Output feedback event-triggered control

A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy in
Automatic Control

by

Mahmoud Abdelrahim

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Thesis defended on 23 July 2014 in presence of the committee:

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To my parents, my brothers, and my sisters

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Contents

Abbreviations

A/D	Analog to digital
D/A	Digital to analog
NCS	Networked control system(s)
LTI	Linear time-invariant
MATI	Maximum allowable transmission interval
LMI	Linear matrix inequality(ies)
ETC	Event-triggered control
PETC	Periodic event-triggered control
ISS	Input-to-state stability
ZOH	Zero-order-hold

Notations

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_{\geq 0}$	the set of non-negative real numbers
$\mathbb{R}^n$	n-dimensional real Euclidean space
$\mathbb{R}^{n \times m}$	the set of $n \times m$ real matrices
$\mathbb{Z}_{\geq 0}$	the set of non-negative integers
$\mathbb{Z}_{>0}$	the set of positive integers
$\mathcal{K}$	a continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is zero at zero, and strictly increasing
$\mathcal{K}_\infty$	a continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$, and $\gamma(s) \to \infty$ as $s \to \infty$
$\mathcal{KL}$	a continuous function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(., t)$ is of class $\mathcal{K}$ and for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s, .)$ is decreasing to zero
$\lambda_{\min}(A)$	the minimum eigenvalue of the real symmetric matrix A
$\lambda_{\max}(A)$	the maximum eigenvalue of the real symmetric matrix A
A^T	the transpose of the matrix A
A^{-T}	the inverse of the transposed and vice versa, $A^{-T} = (A^{-1})^T = (A^T)^{-1}$
diag $(A_1, \cdots, A_N)$	block-diagonal matrix with the entries $A_1, \cdots, A_N$ on the diagonal
$\star$	stands for symmetric blocks in a matrix
$A > 0$	symmetric positive definite matrix A
$A < 0$	symmetric negative definite matrix A
$\|x\|$	the Euclidean norm of the vector $x \in \mathbb{R}^n$, defined as $\|x\| := \sqrt{x^T x}$
$\|A\|$	the 2-norm of the matrix $A \in \mathbb{R}^{n \times m}$, defined as $\|A\| := \sqrt{\lambda_{\max}(A^T A)}$
$\Sigma(Q)$	stands for $Q + Q^T$ for any square matrix Q
$\mathbb{I}_n$	the identity matrix of dimension n

(x, y)	denotes the vector $[x^T, y^T]^T$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$
$\forall$	for all
$:=$	left-hand side is defined by right-hand side
$=:$	right-hand side is defined by left-hand side
$\in$	belongs to
$\subset$	subset
$\cap$	intersection
$\cup$	union
$\rightarrow$	tend to, or mapping to
$\lim$	limit
$\sup$	supremum
$\inf$	infimum
dom ϕ	domain of function ϕ
$T_C(x)$	the tangent cone to the set $C \subset \mathbb{R}^n$ at the point $x \in \mathbb{R}^n$
$\langle \nabla V(x), \upsilon \rangle$	standard directional derivative of the differentiable function V at x along the vector field υ
$V^\circ(x; \upsilon)$	generalized directional derivative of Clarke of the function V at x in the direction υ

Résumé en français

Les systèmes contrôlés via un réseau sont des systèmes dans lesquels le process et le contrôleur communiquent via un canal de communication numérique partagé. L'utilisation d'un réseau offre de nombreux avantages en termes de flexibilité, de complexité de câblage, de coût et de facilité de maintenance, par rapport aux connexions point-à-point conventionnelles. Pour ces raisons, le systèmes commandés via un réseau sont de plus en plus populaires. On peut citer comme champs applicatifs l'industrie chimique, les raffineries, les centrales électriques, les avions, les réseaux de transport d'eau, les usines industrielles, les réseaux d'énergie, le contrôle environnemental, voir [13, 76] par exemple. En contre-partie, réseau induit des contraintes de communication telles que l'échantillonnage irrégulier, la quantification des signaux, des pertes de paquets, des retards de communication variables, voir par exemple [45, 48, 125]. Il est donc nécessaire de construire des lois de commande qui garantissent la stabilité du système tout en prenant en compte ces contraintes. Nous nous concentrons dans le cadre de cette thèse aux limitations induites par l'échantillonnage des signaux, et ignorons les autres effets possibles.

Le commande à transmissions événementielles consiste à définir les instants de transmission selon un critère dépendant de l'état du système et non d'une horloge à l'instar des implantations périodiques. Dans ce dernier cas, la période d'échantillonnage doit être inférieure à une valeur maximale qui dépend du système en question. Bien que cette stratégie soit facile à mettre en œuvre, il n'est pas évident que l'échantillonnage périodique permette une utilisation efficace du réseau. En effet, que le système soit en régime permanent ou transitoire n'a aucun impact sur le nombre de transmissions. Dans le cadre de la commande, il semble plus approprié d'adapter les instants de transmissions à l'état du système d'où l'idée de commande à transmissions événementielles. Ainsi, l'utilisation des ressources de calcul et de communication peut être réduite significativement comparée aux méthodes périodiques. L'idée est de surveiller en permanence l'état du système et de déclencher une transmission (et donc de fermer la boucle de commande) uniquement lorsqu'un critère prédéfini est satisfait, voir [1, 7, 10, 22, 43, 103, 120]. Ce paradigme est actuellement le sujet de nombreux travaux.

La plupart des résultats existants sur le commande à transmissions événementielles supposent que les mesures complètes de l'état sont disponibles. L'application de ces résultats est donc limitée puisqu'ils excluent de facto les lois de commande par retour de sortie. Il s'avère que lorsque seule une sortie du système est mesurée, le problème devient beaucoup plus complexe. Il devient en effet de garantir un temps minimum entre deux transmissions ce qui est nécessaire pour que le contrôleur soit implantable. Il est donc important de développer des commandes par retour de sortie à transmissions événementielles.

Objectifs et contributions

Motivé par les discussions précédentes, dans cette thèse, nous étudions le problème de la commande par retour de sortie à transmissions événementielles. En particulier, nous traitons les sujets suivants :

- Nous développons une conception basée sur l'émulation pour stabiliser une classe de systèmes non linéaires.
- Nous présentons une procédure pour concevoir simultanément la loi de commande et de la condition de déclenchement pour les systèmes linéaires afin de réduire davantage la quantité de transmissions.
- Nous proposons une méthode adaptée aux systèmes non linéaires dont la dynamique ont deux échelles de temps. En particulier, nous nous appuyons uniquement sur la connaissance d'une approximation de la dynamique lente.

Plan de la Thèse

Cette thèse est organisée de la façon suivante.

Chapitre 1 : Introduction

Nous définissons dans un premier temps les systèmes contrôlés via un réseau et nous présentons ensuite les techniques de conception de lois de commande. Ensuite, nous introduisons le principe de la commande à transmissions événementielles et nous expliquons ses avantages par rapport aux implantations périodiques. Nous passons alors en revue les résultats existants dans la littérature sur ce thème et nous montrons les difficultés techniques générées par les commandes par retour de sortie. Enfin, nous présentons nos objectifs et nos contributions.

Chapitre 2 : Conception de lois de commandes à transmission événementielles pour des systèmes non linéaires

Nous proposons une méthode de synthèse de lois de commande événementielle par retour de sorties pour des systèmes non linéaires. Cette approche est dite par émulation, puisqu'un retour de sortie est tout d'abord construit en temps continu, en ignorant l'échantillonnage, puis le critère de transmission est conçu. Le problème est modélisé comme un système hybride. La loi de transmission proposée consiste à combiner une horloge temporelle et un critère dépendant de la sortie afin de garantir l'existence d'un temps minimal entre deux transmissions. Les résultats sont appliqués à deux systèmes physiques, ainsi qu'aux systèmes linéaires stabilisables et détectables. Le cas particulier de la commande par retour d'état est également abordé.

Chapitre 3 : Co-conception pour les systèmes linéaires

Ce chapitre étend les résultats du Chapitre 2 afin de simultanément concevoir le loi de retour de sortie et le critère de transmission pour stabiliser des systèmes linéaires (il ne s'agit plus de synthèse par émulation). Nous présentons des inégalités matricielles linéaires qui permettent la synthèse en question. Nous expliquons ensuite comment exploiter ces résultats pour optimiser l'échantillonnage. Nous proposons d'abord une méthode pour agrandir le temps minimal entre deux transmissions, puis nous présentons une méthode d'optimisation heuristique pour réduire les transmissions.

Chapitre 4 : Systèmes singulièrement perturbés

Dans ce chapitre, nous examinons la synthèse de commande à transmissions événementielles pour la stabilisation de systèmes dont la dynamique évolue selon deux échelles de temps. Notre objectif est de concevoir la commande directement à partir du modèle approximé du système lent (les dynamiques rapides sont ignorées).

Nous suivons l'approche par émulation : nous supposons que nous savons résoudre le problème en absence d'échantillonnage et ensuite nous étudions comment concevoir la règle de transmission en présence des contraintes de communication. Nous proposons dans un premier temps un modèle hybride et nous expliquons qu'une loi de déclenchement que garantit la stabilité et l'existence d'un temps minimal uniforme entre deux transmissions pour le modèle lent ne garantit pas toujours l'existence d'un tel temps pour le système global. Nous présentons ensuite des conditions suffisantes sur le système hybride singulièrement perturbé et nous présentons les résultats principaux. Ensuite, nous montrons que les résultats sont applicables à une classe des systèmes globalement Lipschitziens qui inclue les systèmes linéaires comme un cas particulier. Finalement, avons appliqué les résultats sur un modèle d'avion F-8.

Chapitre 5 : Conclusions

Dans ce chapitre, nous présentons les conclusions générales, nous mettons en évidence les contributions principales et nous fournissons quelques pistes pour la recherche future.

Annexe A : Preuves

Nous fournissons les preuves de résultats du Chapitre 4.

Annexe B : Rappels mathématiques

Nous rappelons quelques préliminaires mathématiques ainsi que les outils fondamentaux requis pour cette thèse.

Publications

Les travaux de cette thèse ont fait l'objet des publications suivantes.

Articles de journal

- M. Abdelrahim, R. Postoyan and J. Daafouz, Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics, *Automatica, accepté*.
- M. Abdelrahim, R. Postoyan, J. Daafouz and D. Nešić, Stabilization of nonlinear systems using event-triggered output feedback laws, *soumis à IEEE Transactions on Automatic Control*.

Articles de conférence

- M. Abdelrahim, R. Postoyan and J. Daafouz, Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics, *In Proceedings of the 9th IFAC*

Symposium on Nonlinear Control Systems, Invited Paper, Toulouse, France, pp. 347-352, 2013.

- M. Abdelrahim, R. Postoyan, J. Daafouz and D. Nešić, Stabilization of nonlinear systems using event-triggered output feedback laws, *In Proceedings of the 21th International Symposium on Mathematics Theory of Networks and Systems, Groningen, Pays-Bas*, pp. 274-281, 2014.

- M. Abdelrahim, R. Postoyan, J. Daafouz and D. Nešić, Co-design of output feedback laws and event-triggering conditions for linear systems, *In Proceedings of the 53rd IEEE Conference on Decision and Control, Los Angeles, Etats-Unis*, 2014.

Presentation

Networked control systems (NCS) are systems in which the plant and the controller communicate with each other via a shared digital communication channel. Traditionally, the control laws are implemented on dedicated platforms and the communication with the plant is performed through point-to-point rigid connections which leads to complex wiring/diagnostic and high cost of maintenance. The utilization of NCS offers attractive benefits in terms of flexibility, reduced complexity in wiring connections, lower cost and ease of maintenance. Due to these advantages, the incorporation of a network in the feedback loop is becoming more and more popular in a wide range of applications. Examples include chemical processes, refineries, power plants, airplanes, water transportation networks, industrial factories, energy collection networks (such as wind farms), environmental monitoring, battlefield and supervision. However, the insertion of a network in the feedback loop induces communication constraints (variable transmission intervals, quantization errors, delays, packet dropouts, and scheduling) which may seriously affect the control objectives. A significant challenge in NCS is therefore to achieve the control objectives (in terms of stability and performance) despite these limitations. In conventional setups, data transmissions are time-driven and two successive transmission instants are constrained to be less than a fixed constant, called the *maximum allowable transmission interval* (MATI). Although time-triggering is appealing from the analysis and implementation point of view, it is not clear that this paradigm is always suitable in the context of NCS. Indeed, the same amount of transmissions per unit of time is generated even when transmissions are not necessary, in view of the control objectives, which may lead to an inefficient and excessive usage of the network. To overcome this shortcoming, event-triggered control has been proposed as an alternative.

Event-triggered control is an implementation technique in which the transmission instants are defined based on a state dependent criterion. The idea is to continuously measure the plant state and to close the feedback loop only when it is needed in view of the stability and/or performance requirements. This may significantly reduce the amount of transmissions compared to the time-triggered paradigm, which is essential for NCS. We note that ETC is also attractive in the context of embedded control systems to reduce the amount of control updates, and therefore computation, and to save the energy of battery powered sensors by reducing the number of transmissions. This paradigm is receiving a considerable interest in the control community nowadays. Most of the existing results assume that the full state measurement is available and can be used for feedback. This is not realistic in many applications. It appears that the design of output feedback event-triggered controllers is much more challenging, in particular because it is more difficult to ensure the existence of a minimum amount of time between two control input updates which is crucial for the controller to be implementable. Few results in the literature have addressed this problem and mostly for linear systems. The purpose of this thesis is to provide methodological tools for the design of output feedback event-triggered controllers for different classes of systems.

Objectives and contributions

The results of this thesis are threefold:

- We develop an emulation-based design for output feedback event-triggered controllers to stabilize a class of nonlinear systems.
- We present a co-design procedure to simultaneously design the output feedback law and the event-triggering condition for linear systems in order to further reduce the amount of transmissions.
- We propose stabilizing event-triggered controllers for nonlinear systems whose dynamics have two-time scales. In particular, we only rely on the knowledge of an approximate model of the slow dynamics.

Outline of the Thesis

The remainder of this thesis is organized as follows.

Chapter 1: Introduction

We provide a brief overview on NCS and ETC. We first start by defining NCS and we then present the main control design approaches. Next, we introduce the basic concept of ETC and we explain its advantages compared to time-triggered implementation. Afterwards, we review the existing results approaches on ETC and we demonstrate the technical difficulties when the full state measurements cannot accessed by the controller. Finally, we present our objectives and contributions.

Chapter 2: Emulation design for nonlinear systems

We study the emulation design of output feedback event-triggered controllers to stabilize a class of nonlinear systems. First, we derive the hybrid model of the NCS and we formally state the problem. Next, we propose sufficient conditions to guarantee an asymptotic stability property for the closed-loop system. We apply the technique on two physical nonlinear examples. We then show how the obtained results can be applied to the particular cases of LTI systems and to state feedback controllers.

Chapter 3: Co-design for LTI systems

This chapter extends the results of Chapter 2 to the joint design of the output feedback law and the event-triggering condition for LTI systems. The required conditions are formulated as LMI conditions which are computationally tractable. Then, we show how the LMI can be used to enlarge the guaranteed minimum inter-transmission time and to heuristically reduce the amount of transmissions. The results are illustrated on a numerical example.

Chapter 4: Singularly perturbed systems

We investigate the stabilization of two-time scales systems by event-triggered controllers based only on an approximate model describing the slow dynamics, by following the emulation approach. We show that a triggering law which guarantees the stability and the existence of a uniform minimum amount of time between two transmissions for the slow model may not ensure the existence of such a time for the overall system. Then, we propose sufficient conditions on the hybrid singularly perturbed system to guarantee the closed-loop stability of the original system as well as the existence of a dwell-time for the inter-transmission times. Next, we show that the results are applicable to a class of globally Lipschitz systems which encompasses LTI systems as a particular case. Finally, we apply the results to the autopilot control of an F-8 aircraft model.

Chapter 5: Conclusions

We give general conclusions, we highlight the main contributions, and we provide directions for future research.

Appendix A: Proofs of Chapter 4

We provide the proofs of some of the results of Chapter 4.

Appendix B: Mathematical review

We recall some mathematical preliminaries and fundamental tools that support the presentation of the technical results.

Publications

The following publications have been accepted or are under review, based on the presented material in this thesis.

Journal papers

- M. Abdelrahim, R. Postoyan and J. Daafouz, Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics, *Automatica, accepted.*
- M. Abdelrahim, R. Postoyan, J. Daafouz and D. Nešić, Stabilization of nonlinear systems using event-triggered output feedback laws, *submitted for publication to IEEE Transactions on Automatic Control.*

Conference articles

- M. Abdelrahim, R. Postoyan and J. Daafouz, Event-triggered control of nonlinear singularly perturbed systems based only on the slow dynamics, *In Proceedings of the 9th IFAC Symposium on Nonlinear Control Systems, Invited Paper, Toulouse, France*, pp. 347-352, 2013.

- M. Abdelrahim, R. Postoyan, J. Daafouz and D. Nešić, Stabilization of nonlinear systems using event-triggered output feedback laws, *In Proceedings of the 21th International Symposium on Mathematics Theory of Networks and Systems, Groningen, The Netherlands*, pp. 274-281, 2014.

- M. Abdelrahim, R. Postoyan, J. Daafouz and D. Nešić, Co-design of output feedback laws and event-triggering conditions for linear systems, *In Proceedings of the 53rd IEEE Conference on Decision and Control, Los Angeles, U.S.A.*, 2014.

Chapter 1

Introduction

1.1 Networked control systems

NCS are feedback systems in which the control loop is closed over a (shared) network. NCS essentially consist of four components, see Figure 1.1:

- sensors to collect plant measurements;
- controller;
- actuators to execute the control inputs;
- communication network to transfer information from the sensors to the controller and/or from the controller to the actuators.

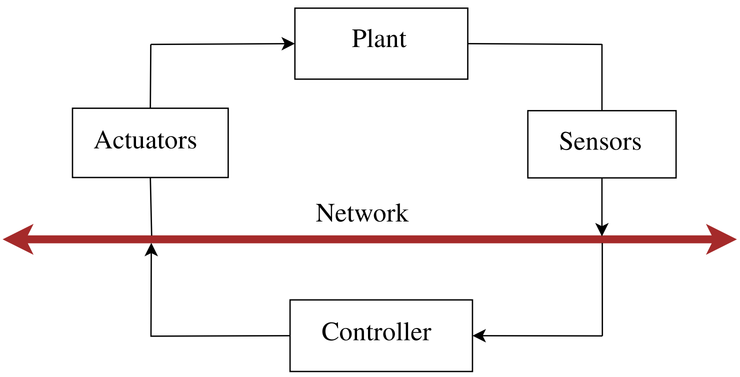

FIGURE 1.1: Block diagram of NCS.

NCS offer great benefits compared to the conventional point-to-point connection in terms of lighter wiring, lower installation costs, greater abilities for diagnosis, flexible reconfiguration and ease of maintenance. However, the insertion of a shared network in the feedback loop induces the following constraints and phenomena on data transmissions ([13], [125]):

- **Variable inter-transmission times.** In NCS, the next transmission instant for each node[1] is usually determined by local control units. These local controllers may have limited processing power and clocks with low accuracy. Hence, the resulted inter-transmission times may become uncertain and time varying.
- **Scheduling.** A rule called protocol is typically used to orchestrate transmissions of the different nodes over the channel. When the plant sensors are distributed for instance, and therefore not assigned to the same node, this implies that the controller only receives a partial knowledge of the plant output at each transmission instant.
- **Quantization errors.** This phenomenon occurs due to the A/D conversion of the plant analog signal since, at each transmission instant, the plant measurement has to be represented by a finite number of bits. The deviation between the analog value and its corresponding binary conversion is known as the quantization error.
- **Packet dropouts.** The sharing of a limited communication bandwidth by many control loops and applications may not allow the sensors to transmit data immediately when it is needed since the network may be busy by other tasks. As a result, some packets may be lost or arrive out of date.
- **Time delays.** Each node may have to wait a certain amount of time before sending its packet. Furthermore, the transmission time over the network may not negligible. These phenomena lead to time delays.

The presence of one or more of these network-induced constraints can damage the closed-loop performance or even lead to instability. Therefore, it is strongly required to develop well-suited control techniques for NCS to handle these issues. In this thesis, we focus on the first communication constraint, *i.e.* variable inter-transmission times.

[1] A node is a component connected to the network.

1.1.1 Control design approaches

Inserting a shared network in the feedback loop increases the design complexity of control systems and requires to develop methodologies specifically adapted to NCS. In fact, the design of NCS combines the domains of control systems, communication networks (and real-time computing). The main control design techniques in the literature are: emulation, co-design, and direct discrete-time. We briefly present these techniques in the following and we refer the reader to [46], [12] for more detailed explanation and literature review.

1.1.1.1 Emulation

The emulation approach consists in first synthesizing the controller in continuous-time, while ignoring the network effects. Then the network is taken into account and sufficient conditions on the latter are derived to maintain the properties ensured by the controller, see *e.g.* [65], [45]. This approach is appealing as it allows to use available tools in continuous-time to design the controller. On the other hand, the conditions imposed on the network are constrained by the initial choice of the controller, and these may be too conservative. In this context, it has been shown in the literature that for digital controllers designed by emulation, the inter-transmission times have to be less than the MATI to maintain stability of NCS, see *e.g.* [113], [112], [78].

1.1.1.2 Co-design

In the emulation approach, the feedback law and the network are separately designed in a sequential manner which may lead to restrictive conditions on the network. For instance, we may need to work with a very small MATI to preserve stability. One solution to avoid this issue is to simultaneously design the feedback law and the network. Note that this co-design strategy is more challenging since it is required to simultaneously take care of the controller design and its implementation which may result in conflicting constraints. For more explanation and literature review on this approach, we refer the reader to [91], [127], [21], [12].

1.1.1.3 Direct discrete-time

An alternative approach is to synthesize the controller directly based on the discrete time model of the plant. The design procedure consists in three steps, see *e.g.* [19], [77], [31]:

1. Obtain a discrete-time model of the plant.

2. Design the controller to stabilize the closed-loop system.

3. Verify the closed-loop stability of the true model, *i.e.* the continuous-time plant with the sampled input.

The main drawback of this strategy is that it is very difficult to obtain the exact discrete-time model of the plant, in particular for nonlinear systems, since we need to know an explicit analytic solution of the differential equation that describes the continuous-time dynamics. On the other hand, if an approximate discrete-time model is used, it is not obvious that the designed controller for the approximate model will stabilize the true model of the NCS, see *e.g.* [80]. Therefore, most existing results on this design strategy are dedicated to linear systems [46].

1.2 Event-triggered control

In convention setups, the feedback laws are implemented in a time-triggered fashion such that two transmission instants are separated by (at most) the MATI. Although this strategy is appealing from the implementation point of view, it is not obvious that time-triggering is appropriate for NCS. First, the transmission interval is usually designed such that the closed-loop stability is guaranteed in all possible situations. To do so, the design has to be carried out based on the worst case scenario which may result in a small MATI bound. The second reason is that the time-triggered approach has a blind nature since the transmission instants are generated regardless the system state. This may lead to an inefficient usage of the computation and the communication resources. Intuitively, if the system has reached a desired equilibrium point and no disturbance is acting on the plant, there is no need to close the loop and to calculate a new control input, but the time-triggered paradigm keeps doing so. In the last decades, many researchers suggested to develop alternative implementation policies such that the amount of transmissions is adapted to the current plant state. This may allow to significantly reduce the usage of the communication and computation resources. ETC has been proposed in this context.

1.2.1 The idea

ETC is a control strategy in which the loop is closed only when a designed state-dependent criterion is violated, see *e.g.* [10], [7], [103], [44], [8], [1], [43] and the references therein.

The event-triggered controller consists of two parts: a triggering-condition which decides the next transmission instant and a feedback law which generates the control input based on the last transmitted value of the plant measurements, see Figure 1.2.

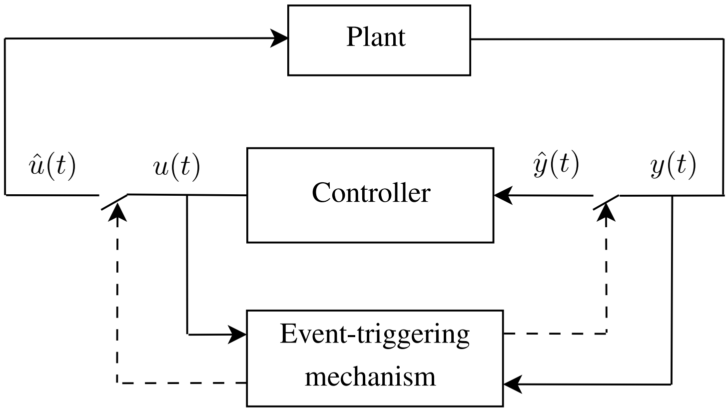

FIGURE 1.2: Event-triggered control schematic [22].

ETC was originally motivated by the following considerations (see [7], [10]):

- **Close to the human behaviour.** Event-triggered implementation is close in nature to the way a human behaves as a controller which only samples and takes a control decision when some events occur, like in the car driving.
- **Natural approach for many applications.** ETC is natural in many contexts, examples include speed control of internal combustion engines, the control of production rate for manufacturing systems, motion control where an angle or a position are measured by encoders, systems with relay feedback and many other examples.
- **Time-triggered implementation may be inefficient or difficult to implement.** In modern distributed control systems, it becomes difficult or inefficient to stick to the time-triggered paradigm, in particular for NCS since the channel may be shared by many processes.
- **Event-triggered control may reduce transmissions.** Since the transmission instants are adapted to the current system state, this may lead to a significant reduction in the amount of communication via the network as shown in *e.g.* [11], [8], [90], [25], [56], [88].

A fundamental issue in ETC is to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. This requirement is essential to prevent the occurrence of Zeno phenomenon, *i.e.* to avoid the generation of an infinite number of transmissions in a finite time. This task induces non trivial difficulties in the stability analysis. Note that the existence of such a lower bound on the inter-transmission times is not only useful to prove stability but also because two triggering instants cannot occur arbitrarily close in time in practice due to the hardware limitations.

Several terminologies in literature are used to refer to the ETC, related to the context where it has been applied. The term *send-on-delta* is used in the context of sensor networks, *e.g.* [73], [102], [86], while the term *level-crossing sampling* has been utilized in context of signal conversion and processing, see *e.g.* [2], [66], [37]. In the control systems community, the terminologies *deadband sampling* [81], [109], *Lebesgue sampling* [11], [9] are also equivalently used to refer to ETC.

1.2.2 Hybrid model

Consider the case where the controller communicates with the plant via a digital channel. In this section, we derive a model of event-triggered control systems in the case where the feedback law only has access to an output of the plant for the sake of generality (like in [22], [30], [87]).

Consider the nonlinear plant model

$$\dot{x}_p = f_p(x_p, u), \qquad y = g_p(x_p), \tag{1.1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output of the plant. Assume that the plant is stabilized by general dynamic controller of the form

$$\dot{x}_c = f_c(x_c, y), \qquad u = g_c(x_c, y), \tag{1.2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state. Note that, by setting $u = g_c(y)$, we obtain a static controller. Since the feedback loop is closed via a digital channel, the plant output and the control input are sent only at some transmission instants $t_i, i \in \mathbb{Z}_{\geq 0}$, see Figure 1.2. At each transmission instant, the plant output is sent to the controller which computes a new control input that is instantaneously transmitted to the plant. We assume that this process is performed in a synchronous manner and we ignore the computation times and the possible transmission delays. In that way, we obtain

$$\left.\begin{array}{rcll} \dot{x}_p & = & f_p(x_p, \hat{u}) & t \in [t_i, t_{i+1}] \\ \dot{x}_c & = & f_c(x_c, \hat{y}) & t \in [t_i, t_{i+1}] \\ u & = & g_c(x_c, \hat{y}) & \\ \dot{\hat{y}} & = & 0 & t \in [t_i, t_{i+1}] \\ \dot{\hat{u}} & = & 0 & t \in [t_i, t_{i+1}] \\ \hat{y}(t_i^+) & = & y(t_i) & \\ \hat{u}(t_i^+) & = & u(t_i), & \end{array}\right\} \tag{1.3}$$

where $\hat{y}$ and $\hat{u}$ respectively denote the last transmitted values of the plant output and of the control input. We assume that zero-order-hold devices are used to generate the sampled values $\hat{y}$ and $\hat{u}$ between two successive transmission instants which leads to $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$ for almost all $t \in [t_i, t_{i+1}], i \in \mathbb{Z}_{\geq 0}$. Other types of holding function can be considered ([64]) but we do not investigate those in this thesis. After each transmission instant, $\hat{y}$ and $\hat{u}$ are reset to the actual values of y and u, respectively. We introduce the network-induced error $e := (e_y, e_u) \in \mathbb{R}^{n_e}$, where

$$\begin{array}{rcl} e_y & := & \hat{y} - y \\ e_u & := & \hat{u} - u, \end{array} \tag{1.4}$$

which are reset to 0 at each transmission instant. Note that when static output feedback controller are considered, we have that $e_u = 0$ and we only consider e_y.

We observe that the closed-loop system is a hybrid dynamical model since it combines continuous-time evolutions, to model the plant and the controller dynamics, and discrete phenomena which model transmissions. Many modeling frameworks have been developed in the literature to capture the hybrid nature of dynamical models. Examples include hybrid dynamical systems [33], [34], mixed logical dynamical (MLD) models [14], complementarity systems [108], [39], hybrid automata [47], hybrid inclusions or impulsive systems [35], [38] and switching systems [59]. Among these frameworks, we choose to model NCS using the hybrid formalism of [34], as in [22], [30], [1]. This choice is justified by the fact that this approach provides an efficient and compact method to describe general hybrid systems. In addition, this formalism allows us to use the elegant concepts of solutions and stability developed in [34]. In this way, the system is modeled as follows

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} f(x,e) \\ g(x,e) \end{pmatrix} \quad (x,e) \in C \qquad \begin{pmatrix} x^+ \\ e^+ \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (x,e) \in D, \tag{1.5}$$

where $x := (x_p, x_c) \in \mathbb{R}^{n_x}$. The functions f, g in (1.5) are given by

$$\begin{aligned} f(x,e) &:= \begin{pmatrix} f_p\Big(x_p, g_c(x_c, y+e_y) + e_u\Big) \\ f_c(x_c, y+e_y) \end{pmatrix} \\ g(x,e) &:= \begin{pmatrix} -\frac{\partial}{\partial x_p} g_p(x_p) f_p\Big(x_p, g_c(x_c, y+e_y) + e_u\Big) \\ -\frac{\partial}{\partial x_c} g_c(x_c, y+e_y) f_c(x_c, y+e_y) \end{pmatrix}. \end{aligned} \tag{1.6}$$

The flow and jump sets, respectively denoted C and D, are defined according to the triggering condition that we will design later. As long as the triggering condition is not violated, the system flows on C where no transmission occurs. Jumps occur only if the triggering condition is verified, *i.e.* $(x,e) \in D$. When $(x,e) \in C \cap D$, the system flows only if flowing keeps (x,e) in C, otherwise the system experiences a jump. The functions f, g in (1.5) are assumed to be continuous and the sets C and D are closed. This will be the case in this thesis to ensure that system (1.5) is well-posed, see Chapter 6 in [34].

1.2.3 Event-triggering mechanisms

The main objective of the ETC problem is to design the flow and the jump sets of system (1.5), *i.e.* the triggering condition, to guarantee the closed-loop stability, to reduce the number of transmissions, and to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. In what follows, we present some common techniques in the literature to design the flow and the jump sets.

1.2.3.1 Static threshold

The evolution of the norm of the network-induced error is restricted to be upper bounded by a positive constant threshold. In this case, the flow and the jump sets for system (1.5) are

$$\begin{aligned} C &= \{(x, e) : |e| \leq \Delta\} \\ D &= \{(x, e) : |e| \geq \Delta\}, \end{aligned} \tag{1.7}$$

where $\Delta > 0$ is a designed constant. In that way, the minimum inter-transmission time is strictly positive and corresponds to the minimum time it takes for $|e|$ to evolve from zero to Δ. However, the achieved stability property with this triggering mechanism is generally not asymptotic, *i.e.* the system state typically converges to some neighbourhood of the origin. This triggering technique is referred to as deadband control [81], send-on-delta [73], non-uniform mechanism [44].

1.2.3.2 State-dependent threshold

In many control system applications, asymptotic stability properties are required for the closed-loop system. To achieve this goal, the triggering condition threshold should be a function of the system state and not a fixed constant as in the previous technique. Most of the existing results on this triggering mechanism assume that the full state measurement can be accessed by the controller, *e.g.* [43, 44, 57, 61, 70, 89, 119] and the references therein. Consequently, the feedback law is designed based on the full state information and we have that, in view of (1.3), $\hat{y} = \hat{x}_p$, $u = g_c(\hat{x}_p)$ and the sampling induced error becomes

$$e_x = \hat{x}_p - x_p. \tag{1.8}$$

In this context, many strategies have been proposed in the literature to construct the flow and jump sets for the hybrid system (1.5). We present here the result in [103] which is one of the common techniques in the literature. Furthermore, we will start from this result to establish our triggering mechanism later. The idea in [103] is to first assume that the state feedback law $u = k(\hat{x}_p)$ renders the closed-loop system

$$\dot{x}_p = f_p(x_p, k(x_p + e_x)) \tag{1.9}$$

input-to-state stable (ISS) (see Definition B.2) with respect to the network-induced error e_x, see *e.g.* [103], [1]. This property is usually ensured by the existence of a smooth, positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^{n_x}$ and all $e \in \mathbb{R}^{n_e}$

$$\frac{\partial V}{\partial x} f(x, k(x+e)) \leq -\alpha(|x|) + \gamma(|e|), \tag{1.10}$$

where $\alpha, \gamma \in \mathcal{K}_\infty$. Hence, if the triggering condition is constructed as

$$\gamma(|e|) \leq \sigma\alpha(|x|) \tag{1.11}$$

for some $\sigma \in (0, 1)$, then (1.10) becomes

$$\frac{\partial V}{\partial x} f(x, k(x+e)) \leq -(1-\sigma)\alpha(|x|) \tag{1.12}$$

which ensures that V strictly decreases along the solution to system (1.9). Hence, the obtained stability property of system (1.9) in the absence of network is preserved. As a consequence, the flow and the jump sets in (1.5) are defined as follows

$$\begin{aligned} C &= \{(x, e) : \gamma(|e|) \leq \sigma\alpha(|x|)\} \\ D &= \{(x, e) : \gamma(|e|) \geq \sigma\alpha(|x|)\}. \end{aligned} \tag{1.13}$$

It is important to highlight here that the construction of (1.13) by itself does not a priori ensure that the minimum inter-transmission time is strictly positive to avoid the Zeno phenomenon. Additional conditions are required to generate this property. In [103] for instance, the functions $f_p, k, \alpha^{-1}, \gamma$ in (1.10) are required to be locally Lipschitz.

Remark 1.1. *Condition (1.11) is only a sufficient condition to guarantee that (1.12) holds. It is also possible to directly define the flow and the jump sets as*

$$\begin{aligned} C &= \{(x, e) : \tfrac{\partial V}{\partial x} f(x, k(x+e)) \leq -(1-\sigma)\alpha(|x|)\} \\ D &= \{(x, e) : \tfrac{\partial V}{\partial x} f(x, k(x+e)) \geq -(1-\sigma)\alpha(|x|)\}, \end{aligned} \tag{1.14}$$

without using (1.10), to potentially further reduce transmissions (see e.g. [96], [1]). □

1.2.3.3 Using additional variables

In some cases, additional variables $\eta \in \mathbb{R}^{n_\eta}$ can be introduced to design the triggering condition. It is shown in [1], [89], [32] that such variables can be used to further reduce transmissions. Moreover, [1], [89] also explain that the technique in [121] can be reinterpreted within the formalism of [34] by adding an appropriate variable η.

1.2.4 Other state-dependent sampling paradigms

Other state-dependent sampling implementations have also been proposed in the literature. Self-triggered control is an implementation approach in which the next transmission instant is determined by the controller itself based on the latest measurements of the state and knowledge on the plant dynamical model, see *e.g.* [110], [67], [89], [5], [117], [68], [3]. The potential advantage of this approach is that we do not need to continuously monitor the plant measurements as in ETC. However, a main challenge in self-triggered control is how to precisely estimate the next transmission instant. Periodic event-triggered control is another alternative in which the plant measurements are sampled periodically and, at each sampling instant, the event-triggering condition is evaluated to decide whether or not to transmit new measurements and control signals, see *e.g.* [7], [44], [27], [40], [42], [87]. The main benefit of this strategy is that the Zeno phenomenon is ensured to be avoided since the periodic sampling interval serves as a guaranteed lower bound on the inter-transmission times. On the other hand, a thorough analysis of this approach is not trivial to design an appropriate sampling period of the triggering mechanism such that the closed-loop stability is preserved and the performance is not degraded. An alternative state-dependent strategy has been proposed in [29] based on a mapping of the state space which is designed offline to reduce the amount of transmissions during the real-time control of the system. In this thesis, we focus on the event-triggering approach.

1.2.5 Output feedback control

The methods presented so far assume that the full state can be measured. In this case, both the feedback law and the event-triggering condition are functions of the full state vector. In practice, we often have access to an output of the plant and not to the full state. It has to be noted that the existing results on state feedback ETC cannot be directly extended to output feedback controllers since the existence of a strictly positive lower bound on the inter-transmission times is no longer guaranteed in this case which induces more technical difficulties in the stability analysis. We clarify this point by recalling Example 2 in [22].

1.2.5.1 Motivating example

Consider the LTI system

$$\begin{aligned} \dot{x}_p &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [-1 \;\; 4]\, x_p, \end{aligned} \tag{1.15}$$

where $x \in \mathbb{R}^2$ is the plant state, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the output of the plant. The system can be stabilized by the following dynamic output feedback controller

$$\begin{aligned} \dot{x}_c &= \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y \\ u &= [1 \;\; -4]\, x_c, \end{aligned} \tag{1.16}$$

where $x_c \in \mathbb{R}^2$ is the state of the dynamic controller. Let the network-induced error defined as, for $t \in [t_i, t_{i+1}]$

$$e_y(t) = y(t_i) - y(t). \tag{1.17}$$

The straightforward extension of the triggering condition (1.11) gives

$$|e_y| \leq \sigma |y| \tag{1.18}$$

for some sufficiently small $\sigma > 0$. Unfortunately, this triggering rule is not suitable since the existence of a uniform strictly positive lower bound is not ensured like with state feedback controllers and hence, Zeno phenomenon may occur. Indeed, when $y = 0$, an infinite number of jumps occurs for any value of x such that $g_p(x_p) = 0$. This situation is shown in Figure 1.3 where we note that the transmission instants accumulate at $t = 1.7674$ which reveals the occurrence of Zeno phenomenon[2].

[2] All the simulations in this thesis have been carried out by using HyEQ toolbox [92].

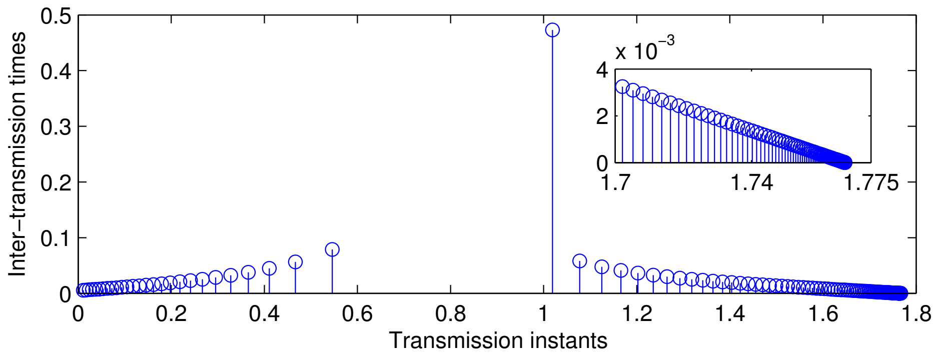

FIGURE 1.3: Inter-transmission times with a zoom-in of the last transmissions.

To overcome this issue, event-triggered controllers have to be developed in order to maintain the closed-loop stability while ensuring that the minimum inter-transmission time is strictly positive. In [22] for instance, this issue was overcome by adding a constant to the triggering condition which leads to

$$
\begin{aligned}
C &= \left\{(x,e) : \left(|e_y|^2 \leq \sigma_y|y|^2 + \varepsilon_y\right) \text{ and } \left(|e_u|^2 \leq \sigma_u|u|^2 + \varepsilon_u\right)\right\} \\
D &= \left\{(x,e) : \left(|e_y|^2 = \sigma_y|y|^2 + \varepsilon_y\right) \text{ or } \left(|e_u|^2 = \sigma_u|u|^2 + \varepsilon_u\right)\right\}
\end{aligned}
\tag{1.19}
$$

for $\sigma_y, \sigma_u, \varepsilon_y, \varepsilon_u \geq 0$, from which a practical stability property is derived, *i.e.* the state trajectory converges to a neighbourhood to the origin whose size depends on the parameters $\varepsilon_y, \varepsilon_u$.

1.2.5.2 Existing results

To the best of our knowledge, the problem of output feedback ETC has been first investigated in [53] and then in [8, 22, 30, 41, 55, 58, 72, 84, 100, 105, 126] for LTI systems and only in [123] for nonlinear systems. We have seen above that the event-triggered controller proposed for LTI systems in [22] guarantees a practical stability property. These controllers are such that the smaller the size of the neighbourhood to the origin, the shorter the guaranteed minimum inter-transmission time. The results in [126], [72] focus on PETC for linear systems. In [8], [100], [104],[58], [41], [30], event-triggered observer-based controllers have been developed. The triggering mechanisms in these architectures are assumed to have access to both the output measurement and the estimated state by the observer. For nonlinear systems, we are only aware of the result in [123] where passivity tools were used to derive triggering conditions to achieve an $\mathcal{L}_2$ stability property. To ensure the existence of a strictly positive lower bound on the inter-transmission times, the authors in [123] require that the output of the plant y belongs to a bounded sector of the state. This requirement seems to be conservative and we show that this condition dos not hold for all the illustrative examples that we have considered in the next two chapters.

