, with $P_i \succ 0$, for $1 \leq i \leq N$ such that (4.7) and (4.8) are verified. Consider $V(x_k, \theta_k) = x_k^T P_{\theta_k} x_k$. Let $\Upsilon_{(\theta_k, x_k)} \triangleq \sum_{j=1}^N \pi_{\theta_k j}(x_k) P_j$, $\mathcal{L}_{\theta_k} \triangleq \sum_{j=1}^N \lambda_{\theta_k j} P_j$, and $\mathcal{M}_{\theta_k} \triangleq \sum_{j=1}^N \mu_{\theta_k j} P_j$. One has

$$\begin{aligned} & \mathbb{E}[V(x_{k+1}, \theta_{k+1}) | \mathcal{F}_k] - V(x_k, \theta_k) \\ &= x_k^T (\tilde{A}_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \tilde{A}_{\theta_k} - P_{\theta_k}) x_k + 2x_k^T \tilde{A}_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \mathbb{E}[w_k | \mathcal{F}_k] + 2\nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \tilde{A}_{\theta_k} x_k \\ & \quad + 2\nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \mathbb{E}[w_k | \mathcal{F}_k] + \nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} B_{\theta_k} \nu_k + \mathbb{E}[w_k^T \Upsilon_{(\theta_k, x_k)} w_k | \mathcal{F}_k], \\ & \leq -\mu \|x_k\|^2 + 2x_k^T \tilde{A}_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \mathbb{E}[w_k | \mathcal{F}_k] + 2\nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \tilde{A}_{\theta_k} x_k \\ & \quad + 2\nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \mathbb{E}[w_k | \mathcal{F}_k] + \alpha_1 \|\nu_k\|^2 + \alpha_2 \mathbb{E}[w_k^T w_k | \mathcal{F}_k], \end{aligned}$$

where $\mu = \min_{1 \leq j \leq 2} \left(\min_{1 \leq i \leq N} (\lambda_{\min}(-\mathcal{E}_i^j)) \right)$,

$\alpha_1 = \max \left(\max_{1 \leq i \leq N} (\lambda_{\max}(B_i^T \mathcal{L}_i B_i)) , \max_{1 \leq i \leq N} (\lambda_{\max}(B_i^T \mathcal{M}_i B_i)) \right)$,

and $\alpha_2 = \max \left(\max_{1 \leq i \leq N} (\lambda_{\max}(\mathcal{L}_i)) , \max_{1 \leq i \leq N} (\lambda_{\max}(\mathcal{M}_i)) \right)$. Because the random vector w_k is

independent of $\mathcal{F}_k$ we obtain

$$\begin{aligned} & \mathbb{E}[V(x_{k+1}, \theta_{k+1}) | \mathcal{F}_k] - V(x_k, \theta_k) \\ & \leq -\mu \|x_k\|^2 + 2x_k^T \tilde{A}_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \mathbb{E}[w_k] + 2\nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \tilde{A}_{\theta_k} x_k + 2\nu_k^T B_{\theta_k}^T \Upsilon_{(\theta_k, x_k)} \mathbb{E}[w_k] \\ & \quad + \alpha_1 \|\nu_k\|^2 + \alpha_2 \mathbb{E}[w_k^T w_k], \\ & \leq -\mu \|x_k\|^2 + 2\|\nu_k\| \|B_{\theta_k}\| \|\Upsilon_{(\theta_k, x_k)}\| \|\tilde{A}_{\theta_k}\| \|x_k\| + \alpha_1 \|\nu_k\|^2 + n\alpha_2. \end{aligned}$$

From the inequality given in lemma A.2,

$$2\|\nu_k\| \|B_{\theta_k}\| \|\Upsilon_{(\theta_k, x_k)}\| \|\tilde{A}_{\theta_k}\| \|x_k\| \leq \kappa \|x_k\|^2 + \kappa^{-1} \beta^2 \|\nu_k\|^2,$$

where $\beta = \left(\max_{1 \leq i \leq N} \|B_i\| \right) \cdot \alpha_2 \cdot \left(\max_{1 \leq i \leq N} \|\tilde{A}_i\| \right)$. This yields

$$\mathbb{E}[V(x_{k+1}, \theta_{k+1}) | \mathcal{F}_k] - V(x_k, \theta_k) \leq -(\mu - \kappa) \|x_k\|^2 + (\alpha_1 + \kappa^{-1} \beta^2) \|\nu_k\|^2 + n\alpha_2.$$

Take $\kappa \in (0, \mu)$. We have

$$\mathbb{E}[V(x_{k+1}, \theta_{k+1}) | \mathcal{F}_k] \leq (1 - c_2^{-1}(\mu - \kappa)) V(x_k, \theta_k) + (\alpha_1 + \kappa^{-1} \beta^2) \|\nu_k\|^2 + n\alpha_2, \quad (4.33)$$

where $c_2 = \max_{1 \leq i \leq N} (\lambda_{\max}(P_i))$. One can show using similar arguments as in (Ji and Chizeck, 1990b) that $0 < (1 - c_2^{-1}(\mu - \kappa)) < 1$. Let $q = 1 - c_2^{-1}(\mu - \kappa)$. Taking expectation on both sides of (4.33), we get

$$\mathbb{E}[V(x_{k+1}, \theta_{k+1})] \leq q \mathbb{E}[V(x_k, \theta_k)] + (\alpha_1 + \kappa^{-1} \beta^2) \mathbb{E}[\|\nu_k\|^2] + n\alpha_2.$$

We obtain recursively

$$\begin{aligned} \mathbb{E}[V(x_k, \theta_k)] & \leq q^k \mathbb{E}[V(x_0, \theta_0)] + (\alpha_1 + \kappa^{-1} \beta^2) \sum_{p=1}^{k-1} q^{k-p-1} \mathbb{E}[\|\nu_p\|^2] + n\alpha_2 \sum_{p=1}^{k-1} q^{k-p-1}, \\ & \leq q^k \mathbb{E}[V(x_0, \theta_0)] + \frac{n\alpha_2 + \delta(\alpha_1 + \kappa^{-1} \beta^2)}{1 - q}. \end{aligned}$$

Finally, we obtain

$$\mathbb{E}[\|x_k\|^2] \leq \frac{c_2}{c_1} q^k \|x_0\|^2 + \frac{n\alpha_2 + \delta(\alpha_1 + \kappa^{-1} \beta^2)}{c_1(1 - q)},$$

where $c_1 = \min_{1 \leq i \leq N} (\lambda_{\min}(P_i))$. Hence the proof is complete. $\square$

Proof of Lemma 4.11:

Consider the given probabilistic constraint (4.13),

$$\Pr\{Gx_{k+1} \leq H | \mathcal{F}_k\} \geq \xi. \quad (4.34)$$

Let $H_k \triangleq H - G(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k)$, then (4.34) is equivalent to

$$\Pr\{Gw_k \leq H_k\} \geq \xi. \quad (4.35)$$

Let $\mathcal{Z} = \{\gamma : G\gamma \leq H_k\}$, then we can rewrite the above condition as $\Pr\{w_k \in \mathcal{Z}\} \geq \xi$,

which is equivalent to

$$\frac{1}{\sqrt{2\pi^n}} \int_{\mathcal{Z}} e^{-\frac{1}{2}\|v\|^2} dv \geq \xi. \quad (4.36)$$

Since $\mathcal{Z}$ is a polyhedron, it is difficult to obtain the closed form of the above integral. So, we consider an inscribed ellipsoidal approximation of $\mathcal{Z}$ (Van Hessem et al., 2001), (Van Hessem and Bosgra, 2002). It is reasonable since the level curves of multivariate Gaussian distribution are ellipsoids. Consider an ellipsoid (since the covariance of w_k is identity matrix),

$$\mathcal{E}_\delta \triangleq \{\gamma : \|\gamma\|^2 \leq \delta^2\}.$$

If $\mathcal{E}_\delta$ is inscribed in the polyhedron $\mathcal{Z}$, i.e;

$$\mathcal{E}_\delta \subset \mathcal{Z}, \quad (4.37)$$

then $\Pr\{w_k \in \mathcal{Z}\} \geq \xi$ is implied by

$$\frac{1}{\sqrt{2\pi^n}} \int_{\mathcal{E}_\delta} e^{-\frac{1}{2}\|v\|^2} dv \geq \xi.$$

The above multiple integral can be converted to single integral by following standard techniques of finding multivariate probability density function as

$$\frac{1}{\sqrt{2^n}\Gamma(n/2)} \int_0^{\delta^2} \chi^{\frac{n-2}{2}} e^{-\frac{1}{2}\chi} d\chi \geq \xi,$$

that can be rewritten as

$$F_{n,Chi}(\delta^2) \geq \xi, \quad (4.38)$$

where $F_{n,Chi}(\cdot)$ denotes the Chi-square cumulative distribution function with n degrees of freedom. The above inequality can be satisfied with δ value of $\sqrt{F_{n,Chi}^{-1}(\xi)}$. By utilizing the maximization of a liner functional over an ellipsoidal set, $\mathcal{E}_\delta \subset \mathcal{Z}$ can be ensured by (Ben-Tal and Nemirovski, 2001), (Van Hessem et al., 2001),

$$\delta \sqrt{G_j G_j^T} \leq H_{kj}, \quad 1 \leq j \leq r, \quad (4.39)$$

$$\delta \|G_j\|_2 \leq H_{kj}, \quad 1 \leq j \leq r, \quad (4.40)$$

for (4.37), where G_j and H_{kj} are j^{th} rows of G and H_k respectively. Thus (4.39) is equivalent to

$$G_j(\tilde{A}_{\theta_k} x_k + B_{\theta_k} \nu_k) \leq H_j - \delta \|G_j\|_2, \quad 1 \leq j \leq r,$$

thus completes the proof. $\square$

4.7 Conclusions

We considered a RHC of discrete-time SDJLSs subject to additive stochastic unbounded disturbance with probabilistic state constraints. We used an affine state-feedback control, and synthesized feedback gains that guarantee the mean square boundedness of the system by solving a LMI off-line. We obtained sufficient deterministic conditions to satisfy probabilistic constraints by utilizing inverse cumulative distribution function, and consequently converted the overall receding horizon problem as a tractable deterministic optimization problem. Although, it is difficult to guarantee the recursive feasibility in the case of stochastic unbounded disturbance, we attempted to resolve this issue with an addition of slack variable to the obtained constraints with a penalty in the objective function. We performed simulations on a numerical example to verify the proposed methodology. We would like to extend the results of this chapter to a case where the state information is not perfectly available. This idea will be addressed in detail in the next chapter.

Chapter 5

A Receding Horizon Control of Discrete-time Markov Jump Linear Systems with Probabilistic State Constraints by Incorporating State Estimation

5.1 Introduction

In the previous chapter, we studied the RHC of discrete-time SDJLSs subject to probabilistic state constraints. In this chapter, we would like to extend the results of the previous chapter to the case of imperfect state availability. However, the unavailability of the perfect state variable makes the state-dependent jump process intractable. Thus, we restrict ourselves to the RHC of discrete-time inhomogeneous MJLSs, a simpler discrete-time inhomogeneous RJLSs, subject to probabilistic state constraints, when the perfect state information is not available. In this case, the underlying random jump process is time inhomogeneous Markov, which can be tractable even though the state is not perfectly available. In Section 2.1, we gave research motivations to study time inhomogeneous MJLSs and also reviewed the existing results related to stability.

In this chapter, we consider a discrete-time inhomogeneous MJLS subject to additive process noise and measurement noise that may have an unbounded support. In the presence of noisy measurements and the availability of mode information at each time, state estimation problem of discrete-time MJLSs has been addressed in (Costa, 1994; Ji and Chizeck, 1990b; Chizeck and Ji, 1988) and references therein and later. Most of these works assumed the availability of mode information at each time, because the unavailability of mode values would lead to an optimal filter obtained from a bank of Kalman

filters, which demands exponentially increasing memory and computing capabilities (Bar-Shalom and Li, 1993). Thus for simplicity of the exposition, we assume the availability of mode information at each time, but not in advance. We also deal with constraints on different variables of the system, which appear in all practical systems.

We discussed in chapter 4 that in the presence of process noise with an unbounded support (e.g. Gaussian), the constraints may not necessarily be satisfied. Also, due to the former reason, it is impossible to ensure bounds on the control input with linear state-feedback policy (Hokayem et al., 2012). A viable alternative can be addressing the constraint satisfaction in a stochastic manner. For discrete-time systems, the RHC problem the RHC with stochastic constraints by incorporating state estimation was addressed in (Yan and Bitmead, 2005; Hokayem et al., 2012) etc. To the best of our knowledge, the RHC of discrete-time inhomogeneous MJLSs with incomplete state information and probabilistic constraints has not been examined. In this chapter, we consider the probabilistic state constraints, in particular individual probabilistic constraints which are simpler to handle and result in convex conditions as compared to joint probabilistic constraints that are non-convex and it is hard to find its feasible region (Prékopa, 1995; Henrion, 2007). A detailed exposition of our choice of individual probabilistic state constraints is given in Section 4.4.1 of chapter 4.

Over all, in this chapter, we address a one-step RHC of discrete-time inhomogeneous MJLSs with additive Gaussian measurement and process noise, and individual probabilistic state constraints. We would like to emphasize several challenges in our problem setup. First, one needs a filter to estimate the state variable with noisy measurements in the RHC framework. Second, in the presence of additive Gaussian process noise, it is not possible to ensure hard bounds on the state, and also on the linear state-feedback control. Third, one ought to address the pre-stabilization of the system before considering the RHC problem. Fourth, one needs to obtain a tractable representation of the RHC problem in the presence of probabilistic state constraints.

Our approach along with main contributions in this chapter can be listed as follows. In our problem setup, the control obtained by solving the RHC problem may not necessarily make the overall system linear, but we show state estimation equations still follow recursions that appear like Kalman filter recursions. We consider the control to follow an affine state-feedback policy (Bemporad and Morari, 1999; Mayne et al., 2005), where linear state-feedback gains are designed off-line for pre-stabilization and admissible offset terms are computed on-line to solve the RHC problem. The admissible offset terms are the reason for possible nonlinearity of the system, because of their possible nonlinear dependence on output measurements, which implicitly depends on the system states. In the presence of additive Gaussian process noise, it is not possible to guarantee hard bounds on the state and the affine state-feedback control, thus we consider the state variable belongs

to individual probabilistic constrained set and the control to be unconstrained, except for the offset term. We replace individual probabilistic state constraints by deterministic constraints in terms of conditional estimates and covariances of the state variable. Finally, the overall RHC problem is approximated as a tractable deterministic RHC problem.

The rest of the chapter is organized as follows: Section 5.2 introduces the problem considered in this chapter. We give pre-stabilization results in Section 5.3. Section 5.4 presents a tractable representation of the over all RHC policy. We show the effectiveness of our approach on vertical take-off and landing vehicle in Section 5.5. To improve readability of the chapter, we give majority of the proofs in Section 5.6. Finally, we give conclusions in Section 5.7.

