Université de Lorraine
UFR S.T.M.I.A.
École Doctorale IAEM
D.F.D. Mathématiques

et

Université du Luxembourg
Faculté des Sciences, de la Technologie
et de la Communication

THESE

présentée le 28 février 2014

pour l'obtention du doctorat de

l'Université de Lorraine
Discipline *Mathématiques*

l'Université du Luxembourg
Discipline *Mathématiques*

par

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Modélisation financière avec des processus de Volterra et applications aux options, aux taux d'intérêt et aux risques de crédit

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Acknowledgements

At this moment of my thesis I would like to thank all those people who made this thesis possible and an unforgettable experience for me.

First of all, I would like to express my deepest sense of gratitude to my supervisors Prof. Anton Thalmaier and Prof. Marco Dozzi for giving me the opportunity to write this thesis at the University of Luxembourg and the University of Lorraine. I thank them for the systematic guidance and great effort they put into training me and answering to my questions and queries.

I would like to thank very much all the members of my thesis committee who have accepted to judge my research work. I am grateful to Prof. Ivan Nourdin and Prof. Giovanni Peccati who have accepted to be part of the jury and to examine my work. I especially express my deep gratitude to this thesis reviewers Prof. Yuliya Mishura and Prof. Francesco Russo who are famous experts on fractional Brownian motion and financial mathematics, for their time and dedication and for their very constructive comments. I am greatly honored that they accepted to be reviewers of my thesis, and to provide their insightful comments to my research study.

Completing this work would have been all the more difficult without the support and friendship provided by the other members of the Mathematics Research Unit in the University of Luxembourg.

Last but not the least, I would like to thank my parents, for giving birth to me at the first place and supporting me throughout my life.

Abstract

The purpose of this work is the analysis of financial models with stochastic processes having memory and eventually discontinuities which are often observed in statistical evaluation of financial data. For a long time the fractional Brownian motion seemed to be a natural tool for modeling continuous phenomena with memory. In contrast with the classical models which are usually formulated in terms of Brownian motion or Lévy processes and which are analyzed with the Itô stochastic calculus, the models with fractional Brownian motion require a different approach and some other advanced methods of analysis. As for example the Malliavin calculus which leads to an adequate stochastic calculus for Gaussian processes. Moreover, one should notice that these techniques have been already successfully applied to the calibration of mathematical models in finance. The questions of pricing under non arbitrage conditions with a fractional Brownian motion cannot be treated by the Fundamental Theorem of Asset Pricing which only holds true in the class of semimartingales.

This work aims also to be a contribution to the stochastic analysis for processes with memory and discontinuities, and in this context we do propose solutions to questions in option pricing, interest rates and credit risk.

In Chapter 2 the stochastic calculus is used to study the classes of jump processes with memory in the jump times (considered as random variables). A class of fractional (or filtered) Lévy processes is considered with different regularity assumptions on the kernel, especially for the one which implies that the fractional Levy process is a semimartingale. An Itô formula is proven and the chaos decomposition is considered. Moreover, filtered doubly stochastic Lévy processes are also studied. The intensity of their jump times is stochastic and it is typically modeled by a filtered Poisson process. Chapters 3 to 5 are devoted to the stochastic analysis of financial models formulated in terms of the processes studied in Chapter 2.

In Chapter 3 the question of the existence of risk neutral probability measures is studied for the fractional Black-Scholes models. Continuous fractional processes having a kernel that is integrated with respect to a Brownian motion and which are semimartingales (typically approximations to fractional Brownian motion), are shown to be stochastic trends rather than diffusions. Mixed processes, containing Brownian motion and a fractional process, are therefore proposed as adequate asset pricing models fitting to the context of the fundamental theorem of asset pricing. In the presence of jump processes (especially filtered doubly stochastic Poisson processes) the risk neutral measures are shown to be not unique.

In Chapter 4 interest rates are modeled as solutions of stochastic differential equations driven by (continuous) mixed processes, in particular the Vasicek, the Cox-Ingersoll-Ross and the Heath-Jarrow-Morton models. For the first two models, the spot rate is evaluated under a risk free probability measure obtained according to a market price of risk. It is shown that arbitrage free pricing of interest rate derivatives with fixed maturity can be carried out by solving explicitly stated partial differential equations. Following the third model an arbitrage free evaluation of the zero coupon bond is given.

Chapter 5 deals with credit risk models. For structural models, the historical and the risk neutral default probabilities are evaluated for mixed processes models as in Chapters 3 and 4.

Introduction

Among the first systematic treatments of the option pricing problem has been the pioneering work of Black, Scholes and Merton [13] who proposed the widely known, and extensively used, Black-Scholes (BS) model. The BS model rests on the assumption that the natural logarithm of the stock price $(S_t)_{t\geqslant 0}$ follows a random walk or diffusion with deterministic drift. However, this classical formula is quite often criticized. Many empirical data reveal that the stock price distribution usually has properties like "fat tails", "volatility clustering", "self similarity", "long range dependence" and many other interesting stylized facts, which contradict the traditional Black-Scholes assumptions. In my dissertation, I was interested in modeling the long range dependence of asset returns using Fractional Brownian motion.