1.3 Objectives and contributions

Motivated by the previous discussions, in this thesis, we address the following problems in the context of output feedback ETC.

Chapter 2: Emulation design for nonlinear systems

We design output feedback ETC for nonlinear systems by emulation, see Section 1.1.1.1. The design objectives are to guarantee a (global) asymptotic stability property and to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. The proposed strategy combines the event-triggering condition of [103] adapted to output measurements and the results on time-driven sampled-data systems in [79]. Indeed, the event-triggering condition is only (continuously) evaluated after T units of times have elapsed since the last transmission, where T corresponds to the MATI given by [79]. This two-step procedure is justified by the fact that the adaption of the event-triggering condition of [103] to output feedback on its own can lead to Zeno phenomenon as we have seen in Section 1.2.5.

Our results rely on similar assumptions as in [79] which allow us to derive both local and global results. Contrary to [30], the approach is applicable to nonlinear systems and the output feedback law is not necessarily based on an observer. Compared to [123], we rely on a different set of assumptions and we conclude a different stability property. In addition, we show that our results are applicable to any LTI systems that are stabilizable and detectable, which is a priori not the case of [123]. Furthermore, we apply our results to the controlled Lorenz model of fluid convection and to a single-link robot arm model, which are nonlinear and which do not satisfy the conditions of [123]. It has to be noted that the event-triggering mechanism that we propose is different from the periodic event-triggered control (PETC) paradigm, see *e.g.* [42], [87], where the triggering condition is verified only at some periodic sampling instants. In our case, the triggering mechanism is *continuously* evaluated, once T units of time have elapsed since the last transmission.

For LTI systems, the required conditions are reformulated as a linear matrix inequality (LMI). The triggering condition parameters are then obtained by solving this LMI condition which is shown to be always feasible for LTI systems that are stabilizable and detectable. We also compare the effectiveness of our proposed triggering mechanism with the existing results on a numerical example. Furthermore, we show how the proposed technique can be fruitfully employed in the context of state feedback control as a special case, to directly tune the lower bound on the inter-transmission times. Although such a lower bound is guaranteed in [103], the obtained value may be subject to some conservatism. More interestingly, the internal structure of our triggering mechanism ensures that the generated amount of transmissions are less than or, at least, equal to those produced by conventional periodic setups using [79].

Chapter 3: Co-design for linear systems

The vast majority of existing event-triggered controllers are designed by emulation, see [1, 43, 120] and the references therein. The potential disadvantage of this technique is that it is difficult to obtain an *optimal* design since we are restricted by the initial choice of the feedback law. To avoid the design constraints imposed by the emulation approach, three directions of research are proposed in the literature: co-design of feedback laws and event-triggering conditions, *e.g.* see [49, 83, 85, 98, 99, 106, 124], joint design of control inputs and self-triggering conditions, *e.g.* [4, 15, 23, 28, 36, 118], and optimal event-triggered control, *e.g.* [6, 74, 75, 90, 97].

We are interested in the first direction where start form the emulation analysis for linear systems in the previous part to develop a co-design procedure, see Section 1.1.1.2, of the output feedback law and the event-triggering condition. To the best of our knowledge, this problem has been only addressed in [126], [72]. The proposed co-design methods in [126], [72] are concerned with periodic event-triggered controllers in which the output measurements are sampled periodically and then it is the task of the triggering condition to decide whether the control input needs to be updated. However, an open question regarding these techniques is how to calculate the appropriate sampling period of the triggering mechanism. This is a key aspect in the construction of PETC since the sampling of the triggering mechanism may deteriorate the closed-loop performance or may require a higher network bandwidth than the available one, see [87].

Unlike [126], [72], we provide a co-design algorithm where the triggering condition is continuously evaluated. The required conditions have been formulated in terms of LMIs and the event-triggered controller is then obtained by solving these LMIs. It is important to note that the results obtained by emulation approach do not allow for co-design because the resulted LMI condition is nonlinear in this case. Furthermore, the encountered nonlinearity cannot be directly handled by congruence transformations like in standard output feedback design problems, which induces non-trivial technical difficulties. We thus needed to introduce an additional LMI constraint to linearize the LMI condition of the emulation case using the tools of [94]. We then take advantage of the flexibility of co-design to enhance the efficiency of the event-triggered controllers in two senses. We first maximize the minimum inter-transmission time which is essential in practice. Indeed, while the existence of dwell-time is typically ensured in emulation results, its value may be very small and may thus violate the hardware constraints. It is therefore important to propose designs which are able to ensure larger minimum times between two transmissions. We then propose a heuristic to reduce the amount of transmissions, whose efficiency is confirmed by simulations.

Chapter 4: Singularly perturbed systems

Singularly perturbed systems are systems whose dynamics involve physical phenomena occurring in two-time scales. The dynamical model of such systems is generally given by, see [52, 54]

$$\begin{aligned} \dot{x} &= f(x, z, u) \\ \epsilon\dot{z} &= g(x, z, u) \\ u &= k(x, z), \end{aligned} \tag{1.20}$$

where $x \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_z}$ are the states, $u \in \mathbb{R}^{n_u}$ is the control input, and $\epsilon > 0$ is a small parameter which determines the degree of separation between the slow and fast modes of the system. Hence, the two-time scale feature comes from the fact that dynamics of z evolves faster than x when ϵ is small (since $\dot{z} = g(x, z, u)/\epsilon$).

The analysis of this class of systems requires careful handling of the two-time scale nature since this property may lead to ill conditioning controllers or/and instability of the closed-loop if ignored. Singular perturbation theory provides powerful tools to design and analyse these two-time scale control systems, see *e.g.* [54], [52]. The cornerstone result of the singular perturbation theory is that the original system (1.20) can be decomposed into two separate approximate slow and fast models of the form

$$\begin{aligned} \dot{x} &= f_s(x, u_s) \\ \epsilon\dot{y} &= g_f(y, u_f), \end{aligned} \tag{1.21}$$

where f_s, g_f are the approximate functions of f, g and y is obtained after changing the variables. Next, reduced order controllers u_s, u_f are designed independently to stabilize each subsystem. Then, under certain conditions, the composite control law $u = u_s + u_f$ guarantees the overall stability of the original system (1.20) in virtue of the singular perturbation theory. In this way, the control design problem is greatly simplified since we only need to stabilize each approximate model separately.

We are interested to design stabilizing event-triggered controllers based only on an approximate model of the slow dynamics. This problem is motivated by the fact that engineers often neglect the fast stable dynamics in practice and design the feedback law based only on the slow model. To the best of our knowledge, this is the first result in that direction. We highlight specific challenges which arise with the ETC of singularly perturbed systems:

- The state of the fast model experiences a jump at each transmission due to the change of variables that is introduced to separate the slow and the fast dynamics using singular perturbation theory. These jumps induce non-trivial difficulties in the stability analysis. That is a feature of the problem which is not present in available results on event-triggered control where only the sampling-induced error is reset to zero at each transmission, see *e.g.* [1, 22, 43, 103, 120];
- The existence of a strictly positive lower bound on the inter-transmission times is no longer ensured due to the fact that we neglect the fast dynamics.

The stability of this type of systems is analysed in [93], [115], [116]. In this chapter, we address a design problem as we construct the flow and jump sets (*i.e.* the triggering condition) and we propose different stability analyses under a different set of assumptions. We propose two classes of event-triggered controllers. The first policy relies on the event-triggering conditions [22, 69], see Section 1.2.5, but it requires to fully modify the stability analysis to handle the features of the problem due to the two-time scale nature of the system. We show that a semiglobal practical stability property holds where the adjustable parameter appears in the event-triggering condition. The second technique combines the event-triggered implementation of [103] with the time-triggered results in [79] like in Chapter 2. We show that a global asymptotic stability property is satisfied in this case, under an additional assumption. The results are shown to be applicable to a class of globally Lipschitz systems, which encompasses stabilizable LTI systems as a particular case. The approach is illustrated on the autopilot control of an F-8 aircraft model.

1.4 Conclusion

In this chapter, we have presented a brief overview on NCS and ETC and we have then explained the objectives and contributions of this thesis.

For NCS, we have started by introducing NCS and we have mentioned the benefits and the phenomena occurring due to the insertion of a shared communication channel in the feedback loop. Then, we have emphasized the main challenge that motivates most existing results on NCS. Afterwards, we have discussed the control design techniques for NCS.

In the second part of this chapter, the event-triggered control has been presented as a suitable implementation strategy of feedback laws for NCS, we refer the reader to [57] for more explanation on ETC. The approach has been first motivated by highlighting the drawbacks of the traditional time-triggered setups then, we have demonstrated the underlying idea of event-based triggering and its advantages compared to the periodic paradigm. The hybrid dynamical model of of the closed-loop system has been then derived, see [63], [34] for more details on hybrid dynamical systems. Next, we have provided some insights on the ETC by exploring the common techniques in the literature where we have drawn the attention to the Zeno phenomenon which has to be avoided in order to make the triggering mechanism implementable in practice. Then, it has been shown that the exclusion of Zeno behaviour becomes more challenging task when the full state measurement is not available.

Our objectives and contributions have been briefly explained in the last part of this chapter. We have first motivated ourselves by the lack of results on output feedback ETC in the literature to develop an appropriate event-triggered mechanism for nonlinear systems by emulation. Then, to allow more flexibility in the design of the event-triggered controller, we will propose a co-design procedure to simultaneously design the output feedback law and the flow and jump sets for LTI systems, interested readers on dynamic output feedback controllers and LMI controller synthesis are referred to Chapter 10 in [20] and Chapters 1,4 in [95]. Finally, event-triggered controllers will be developed to stabilize nonlinear singularly perturbed systems based only on the slow dynamics, see [54] and Chapter 11 in [52].

Chapter 2

Emulation design for nonlinear systems

This chapter addresses the synthesis of output feedback event-triggered controllers for nonlinear systems. We design the controller using the emulation approach (see Section 1.1.1.1). The proposed technique is illustrated on two physical nonlinear systems for which the required conditions are verified. We show that the proposed strategy can be applied to any detectable and stabilizable LTI system. We also explain the interest of the triggering condition in the context of state feedback control.

2.1 Hybrid model

As in Section 1.1.1.1, we first ignore the communication constraints and we consider the nonlinear plant model

$$\dot{x}_p = f_p(x_p, u), \qquad y = g_p(x_p), \tag{2.1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output of the plant. We focus on general dynamic controllers of the form

$$\dot{x}_c = f_c(x_c, y), \qquad u = g_c(x_c, y), \tag{2.2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state. We emphasize that the x_c-system is not necessarily an observer. We assume that the controller (2.2) has been designed to stabilize the closed-loop system (2.1)-(2.2). Next, we consider the case where the feedback law (2.2) is implemented over a network. We define the sampling induced error as in (1.4) and we introduce an additional clock variable $\tau \in \mathbb{R}_{\geq 0}$ to describe the time elapsed since the last transmission, with the dynamics

$$\dot{\tau} = 1 \quad \tau \in C, \qquad \tau^+ = 0 \quad \tau \in D. \tag{2.3}$$

Then, the hybrid model (1.5) is

$$\begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} f(x,e) \\ g(x,e) \\ 1 \end{pmatrix} \quad (x,e,\tau) \in C \qquad \begin{pmatrix} x^+ \\ e^+ \\ \tau^+ \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad (x,e,\tau) \in D, \tag{2.4}$$

where f, g are defined in (1.6).

Our objective is to design the flow and the jump sets of system (2.4) such that a (global) asymptotic stability property is guaranteed and the number of transmissions is reduced, while ensuring the existence of a strictly positive lower bound on the inter-transmission times.

2.2 Main results

We first present the conditions that we impose on system (2.4), then we present the triggering technique and we state the main stability result. Finally, we illustrate the technique on two physical nonlinear examples.

2.2.1 Assumptions

We make the following assumption on system (2.4), which is inspired by [79].

Assumption 2.1. *There exist $\Delta_x, \Delta_e > 0$, locally Lipschitz positive definite functions $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and $W : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$, a continuous function $H : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, real numbers $L \geq 0$, $\gamma > 0$, $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and continuous, positive definite functions $\delta : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$ and $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^{n_x}$*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|), \tag{2.5}$$

for all $|e| \leq \Delta_e$ *and almost all* $|x| \leq \Delta_x$

$$\langle \nabla V(x), f(x,e) \rangle \leq -\alpha(|x|) - H^2(x) - \delta(y) + \gamma^2 W^2(e) \tag{2.6}$$

and for all $|x| \leq \Delta_x$ *and almost all* $|e| \leq \Delta_e$

$$\langle \nabla W(e), g(x,e) \rangle \leq LW(e) + H(x). \tag{2.7}$$

We say that Assumption 2.1 holds globally if (2.6) and (2.7) hold for almost all $x \in \mathbb{R}^{n_x}$ *and* $e \in \mathbb{R}^{n_e}$. □

Conditions (2.5)-(2.6) imply that the system $\dot{x} = f(x,e)$ is $\mathcal{L}_2$-gain stable from W to $(H, \sqrt{\delta})$. This property can be analysed by investigating the robustness property of the closed-loop system (2.1)-(2.2) with respect to input and/or output measurement errors in the absence of sampling. Note that, since W is positive definite and continuous (since it is locally Lipschitz), there exists $\chi \in \mathcal{K}_\infty$ such that $W(e) \leq \chi(|e|)$ (according to Lemma 4.3 in [52]) and hence (2.5), (2.6) imply that the system $\dot{x} = f(x,e)$ is input-to-state stable (ISS). We also assume an exponential growth condition of the e-system on flows in (2.7) which is similarly used in [79].

2.2.2 Event-triggering condition

Under Assumption 2.1, the adaptation of the idea of [103] leads to a triggering condition of the form

$$\gamma^2 W^2(e) \leq \delta(y). \tag{2.8}$$

The problem is that Zeno phenomenon may occur with this type of triggering conditions as explained in Section 1.2.5. We propose instead to evaluate the event-triggering condition only after T units have elapsed since the last transmission, where T corresponds to the MATI given by [79]. In that way, we ensure the existence of a strictly positive lower bound on the inter-transmission times. Although the rationale is intuitive, the analysis is not trivial as we will show. Similar approaches have been followed in [30, 71, 122] to enforce a lower bound on the inter-transmission times in different contexts, mainly for linear systems. Note that the idea of enforcing a given time between two jumps is linked to time regularization techniques, see [50].

We thus redesign the triggering condition as follows

$$\gamma^2 W^2(e) \leq \delta(y) \text{ or } \tau \in [0, T], \tag{2.9}$$

where we recall that $\tau \in \mathbb{R}_{\geq 0}$ is the clock variable introduced in (2.4). Consequently, the flow and jump sets of system (2.4) are

$$\begin{aligned} C &= \Big\{(x, e, \tau) : \gamma^2 W^2(e) \leq \delta(y) \text{ or } \tau \in [0, T]\Big\} \\ D &= \Big\{(x, e, \tau) : \Big(\gamma^2 W^2(e) = \delta(y) \text{ and } \tau \geq T\Big) \text{ or } \Big(\gamma^2 W^2(e) \geq \delta(y) \text{ and } \tau = T\Big)\Big\}. \end{aligned} \tag{2.10}$$

Hence, the inter-jump times are uniformly lower bounded by T. This constant is selected such that $T < \mathcal{T}(\gamma, L)$, where

$$\mathcal{T}(\gamma, L) := \begin{cases} \frac{1}{Lr} \arctan(r) & \gamma > L \\ \frac{1}{L} & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh}(r) & \gamma < L \end{cases} \tag{2.11}$$

with $r := \sqrt{\left|(\frac{\gamma}{L})^2 - 1\right|}$ and L, γ come from Assumption 2.1 as in [79].

2.2.3 Stability results

We are ready to state the main result.

Theorem 2.1. *Suppose that Assumption 2.1 holds and consider system (2.4) with the flow and jump sets (2.10), where the constant T is such that $T \in (0, \mathcal{T}(\gamma, L))$. There exist $\Delta > 0$ and $\beta \in \mathcal{KL}$ such that any solution $\phi = (\phi_x, \phi_e, \phi_\tau)$ with $|(\phi_x(0,0), \phi_e(0,0))| \leq \Delta$ satisfies*

$$|\phi_x(t, j)| \leq \beta(|(\phi_x(0,0), \phi_e(0,0))|, t + j) \quad \forall (t, j) \in \operatorname{dom} \phi, \tag{2.12}$$

furthermore, if ϕ is maximal, then it is complete. If Assumption 2.1 holds globally, then (2.12) holds globally. □

Proof of Theorem 2.1. First, we prove the result when Assumption 2.1 holds globally. Let $\zeta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the solution to the following differential system, like in [17], [79]

$$\dot{\zeta} = -2L\zeta - \lambda(\zeta^2 + 1) \qquad \zeta(0) = \theta^{-1}, \tag{2.13}$$

where $\theta \in (0,1)$, $\lambda := \sqrt{\gamma^2+\eta}$ for some $\eta > 0$ and γ comes from Assumption 2.1. We denote $\widetilde{\mathcal{T}}(\theta,\eta,\gamma,L)$ the time it takes for ζ to decrease from θ^{-1} to θ. By following the same lines as in the proof of Claim 1 in [17], the time $\widetilde{\mathcal{T}}(\theta,\eta,\gamma,L)$ is given by

$$\widetilde{\mathcal{T}}(\theta,\eta,\gamma,L) := \begin{cases} \frac{1}{Lr}\arctan \frac{r(1-\theta)}{2\frac{\theta}{1+\theta}(\frac{\sqrt{\gamma^2+\eta}}{L}-1)+1+\theta} & \sqrt{\gamma^2+\eta} > L \\ \frac{1}{L}\frac{1-\theta}{1+\theta} & \sqrt{\gamma^2+\eta} = L \\ \frac{1}{Lr}\operatorname{arctanh} \frac{r(1-\theta)}{2\frac{\theta}{1+\theta}(\frac{\sqrt{\gamma^2+\eta}}{L}-1)+1+\theta} & \sqrt{\gamma^2+\eta} < L, \end{cases} \tag{2.14}$$

where $r = \sqrt{\left|\left(\frac{\sqrt{\gamma^2+\eta}}{L}\right)^2 - 1\right|}$. We note that the time $\widetilde{\mathcal{T}}(\theta,\eta,\gamma,L)$ is a continuous function of (θ,η) which is decreasing in θ and η (by invoking the comparison principle). On the other hand, we note that $\widetilde{\mathcal{T}}(\theta,\eta,\gamma,L) \to \mathcal{T}(\gamma,L)$ as (θ,η) tends to $(0,0)$ (where $\mathcal{T}(\gamma,L)$ is defined in (2.11)). As a consequence, since $T < \mathcal{T}$, there exists (θ,η) such that $T < \widetilde{\mathcal{T}}(\theta,\eta,\gamma,L)$. We fix the couple (θ,η).

Let $q := (x,e,\tau)$. We define for all $q \in C \cup D$

$$R(q) := V(x) + \max\{0, \lambda\zeta(\tau)W^2(e)\}. \tag{2.15}$$

Let $q \in D$, we obtain, in view of (2.4) and the fact that W is positive definite,

$$\begin{aligned} R(G(q)) &= V(x) + \max\{0, \lambda\zeta(0)W^2(0)\} \\ &= V(x) \leq R(q), \end{aligned} \tag{2.16}$$

where $G(q) := (x,0,0)$.

Let $q \in C$ and suppose that $\zeta(\tau) < 0$. As a consequence,

$$R(q) = V(x) \tag{2.17}$$

and it holds that $\tau > T$. Indeed, $\zeta(\tau)$ is strictly decreasing in τ, in view of (2.13), and $\zeta(T) > \zeta(\widetilde{\mathcal{T}}) = \theta > 0$ as $T < \widetilde{\mathcal{T}}$. Then $\zeta(\tau) < 0$ implies that $\tau > T$. Hence, $\gamma^2 W^2(e) \leq \delta(y)$ in view of (2.10) since $q \in C$. Consequently, in view of page 100 in [107], Lemma 1, Assumption 2.1 and (2.15)

$$\begin{aligned} R^\circ(q;F(q)) &= \langle \nabla V(x), f(x,e)\rangle \\ &\leq -\alpha(|x|), \end{aligned} \tag{2.18}$$

where $F(q) := (f(x,e), g(x,e), 1)$. Hence, by following similar arguments as in the proof of Theorem 1 in [79] since α is continuous and positive definite and V is positive definite and radially unbounded, there exists a continuous positive definite function ρ_1 such that

$$\begin{aligned} R^{\circ}(q; F(q)) &\leq -\rho_1(V(x)) \\ &=: -\rho_1(R(q)). \end{aligned} \tag{2.19}$$

When $q \in C$ and $\zeta(\tau) > 0$, we have

$$R(q) = V(x) + \lambda\zeta(\tau)W^2(e). \tag{2.20}$$

As above, in view of Lemma 1, Assumption 2.1 and (2.13), we obtain

$$\begin{aligned} R^{\circ}(q; F(q)) &= \langle \nabla V(x), f(x,e)\rangle + \lambda\dot{\zeta}(\tau)W^2(e) + 2\lambda\zeta(\tau)W(e)\langle \nabla W(e), g(x,e)\rangle \\ &\leq -\alpha(|x|) - H^2(x) - \delta(y) + \gamma^2 W^2(e) + 2\lambda\zeta(\tau)W(e)(LW(e) + H(x)) \\ &\quad +\lambda W^2(e)\Big(-2L\zeta - \lambda(\zeta^2+1)\Big) \\ &\leq -\alpha(|x|) - H^2(x) - \delta(y) + \gamma^2 W^2(e) + 2\lambda\zeta(\tau)W(e)H(x) \\ &\quad -\lambda^2\zeta^2(\tau)W^2(e) - \lambda^2 W^2(e). \end{aligned} \tag{2.21}$$

Using the fact that $2\lambda\zeta(\tau)W(e)H(x) \leq \lambda^2\zeta^2(\tau)W^2(e) + H^2(x)$, we have that

$$\begin{aligned} R^{\circ}(q; F(q)) &\leq -\alpha(|x|) - \delta(y) + \gamma^2 W^2(e) - \lambda^2 W^2(e) \\ &\leq -\alpha(|x|) + \gamma^2 W^2(e) - \lambda^2 W^2(e). \end{aligned} \tag{2.22}$$

Recall that $\lambda^2 = \gamma^2 + \eta$, it holds that

$$R^{\circ}(q; F(q)) \leq -\alpha(|x|) - \eta W^2(e). \tag{2.23}$$

By using the same argument as in (2.19), we derive that

$$\begin{aligned} R^{\circ}(q; F(q)) &\leq -\rho_1(V(x)) - \eta W^2(e) \\ &= -\rho_1(V(x)) - \frac{\eta\theta}{\lambda}\lambda\theta^{-1}W^2(e) \\ &= -\rho_1(V(x)) - \rho_2(\lambda\theta^{-1}W^2(e)), \end{aligned} \tag{2.24}$$

where $\rho_2 : s \mapsto \frac{\eta\theta}{\lambda} s \in \mathcal{K}_\infty$. Since $\zeta(\tau) \leq \theta^{-1}$ for all $\tau \geq 0$ in view of (2.13), it holds that

$$R^\circ(q; F(q)) \leq -\rho_1(V(x)) - \rho_2(\lambda\zeta(\tau)W^2(e)). \tag{2.25}$$

We deduce that there exists a continuous positive definite function ρ_3 such that

$$\begin{aligned} R^\circ(q; F(q)) &\leq -\rho_3(V(x) + \lambda\zeta(\tau)W^2(e)) \\ &=: -\rho_3(R(q)). \end{aligned} \tag{2.26}$$

When $\zeta(\tau) = 0$, we obtain, in view of (2.19), (2.26) and Lemma 1

$$R^\circ(q; F(q)) \leq \max\{-\rho_1(R(q)), -\rho_3(R(q))\}. \tag{2.27}$$

Consequently, it holds that, for all $q \in C$

$$R^\circ(q; F(q)) \leq -\rho(R(q)), \tag{2.28}$$

where $\rho := \min\{\rho_1, \rho_3\}$ is continuous and positive definite. Let ϕ be a solution to (2.4), (2.10). In view of (2.28) and by definition of the Clarke's derivative (see for instance page 99 in [107]), it holds that, for all j and for almost all $t \in I^j$ (where $I^j = \{t : (t, j) \in \operatorname{dom}\phi\}$)

$$\dot{R}(\phi(t, j)) \leq R^\circ(\phi(t, j); F(\phi(t, j))) \leq -\rho(R(\phi(t, j))). \tag{2.29}$$

Thus, in view of (2.16), (2.29) and since inter-jump times are lower bounded by T in view of (2.10), we conclude that, by following the same lines as in the end of the proof of Theorem 1 in [79], there exists $\bar{\beta} \in \mathcal{KL}$ such that for any solution ϕ to (2.4), (2.10) and any $(t, j) \in \operatorname{dom}\phi$,

$$R(\phi(t, j)) \leq \bar{\beta}(R(\phi(0, 0)), 0.5t + 0.5Tj). \tag{2.30}$$

In view of Assumption 2.1 and since W is continuous (since it is locally Lipschitz) and positive definite, there exists $\overline{\alpha}_W \in \mathcal{K}_\infty$ such that $W(e) \leq \overline{\alpha}_W(|e|)$ for all $e \in \mathbb{R}^{n_e}$ according to Lemma 4.3 in [52]. As a result, in view of Assumption 2.1, (2.13) and (2.15), it holds that, for all

$q \in C \cup D$,

$$\begin{aligned} V(x) \leq R(q) \leq V(x) + \frac{\lambda}{\theta} W^2(e) \\ \underline{\alpha}(|x|) \leq R(q) \leq \overline{\alpha}(|x|) + \frac{\lambda}{\theta} \overline{\alpha}_W(|e|) \\ \underline{\alpha}(|x|) \leq R(q) \leq \overline{\alpha}_R(|(x,e)|), \end{aligned} \tag{2.31}$$

where $\overline{\alpha}_R : s \mapsto \overline{\alpha}(s) + \frac{\lambda}{\theta}\overline{\alpha}_W(s) \in \mathcal{K}_\infty$. Hence, in view of (2.30) and (2.31), we deduce that for any solution ϕ to (2.4), (2.10) and for all $(t,j) \in \operatorname{dom}\phi$

$$\underline{\alpha}(|\phi_x(t,j)|) \leq R(\phi(t,j)) \leq \bar{\beta}\Big(\overline{\alpha}_R(|(\phi_x(0,0),\phi_e(0,0))|), 0.5t + 0.5Tj\Big). \tag{2.32}$$

Consequently,

$$\begin{aligned} |\phi_x(t,j)| &\leq \underline{\alpha}^{-1}\Big(\bar{\beta}\Big(\overline{\alpha}_R(|(\phi_x(0,0),\phi_e(0,0))|), 0.5t + 0.5Tj\Big)\Big) \\ &= \beta(|(\phi_x(0,0),\phi_e(0,0))|, t+j), \end{aligned} \tag{2.33}$$

where $\beta : (s_1, s_2) \mapsto \underline{\alpha}^{-1}(\bar{\beta}(\overline{\alpha}_R(s_1), s_2)) \in \mathcal{KL}$. Thus, (2.12) holds.

We now investigate the completeness of the maximal solutions to system (2.4), (2.10). Let ϕ be a maximal solution to (2.4), (2.10). We first show that ϕ is nontrivial, *i.e.* its domain contains at least two points (see Definition 2.5 in [34]). According to Proposition 6.10 in [34], it suffices for that purpose to prove that $\{F(q)\} \cap T_C(q) \neq \emptyset$ for any $q := (x,e,\tau) \in C \backslash D$, where $T_C(q)$ is the tangent cone to C at q. Let $q \in C\backslash D$. If q is in the interior of C, $T_C(q) = \mathbb{R}^{n_x+n_e+1}$ and the required condition holds. If q is not in the interior of C, necessarily $\tau = 0$ as $q \in C\backslash D$, in this case $T_C(q) = \mathbb{R}^{n_x+n_e} \times \mathbb{R}_{\geq 0}$ and we see that $F(q) \in T_C(q)$, in view of (2.4). Hence, ϕ is nontrivial according to Proposition 6.10 in [34]. In view of (2.4), (2.10) and (2.33), ϕ_x and ϕ_τ cannot explode in finite time. Recall that the network-induced error is $\phi_e = (\phi_{e_y}, \phi_{e_u})$ with $\phi_{e_y} = \phi_y(t_j,j) - \phi_y(t,j)$, $\phi_{e_u} = \phi_u(t_j,j) - \phi_u(t,j)$ for $j > 0$ and $(t,j) \in \operatorname{dom}\phi$ where we write $\operatorname{dom}\phi = \cup_{j\in\{0,\ldots,J\}}([t_j,t_{j+1}],j)$ with some abuse of notation. Hence, in view of (2.1), (2.2), (2.33) and since g_p, g_c are continuous, it holds that, for all $j > 0$ and $(t,j) \in \operatorname{dom}\phi$

$$\begin{aligned} |\phi_{e_y}(t,j)| &= |g_p(\phi_{x_p}(t_j,j)) - g_p(\phi_{x_p}(t,j))| \\ &\leq |g_p(\phi_{x_p}(t_j,j))| + |g_p(\phi_{x_p}(t,j))| \\ &\leq 2 \max_{|z|\leq\beta(|(\phi_x(0,0),\phi_e(0,0))|,0)} |g_p(z)|. \end{aligned} \tag{2.34}$$

Similarly, we obtain, for all $j > 0$ and $(t, j) \in \operatorname{dom} \phi$

$$\begin{aligned}
|\phi_{e_u}(t,j)| &\leq |g_c(\phi_{x_c}(t_j,j), \phi_y(t_j,j))| + |g_c(\phi_{x_c}(t,j), \phi_y(t_j,j))| \\
&= |g_c(\phi_{x_c}(t_j,j), g_p(\phi_{x_p}(t_j,j))| + |g_c(\phi_{x_c}(t,j), g_p(\phi_{x_p}(t_j,j))| \\
&\leq 2 \max_{\substack{|z_1| \leq \beta(|(\phi_x(0,0),\phi_e(0,0))|,0) \\ |z_2| \leq \max|g_p(z_1)|}} |g_c(z_1, z_2)|.
\end{aligned} \tag{2.35}$$

When $j = 0$, we have that $|\phi_{e_y}(t,0)| \leq |\phi_{e_y}(0,0)| + |g_p(\phi_{x_p}(0,0)) - g_p(\phi_{x_p}(t,0))|$ and $|\phi_{e_u}(t,0)| \leq |\phi_{e_u}(0,0)| + |g_c(\phi_{x_c}(0,0), \phi_y(0,0)) - g_c(\phi_{x_c}(t,0), \phi_y(0,0))|$ and we can derive similar bounds on the interval $[0, t_1]$. Thus, in view of (2.34) and (2.35) and since ϕ_e is reset to 0 at each jump, ϕ_e cannot blow up in finite time. As a consequence, ϕ cannot explode in finite time. Let $G(x, e, \tau) := (x, 0, 0)$ denotes the jump map in (2.10). The solutions to (2.4), (2.10) cannot leave the set $C \cup D$ after a jump since $G(D) \subset C$ in view of (2.4), (2.10). Thus, we conclude that maximal solutions to (2.4), (2.10) are complete according to Proposition 6.10 in [34]. Finally, we note that if Assumption 2.1 holds locally, then there exists $\Delta > 0$ such that (2.16) and (2.29) hold on the invariant set $|(x, e)| \leq \Delta$ and consequently (2.12) holds locally. □

Remark 2.1. *We can redesign the triggering condition in (2.10) if the event-triggering mechanism has the access to both the plant output y and the state of the dynamic controller x_c, as considered in [105] for instance, see Figure 2.1. Then, condition (2.6) is modified to be*

$$\langle \nabla V(x), f(x,e) \rangle \leq -\alpha(|x|) - H^2(x) - \delta(y) - \xi(|x_c|) + \gamma^2 W^2(e), \tag{2.36}$$

where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous positive definite function. As a consequence, the flow and jump sets are

$$\begin{aligned}
C &= \left\{(x,e,\tau) : \gamma^2 W^2(e) \leq \delta(y) + \xi(|x_c|) \text{ or } \tau \in [0,T]\right\} \\
D &= \Big\{(x,e,\tau) : \left(\gamma^2 W^2(e) = \delta(y) + \xi(|x_c|) \text{ and } \tau \geq T\right) \text{ or} \\
&\qquad \left(\gamma^2 W^2(e) \geq \delta(y) + \xi(|x_c|) \text{ and } \tau = T\right)\Big\}.
\end{aligned} \tag{2.37}$$

which may yields larger inter-transmission times compared to (2.10). □

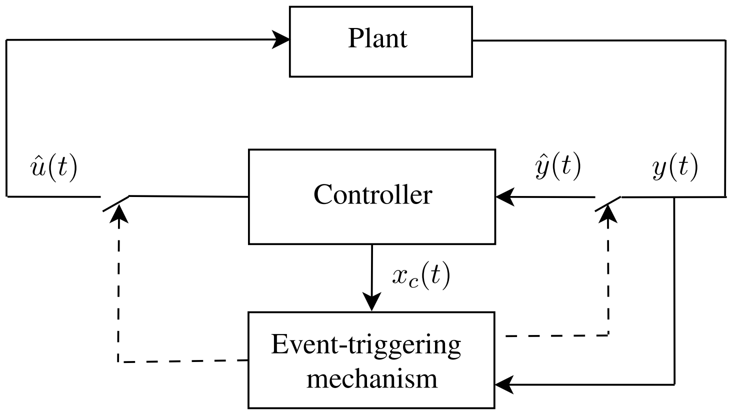

FIGURE 2.1: Event-triggered control schematic [105]

2.2.4 Illustrative examples

2.2.4.1 Controlled Lorenz model of fluid convection

Consider the controlled Lorenz equations which model fluid convection [114]

$$\begin{aligned} \dot{x}_1 &= -ax_1 + ax_2 \\ \dot{x}_2 &= bx_1 - x_2 - x_1x_3 + u \\ \dot{x}_3 &= x_1x_2 - cx_3 \\ y &= x_1, \end{aligned} \tag{2.38}$$

where x_1 is proportional to the intensity of the convective motion, x_2 is proportional to the temperature difference between the ascending and descending currents, x_3 is proportional to the distortion of the vertical temperature profile from being linear, and u corresponds to the tilt angle of a closed-loop of natural convection from the vertical. The three parameters a, b, and c are related to some physical constants and all three are positive, see [111] for more detail.

The static output feedback law $u = -(\frac{p_1}{p_2}a + b)x_1$, where $p_1, p_2 > 0$, globally stabilizes system (2.38), which can be verified as follows. Let $x := (x_1, x_2, x_3)$ and

$$f(x) := \begin{pmatrix} -ax_1 + ax_2 \\ bx_1 - x_2 - x_1x_3 - (\frac{p_1}{p_2}a + b)x_1 \\ x_1x_2 - cx_3 \end{pmatrix}. \tag{2.39}$$

Consider the quadratic function $V(x) = p_1x_1^2 + p_2x_2^2 + p_2x_3^2$. It holds that, for all $x \in \mathbb{R}^3$

$$\begin{aligned}\langle \nabla V(x), f(x)\rangle &= 2p_1x_1(-ax_1 + ax_2) + 2p_2x_2(bx_1 - x_2 - x_1x_3 + u) \\ &\quad +2p_2x_3(x_1x_2 - cx_3) \\ &= -2p_1ax_1^2 - 2p_2x_2^2 - 2p_2cx_3^2 + 2(p_1a + p_2b)x_1x_2 \\ &\quad +2p_2x_2u + (-2p_2 + 2p_2)x_1x_2x_3 \\ &= -2p_1ax_1^2 - 2p_2x_2^2 - 2p_2cx_3^2 + 2(p_1a + p_2b)x_1x_2 + 2p_2x_2u.\end{aligned} \tag{2.40}$$

Hence, by substituting by $u = -(\frac{p_1}{p_2}a + b)x_1$, we have that

$$\langle \nabla V(x), f(x)\rangle = -2p_1ax_1^2 - 2p_2x_2^2 - 2p_2cx_3^2. \tag{2.41}$$

Therefore, the output feedback law $u = -(\frac{p_1}{p_2}a + b)x_1$ globally stabilizes the origin of (2.38). We take into account the network-induced error

$$e = \hat{y} - y = \hat{x}_1 - x_1.$$

Note that it is not necessary to consider the error in u as the controller is static (see Section 1.2.2). As a consequence, the functions $f(x, e)$ and $g(x, e)$ in (2.4) are

$$\begin{aligned} f(x,e) &= \begin{pmatrix} -ax_1 + ax_2 \\ bx_1 - x_2 - x_1x_3 - (\frac{p_1}{p_2}a + b)(x_1 + e) \\ x_1x_2 - cx_3 \end{pmatrix} \\ g(x,e) &= ax_1 - ax_2. \end{aligned} \tag{2.42}$$

Let $W(e) = |e|$. Consequently, for all $x \in \mathbb{R}^3$ and almost all $e \in \mathbb{R}$

$$\langle \nabla W(e), g(x, e)\rangle \leq a(|x_1| + |x_2|). \tag{2.43}$$

Hence, condition (2.7) holds with $L = 0$ and $H(x) = a(|x_1| + |x_2|)$. By following the same lines as in (2.41), we have that

$$\begin{aligned}\langle\nabla V(x), f(x,e)\rangle &= -2p_1ax_1^2-2p_2x_2^2-2p_2cx_3^2+2(p_1a+p_2b)x_1x_2+2p_2x_2\hat{u}\\ &= -2p_1ax_1^2-2p_2x_2^2-2p_2cx_3^2+2(p_1a+p_2b)x_1x_2\\ &\quad -2p_2(\tfrac{p_1}{p_2}a+b)x_2(x_1+e)\\ &\leq -2p_1ax_1^2-2p_2x_2^2-2p_2cx_3^2+2p_2(\tfrac{p_1}{p_2}a+b)x_2e.\end{aligned} \tag{2.44}$$

Using the fact that $2(\frac{p_1}{p_2}a+b)x_2e\leq x_2^2+(\frac{p_1}{p_2}a+b)^2e^2$, it holds that

$$\begin{aligned}\langle\nabla V(x), f(x,e)\rangle &\leq -2p_1ax_1^2-2p_2x_2^2-2p_2cx_3^2+p_2x_2^2+p_2(\tfrac{p_1}{p_2}a+b)^2e^2\\ &= -2p_1ax_1^2-p_2x_2^2-2p_2cx_3^2+p_2(\tfrac{p_1}{p_2}a+b)^2e^2.\end{aligned} \tag{2.45}$$

Adding and subtracting the term $H^2(x)=ax_1^2+ax_2^2+2a|x_1||x_2|\leq 2ax_1^2+2ax_2^2$, we obtain

$$\begin{aligned}\langle\nabla V(x), f(x,e)\rangle &\leq -2p_1ax_1^2-p_2x_2^2-2p_2cx_3^2+p_2(\tfrac{p_1}{p_2}a+b)^2e^2-H^2(x)+ax_1^2\\ &\quad +ax_2^2+2a|x_1||x_2|\\ &\leq -2p_1ax_1^2-p_2x_2^2-2p_2cx_3^2+p_2(\tfrac{p_1}{p_2}a+b)^2e^2-H^2(x)+2ax_1^2\\ &\quad +2ax_2^2\end{aligned} \tag{2.46}$$

$$\begin{aligned}\langle\nabla V(x), f(x,e)\rangle &\leq -2a(p_1-1)x_1^2-(p_2-2a)x_2^2-2p_2cx_3^2-H^2(x)+p_2(\tfrac{p_1}{p_2}a+b)^2e^2\\ &\leq -\min\{a(p_1-1),(p_2-2a),2p_2c\}|x|^2-a(p_1-1)y^2-H^2(x)\\ &\quad +p_2(\tfrac{p_1}{p_2}a+b)^2e^2.\end{aligned} \tag{2.47}$$

By taking $p_1>1$ and $p_2>2a$, condition (2.6) holds with $\alpha(|x|)=\min\{a(p_1-1),(p_2-2a),2p_2c\}|x|^2$, $\delta(y)=a(p_1-1)y^2$ and $\gamma^2=p_2(\frac{p_1}{p_2}a+c)^2$.

We have shown that Assumption 2.1 holds, we can then apply the results of Section 2.2.3. For the parameter values $a=10, b=28, c=8/3$ used in [114], we set $p_1=2$, $p_2=3a$ and we obtain $T=0.01$. Table 2.1 provides the minimum and the average inter-sampling times for the proposed triggering mechanism (2.10) for 200 randomly distributed initial conditions such that $|(x(0,0),e(0,0))|\leq 100$ and $\tau(0,0)=0$. The constant τ_{avg} serves as a measure of the amount of transmissions (the bigger τ_{avg}, the less transmissions). We present simulations for one initial condition $(x(0,0), e(0,0), \tau(0,0)) = (-20, -20, 30, 0, 0)$. Figure 2.2 shows that plant states converge asymptotically to the origin as expected. Figure 2.3 provides an insight of how the proposed triggering mechanism works, in particular the interaction between the event-triggered rule and the time-triggered part. We note that Zeno phenomenon occurs when we remove the latter. It can be noted that the results in [123] are not applicable to this system because condition (3) of Proposition 1 in [123] does not hold.

T	τ_{min}	τ_{avg}
0.01	0.01	0.0109

TABLE 2.1: Minimum and average inter-transmission times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for a simulation time of 10s.