5.2 Problem Description

Consider a discrete-time MJLS:

$$\begin{aligned} x_{k+1} &= A_{\theta_k} x_k + B_{\theta_k} u_k + w_k, \quad k \in \mathbb{N}_{\geq 0}, \\ y_k &= C_{\theta_k} x_k + v_k, \end{aligned} \quad (5.1)$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^r$ is the measured output, the random process θ_k is the system mode, and A_{θ_k} , B_{θ_k} , C_{θ_k} are the system matrices of appropriate dimensions that are assumed to be known. Let $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}} \in S = \{1, 2, \dots, N\}$ be a finite-state Markov chain described by

$$\Pr\{\theta_{k+1} = j | \theta_k = i, k\} = \pi_{ij}(k), \quad (5.2)$$

where $\pi_{ij}(k) \in [0, 1]$ is the transition probability from a state i to a state j , and $\sum_{j=1}^N \pi_{ij}(k) = 1$.

Remark 5.1. For simplicity of the exposition, we consider a time homogeneous Markov chain, that means $\{\theta_k\}_{k \in \mathbb{N}_{\geq 0}} \in S$ can be described by

$$\Pr\{\theta_{k+1} = j | \theta_k = i\} = \pi_{ij}, \quad (5.3)$$

where $\pi_{ij} \in [0, 1]$ is the transition probability from a state i to a state j , and $\sum_{j=1}^N \pi_{ij} = 1$. The results obtained in this chapter may be applicable to a time inhomogeneous case, which requires a further attention.

We make the following two assumptions about the mode and the noise sequences.

Assumption 5.2. We assume that at each time $k \in \mathbb{N}_{\geq 0}$, the mode θ_k is available.

Assumption 5.3. Let the initial state $x_0 \in \mathbb{R}^n$, the process noise $w_k \in \mathbb{R}^n$, and the measurement noise $v_k \in \mathbb{R}^r$ be mutually independent and Gaussian i.e., $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$, $w_k \sim \mathcal{N}(0, W)$, and $v_k \sim \mathcal{N}(0, V)$ with $W \succ 0$ and $V \succ 0$. We assume that the above stochastic processes are independent of the mode θ_k .

For each $k \in \mathbb{N}_{\geq 0}$, we set $\mathcal{H}_k$ the σ -algebra generated by $\{(y_l, \theta_l); 0 \leq l \leq k\}$. At each time $k \in \mathbb{N}_{\geq 0}$, our RHC policy is to consider a one-step horizon objective function with individual probabilistic state constraints as

$$\begin{aligned} \mathcal{P}_1 : \min_{u_k} J_k &= \mathbb{E}[x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + x_{k+1}^T \Xi_{\theta_{k+1}} x_{k+1} | \mathcal{H}_k] \\ \text{s.t. } \Pr \{G_j x_{k+1} \leq H_j | \mathcal{H}_k\} &\geq \xi, \quad 1 \leq j \leq r, \quad \xi \in (0, 1), \end{aligned} \quad (5.4)$$

where $Q_i \succ 0$, $R_i \succ 0$, $\Xi_i \succ 0$ for $i \in S$. The control input u_k follows an affine state-feedback law that will be given later in this section, and depends on $\mathcal{H}_k$. In the individual probabilistic state constraints (5.4), $G_j \in \mathbb{R}^{1 \times n}$ and $H_j \in \mathbb{R}$ are assumed to be known for $1 \leq j \leq r$; ξ denotes the level of constraint satisfaction, which needs to be high enough ($\xi \gg 0$). However, without loss of generality, we assume that ξ may take any values in the interval $(0, 1)$. We can say that $\mathcal{P}_1$ is a stochastic optimization problem, because of the stochastic nature of the objective function and the constraints. In $\mathcal{P}_1$, the objective function J_k is a conditional expectation of a quadratic function of the state and the control with respect to $\mathcal{H}_k$, which is to be minimized in receding horizon manner. The choice of one-step prediction in $\mathcal{P}_1$ is explained in detail in Section 5.4 in perspective of the system and the constraints, which resembles similar explanation from chapter 4.

To evaluate J_k in $\mathcal{P}_1$, one needs the conditional densities $f(x_k | \mathcal{H}_k)$, $f(x_{k+1} | \mathcal{H}_k)$ of the state variables. We provide recursive equations for state estimation in the following.

Proposition 5.4. *Let assumptions 1 and 2 hold. Assume that u_k depends on $\mathcal{H}_k$. Then the conditional densities $f(x_k | \mathcal{H}_k)$ and $f(x_{k+1} | \mathcal{H}_k)$ are Gaussian distributions $\mathcal{N}(\hat{x}_{k|k}, P_{k|k})$ and $\mathcal{N}(\hat{x}_{k+1|k}, P_{k+1|k})$, respectively. The conditional means and covariances can be given by, for $k = -1, 0, 1, \dots$,*

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} (y_{k+1} - C_{\theta_{k+1}} \hat{x}_{k+1|k}), \quad (5.5)$$

$$P_{k+1|k+1} = (\mathbb{I}_n - K_{k+1} C_{\theta_{k+1}}) P_{k+1|k}, \quad (5.6)$$

$$K_{k+1} = P_{k+1|k} C_{\theta_{k+1}}^T (C_{\theta_{k+1}} P_{k+1|k} C_{\theta_{k+1}}^T + V)^{-1}, \quad (5.7)$$

$$\hat{x}_{k+1|k} = A_{\theta_k} \hat{x}_{k|k} + B_{\theta_k} u_k, \quad (5.8)$$

$$P_{k+1|k} = A_{\theta_k} P_{k|k} A_{\theta_k}^T + W, \quad (5.9)$$

$$\hat{x}_{0|-1} = \hat{x}_0, \quad (5.10)$$

$$P_{0|-1} = P_0, \quad (5.11)$$

where $\hat{x}_{k+1|k} \triangleq \mathbb{E}[x_{k+1} | \mathcal{H}_k]$, $\hat{x}_{k+1|k+1} \triangleq \mathbb{E}[x_{k+1} | \mathcal{H}_{k+1}]$, $P_{k+1|k} \triangleq \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k}) \star | \mathcal{H}_k] \succ 0$, and $P_{k+1|k+1} \triangleq \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k+1}) \star | \mathcal{H}_{k+1}] \succ 0$.

Proof. A proof can be an immediate extension of theorem 1 of (Chizeck and Ji, 1988) and a theorem in p.103 of (Kumar and Varaiya, 1986) $\square$

Observe that the recursive equations (5.5–5.11) for state estimation appear like the Kalman filter equations, which are useful in evaluating J_k in $\mathcal{P}_1$.

To be more specific about the terminology in this chapter, when necessary, we call the conditional means and covariances $\hat{x}_{k+1|k+1}$, $\hat{x}_{k|k+1}$, $P_{k+1|k+1}$, $P_{k|k}$ at each $k \in \mathbb{N}_{\geq 0}$, as estimated state, predicted state, estimated covariance matrix, predicted covariance matrix respectively.

In this chapter, we parametrize the control input as affine state-feedback with estimated states (Bemporad and Morari, 1999; Mayne et al., 2005),

$$u_k = F_{\theta_k} \hat{x}_{k|k} + \nu_k, \quad k \in \mathbb{N}_{\geq 0}, \quad (5.12)$$

where mode-dependent linear feedback gains F_{θ_k} , for $\theta_k \in S$, will be designed off-line to pre-stabilize the system. At each time $k \in \mathbb{N}_{\geq 0}$, $\mathcal{P}_1$ may be solved by ν_k that is assumed to belong to a compact set $\mathbb{U} \subset \mathbb{R}^m$ for tractability. Since, we can have the estimated states from proposition 5.4, we could employ them in state-feedback (5.12).

Remark 5.5. Observe that the control input u_k obtained by solving $\mathcal{P}_1$ may have a non-linear dependence on $\mathcal{H}_k$. Thus we cannot assume the state variables to have Gaussian probability distributions as in the case of the standard linear quadratic Gaussian problem. However, the initial error $x_0 - \hat{x}_0$ has zero mean and covariance P_0 , and from recursive equations (5.5–5.11), the error between actual and the predicted states are Gaussian with zero mean and covariance $P_{k+1|k}$ for $k \in \mathbb{N}_{\geq 0}$, i.e; $\tilde{x}_{k+1|k} \triangleq x_{k+1} - \hat{x}_{k+1|k} \sim \mathcal{N}(0, P_{k+1|k})$, for $k \in \mathbb{N}_{\geq 0}$. We denote the error $\tilde{x}_{k+1|k}$ as state estimation error in this chapter. This remark is the key to obtain a tractable representation of the probabilistic constraints (5.4).

Remark 5.6. Notice that the state-feedback control input (5.12) is not bounded a priori. The reason is as follows. From (5.5) and (5.8), the estimated state $\hat{x}_{k|k}$ in (5.12) can expressed as

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} C_{\theta_{k+1}} (x_{k+1} - \hat{x}_{k+1|k}),$$

which is a function of $x_{k+1} - \hat{x}_{k+1|k}$ that is Gaussian from remark 5.5. Since Gaussian distribution has an unbounded support, we cannot assume the a priori boundedness of u_k (5.12).

Thus $\mathcal{P}_1$ with the affine state-feedback can be expressed as

$$\begin{aligned} \mathcal{P}_2 : \min_{\nu_k} J_k &= \mathbb{E}[x_k^T Q_{\theta_k} x_k + u_k^T R_{\theta_k} u_k + x_{k+1}^T \Xi_{\theta_{k+1}} x_{k+1} | \mathcal{H}_k] \\ \text{s.t. } \Pr \{G_j x_{k+1} \leq H_j | \mathcal{H}_k\} &\geq \xi, \quad 1 \leq j \leq r, \quad \xi \in (0, 1), \end{aligned} \quad (5.13)$$

$$u_k = F_{\theta_k} \hat{x}_{k|k} + \nu_k, \quad (5.14)$$

$$\nu_k \in \mathbb{U}. \quad (5.15)$$

At each $k \in \mathbb{N}_{\geq 0}$, $\mathcal{P}_2$ may be solved with respect to ν_k , where the objective function and the constraints contain ν_k implicitly. We proceed to pre-stabilization results in the next section.

5.3 Pre-stabilization Results

In this section, we pre-stabilize the system (5.1) with linear state-feedback gains in (5.12).

In state estimation problems, generally it is necessary to ensure that the conditional covariances of state estimator are bounded that ensures boundedness of the state estimation error. For discrete-time linear time varying systems, the conditional covariances of the Kalman filter are bounded if the system is uniformly detectable (Anderson and Moore, 1981). For discrete-time MJLSs, we use a detectability definition in stochastic framework. Under some stochastic detectability conditions, we first present that the conditional covariance is mean bounded (Costa and Astolfi, 2010). Then, we design mode-dependent linear feedback gains in (5.12) such that the system (5.1) is mean square bounded, where the approach relies on a stochastic version of Lyapunov's second method (Costa et al., 2005; Dragan et al., 2010) by the choice of mode-dependent Lyapunov function.

We begin with mean boundedness of conditional covariance of state estimator (5.5–5.11).

5.3.1 Mean Boundedness of Conditional Covariance Process

In this section, we give sufficient conditions in terms of LMIs, which imply the mean boundedness of conditional covariance of state estimator (5.5–5.11).

From the state estimation recursions (5.5), (5.8),

$$\hat{x}_{k+1|k} = A_{\theta_k} \hat{x}_{k|k-1} + A_{\theta_k} K_k (y_k - C_{\theta_k} \hat{x}_{k|k-1}) + B_{\theta_k} u_k,$$

and the state estimation error $\tilde{x}_{k+1|k}$ can given by

$$\tilde{x}_{k+1|k} = (A_{\theta_k} - L_k C_{\theta_k}) \tilde{x}_{k|k-1} + w_k - L_k v_k,$$

where $L_k = A_{\theta_k} K_k$. Observe that from proposition 5.4,

$$\mathbb{E} [\tilde{x}_{k+1|k} \tilde{x}_{k+1|k}^T | \mathcal{H}_k] = P_{k+1|k}.$$

Since, $P_{k+1|k}$ depends on the random variables $\{\theta_l; 0 \leq l \leq k\}$, it forms a stochastic process. Thus $P_{k+1|k}$, also denoted here as *conditional covariance process*, said to be mean bounded if

$$\mathbb{E} [P_{k+1|k}] = \mathbb{E} [\mathbb{E} [\tilde{x}_{k+1|k} \tilde{x}_{k+1|k}^T | \mathcal{H}_k]] \preceq \Gamma, \quad (5.16)$$

for some $\Gamma \succ 0$. In the sequel, we present a sufficient condition that satisfies (5.16). In that direction, we give a definition of stochastic detectability that can be found in (Costa and Astolfi, 2010; Ji and Chizeck, 1990a) for instance.

Definition 5.7. We say that the pair $\{C_{\theta_k}, A_{\theta_k}\}$ is stochastically detectable if there exist matrices $\{L_k\}_{k \in \mathbb{N}_{\geq 0}}$ of suitable dimension such that

$$\mathbb{E} \left[\sum_{k=0}^{+\infty} \|z_k\|^2 \right] < +\infty,$$

where $z_{k+1} = (A_{\theta_k} - L_k C_{\theta_k}) z_k$ for any initial conditions z_0 and θ_0 .

We utilize a theorem which relates the stochastic detectability and the mean boundedness of $P_{k+1|k}$.

Theorem 5.8. (*Costa and Astolfi, 2010*) Assume that the pair $\{C_{\theta_k}, A_{\theta_k}\}$ is stochastically detectable. Then there exists $\Gamma \succ 0$, such that $\mathbb{E} [P_{k+1|k}] \leq \Gamma$ for $k \in \mathbb{N}_{\geq 0}$.

In (Costa and Astolfi, 2010), the authors showed that the conditional covariance process is mean bounded for discrete-time MJLSs with the mode taking values in a countably infinite-state Markov chain, and the mode taking values in a finite-state Markov chain was regarded as a special case.

A testable condition for stochastic detectability of $\{C_{\theta_k}, A_{\theta_k}\}$ can be given by following lemma.

Lemma 5.9. (*Costa and Do Val, 2002*) The pair $\{C_{\theta_k}, A_{\theta_k}\}$ is stochastically detectable if and only if the following LMIs are satisfied for $1 \leq i \leq N$:

$$\begin{bmatrix} A_i^T X_i A_i + C_i^T S_i^T A_i + A_i^T S_i C_i - H_i & C_i^T S_i^T \\ \star & -X_i \end{bmatrix} \prec 0,$$

$$\sum_{j=1}^N \pi_{ij} H_j \preceq X_i,$$

$$X_i \succ 0, H_i \succ 0, S_i \in \mathbb{R}^{n \times r}.$$

Finally, we give a lemma that shows the boundedness of $\mathbb{E} [P_{k|k}]$ and $\mathbb{E} [\|K_k\|]$ when theorem 5.8 is valid.