The idea that stock returns could exhibit long range dependence was first suggested by Mandelbrot [67]. Mandelbrot and Van Ness [68] introduce the fractional Brownian motion (FBM), a non-semimartingale Gaussian process with long range dependence and dependent increments. Another important feature of this process is its dependence on a Hurst parameter, commonly used as a measure of market predictability. Market predictability may result from the dependence of the returns. Peters [80] suggests that if a stock time series has a high Hurst exponent, then the stock will be less risky.

The application of FBM in finance was abandoned in its early years since this process is not a semimartingale which results in two problems: the first is defining a stochastic calculus with respect to FBM and the second is that arbitrage opportunities cannot be excluded in this model.

Actually there are several methods available which allow to construct a stochastic calculus for this process. The so-called Malliavin calculus [76] is based on a probabilistic and functional analytic notion of derivation and integration and allows a very efficient stochastic calculus for Gaussian processes which has already been successfully applied to mathematical models in finance, see [65] for example. A second method, introduced in [85], defines three types of integrals (namely, forward, backwards and symmetric) and gives a theory of stochastic integration via regularization. The advantage of regularization lies in the fact that this approach is natural and relatively simple, and easily connects to other approaches. A third method is the so-called Wick product for constructing stochastic integrals, see [10].

Concerning the second problem, arbitrage strategies are defined as strategies $\phi \in \mathcal{A}$ which realize a possibly positive gain by starting from a zero initial capital. We notice that the definition of arbitrage depends on the set of admissible strategies $\mathcal{A}$ and on the definition of the stochastic integral. Arbitrage pricing theory is based on a fundamental result of Delbaen and Schachermayer (1994) [31] known as the *Fundamental Theorem of Asset Pricing* which shows in full generality that *"No free lunch with vanishing risk"* (NFLVR) is equivalent to the existence of an equivalent local martingale measure. A similar result has also been established by Ansel and Stricker (1992) [3]. In other words, there is no arbitrage if and only if there exists a martingale measure under which the discounted prices are martingales.

Since fractional Brownian motion is not a semimartingale, a model in which the log prices are described by a fractional Brownian motion is not arbitrage-free, in the sense that there exists a strategy $\phi \in \mathcal{A}$ realizing a possibly non-zero gain by starting from a zero initial capital. Rogers (1997) [84] gives the mathematical argument for existence of arbitrage opportunities for this process.

Several attempts have been proposed in order to exclude arbitrage opportunities in the context of non-semimartingale models like fractional Brownian motion. Cheridito [21] suggests to forbid high frequency trading, and Guasoni [44] suggests to introduce transaction costs. However, none of these method were sufficient to resolve completely the problem. Even if arbitrage opportunities disappear, option pricing with these methods remains an open problem. In the other hand, Coviello et al. [29] proposed to proceed by restricting the class of admissible strategies. In [8] the authors proposed non-semimartingale models and also proceed by defining a class of admissible strategies for which the model is free of arbitrage. In Chapter 3 we give an answer to this problem.

On the other hand, most of the recent literature dealing with modeling of financial assets assumes that the underlying dynamics of equity prices follows a diffusion process plus a jump process or Lévy process in general. This is done to incorporate rare or extreme events not captured in Gaussian models. Fractional Lévy processes (FLP) can be introduced as a natural generalization of the integral representation of fractional Brownian motion:

$$L^K(t) = \int_0^t K_H(t,s)dL(s),$$

where L is a Lévy process, H the Hurst exponent and $K_H(t,s)$ is a kernel verifying the usual properties*. In the literature there are many possible approaches to define such processes, see Bender and Marquardt [69, 6], Tikanmäki and Mishura [95]. Tikanmäki and Mishura showed for instance that this process has almost surely Hölder continuous paths of any order strictly less than $H - 1/2$ for $H > 1/2$, and with positive probability it has discontinuous sample paths for $H < \frac{1}{2}$. Hence, using this process to model jumps in asset returns may not be advantageous. Moreover, we explain in Chapter 2 that the definition of FLP by means of integral representations of fractional Brownian motion is delicate for many reasons, hence we propose a new definition and we develop a stochastic calculus for it.

The thesis is organized as follows:

In Chapter 1 we review previous work upon which our approach is based. We recall basic definitions on stochastic calculus with respect to Lévy and Gaussian processes. In Section 1.3 we review a stochastic calculus via Malliavin calculus with respect to Gaussian semimartingales. We give sufficient conditions for a Volterra process to be a semimartingale and compare two Itô formulas, one with respect to divergence integrals and one with respect to classical Itô calculus. Section 1.4 deals with SDE driven by fractional Brownian motion frequently used in mathematical finance, like the exponential Ornstein-Uhlenbeck model and the Cox-Ingersoll-Ross model.