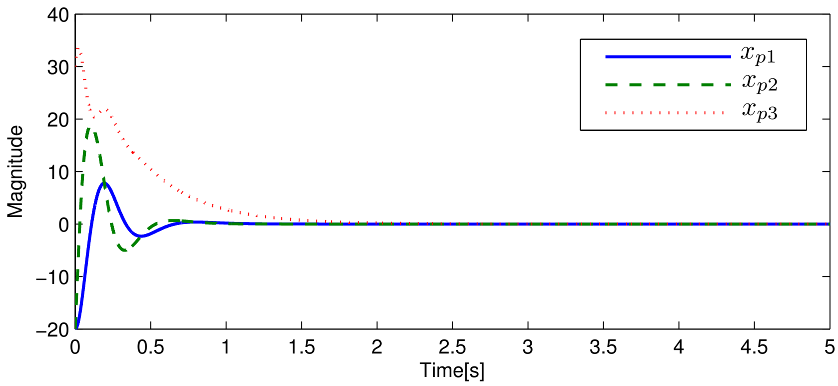

FIGURE 2.2: States of the plant

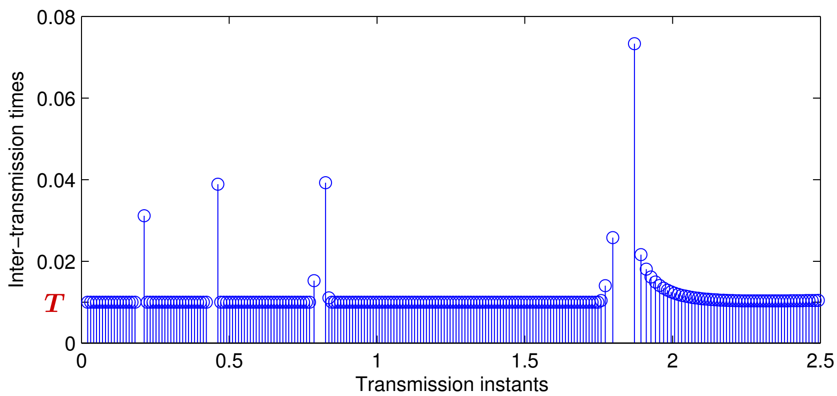

FIGURE 2.3: Inter-transmission times.

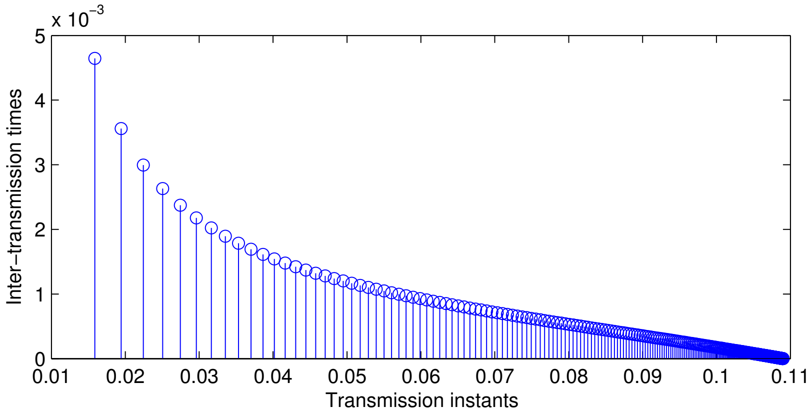

FIGURE 2.4: Inter-transmission times with [103].

2.2.4.2 Single-link robot arm model

Consider the dynamics of a single-link robot arm

$$\begin{aligned} \dot{x}_{p1} &= x_{p2} \\ \dot{x}_{p2} &= -\sin(x_{p1}) + u \\ y &= x_{p1}, \end{aligned} \tag{2.48}$$

where x_{p1} denotes the angle, x_{p2} the rotational velocity and u the input torque. The system can be written as

$$\dot{x}_p = Ax_p + Bu - \phi(y), \qquad y = Cx_p, \tag{2.49}$$

where $x_p := (x_{p1}, x_{p2})$ and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\phi(y) = \begin{bmatrix} 0 \\ \sin(y) \end{bmatrix}$.

In order to stabilize system (2.50), we first construct a state feedback controller of the form $u = Kx_p + B^T\phi(y)$. Hence, system (2.48) reduces to

$$\dot{x}_p = (A + BK)x_p, \qquad y = Cx_p. \tag{2.50}$$

We design the gain K such that the eigenvalues of the closed loop system (2.50) are $(-1, -2)$ (which is possible since the pair (A, B) is controllable). Hence, the gain K is selected to be $K = [-2 \quad -3]$. Next, since only the measurement of y is available, we construct a state-observer of the following form

$$\begin{aligned} \dot{x}_c &= Ax_c + Bu - \phi(y) + M(y - Cx_c) \\ &= (A - MC)x_c + Bu - \phi(y) + My, \end{aligned} \tag{2.51}$$

where $x_c \in \mathbb{R}^2$ is the estimated state and M is the observer gain matrix. We design the gain matrix M such that the eigenvalues of $(A - MC)$ are $(-5, -6)$ (which is possible since the pair (A, C) is observable). Thus, the observer gain is selected to be $M = [11 \quad 30]^T$. As a result, the closed-loop system in the absence of sampling is given by

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu - \phi(y), & y &= Cx_p \\ \dot{x}_c &= (A - MC)x_c + Bu - \phi(y) + My, & u &= Kx_c + B^T\phi(y). \end{aligned} \tag{2.52}$$

We now take into account the effect of the network. We consider the scenario where the controller receives the output measurements only at transmission instants $t_i, i \in \mathbb{Z}_{\geq 0}$ while the controller is directly connected to the plant actuators. We design a triggering condition of the form (2.9). As a consequence, the network-induced error is $e = e_y = \hat{y} - y$ and we obtain, for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \dot{x}_p &= Ax_p + B\Big(Kx_c + B^T\phi(\hat{y})\Big) - \phi(y) \\ \dot{x}_c &= (A - MC + BK)x_c + MCx_p + Me_y. \end{aligned} \tag{2.53}$$

Let $x := (x_p, x_c)$. Then, system (2.53) has the following dynamics on flows

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & BK \\ MC & A - MC + BK \end{pmatrix} \begin{pmatrix} x_p \\ x_c \end{pmatrix} + \begin{pmatrix} 0 \\ M \end{pmatrix} e + \begin{pmatrix} \phi(y+e) - \phi(y) \\ 0 \end{pmatrix} \\ &=: \mathcal{A}x + \mathcal{B}e + \psi(y, e). \end{aligned} \tag{2.54}$$

Since $e = \hat{y} - y$ and in view of (2.48), we have $\dot{e} = -\dot{y} = -x_{p2}$. Hence, the functions f, g in (2.4) are $f(x, e) = \mathcal{A}x + \mathcal{B}e + \psi(y, e)$ and $g(x, e) = -x_{p2}$.

Verification of Assumption 2.1

We now verify Assumption 2.1. Let $W(e) := |e|$ for all $e \in \mathbb{R}$. Consequently, for almost all e and all x

$$\langle \nabla W(e), g(x, e) \rangle \leq |x_{p2}|. \tag{2.55}$$

Hence, condition (2.7) holds with $H(x) = |x_{p2}|$ and $L = 0$. Let $V(x) = x^T Px$, where P is a real positive definite symmetric matrix such that $\mathcal{A}^T P + P\mathcal{A} = -Q$ (such a matrix P always exist since $\mathcal{A}$ is Hurwitz) and Q is real positive definite and symmetric such that $\lambda_{\min}(Q) > 4$. We select Q as a block diagonal matrix with the diagonal elements equal to 4.2, thus $\lambda_{\min}(Q) = 4.2$. Then, we have, for all $e \in \mathbb{R}$ and all $x \in \mathbb{R}^4$

$$\begin{aligned}\langle \nabla V(x), f(x,e)\rangle &= x^T(\mathcal{A}^T P + P\mathcal{A})x + 2x^T P(\mathcal{B}e + \psi(y,e)) \\ &\leq -\lambda_{\min}(Q)|x|^2 + 2|P\mathcal{B}||x||e| + 2|P||x||\psi(y,e)|.\end{aligned} \tag{2.56}$$

In view of (2.54), by applying the mean value theorem and since the sin function is globally Lipschitz, it holds that, for some $c \in [y, y+e]$

$$|\psi(y,e)| = |\phi(y+e) - \phi(y)| = |\sin(y+e) - \sin(y)| = |y+e-y||\cos(c)| \leq |e|. \tag{2.57}$$

As a consequence,

$$\langle \nabla V(x), f(x,e)\rangle \leq -\lambda_{\min}(Q)|x|^2 + 2(|P\mathcal{B}| + |P|)|x||e|. \tag{2.58}$$

Using the fact that $2(|P\mathcal{B}| + |P|)|x||e| \leq \frac{\lambda_{\min}(Q)}{2}|x|^2 + \frac{2(|P\mathcal{B}|+|P|)^2}{\lambda_{\min}(Q)}|e|^2$ and recalling that $\frac{\lambda_{\min}(Q)}{4} > 1$, it holds that

$$\begin{aligned}\langle \nabla V(x), f(x,e)\rangle &\leq -\frac{\lambda_{\min}(Q)}{2}|x|^2 + \frac{2(|P\mathcal{B}|+|P|)^2}{\lambda_{\min}(Q)}|e|^2 \\ &= -\frac{\lambda_{\min}(Q)}{4}|x|^2 - \frac{\lambda_{\min}(Q)}{4}(x_{p1}^2 + x_{p2}^2 + x_c^2) + \frac{2(|P\mathcal{B}|+|P|)^2}{\lambda_{\min}(Q)}|e|^2 \\ &\leq -\frac{\lambda_{\min}(Q)}{4}|x|^2 - |x_{p2}|^2 - \frac{\lambda_{\min}(Q)}{4}y^2 + \frac{2(|P\mathcal{B}|+|P|)^2}{\lambda_{\min}(Q)}|e|^2.\end{aligned} \tag{2.59}$$

Thus, condition (2.6) is verified with $\alpha(|x|) = \frac{\lambda_{\min}(Q)}{4}|x|^2$, $\delta(y) = \frac{\lambda_{\min}(Q)}{4}y^2$ and $\gamma^2 = \frac{2(|P\mathcal{B}|+|P|)^2}{\lambda_{\min}(Q)}$.

Simulation results

We obtain the numerical value $\gamma = 26.5333$, which gives, in view of (2.11), $\mathcal{T} = 0.0592$. We take $T = 0.059$. Figure 2.5 shows that the plant and the estimated state asymptotically converge to the origin as expected. The generated inter-transmission times by the proposed mechanism (2.9) are shown in Figure 2.6 where we can observe the interaction between the time-triggered [79] and the event-triggered [103] techniques. Table 2.2 gives the minimum and the average inter-sampling times for the proposed triggering mechanism (2.10) for 200 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$. Figure 2.7 presents the inter-transmission times with the triggering condition $\gamma^2 W^2(e) \leq \delta(y)$ without enforcing a constant time T between transmissions (*i.e.* $T = 0$ in (2.9), (2.10)). We note that Zeno phenomenon occurs in this case, like in Section 2.2.4.1.

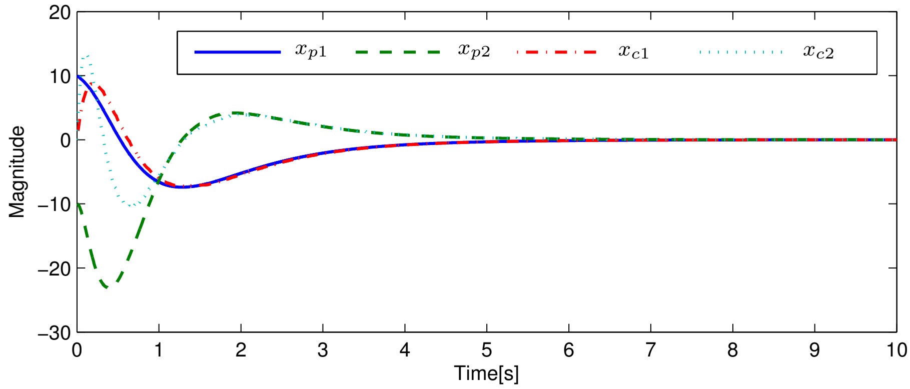

FIGURE 2.5: Actual and estimated states of the plant.

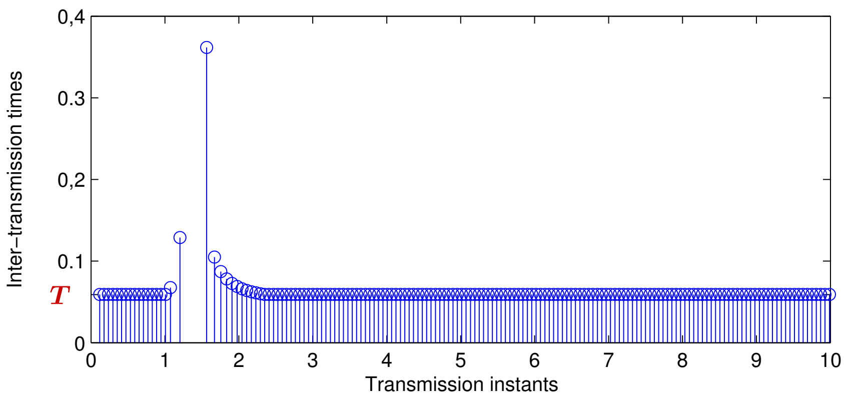

FIGURE 2.6: Inter-transmission times.

T	τ_{min}	τ_{avg}
0.059	0.059	0.0625

TABLE 2.2: Minimum and average inter-transmission times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for a simulation time of 10s.

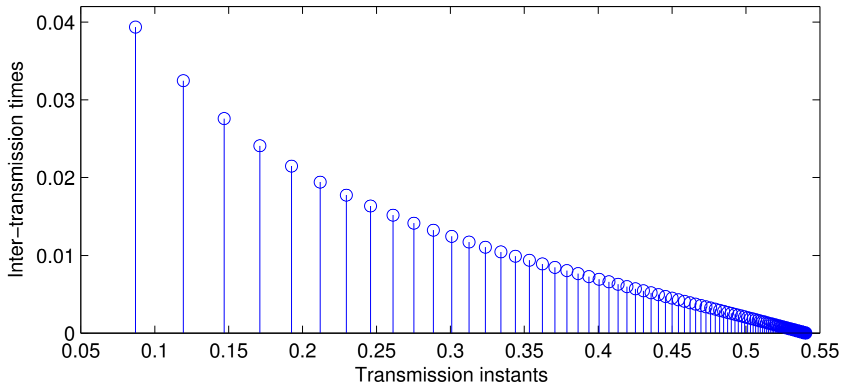

FIGURE 2.7: Inter-transmission times with [103].

2.3 Case studies

In this section, we show how the results can be applied for two important special cases. We first apply our results to LTI systems and we show that the required conditions are always satisfied by LTI systems that are stabilizable and detectable. Then, we illustrate the effectiveness of the proposed strategy on a numerical example. Next, we show how the proposed technique can be exploited in the context of state feedback control. We illustrate the idea on two numerical examples.

2.3.1 LTI systems

We now focus on the particular case of linear systems. We formulate the required conditions in Assumption 2.1 as an LMI constraint. Then, we design the triggering condition by solving this LMI. We finally compare the results with [22] on a numerical example.

2.3.1.1 Analytical results

Consider the LTI plant model

$$\dot{x}_p = A_p x_p + B_p u, \qquad y = C_p x_p, \tag{2.60}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and A_p, B_p, C_p are matrices of appropriate dimensions. We design the following dynamic controller to stabilize (2.60) in the absence of sampling

$$\dot{x}_c = A_c x_c + B_c y, \qquad u = C_c x_c + D_c y, \tag{2.61}$$

where $x_c \in \mathbb{R}^{n_c}$ and A_c, B_c, C_c, D_c are matrices of appropriate dimensions. We introduce the network-induced error as in (1.4), *i.e.*

$$\begin{aligned} e_y &:= \hat{y} - y \\ e_u &:= \hat{u} - u. \end{aligned} \tag{2.62}$$

Then, we obtain, for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \dot{x}_p &= (A_p + B_p D_c C_p) x_p + B_p C_c x_c + B_p D_c e_y + B_p e_u \\ \dot{x}_c &= A_c x_c + B_c C_p x_p + B_c e_y. \end{aligned} \tag{2.63}$$

The dynamics of the network-induced error between two successive transmission instants is given by, for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \dot{e}_y &= -\dot{y} = -C_p \dot{x}_p \\ &= -C_p(A_p + B_p D_c C_p) x_p - C_p B_p C_c x_c - C_p B_p D_c e_y - C_p B_p e_u \end{aligned} \tag{2.64}$$

and

$$\begin{aligned} \dot{e}_u &= -\dot{u} = -C_c \dot{x}_c - D_c \dot{\hat{y}} \\ &= -C_c A_c x_c - C_c B_c C_p x_p - C_c B_c e_y. \end{aligned} \tag{2.65}$$

Let $x = (x_p, x_c) \in \mathbb{R}^{n_x}$ and $e = (e_y, e_u) \in \mathbb{R}^{n_e}$. In view of (2.63)-(2.65), it holds that, between two successive transmission instants

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_c \end{pmatrix} \begin{pmatrix} x_p \\ x_c \end{pmatrix} + \begin{pmatrix} B_p D_c & B_p \\ B_c & 0 \end{pmatrix} \begin{pmatrix} e_y \\ e_u \end{pmatrix} \\ &=: \mathcal{A}_1 x + \mathcal{B}_1 e \end{aligned} \tag{2.66}$$

and

$$\begin{aligned}\dot{e} &= \begin{pmatrix} -C_p(A_p + B_pD_cC_p) & -C_pB_pC_c \\ -C_cB_cC_p & -C_cA_c \end{pmatrix}\begin{pmatrix} x_p \\ x_c \end{pmatrix} + \begin{pmatrix} -C_pB_pD_c & -C_pB_p \\ -C_cB_c & 0 \end{pmatrix}\begin{pmatrix} e_y \\ e_u \end{pmatrix} \\ &=: \mathcal{A}_2x + \mathcal{B}_2e.\end{aligned} \tag{2.67}$$

Hence, the hybrid system (2.4) becomes

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{pmatrix} &= \begin{pmatrix} \mathcal{A}_1x + \mathcal{B}_1e \\ \mathcal{A}_2x + \mathcal{B}_2e \\ 1 \end{pmatrix} \quad (x, e, \tau) \in C \\ \begin{pmatrix} x^+ \\ e^+ \\ \tau^+ \end{pmatrix} &= \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad (x, e, \tau) \in D.\end{aligned} \tag{2.68}$$

We obtain the following result.

Proposition 1. *Consider system (2.68). Suppose that there exist $\varepsilon_1, \varepsilon_2, \mu > 0$ and a positive definite symmetric real matrix P such that*

$$\begin{pmatrix} \mathcal{A}_1^TP + P\mathcal{A}_1 + \varepsilon_1\mathbb{I}_{n_x} + \mathcal{A}_2^T\mathcal{A}_2 + \varepsilon_2\overline{C}_p^T\overline{C}_p & P\mathcal{B}_1 \\ \mathcal{B}_1^TP & -\mu\mathbb{I}_{n_e} \end{pmatrix} \leq 0, \tag{2.69}$$

where $\overline{C}_p = [C_p \quad 0]$ and 0 represents the matrix of zeros of size $n_y \times n_c$. Then Assumption 2.1 globally holds with $V(x) = x^TPx$, $\underline{\alpha}(|x|) = \lambda_{\min}(P)|x|^2$, $\overline{\alpha}(|x|) = \lambda_{\max}(P)|x|^2$, $W(e) = |e|$, $H(x) = |\mathcal{A}_2x|$, $L = |\mathcal{B}_2|$, $\gamma = \sqrt{\mu}$, $\alpha(|x|) = \varepsilon_2|x|^2$ and $\delta(y) = \varepsilon_1|y|^2$. □

Proof of Proposition 1. Let $W(e) = |e|$. Then we have, for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$

$$\langle\nabla W(e), \mathcal{A}_2x + \mathcal{B}_2e\rangle \leq |\mathcal{A}_2x| + |\mathcal{B}_2||e|. \tag{2.70}$$

Hence, condition (2.7) holds with $L = |\mathcal{B}_2|$ and $H(x) = |\mathcal{A}_2x|$. Let $V(x) = x^TPx$. Consequently, condition (2.5) is satisfied with $\underline{\alpha}(|x|) = \lambda_{\min}(P)|x|^2$ and $\overline{\alpha}(|x|) = \lambda_{\max}(P)|x|^2$. It holds that, for all $e \in \mathbb{R}^{n_e}$ and almost all $x \in \mathbb{R}^{n_x}$

$$\langle \nabla V(x), \mathcal{A}_1 x + \mathcal{B}_1 e \rangle \;=\; x^T(\mathcal{A}_1^T P + P\mathcal{A}_1)x + x^T P \mathcal{B}_1 e + e^T \mathcal{B}_1^T P x. \tag{2.71}$$

By post- and pre-multiplying LMI (2.69) respectively by the state vector (x, e) and its transpose, we obtain

$$\begin{aligned} x^T(\mathcal{A}_1^T P + P\mathcal{A}_1)x + x^T P \mathcal{B}_1 e + e^T \mathcal{B}_1^T P x \;&\leq\; -\varepsilon_2 x^T x - x^T \mathcal{A}_2^T \mathcal{A}_2 x - \varepsilon_1 x^T \overline{C}_p^T \overline{C}_p x \\ &\quad +\mu e^T e \end{aligned} \tag{2.72}$$

which implies

$$\begin{aligned} x^T(\mathcal{A}_1^T P + P\mathcal{A}_1)x + x^T P \mathcal{B}_1 e + e^T \mathcal{B}_1^T P x \;&\leq\; -\varepsilon_2 |x|^2 - |\mathcal{A}_2 x|^2 - \varepsilon_1 |\overline{C}_p x|^2 + \mu |e|^2 \\ &=\; -\varepsilon_2 |x|^2 - |\mathcal{A}_2 x|^2 - \varepsilon_1 |y|^2 + \mu |e|^2. \end{aligned} \tag{2.73}$$

As a result, in view of (2.71), (2.73), condition (2.6) is verified with $\alpha(|x|) = \varepsilon_2 |x|^2$, $\delta(y) = \varepsilon_1 |y|^2$ and $\gamma = \sqrt{\mu}$. □

We note that the flow and jump sets (2.10) in the linear case are

$$\begin{aligned} C &= \Big\{ (x, e, \tau) : \mu |e|^2 \leq \varepsilon_1 |y|^2 \text{ or } \tau \in [0, T] \Big\} \\ D &= \Big\{ (x, e, \tau) : \Big(\mu |e|^2 = \varepsilon_1 |y|^2 \text{ and } \tau \geq T \Big) \text{ or } \Big(\mu |e|^2 \geq \varepsilon_1 |y|^2 \text{ and } \tau = T \Big) \Big\}, \end{aligned} \tag{2.74}$$

with T defined in (2.11).

Proposition 1 provides a sufficient condition, namely (2.69), for the verification of Assumption 2.1, which thus allows us to use the results in Section 2.2 for LTI systems. It has to be noted that the LMI (2.69) can always be satisfied when system (2.60) is stabilizable and detectable. Indeed, in this case, we can select the controller (2.61) such that $\mathcal{A}_1$ is Hurwitz. Noting that (2.69) is equivalent to the following inequalities (by using the Schur complement of (2.69), see Section A.5.5 in [16]),

$$\begin{aligned} -\mu \mathbb{I}_{n_e} &\leq 0 \\ \mathcal{A}_1^T P + P\mathcal{A}_1 + \mathcal{A}_2^T \mathcal{A}_2 + \varepsilon_1 \overline{C}_p^T \overline{C}_p + \varepsilon_2 \mathbb{I}_{n_x} + \tfrac{1}{\mu} P \mathcal{B}_1 \mathcal{B}_1^T P &\leq 0. \end{aligned} \tag{2.75}$$

We see that we can select the matrix P such that $\mathcal{A}_1^T P + P\mathcal{A}_1 + \varepsilon_1 \mathbb{I}_{n_x} + \mathcal{A}_2^T \mathcal{A}_2 + \varepsilon_2 \overline{C}_p^T \overline{C}_p$ is negative definite. It then suffices to select μ sufficiently large to ensure the last inequality.

In view of Proposition 1, Assumption 2.1 holds with $\gamma^2 = \mu$. On the other hand, the smaller γ, the larger the upper-bound on T in (2.11). Hence, we can minimize μ under the linear constraint (2.69) to enlarge the constant T. Note that $L = |\mathcal{B}_2|$ is fixed, since $\mathcal{B}_2$ depends on the plant and the controller matrices and the controller is assumed to be known a priori, and hence, we can only play with γ to enlarge T.

2.3.1.2 Illustrative example

We consider the same example in Section 1.2.5.1, *i.e.* the plant model is

$$\begin{aligned} \dot{x}_p &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 4 \end{bmatrix} x_p \end{aligned} \tag{2.76}$$

and the dynamic controller is

$$\begin{aligned} \dot{x}_c &= \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y \\ u &= \begin{bmatrix} 1 & -4 \end{bmatrix} x_c. \end{aligned} \tag{2.77}$$

In view of (2.66), (2.67), we obtain

$$\begin{aligned} \mathcal{A}_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & -4 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & 0 & -5 \end{bmatrix}, & \mathcal{B}_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \mathcal{A}_2 &= \begin{bmatrix} 8 & -11 & -4 & 16 \\ -4 & 16 & 0 & -21 \end{bmatrix}, & \mathcal{B}_2 &= \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}. \end{aligned} \tag{2.78}$$

Hence, $L = |\mathcal{B}_2| = 4$. Then, we obtain the values $\varepsilon_1 = 1.5839$, $\varepsilon_2 = 13.9969$, $\gamma = 89.9666$ by solving the LMI (2.69) using the SEDUMI solver [101] with the YALMIP interface [62]. Consequently, the guaranteed minimum inter-transmission time is $T = 0.017$, by using (2.11). Table 2.3 provides the minimum and the average inter-transmission times, respectively denoted as τ_{min} and τ_{avg}, for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq$

25 and $\tau(0,0) = 0$. The values of τ_{avg} in Table 2.3 indicates that the generated amount of transmissions by the proposed triggering mechanism is approximately 100 times less than the amount given by [22]. Moreover, the stability property achieved in [22] is a practical stability property, while we ensure a global asymptotic stability property. Figures 2.8, 2.9 present the simulations for one initial condition $(x(0,0), e(0,0), \tau(0,0)) = (10, -10, 0, 0, 0, 0, 0)$. We observe that the system state asymptotically converges to the origin as expected and the inter-transmission times in Figure 2.9 clarify the idea of the combined event-triggered and time-triggered techniques. We note that the results in [123] are not applicable to this system because condition (3) of Proposition 1 in [123] is not again satisfied. We do not compare our results with [105] because the triggering mechanism is different and the dynamic controller in [105] is an observer based controller. □

	Guaranteed dwell-time	$\tau_{\min}$	τ_{avg}
Donkers & Heemels [22] $\sigma_1 = \sigma_2 = 10^{-3}$, $\varepsilon_1 = \varepsilon_1 = 10^{-3}$	6.5×10^{-9}	2.103×10^{-6}	1.68×10^{-4}
The proposed triggering mechanism	0.017	0.017	0.0202

TABLE 2.3: Minimum and average inter-transmission times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 25$ and $\tau(0,0) = 0$ for a simulation time of 20 seconds.

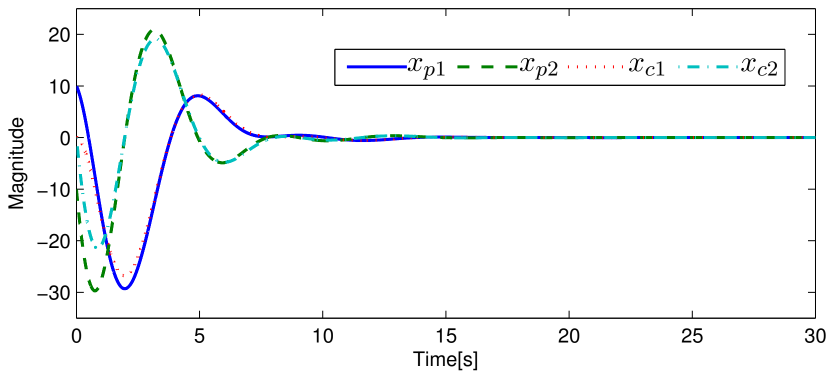

FIGURE 2.8: State trajectories of the plant and the dynamic controller.

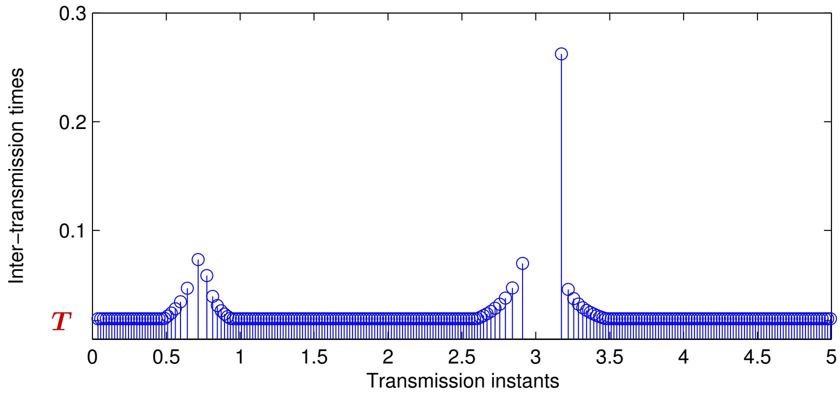

FIGURE 2.9: Inter-transmission times for the first five seconds.

2.3.2 State feedback controllers

The technique proposed in Section 2.2 is also relevant in the context of state feedback control, *i.e.* when $y = x$, as the constant T in (2.10) can be used to directly tune the lower bound on the inter-transmission time (up to $\mathcal{T}$ in (2.11)). Although the existence of a lower bound on the inter-transmission times is guaranteed in [103], the obtained value may be subject to some conservatism and it is not explicitly predetermined as in our triggering mechanism. Furthermore, the generated amount of transmissions by our triggering mechanism are ensured to be less than or, at least, equal to those generated by conventional periodic setups in the sense of [79]. These properties of our proposed triggering mechanism extend its interest to the context of state feedback control.

2.3.2.1 Analytical results

Since the full state measurement is available, we can replace $\gamma^2 W^2(e) \leq \delta(y)$ in (2.9) by $\gamma^2 W^2(e) \leq \sigma(\alpha(|x|) + H^2(x) + \delta(x))$ when Assumption 2.1 holds. Consequently, the flow and jump sets can be taken as

$$\begin{aligned} C &= \Big\{(x,e,\tau) : \gamma^2 W^2(e) \leq \sigma\Big(\alpha(|x|) + H^2(x) + \delta(x)\Big) \text{ or } \tau \in [0,T]\Big\} \\ D &= \Big\{(x,e,\tau) : \Big(\gamma^2 W^2(e) = \sigma\Big(\alpha(|x|) + H^2(x) + \delta(x)\Big) \text{ and } \tau \geq T\Big) \text{ or} \\ &\qquad \Big(\gamma^2 W^2(e) \geq \sigma\Big(\alpha(|x|) + H^2(x) + \delta(x)\Big) \text{ and } \tau = T\Big)\Big\}, \end{aligned} \tag{2.79}$$

where $\sigma \in (0,1)$ and T is such that $T \in (0, \mathcal{T}(\gamma, L))$. Note that, unlike (2.10), we have to introduce σ here to guarantee that $\langle \nabla V(x), f(x,e) \rangle$ in (2.6) is strictly negative. The following result is a direct consequence of Theorem 2.1.

Corollary 2.1. *Suppose that Assumption 2.1 holds and consider system (2.4), (2.79). Then, the conclusions of Theorem 2.1 hold.* □

The proof of Corollary 2.1 follows the same lines as in the proof of Theorem 2.1. We illustrate the benefits of the proposed triggering condition on the following examples.

2.3.2.2 Illustrative examples

Example 1 in [79]. Consider the following family of nonlinear systems

$$\begin{aligned} \dot{x} &= -2x + dx^2 - x^3 - 2e &= f(x,e,d) \\ \dot{e} &= 2e + 2x - dx^2 + x^3 &= g(x,e,d), \end{aligned} \tag{2.80}$$

where $x \in \mathbb{R}$, $e \in \mathbb{R}$ and $|d| \leq 1$ is unknown and possibly time-varying. By following [79], we consider the function $W(e) = |e|$ which satisfies

$$\langle \nabla W(e), g(x,e,d) \rangle \leq 2|e| + |2x - dx^2 + x^3| \tag{2.81}$$

$$= 2W(e) + H(x,d), \tag{2.82}$$

where $H(x,d) = |2x - dx^2 + x^3|$. Hence, condition (2.7) is verified with $L = 2$. We consider also the same Lyapunov function as in [79]

$$V(x) = \mu^2(\nu\frac{x^2}{2} + \beta\frac{x^4}{4}), \tag{2.83}$$

where $\mu, \nu, \beta > 0$. By following similar lines as in [79], we obtain

$$\langle \nabla V(x), f(x,e,d) \rangle \leq -\mu^2\varepsilon|x|^2 - H^2(x,d) + \mu^2(2\nu^2 + 2\beta^2)|e|^2, \tag{2.84}$$

where $\mu = 2, \nu = 0.77, \beta = 0.77, \varepsilon = 0.01$. Thus, condition (2.6) holds with $\alpha(|x|) := \mu^2\varepsilon|x|^2$ and $\gamma = \mu\sqrt{2\nu^2 + 2\beta^2} = 3.08$. By substituting by L, γ in (2.11) we obtain $\mathcal{T} = 0.3689$. Hence, we take $T = 0.36$ and we run simulations with $\sigma = 0.9$ and $d = 0.1$ and by using HyEQ toolbox [92].

The state trajectories and inter-sampling intervals under the event-triggered controller of [103] and our proposed event-triggered controller (with $T = 0.36$) are plotted in Figure 2.10. Table 2.4 shows the obtained values of the minimum and the average inter-jump intervals with two different values of T and for 200 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$. We note that when $T = 0.36$, $\tau_{\text{min}} = \tau_{\text{avg}}$ implies that the transmission instants are typically generated by the time-triggered condition which is not the case for $T = 0.1$.

	[103]	Our proposed mechanism (2.79)	
		$T = 0.1$	$T = 0.36$
τ_{min}	0.115	0.1599	0.36
τ_{avg}	0.1887	0.1902	0.36

TABLE 2.4: Minimum and average inter-execution times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for simulation time of 10 s.

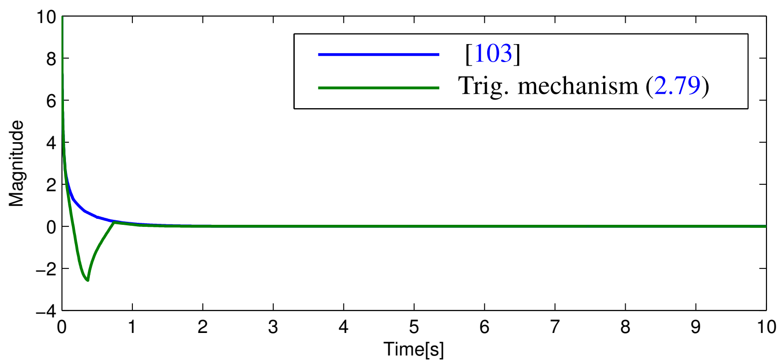

FIGURE 2.10: Closed-loop state trajectories with the triggering mechanisms (2.79) and [103]

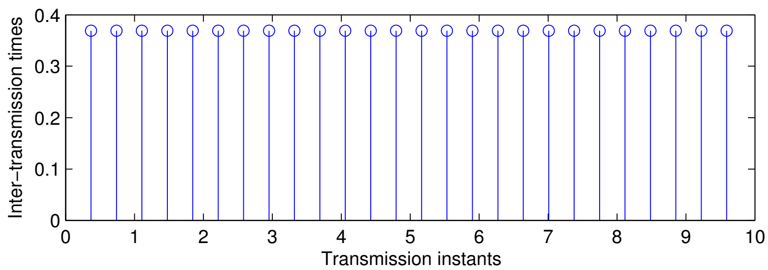

FIGURE 2.11: Inter-transmission times

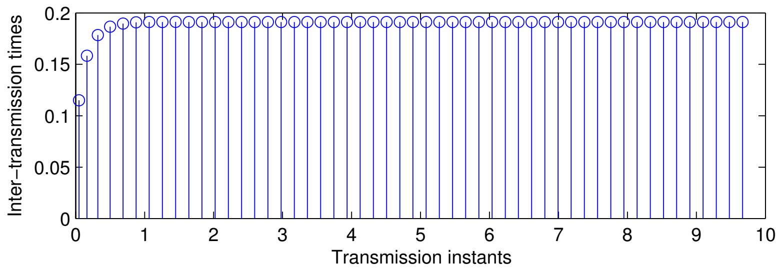

FIGURE 2.12: Inter-transmission times with [103]

Numerical example in [103]. Consider the LTI system

$$\dot{x} = Ax + Bu, \tag{2.85}$$

where $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since the pair (A, B) is stabilizable, we take the control input $u = Kx$ with $K = [1 \ \ -4]$ as in [103]. By following similar lines as in Section 2.3.1, we derive the LMI (2.69) with $\mathcal{A}_1 = \mathcal{A}_2 = A + BK$, $\mathcal{B}_1 = \mathcal{B}_2 = BK$ and $\varepsilon_1 = 0$. Hence, by solving the resulted LMI, we obtain the numerical values $L = 4.1231, \varepsilon_2 = 0.68, \gamma = 17.3495$ which lead to $\mathcal{T} = 0.079$. We set $T = 0.075$ and we compare the generated minimum and average inter-transmission times by both the proposed triggering strategy and the triggering condition in [103], *i.e.* with $T = 0$, as shown in Table 2.5. We note that the proposed mechanism produces larger values of τ_{min}, τ_{avg}. To spotlight the effect of the time-triggered part in the proposed triggering mechanism, the enforced lower bound T is plotted in Figures 2, 3 versus the generated inter-transmission times by both the proposed triggering mechanism and the triggering condition in [103] respectively, for one initial condition. The state trajectories for both cases are plotted in Figure 2.13.

	[103]	Our proposed mechanism (2.79)
τ_{min}	0.0543	0.075
τ_{avg}	0.0659	0.0772

TABLE 2.5: Minimum and average inter-execution times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for a simulation time of 10 s.

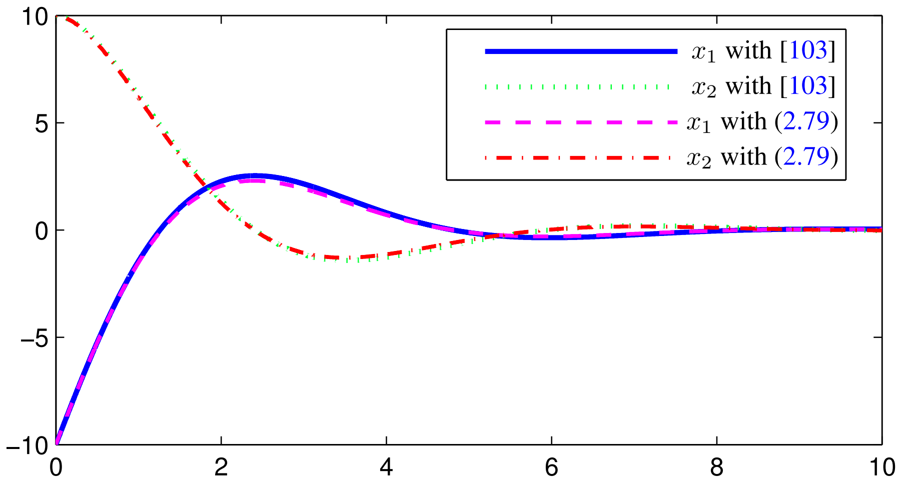

FIGURE 2.13: Closed-loop state trajectories

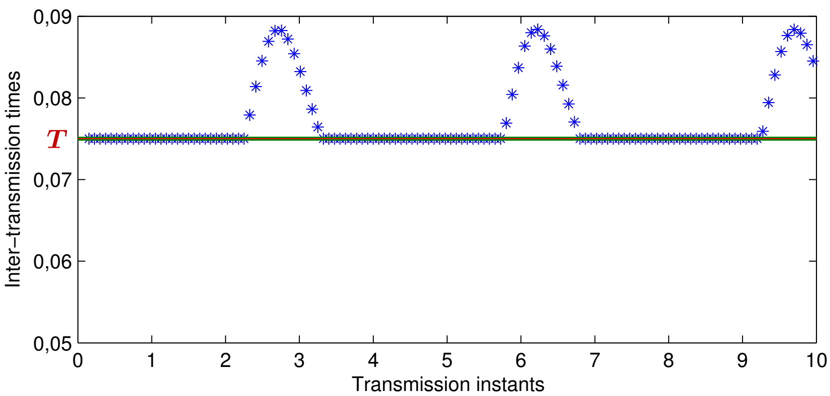

FIGURE 2.14: Inter-transmission times.

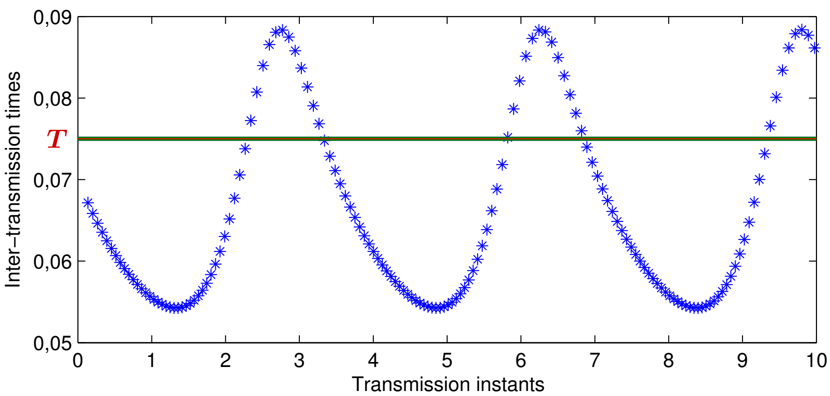

FIGURE 2.15: Inter-transmission times with [103].