Lemma 5.10. Assume that the assumptions of theorem 5.8 is valid. Then

$$\mathbb{E} [P_{k|k}] \preceq \Gamma, \tag{5.17}$$

$$\mathbb{E} [\|K_k\|] \leq \varrho, \tag{5.18}$$

for some $\varrho > 0$.

Proof. Given in Section 5.6. □

In the next section, lemma 5.10 plays a key role in the pre-stabilization result.

5.3.2 Stabilization of the System via Mean Square Boundedness of the System States

In this section, we pre-stabilize the system (5.1) in the sense of mean square boundedness, using the linear state-feedback control law (5.12), by solving a set of LMIs.

Consider the state estimation equations from proposition 5.4 with the control input (5.12),

$$\begin{aligned}\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1} (y_{k+1} - C_{\theta_{k+1}} \hat{x}_{k+1|k}), \\ &= A_{\theta_k} \hat{x}_{k|k} + B_{\theta_k} u_k + K_{k+1} (y_{k+1} - C_{\theta_{k+1}} \hat{x}_{k+1|k}), \\ &= \tilde{A}_{\theta_k} \hat{x}_{k|k} + B_{\theta_k} \nu_k + K_{k+1} (C_{\theta_{k+1}} A_{\theta_k} (x_k - \hat{x}_{k|k}) + C_{\theta_{k+1}} w_k + v_{k+1}), \\ &\triangleq \tilde{A}_{\theta_k} \hat{x}_{k|k} + B_{\theta_k} \nu_k + \varphi_k,\end{aligned}\tag{5.19}$$

where $\tilde{A}_{\theta_k} = A_{\theta_k} + B_{\theta_k} F_{\theta_k}$ and

$$\varphi_k = K_{k+1} (C_{\theta_{k+1}} A_{\theta_k} (x_k - \hat{x}_{k|k}) + C_{\theta_{k+1}} w_k + v_{k+1}).\tag{5.20}$$

As a first step in the pre-stabilization, we give a lemma that establishes the mean square boundedness of φ_k .

Lemma 5.11. *Consider φ_k in (5.20). Then*

$$\mathbb{E} [\|\varphi_k\| | \mathcal{H}_k] \leq \delta,\tag{5.21}$$

for some $\delta > 0$.

Proof. Given in Section 5.6. □

Before proceeding to the main result of stabilization, we give the following notations.

Let $\mathcal{T} \subset \mathbb{R}^{m \times n}$ be a linear subspace of real matrices. Set $\mathcal{T}^N = \mathcal{T} \times \mathcal{T} \times \cdots \times \mathcal{T}$. Also, let $\mathcal{S}_n \subset \mathbb{R}^{n \times n}$ be a linear subspace of real symmetric matrices. Set $\mathcal{S}_n^N = \mathcal{S}_n \times \mathcal{S}_n \times \cdots \times \mathcal{S}_n$.

We give a proposition to pre-stabilize the system (5.1) in the following.

Proposition 5.12. *If the pair $\{C_{\theta_k}, A_{\theta_k}\}$ is stochastically detectable and if there exist matrices $F = (F_1 \ \cdots \ F_N) \in \mathcal{T}^N$, $\Psi = (\Psi_1 \ \cdots \ \Psi_N) \in \mathcal{S}_n^N$, with $\Psi_i \succ 0$, $1 \leq i \leq N$, such that*

$$\tilde{A}_i^T \left(\sum_{j=1}^N \pi_{ij} \Psi_j \right) \tilde{A}_i - \Psi_i = \mathcal{E}_i \prec 0,\tag{5.22}$$

where $\tilde{A}_i = A_i + B_i F_i$. Then

$$\mathbb{E} [\|x_k\|^2] \leq \zeta,\tag{5.23}$$

for all $k \in \mathbb{N}_{\geq 0}$ and for some $\zeta > 0$.

Proof. Given in Section 5.6. □

Observe that, it is not possible to synthesize the state-feedback gains F_{θ_k} , for $\theta_k \in \mathcal{S}$ directly using proposition 5.12, because the inequalities (5.22) are not LMIs. In the following, we provide a lemma to synthesize the state-feedback gains F_{θ_k} , by a reformulation of (5.22) to LMIs.

Lemma 5.13. *If the pair $\{C_{\theta_k}, A_{\theta_k}\}$ is stochastically detectable and if there exist matrices $Y = (Y_1 \ \cdots \ Y_N) \in \mathcal{T}^N$, $X = (X_1 \ \cdots \ X_N) \in \mathcal{S}_n^N$, with $X_i \succ 0$, $1 \leq i \leq N$, such that*

$$\begin{bmatrix} -X_D & \Lambda_i (A_i X_i + B_i Y_i) \\ \star & -X_i \end{bmatrix} \prec 0, \quad (5.24)$$

where

$$\begin{cases} \Lambda_i = \begin{bmatrix} \sqrt{\pi_{i1}} \mathbb{I}_n & \cdots & \sqrt{\pi_{iN}} \mathbb{I}_n \end{bmatrix}^T, \\ X_D = \text{diag}\{X_1, \dots, X_N\}, \end{cases}$$

then the trajectories of the system (5.1) for any v_k satisfy (5.23), and F_i is given by

$$F_i = Y_i X_i^{-1}. \quad (5.25)$$

Proof. If LMIs (5.24) are feasible then the LMIs (5.22) are also feasible (Dragan et al., 2010). Using proposition 5.12, it follows that the trajectories of (5.19) satisfy the estimates (5.23). Hence the proof is complete. $\square$

5.4 A Tractable Representation of the RHC Problem

In this section, we convert the stochastic optimization problem $\mathcal{P}_2$ to a tractable deterministic optimization problem. the individual probabilistic state constraints (5.13) with suitable deterministic constraints. Subsequently, we convert $\mathcal{P}_2$ to a deterministic optimization problem.

5.4.1 Converting Probabilistic Constraints to Deterministic Constraints

In the sequel, we replace the individual probabilistic state constraints (5.13) in $\mathcal{P}_2$ with equivalent deterministic constraints in terms of conditional means and covariances of the state variable as proposed in (Yan and Bitmead, 2005).

Consider (5.13),

$$\begin{aligned} \Pr \{G_j x_{k+1} \leq H_j | \mathcal{H}_k\} \geq \xi &\iff \Pr \{G_j (x_{k+1} - \hat{x}_{k+1|k}) \leq H_j - G_j \hat{x}_{k+1|k} | \mathcal{H}_k\} \geq \xi \\ &\iff \Pr \{G_j \tilde{x}_{k+1|k} \leq H_j - G_j \hat{x}_{k+1|k} | \mathcal{H}_k\} \geq \xi. \end{aligned} \quad (5.26)$$

From remark 5.5, the state estimation error is Gaussian, i.e., $\tilde{x}_{k+1|k} \sim \mathcal{N}(0, P_{k+1|k})$. Thus, $G_j \tilde{x}_{k+1|k} \sim \mathcal{N}(0, G_j P_{k+1|k} G_j^T)$. Denote $(\Upsilon_{k+1|k})_j \triangleq G_j P_{k+1|k} G_j^T$, then $(\Upsilon_{k+1|k})_j^{-\frac{1}{2}} G_j \tilde{x}_{k+1|k} \sim \mathcal{N}(0, 1)$, which is a uni-variate Gaussian distribution. Thus (5.26) is equivalent to

$$\begin{aligned} \Pr \left\{ (\Upsilon_{k+1|k})_j^{-\frac{1}{2}} G_j \tilde{x}_{k+1|k} \leq (\Upsilon_{k+1|k})_j^{-\frac{1}{2}} (H_j - G_j \hat{x}_{k+1|k}) | \mathcal{H}_k \right\} &\geq \xi \\ \iff (\Upsilon_{k+1|k})_j^{-\frac{1}{2}} (H_j - G_j \hat{x}_{k+1|k}) &\geq F_w^{-1}(\xi), \end{aligned} \quad (5.27)$$

where $F_w^{-1}(\cdot)$ is the inverse cumulative distribution function of the normal random variable. Finally (5.27) is simplified to

$$G_j \hat{x}_{k+1|k} \leq H_j - \sqrt{(\Upsilon_{k+1|k})_j} F_w^{-1}(\xi). \quad (5.28)$$

Observe that (5.28) is deterministic in terms of predicted states and covariances, and is equivalent to (5.13).

Remark 5.14. From remark 5.5, since the state estimation error $\tilde{x}_{k+1|k}$ is Gaussian, we could convert the probabilistic constraints (5.13) to deterministic constraints (5.28). Here, we handled only uni-variate Gaussian distribution at the main conversion step (5.27), where given ξ , $F_w^{-1}(\cdot)$ can be easily calculated.

Remark 5.15. In $\mathcal{P}_1$ ($\mathcal{P}_2$), we consider the one-step prediction horizon because of the probabilistic constraints (5.4) and the system (5.1). In this section, we obtained a deterministic equivalence of the constraints (5.4) as (5.28) in terms of the predicted state variable and its covariance. Suppose, if we consider a multi-step prediction in our case, the probabilistic constraints in (5.4) look like $\Pr\{G_j x_{k+N} \leq H_j | \mathcal{H}_k\} \geq \xi$, for $N \geq 2$. At each $k \in \mathbb{N}_{\geq 0}$, the random variables θ_{k+m} , $1 \leq m \leq N-1$ are not exactly known a priori, thus it is not possible to obtain the state estimates for $k+1, \dots, K+N-1$ from proposition 5.4, which imply that we cannot obtain the deterministic equivalent of the constraints by proceeding with the similar approach of this section. Thus, we choose the one-step prediction horizon to obtain a deterministic equivalence of the probabilistic constraints (5.4) for tractability of $\mathcal{P}_1$ ($\mathcal{P}_2$).

Similar to chapter 4, for feasibility of $\mathcal{P}_2$, we modify the constraints (5.28) as

$$G_j \hat{x}_{k+1|k} \leq H_j - \sqrt{(\Upsilon_{k+1|k})_j} F_w^{-1}(\xi) + \rho_k, \quad (5.29)$$

where $\rho_k \geq 0$. The addition of a variable ρ_k can be compensated by adding a variable $\alpha \rho_k$ to the objective function in $\mathcal{P}_2$ with a very high value of $\alpha > 0$.

5.4.2 A Deterministic Representation of the Overall RHC Problem

In this section, we present $\mathcal{P}_2$ as a tractable deterministic problem in terms of predicted states and covariance matrices.

Denote $\tilde{\Xi}_k \triangleq \sum_{j=1}^N \pi_{\theta_k j} \Xi_j$. Then, by simple manipulations, the terms in $\mathcal{P}_2$ can be rewritten as

$$\mathbb{E} \left[x_{k+1}^T \Xi_{\theta_{k+1}} x_{k+1} | \mathcal{H}_k \right] = \text{tr} \left(P_{k+1|k} \tilde{\Xi}_k \right) + \hat{x}_{k+1|k}^T \tilde{\Xi}_k \hat{x}_{k+1|k}, \quad (5.30)$$

and

$$\mathbb{E} \left[x_k^T Q_{\theta_k} x_k | \mathcal{H}_k \right] = \text{tr} \left(P_{k|k} Q_{\theta_k} \right) + \hat{x}_{k|k}^T Q_{\theta_k} \hat{x}_{k|k}. \quad (5.31)$$

By (5.29), (5.30) and (5.31), $\mathcal{P}_2$ can be expressed as

$$\begin{aligned} \mathcal{P}_3 : \min_{\nu_k, \rho_k} J_k &= \text{tr} \left(P_{k|k} Q_{\theta_k} \right) + \hat{x}_{k|k}^T Q_{\theta_k} \hat{x}_{k|k} + \text{tr} \left(P_{k+1|k} \tilde{\Xi}_k \right) + \hat{x}_{k+1|k}^T \tilde{\Xi}_k \hat{x}_{k+1|k} + u_k^T R_{\theta_k} u_k \\ &\quad + \alpha \rho_k \\ \text{s.t. } G_j \hat{x}_{k+1|k} &\leq H_j - \sqrt{\left(\Upsilon_{k+1|k} \right)_j} F_w^{-1}(\xi) + \rho_k, \quad 1 \leq j \leq r, \quad \xi \in (0, 1), \\ u_k &= F_{\theta_k} \hat{x}_{k|k} + \nu_k, \\ \nu_k &\in \mathbb{U}, \\ \rho_k &\geq 0. \end{aligned} \quad (5.32)$$

Observe that, at each time instant $k \in \mathbb{N}_{\geq 0}$, $\mathcal{P}_3$ is a deterministic optimization problem that approximates the original stochastic optimization problem $\mathcal{P}_2$, but in a non-certainty equivalent manner because the constraints in $\mathcal{P}_3$ contain conditional covariances (Van de Water and Willems, 1981). Thus the resulting control is a non-certainty equivalence controller.

When we solve $\mathcal{P}_3$, if ρ_k is zero, for all $k \in \mathbb{N}_{\geq 0}$, then we have solved original $\mathcal{P}_2$. If ρ_k is positive, then we conclude that $\mathcal{P}_2$ was infeasible, in which case at least the variable ρ_k should be as low as possible. Thus we say that $\mathcal{P}_3$ is an approximation of $\mathcal{P}_2$ in general, because of the term ρ_k . We denote J_k^* as an optimal solution of $\mathcal{P}_3$.

5.5 An Illustrative Example of Vertical Take Off and Landing Vehicle

We illustrate the effectiveness of the proposed approach on vertical take-off and landing (VTOL) vehicle dynamical system.