Chapter 2 is devoted to study the class of fractional Lévy process (FLP). Our main contribution in this chapter is adjusting the definition and proposing a stochastic calculus for it. In

*K can for example be the Molchan-Golosov or the Mandelbrot-Van Ness kernel

Sections 2.2 to 2.4 we compare the trajectories of FBM and FLP. We explain why fractional Lévy processes cannot be the natural generalization of fractional Brownian motion. In Section 2.2.4 we explain why the trajectories of a FLP become continuous when the Hurst exponent H is $> \frac{1}{2}$ and discuss the behaviour of this process when the kernel approaches the diagonal. In Section 2.3.2 we recall that the integral form of FBM of Liouville type can be obtained when passing to the limit with a discrete fractional ARIMA process, but we explain that this procedure does not make sense in the case of FLP.

In Section 2.5 and 2.6 we investigate the class of additive processes obtained from Lévy processes by relaxing the condition of stationarity of the increments. The motivation for introducing this more general class is to be eventually able to develop an appropriate definition of a FLP. We mention the method introduced by Solé [92] in order to define a Chaos expansion related to these processes. In Section 2.6 we consider the case of inhomogeneous Poisson processes with cumulative intensity Λ and propose a stochastic calculus in terms of divergence integrals by defining a transformation between the spaces $\tilde{L}^2(\lambda\times\nu^\Lambda)$ and $\tilde{L}^2(\lambda\times\nu^1)$. Section 2.7 is an important part of the chapter. We propose a stochastic calculus in terms of divergence integrals for filtered Lévy processes and derive an Itô formula. For this aim, we extensively use the theory of chaotic representation of Lévy processes and Malliavin calculus with respect to Lévy processes, see for example Nualart and Schoutens (2000) [77] and the book of Di Nunno et al. (2009) [34].

In order to define FLP, we consider in Section 2.8 the class of doubly stochastic Poisson processes (DSPP) in a first step. Cox processes are popular examples of DSPP; they are defined as Poisson processes with a stochastic intensity $\lambda(t,\omega)$. We derive the related stochastic calculus and Itô formula.

In Section 2.10 we propose to define a fractional doubly stochastic Poisson process with intensity

$$\lambda^K(t) = \int_0^t K(t,s)d\lambda(s).$$

This process conserves the properties of a point process. The generalization to the doubly stochastic fractional Lévy case is not difficult and can be treated like the classical case, see [82]. This process is used in our financial models in Chapter 3.

Chapter 3 of the thesis deals with the problem of arbitrage in models based on fractional Brownian motion. We proceed as suggested by Rogers [84] by approximating fractional Brownian motion by semimartingales in order to stay under the framework of the fundamental Theorem of asset pricing. Taking Volterra processes of the form

$$B^K(t) = \int_0^t K(t,s)dW(s),$$

where

- W is a Brownian motion
- K is absolutely continuous on $\mathbb{R}_+$ with respect to t and with a square integrable density K'.

It is shown in [18] and [17] that these processes are semimartingales. In Section 3.4 we show, by considering the natural semimartingale closest to FBM, that the FBM plays more likely the role of a stochastic trend rather than a volatility. In the same section we show that long memory behavior does not depend on the singularity when $t = s$, and deduce that this class of processes is more appropriate to financial modeling than fractional Brownian motion in its original form. As we explain, in this case the process B^K when introduced in the Black-Scholes model is more like a stochastic trend. To correct the model we propose to add a Brownian motion independent of B^K to the model. In this way we obtain a kind of mixed Brownian motion defined as sum of a Brownian motion and a semimartingale Volterra process. Brownian motion plays the role of the diffusion part, while the Volterra is a stochastic trend. All together, we obtain the following model for asset prices

$$dS(t) = \mu S(t)\,dt + \sigma_1 S(t)\,dW(t) + \sigma_2 S(t)\,dB^K(t),$$

where μ, σ_1, σ_2 are deterministic parameters. The mixed fractional Brownian motion which we introduce should not be confused with the one introduced by Cheridito [20] which is a semimartingale only for $H > 3/4$.

Chapter 4 and Chapter 5 treat interest rates models and credit risk model. After solving the problem of arbitrage in Chapter 3, one can now try to apply fractional models to specific topics in finance. In Section 4.2 we show how to price contracts on short interest rates and at the end of the chapter we study a Heath-Jarrow-Morton (HJM) model and give sufficient conditions for the evaluation of arbitrage free coupon bonds. In Chapter 5 finally, we study credit risk models. Credit risk models are often classified as structural or reduced form models. In Section 5.2.1, we study the structural models and in Section 5.2.2 the reduced form models. We currently develop the approach of this chapter further, since empirical studies, for example [58], mention the presence of a significative Hurst exponent in credit spread data.