2.4 Conclusion

In this chapter, we have developed output-based event-triggered controllers for the stabilization of nonlinear systems. The proposed technique ensures an asymptotic stability property and enforces a minimum amount of time between two consecutive transmission instants. The required conditions have been shown to hold for two physical nonlinear systems. Moreover, we have explained that the advantage of the proposed technique can be employed in the context of state feedback control to directly tune the minimum inter-transmission time. For LTI systems, the conditions have been formulated in terms of an LMI which is feasible for any stabilizable and detectable LTI systems. Then, the triggering condition in this case is designed by solving the derived LMI. In the next chapter, we will start from this LMI to develop a co-design procedure to construct the output feedback law and the event-triggering condition.

Chapter 3

Co-design for LTI systems

In the previous chapter, we have assumed that the feedback control law was known in the absence of network, then we synthesized the triggering condition. This sequential order of design may prevent an efficient usage of the computation and communication resources as we are restricted by the initial choice of feedback law. To overcome this issue, in this chapter, we use the triggering condition designed in Chapter 2 for linear systems as a starting point to simultaneously design the event-triggering condition and the feedback law.

3.1 Hybrid model

Consider the LTI system

$$\dot{x}_p = A_p x_p + B_p u, \qquad y = C_p x_p, \tag{3.1}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and A_p, B_p, C_p are matrices of appropriate dimensions. We will design dynamic output feedback laws of the form (we take $D_c = 0$ for simplicity)

$$\dot{x}_c = A_c x_c + B_c y, \qquad u = C_c x_c, \tag{3.2}$$

where $x_c \in \mathbb{R}^{n_c}$ and A_c, B_c, C_c are matrices of appropriate dimensions. We focus on the case where the controller has the same dimension as the plant, *i.e.* $n_c = n_p$. By following the same lines as in Section 2.3.1, we obtain the hybrid model below

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{e} \\ \dot{\tau} \end{pmatrix} &= \begin{pmatrix} \mathcal{A}_1 x + \mathcal{B}_1 e \\ \mathcal{A}_2 x + \mathcal{B}_2 e \\ 1 \end{pmatrix} \quad (x,e,\tau) \in C \\ \begin{pmatrix} x^+ \\ e^+ \\ \tau^+ \end{pmatrix} &= \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad\quad (x,e,\tau) \in D, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{A}_1 &= \begin{pmatrix} A_p & B_p C_c \\ B_c C_p & A_c \end{pmatrix}, & \mathcal{B}_1 &= \begin{pmatrix} 0 & B_p \\ B_c & 0 \end{pmatrix} \\ \mathcal{A}_2 &= \begin{pmatrix} -C_p A_p & -C_p B_p C_c \\ -C_c B_c C_p & -C_c A_c \end{pmatrix}, & \mathcal{B}_2 &= \begin{pmatrix} 0 & -C_p B_p \\ -C_c B_c & 0 \end{pmatrix}. \end{aligned} \tag{3.4}$$

and flow and jump sets are as defined in (2.74)

$$\begin{aligned} C &= \Big\{ (x,e,\tau) : \mu |e|^2 \leq \varepsilon_1 |y|^2 \text{ or } \tau \in [0,T] \Big\} \\ D &= \Big\{ (x,e,\tau) : \Big(\mu|e|^2 = \varepsilon_1 |y|^2 \text{ and } \tau \geq T \Big) \text{ or } \Big(\mu |e|^2 \geq \varepsilon_1 |y|^2 \text{ and } \tau = T \Big) \Big\}, \end{aligned} \tag{3.5}$$

where

$$\mathcal{T}(\gamma, L) := \begin{cases} \frac{1}{Lr} \arctan(r) & \gamma > L \\ \frac{1}{L} & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh}(r) & \gamma < L \end{cases} \tag{3.6}$$

and $r = \sqrt{|(\frac{\gamma}{L})^2 - 1|}$.

Our objective is to design the dynamic controller (3.2) and the flow and the jump sets (3.5) of the hybrid system (3.3) such that the conclusions of Theorem 2.1 hold.

The idea is to start from the LMI (2.69), *i.e.*

$$\begin{pmatrix} \mathcal{A}_1^T P + P \mathcal{A}_1 + \varepsilon_1 \mathbb{I}_{n_x} + \mathcal{A}_2^T \mathcal{A}_2 + \varepsilon_2 \overline{C}_p^T \overline{C}_p & P \mathcal{B}_1 \\ \mathcal{B}_1^T P & -\mu \mathbb{I}_{n_e} \end{pmatrix} \leq 0 \tag{3.7}$$

to establish an LMI-based co-design procedure of both the flow and jump sets (3.5) and the dynamic controller (3.2). It is important to note that the derivation of LMI for co-design from (3.7) is not trivial as the nonlinear term $\mathcal{A}_2^T\mathcal{A}_2$ depends on the controller matrices. This term does not appeared in the classical output feedback design problems and cannot be directly handled by congruence transformations like in standard output feedback design problems [94].

3.2 Global asymptotic stabilization

The following theorem formulates the co-design problem of the output feedback law (3.2) and the parameters of the flow and jump sets (3.5) in terms of LMI. We use boldface symbols to emphasize the LMI decision variables.

Theorem 3.1. *Consider system (3.3) with the flow and jump sets (3.5). Suppose that there exist symmetric positive definite real matrices* $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n_p \times n_p}$, *real matrices* $\boldsymbol{M} \in \mathbb{R}^{n_p \times n_p}, \boldsymbol{Z} \in \mathbb{R}^{n_p \times n_y}, \boldsymbol{N} \in \mathbb{R}^{n_u \times n_p}$ *and* $\boldsymbol{\varepsilon}, \boldsymbol{\mu} > 0$ *such that*

$$\begin{pmatrix} \Sigma(\boldsymbol{Y}A_p + \boldsymbol{Z}C_p) & \star & \star & \star & \star & \star & \star \\ A_p + \boldsymbol{M}^T & \Sigma(A_p\boldsymbol{X} + B_p\boldsymbol{N}) & \star & \star & \star & \star & \star \\ \boldsymbol{Z}^T & 0 & -\boldsymbol{\mu}\mathbb{I}_{n_y} & \star & \star & \star & \star \\ B_p^T\boldsymbol{Y} & B_p^T & 0 & -\boldsymbol{\mu}\mathbb{I}_{n_u} & \star & \star & \star \\ \boldsymbol{Y}A_p + \boldsymbol{Z}C_p & \boldsymbol{M} & 0 & 0 & -\boldsymbol{Y} & \star & \star \\ A_p & A_p\boldsymbol{X} + B_p\boldsymbol{N} & 0 & 0 & -\mathbb{I}_{n_p} & -\boldsymbol{X} & \star \\ C_p & C_p\boldsymbol{X} & 0 & 0 & 0 & 0 & -\boldsymbol{\varepsilon}\mathbb{I}_{n_y} \end{pmatrix} < 0 \tag{3.8}$$

$$\begin{pmatrix} -\mathbb{I}_{n_y} & \star & \star & \star \\ 0 & -\mathbb{I}_{n_u} & \star & \star \\ -C_p^T & 0 & -\boldsymbol{Y} & \star \\ -\boldsymbol{X}C_p^T & -\boldsymbol{N}^T & -\mathbb{I}_{n_p} & -\boldsymbol{X} \end{pmatrix} < 0. \tag{3.9}$$

Take $\gamma = \sqrt{\boldsymbol{\mu}}$, $L = |\mathcal{B}_2|$, $\varepsilon_1 = \boldsymbol{\varepsilon}^{-1}$ *and*

$$\begin{aligned} A_c &= V^{-1}(\boldsymbol{M} - \boldsymbol{Y}A_p\boldsymbol{X} - \boldsymbol{Y}B_p\boldsymbol{N} - \boldsymbol{Z}C_p\boldsymbol{X})U^{-T} \\ B_c &= V^{-1}\boldsymbol{Z}, \quad C_c = \boldsymbol{N}U^{-T}, \end{aligned} \tag{3.10}$$

where $U, V \in \mathbb{R}^{n_p \times n_p}$ are any square and invertible matrices such that[1] $UV^T = \mathbb{I}_{n_p} - \boldsymbol{X}\boldsymbol{Y}$. *Then, there exists $\chi \in \mathcal{KL}$ such that any solution $\phi = (\phi_x, \phi_e, \phi_\tau)$ satisfies*

$$|\phi_x(t,j)| \leq \chi(|(\phi_x(0,0), \phi_e(0,0))|, t+j) \quad \forall (t,j) \in \mathrm{dom}\,\phi \tag{3.11}$$

and, if ϕ is maximal, it is also complete. □

Proof of Theorem 3.1. We define the following matrices

$$S = \begin{pmatrix} X & U \\ U^T & \hat{X} \end{pmatrix}, S^{-1} = \begin{pmatrix} Y & V \\ V^T & \hat{Y} \end{pmatrix}, \Gamma = \begin{pmatrix} Y & \mathbb{I}_{n_p} \\ V^T & 0 \end{pmatrix}, G = \begin{pmatrix} -C_p & 0 \\ 0 & -C_c \end{pmatrix}, \tag{3.12}$$

where $\hat{X}, \hat{Y} \in \mathbb{R}^{n_p \times n_p}$ are symmetric positive definite real matrices of appropriate dimension. Since $SS^{-1} = \mathbb{I}_{2n_p}$, it holds that $XY + UV^T = \mathbb{I}_{n_p}$, $XV + U\hat{Y} = 0$, $U^TY + \hat{X}V^T = 0$ and $U^TV + \hat{X}\hat{Y} = \mathbb{I}_{n_p}$. After some direct calculations, recall that $\overline{C}_p = [C_p \;\; 0]$, we obtain

$$\begin{aligned} S\Gamma &= \begin{pmatrix} \mathbb{I}_{n_p} & X \\ 0 & U^T \end{pmatrix}, \Gamma^T S\Gamma = \begin{pmatrix} Y & \mathbb{I}_{n_p} \\ \mathbb{I}_{n_p} & X \end{pmatrix}, \mathcal{B}_1^T\Gamma = \begin{pmatrix} Z^T & 0 \\ B_p^T Y & B_p^T \end{pmatrix} \\ GS\Gamma &= \begin{pmatrix} -C_p & -C_pX \\ 0 & -N \end{pmatrix}, \Gamma^T \mathcal{A}_1 S\Gamma = \begin{pmatrix} YA_p + ZC_p & M \\ A_p & A_pX + B_pN \end{pmatrix} \\ \overline{C}_p S\Gamma &= (C_p \quad C_pX). \end{aligned} \tag{3.13}$$

[1] In view of the Schur complement of LMI (3.9), we deduce that $\begin{pmatrix} \boldsymbol{Y} & \mathbb{I}_{n_p} \\ \mathbb{I}_{n_p} & \boldsymbol{X} \end{pmatrix} > 0$ which implies that $\boldsymbol{X} - \boldsymbol{Y}^{-1} > 0$ and thus, $\mathbb{I}_{n_p} - \boldsymbol{X}\boldsymbol{Y}$ is nonsingular. Hence, the existence of nonsingular matrices U, V is always ensured.

Consequently, inequalities (3.8), (3.9) can be written as

$$\begin{pmatrix} -\Gamma^T(S\mathcal{A}_1^T+\mathcal{A}_1 S)\Gamma & \star & \star & \star \\ \mathcal{B}_1^T\Gamma & -\mu\mathbb{I}_{n_e} & \star & \star \\ \Gamma^T\mathcal{A}_1 S\Gamma & 0 & -\Gamma^T S\Gamma & \star \\ \overline{C}_p S\Gamma & 0 & 0 & -\varepsilon\mathbb{I}_{n_y} \end{pmatrix} < 0$$

$$\begin{pmatrix} -\mathbb{I}_{n_e} & GS\Gamma \\ \Gamma^T S G^T & -\Gamma^T S\Gamma \end{pmatrix} < 0. \tag{3.14}$$

By pre and post multiplying the first LMI respectively by diag $(\mathbb{I}_{n_x}, \mathbb{I}_{n_e}, G\Gamma^{-T}, \mathbb{I}_{n_y})$ and its transpose and by using the Schur complement of the second LMI, we obtain

$$\begin{pmatrix} -\Gamma^T(S\mathcal{A}_1^T+\mathcal{A}_1 S)\Gamma & \star & \star & \star \\ \mathcal{B}_1^T\Gamma & -\mu\mathbb{I}_{n_e} & \star & \star \\ G\mathcal{A}_1 S\Gamma & 0 & -GSG^T & \star \\ \overline{C}_p S\Gamma & 0 & 0 & -\varepsilon\mathbb{I}_{n_y} \end{pmatrix} < 0 \tag{3.15}$$

and

$$-\mathbb{I}_{n_e} < -GSG^T. \tag{3.16}$$

As a consequence, it holds that

$$\begin{pmatrix} -\Gamma^T(S\mathcal{A}_1^T+\mathcal{A}_1 S)\Gamma & \star & \star & \star \\ \mathcal{B}_1^T\Gamma & -\mu\mathbb{I}_{n_e} & \star & \star \\ G\mathcal{A}_1 S\Gamma & 0 & -\mathbb{I}_{n_e} & \star \\ \overline{C}_p S\Gamma & 0 & 0 & -\varepsilon\mathbb{I}_{n_y} \end{pmatrix} < 0. \tag{3.17}$$

Let $P = S^{-1}$ and pre and post multiply (3.17) respectively by diag $(P\Gamma^{-T}, \mathbb{I}_{n_e}, \mathbb{I}_{n_e}, \mathbb{I}_{n_y})$ and its transpose. Then, we have (note that $\mathcal{A}_2 = G\mathcal{A}_1$)

$$\left(\begin{array}{cc|cc} A_1^T P + P\mathcal{A}_1 & \star & \star & \star \\ \mathcal{B}_1^T P & -\mu\mathbb{I}_{n_e} & \star & \star \\ \hline \mathcal{A}_2 & 0 & -\mathbb{I}_{n_e} & \star \\ \overline{C}_p & 0 & 0 & -\varepsilon\mathbb{I}_{n_y} \end{array}\right) =: \left(\begin{array}{c|c} T_1 & T_2^T \\ \hline T_2 & T_3 \end{array}\right) < 0. \tag{3.18}$$

By using the Schur complement of (3.18), we obtain

$$\begin{pmatrix} \mathcal{A}_1^T P + P\mathcal{A}_1 + \mathcal{A}_2^T\mathcal{A}_2 + \varepsilon_1 \overline{C}_p^T \overline{C}_p & P\mathcal{B}_1 \\ \mathcal{B}_1^T P & -\mu\mathbb{I}_{n_e} \end{pmatrix} < 0, \tag{3.19}$$

where $\varepsilon_1 := \varepsilon^{-1}$. Hence, it holds that there exists $\varepsilon_2 > 0$ sufficiently small such that

$$\begin{pmatrix} \mathcal{A}_1^T P + P\mathcal{A}_1 + \mathcal{A}_2^T\mathcal{A}_2 + \varepsilon_1 \overline{C}_p^T \overline{C}_p + \varepsilon_2\mathbb{I}_{n_x} & P\mathcal{B}_1 \\ \mathcal{B}_1^T P & -\mu\mathbb{I}_{n_e} \end{pmatrix} \leq 0. \tag{3.20}$$

Thus, Theorem 3.1 holds in virtue of Proposition 1. □

We note that LMI (3.8), (3.9) are computationally tractable and can be solved using the SEDUMI solver [101] with the YALMIP interface [62]. Hence, by solving (3.8) and (3.9), we obtain the feedback law, see (3.10), and the parameters of the flow and jump sets (3.5) μ and ε_1. Note that T is also obtained by substituting in (3.6) with $\gamma = \sqrt{\mu}$ and $L = |\mathcal{B}_2|$.

We note also that the nonstandard term $\mathcal{A}_2^T\mathcal{A}_2$ in (3.7) is the reason why the constructed LMI (3.8) differs from the classical one and why the additional convex constraint (3.9) is needed in Theorem 3.1.

3.3 Optimization problems

The flexibility of the co-design procedure proposed in Section 2.2 can be exploited in many ways. In this section, we explain how to use the LMI conditions (3.8) and (3.9) to enlarge the guaranteed minimum amount of time between any two transmissions. We then propose a heuristic method to reduce the amount of transmissions. The efficiency of these methods is illustrated by simulations in Section 3.4.

3.3.1 Enlarging the guaranteed minimum inter-transmission time

A key challenge in the design of output feedback event-triggered controllers is to ensure the existence of a uniform strictly positive lower bound on the inter-transmission times. Although the existence of that lower bound is guaranteed by different techniques in the literature, the available expressions are often subject to some conservatism. It is therefore unclear whether the event-triggered controller has a dwell-time which is compatible with the hardware limitations. We investigate in this section how to employ the LMI conditions (3.8), (3.9) to maximize the guaranteed minimum inter-transmission time. We first state the following lemma to motivate our approach.

Lemma 3.1. *Let $\mathcal{S}$ be the set of solutions to system (3.3), (3.5). It holds that*

$$T = \inf_{\phi\in\mathcal{S}}\{t'-t : \exists j \in \mathbb{Z}_{>0},\ (t,j),(t,j+1),(t',j+1),(t',j+2) \in \operatorname{dom}\phi\}. \tag{3.21}$$

□

Proof of Lemma 3.1. Let $T^* := \inf_{\phi\in\mathcal{S}}\{t'-t : \exists j \in \mathbb{Z}_{>0},\ (t,j),(t,j+1),(t',j+1),(t',j+2) \in \operatorname{dom}\phi\}$. The definitions of the flow and jump sets in (3.5) guarantee that $T^* \geq T$. We now show that $T^* \leq T$. Let $\widetilde{\phi} = (\widetilde{\phi}_x, \widetilde{\phi}_e, \widetilde{\phi}_\tau) \in \mathcal{S}$ be such that $\widetilde{\phi}_x(0,0) = 0, \widetilde{\phi}_e(0,0) = 0, \widetilde{\phi}_\tau(0,0) = 0$. Then, $\widetilde{\phi}_x(t,j) = 0, \widetilde{\phi}_e(t,j) = 0$ for all $(t,j) \in \operatorname{dom}\widetilde{\phi}$, in view of (3.3). As a consequence, $\gamma^2|\widetilde{\phi}_e(t,j)|^2 = \sigma\varepsilon_1|\widetilde{\phi}_y(t,j)|^2$ where $\widetilde{\phi}_y(t,j) = \overline{C}_p\widetilde{\phi}_x(t,j)$ for all $(t,j) \in \operatorname{dom}\widetilde{\phi}$ and two successive jumps are separated by T units of time. We have that $T = \inf\{t'-t : \exists j \in \mathbb{Z}_{>0},\ (t,j),(t,j+1),(t',j+1),(t',j+2) \in \operatorname{dom}\widetilde{\phi}\} \geq T^*$. Consequently $T = T^*$. □

Lemma 3.1 implies that the lower bound T on the inter-transmission times guaranteed by (3.5) corresponds to the actual minimum inter-transmission time as defined by the right-hand side of (3.21). Hence, by maximizing T, we enlarge the minimum inter-transmission time.

To maximize T, we will maximize $\mathcal{T}(\gamma, L)$ in (3.6). We see that $\mathcal{T}$ increases as γ and L decrease. Hence, our objective is to minimize γ and L. Since γ corresponds to $\sqrt{\mu}$ and μ enters linearly in the LMI (3.8), we can directly minimize γ under the LMI (3.8), (3.9). The minimization of L, on the other hand, requires more attention. We recall that $L = |\mathcal{B}_2| = \sqrt{\lambda_{\max}(\mathcal{B}_2^T\mathcal{B}_2)}$, where

$$\mathcal{B}_2^T\mathcal{B}_2 = \begin{pmatrix} B_c^T C_c^T C_c B_c & 0 \\ 0 & B_p^T C_p^T C_p B_p \end{pmatrix} \tag{3.22}$$

hence,

$$L = \max\left(\sqrt{\lambda_{\max}(B_c^T C_c^T C_c B_c)}, \sqrt{\lambda_{\max}(B_p^T C_p^T C_p B_p)}\right). \tag{3.23}$$

Therefore, L can be minimized up to $\sqrt{\lambda_{\max}(B_p^T C_p^T C_p B_p)}$ which is fixed as it only depends on the plant matrices. In view of (3.10), we have that

$$B_c^T C_c^T C_c B_c = \boldsymbol{Z}^T V^{-T} U^{-1} \boldsymbol{N}^T \boldsymbol{N} U^{-T} V^{-1} \boldsymbol{Z}. \tag{3.24}$$

Thus, L depends nonlinearly on the LMI variables $\boldsymbol{N}$ and $\boldsymbol{Z}$ and it can a priori not be directly minimized. To overcome this issue, we impose the following upper bound

$$B_c^T C_c^T C_c B_c < \alpha\beta\mathbb{I}_{n_y} \tag{3.25}$$

for some $\alpha, \beta > 0$. As a result, minimizing α and β may help to minimize L as we will show on an example in Section 3.4. We translate inequality (3.25) into an LMI and we state the following claim.

Claim 3.1. *Assume that LMI (3.8), (3.9) are verified. Then, there exist $\boldsymbol{\alpha}, \boldsymbol{\beta} > 0$ such that*

$$\begin{pmatrix} \boldsymbol{\alpha}\mathbb{I}_{n_y} & \star & \star & \star \\ 0 & \boldsymbol{\beta}\mathbb{I}_{n_u} & \star & \star \\ 0 & \boldsymbol{N}^T & \boldsymbol{X} & \star \\ \boldsymbol{Z} & 0 & \mathbb{I}_{n_p} & \boldsymbol{Y} \end{pmatrix} > 0 \tag{3.26}$$

which implies that inequality (3.25) holds. ■

Proof of Claim 3.1. By using Schur complement of (3.26), we deduce that

$$\begin{pmatrix} \alpha\mathbb{I}_{n_y} - Z^T Y^{-1} Z & \star & \star \\ 0 & \beta\mathbb{I}_{n_u} & \star \\ -Y^{-1}Z & N^T & X - Y^{-1} \end{pmatrix} > 0. \tag{3.27}$$

Re-applying the Schur complement of the last inequality yields

$$X - Y^{-1} > 0$$

$$\begin{pmatrix} \alpha\mathbb{I}_{n_y} - Z^TY^{-1}Z - Z^TY^{-1}(X-Y^{-1})^{-1}Y^{-1}Z & \star \\ N(X-Y^{-1})^{-1}Y^{-1}Z & \beta\mathbb{I}_{n_u} - N(X-Y^{-1})^{-1}N^T \end{pmatrix} > 0. \tag{3.28}$$

Using the fact that

$$(Y - X^{-1})^{-1} = Y^{-1} + Y^{-1}(X-Y^{-1})^{-1}Y^{-1} \tag{3.29}$$

and since $(Y - X^{-1})^{-1} > 0$ and $(X - Y^{-1})^{-1} > 0$, in view of the Schur complement of (3.9), inequality (3.28) implies that

$$\begin{pmatrix} \alpha\mathbb{I}_{n_y} & \star \\ N(X-Y^{-1})^{-1}Y^{-1}Z & \beta\mathbb{I}_{n_u} \end{pmatrix} > 0. \tag{3.30}$$

It holds that

$$\begin{aligned} (X-Y^{-1})^{-1}Y^{-1} &= (Y(X-Y^{-1}))^{-1} = (YX - \mathbb{I}_{n_p})^{-1} \\ &= -(\mathbb{I}_{n_p} - YX)^{-1}. \end{aligned} \tag{3.31}$$

As a consequence

$$\begin{pmatrix} \alpha\mathbb{I}_{n_y} & \star \\ -N(\mathbb{I}_{n_p} - YX)^{-1}Z & \beta\mathbb{I}_{n_u}, \end{pmatrix} > 0 \tag{3.32}$$

which implies that

$$Z^T(\mathbb{I} - YX)^{-T}N^TN(\mathbb{I} - YX)^{-1}Z < \alpha\beta\mathbb{I}_{n_y}. \tag{3.33}$$

On the other hand, in view (3.10), we have

$$\begin{aligned} C_cB_c = NU^{-T}V^{-1}Z = N(UV^T)^{-T}Z \\ = N(\mathbb{I}_{n_p} - XY)^{-T}Z = N(\mathbb{I}_{n_p} - YX)^{-1}Z. \end{aligned} \tag{3.34}$$

As a result, in view of (3.33), (3.34), it holds that

$$B_c^TC_c^TC_cB_c < \alpha\beta\mathbb{I}_{n_y}. \tag{3.35}$$

Thus, Claim 3.1 is verified. ■

We note that (3.26) does not introduce additional constraints on system (3.3) compared to (3.8), (3.9). This comes from the fact that there always exist $\alpha, \beta > 0$ (eventually large) such that (3.26) holds, in view of Schur complement of (3.26).

In conclusion, we formulate the problem as a multiobjective optimization problem as we want to minimize μ, α, β under the constraint (3.8), (3.9) and (3.26). Several approaches have been proposed in the literature to handle such problems, see *e.g.* [26]. We choose the weighted sum strategy among others and we formulate the LMI optimization problem as follows

$$\begin{array}{c} \min \lambda_1\mu + \lambda_2\alpha + \lambda_3\beta \\ \text{subject to (3.8), (3.9), (3.26)} \end{array} \tag{3.36}$$

for some weights $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

3.3.2 Reducing the amount of transmissions

We present a heuristic to reduce the amount of transmissions generated by the triggering mechanism. This goal can be achieved by optimizing the parameters of the event-triggered rule such that the triggering condition is violated after the longest possible time since the last transmission. In view of (3.5) and Theorem 3.1, since $\gamma = \sqrt{\mu}, \varepsilon_1 = \varepsilon^{-1}$, the event-triggering condition is given by

$$\mu|e|^2 \leq \varepsilon^{-1}|y|^2 \text{ or } \tau \in [0, T]. \tag{3.37}$$

As a consequence, in order to reduce the number of instants at which the rule (3.37) is not satisfied, we need to minimize the parameters μ and ε. More precisely, we need to minimize the product $\varepsilon\mu$. Since the product $\varepsilon\mu$ is nonlinear, we simply minimize the weighted sum of the two parameters to maintain the convexity property. Moreover, we need to take into account the evolution of the e-variable. Indeed, it is not because $\varepsilon\mu$ is minimized that less transmissions will occur because the variable e may more rapidly reach the threshold in (3.37) in this case. To address this point, we notice that, in view of Assumption 1 and Proposition 1, the variable e satisfies, for all $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$

$$\langle \nabla|e|, \mathcal{A}_2 x + \mathcal{B}_2 e\rangle \leq L|e| + |\mathcal{A}_2 x|. \tag{3.38}$$

Thus, minimizing L may lead to the reduction of the rate of growth of the norm of the error. Indeed, the rate of growth of $|e|$ is also affected by the matrix $\mathcal{A}_2$. However, the handling of $\mathcal{A}_2$ does not seem tractable since it depends nonlinearly on all parameters of the controller, see(3.4), and we may investigate in a future work.

To summarize, the optimization problem below may be used to reduce the amount of transmissions

$$\begin{array}{c} \min \lambda_1\mu + \lambda_2\alpha + \lambda_3\beta + \lambda_4\varepsilon \\ \text{subject to (3.8), (3.9), (3.26)} \end{array} \tag{3.39}$$

for some weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

3.4 Illustrative example

We revisit Example 2 in [22] studied in Section 2.3.1.2 where the plant model is given by

$$\begin{aligned} \dot{x}_p &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 4 \end{bmatrix} x_p. \end{aligned} \tag{3.40}$$

First, we solve the optimization problem (3.36) to seek for the largest possible lower bound on the inter-transmission times. We set $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and we obtain

$$\begin{aligned} T &= 0.0114, & \mu &= 18433, & \varepsilon &= 2.7709 \times 10^6 \\ L &= 4.0586, & \alpha &= 4681.5, & \beta &= 4.6599 \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} A_c &= \begin{bmatrix} 1.0919 & -1.1422 \\ 4.9734 & -6.1425 \end{bmatrix}, \quad B_c = \begin{bmatrix} 16.7501 \\ 64.6472 \end{bmatrix}, \\ C_c &= \begin{bmatrix} 0.1157 & -0.0928 \end{bmatrix}. \end{aligned} \tag{3.42}$$

We note that, in view of (3.23), (3.40), (3.42), $L = \max\{4.0855, 4\} = 4.0855$. Table 3.1 gives the minimum and the average inter-sampling times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 25$ and $\tau(0,0) = 0$. We observe from the corresponding entries in Table 3.1 that $\tau_{\text{min}} = \tau_{\text{avg}}$ which implies that generated transmission instants are periodic. This may be explained by the fact that the product $\varepsilon\mu = 5.1075 \times 10^{10}$ is very big and thus the output-dependent part in (3.37) is 'quickly' violated. To avoid that phenomenon, we optimize the parameters of the event-triggering condition such that the rule is violated after the longest possible time since the last transmission instant, as discussed in Section 3.3.2. Thus, we minimize the weighted sum $\lambda_1\mu + \lambda_2\alpha + \lambda_3\beta + \lambda_4\varepsilon$ subject to (3.8), (3.9), (3.26). We take $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$ and we obtain

$$\begin{array}{lllllllll} T &=& 0.0113, & \mu &=& 18455, & \varepsilon &=& 28.6475 \\ L &=& 4.0624, & \alpha &=& 4687.7, & \beta &=& 4.6669 \end{array} \tag{3.43}$$

and the dynamic controller matrices are

$$\begin{aligned} A_c &= \begin{bmatrix} 1.0927 & -1.1423 \\ 4.9809 & -6.1477 \end{bmatrix}, \quad B_c = \begin{bmatrix} 16.7530 \\ 64.7121 \end{bmatrix}, \\ C_c &= \begin{bmatrix} 0.1158 & -0.0927 \end{bmatrix}. \end{aligned} \tag{3.44}$$

	Guaranteed dwell-time	τ_{min}	τ_{avg}
[22] $\sigma_1 = \sigma_2 = 10^{-3}$, $\varepsilon_1 = \varepsilon_1 = 10^{-3}$	6.5×10^{-9}	4.8055×10^{-6}	2.2905×10^{-4}
Optimization problem (3.36) $\lambda_1 = \lambda_2 = \lambda_3 = 1$	0.0114	0.0114	0.0114
Optimization problem (3.39) $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 1$	0.0113	0.0113	0.0116
Optimization problem (3.39) $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 10^4$	0.0109	0.0109	0.0261

TABLE 3.1: Minimum and average inter-transmission times for 100 randomly distributed initial conditions such that $|(x(0,0), e(0,0))| \leq 25$ and $\tau(0,0) = 0$ for a simulation time of 20s.

We note from the corresponding entries in Table 3.1 that the guaranteed dwell-time T is slightly smaller than the previous one but the average inter-transmission time τ_{avg} is larger than the previous value (in this case $\varepsilon\mu = 5.2869 \times 10^5$). Furthermore, we can play with the weight coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to further reduce transmissions. Since we know that L cannot become less than 4 and that the value obtained above is already close to this lower bound, we will give ε the most relative importance by increasing the weight λ_4 to further decrease the magnitude of

$\varepsilon\mu$. We found that the minimum value of $\varepsilon\mu = 8049$ is obtained with $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 10^4$ which yield

$$\begin{aligned} T &= 0.0109, & \mu &= 19856, & \varepsilon &= 0.4054 \\ L &= 4.3801, & \alpha &= 8757, & \beta &= 4418.3 \end{aligned} \tag{3.45}$$

and the dynamic controller matrices are

$$\begin{aligned} A_c &= \begin{bmatrix} 1.1684 & -1.1627 \\ 5.6744 & -6.6241 \end{bmatrix}, \quad B_c = \begin{bmatrix} 16.9843 \\ 70.3309 \end{bmatrix}, \\ C_c &= \begin{bmatrix} 0.1182 & -0.0908 \end{bmatrix}. \end{aligned} \tag{3.46}$$

We note that τ_{avg} is twice bigger than with the controller (3.43), (3.44) in this case and the guaranteed minimum inter-transmission time T is of the same order of magnitude compared to the previous values, as shown in Table 3.1. It can be noticed in Table 3.1 that, for all cases, the guaranteed lower bound T corresponds to the minimum inter-transmission time $\tau_{\min}$ generated by the triggering mechanism.

In comparison, the guaranteed lower bound on the inter-transmission times in [22] is 6.5×10^{-9} while the observed lower bound and the average inter-transmission time during the simulations respectively are 4.8055×10^{-6} and 2.2905×10^{-4}, as shown in Table 3.1. Moreover, the stability property achieved in [22] is a practical stability property, while we ensure a global asymptotic stability property. These observations justify the potential of the proposed co-design technique to reduce transmissions. In [72], the guaranteed and the the simulated lower bounds on the inter-transmission times are found to be the sampling period $h = 10^{-4}$, which is 100 times smaller than those we ensure.

We provide in Figures 3.1, 3.2 the state trajectories and the inter-transmission times for the initial condition $(x(0,0), e(0,0), \tau(0,0)) = (10, -10, 0, 0, 0, 0, 0)$. The impact of the time-triggered rule on the triggering instants is clearly shown in Figure 3.2 where a lower bound T is enforced on the inter-transmission times.

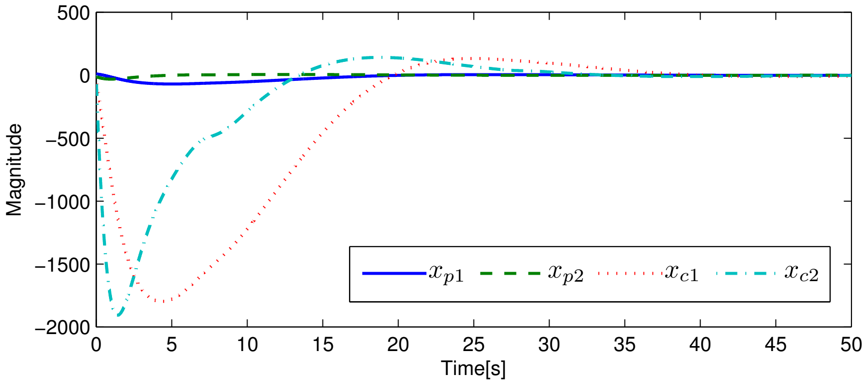

FIGURE 3.1: State trajectories of the plant and the controller.

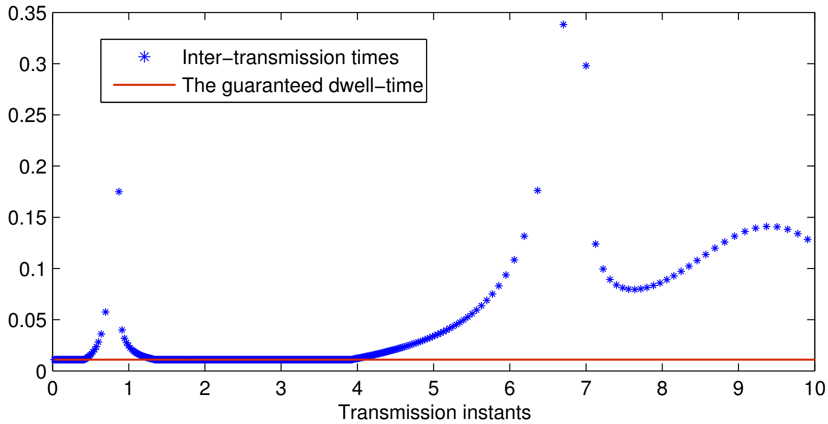

FIGURE 3.2: Inter-transmission times.

3.5 Conclusion

A co-design procedure of output feedback laws and event-triggering conditions for LTI systems has been presented. The proposed scheme guarantees a global asymptotic stability property for the closed-loop and enforces a strictly positive lower bound on the inter-transmission times. The required conditions have been formulated in terms of LMI. Then, the event-triggered controller and the flow and jump sets are synthesized by solving these LMI. Next, we took advantage of the flexibility of co-design to enhance the efficiency of the event-triggered controllers in two senses. We first demonstrated how the guaranteed lower bound on the inter-transmission times can be enlarged which can be useful in practice to help satisfying the hardware constraints. We then presented a heuristic to reduce the amount of transmissions, whose efficiency is confirmed by simulations.

Chapter 4

Singularly perturbed systems

4.1 Introduction

Many industrial control systems involve dynamical phenomena occurring in two separate time scales, known as *slow* and *fast* dynamics. These systems are usually referred to as singularly perturbed systems or two-time scale systems. Examples include motor control systems, convection-diffusion systems, power systems, magnetic-ball suspension systems, economic models, and many others. In this chapter, we design event-triggered controllers for nonlinear singularly perturbed systems. In particular, we focus on the scenario where the triggering condition is synthesized based only on the slow dynamics while we ignore the fast model, which is assumed to be stable.

It is well established in the literature that standard control methods cannot be directly applied to singularly perturbed systems since the two-time scale dynamical behaviour may lead to ill-conditioned controllers and/or closed-loop instability. To handle these issues, the control design and the stability analysis problems are usually addressed within the framework of singular perturbation, see [54], [52]. The basic idea in this framework is to reduce the complexity of the system through suitable approximations of the slow and fast dynamics by means of Tikhonov's theorem. In particular, if the approximate fast model is asymptotically stable, it is possible to design the controller based only on the approximate slow dynamics and to guarantee the stability of the overall system under certain conditions. This approach is often followed by engineers and the purpose of this chapter is to investigate whether it still applies in the context of ETC.

We design the event-triggered controllers by emulation like in Chapter 2 and we first develop a triggering mechanism to achieve a practical stability property. Then, we adapt the triggering mechanism synthesized in Chapter 2 to this context. Note that the results in Chapter 2 are no longer valid for this class of systems for the reasons mentioned above. We need to greatly revisit the stability analysis in order to ensure the desired asymptotic stability property. The results are shown to be applicable to a class of globally Lipschitz systems, which encompasses stabilizable LTI systems as a particular case.

4.2 Approximate models

We first recall the results of Chapter 11 in [52] for continuous-time systems to derive the approximate models. Consider the following nonlinear time-invariant singularly perturbed system

$$\dot{x} = f(x, z, u) \tag{4.1}$$

$$\epsilon\dot{z} = g(x, z, u) \tag{4.2}$$

$$u = k(x, z), \tag{4.3}$$

where $x \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_z}$ are the states, $u \in \mathbb{R}^{n_u}$ is the control input and $\epsilon > 0$ is a small parameter. We use singular perturbation theory to approximate the slow and the fast dynamics. We rely on the following standard assumption (see (11.3)-(11.4) in [52]).

Assumption 4.1. *The equation $g(x, z, u) = 0$ has $n \geq 1$ isolated real roots*

$$z = h_i(x, u), \quad i = 1, 2, ..., n \tag{4.4}$$

where h_i is continuously differentiable. □

In that way, the substitution of the ith-root $z = h(x, u)$ into (4.1) yields the corresponding approximate slow model

$$\dot{x} = f(x, h(x, u), u). \tag{4.5}$$

To investigate stability, it is more convenient to write system (4.1)-(4.2) with the coordinates (x, y) where

$$y := z - h(x, u) \tag{4.6}$$

is introduced to shift the quasi-steady-state of z to the origin. In the new coordinates (x, y), system (4.1) becomes

$$\dot{x} = f(x, y + h(x, u), u). \tag{4.7}$$

In view of (4.2), (4.6), we have that

$$\begin{aligned}\epsilon\dot{y} &= \epsilon\dot{z} - \epsilon\frac{d}{dt}h(x, u)\\ &= g(x, z, u) - \epsilon\Big(\frac{\partial h}{\partial x}f(x, z, u) + \frac{\partial h}{\partial u}\frac{\partial u}{\partial x}f(x, z, u) + \frac{1}{\epsilon}\frac{\partial h}{\partial u}\frac{\partial u}{\partial z}g(x, z, u)\Big)\\ &= (1 - \frac{\partial h}{\partial u}\frac{\partial u}{\partial z})g(x, z, u) - \epsilon\Big(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{\partial u}{\partial x}\Big)f(x, z, u).\end{aligned} \tag{4.8}$$

We introduce a new time variable $\tau = (t - t_0)/\epsilon$, then

$$\epsilon\frac{dy}{dt} = \frac{dy}{d\tau}.$$

In the new time scale τ, (4.8) is represented by

$$\frac{dy}{d\tau} = (1 - \frac{\partial h}{\partial u}\frac{\partial u}{\partial z})g(x, z, u) - \epsilon\Big(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{\partial u}{\partial x}\Big)f(x, z, u).$$

Then the fast dynamic is obtained by setting $\epsilon = 0$, see Chapter 11 in [52],

$$\frac{dy}{d\tau} = (1 - \frac{\partial h}{\partial u}\frac{\partial u}{\partial z})g(x, z, u). \tag{4.9}$$

The origin of system (4.7)-(4.8) is usually stabilized thanks to a controller of the form $u_s + u_f$, where $u_s = k_s(x)$ and $u_f = k_f(y)$ are respectively designed to stabilize the approximate models (4.5), (4.9). In that way, it is possible to ensure stability properties for system (4.7)-(4.8) under some conditions on the interconnection of system (4.7)-(4.8). In particular, when the origin is globally asymptotically stable for the fast dynamics (4.9), it is possible to take the controller to be $u = u_s$ in some cases, like for LTI systems (see Chapter 3 in [54]), some classes of nonlinear systems (in view of Chapter 11 in [52]), and LTI sampled-data systems (see [82]).

In this study, we want to know whether a similar approach is applicable in the context of ETC. Hence, we concentrate on the case where the approximate fast dynamics (4.9) is stable and we aim at designing the feedback law based only on the slow model (4.5).