Let a discretized version of VTOL vehicle dynamics (Aberkane et al., 2008), with zero order hold and sampling time of 0.2 seconds, be given by (5.1). Here $x_k \in \mathbb{R}^4$, $u_k \in \mathbb{R}^2$, $y_k \in \mathbb{R}^2$, $x_k(1)$, $x_k(2)$, $x_k(3)$, $x_k(4)$ denote horizontal velocity, vertical velocity, pitch rate, pitch angle respectively, and $u_k(1)$, $u_k(2)$ denote thrust along the aircraft axis, pitch moment respectively. Let $\theta_k \in \{1, 2, 3\}$ denote different air speeds, where the modes 1, 2, 3 denote the air speeds of 135, 60, 170 knots respectively. Consider the following

initial conditions

$$\hat{x}_{0|0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 20 \end{bmatrix}^T, P_0 = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, W = 0.3 * \mathbb{I}_4, V = 0.1 * \mathbb{I}_2.$$

The transition probability matrix Π along with the mode parameters are given as

$$\Pi = \begin{bmatrix} 0.9821 & 0.0132 & 0.0047 \\ 0.0132 & 0.9868 & 0 \\ 0.0047 & 0 & 0.9953 \end{bmatrix},$$

$\theta_k(\text{knots})$	$a_{32}(\theta_k)$	$a_{34}(\theta_k)$	$b_{21}(\theta_k)$
135	0.0627	0.2398	0.6690
60	0.0112	0.0167	0.2043
170	0.0865	0.4452	0.9526

The system matrices are as follows

$$A_{\theta_k} = \begin{bmatrix} 0.9927 & 0.0048 & -0.0053 & -0.0931 \\ 0.0082 & 0.8154 & -0.0717 & -0.7351 \\ 0.0191 & a_{32}(\theta_k) & 0.8924 & a_{34}(\theta_k) \\ 0.0019 & 0.0066 & 0.1882 & 1.0254 \end{bmatrix}, B_{\theta_k} = \begin{bmatrix} 0.0912 & 0.0302 \\ b_{21}(\theta_k) & -1.3961 \\ -1.0145 & 0.7951 \\ -0.1042 & 0.0827 \end{bmatrix}, C_{\theta_k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T.$$

One can verify that lemma 5.9 can be satisfied by

$$H_1 = \begin{bmatrix} 0.8841 & 0.0091 & 0.0584 & 0.0718 \\ 0.0091 & 0.9933 & 0.4994 & 0.6143 \\ 0.0584 & 0.4994 & 1.3448 & -0.0875 \\ 0.0718 & 0.6143 & -0.0875 & 1.2891 \end{bmatrix}, H_2 = \begin{bmatrix} 0.8816 & -0.0117 & 0.0233 & 0.0716 \\ -0.0117 & 0.8167 & 0.2123 & 0.6055 \\ 0.0233 & 0.2123 & 1.4405 & -0.0554 \\ 0.0716 & 0.6055 & -0.0554 & 1.3839 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0.8874 & 0.0365 & 0.0785 & 0.0692 \\ 0.0365 & 1.2258 & 0.6647 & 0.6006 \\ 0.0785 & 0.6647 & 1.1455 & -0.1454 \\ 0.0692 & 0.6006 & -0.1454 & 1.2200 \end{bmatrix},$$

and

$$X_1 = \begin{bmatrix} 1.3914 & -0.0111 & 0.0591 & 0.0895 \\ -0.0111 & 1.3828 & 0.5272 & 0.7599 \\ 0.0591 & 0.5272 & 1.5641 & -0.1974 \\ 0.0895 & 0.7599 & -0.1974 & 1.5616 \end{bmatrix}, X_2 = \begin{bmatrix} 1.3890 & -0.0315 & 0.0217 & 0.0901 \\ -0.0315 & 1.2100 & 0.2241 & 0.7598 \\ 0.0217 & 0.2241 & 1.6696 & -0.1158 \\ 0.0901 & 0.7598 & -0.1158 & 1.6318 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 1.3947 & 0.0167 & 0.0816 & 0.0853 \\ 0.0167 & 1.6169 & 0.7103 & 0.7316 \\ 0.0816 & 0.7103 & 1.3451 & -0.2845 \\ 0.0853 & 0.7316 & -0.2845 & 1.5264 \end{bmatrix}.$$

Thus, the given system is stochastically detectable and the conditional covariance process is mean bounded from theorem 5.8. The boundedness of the conditional covariance matrices can be observed from Figure 5.1, which is obtained from Monte Carlo simulations for 100 runs. Consider the affine state-feedback control (5.12) with $\nu_k := [\nu_k(1) \ \nu_k(2)]^T$ and $|\nu_k(1)| \leq 20$, $|\nu_k(2)| \leq 30$. The system can be pre-stabilized in the sense of (5.23),

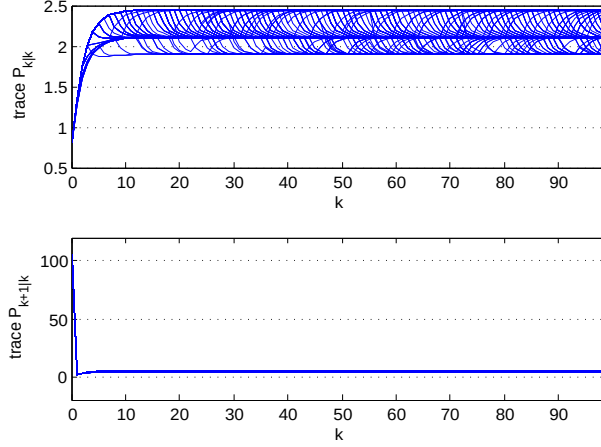


Figure 5.1: Conditional covariances

and the pre-stabilizing gains (5.12) can be obtained from lemma 5.13 as

$$K_1 = \begin{bmatrix} -0.5813 & 0.7624 & 1.2097 & 0.4564 \\ -0.3601 & 0.9397 & 0.5160 & -0.2989 \end{bmatrix}, K_2 = \begin{bmatrix} -0.3654 & 0.5155 & 0.8423 & 0.0882 \\ -0.1196 & 0.6548 & 0.0652 & -0.4960 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.8563 & 1.0285 & 1.6354 & 0.9261 \\ -0.6849 & 1.2691 & 1.0429 & 0.1025 \end{bmatrix}.$$

Consider the individual probabilistic constraints (5.4) on horizontal and vertical velocity with

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \quad \xi = 0.9,$$

where G_j and H_j for $1 \leq j \leq 4$ denote j^{th} rows of G and H respectively.

Let $\Xi_1 = \Xi_2 = \Xi_3 = \mathbb{I}_4$, $Q_1 = Q_2 = Q_3 = 2 * \mathbb{I}_4$, $R_1 = R_2 = R_3 = \mathbb{I}_2$, and $\alpha = 1000$. Using these parameters, we solve $\mathcal{P}_3$ at each time k and performed Monte Carlo simulations for 100 runs. Figure 5.2a shows the evolution of the corresponding actual state values, which satisfies the individual probabilistic state constraints (5.4) qualitatively. The red colored rectangle in Figure 5.2a indicate the hard constraints $G_j x_k \leq H_j$ for $1 \leq j \leq 4$ in (5.4) and blue colored dots indicate different values of the actual state variables (horizontal velocity and vertical velocity) obtained from Monte Carlo simulations. Since we are dealing with probabilistic constraints, constraint violations can be observed from the figure.

To compare probabilistic constraints (5.32), with different values of ξ qualitatively, we perform Monte Carlo simulations of 50 runs for values of $\xi = 0.6$, $\xi = 0.3$ separately by keeping the remaining parameters same as above, and the actual state variables (horizontal velocity and vertical velocity) are shown in Figure 5.2b and Figure 5.2c. One can observe that the larger the level of constraint satisfaction, the better the state variables meet the

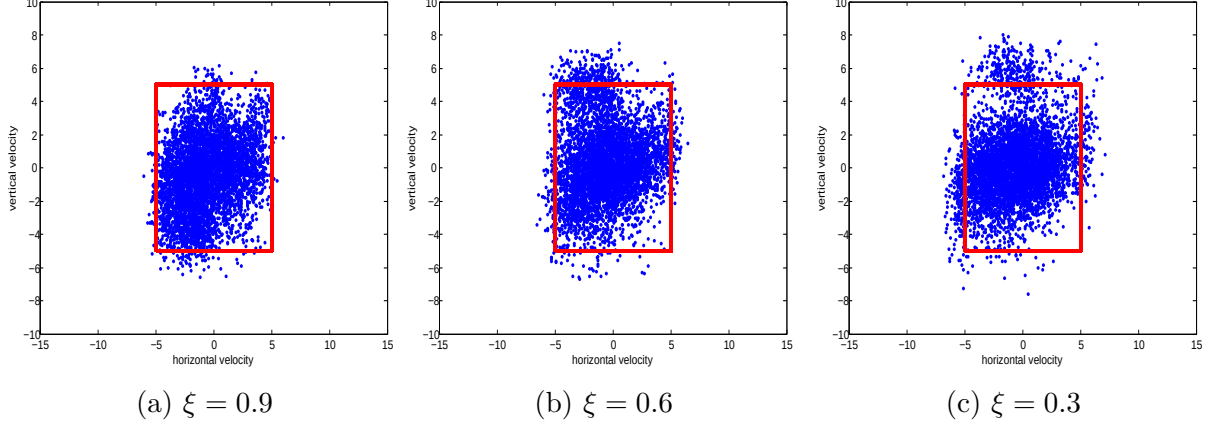


Figure 5.2: Actual state values for different values of ξ

required target. In these three experiments ($\xi = 0.9$, $\xi = 0.6$, $\xi = 0.3$), we obtain the mean values of ρ_k as 0.1152, 0.0127 and 0.0538, which makes the constraints (5.32) feasible with very low values of ρ_k . It implies that the higher the level of constraint satisfaction ξ , the more the chance of becoming the original constraints (5.28) infeasible; subsequently the values of slack variables ρ_k tend to increase with ξ for the feasibility of (5.32).

5.6 Proofs

Proof of Lemma 5.10:

From proposition 5.4, $P_{k+1|k+1}$ can be expressed as

$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} C_{\theta_{k+1}}^T (C_{\theta_{k+1}} P_{k+1|k} C_{\theta_{k+1}}^T + V)^{-1}. \quad (5.34)$$

Using matrix inversion lemma given in (Simon, 2006), an alternative form of (5.34) can be obtained as

$$P_{k+1|k+1}^{-1} = P_{k+1|k}^{-1} + (C_{\theta_{k+1}} V^{-1} C_{\theta_{k+1}}). \quad (5.35)$$

Thus

$$P_{k+1|k+1} \leq P_{k+1|k}. \quad (5.36)$$

Since theorem 5.8 is valid, by taking expectation on both sides (5.36) satisfy (5.17). Similarly an alternative form of K_{k+1} can be obtained as

$$K_{k+1} = P_{k+1|k} C_{\theta_{k+1}}^T V^{-1},$$

thus

$$\|K_{k+1}\| \leq \|P_{k+1|k}\| \|C\| \|V^{-1}\|, \quad (5.37)$$

where $\|C\| = \sup_{i \in S} \|C_i\|$. Since theorem 5.8 is valid, and $V \succ 0$, by taking expectation on both sides of (5.37) and by setting ϱ to be $\mathbb{E} [\|P_{k+1|k}\|] \|C\| \|V^{-1}\|$ satisfy (5.18). $\square$

Proof of Lemma 5.11:

Consider

$$\|\varphi_k\| \leq \|K_{k+1}\| \left(\|C\| \|A\| \|x_k - \hat{x}_{k|k}\| + \|C\| \|w_k\| + \|v_{k+1}\| \right), \quad (5.38)$$

where $\|A\| = \sup_{i \in S} \|A_i\|$, $\|C\| = \sup_{i \in S} \|C_i\|$. Taking expectation with respect to $\mathcal{H}_k$ on both sides of (5.38) and by setting δ to be $\varrho \left(\|C\| \|A\| \sqrt{\text{tr}(\Gamma)} + \|C\| \sqrt{\text{tr}(W)} + \sqrt{\text{tr}(V)} \right)$, one obtains (5.21). $\square$

Proof of Proposition 5.12:

Consider

$$\begin{aligned} \mathbb{E} [\|x_k\|^2] &= \mathbb{E} [\|x_k - \hat{x}_{k|k} + \hat{x}_{k|k}\|^2], \\ &\leq \mathbb{E} [\|x_k - \hat{x}_{k|k}\|^2] + \mathbb{E} [\|\hat{x}_{k|k}\|^2]. \end{aligned}$$

The boundedness of $\mathbb{E} [\|x_k - \hat{x}_{k|k}\|^2]$ follows from lemma 5.10.

One knows that there exist matrices $F_i \in \mathcal{T}^N$ and $\Psi_i \in \mathcal{S}_n^N$, with $\Psi_i \succ 0$, for $1 \leq i \leq N$ such that (5.22) is verified. Consider $V(\hat{x}_{k|k}, \theta_k) \triangleq \hat{x}_{k|k}^T \Psi_{\theta_k} \hat{x}_{k|k}$. Denote $\tilde{\Psi}_{\theta_k} \triangleq \sum_{j=1}^N \pi_{\theta_k j} \Psi_j$. Consider

$$\begin{aligned} &\mathbb{E} [V(\hat{x}_{k+1|k+1}, \theta_{k+1}) | \mathcal{H}_k] - V(\hat{x}_{k|k}, \theta_k) \\ &= \mathbb{E} [\hat{x}_{k+1|k+1}^T \Psi_{\theta_{k+1}} \hat{x}_{k+1|k+1} | \mathcal{H}_k] - \hat{x}_{k|k}^T \Psi_{\theta_k} \hat{x}_{k|k}, \\ &= \mathbb{E} \left[\left(\tilde{A}_{\theta_k} \hat{x}_{k|k} + B_{\theta_k} \nu_k + \varphi_k \right)^T \Psi_{\theta_{k+1}} \left(\tilde{A}_{\theta_k} \hat{x}_{k|k} + B_{\theta_k} \nu_k + \varphi_k \right) | \mathcal{H}_k \right] - \hat{x}_{k|k}^T \Psi_{\theta_k} \hat{x}_{k|k}, \\ &= \hat{x}_{k|k}^T \left(\tilde{A}_{\theta_k}^T \tilde{\Psi}_{\theta_k} \tilde{A}_{\theta_k} - \Psi_{\theta_k} \right) \hat{x}_{k|k} + 2E \left[\hat{x}_{k|k}^T \tilde{A}_{\theta_k}^T \Psi_{\theta_{k+1}} B_{\theta_k} \nu_k | \mathcal{H}_k \right] \\ &\quad + \mathbb{E} \left[\nu_k^T B_{\theta_k}^T \Psi_{\theta_{k+1}} B_{\theta_k} \nu_k | \mathcal{H}_k \right] + \mathbb{E} \left[\varphi_k^T \Psi_{\theta_{k+1}} \varphi_k | \mathcal{H}_k \right], \\ &\leq -\mu \|\hat{x}_{k|k}\|^2 + 2\hat{x}_{k|k}^T \tilde{A}_{\theta_k}^T \tilde{\Psi}_{\theta_k} B_{\theta_k} \nu_k + \nu_k^T B_{\theta_k}^T \tilde{\Psi}_{\theta_k} B_{\theta_k} \nu_k + \alpha_1 \mathbb{E} [\|\varphi_k\|^2 | \mathcal{H}_k], \\ &\leq -\mu \|\hat{x}_{k|k}\|^2 + 2\|\hat{x}_{k|k}\| \|\tilde{A}_{\theta_k}\| \|\tilde{\Psi}_{\theta_k}\| \|B_{\theta_k}\| \|\nu_k\| + \alpha_2 \|\nu_k\|^2 + \alpha_1 \gamma \delta^2, \end{aligned}$$

where $\mu = \min_{1 \leq i \leq N} (\lambda_{\min}(-\mathcal{E}_i))$, $\alpha_1 = \max_{1 \leq i \leq N} (\lambda_{\max}(\tilde{\Psi}_i))$, $\alpha_2 = \max_{1 \leq i \leq N} (\lambda_{\max}(B_i^T \tilde{\Psi}_i B_i))$ and a sufficiently high $\gamma > 0$. From inequality given in lemma A.2,

$$2\|\hat{x}_{k|k}\| \|\tilde{A}_{\theta_k}\| \|\tilde{\Psi}_{\theta_k}\| \|B_{\theta_k}\| \|\nu_k\| \leq \kappa \|\hat{x}_{k|k}\|^2 + \kappa^{-1} \beta^2 \|\nu_k\|^2,$$

where $\beta = \left(\max_{1 \leq i \leq N} \|\tilde{A}_i\| \right) \cdot \alpha_1 \cdot \left(\max_{1 \leq i \leq N} \|B_i\| \right)$. This yields

$$\mathbb{E} [V(\hat{x}_{k+1|k+1}, \theta_{k+1}) | \mathcal{H}_k] - V(\hat{x}_{k|k}, \theta_k) \leq -(\mu - \kappa) \|\hat{x}_{k|k}\|^2 + (\alpha_2 + \kappa^{-1} \beta^2) \|\nu_k\|^2 + \alpha_1 \gamma \delta^2.$$