Résumé des résultats

Parmi les premiers traitements systématiques du problème d'évaluation des options fût le travail de Black, Scholes [13], qui ont proposé le célèbre modèle connu sous le nom de Black-Scholes (BS). Ce modèle repose sur l'hypothèse que le logarithme naturel du prix d'achat d'actions $(S_t)_{t\geqslant 0}$ suit une marche aléatoire ou une diffusion avec dérive déterministe. Cependant, cette formule classique est très souvent critiquée. Beaucoup de données empiriques révèlent que la distribution du prix des actions possède des propriétés comme "queues épaisses", "volatilité altérnée"[†], "auto similarité", "mémoire longue" et bien d'autres faits intéressants, qui contredisent les hypothèses traditionnelles de Black-Scholes. Dans ce travail, je me suis intéressé à la modélisation de la dépendance à longue mémoire des rendements des actifs en utilisant le mouvement Brownien fractionnaire et des généralisations fractionnaires des processus de Lévy.

L'idée que les rendements boursiers peuvent présenter une dépendance à longue mémoire était suggérée par Mandelbrot [67]. Mandelbrot et Van Ness [68] introduisirent le mouvement Brownien fractionnaire (MBF), un processus gaussien non-semimartingale à longue mémoire et à accroissements dépendants. Une autre caractéristique importante de ce processus est sa dépendance à l'égard d'un paramètre de Hurst, couramment utilisé en tant que mesure de la prévisibilité du marché. La prévisibilité du marché peut entraîner la dépendance des rendements. Peters [80] suggère que si la série temporelle du prix des actifs donne un exposant de Hurst élevé, l'actif sera moins risqué.

L'application de MBF en finance a été abandonnée dans ses premières années étant donné que ce processus n'est pas une semimartingale, ce qui se traduit par deux problèmes: le premier est de définir un calcul stochastique par rapport au MBF et le second est que les possibilités d'arbitrage ne peuvent pas être exclues avec ce modèle.

En fait, il existe désormais plusieurs méthodes qui permettent de construire un calcul stochastique pour ce processus. Le soi-disant calcul de Malliavin [76] est basé sur une notion probabiliste et fonctionnelle de la dérivée et de l'intégrale et permet un calcul stochastique efficace pour les processus gaussiens et a déjà été appliqué avec succès à la modélisation mathématique en finance, voir [65] par exemple. Une deuxième méthode, introduite dans [85], définit trois types d'intégrales (forward, backward et symétrique) et donne une intégration stochastique par régularisation. L'avantage de cette méthode de régularisation réside dans le fait que cette approche est naturelle et relativement simple et se connecte facilement à d'autres approches. Une troisième méthode, est definie par l'intermédiaire du produit de Wick pour construire une intégrale stochastique, voir [10].

Concernant le deuxième problème, les stratégies d'arbitrage sont définies comme des stratégies $\phi \in \mathcal{A}$ qui peuvent réaliser un gain positif à partir d'un capital initial nul. Nous remarquons que la définition de l'arbitrage dépend de l'ensemble des stratégies admissibles $\mathcal{A}$ et de la définition de l'intégrale stochastique. La théorie d'évaluation par arbitrage est basée sur un résultat fondamental de Delbaen et Schachermayer (1994) [31] connu comme le *fundamental theorem of asset pricing* qui montre en toute généralité que *"no free lunch with vanishing risk"* est équivalent à l'existence d'une mesure équivalente à une martingale locale. Un résultat similaire a également été introduit par Ansel et Stricker (1992) [3]. En d'autres termes, il n'y a pas d'arbitrage si et seulement s'il existe une mesure martingale sous laquelle les prix actualisés sont des martingales.

Puisque le mouvement Brownien fractionnaire n'est pas une semimartingale, un modèle suiv-

[†]périodes de forte volatilité alternées avec des périodes de faible volatilité

ant lequel les logarithmes des prix sont décrits par un mouvement Brownien fractionnaire n'est pas sans arbitrage, dans le sens qu'il existe une stratégie $\phi \in \mathcal{A}$ qui éventuellement réalise un gain non nul en partant d'un capital nul initial. Rogers (1997) [84] donne l'argument mathématique qui montre l'existence des opportunités d'arbitrage avec ce processus.

Plusieurs tentatives ont étés proposées afin d'exclure les opportunitées d'arbitrage dans le contexte des modèles non-semimartingale comme le mouvement Brownien fractionnaire. Cheridito [21] suggère d'interdire le trading haute fréquence. Guasoni [44] suggère d'introduire les coûts de transaction. Cependant, aucune de ces méthodes n'est suffisante pour résoudre complètement le problème. Même si les opportunités d'arbitrage disparaissent, l'évaluation des prix des options avec ces méthodes reste un problème ouvert. En revanche, Coviello et al. [29] propose de procéder en restreignant la classe des stratégies admissibles. Dans le chapitre 3, nous donnons une réponse à ce problème.