4.3 Hybrid model

We follow an emulation-like approach as we first assume that the slow model (4.5) can be stabilized by a controller of the form $u = k(x)$. Afterwards, we take into account the communication constraints and we synthesize appropriate triggering conditions.

The controller receives the state measurements only at the transmission instants $t_i, i \in \mathbb{Z}_{\geq 0}$ and we consider zero-order-hold devices. In that way, we have that, for almost all $t \in [t_i, t_{i+1}]$,

$$u(t) = k(x(t_i)). \tag{4.10}$$

The sequence of transmission instants $t_i, i \in \mathbb{Z}_{\geq 0}$ is defined by the event-triggering condition we will design. We introduce the sampling-induced error $e \in \mathbb{R}^{n_x}$, for almost all $t \in [t_i, t_{i+1}]$

$$e(t) = x(t_i) - x(t), \tag{4.11}$$

which is reset to zero at each transmission instant. The state feedback controller (4.10) is therefore given by

$$u = k(x + e). \tag{4.12}$$

Hence, the slow model (4.5) becomes

$$\dot{x} = f\Big(x, h(x, k(x + e)), k(x + e)\Big) =: f_s(x, e) \tag{4.13}$$

and, in view of (4.6), the variable y is

$$y = z - h(x, k(x + e)). \tag{4.14}$$

We note that the variable y experiences a jump after each transmission as e is reset to zero at each $t_i, i \in \mathbb{Z}_{\geq 0}$. Consequently, system (4.7) is, for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \dot{x} &= f\Big(x, y + h(x, k(x + e)), k(x + e)\Big) \\ &=: f_x(x, y, e) \end{aligned} \tag{4.15}$$

and we have

$$x(t_{i+1}^+) = x(t_{i+1}). \tag{4.16}$$

On the other hand, we obtain from (4.14), for almost all $t \in [t_i, t_{i+1}]$

$$\begin{aligned}\epsilon\dot{y} &= \epsilon\dot{z} - \epsilon\frac{d}{dt}h(x,u)\\ &= g(x,z,u) - \epsilon\Big(\frac{\partial h}{\partial x}f_x(x,y,e) + \frac{\partial h}{\partial u}\frac{\partial u}{\partial x}f_x(x,y,e) - \frac{\partial h}{\partial u}\frac{\partial u}{\partial e}f_x(x,y,e)\Big),\end{aligned} \tag{4.17}$$

where $\dot{e}_x = -\dot{x} = -f_x(x,y,e)$ by (4.11), then

$$\epsilon\dot{y} = g\Big(x, y + h(x,k(x+e)), k(x+e)\Big) - \epsilon\frac{\partial h}{\partial x}f_x(x,y,e) \tag{4.18}$$

$$=: f_y(x,y,e), \tag{4.19}$$

then we obtain the fast model by setting $\epsilon = 0$

$$\begin{aligned}\frac{dy}{d\tau} &= g\Big(x, y + h(x,k(x+e)), k(x+e)\Big)\\ &=: g_f(x,y,e).\end{aligned} \tag{4.20}$$

and we have

$$\begin{aligned}y(t_{i+1}^+) &= z(t_{i+1}^+) - h\Big(x(t_{i+1}^+), k(x(t_{i+1}^+) + e(t_{i+1}^+))\Big)\\ &= z(t_{i+1}) - h\Big(x(t_{i+1}), k(x(t_{i+1}) + 0)\Big)\\ &= y(t_{i+1}) + h\Big(x(t_{i+1}), k(x(t_{i+1}) + e(t_{i+1}))\Big) - h\Big(x(t_{i+1}), k(x(t_{i+1}))\Big)\\ &=: h_y(x(t_{i+1}), y(t_{i+1}), e(t_{i+1})).\end{aligned} \tag{4.21}$$

We note that the state variable y experiences a jump at each transmission which is an important difference with the model presented in Section 2.1.

Let $q = (x, y, e, \tau) \in \mathbb{R}^{n_q}$, where $n_q = 2n_x + n_y + 1$ and $\tau \in \mathbb{R}_{\geq 0}$ is a clock variable which describes the time elapsed since the last jump as in (2.3). In view of (4.15)-(4.21), the system is modeled as follows

$$\begin{aligned}\dot{q} &= F(q) \quad q \in C\\ q^+ &= G(q) \quad q \in D,\end{aligned} \tag{4.22}$$

where

$$F(q) := \begin{pmatrix} f_x(x,y,e) \\ \frac{1}{\epsilon} f_y(x,y,e) \\ -f_x(x,y,e) \\ 1 \end{pmatrix}, \quad G(q) := \begin{pmatrix} x \\ h_y(x,y,e) \\ 0 \\ 0 \end{pmatrix}. \tag{4.23}$$

The flow and the jump maps are assumed to be continuous and the sets C and D will be closed.

Our objective is to design the flow and jump sets (4.23) based only on the approximate model of the slow dynamics and such that the overall stability of system (4.22) is guaranteed and the existence of a strictly positive amount of time between two jumps is ensured.

4.4 Assumptions

We present the assumptions made on system (4.22). First, we assume that the slow system (4.13) is input-to-state stable (ISS) with respect to e.

Assumption 4.2. *There exist a continuously differentiable function $V_x : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_x, \overline{\alpha}_x, \gamma_1$ with γ_1 continuously differentiable and $\alpha_1 > 0$ such that for all $(x,e) \in \mathbb{R}^{2n_x}$ the following is satisfied*

$$\begin{aligned} \underline{\alpha}_x(|x|) &\leq V_x(x) \leq \overline{\alpha}_x(|x|) \\ \frac{\partial V_x}{\partial x} f_s(x,e) &\leq -\alpha_1 V_x(x) + \gamma_1(|e|). \end{aligned} \tag{4.24}$$

□

To guarantee the overall stability of the closed-loop system, we need to make some assumptions on the stability of the fast model (4.20) in the presence of the communication constraints of the slow system. In particular, we assume that the following stability property holds for the fast dynamics like in [52].

Assumption 4.3. *There exist a continuously differentiable function $V_y : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}_y, \overline{\alpha}_y$ and $\alpha_2 > 0$ such that for all $(x,y,e) \in \mathbb{R}^{2n_x+n_y}$*

$$\begin{aligned} \underline{\alpha}_y(|y|) &\leq V_y(x,y) \leq \overline{\alpha}_y(|y|) \\ \frac{\partial V_y}{\partial y} g_f(x,y,e) &\leq -\alpha_2 V_y(x,y). \end{aligned} \tag{4.25}$$

□

Assumption 4.3 implies that the origin of the fast dynamics (4.20) is globally asymptotically stable. Note that Assumption 4.3 does not imply that the origin of the fast dynamics (4.20) is globally exponentially stable as the functions $\underline{\alpha}_y, \overline{\alpha}_y$ can be nonlinear. We impose the following conditions on the interconnections between the slow and fast dynamics (4.13), (4.20).

Assumption 4.4. *There exist a class $\mathcal{K}_\infty$ function γ_2 and $\beta_1, \beta_2, \beta_3 > 0$ such that for all $(x, y, e) \in \mathbb{R}^{2n_x+n_y}$ the following hold*

$$\begin{array}{rcl} \frac{\partial V_x}{\partial x}\left[f_x(x,y,e) - f_s(x,e)\right] & \leq & \beta_1\sqrt{V_x(x)V_y(x,y)} \\ \left[\frac{\partial V_y}{\partial x} - \frac{\partial V_y}{\partial y}\frac{\partial h}{\partial x}\right] f_x(x,y,e) & \leq & \beta_2\sqrt{V_x(x)V_y(x,y)} + \beta_3 V_y(x,y) + \gamma_2(|e|), \end{array} \tag{4.26}$$

where V_x and V_y come from Assumptions 4.2 and 4.3 respectively. In addition, there exists $L > 0$ such that, for all $s \geq 0$

$$\gamma_2 \circ \gamma_1^{-1}(s) \leq Ls, \tag{4.27}$$

where γ_1 comes from Assumption 4.2. □

Conditions (4.26) represent the effect of the deviation of the original system (4.22) from the slow and fast models (4.13), (4.20) respectively and are related to (11.43) and (11.44) in [52].

Finally, we assume that the dynamics of V_y along jumps of system (4.22) satisfies the following condition.

Assumption 4.5. *There exist $\lambda_1, \lambda_2 > 0$ such that for all $(x, y, e) \in \mathbb{R}^{2n_x+n_y}$*

$$V_y(x, h_y(x,y,e)) \leq V_y(x,y) + \lambda_1\gamma_1(|e|) + \lambda_2\sqrt{\gamma_1(|e|)V_y(x,y)}, \tag{4.28}$$

where V_x, γ_1 and V_y come from Assumptions 4.2 and 4.3 respectively. □

Assumption 4.5 is an algebraic condition which only requires the knowledge of h_y (which is defined in (4.21)) and γ_1 and V_y from Assumptions 4.2 and 4.3 respectively: we do not need to know the triggering condition to check it.

Remark 4.1. *Assumptions 4.3, 4.4 require (4.25), (4.26) to hold regardless the magnitude of the sampling-induced error e. We show in Section 4.6 that all these conditions are satisfied by a class of globally Lipschitz systems which encompasses LTI systems as a particular case.* □

4.5 Main results

Before presenting the main results, we show that the design of triggering conditions of the same form as in [103] for the slow model may not ensure the existence of a strictly positive minimum amount of time between two jumps for the overall system.

4.5.1 A first observation

We have seen in Section 1.2.5 that the triggering mechanism in [103] cannot be directly extended to the output feedback case since the Zeno phenomenon will occur. A similar situation is encountered here since we aim to ignore the fast state and to synthesize the triggering condition based only on the approximate slow model. To be more precise, in view of Assumption 4.2, a first attempt would be to define a triggering condition of the form $\gamma_1(|e|) \geq \sigma\alpha_1 V_x(x)$ where $\sigma \in (0,1)$ like in [103]. The flow and jump sets are in this case

$$\begin{aligned} C &= \{q : \gamma_1(|e|) \leq \sigma\alpha_1 V_x(x)\} \\ D &= \{q : \gamma_1(|e|) = \sigma\alpha_1 V_x(x)\}. \end{aligned} \tag{4.29}$$

The results in [103] guarantee a global asymptotic stability property for the origin of the slow model (4.13) and the existence of a uniform (semiglobal) amount of time between two jumps (under some conditions). However, this triggering rule no longer ensures a minimum time of flow between two jumps for system (4.22). Indeed $G(D) \cap D = \{q : x = e = 0\} \neq \emptyset$. Thus, any solution in $G(D) \cap D$ may jump an infinite number of times, which makes the controller not implementable in practice. In the sequel, we first apply existing strategies in order to overcome this issue and we investigate how to modify the stability analysis and what kind of stability property one may expect. We also propose another strategy that allows to guarantee a global asymptotic stability property.

4.5.2 Semiglobal practical stabilization

The most straightforward approach to enforce a lower bound on the inter-jumps for system (4.22) is to add a dead-zone to the triggering condition (4.29), *i.e.*

$$\gamma_1(|e|) \geq \max\{\sigma\alpha_1 V_x(x), \rho\}, \tag{4.30}$$

where $\rho > 0$ is a design parameter. The flow and jump sets in (4.22) are then

$$C = \{q : \gamma_1(|e|) \leq \max\{\sigma\alpha_1 V_x(x), \rho\}\}$$
$$D = \{q : \gamma_1(|e|) = \max\{\sigma\alpha_1 V_x(x), \rho\}\} \tag{4.31}$$

with $q = (x, y, e)$. Although this type of triggering conditions has already been used in [22], [69] for example, the fact that the state y experiences jumps and that we rely on different assumptions require to fully modify the stability analysis and leads to the following result.

Theorem 4.1. *Consider system (4.22) with the flow and jump sets defined in (4.31). Suppose that Assumptions 4.1-4.5 hold. Then, for any $\Delta, \rho > 0$, there exist $\beta \in \mathcal{KL}$, $\kappa \in \mathcal{K}_\infty$ and $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$ and any solution $\phi = (\phi_x, \phi_y, \phi_e)$ with $|\phi(0,0)| \leq \Delta$,*

$$|\phi(t,j)| \leq \beta(|\phi(0,0)|, t + j) + \kappa(\rho) \quad \forall (t,j) \in \text{dom}\, \phi, \tag{4.32}$$

and all inter-transmission times are lower-bounded by a strictly positive constant $\frac{\rho}{\xi(\Delta)}$, where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a continuous increasing function, i.e. for all $j \in \mathbb{Z}_{\geq 0}$ $\sup I^j - \inf I^j \geq \frac{\rho}{\xi(\Delta)}$, where $I^j = \{t : (t,j) \in \text{dom}\, \phi\}$. Furthermore, all maximal solutions to (4.22) are complete. □

The proof of Theorem 4.1 is provided in Appendix A. Theorem 4.1 ensures a semiglobal practical stability property for system (4.22). Indeed, given an arbitrary (large) ball of initial conditions centered at the origin and of radius Δ and any constant ρ, there exists ϵ sufficiently small such that solutions to (4.22), (4.31) converge towards a neighbourhood of the origin whose 'size' can be rendered arbitrarily small by reducing ρ (at the price of shorter inter-transmission intervals, typically).

4.5.3 Global asymptotic stabilization

We propose another strategy to design the event-triggering condition to ensure a global asymptotic stability property under an extra assumption. We borrow the idea presented in Chapter 2 to combine the event-triggered technique of [103] with the time-triggered results of [79] such that we allow transmissions only after a fixed amount of time T^* has elapsed since the last one.

We suppose that Assumptions 4.1-4.5 are satisfied with $\gamma_1(s) = \bar{\gamma}_1 s^2$ and $\gamma_2(s) = \bar{\gamma}_2 s^2$ for some $\bar{\gamma}_1, \bar{\gamma}_2 \geq 0$ and for $s \geq 0$. We define the flow and jump sets as follows

$$C := \{q : \bar{\gamma}_1|e|^2 \leq \sigma\alpha_1 V_x(x) \text{ or } \tau \in [0, T^*]\}$$
$$D := \left\{q : \left(\bar{\gamma}_1|e|^2 = \sigma\alpha_1 V_x(x) \text{ and } \tau \geq T^*\right) \text{ or } \left(\bar{\gamma}_1|e|^2 \geq \sigma\alpha_1 V_x(x) \text{ and } \tau = T^*\right)\right\}. \tag{4.33}$$

Inspired by [79], we make the following additional assumption on system (4.22).

Assumption 4.6. *There exist $M, N \geq 0$ such that, for all $(x, y) \in \mathbb{R}^{n_x+n_y}$ and for almost all $e \in \mathbb{R}^{n_x}$*

$$\langle \nabla|e|, -f_x(x, y, e)\rangle \leq M|e| + N(\sqrt{V_x(x)} + \sqrt{V_y(x, y)}),$$

where V_x and V_y come from Assumptions 4.2 and 4.3 respectively. □

The constant T^* in (4.33) is selected such that $T^* < \mathcal{T}(\alpha_1, \bar{\gamma}_1, M, N)$, like in [79], where

$$\mathcal{T}(\alpha_1, \bar{\gamma}_1, M, N) := \begin{cases} \frac{1}{Mr}\arctan(r) & M^2 < \frac{\bar{\gamma}_1 N^2}{\alpha_1} \\ \frac{1}{M} & M^2 = \frac{\bar{\gamma}_1 N^2}{\alpha_1} \\ \frac{1}{Mr}\operatorname{arctanh}(r) & M^2 > \frac{\bar{\gamma}_1 N^2}{\alpha_1} \end{cases} \tag{4.34}$$

with

$$r := \sqrt{\left|\frac{\bar{\gamma}_1 N^2}{\alpha_1 M^2} - 1\right|}, \tag{4.35}$$

where M, N come from Assumption 4.6 and $\alpha_1, \bar{\gamma}_1$ come from Assumption 4.2. We obtain the following result.

Theorem 4.2. *Consider system (4.22) with the flow and jump sets defined in (4.33) and suppose the following hold.*

1. *Assumptions 4.1-4.6 hold with $\gamma_1(s) = \bar{\gamma}_1 s^2$ and $\gamma_2(s) = \bar{\gamma}_2 s^2$ with $\bar{\gamma}_1, \bar{\gamma}_2 \geq 0$, for $s \geq 0$.*
2. *The constant T^* in (4.33) is such that $T^* \in (0, \mathcal{T})$.*

Then there exist $\beta \in \mathcal{KL}$ and $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon})$ and any solution $\phi = (\phi_x, \phi_y, \phi_e, \phi_\tau)$

$$|(\phi_x(t, j), \phi_y(t, j))| \leq \beta(|\phi(0, 0)|, t + j) \quad \forall (t, j) \in \operatorname{dom}\phi. \tag{4.36}$$

Moreover, all maximal solutions to (4.22) are complete. □

The proof of Theorem 4.2 is given in Appendix A. We see that Theorem 4.2 ensures a global asymptotic stability property and that it requires an additional condition to hold, namely Assumption 4.6, compared to Section 4.5.2.

4.6 Case studies

4.6.1 A class of globally Lipschitz systems

In this section, we show that all the conditions of Section 4.2 are verified by a class of globally Lipschitz systems, which includes LTI systems as a particular case. We assume that the vector fields f, g and k are globally Lipschitz and that the stability of the slow and the fast model can be verified using quadratic functions V_x and V_y. Under these conditions, the proposition below states that Assumptions 4.1-4.6 hold. Hence, the triggering rules presented in Sections 4.5.2 and 4.5.3 can be applied.

Proposition 2. *Consider system (4.1)-(4.2), (4.12). Suppose the following hold.*

1. *Assumption 4.1 is satisfied and there exists $L_h > 0$ such that*

$$|h(x_1, e_1) - h(x_2, e_2)| \quad \leq \quad L_h(|x_1 - x_2| + |e_1 - e_2|). \tag{4.37}$$

2. *There exist $L_f, L_s > 0$ such that the functions $f(x, y, e)$ in (4.15), (4.13) verifies that*

$$\begin{aligned} |f(x_1, y_1, e_1) - f(x_2, y_2, e_2)| &\leq L_f(|x_1 - x_2| + |y_1 - y_2| + |e_1 - e_2|) \\ |f_s(x, e) - f_s(x, 0)| &\leq L_s|e|. \end{aligned} \tag{4.38}$$

3. *There exist positive definite and symmetric real matrices P_1, P_2 such that the functions $V_x : x \mapsto x^T P_1 x$ and $V_y : y \mapsto y^T P_2 y$ satisfy, for all $(x, y, e) \in \mathbb{R}^{2n_x + n_y}$*

$$\frac{\partial V_x}{\partial x} f_s(x, 0) \leq -\bar{\alpha}_1 V_x(x) \tag{4.39}$$

$$\frac{\partial V_y}{\partial y} g_f(x, y, e) \leq -\bar{\alpha}_2 V_y(x, y), \tag{4.40}$$

where $\bar{\alpha}_1, \bar{\alpha}_2 > 0$. Then, Assumptions 4.2-4.6 are satisfied with

$$\begin{array}{rclrcl}
\alpha_1 & = & \frac{\bar{\alpha}_1}{2}, & \gamma_1(|e|) & = & \frac{2L_s^2|P_1|^2}{\bar{\alpha}_1\lambda_{\min}(P_1)}|e|^2 \\
\alpha_2 & = & \bar{\alpha}_2, & \beta_1 & = & \frac{2L_f|P_1|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}} \\
\beta_2 & = & \frac{2L_hL_f|P_2|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}, & \beta_3 & = & \frac{3L_hL_f|P_2|}{\lambda_{\min}(P_2)} \\
\gamma_2(|e|) & = & L_hL_f|P_2||e|^2, & L & = & \frac{\bar{\alpha}_1L_hL_f\lambda_{\min}(P_1)|P_2|}{2L_s^2|P_1|^2} \\
\lambda_1 & = & \frac{\bar{\alpha}_1L_h^2\lambda_{\min}(P_1)|P_2|}{2L_s^2|P_1|^2}, & \lambda_2 & = & \sqrt{\frac{\bar{\alpha}_1L_h^2\lambda_{\min}(P_1)|P_2|^2}{L_s^2\lambda_{\min}(P_2)|P_1|^2}} \\
M & = & L_f, & N & = & \frac{L_f}{\min\{\sqrt{\lambda_{\min}(P_1)},\sqrt{\lambda_{\min}(P_2)}\}}.
\end{array} \tag{4.41}$$

□

Proof of Proposition 2.

- Assumption 4.2: In view of (4.13) and item (3) of Proposition 2, we have, for all $(x,e) \in \mathbb{R}^{2n_x}$,

$$\begin{array}{rcl}
\frac{\partial V_x}{\partial x}f_s(x,e) & = & \frac{\partial V_x}{\partial x}f_s(x,0) + \frac{\partial V_x}{\partial x}(f_s(x,e) - f_s(x,0)) \\
& \leq & -\bar{\alpha}_1V_x(x) + 2x^TP_1(f_s(x,e) - f_s(x,0)) \\
& \leq & -\bar{\alpha}_1V_x(x) + 2|x||P_1||f_s(x,e) - f_s(x,0)|.
\end{array} \tag{4.42}$$

  As a consequence, in view of item (1) in Proposition 2,

$$\frac{\partial V_x}{\partial x}f_s(x,e) \leq -\bar{\alpha}_1V_x(x) + 2L_s|P_1||x||e|. \tag{4.43}$$

  Using the fact that

$$2L_s|P_1||x||e| \leq \frac{\bar{\alpha}_1\lambda_{\min}(P_1)}{2}|x|^2 + \frac{2}{\bar{\alpha}_1\lambda_{\min}(P_1)}L_s^2|P_1|^2|e|^2 \tag{4.44}$$

  and the fact that $\lambda_{\min}(P_1)|x|^2 \leq V_x(x)$, since P_1 is positive definite and symmetric, it holds that

$$\begin{aligned}
\frac{\partial V_x}{\partial x} f_s(x,e) &\leq -\bar{\alpha}_1 V_x(x) + \frac{\bar{\alpha}_1 \lambda_{\min}(P_1)}{2}|x|^2 + \frac{2L_s^2|P_1|^2}{\bar{\alpha}_1 \lambda_{\min}(P_1)}|e|^2 \\
&\leq -\bar{\alpha}_1 V_x(x) + \frac{\bar{\alpha}_1}{2} V_x(x) + \frac{2L_s^2|P_1|^2}{\bar{\alpha}_1 \lambda_{\min}(P_1)}|e|^2 \\
&\leq -\frac{\bar{\alpha}_1}{2} V_x(x) + \frac{2L_s^2|P_1|^2}{\bar{\alpha}_1 \lambda_{\min}(P_1)}|e|^2.
\end{aligned}$$

Hence Assumption 4.2 holds with $\alpha_1 = \frac{\bar{\alpha}_1}{2}$ and $\gamma_1(|e|) = \frac{2L_s^2|P_1|^2}{\bar{\alpha}_1 \lambda_{\min}(P_1)}|e|^2$.

- Assumption 4.3 follows directly from item (3) of Proposition 2 with $\alpha_2 = \bar{\alpha}_2$.
- Assumption 4.4: In view of items (1), (3) of Proposition 2 and since $\lambda_{\min}(P_1)|x|^2 \leq V_x(x)$ and $\lambda_{\min}(P_2)|y|^2 \leq V_y(x,y)$, it holds that, for all $(x,y,e) \in \mathbb{R}^{2n_x+n_y}$

$$\begin{aligned}
\frac{\partial V_x}{\partial x}\left[f_x(x,y,e) - f_s(x,e)\right] &= 2x^T P_1(f_x(x,y,e) - f_s(x,e)) \\
&\leq 2|P_1||x||f_x(x,y,e) - f_s(x,e)| \\
&\leq 2L_f|P_1||x||y| \\
&= \frac{2L_f|P_1|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}\sqrt{\lambda_{\min}(P_1)|x|^2}\sqrt{\lambda_{\min}(P_2)|y|^2} \\
&\leq \frac{2L_f|P_1|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}\sqrt{V_x(x)V_y(x,y)}.
\end{aligned} \tag{4.45}$$

Thus, the first condition in Assumption 4.4 is verified with $\beta_1 = \frac{2L_f|P_1|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}$.

On the other hand, in view of items (1), (3) of Proposition 2 and using the fact that $2|e||y| \leq |y|^2 + |e|^2$ and using that $f_x(0,0,0) = 0$ since the origin of system (4.13) is asymptotically stable in view of (4.39), it holds that, for all $(x,y,e) \in \mathbb{R}^{2n_x+n_y}$

$$\begin{aligned}
\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_y}{\partial y}\frac{\partial h}{\partial x}\right] f_x(x,y,e) &\leq 2|P_2||y||\frac{\partial h}{\partial x}||f_x(x,y,e)| \\
&\leq 2|P_2||y|L_h L_f(|x| + |y| + |e|) \\
&= 2|P_2|L_h L_f|x||y| + 2|P_2|L_h L_f|y|^2 + 2|P_2|L_h L_f|y||e| \\
&\leq 2|P_2|L_h L_f|x||y| + 2|P_2|L_h L_f|y|^2 + |P_2|L_h L_f(|y|^2 + |e|^2)
\end{aligned} \tag{4.46}$$

$$\begin{aligned}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_y}{\partial y}\frac{\partial h}{\partial x}\right] f_x(x,y,e) &\le \frac{2|P_2|L_h L_f}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}\sqrt{V_x(x)V_y(x,y)} \\ &+ \frac{3|P_2|L_h L_f}{\lambda_{\min}(P_2)} V_y(x,y) + |P_2|L_h L_f|e|^2.\end{aligned} \tag{4.47}$$

Hence, the second condition in Assumption 4.4 holds with $\beta_2 = \frac{2|P_2|L_h L_f}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}$, $\beta_3 = \frac{3|P_2|L_h L_f}{\lambda_{\min}(P_2)}$ and $\gamma_2(|e|) = |P_2|L_h L_f|e|^2$. Since $\gamma_1(|e|) = \frac{2L_s^2|P_1|^2}{\bar{\alpha}_1\lambda_{\min}(P_1)}|e|^2$, it holds that

$$\gamma_1^{-1}(|e|) = \frac{\sqrt{\bar{\alpha}_1\lambda_{\min}(P_1)|e|}}{\sqrt{2L_s^2|P_1|^2}}. \tag{4.48}$$

As a consequence,

$$\begin{aligned}\gamma_2 \circ \gamma_1^{-1}(|e|) &= |P_2|L_h L_f \frac{\bar{\alpha}_1\lambda_{\min}(P_1)|e|}{2L_s^2|P_1|^2} \\ &= \frac{\bar{\alpha}_1\lambda_{\min}(P_1)L_h L_f|P_2|}{2L_s^2|P_1|^2}|e|\end{aligned} \tag{4.49}$$

Then, the third condition in Assumption 4.4 is satisfied with $L = \frac{\bar{\alpha}_1\lambda_{\min}(P_1)L_h L_f|P_2|}{2L_s^2|P_1|^2}$.

- Assumption 4.5: In view of (4.21) and the definition of V_y,

$$\begin{aligned}&V_y(x,h_y(x,y,e)) = h_y^T(x,y,e)P_2 h_y(x,y,e) \\ &= \Big(y + h(x,k(x+e)) - h(x,k(x))\Big)^T P_2\Big(y + h(x,k(x+e)) - h(x,k(x))\Big) \\ &= y^T P_2 y + (h(x,k(x+e)) - h(x,k(x)))^T P_2(h(x,k(x+e)) - h(x,k(x))) \\ &\quad + y^T P_2(h(x,k(x+e)) - h(x,k(x))) + (h(x,k(x+e)) - h(x,k(x)))^T P_2 y \\ &\le V_y(x,y) + |P_2||h(x,k(x+e)) - h(x,k(x))|^2 \\ &\quad + 2|P_2||y||h(x,k(x+e)) - h(x,k(x))|.\end{aligned} \tag{4.50}$$

Since h is globally Lipschitz, it holds that

$$\begin{aligned}V_y(x,h_y(x,y,e)) &\le V_y(x,y) + |P_2|L_h^2|e|^2 + 2|P_2||y|L_h|e| \\ &\le V_y(x,y) + |P_2|L_h^2\frac{\bar{\alpha}_1\lambda_{\min}(P_1)}{2L_s^2|P_1|^2}\gamma_1(|e|) \\ &\quad + 2|P_2|L_h\frac{\sqrt{\bar{\alpha}_1\lambda_{\min}(P_1)}}{\sqrt{2L_s^2|P_1|^2}}\frac{1}{\sqrt{\lambda_{\min}(P_2)}}\sqrt{\gamma_1(|e|)V_y(x,y)}\end{aligned} \tag{4.51}$$

$$V_y(x, h_y(x, y, e)) \leq V_y(x, y) + \frac{\bar{\alpha}_1 L_h^2 \lambda_{\min}(P_1)|P_2|}{2L_s^2|P_1|^2}\gamma_1(|e|) + \frac{\sqrt{2L_h^2\bar{\alpha}_1\lambda_{\min}(P_1)|P_2|^2}}{\sqrt{\lambda_{\min}(P_2)L_s^2|P_1|^2}}\sqrt{\gamma_1(|e|)V_y(x, y)}. \tag{4.52}$$

Consequently, Assumption 4.5 is satisfied with $\lambda_1 = \frac{\bar{\alpha}_1 L_h^2 \lambda_{\min}(P_1)|P_2|}{2L_s^2|P_1|^2}$ and $\lambda_2 = \frac{\sqrt{2L_h^2\bar{\alpha}_1\lambda_{\min}(P_1)|P_2|^2}}{\sqrt{\lambda_{\min}(P_2)L_s^2|P_1|^2}}\sqrt{\gamma_1(|e|)V_y(x, y)}$.

- Assumption 4.6: In view of (4.22)-(4.23) and item (1) of Proposition 2 and using that $f_x(0, 0, 0) = 0$, it holds that, for all $(x, y) \in \mathbb{R}^{n_x+n_y}$ and for almost all $e \in \mathbb{R}^{n_x}$

$$\begin{aligned}\langle \nabla|e|, -f_x(x, y, e)\rangle &\leq L_f(|x| + |y| + |e|) \\ &= L_f|e| + L_f(\frac{1}{\sqrt{\lambda_{\min}(P_1)}}\sqrt{V_x(x)} + \frac{1}{\sqrt{\lambda_{\min}(P_2)}}\sqrt{V_y(x, y)}).\end{aligned} \tag{4.53}$$

Hence, Assumption 4.6 is verified with $M = L_f$ and $N = \frac{L_f}{\min\{\sqrt{\lambda_{\min}(P_1)}, \sqrt{\lambda_{\min}(P_2)}\}}$. □

4.6.2 Application to LTI systems

The results in Section 4.6.1 can be directly applied to LTI systems. However, we can obtain for this class of systems less conservative values for the parameters in Assumptions 4.2-4.6 than those derived in Section 4.6.1. We first derive the approximate models as in Section 4.3 then we state the result.

Consider the LTI singularly perturbed systems

$$\dot{x} = A_{11}x + A_{12}z + B_1u = f(x, z, u) \tag{4.54}$$

$$\epsilon\dot{z} = A_{21}x + A_{22}z + B_2u = g(x, z, u) \tag{4.55}$$

$$u = Kx \tag{4.56}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $u \in \mathbb{R}^u$ and $\epsilon > 0$. We assume that A_{22} is invertible and Hurwitz. Hence, Assumption 4.1 holds with

$$h(x, u) = -A_{22}^{-1}(A_{21}x + B_2u). \tag{4.57}$$

By introducing the sampling error e and applying the change of variables $y = z - h(x, u)$, the

x-system (4.54) becomes

$$\begin{aligned}\dot{x} &= A_{11}x + A_{12}(y - A_{22}^{-1}A_{21}x - A_{22}^{-1}B_2u) + B_1u \\ &= (A_{11} - A_{12}A_{22}^{-1}A_{11})x + A_{12}y + (B_1 - A_{12}A_{22}^{-1}B_2)K(x+e) \\ &= (A_0 + B_0K)x + A_{12}y + B_0Ke \\ &= f_x(x,y,e),\end{aligned} \tag{4.58}$$

where

$$\begin{aligned}A_0 &:= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ B_0 &:= B_1 - A_{12}A_{22}^{-1}B_2.\end{aligned} \tag{4.59}$$

Let $\Lambda := A_0 + B_0K$ and by setting $y = 0$, we obtain the approximate slow model

$$\begin{aligned}\dot{x} &= \Lambda x + B_0Ke \\ &= f_s(x,e).\end{aligned} \tag{4.60}$$

Assuming that the pair (A_0, B_0) is stabilizable, we take K such that Λ is Hurwitz. By following similar lines as in (4.18), the y-system becomes

$$\begin{aligned}\epsilon\dot{y} &= \epsilon\dot{z} - \epsilon\tfrac{\partial h}{\partial x}\dot{x} \\ &= A_{21}x + A_{22}(y - A_{22}^{-1}A_{21}x - A_{22}^{-1}B_2u) + B_2u + \epsilon A_{22}^{-1}A_{21}(\Lambda x + A_{12}y + B_0Ke) \\ &= A_{22}y + \epsilon A_{22}^{-1}A_{21}(\Lambda x + A_{12}y + B_0Ke) \\ &= f_y(x,y,e)\end{aligned} \tag{4.61}$$

By introducing the time variable $\tau = \frac{t}{\epsilon}$ and yb setting $\epsilon = 0$, We derive the approximate fast model

$$\frac{dy}{d\tau} = A_{22}y = g_f(x,y,e). \tag{4.62}$$

Hence, the hybrid model of (4.22) is, recall that $q = (x,y,e,\tau) \in \mathbb{R}^{n_q}$,

$$\begin{aligned}\dot{q} &= F(q) \quad q \in C \\ q^+ &= G(q) \quad q \in D,\end{aligned} \tag{4.63}$$

and the flow and jump maps are given by

$$F(q) = \begin{pmatrix} \Lambda x + A_{12}y + B_0 K e \\ \Gamma\Lambda x + (\Gamma A_{12} + \frac{1}{\epsilon}A_{22})y + \Gamma B_0 K e \\ -\Lambda x - A_{12}y - B_0 K e \\ 1 \end{pmatrix}, \qquad G(q) = \begin{pmatrix} x \\ y - \Gamma_1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.64}$$

where $\Gamma := A_{22}^{-1}A_{21}$ and $\Gamma_1 := A_{22}^{-1}B_2 K$.

Proposition 3. *Consider system (4.54)-(4.56). Suppose that A_{22} is invertible and Hurwitz and the pair (A_0, B_0) is stabilizable. Let P_1, P_2 be real positive definite and symmetric matrices such that $\Lambda^T P_1 + P_1\Lambda = -\mathbb{I}_n$ and $A_{22}^T P_2 + P_2 A_{22} = -\mathbb{I}_m$. Then, Assumptions 4.2-4.6 are satisfied with*

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2\lambda_{\max}(P_1)}, & \gamma_1(|e|) &= 2|P_1 B_0 K|^2|e|^2 \\
\alpha_2 &= \tfrac{1}{\lambda_{\max}(P_2)}, & \beta_1 &= \tfrac{2|P_1 A_{12}|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}} \\
\beta_2 &= \tfrac{2|P_2\Gamma\Lambda|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}, & \beta_3 &= \tfrac{2|P_2\Gamma A_{12}|+|P_2\Gamma B_0 K|^2}{\lambda_{\min}(P_2)} \\
\gamma_2(|e|) &= |e|^2, & L &= \tfrac{1}{2|P_1 B_0 K|^2} \\
\lambda_1 &= \tfrac{|\Gamma_1^T P_2 \Gamma_1|}{2|P_1 B_0 K|^2}, & \lambda_2 &= \tfrac{2|\Gamma_1^T P_2|}{|P_1 B_0 K|\sqrt{2\lambda_{\min}(P_2)}} \\
M &= |B_0 K|, & N &= \max\{\tfrac{|\Lambda|}{\sqrt{\lambda_{\min}(P_1)}}, \tfrac{|A_{12}|}{\sqrt{\lambda_{\min}(P_2)}}\}.
\end{aligned} \tag{4.65}$$

□

Proof of Proposition 3.

- Assumption 4.2: let $V_x(x) = x^T P_1 x$. Hence, the first condition of Assumption 4.2 is verified with $\underline{\alpha}_x(|x|) = \lambda_{\min}(P_1)|x|^2$, $\overline{\alpha}_x(|x|) = \lambda_{\max}(P_1)|x|^2$. It holds that, for all $x \in \mathbb{R}^{n_x}$

$$\begin{aligned}
\langle \nabla V_x(x), f_s(x,e)\rangle &= x^T(\Lambda^T P_1 + P_1\Lambda)x + 2x^T P_1 B_0 K e \\
&= -x^T \mathbb{I}_n x + 2x^T P_1 B_0 K e \\
&\leq -|x|^2 + 2|P_1 B_0 K||x||e|.
\end{aligned} \tag{4.66}$$

Using the fact that $2|P_1B_0K||x||e| \leq \frac{1}{2}|x|^2 + 2|P_1B_0K|^2|e|^2$, we obtain

$$\begin{aligned}\langle \nabla V_x(x), f_s(x,e)\rangle &\leq -\frac{1}{2}|x|^2 + 2|P_1B_0K|^2|e|^2 \\ &\leq -\frac{1}{2\lambda_{\max}(P_1)}V_x(x) + 2|P_1B_0K|^2|e|^2. \end{aligned} \tag{4.67}$$

Hence, Assumption 4.2 is verified with $\alpha_1 = \frac{1}{2\lambda_{\max}(P_1)}$ and $\gamma_1(|e|) = 2|P_1B_0K|^2|e|^2$.

- Assumption 4.3: let $V_y(x,y) = y^TP_2y$. It holds that, for all $y \in \mathbb{R}^{n_y}$

$$\begin{aligned}\langle \nabla V_y(x,y), g_f(x,y,e)\rangle = A_{22}^TP_2 + P_2A_{22} = -y^T\mathbb{I}_my \leq -|y|^2 \\ \leq -\frac{1}{\lambda_{\max}(P_2)}V_y(x,y). \end{aligned} \tag{4.68}$$

Thus, Assumption 4.3 holds with $\underline{\alpha}_y(|y|) = \lambda_{\min}(P_2)|y|^2$, $\overline{\alpha}_y(|y|) = \lambda_{\max}(P_2)|y|^2$ and $\alpha_2 = \frac{1}{\lambda_{\max}(P_2)}$.

- Assumption 4.4: In view of (4.58), (4.60), it holds that

$$\begin{aligned}\frac{\partial V_x}{\partial x}\left[f_x(x,y,e) - f_s(x,e)\right] &= 2x^TP_1A_{12}y \\ &\leq 2|P_1A_{12}||x||y| \\ &\leq \frac{2|P_1A_{12}|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}\sqrt{V_x(x)V_y(x,y)} \end{aligned} \tag{4.69}$$

and

$$\begin{aligned}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_y}{\partial y}\frac{\partial h}{\partial x}\right] f_x(x,y,e) &= -2y^TP_2(-\Gamma)(\Lambda x + A_{12}y + B_0Ke) \\ &\leq 2|P_2\Gamma\Lambda||x||y| + 2|P_2\Gamma A_{12}||y|^2 \\ &\quad + 2|P_2\Gamma B_0K||e||y|. \end{aligned} \tag{4.70}$$

Using the fact that $2|P_2\Gamma B_0 K||e||y| \leq |e|^2 + |P_2\Gamma B_0 K|^2|y|^2$, it holds

$$\begin{aligned}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_y}{\partial y}\frac{\partial h}{\partial x}\right] f_x(x,y,e) &\leq 2|P_2\Gamma\Lambda||x||y| + (2|P_2\Gamma A_{12}| + |P_2\Gamma B_0 K|^2)|y|^2 \\ &\quad + |e|^2 \\ &\leq \frac{2|P_2\Gamma\Lambda|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}\sqrt{V_x(x)V_y(x,y)} + |e|^2 \\ &\quad + (2|P_2\Gamma A_{12}| + |P_2\Gamma B_0 K|^2)\frac{1}{\lambda_{\min}(P_2)}V_y(x,y).\end{aligned} \tag{4.71}$$

Hence, in view of (4.69), (4.71), conditions (4.26) in Assumption 4.4 are satisfied with $\beta_1 = \frac{2|P_1 A_{12}|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}$, $\gamma_2(|e|) = |e|^2$, $\beta_2 = \frac{2|P_2\Gamma\Lambda|}{\sqrt{\lambda_{\min}(P_1)\lambda_{\min}(P_2)}}$, and $\beta_3 = \frac{2|P_2\Gamma A_{12}|+|P_2\Gamma B_0 K|^2}{\lambda_{\min}(P_2)}$. Since $\gamma_1(|e|) = 2|P_1 B_0 K|^2|e|^2$, we have that $\gamma_1^{-1}(|e|) = \sqrt{\frac{|e|}{2|P_1 B_0 K|^2}}$. consequently,

$$\gamma_2 \circ \gamma_1^{-1}(|e|) = \frac{1}{2|P_1 B_0 K|^2}|e|. \tag{4.72}$$

As a result, condition (4.27) in Assumption 4.4 is verified with $L = \frac{1}{2|P_1 B_0 K|^2}$.