Take $\kappa \in (0, \mu)$. We have

$$\mathbb{E} [V(\hat{x}_{k+1|k+1}, \theta_{k+1}) | \mathcal{H}_k] \leq (1 - c_2^{-1}(\mu - \kappa)) V(\hat{x}_{k|k}, \theta_k) + (\alpha_2 + \kappa^{-1} \beta^2) \|\nu_k\|^2 + \alpha_1 \gamma \delta^2, \quad (5.39)$$

where $c_2 = \max_{1 \leq i \leq N} (\lambda_{\max}(\Psi_i))$ with the constraint $0 < (1 - c_2^{-1}(\mu - \kappa)) < 1$. Let $q = 1 - c_2^{-1}(\mu - \kappa)$. Taking the expectation on both sides of (5.39), we get

$$\mathbb{E} [V(\hat{x}_{k+1|k+1}, \theta_{k+1})] \leq q E [V(\hat{x}_{k|k}, \theta_k)] + (\alpha_2 + \kappa^{-1} \beta^2) \mathbb{E} [\|\nu_k\|^2] + \alpha_1 \gamma \delta^2.$$

We obtain recursively

$$\begin{aligned}\mathbb{E} [V(\hat{x}_{k|k}, \theta_k)] &\leq q^k \mathbb{E} [V(\hat{x}_{0|0}, \theta_0)] + (\alpha_2 + \kappa^{-1} \beta^2) \sum_{p=1}^{k-1} q^{k-p-1} \mathbb{E} [\|\nu_p\|^2] \\ &\quad + \alpha_1 \gamma \delta^2 \sum_{p=1}^{k-1} q^{k-p-1}, \\ &\leq q^k \mathbb{E} [V(\hat{x}_{0|0}, \theta_0)] + \frac{\alpha_1 \gamma \delta^2 + \tau(\alpha_2 + \kappa^{-1} \beta^2)}{1 - q}.\end{aligned}$$

Finally, we obtain

$$\mathbb{E} [\|\hat{x}_{k|k}\|^2] \leq \frac{c_2}{c_1} q^k \|\hat{x}_{0|0}\|^2 + \frac{\alpha_1 \gamma \delta^2 + \tau(\alpha_2 + \kappa^{-1} \beta^2)}{1 - q},$$

where $c_1 = \min_{1 \leq i \leq N} (\lambda_{\min}(\Psi_i))$. Hence the proof is complete. $\square$

5.7 Conclusions

We considered a one-step RHC of discrete-time MJLSs with process and measurement noise, and individual probabilistic state constraints. Although the receding horizon controller depends on the measurements in a nonlinear manner, under the assumption of availability of the jump process, we obtained state estimates that resemble Kalman filter equations. We used an affine state-feedback control with an estimated state, and synthesized feedback gains off-line that guarantee the mean square boundedness of the system. We replaced the probabilistic constraints by deterministic constraints in terms of state predictions and covariances, and consequently approximated the overall receding horizon problem as a tractable deterministic optimization problem. Since the constraints contain the state covariance terms, we obtained a non-certainty equivalence controller. We illustrated the proposed approach on the dynamics of vertical take-off and landing vehicle.

Chapter 6

Conclusions and Future Work

In conclusion, we summarise the main results of this thesis and outline suggestions for possible future directions.

6.1 Summary of the Thesis

In this thesis, we considered random jump linear systems with the underlying random jump process as a time inhomogeneous process and addressed the problems of stability and control subject to constraints. We give a summary of the main contributions of this thesis in the following.

- Part I: Stability and stabilization of RJLSs
 - In **chapter 2**, we addressed second moment stability and state-feedback stabilization of discrete-time inhomogeneous MJLS with interval TPM. We obtained sufficient conditions for second moment stability in terms of spectral radius of a suitably defined matrix by using the results of interval analysis and Kronecker products, in terms of LMIs by using stochastic version of Lyapunov's second method. Using a sufficient condition of LMIs, we synthesized a state-feedback stabilizing controller.
 - In **chapter 3**, we addressed second moment stability and $(\mathcal{H}_\infty)$ state-feedback stabilization of continuous-time SDJLS. By using arguments from measure theory, stochastic version of Lyapunov's second method and Dynkin's formula, we obtained a sufficient condition for second stability in terms of LMIs. Using the sufficient condition, we synthesized a $(\mathcal{H}_\infty)$ state-feedback stabilizing controller in terms of LMIs and illustrated our results on a macroeconomic system.
- Part II: Control of RJLSs subject to constraints

- In **chapter 4**, we addressed a one-step RHC scheme for discrete-time SDJLS with process noise and perfect state availability with probabilistic state constraints. We converted the overall RHC problem as a tractable deterministic optimization problem and illustrated our approach on a numerical example subsequently.
- In **chapter 5**, we addressed a one-step RHC scheme for discrete-time inhomogeneous MJLS with process noise and noisy measurements with probabilistic state constraints. We followed the similar approach of chapter 4 in dealing the RHC problem and illustrated our approach on dynamics of a vertical takeoff landing vehicle.

6.2 Directions for Future Research

In this section, we outline some possible directions for future research.

- In chapter 2, we obtained two alternative sufficient conditions for second moment stability. However, we could not conclude which one is the better condition in terms of conservativeness. This will be addressed in future, where we would like to compare or relate both the obtained conditions.
- As an extension of chapter 2, using the obtained LMIs, we can also address the robust control problem (De Farias et al., 2002; Sun et al., 2006) in the presence of uncertainties in system parameters. A similar extension can be considered in chapter 3, where we would like to address the robust control problem in addition to the $\mathcal{H}_\infty$ state-feedback stabilization.
- In chapter 4 and chapter 5, we could not place hard bounds on the control input due to the presence of unbounded noise. One remedy could be to consider an affine disturbance feedback policy (Hokayem et al., 2010, 2012), which is a causal feedback policy, to handle bounded controls in our problem setup.
- Also, the issue of recursive feasibility needs a further attention. We used slack variable approach for recursive feasibility in chapter 4 and chapter 5, which adds an additional term in the cost. An alternative approach to treat recursive feasibility may be to employ randomized algorithms approach (Tempo et al., 2013). Computational complexity may not be an issue because recursive feasibility is checked off-line in general.
- In chapter 4 and chapter 5, we considered a one-step RHC scheme which facilitated us to convert the probabilistic state constraints to tractable deterministic constraints. Employing a multi-step RHC scheme in this context is a challenging

problem because one needs to obtain tractable constraints to solve the RHC problem. This problem will be studied in depth future.

- An interesting extension results of chapter 2 would be to consider second moment stability and stabilization of fault tolerant control systems (FTCS) (Shi and Boukas, 1997; Mahmoud et al., 2001). In FTCS, the fault occurrence process and the fault detection process are usually modelled as two separate random jump processes with time homogeneous Markov chain. For instance, we can consider that the failure rate of a component depends on the present state of the component (Giorgio et al., 2010; Kim et al., 2008), and consider the FTCS with state-dependent failure occurrence process. As a first step, we provide a mathematical model¹ in this setup in (Chitraganti et al., 2014b).
- In this thesis, we focused on centralized RHC (or MPC) techniques applied to a numerical example and VTOL vehicle, which were modelled as RJLSs. We would like to extend the centralized RHC techniques to distributed RHC cases (Camponogara et al., 2002; Scattolini, 2009) for RJLSs given by networked agent systems, which are distributed in nature with an objective to reach an agreement regarding a certain quantity of interest that depends on all agents or subsystems. In distributed RHC, there is not a single RHC but instead there are multiple receding horizon controllers, each for a particular subsystem. Each local RHC acts on its subsystem (agent) but now not only considering dynamics, constraints, objectives, and disturbances of the subsystem under consideration but also consider the interactions among the subsystems. As an illustration, for example, a distributed MPC scheme with three agent systems is shown in Figure 6.1.

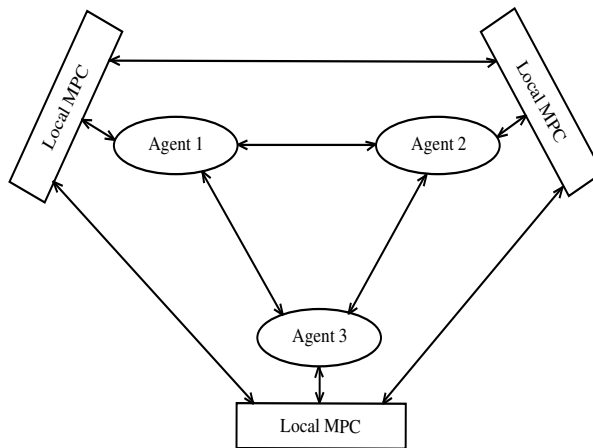


Figure 6.1: An illustration of distributed MPC of networked agent systems

In particular, we would like to consider the following networked agent systems:

¹Please see Appendix C for a brief description

- ✓ an economic system where multiple decision makers influence the overall economy,
- ✓ distributed vehicle networks,

and apply the distributed MPC approach to them. We can consider the SDJLS modeling of an economic system with different economic situations. Also, in distributed vehicle networks, we can consider the transmission delays of information between different vehicles depend on the distance between the vehicles, where distance is part of the state of the system. Thus, we can consider the state-dependent transmission delays, and consider the SDJLS formulation of the distributed vehicle network to apply the distributed RHC approach. This problem forms an interesting direction, which we would like to address in near future.

Appendix A

Mathematical Preliminaries

In this chapter, we review some of the preliminaries which are used throughout the thesis. The material presented in this chapter is fairly standard and available in many references, for example (Costa et al., 2005, 2013; Maciejowski, 2002; Boyd, 1994; Lancaster and Tismenetsky, 1985; Alefeld and Herzberger, 1983; Mao and Yuan, 2006; Kumar and Varaiya, 1986; Gikhman and Skorokhod, 2004; Boukas and Liu, 2002).

A.1 Elements of probability theory

Probability space: Let $(\Omega, \mathcal{F}, \Pr)$ be the probability space, where Ω is the sample space, $\mathcal{F}$ is the sigma-algebra of the events, and $\Pr$ is the probability measure. Then the probability measure $\Pr : \mathcal{F} \rightarrow [0, 1]$ is a function on $\mathcal{F}$ such that

1. $\Pr \{\Omega\} = 1$.
2. If the events $\{S_i\} \subseteq \mathcal{F}$ are mutually exclusive ($S_i \cap S_k = \emptyset$ for $i \neq k$), then

$$\Pr \{\cup_{i \in \mathcal{I}} S_i\} = \sum_{i \in \mathcal{I}} \Pr \{S_i\},$$

where $\mathcal{I}$ is a countable set of positive integers.

Random variables, distribution and density functions: Let $\mathbb{R}$ denote the set of real numbers. A function $f : \Omega \rightarrow \mathbb{R}$ is said to be measurable with respect to σ -algebra $\mathcal{F}$ if $f^{-1}(A) \in \mathcal{F}$ for every borel set $A \subseteq \mathbb{R}$. A real random variable x defined on a probability space $(\Omega, \mathcal{F}, \Pr)$ is a measurable function mapping Ω into $\mathcal{Y} \in \mathbb{R}$. The cumulative distribution function of a random variable x is defined as

$$F_x(\alpha) = \Pr \{x \leq \alpha\}.$$

The function $F_x(\alpha)$ is non-decreasing, right continuous, and $F_x(\alpha) \rightarrow 0$ for $\alpha \rightarrow -\infty$, $F_x(\alpha) \rightarrow 1$ for $\alpha \rightarrow +\infty$. We define a closely related concept, inverse cumulative distri-

bution of a random variable as

$$F_x^{-1}(\beta) = \inf \{ \alpha \in \mathbb{R} : F_x(\alpha) \geq \beta \},$$

with $\beta \in (0, 1)$.

For continuous-time random variable x , the probability density function is given by

$$f_x(\alpha) = \frac{dF_x(\alpha)}{d\alpha}.$$

For discrete-time random variable x , the counter part is the mass density that is defined as

$$f_x(\alpha) = \lim_{\epsilon \rightarrow 0} F_x(\alpha + \epsilon) - F_x(\alpha - \epsilon) = \Pr\{x = \alpha\}.$$

A random matrix contain random variables as its elements.

Expected value and covariance: Let X be a real random matrix and let $\mathcal{M}$ be a Lebesgue measurable function of appropriate dimensions, then expected value of the random matrix $\mathcal{M}(X)$ is defined as

$$\mathbb{E}[\mathcal{M}(X)] = \int_{\mathcal{Y}} \mathcal{M}(\alpha) f_X(\alpha) d\alpha,$$

where $\mathcal{Y}$ is the support of X .

If X is a real discrete-time random matrix with countable support $\mathcal{Y} = \{\alpha_1, \alpha_2, \dots\}$ and $Y \in \mathcal{Y}$, then

$$\Pr\{X \in Y\} = \sum_{\alpha_i \in Y} f_X(\alpha_i) = \sum_{\alpha_i \in Y} \Pr\{X = \alpha_i\},$$

and the expected value of $\mathcal{M}(X)$ can be defined as

$$\mathbb{E}[\mathcal{M}(X)] = \sum_{\alpha_i \in Y} \mathcal{M}(\alpha_i) f_X(\alpha_i).$$

The covariance matrix of $x \in \mathbb{R}^n$, where $\mathbb{R}^n$ is the n -dimensional real Euclidean space, is defined as

$$\text{cov}(x) = \mathbb{E}[(x - \mathbb{E}[x])(x - \mathbb{E}[x])^T],$$

where x^T is the transpose of x . The covariance of $x \in \mathbb{R}$ is called the variance and is given by

$$\text{var}(x) = \mathbb{E}[(x - \mathbb{E}(x))^2].$$

A.2 Matrix operations

Permutation matrix: A $n \times n$ matrix P is a permutation matrix if it is obtained by permuting the rows or columns of an $n \times n$ identity matrix according to some permutation of the numbers 1 to n . Permutation matrices are orthogonal (hence, $P^{-1} = P^T$).