D'autre part, la plupart des travaux récents traitant la modélisation des actifs financiers suppose que les dynamiques des prix des actions suivent un processus de diffusion plus un processus à sauts ou un processus de Lévy en général. Ceci est fait pour intégrer des évènements rares ou extrêmes pas capturés dans les modèles gaussiens. Les processus de Lévy fractionnaires peuvent être introduits comme une généralisation naturelle de la représentation de l'intégrale du mouvement Brownien fractionnaire:

$$L^K(t) = \int_0^t K_H(t,s)dL(s),$$

où L est un processus de Lévy, H l'exposant de Hurst et $K_H(t,s)$ est un noyau vérifiant les propriétés habituelles ‡. Dans la littérature, il existe de nombreuses approches possibles pour définir ces processus, voir Bender et Marquardt [69, 6], Tikanmäki et Mishura [95]. Tikanmäki et Mishura ont montré par exemple que ce processus a presque sûrement des trajectoires continues Hölderiennes de tout ordre strictement inférieur à $H - \frac{1}{2}$ pour $H > \frac{1}{2}$, et a des trajectoires discontinues avec une probabilité positive pour $H < \frac{1}{2}$. Ainsi, utiliser ce processus pour modéliser des sauts dans les rendements des actifs pourraient ne pas être avantageux. En outre, nous expliquons dans le chapitre 2 que la définition du processus de Lévy fractionnaire comme représentation par l'intégrale du Brownien fractionnaire est un peu délicate pour de nombreuses raisons. Nous proposons une nouvelle définition et nous développons un calcul stochastique en liaison.

La thèse est organisée comme suite:

Dans le chapitre 1, nous examinons les travaux antérieurs sur lesquels se fonde notre approche. Nous rappelons les définitions de base sur le calcul stochastique par rapport aux processus de Lévy et les processus gaussiens.

Le chapitre 2 est consacré à l'étude de la classe des processus de Lévy fractionnaire. La principale contribution dans ce chapitre est d'ajuster la définition et de proposer un calcul stochastique en liaison. Principalement, une formule d'Itô pour les processus Lévy fractionnaires et pour les processus doublement stochastique est prouvée.

Le chapitre 3 de la thèse traite du problème d'arbitrage dans les modèles fractionnaires. Nous procédons comme suggéré par Rogers [84] en approximant le mouvement Brownien fractionnaire par des semimartingales afin de rester dans le domaine de validité du théorème fondamental de

‡ K peut être défini comme le noyau de Molchan-Golosov ou le noyau de Mandelbrot-Van Ness.

la finance. Nous terminerons le chapitre par le calcul des probabilités risque neutre pour un modele de Black-Scholes avec des discontinuitées qui sont modélisées dans le chapitre 2.

Chapitres 4 et 5, traitent les modèles de taux d'intérêts et les modèles de risque de crédit. Après avoir résolu le problème d'arbitrage au Chapitre 3, on peut maintenant essayer d'appliquer les modèles fractionnaires à des sujets spécifiques dans la finance. Nous montrons comment évaluer les prix des contrats sur taux à court terme et à la fin du chapitre, nous étudions un modèle Heath-Jarrow-Morton et nous donnons les conditions suffisantes pour l'évaluation en absence d'arbitrages des obligations à coupons. Dans le chapitre 5, nous étudions les modèles de risque de crédit avec des processus fractionnaires.

Nous donnons dans la suite un résumé des résultats principaux.

Chapitre 1 – Processus de Lévy et semimartingales gaussiennes

Processus de Lévy: définition

Considerons un espace de probabilité filtré $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. Un processus de Lévy $(L_t)_{t\geqslant 0}$ est un processus stochastique à valeurs dans $\mathbb{R}$ et à accroissements indépendants et stationnaires, et continu à droite en probabilité (pour tout $\varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|L_{t+h} - L_t| > \varepsilon) = 0$). On choisira toujours la version càdlàg.

Fixons un intervalle $[0, T]$. On associe à un processus de Lévy son triplet caractéristique (μ, σ, ν). Dans la suite on considère également des processus où μ_s et σ_s aléatoires (μ_s, σ_s: $[0, T] \times \Omega \to \mathbb{R}$, $\mathcal{F}_s$-adaptés tel que $\mathbb{E}\left[\int_0^T \{\mu_s^2 + \sigma_s^2\} ds\right] < \infty$).

Pour $(\gamma_t(x))_{t\in[0,T]}$ une fonction aléatoire adaptée, continue à gauche en t, measurable en x, telle que

$$\mathbb{E}\left[\int_0^T \int_{|x|>1} |\gamma_t(x)| \nu(dx)\, dt\right] < \infty; \qquad \mathbb{E}\left[\int_0^T \int_{|x|\leqslant 1} \gamma_t^2(x) \nu(dx)\, dt\right] < \infty,$$

on définit un processus d'Itô-Lévy:

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|x|>1} \gamma_s(x) N(ds \times dx) + \int_0^t \int_{|x|\leqslant 1} \gamma_s(x) \tilde{N}(ds \times dx),$$

où N est une $\mathcal{F}_t$-mesure aléatoire de Poisson d'intensité $\nu(dx)\, dt$. Notons par $\tilde{N}$ le processus compensé.