- Assumption 4.5: the dynamics of V_y along jumps of the trajectories of system (4.63) is given by

$$\begin{aligned}V_y(x, h_y(x,y,e)) &= V_y(x, y - \Gamma_1 e_x) \\ &= (y - \Gamma_1 e)^T P_2 (y - \Gamma_1 e) \\ &= y^T P_2 y + e^T \Gamma_1^T P_2 \Gamma_1 e - y^T P_2 \Gamma_1 e - e^T \Gamma_1^T P_2 y \\ &\leq V_y(x,y) + |\Gamma_1^T P_2 \Gamma_1||e|^2 + 2|\Gamma_1^T P_2||e||y| \\ &\leq V_y(x,y) + \frac{|\Gamma_1^T P_2 \Gamma_1|}{2|P_1 B_0 K|^2}\gamma_1(|e|) \\ &\quad + \frac{2|\Gamma_1^T P_2|}{\sqrt{2|P_1 B_0 K|^2 \lambda_{\min}(P_2)}}\sqrt{\gamma_1(|e|)V_y(x,y)}\end{aligned} \tag{4.73}$$

$$V_y(x, h_y(x, y, e)) \leq V_y(x, y) + \frac{|\Gamma_1^T P_2 \Gamma_1|}{2|P_1 B_0 K|^2}\gamma_1(|e|)$$
$$+ \frac{2|\Gamma_1^T P_2|}{|P_1 B_0 K|\sqrt{2\lambda_{\min}(P_2)}}\sqrt{\gamma_1(|e|)V_y(x, y)}. \tag{4.74}$$

Thus, Assumption 4.5 holds with $\lambda_1 = \frac{|\Gamma_1^T P_2 \Gamma_1|}{2|P_1 B_0 K|^2}$ and $\lambda_2 = \frac{2|\Gamma_1^T P_2|}{|P_1 B_0 K|\sqrt{2\lambda_{\min}(P_2)}}$.

- Assumption 4.6: in view of (4.63), it holds that

$$\begin{aligned}\langle \nabla |e|, -f_x(x, y, e)\rangle &\leq |\Lambda||x| + |A_{12}||y| + |B_0 K||e| \\ &\leq |B_0 K||e| + \frac{|\Lambda|}{\sqrt{\lambda_{\min}(P_1)}}\sqrt{V_x(x)} + \frac{|A_{12}|}{\sqrt{\lambda_{\min}(P_2)}}\sqrt{V_y(x, y)} \\ &\leq M|e| + N(\sqrt{V_x(x)} + \sqrt{V_y(x, y)})\end{aligned} \tag{4.75}$$

where $M = |B_0 K|$ and $N = \max\{\frac{|\Lambda|}{\sqrt{\lambda_{\min}(P_1)}}, \frac{|A_{12}|}{\sqrt{\lambda_{\min}(P_2)}}\}$. Hence, Assumption 4.6 is verified. □

4.7 Autopilot control of an F-8 aircraft

We apply the results developed to the autopilot control of the longitudinal motion of an F-8 aircraft. We borrow the model from Chapter 4 in [54]

$$\dot{x} = A_{11}x + A_{12}z + B_1 u \tag{4.76}$$
$$\epsilon \dot{z} = A_{21}x + A_{22}z + B_2 u \tag{4.77}$$

where $x \in \mathbb{R}^2$ represents the slow 'phugoid mode' and $z \in \mathbb{R}^2$ represents the fast 'short period mode' of the longitudinal motion of an airplane. The parameter ϵ is equal to 0.0336 and the coefficient matrices are given by

$$A_{11} = \begin{bmatrix} -0.195378 & -0.676469 \\ 1.478265 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.917160 & 0.109033 \\ 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -0.051601 & 0 \\ 0.013579 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.367954 & 0.438041 \\ -2.102596 & -0.214640 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.023109 \\ -16.945030 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.048184 \\ -3.810954 \end{bmatrix}$$

We notice that A_{22} is invertible and Hurwitz with the eigenvalues $-8.6696 \pm 28.4712i$ and the pair (A_0, B_0) is controllable where A_0, B_0 are defined in (4.59). Thus, the conditions of Section 4.6.2 are satisfied. The origin of the open-loop system is globally exponentially stable. Nevertheless, the eigenvalues of the slow system are such that the overall system solutions exhibit large oscillations and a slow convergence, see Figure 4.1. Hence, we design the controller $u = Kx$ to improve the closed-loop response. The gain K is selected to place the eigenvalues of the slow system at $(-2, -3)$.

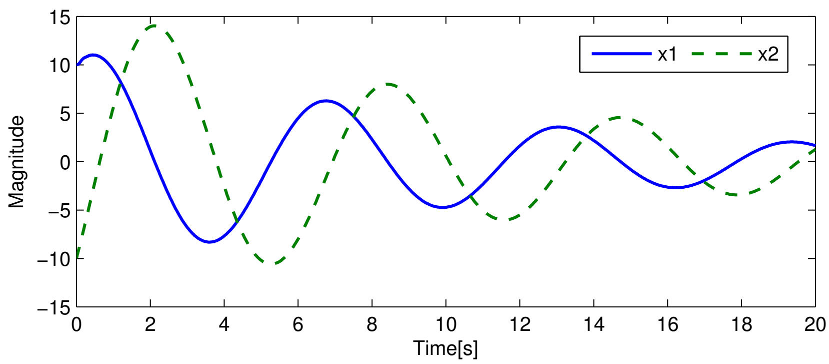

FIGURE 4.1: Open-loop state trajectories of the slow dynamics

We obtain $\gamma_1(|e|) = 1.7795|e|^2$, $\alpha_1 = 0.3104$, $P_1 = \begin{bmatrix} 1.6063 & 0.0826 \\ 0.0826 & 0.1112 \end{bmatrix}$ and we set $\sigma = 0.05$ small in order not to deteriorate the continuous-time closed-loop performance. We run simulations for the initial condition $(x(0,0), z(0,0), e(0,0), \tau(0,0)) = (10, -10, 5, 5, 0, 0, 0)$. For the triggering condition in (4.31), we take $\rho = 0.0001$ and for the triggering condition in (4.33), we obtain $T^* = 0.021$ by using (4.34).

Simulation results with the triggering mechanism in [103] To justify the discussion in Section 4.5.1, we first apply the triggering condition (4.29). Simulations have shown that any initial condition satisfying $x(0,0) = 0$ and $z(0,0) \neq 0$ leads to infinite jumps at $t = 0$ which supports the conclusion in Section 4.5.1 and motivates our proposed triggering conditions.

Simulation results of the triggering mechanism (4.31) Figure 4.2 shows the norm of the state vector and Figure 4.3 shows that it converges to a neighbourhood of the origin. The two-time scale dynamics can be observed in Figures 4.4, Figure 4.5, where the state y converges to the origin faster than the state x. The evolution of the sampling induced error is provided in Figure 4.6 where it can be noted that e is reset when $\bar{\gamma}_1|e|^2$ hits the maximum of $\sigma\alpha_1 V_x(x)$ and ρ. The generated inter-transmission times are given in Figure 4.7.

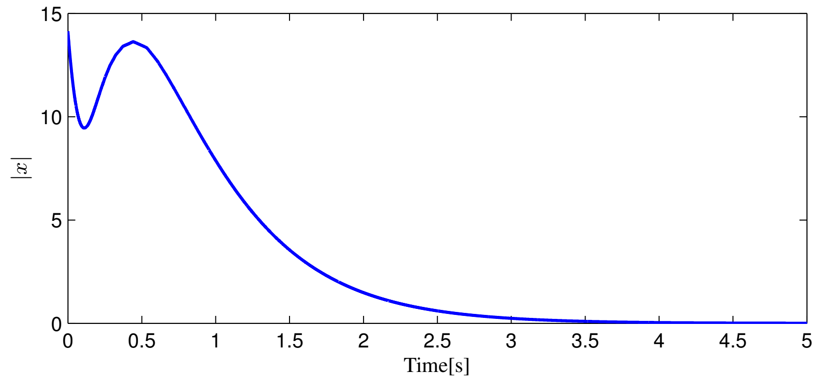

FIGURE 4.2: Norm of the state vector during the first 5 seconds.

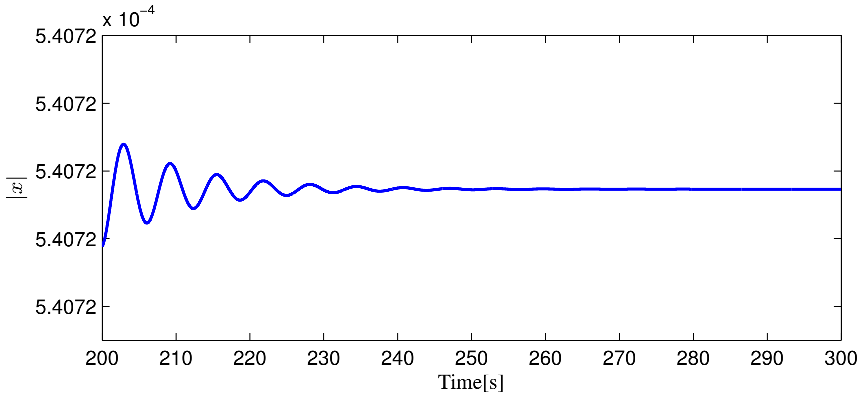

FIGURE 4.3: Norm of the state vector after 200 seconds.

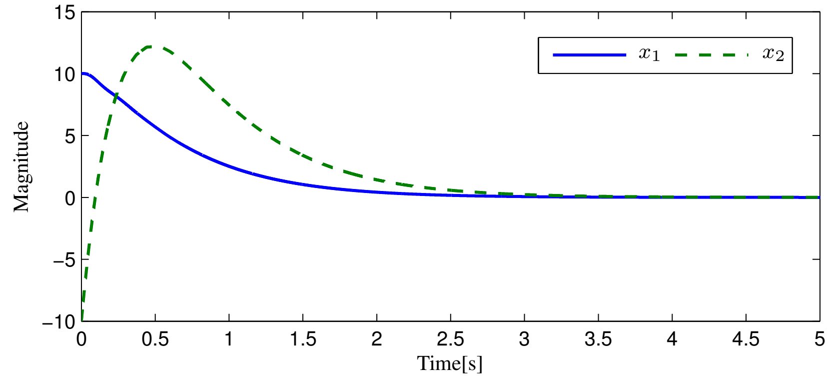

FIGURE 4.4: State trajectories of the slow model.

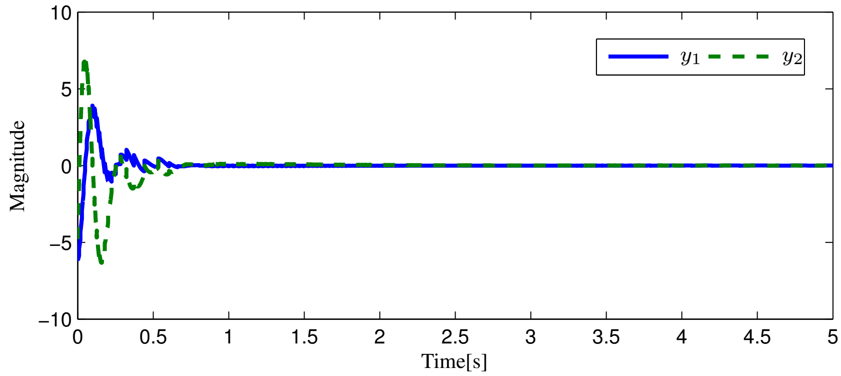

FIGURE 4.5: State trajectories of the fast model.

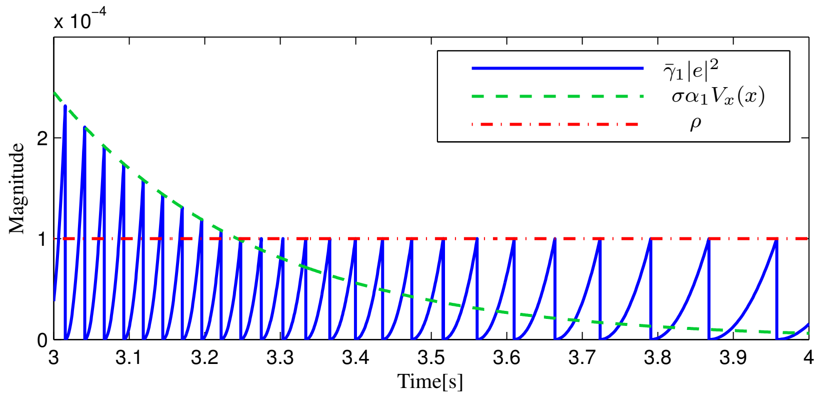

FIGURE 4.6: Evolution of the sampling error.

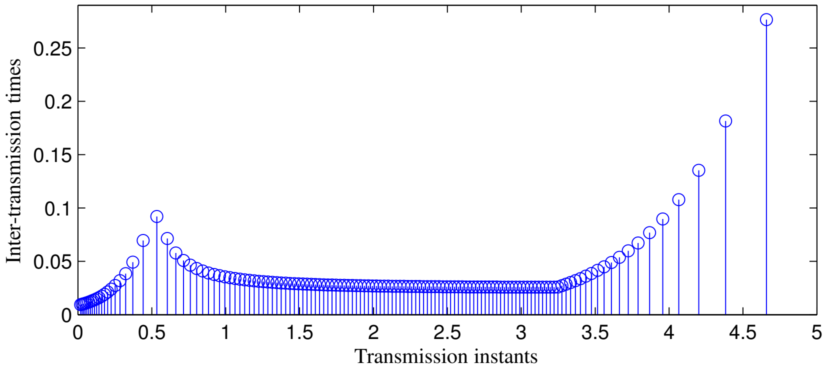

FIGURE 4.7: Inter-transmission times.

Simulation results with the triggering mechanism (4.33) Figure 4.8 shows the norm of the state vector and Figure 4.9 shows its asymptotic convergence to the origin. The state trajectories of the slow and fast states are given in Figures 4.10, Figure 4.11. The internal structure of the triggering mechanism (4.33) is revealed in Figures 4.12, 4.13. We observe that the time-triggered part enforces the lower bound T while the event-triggered part allows for larger inter-transmission times than T.

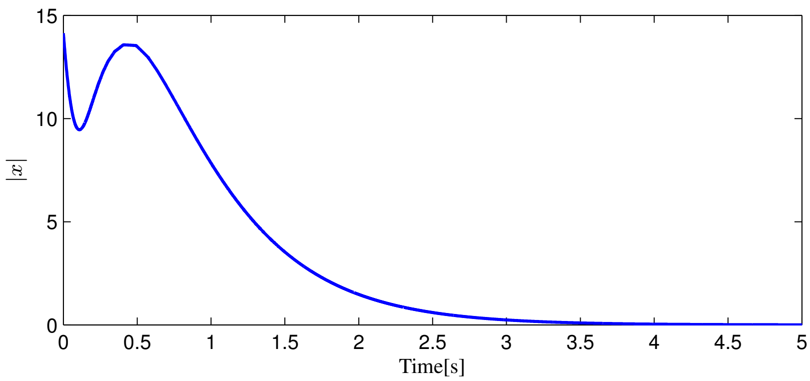

FIGURE 4.8: Norm of the state vector at the first 5 seconds.

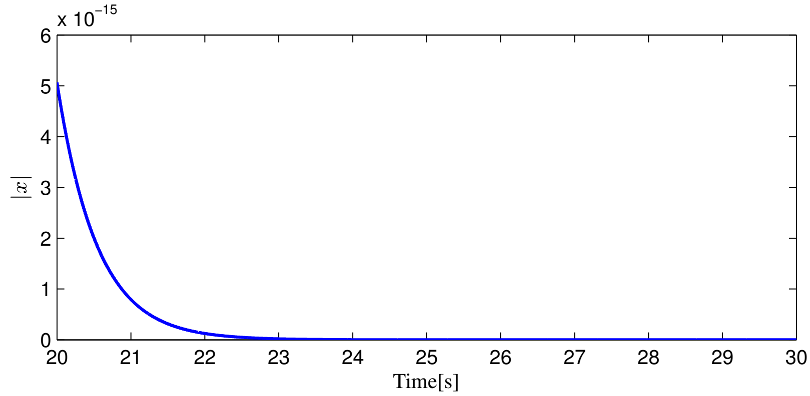

FIGURE 4.9: Norm of the state vector after 30 seconds.

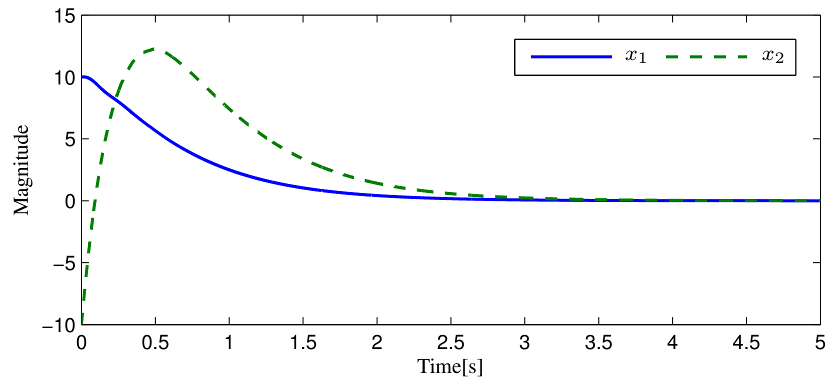

FIGURE 4.10: State trajectories of the slow model.

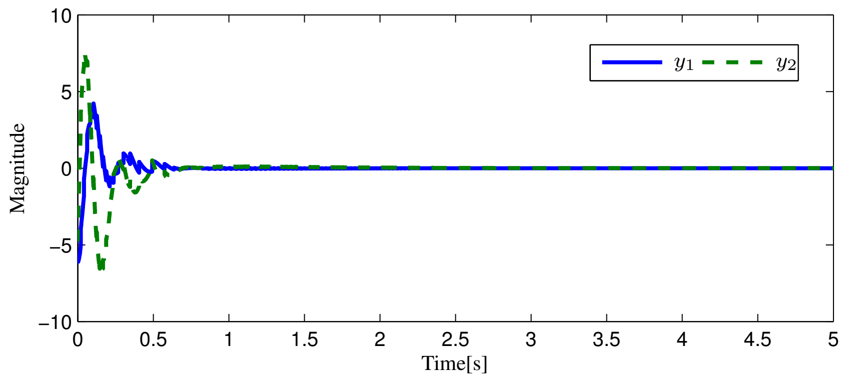

FIGURE 4.11: State trajectories of the fast model.

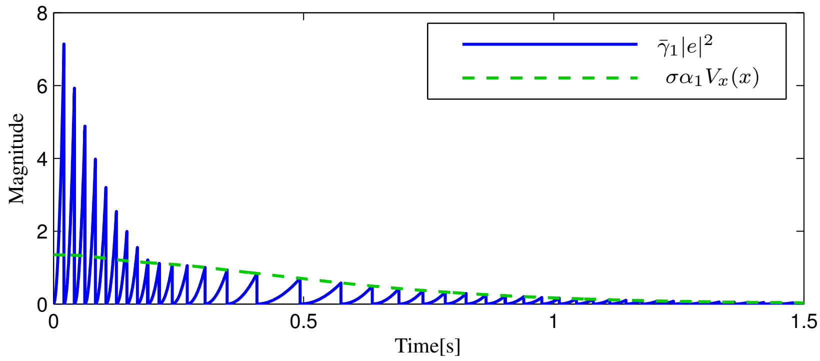

FIGURE 4.12: Evolution of the sampling error.

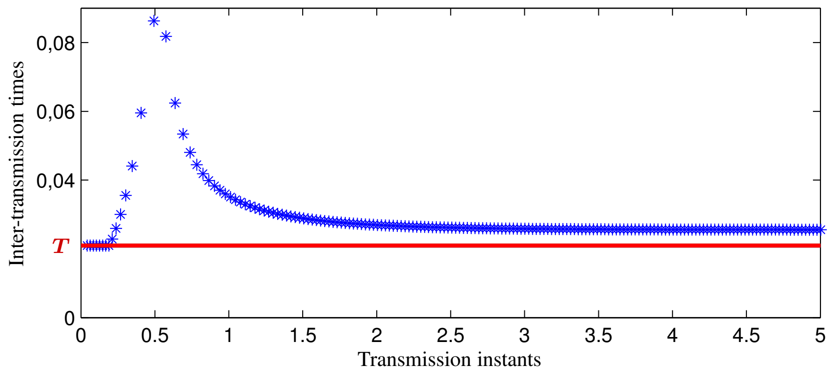

FIGURE 4.13: Inter-transmission times.

We compare the average inter-transmission interval of the proposed event-triggered solutions. Table 4.1 shows the average inter-execution times with a simulation time of 5 seconds for 200 randomly distributed initial conditions such that $|(x(0,0), y(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$. We note in this example that the triggering mechanism (4.33) ensures larger minimum inter-transmission time while the triggering mechanism (4.31) generates less amount of transmissions.

	Triggering mechanism (4.31)	Triggering mechanism (4.33)
τ_{min}	0.0017	0.021
τ_{avg}	0.0306	0.028

TABLE 4.1: Minimum and average inter-execution times for 100 randomly distributed initial conditions such that $|(x(0,0), y(0,0), e(0,0))| \leq 100$ and $\tau(0,0) = 0$ for a simulation time of 5 s.

4.8 Conclusion

We have investigated the event-triggered stabilization of nonlinear singularly perturbed systems based only on the slow dynamics. Two classes of controllers have been developed which ensure different asymptotic stability properties. The first triggering strategy consists of adding some positive constant to the triggering condition based on [103] to ensure that the minimum inter-transmission time is strictly positive. We have shown that a practical stability property is achieved in this case. In the second triggering policy, we have developed an event-triggering condition by combining results from event-triggered and time-triggered techniques. The idea is to turn on the event-triggered part only after a fixed amount of time has been elapsed since the last transmission instant, like in Chapter 2. The proposed mechanism allows to guarantee asymptotic stability property under an additional assumption. The results are applicable to a class of globally Lipschitz systems which encompasses LTI systems as a particular case.

Chapter 5

Conclusions

5.1 Conclusions

We have investigated the synthesis of stabilizing output feedback event-triggered controllers for both nonlinear and linear systems. In particular, we have addressed the following problems:

- In Chapter 2, we have developed output feedback event-triggered controllers to stabilize a general class of nonlinear systems by following the emulation design approach. The proposed triggering mechanism combines the event-triggered [103] and the time-triggered [79] results to enforce a strictly positive amount of time between two transmissions. This minimum time is designed as the MATI given in [79]. Our results rely on similar assumptions as in [79] which allow us to derive both local and global results. The obtained results have been applied to two physical nonlinear systems for which the required conditions have been proved to hold. We have also shown that the required conditions are always verified by LTI systems that are stabilizable and detectable, in which case these were reformulated as an LMI. Moreover, we have explained that the benefit of our proposed triggering mechanism can be nicely transferred to the context of state feedback control to allow the user to directly tune the guaranteed lower bound on the inter-transmission times.
- In Chapter 3, to overcome the design constraints induced by the emulation approach, we have proposed an LMI-based co-design algorithm for LTI systems to simultaneously construct the feedback law and the event-triggering condition. We have then discussed how the resulted LMI can be exploited to optimize the event-triggered condition in two

senses. The first optimization procedure aims to enlarge the guaranteed lower bound on the inter-transmission times. The second optimization problem allows to heuristically reduce the amount of transmissions generated by the event-triggering mechanism. The effectiveness of the approach has been demonstrated on a numerical example.

- In Chapter 4, we have studied the event-triggered stabilization of nonlinear singularly perturbed systems based only on the slow dynamics. The event-triggered controllers have been designed by emulation within the framework of singular perturbation to capture the two-time scale phenomena exhibited by such systems. We have first decomposed the original system into two approximate slow and fast models. Then, we have synthesised two appropriate triggering mechanisms based only on the approximate slow dynamics. The first triggering mechanism relies on existing techniques on event-triggered control and achieves a practical stability property for the closed-loop. The second proposed mechanism adapts the presented technique in Chapter 2 to singularly perturbed systems and leads to an asymptotic stability property, under additional assumptions.

5.2 Contributions

The contributions of this thesis are summarized as follows.

- Few results in the literature address the output feedback event-triggered stabilization problem and most of them are dedicated to linear systems. This problem has been only studied in [123] for nonlinear systems, to the best for our knowledge. We have proposed an alternative design as well as an alternative analysis, which seems to rely on different conditions compared to [123]. We have notably seen in Chapter 2 that all the considered examples violate the conditions imposed in [123].

- An interesting question in practice is whether the event-triggered implementation will achieve less amount of transmissions than those produced by traditional periodic setups. The idea of the proposed triggering mechanism in Chapter 2 provides a qualitative answer to this question.

- Very few results are available in the literature for the co-design of the feedback law and the event-triggering condition and only for specific types of implementations using state-feedbacks. No available results exist for the case where only an output of the plant is continuously monitored. In Chapter 3, we have presented a co-design procedure based on output measurements which is easy to use and may further reduce the amount of transmissions as demonstrated on examples.

- To the best of our knowledge, the results in Chapter 4 on the event-triggered stabilization of singularly perturbed systems are the first ones in this direction. These results are useful in practice since many control systems exhibit two-time scale dynamics and engineers usually design the controller based only on the slow model.

5.3 Recommendations for future research

We think that the obtained results in this thesis can be further extended in several directions.

- An interesting future research direction is to investigate the robustness of our proposed triggering mechanism in Chapter 2 with respect to the measurement errors (and model uncertainties). These phenomena are usually encountered in practice and may have a significant impact on the closed-loop stability and performance and may lead to the Zeno phenomenon. We believe that the fact that transmissions cannot occur before T units of time have elapsed can be useful in this context to avoid the occurrence of Zeno.
- In [95], it has been developed a set of useful LMI tools to synthesize stabilizing output feedback controllers while achieving a desired level of performance in terms of disturbance rejection, peak output amplitude, H_2, H_∞, and other properties. In the same spirit, it would be interesting to improve our proposed co-design procedure to satisfy some performance requirements on the closed-loop system, in virtue of [95].
- A possible extension of the results on singularly perturbed systems is to investigate the general case where the fast dynamics is not necessarily stable. As a consequence, the fast model cannot be ignored and two triggering conditions should be synthesized to stabilize both the approximate slow and fast subsystems a priori.
- It would be interesting to investigate whether the design approach in Chapter 4 can be transferred to other classes of nonlinear systems to simplify the control design problem. In other words, it would be useful in practice if we can synthesize event-triggered controllers for nonlinear systems, that are not necessarily singularly perturbed, based on an approximate model obtained by other means like model reduction or averaging.

- It is also of interest to develop the triggering condition to consider other communication constraints like signal quantization, as in [123]. This is an interesting direction in practice since the asymptotic convergence of solutions can be lost near the equilibrium in the presence of quantization errors as the difference between the current and the desired values of the state becomes small.

Appendix A

Proofs of Chapter 4

We present here the proofs of Theorems 4.1, 4.2 in Chapter 4.

A.1 Proof of Theorem 4.1

We define the function (like in the proof of Theorem 1 in [93])

$$V(q) := V_x(x) + \sqrt{\epsilon} V_y(x, y) \qquad \forall q \in \mathbb{R}^{n_q} \tag{A.1}$$

with $\epsilon \in (0, \epsilon^*)$ where $\epsilon^* > 0$ will be defined in the following. Let $q \in C$, it holds that, in view of (4.15) and (4.18)

$$\begin{aligned} \langle \nabla V(q), F(q) \rangle &= \tfrac{\partial V_x}{\partial x} f_x(x, y, e) + \sqrt{\epsilon} \tfrac{\partial V_y}{\partial x} f_x(x, y, e) + \tfrac{\sqrt{\epsilon}}{\epsilon} \tfrac{\partial V_y}{\partial y} f_y(x, y, e) \\ &= \tfrac{\partial V_x}{\partial x} f_s(x, e) + \tfrac{\partial V_x}{\partial x} \left[f_x(x, y, e) - f_{x_s}(x, e) \right] + \tfrac{1}{\sqrt{\epsilon}} \tfrac{\partial V_y}{\partial y} g_f(x, y, e) \\ &\quad + \sqrt{\epsilon} \left[\tfrac{\partial V_y}{\partial x} - \tfrac{\partial V_y}{\partial y} \tfrac{\partial h}{\partial x} \right] f_x(x, y, e). \end{aligned} \tag{A.2}$$

In view of the definition of the set C, we have that

$$\gamma_1(|e|) \leq \max\{\sigma \alpha_1 V_x(x), \rho\} \tag{A.3}$$

and, since $\gamma_2(.)$ is increasing, it holds that

$$\gamma_2(|e|) \leq \gamma_2 \circ \gamma_1^{-1}\big(\max\{\sigma \alpha_1 V_x(x), \rho\}\big). \tag{A.4}$$

The condition (4.27) ensures that

$$\gamma_2(|e|) \leq L \max\{\sigma\alpha_1 V_x(x), \rho\}. \tag{A.5}$$

Using Assumptions 4.2-4.5, we derive that

$$\begin{aligned}
\langle \nabla V(q), F(q)\rangle &\leq -\alpha_1 V_x(x) + \gamma_1(|e|) + \beta_1\sqrt{V_x(x)V_y(x,y)} - \tfrac{\alpha_2}{\sqrt{\epsilon}} V_y(x,y) \\
&\quad + \sqrt{\epsilon}(\beta_2\sqrt{V_x(x)V_y(x,y)} + \beta_3 V_y(x,y) + \gamma_2(|e|)) \\
&\leq -\alpha_1 V_x(x) + \max\{\sigma\alpha_1 V_x(x), \rho\} - (\tfrac{\alpha_2}{\sqrt{\epsilon}} - \sqrt{\epsilon}\beta_3)V_y(x,y) \\
&\quad + (\beta_1 + \sqrt{\epsilon}\beta_2)\sqrt{V_x(x)V_y(x,y)} + \sqrt{\epsilon}L\max\{\sigma\alpha_1 V_x(x), \rho\} \qquad \text{(A.6)} \\
&= -\alpha_1(1-\sigma(1+\sqrt{\epsilon}L))V_x(x) - (\tfrac{\alpha_2}{\sqrt{\epsilon}} - \sqrt{\epsilon}\beta_3)V_y(x,y) \\
&\quad + (\beta_1 + \sqrt{\epsilon}\beta_2)\sqrt{V_x(x)V_y(x,y)} + (1+\sqrt{\epsilon}L)\rho \\
&= -\chi^T \mathcal{A}\chi + (1+\sqrt{\epsilon}L)\rho,
\end{aligned}$$

where $\chi := (\sqrt{V_x(x)}, \sqrt{V_y(x,y)})$ and

$$\mathcal{A} := \begin{pmatrix} \alpha_1(1-\sigma(1+\sqrt{\epsilon}L)) & -(\beta_1+\sqrt{\epsilon}\beta_2)/2 \\ -(\beta_1+\sqrt{\epsilon}\beta_2)/2 & \frac{\alpha_2}{\sqrt{\epsilon}} - \sqrt{\epsilon}\beta_3 \end{pmatrix}. \tag{A.7}$$

Let $\mu > 0$ defined as follows

$$\mu \in (0, \alpha_1(1-\sigma)) \tag{A.8}$$

The following conditions ensure that $\mathcal{A} \geq \mu\text{diag}(1,\sqrt{\epsilon})$, i.e. $\mathcal{A} - \mu\text{diag}(1,\sqrt{\epsilon})$ is positive definite, where $\text{diag}(1,\sqrt{\epsilon})$ is the diagonal matrix with elements $(1,\sqrt{\epsilon})$ on the diagonal,

$$\begin{cases} \alpha_1(1-\sigma(1+\sqrt{\epsilon}L)) \geq \mu \\ \Big(\alpha_1(1-\sigma(1+\sqrt{\epsilon}L)) - \mu\Big)\Big(\frac{\alpha_2}{\sqrt{\epsilon}} - \sqrt{\epsilon}\beta_3 - \sqrt{\epsilon}\mu\Big) \geq (\beta_1+\sqrt{\epsilon}\beta_2)^2/4 \end{cases} \tag{A.9}$$

The inequalities in (A.9) are always satisfied for $\epsilon \in (0, \epsilon^*)$, where $\epsilon^* > 0$ is sufficiently small. Consequently

$$\begin{aligned}\langle \nabla V(q), F(q)\rangle &\leq -\mu\chi^T \text{diag}(1, \sqrt{\epsilon})\chi + (1+\sqrt{\epsilon}L)\rho \\ &= -\mu V(q) + (1+\sqrt{\epsilon}L)\rho \\ &= -\tfrac{\mu}{2}V(q) - \tfrac{\mu}{2}V(q) + (1+\sqrt{\epsilon}L)\rho\end{aligned} \tag{A.10}$$

Hence, if $\frac{\mu}{2}V(q) \geq (1+\sqrt{\epsilon}L)\rho$, it holds that

$$\langle \nabla V(q), F(q)\rangle \leq -\tfrac{\mu}{2}V(q) \tag{A.11}$$

implying that, by invoking standard comparison principle,

$$V(q) \leq e^{-\frac{\mu}{2}(t-t_0)}V(q(0,0)). \tag{A.12}$$

On the other hand, if $\frac{\mu}{2}V(q) \leq (1+\sqrt{\epsilon}L)\rho$, then $V(q) \leq \frac{2(1+\sqrt{\epsilon}L)\rho}{\mu}$. Thus, the Lyapunov function $V(q)$ satisfies on flows

$$\begin{aligned}V(q) &\leq \max\{e^{-\frac{\mu}{2}(t-t_0)}V(q(0,0)), \tfrac{2(1+\sqrt{\epsilon}L)\rho}{\mu}\} \\ &\leq \max\{e^{-\frac{\mu}{2}(t-t_0)}V(q(0,0)), \tfrac{2(1+L)\rho}{\mu}\},\end{aligned} \tag{A.13}$$

where we have used the fact ϵ^* is sufficiently small such that $\epsilon^* \leq 1$.

Let $q \in D$,

$$V(G(q)) = V_x(x) + \sqrt{\epsilon}V_y(x, h_y(x, y, e)). \tag{A.14}$$

In view of Assumption 4.5 and the definition of the set D,

$$\begin{aligned}V_y(x, h_y(x, y, e)) &\leq V_y(x, y) + \lambda_1\gamma_1(|e|) + \lambda_2\sqrt{\gamma_1(|e|)V_y(x, y)} \\ &\leq V_y(x, y) + \lambda_1 \max\{\sigma\alpha_1 V_x(x), \rho\} \\ &\quad +\lambda_2\sqrt{\max\{\sigma\alpha_1 V_x(x), \rho\}V_y(x, y)}.\end{aligned} \tag{A.15}$$

Using that

$$\begin{aligned}\sqrt{\max\{\sigma\alpha_1 V_x(x),\rho\}V_y(x,y)} &\leq \epsilon^{-\frac{1}{4}}\max\{\sigma\alpha_1 V_x(x),\rho\}+\epsilon^{\frac{1}{4}}V_y(x,y)\\ &\leq \epsilon^{-\frac{1}{4}}\sigma\alpha_1 V_x(x)+\epsilon^{-\frac{1}{4}}\rho+\epsilon^{\frac{1}{4}}V_y(x,y),\end{aligned} \tag{A.16}$$

we deduce that

$$\begin{aligned}V_y(x,h_y(x,y,e)) \leq\ & V_y(x,y)+\lambda_1\sigma\alpha_1 V_x(x)+\lambda_1\rho+\epsilon^{-\frac{1}{4}}\lambda_2\sigma\alpha_1 V_x(x)\\ &+\epsilon^{-\frac{1}{4}}\lambda_2\rho+\epsilon^{\frac{1}{4}}\lambda_2 V_y(x,y).\end{aligned} \tag{A.17}$$

In view of (A.14), (A.17) and using the fact that $\epsilon^{\frac{1}{2}}\leq\epsilon^{\frac{1}{4}}$, since ϵ is sufficiently small, we obtain

$$\begin{aligned}V(G(q)) &= V_x(x)+\sqrt{\epsilon}\Big(V_y(x,y)+\lambda_1\sigma\alpha_1 V_x(x)+\lambda_1\rho+\epsilon^{-\frac{1}{4}}\lambda_2\sigma\alpha_1 V_x(x)+\epsilon^{-\frac{1}{4}}\lambda_2\rho\\ &\qquad +\lambda_2\epsilon^{\frac{1}{4}}V_y(x,y)\Big)\\ &= (V_x(x)+\sqrt{\epsilon}V_y(x,y))+\sqrt{\epsilon}\lambda_1\sigma\alpha_1 V_x(x)+\sqrt{\epsilon}\lambda_1\rho+\epsilon^{\frac{1}{4}}\lambda_2\sigma\alpha_1 V_x(x)\\ &\quad +\epsilon^{\frac{1}{4}}\lambda_2\rho+\epsilon^{\frac{1}{4}}\sqrt{\epsilon}\lambda_2 V_y(x,y)\\ &\leq V(q)+\epsilon^{\frac{1}{4}}\sigma\alpha_1(\lambda_1+\lambda_2)V_x(x)+\epsilon^{\frac{1}{4}}(\lambda_1+\lambda_2)\sqrt{\epsilon}V_y(x,y)+\epsilon^{\frac{1}{4}}(\lambda_1+\lambda_2)\rho\\ &\leq V(q)+\epsilon^{\frac{1}{4}}(\lambda_1+\lambda_2)\max\{\sigma\alpha_1,1\}(V_x(x)+\sqrt{\epsilon}V_y(x,y))+\epsilon^{\frac{1}{4}}(\lambda_1+\lambda_2)\rho\\ &\leq V(q)+\epsilon^{\frac{1}{4}}\lambda V(q)+\epsilon^{\frac{1}{4}}\lambda\rho,\end{aligned} \tag{A.18}$$

where

$$\lambda := (\lambda_1+\lambda_2)\max\{\sigma\alpha_1,1\}. \tag{A.19}$$

As a consequence

$$V(G(q)) \leq (1+2\epsilon^{\frac{1}{4}}\lambda)\max\{V(q),\rho\}. \tag{A.20}$$

We note that properties (A.13), (A.20) are not sufficient to conclude about the asymptotic stability of the origin for the system (4.22) as $V(q)$ may increase at jumps in view of (A.20). Nevertheless, Proposition 3.29 in [34] allows to show that (4.32) is satisfied provided that solutions to (4.22) have a sufficiently long dwell-time. The claim below formalizes this result. It has to be noted that β in (4.32) depends on the ball size of initial conditions Δ which is not the case in [60] and thus, the proof requires a particular careful to handle this point.

Claim A.1. *Let $\phi=(\phi_x,\phi_y,\phi_e)$ be a solution to (4.22), (4.31) with $|\phi(0,0)|\leq\Delta$. If the* *parameter ϵ is sufficiently small such that $\epsilon \in (0, \epsilon^*)$ with*

$$\epsilon^* := \min\left\{\left(\frac{1}{2}(e^{\frac{4\rho}{\mu\xi(\Delta)}} - 1)\right)^{-\frac{1}{4}}, 1\right\}, \tag{A.21}$$

where ρ, μ come from (4.31), (A.8) respectively and $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a continuous increasing function. Then, all inter-transmission times are lower bounded by $\tau(\Delta)$, where

$$\tau(\Delta) \quad := \quad \tfrac{4}{\mu}\ln(1 + 2\epsilon^{\frac{1}{4}}\lambda) \tag{A.22}$$

and λ is defined in (A.19). Furthermore, any solution ϕ to (4.22) satisfies, for some $\psi > 0$

$$V(\phi(t,j)) \quad \leq \quad \max\left\{e^{-\psi(t+j)}V(\phi(0,0)), \theta\rho\right\} \qquad \forall (t,j) \in \operatorname{dom}\phi, \tag{A.23}$$

where $\theta := (1 + 2\lambda)\max\{2\frac{(1+L)}{\mu}, 1\}$. ■

The proof of Claim A.1 is given after the proof of Theorem 4.1.

We now show that the stability property (4.32) holds. In view of (4.24) and (4.25), for any $(t, j) \in \operatorname{dom}\phi$,

$$\begin{aligned} \underline{\alpha}_x(|\phi_x(t,j)|) \quad &\leq \quad \max\left\{e^{-\gamma(t+j)}\overline{\alpha}(\phi(0,0)), \theta\rho\right\} \\ |\phi_x(t,j)| \quad &\leq \quad \underline{\alpha}_x^{-1}\left(\max\left\{e^{-\gamma(t+j)}\overline{\alpha}(|\phi(0,0)|), \theta\rho\right\}\right). \end{aligned} \tag{A.24}$$

Using that $\gamma_1(|e|) \leq \max\{\sigma\alpha_1 V(x), \rho\}$ for any $q \in C \cup D \cup G(D)$, we deduce that for any $(t, j) \in \operatorname{dom}\phi$

$$|\phi_e(t,j)| \quad \leq \quad \max\left\{\beta_e(|\phi(0,0)|, t+j), \vartheta_e(\rho)\right\} \tag{A.25}$$

for some $\beta_e \in \mathcal{KL}$ and $\theta_e \in \mathcal{K}_\infty$. We are left with the y-component of ϕ. In view of Assumptions 4.3-4.4, it holds that

$$\begin{aligned} \langle\nabla V_y(x,y), (f_x, f_y)\rangle \quad &= \quad \tfrac{1}{\epsilon}\tfrac{\partial V_y}{\partial y}g + \left[\tfrac{\partial V_y}{\partial x} - \tfrac{\partial V_y}{\partial y}\tfrac{\partial h}{\partial x}\right] f_x \\ &\leq \quad -\tfrac{\alpha_2}{\epsilon}V_y(x,y) + \beta_2\sqrt{V_x(x)V_y(x,y)} + \beta_3 V_y(x,y) + \gamma_2(|e|) \\ &\leq \quad -(\tfrac{\alpha_2}{\epsilon} - \beta_2 - \beta_3)V_y(x,y) + \beta_2 V_x(x) + \gamma_2(|e|) \end{aligned} \tag{A.26}$$

and, as shown before,

$$\begin{aligned} V_y(x, h_y(x,y,e)) &\leq V_y(x,y) + \lambda_1\gamma_1(|e|) + \lambda_2\sqrt{\gamma_1(|e|)V_y(x,y)} \\ &\leq (1+\epsilon^{\frac{1}{4}}\lambda_2)V_y(x,y) + (\lambda_1 + \epsilon^{-\frac{1}{4}}\lambda_2)\gamma_1(|e|) \end{aligned} \tag{A.27}$$

By following similar lines as above, we deduce that, by taking ϵ^* sufficiently small, the y-system is ISS with respect to x and e. As a consequence, in view of (A.24), (A.26) and (A.27) we derive that

$$|\phi_y(t,j)| \leq \max\left\{\beta_y(|\phi(0,0)|, t+j), \vartheta_y(\rho)\right\} \tag{A.28}$$

for some $\beta_y \in \mathcal{KL}$ and $\theta_y \in \mathcal{K}_\infty$. The property (4.32) then follows from (A.24), (A.25) and (A.28). Equations (A.24), (A.25) and (A.28) ensure that ϕ cannot explode in finite time, neither it can flow out of $C \cup D$ since $G(D) \subset C$. Noting that system (4.22), (4.31) does not admit trivial solution[1], we conclude that maximal solutions to (4.22), (4.31) are complete according to Proposition 6.10 in [34]. □

Proof of Claim A.1. First, we assume that (A.22) holds and we derive (A.23) by induction. Then, we show that (A.22) is always satisfied for all solutions ϕ. We start by studying the dynamics of $V(q)$ in the first transmission time, *i.e.* $j = 0, t \in [0, t_1]$, where t_1 denotes the first transmission instant.