Kronecker products:

Let $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{p \times q}$. Then the Kronecker product (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp \times nq}.$$

For $A = (a_{ij})_{m \times n}$, the linear operator $\text{vec}(A)$ is defined by

$$\text{vec}(A) = [a_{11} \ \cdots \ a_{m1} \ a_{12} \ \cdots \ a_{m2} \ a_{1n} \ \cdots \ a_{mn}]^T.$$

A lemma about basic properties of Kronecker product is as follows:

Lemma A.1. *Let A, B, C and D be matrices of appropriate dimensions. Then,*

- a) $(AB \otimes CD) = (A \otimes C)(B \otimes D)$,
- b) $(A \otimes B)^T = (A^T \otimes B^T)$,
- c) $\text{vec}(A + B) = \text{vec}(A) + \text{vec}(B)$,
- d) $\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)$.

A.3 Inequalities

Lemma A.2. *Let X, Y be real constant matrices of compatible dimensions. Then*

$$X^T Y + Y^T X \leq \xi X^T X + \frac{1}{\xi} Y^T Y$$

holds for any $\xi > 0$.

Linear matrix inequalities: A linear matrix inequality (LMI) is an expression in terms of matrices of form

$$A(y) = A_0 + \sum_{i=1}^m y_i A_i \succ 0,$$

where for $i = 1, \dots, m$, y_i , A_i are real numbers and symmetric matrices respectively. Here $A(y)$ is said to be positive definite matrix. In general, the inequality can be replaced by $\succeq, \prec$ or $\preceq$.

Let U be a non-singular matrix, then

$$A \succ 0 \iff U^T A U \succ 0,$$

where $U^T A U$ is called the congruent transformation of A by U .

For a suitable matrices A, B, C and D , the LMI

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \succ 0,$$

where $A = A^T$, $D = D^T$ is equivalent to

$$D \succ 0, A - BD^{-1}C \succ 0.$$

Here $A - BD^{-1}C$ is called as the Schur complement of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with respect to invertible D .

Appendix B

A Non-exponential Sojourn Time Distribution for Change Detection of Exponentially Weighted Moving Average Algorithm

B.1 Introduction

An active fault tolerant control system (AFTCS) is a control system that have the capability to accommodate system component failures automatically by maintaining overall system stability and acceptable performance. Typically AFTCS consists of a Fault Detection and Isolation (FDI) module and a reconfigurable control scheme. On the occurrence of fault, FDI module set off an alarm which triggers the reconfiguration controller to necessitate the appropriate action. Despite the evident interaction between FDI and reconfiguration algorithms, the research on FDI and reconfiguration methods has often evolved separately. Typically, in the reconfiguration literature, it is usually assumed that a perfect FDI device is available (*i.e.* no detection delays, no false alarms, no missed detection, *etc.*). On the other hand, an FDI algorithm is judged satisfactory if it detects failures quickly and generates few false alarms, regardless of the performance of the closed-loop reconfigured system. However, as shown initially in (Mariton, 1989, 1990) when the FDI output is feedback into a reconfiguration closed-loop control, a short detection delay and a low false alarm rate do not guarantee stable operation of the reconfigurable system. This makes the synthesis of reconfigurable controllers that take explicitly into account the inherent imperfections of the FDI process of great importance. A first model that includes in the same analysis framework the FDI and reconfiguration algorithms was initially proposed by (Srichander and Walker, 1993). The proposed model belongs to the class of MJLSs. Two Markov processes were defined: the first Markov process represents

system components failures and the second Markov process represents the FDI process used to reconfigure the control law. Most of the evaluation methods for FTCS based on Markov modeling have found success because the Markovian hypothesis is very intuitive and that the underlying computations are quite simple.

Modeling the FDI process by Markov chain assumes that the sojourn time between successive transitions of FDI modes has an exponential distribution, which in turn means constant detection rate and constant false alarm rate. It also implies that the future outcomes of FDI tests are independent of the past outcomes, so FDI tests are memory less. However, in order to minimize false alarm rates, sequential FDI tests are often used. In this case, the Markov jump process modeling technique breaks down. This is essentially due to the fact that the Markovian hypothesis imposes restrictions on the distribution of the sojourn time between consecutive states, which should be exponentially distributed (continuous-time case) or geometrically distributed (discrete-time case). This is the main drawback when applying Markov processes in real applications. In many of the practical situations, however, the FDI decision behavior can be modeled by a finite state semi-Markov process conditioned on the failure behavior. The main advantage of such models is that they allow the sojourn time to follow any distribution. So, the sequential change detection algorithms, which allow non-exponential sojourn time distributions also, are useful in semi-Markov modeling of FDI process.

Specific sequential change detection algorithms are the exponential weighted moving average algorithm (EWMA), the cumulative sum algorithm (CUSUM) or the generalized likelihood ratio (GLR) algorithm *etc* (Nikiforov, 1995). Statistical properties of fault detection scheme based on CUSUM test are investigated by Zhao et al. (Zhao and Kinnaert, 2009), in which explicit formulae for the probability densities of fault detection delay and time between false alarms are obtained. Similar study for GLR method is studied by Yang et al. (Yang and Zhao, 2011). EWMA, like CUSUM algorithm, is effective in detecting small shifts which may be critical, and also EWMA is effective than CUSUM in estimating the process mean at any given time (Ong et al., 2004). EWMA method use an exponentially based weighting procedure to give less weightage to previous information and to emphasize the recent data. Srivastava et al. (Srivastava and Wu, 1993) obtained stationary average fault detection delay for EWMA. In this chapter, the probability densities of fault detection delay and the time between false alarms of EWMA method are presented.

To investigate the statistical properties of EWMA, the observation process is considered as a piecewise constant signal disturbed by white noise. We study the fault detection delay and the time between false alarms when there is a positive shift in the mean of the observation process at an unknown time.

We begin with problem description.

B.2 Problem Description

In this section, we will first present briefly the main ingredients of the EWMA algorithm and then describe the problem considered in this chapter.

EWMA algorithm: Consider an *observation* process $\{z_k\}_{k \in \mathbb{N}_{\geq 0}}$, which is a piecewise constant signal disturbed by a white noise. Let $\{w_k\}_{k \in \mathbb{N}_{\geq 0}}$, be a sequence of independent white Gaussian variables with zero mean and unity variance. Hence, $\{z_k\}_{k \in \mathbb{N}_{\geq 0}}$ can be represented as follows

$$z_k = w_k + \mu_0 I_{[k < \theta]} + \mu_1 I_{[k \geq \theta]}, \quad (\text{B.1})$$

where θ is the unknown change-occurrence time, μ_0 represents the mean of $\{z_k\}_{0 \leq k < \theta}$ (before change occurrence), μ_1 the mean of $\{z_k\}_{k \geq \theta}$ (after change occurrence) and $I_{[g(.)]}$ is a standard indicator function defined as

$$I_{[g(.)]} = \begin{cases} 1, & \text{if } g(.) \text{ is true,} \\ 0, & \text{if } g(.) \text{ is false.} \end{cases} \quad (\text{B.2})$$

The decision signal y_k for EWMA algorithm is given by

$$y_k = \lambda \sum_{i=0}^{k-1} (1 - \lambda)^i z_{k-i} + (1 - \lambda)^k z_0, \quad 0 < \lambda < 1,$$

where y_0 in the above formula generally being set to the estimate of the process mean. Since $0 < \lambda < 1$, it is apparent that the weights of y_k decrease geometrically and can be shown that they sum up to unity. The EWMA decision signal can be written in a recursive form as

$$y_k = (1 - \lambda)y_{k-1} + \lambda z_k, \quad 0 < \lambda < 1, \quad k \geq 1, \quad (\text{B.3})$$

Let the observation process z_k be described as in (B.1), and the decision signal y_k for EWMA method be given by (B.3). The *alarm time* of EWMA method can be defined by (Basseville and Nikiforov, 1993),

Definition B.1. A random variable T is called an alarm time for EWMA method with respect to a decision signal y_k if

$$T = \inf\{k : y_k \geq h\}, \quad k \geq 1, \quad (\text{B.4})$$

where h is the threshold of EWMA method.

One of the most important characteristic variables of change detection algorithms is the *time delay for detection* (T_d), which is the time taken by the change detection algorithm to set an alarm when a change actually occurs (Blanke et al., 2006). We used the time delay for detection, detection delay and the fault detection delay interchangeably in this chapter, where they hold the same meaning. Another significant index for change detection algorithms is the *time between false alarms* (T_f). False alarm is the reporting of

change detection by a change detection algorithm, even though the shift is not observed in the process. This phenomenon mainly occurs as a consequence of low thresholds and the presence of high noise during input measurements (Blanke et al., 2006).

The problem treated in this chapter can be resumed as follows: Find the probability density functions of T_d and T_f for EWMA method.

The time delay for detection and the time delay between false alarms for EWMA method will be characterized using (B.4) as shown in the next section.

B.3 Main Results

Before giving the statistical characterization of T_d and T_f , we will first recall some basic results which will be used in the sequel.

B.3.1 Mathematical Preliminaries

Let $(B_t)_{t \geq 0}$ be the standard Brownian motion. The related Ornstein-Uhlenbeck (OU) process $(U_t)_{t \geq 0}$ is defined to be the unique solution of the stochastic differential equation (Doob, 1942)

$$dU_t = -\beta U_t dt + dB_t, \quad U_0 = a \in \mathbb{R}, \quad \beta \in \mathbb{R}. \quad (\text{B.5})$$

The first passage time or hitting time $T_{a \rightarrow b}$ for OU process is denoted as

$$T_{a \rightarrow b} = \inf\{t > 0 : U_t = b | U_0 = a\}. \quad (\text{B.6})$$

It is now possible to describe the probability of the hitting time $T_{a \rightarrow b}$ within a time length t . Let $f_{a \rightarrow b}(t)$ be the probability density function of $T_{a \rightarrow b}$. In a special case when $a < b$ and $b = 0$, (Alili et al., 2005; Pitman and Yor, 1981) obtained an explicit formula for $f_{a \rightarrow 0}(t)$ as

$$f_{a \rightarrow 0}(t) = \frac{|a|}{\sqrt{2\pi}} \left(\frac{\beta}{\sinh(\beta t)} \right)^{3/2} \exp \left(-\frac{\beta a^2 e^{-\beta t}}{2 \sinh(\beta t)} + \frac{\beta t}{2} \right). \quad (\text{B.7})$$

Also for any $a < b$, $\beta \in \mathbb{R}$, (Alili et al., 2005) obtained the integral representation of $f_{a \rightarrow b}(t)$ as

$$f_{a \rightarrow b}(t) = \frac{2\beta}{\pi} \int_0^\infty \cos(\nu \beta t) \frac{Hr_{-\nu}(-\sqrt{\beta}b)Hr_{-\nu}(-\sqrt{\beta}a) + Hi_{-\nu}(-\sqrt{\beta}a)Hi_{-\nu}(-\sqrt{\beta}b)}{Hr_{-\nu}^2(-\sqrt{\beta}b) + Hi_{-\nu}^2(-\sqrt{\beta}b)} d\nu, \quad (\text{B.8})$$

where $Hr_{-\nu}(\cdot)$ and $Hi_{-\nu}(\cdot)$ are specified by

$$\begin{aligned} Hr_{-\nu}(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \cos \left(\frac{-\nu}{2} \log \left(1 + \left(\frac{z}{s} \right)^2 \right) \right) ds, \\ Hi_{-\nu}(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \sin \left(\frac{-\nu}{2} \log \left(1 + \left(\frac{z}{s} \right)^2 \right) \right) ds. \end{aligned}$$

Also note that, in addition to the integral representation of $f_{a \rightarrow b}(t)$, equivalent series representation and the Bessel bridge representation formulae for $f_{a \rightarrow b}(t)$ are presented in (Alili et al., 2005), which are not discussed here.

B.3.2 Characterization of Fault Detection Delay for EWMA Algorithm

Consider the observation process $\{z_k\}_{k \in \mathbb{N}_{\geq 0}}$ given by (B.1). The EWMA decision function given by (B.3) can be rewritten as

$$y_k = (1 - \lambda)y_{k-1} + \lambda\{w_k + \mu_0 I_{[k < \theta]} + \mu_1 I_{[k \geq \theta]}\}, \quad (\text{B.9})$$

where y_0 can be set to the mean of observation process until the change (De Vargas et al., 2004), *i.e.* $y_0 = \mu_0$. Equation (B.9) can be reformulated as follows

$$\begin{cases} y_k - y_{k-1} = -\lambda y_{k-1} + \lambda\{w_k + \mu_0 I_{[k < \theta]} + \mu_1 I_{[k \geq \theta]}\}, \\ y_0 = \mu_0. \end{cases}$$

To explore the fault detection delay of EWMA, consider the threshold as the horizontal level μ_1 . After the actual occurrence of change, the time delay for detection T_d , with the threshold as μ_1 , is characterized by the following equations

$$T_d = \inf\{k : y_k \geq \mu_1, k \geq \theta\}, \quad (\text{B.10})$$

where

$$y_k - y_{k-1} = \lambda(\mu_1 - y_{k-1}) + \lambda w_k, \quad 0 < \lambda < 1, \quad k \geq \theta. \quad (\text{B.11})$$

To obtain the analytical expressions for probability densities of T_d , the continuous EWMA process analogous to y_k is considered. Let T_s be the sampling period, then the decision process y_k can often conveniently be thought of as a continuous-time random process $\mathcal{Y}_t$ evaluated at time $t = kT_s$. By letting (without loss of generality) sampling period to be unity, the continuous time process corresponding to (B.10), (B.11) can be represented as

$$T_d = \inf\{t > 0 : \mathcal{Y}_t \geq \mu_1\} \triangleq \inf\{t > 0 : \mathcal{Y}_t = \mu_1\}, \quad (\text{B.12})$$

$$d\mathcal{Y}_t = \lambda(\mu_1 - \mathcal{Y}_t)dt + \lambda dB_t, \quad \mathcal{Y}_0 = \mu_0, \quad 0 < \lambda < 1, \quad (\text{B.13})$$

where B_t is the standard Brownian motion. Solution to the stochastic differential equation (B.13), $\mathcal{Y}_t$, is an OU process (Doob, 1942). Consider a transformed OU process $\tilde{\mathcal{Y}}_t = \frac{\mathcal{Y}_t - \mu_1}{\lambda}$, then (B.12), (B.13) can be rewritten as

$$T_d = \inf\{t > 0 : \tilde{\mathcal{Y}}_t = 0\}, \quad (\text{B.14})$$

$$d\tilde{\mathcal{Y}}_t = -\lambda\tilde{\mathcal{Y}}_t dt + dB_t, \quad \tilde{\mathcal{Y}}_0 = \frac{\mu_0 - \mu_1}{\lambda}, \quad 0 < \lambda < 1. \quad (\text{B.15})$$

In this case, the OU process $\tilde{\mathcal{Y}}_t$ is similar to the OU process U_t in (B.5), and T_d is similar to $T_{a \rightarrow b}$ in (B.6). Then, one can right away utilize the results of first passage time for an

OU process reviewed in Section B.5 at the end. Consider $\tilde{\mathcal{Y}}_0 = \frac{\mu_0 - \mu_1}{\lambda} \triangleq \alpha$, hence using (B.7), the probability density of time delay for detection can be obtained as

$$f_{T_d}(t) = \frac{d}{dt} \Pr\{T_d \in (0, t]\},$$

$$\triangleq f_{\alpha \rightarrow 0}(t) = \frac{|\alpha|}{\sqrt{2\pi}} \left(\frac{\lambda}{\sinh(\lambda t)} \right)^{3/2} \exp \left(-\frac{\lambda \alpha^2 e^{-\lambda t}}{2 \sinh(\lambda t)} + \frac{\lambda t}{2} \right).$$

In the above scenario, the threshold is a constant level μ_1 , which is the piecewise constant mean of y_k after the change. If one may want to select a general threshold $h > \mu_0$, then similar to equations (B.12), (B.13), one gets

$$\begin{cases} T_d = \inf\{t > 0 : \mathcal{Y}_t = h\}, \\ d\mathcal{Y}_t = \lambda(\mu_1 - \mathcal{Y}_t)dt + \lambda dB_t, \quad \mathcal{Y}_0 = \mu_0, \quad 0 < \lambda < 1, \end{cases} \quad (\text{B.16})$$

or by letting $\tilde{\mathcal{Y}}_t = \frac{\mathcal{Y}_t - \mu_1}{\lambda}$,

$$\begin{cases} T_d = \inf\{t > 0 : \tilde{\mathcal{Y}}_t = \frac{h - \mu_1}{\lambda}\}, \\ d\tilde{\mathcal{Y}}_t = -\lambda \tilde{\mathcal{Y}}_t dt + dB_t, \quad \tilde{\mathcal{Y}}_0 = \frac{\mu_0 - \mu_1}{\lambda}, \quad 0 < \lambda < 1. \end{cases} \quad (\text{B.17})$$

Since $(\frac{\mu_0 - \mu_1}{\lambda}) < (\frac{h - \mu_1}{\lambda})$, one can readily use the (B.8) to obtain $f_{T_d}(t)$. In perspective of computations, threshold set to μ_1 lead to less computations than for the threshold h .