X est une semimartingale avec variation quadratique

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds + \int_0^t \int_{\mathbb{R}} \gamma_s^2(x) \nu(dx)\, ds,$$

où on assume que la dernière intégrale est finie.

Pour $f \in C^2(\mathbb{R}_+, \mathbb{R})$, on la formule d'Itô suivante:

$$\begin{aligned} f(t,X_t) - f(0,X_0) = & \int_0^t \frac{\partial}{\partial t} f(s,X_s)ds + \int_0^t \frac{\partial}{\partial x} f(s,X_s)\left[\mu_s ds + \sigma_s dW_s\right] + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f\left(s,X_s\right)\sigma_s^2 ds \\ & + \int_0^t \int_{|x|<1} \left\{ f(s,X_{s-} + \gamma_s(z)) - f(s,X_{s-}) - \frac{\partial}{\partial x} f(s,X_{s-})\gamma_s(x) \right\} \nu(dx)ds \\ & + \int_0^t \int_{|x|<1} \left\{ f(s,X_{s-} + \gamma_s(z)) - f(s,X_{s-}) \right\} \tilde{N}(ds \times dx) \\ & + \int_0^t \int_{|x|\geqslant 1} \left\{ (f(s,X_{s-} + \gamma_s(z)) - f(s,X_{s-}) \right\} N(ds \times dx). \end{aligned}$$

Formule de Girsanov

Soit $\mathbb{Q}$ une probabilité sur l'espace de probabilité filtré $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. Notons $\mathbb{Q}_t$ la restriction de $\mathbb{Q}$ par rapport à la tribu $\mathcal{F}_t$.

Soit $(X_t)_{t\in[0,T]}$ un processus de Lévy satisfaisant la condition du corollaire 1.1.7. Supposons que $\exp(X_t)$, $(t \in [0,T])$ est une martingale et que

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \exp(X_t).$$

Soit K un processus prévisible tel que

$$\int_0^T \int_{\mathbb{R}} \mathbb{E}[|K_t(x)|^2]\, \nu(dx)dt < +\infty.$$

Définissons

$$M_t = \int_0^t \int_{\mathbb{R}_0} K_s\left(x\right) \tilde{N}(ds,dx), \qquad U(s,x) = e^{\gamma_s(x)} - 1,$$

et supposons que

$$\int_0^T \int_{\mathbb{R}_0} \left(e^{\gamma_s(x)} - 1\right)^2 \nu\left(dx\right) dt < +\infty.$$

Finalement, posons

$$B_t = W_t - \int_0^t \sigma_s ds,$$

et

$$\begin{aligned} \tilde{N}_t^{\mathbb{Q}} &= M_t - \int_0^t \int_{\mathbb{R}_0} K_s\left(x\right) U\left(s,\, x\right) \nu\left(dx\right) ds \\ &= \int_0^t \int_{\mathbb{R}_0} K_s\left(x\right) N\left(dx, ds\right) - \int_0^t \int_{\mathbb{R}_0} K_s\left(x\right) e^{\gamma_s(x)} \nu\left(dx\right) ds. \end{aligned}$$

Alors, sous $\mathbb{Q}$, le processus $(B_t)_{t\in[0,T]}$ est un mouvement Brownien, $(\tilde{N}_t^{\mathbb{Q}})_{t\in[0,T]}$ est une $\mathbb{Q}$-martingale, $\nu^{\mathbb{Q}} = e^{\gamma}\nu$ est le $\mathbb{Q}$-compensateur de N.

Semimartingales gaussiennes

Soit ϕ absolument continue sur $\mathbb{R}_+$ avec dérivée ϕ' de carré intégrable. Posons

$$B_t = \int_0^t \phi(t-s)\, dW_s.$$

Alors,

1. $(B_t)_{t\in[0,T]}$ est une semimartingale gaussienne et possède la décomposition suivante

$$B_t = \phi(0)W_t + \int_0^t \int_0^u \phi'(u-s)dW_s du.$$

2. La fonction d'autocorrélation est donnée par

$$\begin{aligned} R(t,s) = \mathbb{E}[B_t B_s] &= \int_0^{t\wedge s} \phi(t-u)\,\phi(s-u)du \\ &= \phi^2(0)\,(t\wedge s) \\ &\quad + \phi(0)\int_0^t \int_0^{r\wedge s} \phi'(r-u)\, dudr \\ &\quad + \phi(0)\int_0^s \int_0^{r\wedge t} \phi'(r-u)\, dudr \\ &\quad + \int_0^t \int_0^s \int_0^{r\wedge v} \phi'(r-u)\,\phi'(v-u)\, dudrdv. \end{aligned}$$