$\forall (t,0) \in \mathrm{dom}\,\phi$

Assume without loss of generality[2] that $\phi_e(0,0) = 0$. In view of (A.13), we have

$$V(\phi(t,0)) \leq \max\left\{e^{-\frac{\mu}{2}t}V(\phi(0,0)), 2\frac{(1+L)}{\mu}\rho\right\}. \tag{A.29}$$

At $t = t_1$, we obtain

$$V(\phi(t_1,0)) \leq \max\{e^{-\frac{\mu}{2}t_1}V(\phi(0,0)), 2\frac{(1+L)}{\mu}\rho\}. \tag{A.30}$$

[1] This comes from the fact that $C\backslash D$ is the interior of C. Hence, the tangent cone (see Section B.3 in Appendix B) is $\mathbb{R}^n$ and (VC) in Proposition 6.10 in [34] holds for any point in $C\backslash D$

[2] If that is not the case, the inequality obtained later in (A.23) will hold for any $(t,j) \in \mathrm{dom}\,\phi$ with $j \geq 1$. A bound on $V(\phi)$ on the interval $[0, t_1]$ can then be derived using (A.13) and (A.20) to upper-bound on $V(\phi)$ on the whole domain $\mathrm{dom}\,\phi$. Note that if ϕ never jumps, the bound on the inter-jump times used in (A.22) trivially holds and (A.23) will be verified.

Next, we study the dynamics of $V(q)$ in the second transmission time, *i.e.* $j = 1, t \in [t_1, t_2]$.

$\forall(t, 1) \in \text{dom}\,\phi$

In view of (A.20), (A.30), we have

$$\begin{aligned} V(\phi(t_1,1)) &\leq (1+2\epsilon^{\frac{1}{4}}\lambda)\max\{V(\phi(t_1,0)),\rho\} \\ &\leq (1+2\epsilon^{\frac{1}{4}}\lambda)\max\{e^{-\frac{\mu}{2}t_1}V(\phi(0,0)), 2\frac{(1+L)}{\mu}\rho,\rho\} \\ &= \max\{(1+2\epsilon^{\frac{1}{4}}\lambda)e^{-\frac{\mu}{2}t_1}V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)\frac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho\}. \end{aligned} \tag{A.31}$$

In view of (A.13), we obtain

$$\begin{aligned} V(\phi(t,1)) &\leq \max\left\{\, e^{-\frac{\mu}{2}(t-t_1)}V(\phi(t_1,1)), \frac{2(1+L)}{\mu}\rho\right\} \\ &\leq \max\left\{\, (1+2\epsilon^{\frac{1}{4}}\lambda)e^{-\frac{\mu}{2}(t-t_1)}e^{-\frac{\mu}{2}t_1}V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)\frac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t-t_1)},\right. \\ &\qquad \left.(1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\frac{\mu}{2}(t-t_1)}, \frac{2(1+L)}{\mu}\rho\right\} \\ &= \max\left\{\, (1+2\epsilon^{\frac{1}{4}}\lambda)e^{-\frac{\mu}{2}t}V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)\frac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t-t_1)},\right. \\ &\qquad \left.(1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\frac{\mu}{2}(t-t_1)}, \frac{2(1+L)}{\mu}\rho\right\}. \end{aligned} \tag{A.32}$$

At $t = t_2$, we have

$$\begin{aligned} V(\phi(t_2,1)) \leq \max\left\{\, (1+2\epsilon^{\frac{1}{4}}\lambda)e^{-\frac{\mu}{2}t_2}V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)\frac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_2-t_1)},\right. \\ \left.(1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\frac{\mu}{2}(t_2-t_1)}, 2\frac{(1+L)}{\mu}\rho\right\}. \end{aligned} \tag{A.33}$$

similarly, we study the third inter-transmission time, *i.e.* $j = 2, t \in [t_2, t_3]$.

$\forall(t, 2) \in \text{dom}\,\phi$

In view of (A.20), (A.33), we obtain

$$\begin{aligned} V(\phi(t_2,2)) &\leq (1+2\epsilon^{\frac{1}{4}}\lambda)\max\{V(\phi(t_2,1)),\rho\} \\ &\leq \max\left\{(1+2\epsilon^{\frac{1}{4}}\lambda)^2e^{-\frac{\mu}{2}t_2}V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)^2\frac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_2-t_1)},\right. \\ &\qquad \left.(1+2\epsilon^{\frac{1}{4}}\lambda)^2\rho e^{-\frac{\mu}{2}(t_2-t_1)}, (1+2\epsilon^{\frac{1}{4}}\lambda)\frac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho\right\}. \end{aligned} \tag{A.34}$$

In view of (A.13), we have

$$\begin{aligned} V(\phi(t,2)) \leq \max\{ \; & e^{-\frac{\mu}{2}(t-t_2)}V(\phi(t_2,2)), \tfrac{2(1+L)}{\mu}\rho\} \\ \leq \max\{ \; & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 e^{-\frac{\mu}{2}t_2} e^{-\frac{\mu}{2}(t-t_2)} V(\phi(0,0)), \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_2-t_1)} e^{-\frac{\mu}{2}(t-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \rho e^{-\frac{\mu}{2}(t_2-t_1)} e^{-\frac{\mu}{2}(t-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda) \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\frac{\mu}{2}(t-t_2)}, \tfrac{2(1+L)}{\mu}\rho\} \end{aligned} \tag{A.35}$$

$$\begin{aligned} V(\phi(t,2)) \leq \max\{ \; & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 e^{-\frac{\mu}{2}t} V(\phi(0,0)), \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_2-t_1)} e^{-\frac{\mu}{2}(t-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \rho e^{-\frac{\mu}{2}(t_2-t_1)} e^{-\frac{\mu}{2}(t-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda) \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\frac{\mu}{2}(t-t_2)}, \tfrac{2(1+L)}{\mu}\rho\}. \end{aligned} \tag{A.36}$$

At $t = t_3$, we have

$$\begin{aligned} V(\phi(t_3,2)) \leq \max\{ \; & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 e^{-\frac{\mu}{2}t_3} V(\phi(0,0)), \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_2-t_1)} e^{-\frac{\mu}{2}(t_3-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \rho e^{-\frac{\mu}{2}(t_2-t_1)} e^{-\frac{\mu}{2}(t_3-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda) \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_3-t_2)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\frac{\mu}{2}(t_3-t_2)}, \tfrac{2(1+L)}{\mu}\rho\} \end{aligned} \tag{A.37}$$

In view of (A.20), (A.37), we obtain

$$
\begin{aligned}
V(\phi(t_3,3)) &\leq (1+2\epsilon^{\frac{1}{4}}\lambda)\max\{V(\phi(t_3,2)),\rho\} \\
&\leq \max\big\{(1+2\epsilon^{\frac{1}{4}}\lambda)^3 e^{-\frac{\mu}{2}t_3}V(\phi(0,0)), \\
&\qquad (1+2\epsilon^{\frac{1}{4}}\lambda)^3\tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_2-t_1)}e^{-\frac{\mu}{2}(t_3-t_2)}, \\
&\qquad (1+2\epsilon^{\frac{1}{4}}\lambda)^3\rho e^{-\frac{\mu}{2}(t_2-t_1)}e^{-\frac{\mu}{2}(t_3-t_2)}, \\
&\qquad (1+2\epsilon^{\frac{1}{4}}\lambda)^2\tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}(t_3-t_2)}, \\
&\qquad (1+2\epsilon^{\frac{1}{4}}\lambda)^2\rho e^{-\frac{\mu}{2}(t_3-t_2)}, (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, \rho\}.
\end{aligned}
\tag{A.38}
$$

Since $\tau(\Delta)$ is a dwell-time, then $t_{j+1}-t_j \geq \tau(\Delta)$ and $t_j \geq j\tau(\Delta)$. Hence, it holds that

$$
\begin{aligned}
V(\phi(t_3,3)) \leq \max\big\{&(1+2\epsilon^{\frac{1}{4}}\lambda)^3 e^{-\frac{\mu}{2}3\tau(\Delta)}V(\phi(0,0)), \\
&(1+2\epsilon^{\frac{1}{4}}\lambda)^3\tfrac{2(1+L)}{\mu}\rho e^{-2\tau(\Delta)}, (1+2\epsilon^{\frac{1}{4}}\lambda)^3\rho e^{-2\tau(\Delta)}, \\
&(1+2\epsilon^{\frac{1}{4}}\lambda)^2\tfrac{2(1+L)}{\mu}\rho e^{-\tau(\Delta)}, (1+2\epsilon^{\frac{1}{4}}\lambda)^2\rho e^{-\tau(\Delta)}, \\
&(1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, \rho\}.
\end{aligned}
\tag{A.39}
$$

The inequality (A.22) ensures that

$$
\tau(\Delta) \quad \geq \quad \tfrac{2}{\mu}\ln(1+2\epsilon^{\frac{1}{4}}\lambda) \tag{A.40}
$$

implying that

$$
(1+2\epsilon^{\frac{1}{4}}\lambda)e^{-\frac{\mu}{2}\tau(\Delta)} \leq 1. \tag{A.41}
$$

Consequently, $(1+2\epsilon^{\frac{1}{4}}\lambda)^n e^{-\frac{\mu}{2}n\tau(\Delta)} \leq 1$ for any $n > 1$. As a result, (A.39) verifies

$$
\begin{aligned}
V(\phi(t_3,3)) &\leq \max\big\{V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho, \\
&\qquad (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, \rho\} \\
&= \max\big\{V(\phi(0,0)), (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho\}.
\end{aligned}
\tag{A.42}
$$

Thus, by induction, we deduce that, for any $(t_j,j)\mathrm{dom}\,\phi$

$$
V(\phi(t_j,j)) \leq \max\big\{V(\phi(0,0)), \theta\rho\}, \tag{A.43}
$$

where $\theta = (1+2\lambda)\max\{2\frac{(1+L)}{\mu}, 1\}$ and by using the fact that $\epsilon^{\frac{1}{4}} < 1$ since $\epsilon \in (0, \epsilon^*)$ is sufficiently small. In view of (A.36), (A.41), it holds that

$$\begin{aligned} V(\phi(t,2)) \leq \max\{ & \ (1+2\epsilon^{\frac{1}{4}}\lambda)^2 e^{-\frac{\mu}{2}t} V(\phi(0,0)), \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \tfrac{2(1+L)}{\mu}\rho e^{-\frac{\mu}{2}\tau(\Delta)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)^2 \rho e^{-\tau(\Delta)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho e^{-\tau(\Delta)}, \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\rho e^{-\tau(\Delta)}, \tfrac{2(1+L)}{\mu}\rho\} \\ \leq \max\{ & \ (1+2\epsilon^{\frac{1}{4}}\lambda)^2 e^{-\frac{\mu}{2}t} V(\phi(0,0)), \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho, \\ & \tfrac{2(1+L)}{\mu}\rho, \rho, \tfrac{2(1+L)}{\mu}\rho\} \end{aligned} \tag{A.44}$$

$$\begin{aligned} V(\phi(t,2)) \leq \max\{ & \ (1+2\epsilon^{\frac{1}{4}}\lambda)^2 e^{-\frac{\mu}{2}t} V(\phi(0,0)), \\ & (1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{2(1+L)}{\mu}\rho, (1+2\epsilon^{\frac{1}{4}}\lambda)\rho\}. \end{aligned} \tag{A.45}$$

Hence, by induction, we deduce that, for any $(t,j) \in \operatorname{dom}\phi$

$$\begin{aligned} V(\phi(t,j)) &\leq \max\left\{(1+2\epsilon^{\frac{1}{4}}\lambda)^j e^{-\frac{\mu}{2}t} V(\phi(0,0)), \theta\rho\right\} \\ &= \max\left\{e^{\ln(1+2\epsilon^{\frac{1}{4}}\lambda)j - \frac{\mu}{2}t} V(\phi(0,0)), \theta\rho\right\} \end{aligned} \tag{A.46}$$

We now use similar arguments as in Proposition 3.29 [34] to conclude. Let $(t,j) \in \operatorname{dom}\phi$. We want to show that $e^{\ln(1+2\epsilon^{\frac{1}{4}}\lambda)j - \frac{\mu}{2}t} \leq e^{-\psi(t+j)}$, where $\psi > 0$, which is equivalent to show that

$$\ln(1+2\epsilon^{\frac{1}{4}}\lambda)j - \tfrac{\mu}{2}t \leq -\psi(t+j). \tag{A.47}$$

By re-arranging the terms

$$(\ln(1+2\epsilon^{\frac{1}{4}}\lambda) + \psi)j \quad \leq \quad (\tfrac{\mu}{2} - \psi)t. \tag{A.48}$$

Since $\tau(\Delta)$ is a dwell-time by assumption, it holds that, for any $(t,j) \in \operatorname{dom}\phi$, $t \geq \tau(\Delta)j$, *i.e.*

$j \leq \frac{t}{\tau(\Delta)}$. As a result, the following inequality ensures (A.48)

$$\begin{aligned}(\ln(1+2\epsilon^{\frac{1}{4}}\lambda)+\psi)\tfrac{t}{\tau(\Delta)} &\leq (\tfrac{\mu}{2}-\psi)t \\ (\ln(1+2\epsilon^{\frac{1}{4}}\lambda)+\psi)\tfrac{1}{\tau(\Delta)} &\leq \tfrac{\mu}{2}-\psi \\ \psi(\tfrac{1}{\tau(\Delta)}+1) &\leq \tfrac{\mu}{2}-\ln(1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{1}{\tau(\Delta)}.\end{aligned} \tag{A.49}$$

We therefore see that it suffices to have $\frac{\mu}{2}-\ln(1+2\epsilon^{\frac{1}{4}}\lambda)\frac{1}{\tau(\Delta)} > 0$ to guarantee the existence of $\psi > 0$, which can be written as

$$\frac{\mu}{2} > \ln(1+2\epsilon^{\frac{1}{4}}\lambda)\frac{1}{\tau(\Delta)}. \tag{A.50}$$

Using the definition of $\tau(\Delta)$, we obtain

$$\begin{aligned}\tfrac{\mu}{2} &> \ln(1+2\epsilon^{\frac{1}{4}}\lambda)\tfrac{\mu}{4\ln(1+2\sqrt{d(\Delta)}\lambda)} \\ \tfrac{\mu}{2} &> \tfrac{\mu}{4}\end{aligned} \tag{A.51}$$

which is always true since $\mu > 0$. Hence, we take

$$\psi \in \left(0, \frac{\frac{\mu}{2}-\ln(1+2\epsilon^{\frac{1}{4}}\lambda)\frac{1}{\tau(\Delta)}}{1+\frac{1}{\tau(\Delta)}}\right). \tag{A.52}$$

As a consequence, for any $(t,j) \in \operatorname{dom}\phi$,

$$V(\phi(t,j)) \leq \max\left\{e^{-\gamma(t+j)}V(\phi(0,0)), \theta\rho\right\}. \tag{A.53}$$

To finish the proof of Claim A.1, we need to show that (A.22) is always satisfied. For that purpose, we study the dynamics of the sampling error function $\gamma_1(|e|)$ as follows. The length of the inter-jump interval is lower bounded by the time it takes for $\gamma_1(|\phi_e|)$ to grow from 0 to ρ in view of (4.31). In view of (4.24), (4.25), (A.29), it holds that

$$\begin{aligned}V(\phi(t,0)) &\leq \max\left\{e^{-\frac{\mu}{2}t}V(\phi(0,0)), 2\tfrac{(1+L)}{\mu}\rho\right\} \\ &\leq \max\left\{V(\phi(0,0)), \theta\rho\right\} \\ &\leq \max\left\{\overline{\alpha}_x(|\phi_x(0,0)|)+\sqrt{\epsilon}\overline{\alpha}_y(|\phi_y(0,0)|), \theta\rho\right\} \\ &\leq \max\left\{\overline{\alpha}(|(\phi_x(0,0),\phi_y(0,0))|), \theta\rho\right\},\end{aligned} \tag{A.54}$$

where $\overline{\alpha}(|(\phi_x(t,j),\phi_y(t,j))|) = \overline{\alpha}_x(|\phi_x(t,j)|) + \overline{\alpha}_y(|\phi_y(t,j)|)$ and using the fact that $\epsilon^* \leq 1$. Consequently, $(\phi_x(t,0),\phi_y(t,0))$ lie in the compact set, for all $(t,0) \in \operatorname{dom}\phi$

$$\mathcal{S}(\Delta) := \{(x,y)\, :\, V(x,y,0) \leq \max\{\overline{\alpha}(\Delta),\theta\rho\}\}. \tag{A.55}$$

Since $\phi_e(t,0) = \phi_x(0,0) - \phi_x(t,0)$, we deduce that $\phi(t,0)$ lie in the compact set $\mathcal{S}(\Delta)$ for all $(t,0) \in \operatorname{dom}\phi$. Since γ_1 is continuously differentiable by assumption, ϕ is continuous between two jump instants, f_x is continuous and $\mathcal{S}(\Delta)$ is compact

$$\begin{aligned} \tfrac{d}{dt}\gamma_1(|\phi_e(t,0)|) &\leq \partial\gamma_1(|\phi_e(t,0)|)|f_x(\phi(t,0))| \\ &\leq \sup_{q\in\mathcal{S}(\Delta)}\{\partial\gamma_1(|e|)|f_x(x,y,e)|\} \\ &< \xi(\Delta), \end{aligned} \tag{A.56}$$

for some $\xi(\Delta) > 0$, which ensures the property on the inter-jump intervals stated below (4.32). Hence $\tau(\Delta) \geq \frac{\rho}{\xi(\Delta)}$. The first jump instant t_1 is lower bounded by the time it takes for $\gamma_1(|e|)$ to grow form 0 to ρ which in return is lower bounded by the time it takes for $t \mapsto \xi(\Delta)$ to grow from 0 to ρ, i.e.

$$t_1 \geq \frac{\rho}{\xi(\Delta)}. \tag{A.57}$$

By following similar lines as above, we deduce that $t_{j+1} - t_j \geq \frac{\rho}{\xi(\Delta)}$ for all $(t,j) \in \operatorname{dom}\phi$. To satisfy dwell-time condition (A.22), the following must hold, for all $(t,j) \in \operatorname{dom}\phi$

$$t_{j+1} - t_j \geq \frac{\rho}{\xi(\Delta)} \geq \frac{4}{\mu}\ln(1 + 2\epsilon^{\frac{1}{4}}\lambda) \tag{A.58}$$

which is verified for $\epsilon \in (0,\epsilon^*)$, where

$$\epsilon^* := \min\left\{\left(\frac{1}{2}(e^{\frac{4\rho}{\mu\xi(\Delta)}} - 1)\right)^{-\frac{1}{4}}, 1\right\} \tag{A.59}$$

which completes the proof of Claim A.1. ■

A.2 Proof of Theorem 4.2

The proof uses elements of the proofs of Theorems 2.1, 4.1 and Theorem 1 in [79]. We first build a differential equation from which the value of $\mathcal{T}$ in (4.34) is obtained. Second, we construct a Lyapunov function candidate R for system (4.22). Third, we study the evolution of R along flows and jumps. Finally, we apply Proposition 3.29 in [34] to deduce (4.36).

Let the function $\varphi : [0, \overline{\mathcal{T}}] \to \mathbb{R}$ be the solution of the differential equation, see [17],

$$\dot{\varphi} = -1 - 2M\varphi - \frac{\bar{\gamma}_1 N^2}{\alpha_1}\varphi(\tau)^2 \quad \varphi(0) = \vartheta^{-1} \quad \varphi(\overline{\mathcal{T}}) = \vartheta, \tag{A.60}$$

where $\vartheta \in (0,1)$, M, N come from Assumption 4.6 and $\bar{\gamma}_1$ is defined in condition (1) in Theorem 4.2. The time $\overline{\mathcal{T}}$ is the time it takes for the φ to decrease from ϑ^{-1} to ϑ and is given by the following claim.

Claim A.2. *(i) For all $\tau \in [0, \overline{\mathcal{T}}]$ we have $\varphi(\tau) \in [\vartheta, \vartheta^{-1}]$ with*

$$\overline{\mathcal{T}}(\vartheta, \alpha_1, \bar{\gamma}_1, M, N) := \begin{cases} \frac{1}{Mr}\arctan \frac{r(1-\vartheta^2)}{1+\vartheta(\frac{2N^2}{\alpha_1}\frac{\bar{\gamma}_1}{M}+\vartheta)+\frac{\alpha_1}{2N^2}\frac{M}{\bar{\gamma}_1}\vartheta(1+r^2)} & M^2 < \frac{\bar{\gamma}_1 N^2}{\alpha_1} \\ \frac{1}{M}\frac{(1-\vartheta^2)}{(1+\vartheta^2)+\frac{\alpha_1}{2N^2}\frac{M}{\bar{\gamma}_1}\vartheta(1+\frac{2N^2}{\alpha_1})} & M^2 = \frac{\bar{\gamma}_1 N^2}{\alpha_1} \\ \frac{1}{Mr}\operatorname{arctanh} \frac{r(1-\vartheta^2)}{1+\vartheta(\frac{2N^2}{\alpha_1}\frac{\bar{\gamma}_1}{M}+\vartheta)+\frac{\alpha_1}{2N^2}\frac{M}{\bar{\gamma}_1}\vartheta(1-r^2)} & M^2 > \frac{\bar{\gamma}_1 N^2}{\alpha_1}, \end{cases} \tag{A.61}$$

where r is defined in (4.35).

(ii) $\overline{\mathcal{T}}(\vartheta, \alpha_1, \bar{\gamma}_1, M, N) \to \mathcal{T}(\alpha_1, \bar{\gamma}_1, M, N)$ when $\vartheta \to 0$, where $\mathcal{T}(\alpha_1, \bar{\gamma}_1, M, N)$ defined in (4.34).

■

The proof of Claim A.2 is given after the proof of Theorem 4.2.

We now define the following differential system

$$\begin{aligned} \dot{\zeta} &= -1 - 2M\zeta(\tau) - \eta - \left(\eta\zeta(\tau) + \tfrac{\bar{\gamma}_1}{\alpha_1 - \eta}(N\zeta(\tau))^2\right) \\ &=: f_\zeta(\tau) \end{aligned} \tag{A.62}$$

with $\zeta(0) = \vartheta^{-1}$, $\vartheta \in (0,1)$ and $\eta \in (0, \alpha_1)$. Let $\widetilde{\mathcal{T}}(\eta, \vartheta)$ denotes the time it takes for ζ to decrease from ϑ^{-1} to ϑ. We note that this time $\widetilde{\mathcal{T}}(\eta, \vartheta)$ is a continuous function of η, ϑ which is decreasing in η, ϑ (by invoking the comparison principle). On the other hand, we note that $\widetilde{\mathcal{T}}(\eta, \vartheta) \to \mathcal{T}$ as (η, ϑ) tends to $(0,0)$ by following similar lines as in the proof of Claim 1 in [79], where $\mathcal{T}$ is defined in (4.34). As a consequence, since $T^* < \mathcal{T}$, there exist η, ϑ such that $T^* \leq \widetilde{\mathcal{T}}(\eta, \vartheta)$ which we fix.

We define

$$R(q) := V_x(x) + dV_y(x,y) + \max\{0, \bar{\gamma}_1\zeta(\tau)|e|^2\} \;\; \forall q \in \mathbb{R}^{n_q}, \tag{A.63}$$

where

$$d \in \left(0, \min\{\frac{\bar{\gamma}_1}{\bar{\gamma}_2}\eta, \frac{1-\sigma}{\sigma}\frac{\bar{\gamma}_1}{\bar{\gamma}_2}, \frac{\vartheta^2}{(\lambda_1+\lambda_2)^2}, \frac{(e^{\eta T^*}-1)^2}{\lambda^2}, 1\}\right) \tag{A.64}$$

and

$$\lambda := \max\{\lambda_2, (\lambda_1+\lambda_2)\sigma\alpha_1\}. \tag{A.65}$$

Let $q \in C$ and consider the case where $\zeta(\tau) > 0$. In view of Assumptions 4.2-4.6 and Lemma 1

$$R^\circ(q; F(q)) \;\; \leq \;\; -(\chi, |e|)^T \mathcal{A}_1 (\chi, |e|), \tag{A.66}$$

where $\chi := (\sqrt{V_x(x)}, \sqrt{V_y(x,y)})$,

$$\mathcal{A}_1 := \begin{pmatrix} \alpha_1 & -\frac{1}{2}(\beta_1 + d\beta_2) & -\bar{\gamma}_1 N\zeta(\tau) \\ * & \frac{d}{\epsilon}\alpha_2 - d\beta_3 & -\bar{\gamma}_1 N\zeta(\tau) \\ * & * & \upsilon(\tau) \end{pmatrix} \tag{A.67}$$

and

$$\upsilon(\tau) := -\bar{\gamma}_1 - d\bar{\gamma}_2 - \bar{\gamma}_1 f_\zeta(\tau) - 2\bar{\gamma}_1 M\zeta(\tau). \tag{A.68}$$

The following conditions ensure that, according to Sylvester's criterion,

$$\mathcal{A}_1 \geq \eta \text{diag}(1, d, \bar{\gamma}_1\zeta(\tau)). \tag{A.69}$$

As a consequence,

$$\begin{cases} 0 \;\; \leq \;\; \alpha_1 - \eta \\ 0 \;\; \leq \;\; (\alpha_1 - \eta)d(\frac{1}{\epsilon}\alpha_2 - \beta_3 - \eta) \geq \frac{1}{4}(\beta_1 + d\beta_2)^2 \\ 0 \;\; \leq \;\; (\alpha_1 - \eta)\left\{d(\frac{1}{\epsilon}\alpha_2 - \beta_3 - \eta)(\upsilon(\tau) - \eta\bar{\gamma}_1\zeta(\tau)) - (\bar{\gamma}_1\zeta(\tau)N)^2\right\} \\ \quad +\frac{1}{2}(\beta_1 + d\beta_2)\left\{-\frac{1}{2}(\beta_1 + d\beta_2)(\upsilon(\tau) - \eta\bar{\gamma}_1\zeta(\tau)) - (\bar{\gamma}_1\zeta(\tau)N)^2\right\} \\ \quad -\bar{\gamma}_1 N\zeta(\tau)\left\{\frac{1}{2}(\beta_1 + d\beta_2)\bar{\gamma}_1\zeta(\tau)N + \bar{\gamma}_1 N\zeta(\tau)d(\frac{1}{\epsilon}\alpha_2 - \beta_3 - \eta)\right\}. \end{cases} \tag{A.70}$$

The first two inequalities above are respectively verified by definition of η and by taking ϵ

sufficiently small. For the last inequality to hold, it suffices to select ϵ sufficiently small provided that

$$\frac{d}{\epsilon}\alpha_2\left((\alpha_1-\eta)(\upsilon-\eta\bar{\gamma}_1\zeta(\tau))-(\bar{\gamma}_1 N\zeta(\tau))^2\right)>0 \tag{A.71}$$

which is equivalent to, by definition of υ and definition of f_ζ in (A.62),

$$(\alpha_1-\eta)(\bar{\gamma}_1\eta-d\bar{\gamma}_2)>0 \tag{A.72}$$

which holds by definition of d and η. Consequently, by selecting ϵ sufficiently small

$$R^\circ(q;F(q)) \quad \leq \quad -\eta R(q). \tag{A.73}$$

Suppose now that $\zeta(\tau)<0$, hence $\bar{\gamma}_1|e|^2\leq\sigma\alpha_1 V_x(x)$ in view of the definition of the set C. Using Assumptions 4.2-4.6 and Lemma 1,

$$R^\circ(q;F(q)) \quad \leq \quad -\chi^T\mathcal{A}_2\chi, \tag{A.74}$$

where

$$\mathcal{A}_2:=\begin{pmatrix}\alpha_1\left(1-\sigma(1+d\bar{\gamma}_2\bar{\gamma}_1^{-1})\right) & -\frac{1}{2}(\beta_1+d\beta_2)\\ -\frac{1}{2}(\beta_1+d\beta_2) & \frac{d}{\epsilon}\alpha_2-d\beta_3\end{pmatrix}. \tag{A.75}$$

By following similar arguments as above and since $d<\frac{1-\sigma}{\sigma}\frac{\bar{\gamma}_1}{\bar{\gamma}_2}$ and $R(q)=V_x(x)+dV_y(x,y)$ in this case, we derive that (A.73) holds by selecting ϵ sufficiently small.

When $\zeta(\tau)=0$, (A.73) is verified in view of Lemma 1 and the results obtained for the cases where $\zeta(\tau)>0$ and $\zeta(\tau)<0$.

Let $q\in D$. Suppose that $\tau=T^*$ (note that $\bar{\gamma}_1|e|^2\geq\sigma\alpha_1 V_x(x)$ in this case). In view of Assumption 4.5

$$\begin{aligned}R(G(q)) &= V_x(x)+dV_y(x,h_y(x,y,e))\\ &\leq V_x(x)+d\Big(V_y(x,y)+\lambda_1\bar{\gamma}_1|e|^2+\lambda_2\sqrt{\bar{\gamma}_1|e|^2V_y(x,y)}\Big).\end{aligned} \tag{A.76}$$

Using that

$$\sqrt{\bar{\gamma}_1|e|^2V_y(x,y)}\leq\frac{1}{\sqrt{d}}\bar{\gamma}_1|e|^2+\sqrt{d}V_y(x,y) \tag{A.77}$$

and since $d \leq \sqrt{d} \leq 1$, it holds that

$$\begin{aligned} R(G(q)) &\leq V_x(x) + dV_y(x,y) + \sqrt{d}(\lambda_1+\lambda_2)\bar{\gamma}_1|e|^2 + d\lambda_2\sqrt{d}V_y(x,y) \\ &\leq V_x(x) + dV_y(x,y) + \sqrt{d}(\lambda_1+\lambda_2)\bar{\gamma}_1|e|^2 + \lambda_2\sqrt{d}(V_x(x) + dV_y(x,y)). \end{aligned} \tag{A.78}$$

We take d sufficiently small such that (since $\zeta(\widetilde{\mathcal{T}}(\eta,\vartheta)) = \vartheta$)

$$\begin{aligned} \sqrt{d}(\lambda_1+\lambda_2)\bar{\gamma}_1|e|^2 &\leq \bar{\gamma}_1\zeta(\widetilde{\mathcal{T}}(\eta,\vartheta))|e|^2 \\ &= \bar{\gamma}_1\vartheta|e|^2 \end{aligned} \tag{A.79}$$

As a consequence

$$R(G(q)) \leq (1+\lambda_2\sqrt{d})(V_x(x) + dV_y(x,y) + \bar{\gamma}_1\zeta(\widetilde{\mathcal{T}}(\eta,\vartheta))|e|^2). \tag{A.80}$$

Since in this case we transmit at $\tau = T^* \leq \widetilde{\mathcal{T}}(\eta,\vartheta)$, then $\zeta(\tau) \geq \zeta(\widetilde{\mathcal{T}}(\eta,\vartheta))$, as $\zeta(\tau)$ is a decreasing function, and we obtain

$$R(G(q)) \leq (1+\lambda_2\sqrt{d})R(q). \tag{A.81}$$

When $\tau > T^*$, it holds that $\bar{\gamma}_1|e|^2 = \sigma\alpha_1 V_x(x)$ in view of (4.33). Hence, by following similar lines as above, we deduce that

$$R(G(q)) \leq (1+\lambda\sqrt{d})R(q), \tag{A.82}$$

where $\lambda = \max\{\lambda_2, (\lambda_1+\lambda_2)\sigma\alpha_1\}$. Thus, (A.82) holds for all $q \in D$ (since $\lambda_2 \leq \lambda$).

Finally, we use similar arguments as in Proposition 3.29 in [34] to conclude. In view of (A.73) and (A.82), the property (3.10) in Proposition 3.29 holds with $\lambda_c = -\eta$ and $e^{\lambda_d} = (1+\lambda\sqrt{d})$. Let $\psi > 0$ and $(t,j) \in \operatorname{dom}\phi$. To satisfy the last condition of Proposition 3.29, we need to show that

$$\ln(1+\lambda\sqrt{d})j - \eta t \leq -\psi(t+j). \tag{A.83}$$

Since $j \leq \frac{t}{T^*}$ in view of (4.33), it suffices to show that

$$\ln(1+\lambda\sqrt{d})\frac{t}{T^*} - \eta t \leq -\psi(t+\frac{t}{T^*}) \tag{A.84}$$

which is equivalent to

$$(\ln(1+\lambda\sqrt{d})+\psi)\frac{t}{T^*} \leq (\eta-\psi)t \tag{A.85}$$

i.e.

$$\psi(\frac{1}{T^*}+1) \leq \eta - \ln(1+\lambda\sqrt{d})\frac{1}{T^*}. \tag{A.86}$$

Hence, we take $d \leq (\frac{e^{\eta T^*}-1}{\lambda})^2$ which ensures that

$$\eta - \ln(1+\lambda\sqrt{d})\frac{1}{T^*} > 0. \tag{A.87}$$

It then suffices to take

$$\psi \in (0, \frac{\eta - \ln(1+\lambda\sqrt{d})\frac{1}{T^*}}{\frac{1}{T^*}+1}) > 0. \tag{A.88}$$

As a result, like in the proof of Proposition 3.29 in [34], we obtain, for all $(t, j) \in \operatorname{dom}\phi$

$$R(\phi(t,j)) \leq e^{-\psi(t+j)}R(\phi(0,0)). \tag{A.89}$$

By using Assumptions 4.2-4.3 and the fact that $\zeta(\tau) \in [\vartheta, \vartheta^{-1}]$, we deduce from (A.89) that (4.36) holds.

Let $\phi = (\phi_x, \phi_y, \phi_e, \phi_\tau)$ be a maximal solution to (4.22)-(4.33). We note that ϕ is non-trivial by using similar arguments as in the proof of Theorem 4.1. In view of (A.89), ϕ_x and ϕ_y cannot explode in finite time. Since $\phi_e(t,j) = \phi_x(t_j,j) - \phi_x(t,j)$ for any $(t_j,j),(t,j) \in \operatorname{dom}\phi$ and $j \geq 1$, ϕ_e cannot explode in finite time. The same conclusion holds for ϕ_τ in view of its dynamics, see (4.22). Hence, ϕ cannot explode in finite-time. In addition, $G(D) \subset C$. As a consequence, ϕ is complete according to Proposition 6.10 in [34]. □

Now we provide the proof of Claim A.2.

Proof of Claim A.2: Let $\beta := \frac{\bar{\gamma}_1 N^2}{\alpha_1}$ for the sake of simplicity. We follow similar lines as in the proof of Lemma 2 in [17]. In view of (A.60), it holds that

$$\mathcal{T} = -\int_{\frac{1}{\vartheta}}^{\vartheta} \frac{d\phi}{\beta\phi^2 + 2M\phi + 1}. \tag{A.90}$$

We define $s := \phi + \frac{M}{\beta}$. Hence, (A.90) in terms of s becomes

$$\begin{aligned} \mathcal{T} &= -\int_{\frac{1}{\vartheta}+\frac{M}{\beta}}^{\vartheta+\frac{M}{\beta}} \frac{ds}{\beta s^2 - \frac{M^2}{\beta}+1} \\ &= -\frac{1}{\beta}\int_{\frac{1}{\vartheta}+\frac{M}{\beta}}^{\vartheta+\frac{M}{\beta}} \frac{ds}{s^2-(\frac{M}{\beta})^2+\frac{1}{\beta}}. \end{aligned}$$

In view of (4.35), $r = \sqrt{\left|\frac{\beta}{M^2}-1\right|}$. Hence, (A.91) can be written as

$$\mathcal{T} = -\frac{1}{\beta}\int_{\frac{1}{\vartheta}+\frac{M}{\beta}}^{\vartheta+\frac{M}{\beta}} \frac{ds}{s^2 - \operatorname{sgn}(M^2-\beta)(\frac{Mr}{\beta})^2}, \tag{A.91}$$

where $\operatorname{sgn}(.)$ is the sign function with $\operatorname{sgn}(0) = 0$.

When $M^2 = \beta$, using the fact that $-\frac{1}{\beta}\int_a^b \frac{ds}{s^2} = \frac{1}{\beta}(\frac{1}{b} - \frac{1}{a})$, we have

$$\begin{aligned} \mathcal{T} &= \frac{1}{\beta}\left(\frac{\beta}{\beta\vartheta+M} - \frac{\beta\vartheta}{\beta+M\vartheta}\right) \\ &= \frac{1}{\beta\vartheta+M} - \frac{\vartheta}{\beta+M\vartheta} \\ &= \frac{\beta+M\vartheta-\vartheta(\beta\vartheta+M)}{(\beta\vartheta+M)(\beta+M\vartheta)} \\ &= \frac{\beta(1-\vartheta^2)}{\beta^2\vartheta+\beta M\vartheta^2+\beta M+M^2\vartheta} \end{aligned}$$

since $M^2 = \beta$, it holds that

$$\begin{aligned} \mathcal{T} &= \frac{\beta(1-\vartheta^2)}{\beta M^2\vartheta+\beta M\vartheta^2+\beta M+M^2\vartheta} \\ &= \frac{1}{M}\frac{\beta(1-\vartheta^2)}{\beta M\vartheta+\beta\vartheta^2+\beta+M\vartheta} \\ &= \frac{1}{M}\frac{\beta(1-\vartheta^2)}{\beta(1+\vartheta^2)+M\vartheta(1+\beta)} \\ &= \frac{1}{M}\frac{(1-\vartheta^2)}{(1+\vartheta^2)+\frac{M\vartheta}{\beta}(1+\beta)}. \end{aligned}$$

When $M^2 < \beta$, using the fact that $-\frac{1}{\beta}\int_a^b \frac{ds}{s^2+(\frac{Mr}{\beta})^2} = -\frac{1}{Mr}\Big(\arctan(\frac{b\beta}{Mr}) - \arctan(\frac{a\beta}{Mr})\Big)$ and that for all $n_1, n_2 \geq 0$ we have $\arctan(n_1) - \arctan(n_2) = \arctan(\frac{n_1-n_2}{1+n_1n_2})$, we obtain

$$\begin{aligned}
\mathcal{T} &= -\tfrac{1}{Mr}\left(\arctan\left(\tfrac{\beta\vartheta+M}{\beta}\tfrac{\beta}{Mr}\right) - \arctan\left(\tfrac{\beta+M\vartheta}{\beta\vartheta}\tfrac{\beta}{Mr}\right)\right)\\
&= -\tfrac{1}{Mr}\left(\arctan\left(\tfrac{\beta\vartheta+M}{Mr}\right) - \arctan\left(\tfrac{\beta+M\vartheta}{Mr\vartheta}\right)\right)\\
&= -\tfrac{1}{Mr}\arctan\left(\frac{\frac{\beta\vartheta+M}{Mr}-\frac{\beta+M\vartheta}{Mr\vartheta}}{1+\frac{(\beta\vartheta+M)(\beta+M\vartheta)}{M^2r^2\vartheta}}\right)\\
&= -\tfrac{1}{Mr}\arctan\left(\frac{\frac{\beta\vartheta^2+M\vartheta-\beta-M\vartheta}{Mr\vartheta}}{\frac{M^2r^2\vartheta+\beta^2\vartheta+\beta M\vartheta^2+\beta M+M^2\vartheta}{M^2r^2\vartheta}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{T} &= -\tfrac{1}{Mr}\arctan\left(\tfrac{\beta Mr(\vartheta^2-1)}{M^2r^2\vartheta+\beta^2\vartheta+\beta M\vartheta^2+\beta M+M^2\vartheta}\right)\\
&= -\tfrac{1}{Mr}\arctan\tfrac{\beta Mr(\vartheta^2-1)}{\beta M(1+\frac{\beta}{M}\vartheta+\vartheta^2)+M^2\vartheta(1+r^2)}
\end{aligned}$$

using the fact that $-\arctan(x) = \arctan(-x)$, it holds that

$$\mathcal{T} = \tfrac{1}{Mr}\arctan\tfrac{r(1-\vartheta^2)}{1+\vartheta(\frac{\beta}{M}+\vartheta)+\frac{M\vartheta}{\beta}(1+r^2)}. \tag{A.92}$$

When $M^2 > \beta$, using the fact that $\frac{1}{\beta}\int_a^b \frac{ds}{(\frac{Mr}{\beta})^2-s^2} = \frac{1}{Mr}\Big(\operatorname{arctanh}(\frac{b\beta}{Mr}) - \operatorname{arctanh}(\frac{a\beta}{Mr})\Big)$ and that for all $n_1, n_2 \geq 0$ we have $\operatorname{arctanh}(n_1) - \operatorname{arctanh}(n_2) = \operatorname{arctanh}(\frac{n_1-n_2}{1-n_1n_2})$ and in the light of (A.92), we obtain

$$\begin{aligned}
\mathcal{T} &= \tfrac{1}{Mr}\operatorname{arctanh}\left(\tfrac{\beta Mr(\vartheta^2-1)}{M^2r^2\vartheta-\beta^2 2\vartheta-\beta M\vartheta^2-\beta M-M^2\vartheta}\right)\\
&= \tfrac{1}{Mr}\operatorname{arctanh}\tfrac{-\beta Mr(1-\vartheta^2)}{-\beta M(1+\frac{\beta}{M}\vartheta+\vartheta^2)-M^2\vartheta(1-r^2)}\\
&= \tfrac{1}{Mr}\operatorname{arctanh}\tfrac{\beta Mr(1-\vartheta^2)}{\beta M(1+\frac{\beta}{M}\vartheta+\vartheta^2)+M^2\vartheta(1-r^2)}\\
&= \tfrac{1}{Mr}\operatorname{arctanh}\tfrac{r(1-\vartheta^2)}{1+\vartheta(\frac{\beta}{M}+\vartheta)+\frac{M\vartheta}{\beta}(1-r^2)}
\end{aligned}$$

recall that $r^2 < 1$ since $M^2 > \beta$. Thus, in view of (A.92), (A.92) and (A.93), Claim A.2 holds.