B.3.3 Characterization of the Time Between False Alarms

When the threshold is low and the system is too noisy, EWMA decision signal y_k crosses the threshold even though there is no change in the observation process, which may trigger the false alarms. Considering the EWMA decision function without any change in the observation process z_k , the time between false alarms can be characterized as

$$\begin{cases} T_f = \inf\{k : y_k \geq \zeta\}, \\ y_k - y_{k-1} = \lambda(\mu_0 - y_{k-1}) + \lambda w_k, \quad y_0 = \mu_0, \quad 0 < \lambda < 1, \end{cases} \quad (\text{B.18})$$

where ζ is the threshold. When the system is too noisy and $(\zeta - \mu_0)$ is small in magnitude, false alarms may occur. Decision signal y_k is set to zero after each detection and the EWMA algorithm starts again, so the time before false alarm (T_f) is identical to the time between successive false alarms (Zhao and Kinnaert, 2009).

Similar to previous section, the continuous analogous problem to (B.18) can be posed as

$$\begin{cases} T_f = \inf\{t > 0 : \mathcal{Y}_t = \zeta\}, \quad \zeta > \mu_0, \\ d\mathcal{Y}_t = \lambda(\mu_0 - \mathcal{Y}_t)dt + \lambda dB_t, \quad \mathcal{Y}_0 = \mu_0, \quad 0 < \lambda < 1. \end{cases} \quad (\text{B.19})$$

Solution to the above stochastic differential equation, $\mathcal{Y}_t$, is an OU process. Let $\bar{Y}_t = \frac{\mathcal{Y}_t - \mu_0}{\lambda}$,

then the above equations can be written as

$$\begin{cases} T_f &= \inf\{t > 0 : \bar{Y}_t = \frac{\zeta - \mu_0}{\lambda}\}, \quad \zeta > \mu_0, \\ d\bar{Y}_t &= -\lambda \bar{Y}_t dt + dB_t, \quad \bar{Y}_0 = 0. \end{cases} \quad (\text{B.20})$$

Probability density function of T_f , $f_{T_f}(t)$ can be obtained by substituting $a = 0$, $b = \frac{\zeta - \mu_0}{\lambda}$, $\beta = \lambda$, in (B.8) as follows

$$\begin{aligned} f_{T_f}(t) &= \frac{d}{dt} \Pr\{T_f \in (0, t]\}, \\ &\triangleq f_{0 \rightarrow b}(t) \\ &= \frac{2\lambda}{\pi} \int_0^\infty \cos(\nu \lambda t) \frac{Hr_{-\nu}(-\sqrt{\lambda}b)}{Hr_{-\nu}^2(-\sqrt{\lambda}b) + Hi_{-\nu}^2(-\sqrt{\lambda}b)} d\nu, \end{aligned} \quad (\text{B.21})$$

where $Hr_{-\nu}(\cdot)$ and $Hi_{-\nu}(\cdot)$ are specified by

$$\begin{aligned} Hr_{-\nu}(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \cos\left(\frac{-\nu}{2} \log\left(1 + \left(\frac{z}{s}\right)^2\right)\right) ds, \\ Hi_{-\nu}(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \sin\left(\frac{-\nu}{2} \log\left(1 + \left(\frac{z}{s}\right)^2\right)\right) ds. \end{aligned}$$

Here ν , s are variables of integration. Equation (B.21) can be approximated by the trapezoidal rule given in (B.22).

B.4 Numerical Experiments

To demonstrate the validity of the analytical distributions given above, simulations are carried out in MATLAB and MATHEMATICA (for (B.22)) to compare the probability density functions of fault detection delay and the time between the false alarms with the normalized experimental histograms. As discussed above, only the positive change in the mean is considered. The observation process z_k which is a piecewise constant signal disturbed by white noise with mean 2 and unit variance (*i.e.* $z_k \sim \mathcal{N}(2, 1)$) without fault, and $z_k \sim \mathcal{N}(4, 1)$ in the presence of fault. Select $\lambda = 0.2$, and $\mu_0 = 2$, which leads to $g_0 = 2$. With $\mu_0 = 2$, $\lambda = 0.2$, the threshold set to 4, $\alpha = -10$, and using (B.16), the probability density function of Fault detection delay ($f_{T_d}(t)$) is shown in Figure B.1.

Similarly, with the parameters $\mu_0 = 2$, $\zeta = 2.2$, $\lambda = 0.2$, the probability density function of the time between the false alarms ($f_{T_f}(t)$) can be obtained using analytical formula (B.21). We use the approximation of $f_{T_f}(t)$ by trapezoidal rule given in (B.22). The approximate $f_{T_f}(t)$ along with the normalized experimental histogram are shown in Figure B.1. The tests for fault detection delay and false alarms with the given parameters are carried out for 10000 times in both cases to obtain the corresponding normalized histograms. The analytical distributions are matching with the normalized histograms as observed in Figure B.1.

In EWMA method, λ is the most important parameter, so the effect of λ on the detection

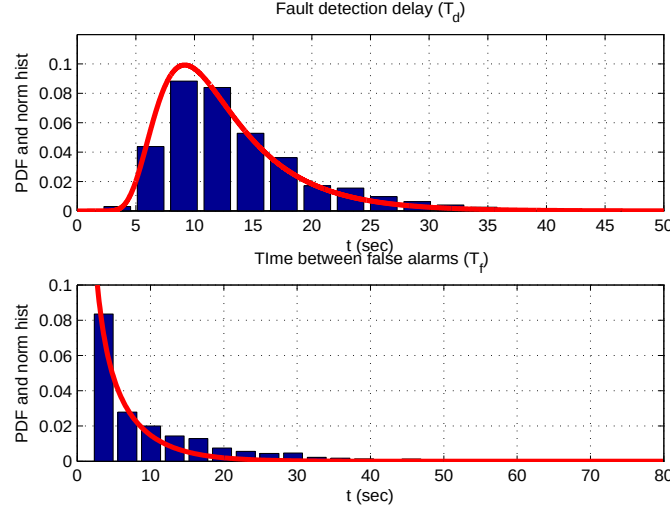


Figure B.1: PDF of fault detection delay ($f_{T_d}(t)$) and the time between false alarms ($f_{T_f}(t)$) and the normalized histograms

delay and the time between false alarms is examined here. The variation of λ on the mean fault detection delay, $\mathbb{E}[T_d]$ is shown in Figure B.2 with parameter $\mu_0 = 2$, $\mu_1 = 4$, and the threshold set to μ_1 . Since computing $\mathbb{E}[T_d]$ is quite cumbersome, the densities are shown in Figure B.2 to observe the variation of $\mathbb{E}[T_d]$. From the figure, as λ increases $\mathbb{E}[T_d]$ decreases, which is the desirable quality of any change detection algorithm. But the increase in λ affect the false alarms which is demonstrated in the following. Also, the effect of variation of the shift of the observation process $\mu_1 - \mu_0$ on the mean fault detection delay is observed in Figure B.2. With $\mu_0 = 2$, $\lambda = 0.2$, and threshold set to μ_1 , one can notice that, as μ_1 increases $\mathbb{E}[T_d]$ increases, which depicts the characteristic of EWMA algorithm which is mainly used for detecting small changes.

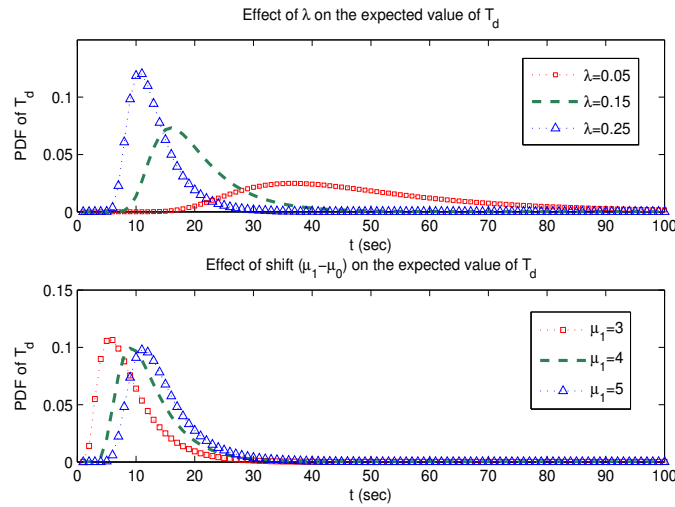


Figure B.2: Effect of λ , ($\mu_1 - \mu_0$) on the expectation of T_d

To observe the effect of variation of λ on the mean time between false alarms, a qualitative

analysis is carried out by plotting the PDF of time between false alarms with λ , and also the dependence of variance of y_k on λ . With $\mu_0 = 2$, $\zeta = 2.2$, the Probability density functions of time between false alarms with different values of λ are shown in Figure B.3. One can notice that as λ increases, the probability of false alarms is relatively high. The same can be justified from Figure B.4 where the variance of y_k , which is obtained as $\frac{\lambda}{2-\lambda}$ as given Figure B.6, is shown with λ . If λ increases, then the variance of y_k also increases, which can cause more false alarms. From Figure B.2, Figure B.3 and Figure B.4, it is clear that the increase in λ causes lesser detection delay, but the false alarms increase. So, the choice of λ is essentially a compromise between detection delay and the false alarms.

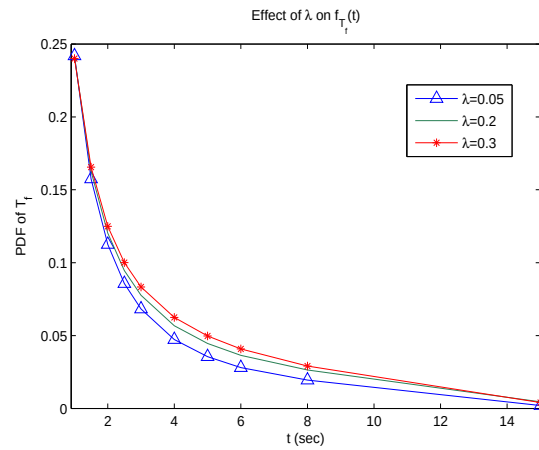


Figure B.3: Effect of λ on the false alarms

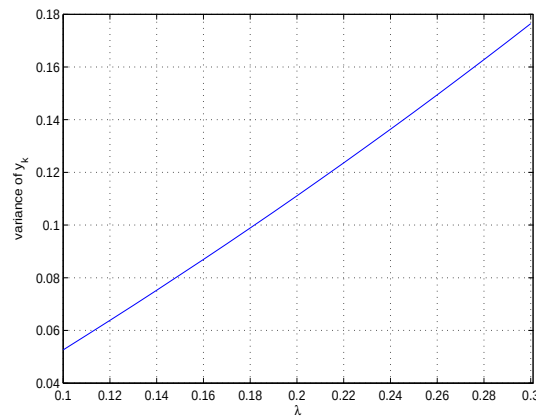


Figure B.4: Effect of λ on the variance of y_k

B.5 Formulae

Consider EWMA decision function y_k

$$y_k = (1 - \lambda)y_{k-1} + \lambda z_k,$$

where z_k is given by (B.1). The variance of y_k is given by (Montgomery, 1997; Nong et al., 2004)

$$\text{var}(y_k) = \text{var}(z_k) \frac{\lambda}{2 - \lambda}.$$

Since $\text{var}(z_k)$ is unity, one has

$$\text{var}(y_k) = \frac{\lambda}{2 - \lambda}.$$

The integral given in (B.8) can be approximated by the trapezoidal rule as follows

$$\begin{aligned} f_{a \rightarrow b}(t) = & \frac{e^{A/2}}{2t} \frac{H_{-A/(2\lambda t)}(-a\sqrt{\lambda})}{H_{-A/(2\lambda t)}(-b\sqrt{\lambda})} \\ & + \frac{e^{A/2}}{t} \sum_{k=1}^N (-1)^k \text{Re} \left(\frac{H_{-A/(2\lambda t) - k\pi i/(\lambda t)}(-a\sqrt{\lambda})}{H_{-A/(2\lambda t) - k\pi i/(\lambda t)}(-b\sqrt{\lambda})} \right) \end{aligned} \quad (\text{B.22})$$

where $H(\cdot)$ is the Hermite function, $A=18.1$ and N is chosen as 500 as given in (Alili et al., 2005). In the context of the chapter, β in (B.5) is equivalent to $\lambda \in (0, 1)$, which is the parameter of the EWMA method.

B.6 Conclusions

Exponentially weighted moving average algorithm is considered for detecting changes in the mean of a signal. In particular, we studied the abrupt changes in a signal which is white Gaussian in nature. Analytical expressions for probability density functions of the detection delay and the time between false alarms are obtained using the expressions of hitting time for Ornstein-Uhlenbeck process, and are validated by the experimental normalized histograms. Since λ is the most important parameter of EWMA method, we also studied the effect of λ on the detection scheme. It is demonstrated that the increase in λ decreases the detection delay, but causes more false alarms, so the choice of λ is a compromise between detection delay and the false alarms occurrence.