Intégration stochastique

Fixons un intervalle $[0,T]$. Notons par $\mathcal{E}$ l'ensemble des fonctions simples sur $[0,T]$. Soit $\mathcal{H}$ l'espace de Hilbert défini comme la fermeture de $\mathcal{E}$ par rapport au produit scalaire

$$\left\langle 1_{[0,t]}, 1_{[0,s]}\right\rangle_{\mathcal{H}} = R(t,s).$$

La fonction $1_{[0,t]} \to B_t$ définit une isométrie entre $\mathcal{H}$ et le premier chaos H_1, un sous-espace fermé de $L^2(\Omega)$ généré par B. La variable $B(\varphi)$ représente l'image dans H_1 d'un élement $\varphi \in \mathcal{H}$.

Soit $\mathcal{S}$ l'ensemble des variables aléatoires cylindriques de la forme $F = f(B(\varphi_1), \ldots, B(\varphi_n))$ où $n \geq 1,\ f \in C_b^\infty(\mathbb{R}^n)$ (f et toutes ses dérivées sont bornées) et $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$. La dérivée de F est l'élément de $L^2(\Omega, \mathcal{H})$ définie par

$$D^B F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\varphi_1), \ldots, B(\varphi_n))\,\varphi_j.$$

Comme d'habitude, $\mathbb{D}_B^{1,2}$ est la fermeture de l'ensemble des variables cylindriques F par rapport à la norme

$$\|F\|_{1,2}^2 = \mathbb{E}\left[|F|^2\right] + \mathbb{E}\left[\left\|D^B F\right\|_{\mathcal{H}}^2\right].$$

L'opérateur de divergence δ^B est défini come l'adjoint de l'opérateur de dérivation. Si une variable aléatoire $u \in L^2(\Omega, \mathcal{H})$ appartient au domaine de l'opérateur de divergence, alors $\delta^B(u)$ est le dual de l'opérateur de dérivation

$$\mathbb{E}\left[F\delta^B(u)\right] = \mathbb{E}\left[\left\langle D^B F, u\right\rangle_{\mathcal{H}}\right]$$

pour tout $F \in \mathbb{D}_B^{1,2}$.

Chapitre 2 – Calcul stochastique pour le processus Lévy fractionnaire

Dans ce chapitre, nous discutons des approches différentes pour définir un processus de Lévy fractionnaire et nous proposons une définition basée sur une généralisation fractionnaire du processus de Cox en laissant l'intensité devenir fractionnaire. Nous rappelons qu'il y a des ambiguités concernant les définitions déjà existantes. Nous proposons un calcul stochastique anticipatif par le calcul de Malliavin et nous établissons une formule d'Itô en utilisant la relation entre l'intégrale trajectorielle et l'intégrale de divergence.

Les processus de Lévy fractionnaires sont définis comme une géneralisation du mouvement Brownien fractionnaire:

$$L_t^H = \int_0^t Z_H(t,s)dL_s, \tag{0.0.1}$$

où $H \in]0,1[$ et

$$Z_H(t,s) = C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} \, du \right] \quad \text{pour } t > s,$$

$$C_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right]^{\frac{1}{2}}.$$

Ici β est la fonction beta.

Il a était montré dans [95] que:

1. Pour $H > 1/2$, L^H a presque surement des trajectoires Hölderiènnes d'ordre strictement inferieure à $H - 1/2$.
2. Pour $H < 1/2$, L^H a des trajectoires discontinues non bornées avec probabilité 1.

Processus de Lévy filtré, calcul stochastique et formule d'Itô

Un processus de Lévy filtré (pure) est défini comme

$$L^K(t) = \int_0^t \int_{\mathbb{R}_0} K(t,s) z L(ds,dz).$$

En compensant le processus, on définit

$$\begin{aligned}
\tilde{L}^K(t) &= \int_0^t \int_{\mathbb{R}_0} K(t,s) z \tilde{L}(ds,dz) \\
&= \int_0^t \int_{\mathbb{R}_0} K(t,s) z L(ds,dz) - \int_0^t \int_{\mathbb{R}_0} K(t,s) z \nu(ds,dz) \\
&= \int_0^t \int_{\mathbb{R}_0} K(t,s) z L(ds,dz) - \int_0^t \int_{\mathbb{R}_0} z \nu^K(ds,dz).
\end{aligned}$$

Dans ce qui suit, on considère des noyaux $K(t,s)$ réguliers [1] définis sur $[0,T] \times [0,T]$. On impose la condition suivante sur $K(t,.)$:

(C) Pour tout $s \in [0,T), K(\cdot,s)$ est à variation bornée sur $(s,T]$ et

$$\int_0^T |K|\left((s,T),s\right)^2 ds < \infty.$$

Remarque.