■

Appendix B

Mathematical review

In this appendix, we provide some mathematical preliminaries and fundamental tools that have been used to obtain the technical results.

B.1 Fundamental properties

The following concepts are useful to investigate the existence and uniqueness of solutions of differential equations.

Definition B.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally Lipschitz in x if there exist a neighbourhood $D \subset \mathbb{R}^n$ of x and a constant $L \geq 0$, called the Lipschitz constant, such that*

$$|f(y) - f(x)| \leq L|y - x| \qquad \forall x, y \in D. \tag{B.1}$$

If $D = \mathbb{R}^n$, we say that the function f is globally Lipschitz. □

The following lemma shows that a continuously differentiable function is locally Lipschitz.

Lemma B.1 (Lemma 3.1 [52]). *Let $f : [a, b] \times D \to \mathbb{R}^m$ be continuous for some domain $D \subset \mathbb{R}^n$. Suppose that $[\frac{\partial f}{\partial x}]$ exists and is continuous on $[a, b] \times D$. If, for a convex subset $W \subset D$, there is a constant $L \geq 0$ such that*

$$\left| \frac{\partial f}{\partial x}(t, x) \right| \leq L \tag{B.2}$$

on $[a, b] \times W$*, then*

$$|f(t, x) - f(t, y)| \leq L|x - y| \tag{B.3}$$

for all $t \in [a, b]$*,* $x \in W$*, and* $y \in W$*.* □

The local Lipschitz property of a function is stronger than continuity and weaker than continuous differentiability. The absolute value function for instance is globally Lipschitz but not (continuously) differentiable everywhere.

Functions properties

A function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$.

The sign function of a real number x, denoted as $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$, is defined as

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if} \quad x < 0; \\ 0 & \text{if} \quad x = 0; \\ 1 & \text{if} \quad x > 0. \end{cases} \tag{B.4}$$

The following properties of comparison functions are useful and have been used within the proofs of our results.

Lemma B.2 (Lemma 4.2 [52])**.** *Let* α_1 *and* α_2 *be class* $\mathcal{K}$ *functions on* $[0, a)$ *for some* $a > 0$*,* α_3 *and* α_4 *be class* $\mathcal{K}_\infty$ *functions, and* β *be a class* $\mathcal{KL}$ *function. Denote the inverse of* α_i *by* α_i^{-1}*. then*

- α_1^{-1} *is defined on* $[0, \alpha_1(a))$ *and is of class* $\mathcal{K}$*;*
- α_3^{-1} *is defined on* $[0, \infty)$ *and is of class* $\mathcal{K}_\infty$*;*
- $\alpha_1 \circ \alpha_2$ *is of class* $\mathcal{K}$*;*
- $\alpha_3 \circ \alpha_4$ *is of class* $\mathcal{K}_\infty$*;*
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ *is of class* $\mathcal{KL}$*.* □

Lemma B.3 (Remark 2.3 in [24]). *For any $\alpha_1, \alpha_2 \in \mathcal{K}$, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}$ such that*

$$\underline{\alpha}(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2) \leq \overline{\alpha}(s_1 + s_2), \quad \forall s_1, s_2 \geq 0. \tag{B.5}$$

In particular, we can take $\underline{\alpha}(s) := \min\{\alpha_1(\frac{s}{2}), \alpha_2(\frac{s}{2})\}$ and $\overline{\alpha}(s) := \max\{2\alpha_1(s), 2\alpha_2(s)\}$ for all $s_1 \geq 0, s_2 \geq 0$. □

Lemma B.4 (Lemma 4.3 [52], [51]). *Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{B.6}$$

for all $x \in B_r$. If $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the above inequality holds for all $x \in \mathbb{R}^n$. □

Derivative of locally Lipschitz functions

We consider locally Lipschitz Lyapunov functions that are not necessarily differentiable everywhere. Therefore, we use the generalized directional derivative of Clarke which inherits some useful properties when dealing with locally Lipschitz functions, see Proposition 2.1.1 in [18]. For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a vector $\upsilon \in \mathbb{R}^n$,

$$V^\circ(x; \upsilon) := \limsup_{h \to 0^+,\, y \to x} \frac{V(y + h\upsilon) - V(y)}{h}. \tag{B.7}$$

For a continuously differentiable function V, $V^\circ(x; \upsilon)$ reduces to the standard directional derivative $\langle \nabla V(x), \upsilon \rangle$, where $\nabla V(x)$ is the (classical) gradient.

Note that, if $\dot{x} = \upsilon(t)$ for almost all t, then $\frac{d}{dt}V(x(t))$ is defined for almost all t and equals the usual one-sided directional derivative, *i.e.*, for almost all t,

$$\frac{d}{dt}V(x(t)) = \lim_{h \to 0^+} \frac{V(x(t) + h\upsilon(t)) - V(x(t))}{h}. \tag{B.8}$$

Comparing (B.8) with (B.7), we see that the generalized directional derivative upper bounds the usual directional derivative. Moreover, the generalized directional derivative offers the following convenient property

- If $f(x,d)$ and $\tilde{\alpha}(x,d)$ are continuous and

$$\frac{\partial V}{\partial x}(x)f(x,d) \leq \tilde{\alpha}(x,d), \quad \forall d, x \notin \Omega, \tag{B.9}$$

where Ω is a set of measure zero containing the set where V is not differentiable, then

$$V^{\circ}(x; f(x,d)) \leq \tilde{\alpha}(x,d), \quad \forall (x,d). \tag{B.10}$$

Fore more detail, see pages 99, 100 in [107].

The proofs of the previous chapters often involve a Lyapunov function which is defined by the maximum of two locally Lipschitz functions. To deal with such functions, we invoke the following result, see Lemma II.1 in [60].

Lemma 1. *Consider two functions $U_1 : \mathbb{R}^n \to \mathbb{R}$ and $U_2 : \mathbb{R}^n \to \mathbb{R}$ that have well-defined Clarke derivatives for all $x \in \mathbb{R}^n$ and $\upsilon \in \mathbb{R}^n$. Introduce three sets $A := \{x : U_1(x) > U_2(x)\}$, $B := \{x : U_1(x) < U_2(x)\}$, $\Gamma := \{x : U_1(x) = U_2(x)\}$. Then, for any $\upsilon \in \mathbb{R}^n$, the function $U(x) := \max\{U_1(x), U_2(x)\}$ satisfies*

(i) $U^{\circ}(x;\upsilon) = U_1^{\circ}(x;\upsilon)$ *for all* $x \in A$*;*

(ii) $U^{\circ}(x;\upsilon) = U_2^{\circ}(x;\upsilon)$ *for all* $x \in B$*;*

(iii) $U^{\circ}(x;\upsilon) \leq \max\{U_1^{\circ}(x;\upsilon), U_2^{\circ}(x;\upsilon)\}$ *for all* $x \in \Gamma$. □

B.2 Input-to-state stability

Consider the following system

$$\dot{x} = f(x,u), \tag{B.11}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m$ is locally Lipschitz in x and u.

Definition B.2 (Input-to-state stability (ISS), Definition 4.7 in [52]). *The system (B.11) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(0)$ and any bounded input u, the corresponding solution x exists for all*

$t \geq t_0$ and satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma \left(\sup_{t_0 \leq \tau \leq t} |u(\tau)| \right).$$

□

ISS is usually ensured using the following Lyapunov characterization.

Theorem B.1 (Lyapunov conditions for ISS, Theorem 4.19 in [52]). *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ \frac{\partial V}{\partial x} f(x,u) &\leq -\alpha(x), \quad \forall |x| \geq \rho(|u|) > 0 \end{aligned} \tag{B.12}$$

for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$ and $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous positive definite function on $\mathbb{R}^n$. Then, the system is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

□

B.3 Hybrid dynamical systems

Throughout the thesis, we model the event-triggered controlled systems by using the hybrid formalism of [34] which allows us to use the well-defined notion of solutions and the tools developed to analyse the stability in [34]. Hence, we consider hybrid systems of the following form

$$\dot{x} = F(x) \quad x \in C, \qquad x^+ = G(x) \quad x \in D, \tag{B.13}$$

where $x \in \mathbb{R}^n$ is the state, $C, D \in \mathbb{R}^n$ and F, G are single-valued functions. This model suggests that the state x of the hybrid system evolves according to the differential equation $\dot{x} = F(x)$ as long as $x \in C$, and it experiences an instantaneous change according to the difference equation $x^+ = G(x)$ when $x \in D$. When $x \in C \cap D$, the system behaves according to the differential equation $\dot{x} = F(x)$ only if this evolution keeps x in C, otherwise the system experiences a discrete transition. To shorthand the notation, we will refer to the continuous behaviour described by a differential equation as *flow* and the discrete behaviour described by a difference equation as *jump*. Consequently, from now on, the elements of hybrid model (B.13) will be named as follows

- F is the flow map;
- C is the flow set;
- G is the jump map;
- D is the jump set.

Assumption 6.5 in [34] provides sufficient conditions on the hybrid model to ensure that model is well-posed, see Section 5.4 in [63] for more detail on the well-posedness of hybrid dynamical models. In the case of single-valued functions F, G, these sufficient conditions reduce to

(i) $C, D \subset \mathbb{R}^n$ are closed sets;

(ii) F, G are continuous functions.

From now on, we assume that conditions (i) and (ii) hold.

Notion of solution

The solutions to system (B.13) are defined on the so-called hybrid time domains. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid time domain* if $E = \bigcup_{j \in \{0,...,J-1\}} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq ... \leq t_J$ and it is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ..., J\})$ is a compact hybrid time domain. A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{Z}_{\geq 0}, t \mapsto \phi(t, j)$ is locally absolutely continuous on $I^j := \{t : (t, j) \in E\}$. A hybrid arc ϕ is a solution to system (B.13) if $\phi(0, 0) \in C \cup D$ and

(i) for every $j \in \mathbb{Z}_{\geq 0}$,

$$\begin{array}{rcl} \phi(t,j) & \in & C, \\ \dot{\phi}(t,j) & = & F(\phi(t,j)), \end{array} \quad \text{for almost all } t \in I^j; \tag{B.14}$$

(ii) for every $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$,

$$\begin{array}{rcl} \phi(t,j) & \in & D, \\ \phi(t,j+1) & = & G(\phi(t,j)), \end{array} \quad \text{for almost all } t \in I^j. \tag{B.15}$$

Definition 1 (Types of hybrid arcs). *A hybrid arc ϕ is called:*

- *nontrivial if* $\operatorname{dom}\phi$ *contains at least two points;*

- *maximal if* $\operatorname{dom}\phi$ *cannot be extended;*
- *complete if* $\operatorname{dom}\phi$ *is unbounded;*
- *Zeno if it is complete and* $\sup_t \operatorname{dom}\phi < \infty$. □

The following notion of tangent cone is useful to study the existence of nontrivial solutions for system (B.13).

Definition 2 (Tangent cone)**.** *The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted $T_S(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exists $x_i \in S$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \searrow 0$, and*

$$w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}. \tag{B.16}$$

□

Proposition 1 (Basic existence of solutions, Proposition 6.10 in [34])**.** *Let $\mathcal{H} = (C, F, D, G)$. Take any arbitrary $\xi \in C \cup D$. If $\xi \in D$ or*

(VC) there exists a neighborhood U of ξ such that for every $x \in U \cap C$,

$$F(x) \in T_C(x),$$

then there exists a nontrivial solution ϕ to $\mathcal{H}$ with $\phi(0,0) = \xi$. If (VC) holds for every $\xi \in C \backslash D$, then there exists a nontrivial solution to $\mathcal{H}$ form every initial point in $C \cup D$, and every $\phi \in \mathcal{S}_{\mathcal{H}}$ satisfies exactly one of the following conditions:

(a) ϕ *is complete;*

(b) $\operatorname{dom}\phi$ *is bounded and the interval I^J, where $J = \sup_j \operatorname{dom}\phi$, has nonempty interior and $t \mapsto \phi(t, J)$ is a maximal solution to $\dot{z} \in F(z)$, in fact $\lim_{t\to T} |\phi(t, J)| = \infty$, where $T = \sup_t \operatorname{dom}\phi$;*

(c) $\phi(T, J) \notin C \cup D$, *where* $(T, J) = \sup \operatorname{dom}\phi$.

Furthermore, if $G(D) \subset C \cup D$, then (c) above does not occur. □

Stability

We introduce here the definitions of some stability properties and alternative characterizations. We start by defining the uniform global pre-asymptotic stability (UGpAS) of a closed set. This property entails that the distance of each solution to a given set is bounded by a function of two quantities: the initial condition's distance to the set and the amount of elapsed time at which the solution is evaluated; moreover, this bound tends to zero as the initial condition's distance to the set tends to zero or the amount of elapsed hybrid time tends to infinity.

Definition 3 (Distance to a closed set, Definition 3.5 in [34]). *Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, the distance of x to $\mathcal{A}$ is denoted as $|x|_{\mathcal{A}}$ and is defined by*

$$|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|. \tag{B.17}$$

□

The UGpAS property is formally defined as follows.

Definition 4 (Uniform global pre-asymptotic stability (UGpAS), Definition 3.6 in [34]). *Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be*

- *uniformly globally stable for $\mathcal{H}$ if there exists a class-$\mathcal{K}_\infty$ function α such that any solution ϕ to $\mathcal{H}$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \alpha(|\phi(0,0)|_{\mathcal{A}})$ for all $(t,j) \in$ dom ϕ;*
- *uniformly globally pre-attractive for $\mathcal{H}$ if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution ϕ to $\mathcal{H}$ with $(t,j) \in$ dom ϕ and $t+j \geq T$ imply $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$;*
- *uniformly globally pre-asymptotically stable for $\mathcal{H}$ if it is both uniformly globally stable and uniformly globally pre-attractive.* □

The prefix "Pre" indicates that maximal solutions are not required to be complete. We remove this prefix when maximal solutions are complete. The following theorem is an equivalent characterization of UGpAS.

Theorem 1 (Equivalence of UGpAS and a $\mathcal{KL}$ bound, Theorem 3.40 in [34]). *Let $\mathcal{H}$ be a hybrid system and $\mathcal{A} \subset \mathbb{R}^n$ be closed. The following statements are equivalent:*

(a) The set $\mathcal{A}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}$;

(b) There exists a $\mathcal{KL}$ function β such that any solution ϕ to $\mathcal{H}$ satisfies

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta(|\phi(0,0)|_{\mathcal{A}}, t+j) \qquad \forall (t,j) \in \text{dom}\,\phi. \tag{B.18}$$

□

B.4 Miscellaneous

We provide below a review of some diverse elements of mathematical analysis that have been used in this thesis.

Definition B.3 (Schur complement, Appendix A.5.5 [16]). *Consider a real symmetric matrix* $X \in \mathbb{R}^{n \times n}$ *partitioned as*

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where $A \in \mathbb{R}^{k \times k}$. *If* $\det(A) \neq 0$, *the matrix*

$$S = C - B^T A^{-1} B$$

is called the Schur complement of A *in* X. *The following characterizations of positive definiteness or semidefiniteness of the block matrix* X *hold:*

- $X > 0$ *if and only if* $A > 0$ *and* $S > 0$;
- *If* $A > 0$, *then* $X \geq 0$ *if and only if* $S \geq 0$. □

Lemma B.5 (Comparison lemma, Lemma 3.4 [52]). *Consider the scalar differential equation*

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0$$

where $f(t, u)$ *is continuous in* t *and locally Lipschitz in* u, *for all* $t \geq 0$ *and all* $u \in J \subset R$. *Let* $[t_0, T)$ *(*T *could be infinity) be the maximal interval of existence of the solution* $u(t)$, *and suppose* $u(t) \in J$ *for all* $t \in [t_0, T)$. *Let* $v(t)$ *be a continuous function whose upper right-hand derivative* $D^+ v(t)$ *satisfies the differential inequality*

$$D^+ v(t) \leq f(t, v(t)), \qquad v(t_0) \leq u_0$$

with $v(t) \in J$ *for all* $t \in [t_0, T)$. *Then,* $v(t) \leq u(t)$ *for all* $t \in [t_0, T)$. □

Theorem B.2 (Mean Value Theorem, page 651 in [52]). *Assume that* $f : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable at at each point* x *of an open set* $S \subset \mathbb{R}^n$. *Let* x *and* y *be two points of* S

such that the line segment $L(x,y) \subset S$. Then there exists a point z of $L(x,y)$ such that

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x),$$

where the line segment $L(x,y)$ joining two distinct points $x, y \in \mathbb{R}^n$ is

$$L(x,y) = \{z | z = \theta x + (1-\theta) y,\ 0 < \theta < 1\}.$$

□

Lemma B.6. *For any $a, b \in \mathbb{R}$ and $\eta \in \mathbb{R}_{>0}$, $2ab \leq \frac{1}{\eta} a^2 + \eta b^2$.* □

Lemma B.7. *For any $a, b \in \mathbb{R}_{\geq 0}$, $\max\{a, b\} \leq a + b$.* □

Lemma B.8. *For any real symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and for any $x \in \mathbb{R}^n$,*

$$\lambda_{\min}(P)|x|^2 \leq x^T P x \leq \lambda_{\max}(P)|x|^2.$$

□

Bibliography

[1]

[2] F. Akopyan, R. Manohar, and A.B. Apsel. A level-crosing flash asynchronous analog-to-digital converter. *In Proceedings of the IEEE International Symposium on Asynchronous Circuits and Systems, Grenoble, France*, pages 12–22, 2006.

[3] J. Almeida, C. Silvestre, and A. Pascoal. Self-triggered output feedback control of linear plants. *In Proceedings of the American Control Conference, Los Angeles, U.S.A.*, pages 2831–2836, 2011.

[4] A. Anta and P. Tabuada. On the minimum attention and anytime attention problems for nonlinear systems. *In Proceedings of the IEEE Conference on Decision and Control, Atlanta, U.S.A.*, pages 3234–3239, 2010.

[5] A. Anta and P. Tabuada. To sample or not to sample: Self-triggered control for nonlinear systems. *IEEE Transactions on Automatic Control*, 55(9):2030–2042, 2010.

[6] D. Antunes, W.P.M.H. Heemels, and P. Tabuada. Dynamic programming formulation of periodic event-triggered control: Performance guarantees and co-design. *In Proceedings of the IEEE Conference on Decision and Control, Hawaii, U.S.A.*, pages 7212–7217, 2012.

[7] K.E. Årzén. A simple event-based PID controller. *In Proceedings of the 14th IFAC World Congress, Beijing, China*, 18:423–428, 1999.

[8] K.J. Åström. Event based control. *In A. Astolfi and L. Marconi (editors)*, Analysis and Design of Nonlinear Control Systems:127–147, 2008.

[9] K.J. Åström and B. Bernhardsson. Systems with lebesgue sampling. *In Proceedings of the Directions in Mathematical Systems Theory and Optimization, Springer: Berlin, Germany*, pages 1–13, 2003.

[10] K.J. Åström and B.M. Bernhardsson. Comparison of periodic and event based sampling for first order stochastic systems. *In Proceedings of the 14th IFAC World Congress, Beijing, China*, pages 301–306, 1999.

[11] K.J. Åström and B.M. Bernhardsson. Comparison of riemann and lebesgue sampling for first order stochastic systems. *In Proceedings of the IEEE Conference on Decision and Control, Nevada, U.S.A.*, 2:2011–2016, 2002.

[12] C. Aubrun, D. Simon, and Y. Song. *Co-design Approaches for Dependable Networked Control Systems*. John Wiley & Sons, Inc., 2010.

[13] J. Baillieul and P.J. Antsaklis. Control and communication challenges in networked real-time systems. *Proceedings of the IEEE*, 95(1):9–28, 2007.

[14] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.

[15] J.D.J.B Berglind, T.M.P. Gommans, and W.P.M.H. Heemels. Self-triggered MPC for constrained linear systems and quadratic costs. *IFAC Nonlinear Model Predictive Control Conference, Noordwijkerhout, The Netherlands*, 4(1):342–348, 2012.

[16] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[17] D. Carnevale, A.R. Teel, and D. Nešić. A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems. *IEEE Transactions on Automatic Control*, 52(5):892–897, 2007.

[18] F.H. Clarke. *Optimization and Nonsmooth Analysis*. Society for Industrial and Applied Mathematics, 1990.

[19] M.B.G. Cloosterman, L. Hetel, N. van de Wouw, W.P.M.H. Heemels, J. Daafouz, and H. Nijmeijer. Controller synthesis for networked control systems. *Automatica*, 46(10):1584–1594, 2010.

[20] M.J. Corless and A.E. Frazho. *Linear Systems and Control: an Operator Perspective*. Marcel Dekker, Inc., 2003.

[21] D.B. Dačić and D. Nešić. Quadratic stabilization of linear networked control systems via simultaneous protocol and controller design. *Automatica*, 43(7):1145–1155, 2007.

[22] M.C.F. Donkers and W.P.M.H. Heemels. Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralised event-triggering. *IEEE Transactions on Automatic Control*, 57(6):1362–1376, 2012.

[23] M.C.F. Donkers, P. Tabuada, and W.P.M.H. Heemels. On the minimum attention control problem for linear systems: A linear programming approach. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A.*, pages 4717–4722, 2011.

[24] D.S. Laila and D. Nešić. Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems. *In Proceedings of the IEEE Conference on Decision and Control, Las Vegas, U.S.A.*, pages 2292–2297, 2002.

[25] S. Durand, N. Marchand, and F.G. Castellanos. Simple lyapunov sampling for event-driven control. *In Proceedings of the 18th IFAC World Congress, Milano, Italy*, pages 8724–8730, 2011.

[26] M. Ehrgott. *Multicriteria Optimization*. Springer, 2nd edition, 2005.

[27] A. Eqtami, D.V. Dimarogonas, and K.J. Kyriakopoulos. Event-triggered control for discrete-time systems. *In Proceedings of the American Control Conference, Maryland, U.S.A.*, pages 4719–4724, 2010.

[28] A. Eqtami, S. Heshmati-Alamdari, D.V. Dimarogonas, and K.J. Kyriakopoulos. Self-triggered model predictive control for nonholonomic systems. *In Proceedings of the European Control Conference, Zurich, Switzerland*, pages 638–643, 2013.

[29] C. Fiter, L.Hetel, W. Perruquetti, and J. Richard. A state dependent sampling for linear state feedback. *Automatica*, 48(8):1860–1867, 2012.

[30] F. Forni, S. Galeani, D. Nešić, and L. Zaccarian. Event-triggered transmission for linear control over communication channels. *Automatica*, 50(2):490–498, 2014.

[31] H. Fujioka. Stability analysis for a class of networked/embedded control systems: A discrete-time approach. *In Proceedings of the American Control Conference, Washington, U.S.A.*, pages 4997–5002, 2008.

[32] A. Girard. Dynamic triggering mechanisms for event-triggered control. 2014. preprint, `http://arxiv.org/abs/1301.2182`.

[33] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems: robust stability and control for systems that combine continuous-time and discrete-time dynamics. *IEEE Control Systems Magazine*, April:28–83, 2009.

[34] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[35] R. Goebel and A.R. Teel. Solution to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573—587, 2006.

[36] T.M.P. Gommans, D. Antunes, M.C.F. Donkers, P. Tabuada, and W.P.M.H. Heemels. Self-triggered linear quadratic control. *Automatica*, to appear.

[37] K.M. Guan, S.S. Kozat, and A.C. Singer. Adaptive reference levels in a level-crossing analog-to-digital converter. *EURASIP Journal on Advances in Signal Processing*, 2008(513706), 2008.

[38] W.M. Haddad, V. Chellaboina, and S.G. Nersesov. Dissipativity and control. 2006.

[39] W.P.M.H. Heemels. *Linear Complementarity Systems: A Study in Hybrid Dynamics*. PhD thesis, Eindhoven University of Technology, 1999.

[40] W.P.M.H. Heemels, M.C.F. Donkers, and A.R. Teel. Periodic event-triggered control based on state feedback. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A.*, pages 2571–2576, 2011.

[41] W.P.M.H. Heemels, M.C.F. Donkers, and A.R. Teel. Model-based periodic event-triggered control for linear systems. *Automatica*, 49(3):698–711, 2013.

[42] W.P.M.H. Heemels, M.C.F. Donkers, and A.R. Teel. Periodic event-triggered control for linear systems. *IEEE Transactions on Automatic Control*, 58(4):847–861, 2013.

[43] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. *In Proceedings of the IEEE Conference on Decision and Control, Hawaii, U.S.A.*, pages 3270–3285, 2012.

[44] W.P.M.H. Heemels, J.H. Sandee, and P.P.J. van den Bosch. Analysis of event-driven controllers for linear systems. *International Journal of Control*, 81(4):571–590, 2008.

[45] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, and D. Nešić. Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance. *IEEE Transactions on Automatic Control*, 55(8):1781–1796, 2010.

[46] W.P.M.H. Heemels and N. van de Wouw. Stability and stabilization of networked control systems. *In Networked Control Systems, Springer London*, 46:203–253, 2010.

[47] T.A. Henzinger. The theory of hybrid automata. *In Proceedings of the 11th Annual Symposium on Logic in Computer Science, New Jersey, U.S.A.*, pages 278–292, 1996.

[48] D. Hristu-Varsakelis and W.S. Levine. *Handbook of Networked and Embedded Control Systems*. Birkhäuser Boston, 2005.

[49] S. Hu and D. Yue. Event-triggered control design of linear networked systems with quantizations. *ISA Transactions*, 51(1):153–162, 2012.

[50] K.H. Johansson, J. Lygeros, S. Sastry, and M. Egerstedt. Simulation of zeno hybrid automata. *In Proceedings of the IEEE Conference on Decision and Control, Arizona, U.S.A.*, pages 3538–3543, 1999.

[51] H.K. Khalil. Errata list for nonlinear systems - third edition. `http://www.egr.msu.edu/~khalil/NonlinearSystems/`.

[52] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[53] E. Kofman and J.H. Braslavsky. Level crossing sampling in feedback stabilization under data-rate constraints. *In Proceedings of the IEEE Conference on Decision and Control, San Diego, U.S.A.*, pages 4423–4428, 2006.

[54] P. Kokotović, H.K. Khalil, and J. O'Reilly. *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, 1986.

[55] D. Lehmann and J. Lunze. Event-based output-feedback control. *In Proceedings of the 19th Mediterranean Conference on Control and Automation, Corfu, Greece*, pages 982–987, 2011.

[56] D. Lehmann, J. Lunze, and K.H. Johansson. Comparison between sampled-data control, deadband control and model-based event-triggered control. *In Proceedings of the IFAC Conference on Analysis and Design of Hybrid Systems, Eindhoven, Netherlands*, pages 7–12, 2012.

[57] M. Lemmon. Event-triggered feedback in control, estimation, and optimization. *In Networked Control Systems, Springer*, pages 293–358, 2011.

[58] L.L. Li and M. Lemmon. Weakly coupled event triggered output feedback system in wireless networked control systems. *Discrete Event Dynamic System*, 2013.

[59] D. Liberzon. *Switching in Systems and Control*. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2003.

[60] D. Liberzon, D. Nešić, and A.R. Teel. Lyapunov-based small-gain theorems for hybrid systems. *IEEE Transactions on Automatic Control*, 59(6):1395–1410, 2014.

[61] Q. Liu, Z. Wang, X. He, and D.H. Zhou. A survey of event-based strategies on control and estimation. *Systems Science & Control Engineering*, 2(1):90–97, 2014.

[62] J. Löfberg. YALMIP: A toolbox for modeling and optimization in MATLAB. *In Proceedings of the CACSD Conference, Taipei, Taiwan*, 2004.

[63] J. Lunze and F. Lamnabhi-Lagarrigue. *Handbook of hybrid systems control: theory, tools, applications*. Cambridge University Press, 2009.

[64] J. Lunze and D. Lehmann. A state-feedback approach to event-based control. *Automatica*, 46:211–215, 2010.

[65] D. Nesic M. Tabbara and A.R. Teel. *Networked control systems: emulation based design*. In Networked Control Systems (Eds. D. Liu and F.-Y. Wang). World Scientific, 2007.

[66] M. MalmirChegini and F. Marvasti. Performance improvement of level-crossing a/d converters. *In Proceedings of the IEEE International Conference on Telecommunications, Penang, Malaysia*, pages 438–441, 2007.

[67] M. Mazo Jr., A. Anta, and P. Tabuada. On self-triggered control for linear systems: Guarantees and complexity. *In Proceedings of the European Control Conference*, 2009.

[68] M. Mazo Jr., A. Anta, and P. Tabuada. An ISS self-triggered implementation of linear controllers. *Automatica*, 46:1310–1314, 2010.

[69] M. Mazo Jr. and M. Cao. Decentralized event-triggered control with one bit communications. *In Proceedings of the 4^{th} IFAC Conf. on Anal. and Des. of Hyb. Sys., Eindhoven, The Netherlands*, pages 52–57, 2012.

[70] M. Mazo Jr. and P. Tabuada. On event-triggered and self-triggered control over sensor/actuator networks. *In Proceedings of the IEEE Conference on Decision and Control, Cancún, Mexico*, pages 435–440, 2008.

[71] M. Mazo Jr. and P. Tabuada. Decentralized event-triggered control over wireless sensor/actuator networks. *IEEE Transactions on Automatic Control*, 56(10):2456–2461, 2011.

[72] X. Meng and T. Chen. Event detection and control co-design of sampled-data systems. *International Journal of Control*, 78(4):777–786, 2014.

[73] M. Miskowicz. Send-on-delta concept: an event-based data reporting strategy. *Sensors*, 6:49–63, 2006.

[74] A. Molin and S. Hirche. Optimal event-triggered control under costly observations. *In Proceedings of the International Symposium on Mathematical Theory of Networks and Systems, Budapest, Hungary*, pages 2203–2208, 2010.

[75] A. Molin, H. Tischer, and S. Hirche. Order reduction in optimal event-triggered control design for linear stochastic systems. *In Proceedings of the American Control Conference, Los Angeles, U.S.A.*, pages 2222–2227, 2011.

[76] R.M. Murray, K.J. Åström, S.P. Boyd, R.W. Brockett, and G. Stein. Future directions in control in an information-rich world. *Control Systems Magazine*, pages 20–33, 2003.

[77] D. Nešić and A. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Transactions on Automatic Control*, 49(7):1103–1034, 2004.

[78] D. Nešić and A. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.

[79] D. Nešić, A.R. Teel, and D. Carnevale. Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control*, 54(3):619–624, 2009.

[80] D. Nešić, A.R. Teel, and P.V. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Systems & Control Letters*, 38(4-5):259—270, 1999.

[81] P.G. Otanez, J.R. Moyne, and D.M. Tilbury. Using deadbands to reduce communication in networked control systems. *In Proceedings of the American Control Conference*, 4:3015–3020, 2002.

[82] Z. Pan and T. Başar. H^{∞}-optimal control for singularly perturbed systems with sampled-state measurements. *In T. Başar and A. Haurie, editors, Advances in Dynamic Games and Applications*, 1:23–55, Birkhäuser, 1994.

[83] C. Peng and Q. Han. A novel event-triggered transmission scheme and $\mathcal{L}_2$ control co-design for sampled-data control systems. *IEEE Transactions on Automatic Control*, 58(10):2620–2626, 2013.

[84] C. Peng and Q. Han. Output-based event-triggered $\mathcal{H}_\infty$ control for sampled-data control systems with nonuniform sampling. *In Proceedings of the American Control Conference, Washington, U.S.A.*, pages 1727–1732, 2013.

[85] C. Peng and T. Yang. Event-triggered communication and $\mathcal{H}_\infty$ control co-design for networked control systems. *Automatica*, 49(5):1326–1332, 2013.

[86] J. Plönnigs, M. Neugebauer, and K. Kabitzsch. A traffic model for networked devices in the building automation. *In Proceedings of the IEEE International Workshop on Factory Communication Systems, Vienna, Austria*, pages 137–145, 2004.

[87] R. Postoyan, A. Anta, W.P.M.H. Heemels, P. Tabuada, and D. Nešić. Periodic event-triggered control for nonlinear systems. *In Proceedings of the IEEE Conference on Decision and Control, Florence, Italy*, pages 7397–7402, 2013.

[88] R. Postoyan, M.C. Bragagnolo, E. Galbrun, J. Daafouz, D. Nešić, and E.B. Castelan. Nonlinear event-triggered tracking control of a mobile robot: design, analysis and experimental results. *In Proceedings of the IFAC Symposium on Nonlinear Control systems, invited paper, Toulouse, France*, pages 318–323, 2013.

[89] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta. Event-triggered and self-triggered stabilization of distributed networked control systems. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A.*, pages 2565–2570, 2011.

[90] M. Rabi, K.H. Johansson, and M. Johansson. Optimal stopping for event-triggered sensing and actuation. *In Proceedings of the IEEE Conference on Decision and Control, Cancun, Mexico*, pages 3607–3612, 2008.

[91] H. Rehbinder and M. Sanfridson. Scheduling of a limited communication channel for optimal control. *Automatica*, 40(3):491–500, 2004.

[92] R.G. Sanfelice, D.A. Copp, and P. Ñañez. A toolbox for simulation of hybrid systems in matlab/simulink: Hybrid equations (HyEQ) toolbox. *In Proceedings of Hybrid Systems: Computation and Control Conference*, pages 101–10, 2013.

[93] R.G. Sanfelice and A.R. Teel. On singular perturbations due to fast actuators in hybrid control systems. *Automatica*, 47:692—701, 2011.

[94] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42(7):896–911, 1997.

[95] C. Scherer and S. Weiland. *Linear Matrix Inequalities in Control*. Springer-Verlag, 2000.

[96] A. Seuret and C. Prieur. Event-triggered sampling algorithms based on a Lyapunov function. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A.*, pages 6125–6133, 2011.

[97] A. Seuret, C. Prieur, S. Tarbouriech, and L. Zaccarian. Event-triggered control with LQ optimality guarantees for saturated linear systems. *In Proceedings of the IFAC Symposium on Nonlinear Control, Toulouse, France*, 9(1):341–346, 2013.

[98] L. Shanbin and X. Bugong. Event-triggered control for discrete-time uncertain linear parameter-varying systems. *In Proceedings of the 32nd Chinese Control Conference, Xi'an, China*, pages 273–278, 2013.

[99] LI Shanbin and XU Bugong. Co-design of event generator and controller for event-triggered control system. *In Proceedings of the Chinese Control Conference, Yantai, China*, pages 175–179, 2011.

[100] J. Sijs, M. Lazar, and W.P.M.H. Heemels. On integration of event-based estimation and robust MPC in a feedback loop. *Hybrid Systems: Computation and control, Stockholm, Sweden*, pages 31–41, 2010.

[101] J. Sturm. Using Sedumi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12, Special issue on Interior Point Methods:625—-653, 1999.

[102] Y.S. Suh. Send-on-delta sensor data transmission with a linear predictor. *sensors*, 7(4):537–547, 2007.

[103] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

[104] P. Tallapragada and N. Chopra. Event-triggered decentralized dynamic output feedback control for LTI systems. *In Estimation and Control of Networked Systems*, 3(1):31–36, 2012.

[105] P. Tallapragada and N. Chopra. Event-triggered dynamic output feedback control for LTI systems. *In Proceedings of the IEEE Conference on Decision and Control, Hawaii, U.S.A.*, pages 6597–6602, 2012.

[106] P. Tallapragada and N. Chopra. On co-design of event trigger and quantizer for emulation based control. *In Proceedings of the American Control Conference, Quebec, Canada*, pages 3772–3777, 2012.

[107] A.R. Teel and L. Praly. On assigning the derivative of a disturbance attenuation control Lyapunov function. *Mathematics of Control, Signal and Systems*, 13(2):95–124, 2000.

[108] A.J. van der Schaft and J.M. Schumacher. Complementarity modeling of hybrid systems. *In Proceedings of the IEEE International Conference on Industrial Informatics, Vienna, Austria*, 43(4):483–490, 1998.

[109] V. Vasyutynskyy and K. Kabitzsch. Deadband sampling in pid control. *In Proceedings of the IEEE International Conference on Industrial Informatics, Vienna, Austria*, pages 45–50, 2007.

[110] M. Velasco, J. Fuertes, and P. Marti. The self triggered task model for real-time control systems. *In Proceedings of the 24th IEEE Real-Time Systems Symposium (RTSS WIP 2003)*, pages 67–70, 2003.

[111] T.L. Vincent and J. Yu. Control of a chaotic system. *Dynamics and Control*, 1:35–52, 1991.

[112] G. Walsh, H. Ye, and L. Bushnell. Stability analysis of networked control systems. *IEEE Transactions on Control System Technology*, 10(3):438–46, 2002.

[113] G.C Walsh, O. Beldiman, and L.G. Bushnell. Asymptotic behavior of nonlinear networked control systems. *IEEE Transactions on Control System Technology*, 46(7):1093–1097, 2001.

[114] C.J. Wan and D.S. Bernstein. Nonlinear feedback control with global stabilization. *Dynamics and Control*, 5:321–346, 1995.

[115] W. Wang, A.R. Teel, and D. Nešić. Analysis for a class of singularly perturbed hybrid systems via averaging. *Automatica*, 48(6):1057–1068, 2012.

[116] W. Wang, A.R. Teel, and D. Nešić. Averaging in singularly perturbed hybrid systems with hybrid boundary layer systems. *In Proceedings of the IEEE Conference on Decision and Conttrol, Hawaii, U.S.A.*, pages 6855–6860, 2012.

[117] X. Wang and M. Lemmon. Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability. *IEEE Transactions on Automatic Control*, 45:452–467, 2009.

[118] X. Wang and M. Lemmon. Attentively efficient controllers for event-triggered feedback systems. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Florida, U.S.A.*, pages 4698–4703, 2011.

[119] X. Wang and M.D. Lemmon. Event design in event-triggered feedback control systems. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Cancún, Mexico*, pages 2105–2110, 2008.

[120] X. Wang and M.D. Lemmon. Event-triggering in distributed networked control systems. *IEEE Transactions on Automatic Control*, 56(3):586–601, 2011.

[121] X. Wang and M.D. Lemmon. On event design in event-triggered feedback systems. *Automatica*, 47(10):2319–2322, 2012.

[122] X. Wang, Y. Sun, and N. Hovakimyan. Asynchronous task execution in networked control systems using decentralized event-triggering. *Systems & Control Letters*, 61(9):936–944, 2012.

[123] H. Yu and P.J. Antsaklis. Event-triggered output feedback control for networked control systems using passivity: Achieving $\mathcal{L}_2$ stability in the presence of communication delays and signal quantization. *Automatica*, 49(1):30–38, 2013.

[124] D. Yue, E. Tian, and Q. Han. A delay system method to design of event-triggered control of networked control systems. *In Proceedings of the IEEE Conference on Decision and Control and European Control Conference, Orlando, U.S.A.*, pages 1668–1673, 2011.

[125] L. Zhang, H. Gao, and O. Kaynak. Network-induced constraints in networked control systems–a survey. *IEEE Transactions On Industrial Informatics*, 9(1):403–4016, 2013.

[126] X. Zhang and Q. Han. Event-based dynamic output feedback control for networked control systems. *In Proceedings of the American Control Conference, Washington, U.S.A.*, pages 3008–3013, 2013.

[127] X. Zhang and D. Hristu-Varsakelis. Stabilization of networked control systems under feedback based communication. *In Proceedings of the American Control Conference*, pages 2933–2938, 2005.

Résumé

La commande à transmissions événementielles est une approche dans laquelle les instants de transmission sont définis selon un critère dépendant de l'état du système et non plus d'une horloge à l'instar des implantations périodiques. Dans cette thèse, nous nous concentrons sur la synthèse de telles lois de commande par retour de sortie. Les contributions sont les suivantes : (i) nous proposons une méthode de synthèse dite par émulation pour des systèmes non linéaires; (ii) nous présentons une méthode de synthèse jointe de la loi de commande et de la condition de déclenchement pour les systèmes linéaires; (iii) nous nous intéressons au cas de systèmes non linéaires singulièrement perturbés et nous construisons le contrôleur à partir d'approximation de la dynamique lente uniquement.

Mots clés : Commande à transmissions événementielles; Systèmes contrôlés via un réseau; Systèmes singulièrement perturbés; Systèmes dynamiques hybrides; Systèmes non linéaires.

Summary

Event-triggered control is a sampling paradigm in which the sequence of transmission instants is determined based on the violation of a state-dependent criterion and not a time-driven clock. In this thesis, we deal with event-triggered output-based controllers to stabilize classes of nonlinear systems. The contributions of the presented material are threefold: (i) we stabilize a class of nonlinear systems by using an emulation-based approach; (ii) we develop a co-design procedure to simultaneously design the output feedback law and the event-triggering condition for linear systems; (iii) we propose stabilizing event-triggered controllers for nonlinear systems whose dynamics have two-time scales (in particular, we only rely on the knowledge of an approximate model of the slow dynamics).

Keywords: Event-triggered control; Networked control systems; Singularly perturbed systems; Hybrid dynamical systems; Nonlinear systems.