Appendix C

A Novel Mathematical Setup for Fault Tolerant Control Systems with State-dependent Failure Process

C.1 Introduction

In fault tolerant control system (FTCS), apart from the actuator, plant, sensors, input, outputs, there are two major modules in general: a fault detection and identification (FDI) module and a reconfiguration controller. In the event of faulty behaviour reported by the FDI module, the reconfiguration controller provides a necessary feedback action such that over all control system is fault tolerant, which is the reason for naming it as FTCS. One can find myriads of contributions of FTCS in the literature for instance in (Blanke et al., 2006; Zhang and Jiang, 2008) etc.

In the presence of abrupt changes or multiple modes that involve randomness, FTCS with Markov processes have been investigated in (Srichander and Walker, 1993; Mahmoud et al., 2001; Tao and Zhao, 2007; Huang, 2013) and the references therein. In these works, the authors primarily assumed different state spaces for a failure process and an FDI process to incorporate fault detection delays, false alarms that may result from various errors in the FDI module. FTCS modelling in this situation can be given by

$$\left. \begin{aligned} \dot{x}(t) &= A_{\theta(t)}x(t) + B_{\theta(t)}u(\eta(t), x(t), t), \\ x(0) &= x_0, \end{aligned} \right\} \quad (\text{C.1})$$

with the state vector $x(t) \in \mathbb{R}^n$, the initial state x_0 , the control input $u(., ., .) \in \mathbb{R}^m$, and the initial state $x_0 \in \mathbb{R}^n$. Let $A_{\theta(t)} \in \mathbb{R}^{n \times n}$, $B_{\theta(t)} \in \mathbb{R}^{n \times m}$, be the system matrices that depend on the mode $\theta(t)$, and are assumed to be known. Here $\{\theta(t), t \geq 0\} \in S$ and $\{\eta(t), t \geq 0\} \in T$ represent the failure process and the FDI process with finite state-spaces $S := \{1, 2, \dots, M\}$ and $T := \{1, 2, \dots, N\}$ respectively. An illustration of the

overall FTCS is given in Figure C.1, where the controller reconfiguration depends on the FDI process at each time. In an ideal scenario of FDI, the FDI process $\{\eta(t), t \geq 0\}$ should match exactly with the failure process $\{\theta(t), t \geq 0\}$, which is not the case in practise, thus the reason for different state spaces S and T for $\{\theta(t), t \geq 0\}$ and $\{\eta(t), t \geq 0\}$.

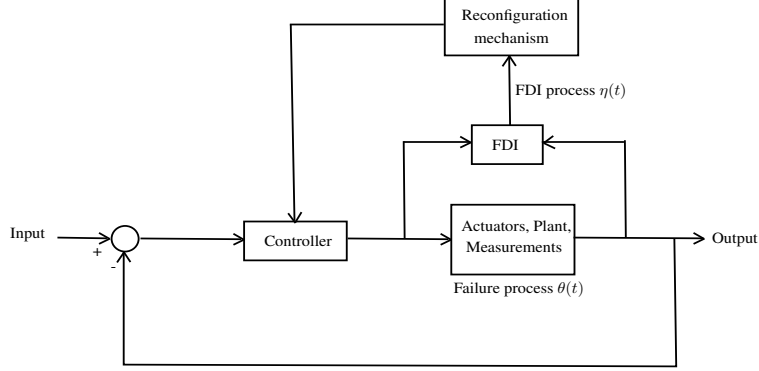


Figure C.1: An illustration of the FTCS

We begin with the failure process.

In the presence of abrupt changes in system parameters that may cause failures in various components, the failure process $\{\theta(t), t \geq 0\}$ has been modelled as time-homogeneous Markov Process, which implies the constant failure rates, which is a quite restrictive assumption. In practice, the failure rates depend on the usage and/or the current state of the component. For example, it was examined in (Bergman, 1978) that the failure rate of a component is dependent on its state at age t , where the state variable may be an amount of wear, stress etc. Also, for instance, in (Giorgio et al., 2010), a state-dependent Markov process was used to describe the random breakdown of cylinder lines in a heavy-duty marine diesel engine. Thus, we consider that the failure process $\theta(t)$ depends on the state variable. Next we move to the FDI process.

In an ideal case, by using various measurements of the FTCS, the FDI process should give accurate results whether a faulty behaviour has occurred in the system, which is used as a feedback to reconfigure the overall the FTCS to correct faulty behaviour. Thus, the FDI process depend directly on the failure process. But, in the presence of noisy measurements, it is difficult for the FDI process to provide accurate behaviour of the system. If the measurement process contain white noise, and memoryless tests are used to characterize the FDI process, then the FDI process can be modelled as a Markov jump process with finite space as examined in (Srichander and Walker, 1993; Mahmoud et al., 2001) etc. In our case, for simplicity, we consider that the FDI process is Markovian. It is a reasonable assumption, because even if the FTCS contain state-dependent failure process, one can still employ memoryless tests for FDI process, which just depend on the measurements from the system. Thus, we consider the FDI process, which is Markovian.

Thus, we consider state-dependent failure process, which is also Markovian in some suitably defined sets, and the FDI process as a pure Markov process which just depend on the failure process. The presence of state-dependent failure process makes the FTCS modelling (C.1) far more complex. To the best of our knowledge, a tractable the FTCS modelling (C.1) with state-dependent failure process has not been addressed, which forms a theme of this chapter.

In the next section, we provide a tractable mathematical model of FTCS with state-dependent failure process.

C.2 Mathematical model of the FTCS with state-dependent failure process

Consider a mathematical model of FTCS:

$$\left. \begin{aligned} \dot{x}(t) &= A_{\theta(t)}x(t) + B_{\theta(t)}u(\eta(t), x(t), t), \\ x(0) &= x_0. \end{aligned} \right\} \quad (\text{C.2})$$

Since we assume the Markovian modelling of the FDI process $\{\eta(t), t \geq 0\} \in T$ that depends on the failure process $\{\theta(t), t \geq 0\} \in S$, its evolutions for $m, n \in T$, $m \neq n$ and $z \in S$ can be considered as:

$$\Pr\{\eta(t + \Delta) = n | \eta(t) = m, \theta(t) = z\} = \delta_{mn,z}\Delta + o(\Delta), \quad (\text{C.3})$$

where $\Delta > 0$ and $\lim_{\Delta \rightarrow 0} \frac{o(\Delta)}{\Delta} = 0$. Regarding the transition rates of $\eta(t)$, we make the following assumption:

Assumption C.1. For each $z \in S$, $m, n \in T$, $\delta_{mn,z}$ is the transition rate of $\eta(t)$ from mode m to mode n with $\delta_{mn,z} \geq 0$ for $m \neq n$, and $\delta_{mm,z} = -\sum_{n=1, n \neq m}^N \delta_{mn,z}$. We further assume that $\delta_{mn,z}$ is bounded for all $z \in S$, $m, n \in T$.

Before proceeding with the state-dependent failure process $\{\theta(t), t \geq 0\}$, we make the following assumption regarding the different sets in $\mathbb{R}^n$.

Assumption C.2. Let $\mathcal{K} = \{1, 2, \dots, K\}$. We assume that the sets $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K \subseteq \mathbb{R}^n$, $\cup_{i=1}^K \mathcal{C}_i = \mathbb{R}^n$ and $\mathcal{C}_p \cap \mathcal{C}_q = \emptyset$ for any $p, q \in \mathcal{K}$, $p \neq q$. At any time $t \geq 0$, we assume that $x(t)$ belongs to exactly one of the sets $\mathcal{C}_l$, $l \in \mathcal{K}$. By its definition, $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K$ are Borel sets in $\mathbb{R}^n$.

Now, the evolutions of the state-dependent failure process $\{\theta(t), t \geq 0\}$ for $i, j \in S$, $i \neq j$

can be governed by:

$$\Pr\{\theta(t + \Delta) = j | \theta(t) = i, x(t)\} = \begin{cases} \lambda_{ij}^1 \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_1, \\ \lambda_{ij}^2 \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_2, \\ \vdots & \\ \lambda_{ij}^K \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_K. \end{cases} \quad (\text{C.4})$$

Regarding the transition rates of $\theta(t)$, we make the following assumption:

Assumption C.3. Let $\mathcal{K} \triangleq \{1, 2, \dots, K\}$. For each $m \in \mathcal{K}$, λ_{ij}^m is the transition rate of $\theta(t)$ from mode i to mode j with $\lambda_{ij}^m \geq 0$ for $i \neq j$, and $\lambda_{ii}^m = -\sum_{j=1, j \neq i}^N \lambda_{ij}^m$. We further assume that λ_{ij}^m is bounded for all $m \in \mathcal{K}$.

Remark C.1: Notice that the evolution of $\theta(t)$ in (C.4) depends on the state variable $x(t)$, hence $\theta(t)$ is called as a state-dependent failure process. Also observe that the failure rates are different depending on which set the state of the system belongs to. It is a valid assumption, because, in this setup, one can predefine suitable sets $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K$ and assign different failure rates in a suitable fashion, thus it is tractable. A more general jump system with state-dependent jump process without any other jump process (for example the FDI process in our case) is considered in this thesis similar to (C.4), where sufficient conditions for stability and stabilizing controller in terms of linear matrix inequalities (LMIs) are obtained by utilizing techniques of probability theory and stochastic version of Lyapunov's second method.

C.3 Challenges and future work

We would like to highlight a couple of challenges in the FTCS with state-dependent failure setup given above.

1. Since the overall FTCS (C.2) is nonlinear, it is necessary to provide its flow as a function of time.
2. For a simple state-feedback reconfiguration control of the FTCS, usually one design state-feedback gains obtained from Lyapunov's second method, which requires the assumption of Markovian process of the joint variables of the Lyapunov candidate (Boukas, 2006).

For point 1 above, one can describe the flows of the FTCS (C.2) when $x(t) \in \mathcal{C}_m$ for each $m \in \mathcal{K}$, which can be characterized in terms of suitably defined first exit times from different sets $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K$. For point 2 above, one can prove that the joint process formed by $(x(t), \theta(t), \eta(t))$ is Markov process by using arguments of probability theory.

Both of these issues has already been investigated in this thesis for a more general system without FTCS configuration, which we would like to extend to the FTCS framework,

which can provide answers to the reconfiguration control in FTCS with state-dependent failure process. This forms a future extension of this chapter.

C.4 Conclusions

We provided a tractable mathematical model for FTCS with state-dependent failure process. For the underlying state-dependent failure process, we explicitly considered different failure rates depending on the set to which the state of the system belongs. Finally, we presented a future course of direction.

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Résumé

Les systèmes dynamiques linéaires à sauts aléatoires représentent une classe de systèmes hybrides stochastiques. La commutation entre les différents modes de ce type de systèmes est régie par un processus stochastique à sauts. En particulier, nous considérons dans le cadre de cette thèse que le processus à sauts aléatoires est non-homogène en temps i.e. que les probabilités (ou taux) de transition entre les différents modes du système sont à temps variant. Il s'agit ici d'une modélisation motivée par des considérations d'ordre pratique.

Cette thèse se décompose en deux parties principales : la première partie est dédiée au problème de la stabilité/stabilisation des systèmes dynamiques linéaires à sauts aléatoires non-homogènes en temps. La seconde partie concerne le problème de la commande sous contraintes de cette classe de systèmes.

Dans la première partie, nous nous intéressons d'une part à l'analyse de la stabilité en moyenne quadratique des systèmes linéaires à sauts Markoviens non-homogènes en temps, et ceci en temps discret. En particulier, nous considérons le cas où la matrice de probabilité de transition de la chaîne de Markov non-homogène varie dans un intervalle. En nous appuyons sur des outils issus de l'analyse par intervalles ainsi que de la théorie des graphes, nous obtenons une condition suffisante de stabilité en moyenne quadratique de cette classe de systèmes. Ces conditions sont exprimées à partir de l'expression du rayon spectrale d'une matrice définie de façon appropriée. D'autre part, nous considérons le problème de stabilité stochastique de systèmes linéaires à temps continu et à sauts aléatoires dépendant de l'état du système i.e. lorsque les taux de transition du processus de saut dépendent de la variable d'état. En utilisant une version stochastique de la deuxième méthode de Lyapunov et la formule de Dynkin adaptée à cette classe de systèmes, nous établissons des conditions suffisantes de stabilité en termes d'inégalités matricielles linéaires.

Dans la seconde partie de cette thèse, nous étudions le problème de commande prédictive des systèmes à sauts aléatoires sujets à des contraintes de type probabilistes. Nous étudions dans un premier temps un problème de commande à horizon glissant d'un système linéaire discret à sauts dépendant de l'état, soumis à des perturbations aléatoires éventuellement non bornées et à des contraintes de type probabilistes sur l'état du système. Dans un deuxième temps, nous relâçons l'hypothèse d'accessibilité de l'état du système et considérons le problème de commande à horizon glissant basée sur l'estimation de l'état d'un système linéaire discret à sauts Markovien soumis à des perturbations aléatoires éventuellement non bornées et à des contraintes de type probabilistes sur l'état du système.

Abstract

We address stability and control problems of random jump linear systems, a kind of stochastic hybrid systems that consists of a set of linear systems and the switching among them is governed by a random jump process. Broadly, we categorize the thesis into two parts : stability and control subject to constraints.

In the first part, we investigate second moment stability and stabilization of random jump linear systems. We first consider discrete-time inhomogeneous Markov jump linear systems, where the underlying transition probability matrix of the inhomogeneous Markov Chain is varying in an interval and obtain a sufficient condition for second moment stability in terms of a spectra radius of a suitably defined matrix by using results of interval analysis and graph theory . Alternatively, we obtain a convex hull representation of the interval transition probability matrix and give a sufficient condition for second moment stability in terms of linear matrix inequalities, using which we deal with stabilization. Next, we consider a continuous-time state-dependent jump linear system, where the transition rates of the underlying random jump process depend on the state variable and address the problem of stochastic stability and stabilization. Using a stochastic version of Lyapunov's second method and the Dynkin's formula, we give sufficient conditions for stochastic stability and state-feedback stabilization in terms of linear matrix inequalities. Subsequently, for the same system, we synthesize a state-feedback $\mathcal{H}_\infty$ controller in the presence of external disturbances that guarantee disturbance rejection while ensuring stochastic stability.

In the second part, we investigate control of random jump linear systems subject to constraints. We use model predictive control, also called receding horizon control to handle constraints. We first investigate a receding horizon control of discrete-time state-dependent jump linear systems subject to possibly unbounded random disturbances and probabilistic state constraints. Due to a nature of the dynamical system and the constraints, we consider a one-step receding horizon. We consider the receding horizon control law to have a linear state-feedback and an admissible offset term, where state-feedback terms are used for stabilization and offset terms are used for obtaining receding horizon control. For the same system, we consider the presence of noisy measurements and would like to apply the receding horizon approach afore mentioned. However, the unavailability of the state variable makes the formulation of state-dependent jump process intractable. Thus we confine ourselves to discrete-time inhomogeneous Markov jump linear systems, a simpler random jump linear systems, with process noise and noisy measurements and address the receding horizon control problem.