Si on impose la condition que la dérivée est de carré integrable

$$\int_0^T \int_{\mathbb{R}_0} \left(\frac{\partial}{\partial t} K(t,s) z\right)^2 \nu(dz,ds) < \infty, \tag{0.0.2}$$

alors

$$\begin{aligned} L_t^K &= \int_0^t \int_{\mathbb{R}_0} K(t,s) z L(ds,dz) \\ &= \int_0^t \int_{\mathbb{R}_0} K(s,s) z L(ds,dz) + \int_0^t \int_{\mathbb{R}_0} \int_s^t \frac{\partial}{\partial r} K(r,s) z dr L(ds,dz) \\ &= \int_0^t \int_{\mathbb{R}_0} K(s,s) z L(ds,dz) + \int_0^t \int_{\mathbb{R}_0} \int_0^r \frac{\partial}{\partial r} K(r,s) z L(ds,dz) dr. \end{aligned} \tag{0.0.3}$$

Le calcul stochastique d'Itô est valide dans ce cas.

D'apres [34], notons $\mathcal{M}^K$ l'ensemble des fonctions stochastiques $\theta(t,z),\ t \in [0,T],\ z \in \mathbb{R}_0$, telles que

1. $\theta(t,z) \equiv \theta(\omega,t,z)$ est adapté

$$\begin{aligned} \|\theta\|^2_{L^{2,K}(\Omega\times\lambda\times\nu)} &:= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(s,z)(K(s,s))^2 \nu(dz) ds\right] \\ &\quad + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_s^T \theta(s,z) K(dr,s)\right)^2 \nu(dz) ds\right] < \infty. \end{aligned}$$

2. $D_{t^+,z}\theta(t,z) := \lim_{s\to t^+} D_{s,z}\theta(t,z)$ existe dans $L^{2,K}(\Omega\times\lambda\times\nu)$.
3. $\theta(t,z) + D_{t^+,z}\theta(t,z)$ est dans le domaine de l'opérateur de divergence.

Soit $\mathbb{M}^K_{1,2}$ la fermeture de l'espace lineaire $\mathcal{M}^K$ par rapport à la norme

$$\|\theta\|^2_{\mathbb{M}^K_{1,2}} := \|\theta\|^2_{L^{2,K}(\Omega\times\lambda\times\nu)} + \|D_{t^+,z}\theta(t,z)\|^2_{L^{2,K}(\Omega\times\lambda\times\nu)}. \tag{0.0.4}$$

Définition. L'intégrale forward $X(t),\ t \in [0,T]$ d'une fonction stochastique $\theta(t,z),\ t \in [0,T]$, $z \in \mathbb{R}_0$ par rapport à $\tilde{L}^K$ est défini par

$$\begin{aligned} X(t) &= \int_0^t \int_{\mathbb{R}_0} \theta(s,z) \tilde{L}^K(d^-s,dz) \\ &= \lim_{m\to\infty} \sum_{0\leqslant s_i\leqslant t} \Delta X_{s_i} 1_{U_m} = \lim_{m\to\infty} \sum_{0\leqslant s_i\leqslant t} 1_{U_m} \theta(s_i,z) \Delta\tilde{L}^K(d^-s_i,dz), \end{aligned} \tag{0.0.5}$$

si la limite existe dans $L^2(\Omega)$. Ici, U_m, $m = 1, 2, \ldots$, est une suite croissante d'ensemble compacts $U_m \subseteq \mathbb{R}_0$ tel que $\nu(U_m) < \infty$.

Lemme. Si $\theta \in \mathbb{M}^K_{1,2}$, alors son intégrale forward existe et

$$\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{L}^K(d^-s,dz) = \int_0^t \int_{\mathbb{R}_0} D_{s^+,z}\theta(s,z)\nu^K(ds,dz) + \int_0^t \int_{\mathbb{R}_0} \left[\theta(s,z) + D_{s^+,z}\theta(s,z)\right] \tilde{L}^K(\delta s,dz). \tag{0.0.6}$$

Lemme. Soit f une fonction de classe $C^2(\mathbb{R})$. Soit $L^K(t), t \in [0,T]$, un processus Lévy filtré avec $K(t,.)$ satisfaisant la condition de la remarque. Alors la formule d'Itô au sens forward a la forme suivante

$$\begin{aligned} f\left(\tilde{L}^K(t)\right) &= f\left(\tilde{L}^K(0)\right) + \int_0^t \int_{\mathbb{R}_0} \left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\right]\tilde{L}(d^-s,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big) - f'\big(\tilde{L}^K(s-)\big)\right]\nu(ds,dz) \\ &+ \int_0^t f'\big(\tilde{L}^K(s-)\big)d\tilde{A}(s) \end{aligned} \tag{0.0.7}$$

où

$$\tilde{A}(t) = \tilde{L}^K(t) - \int_0^t \int_{\mathbb{R}_0} K(s,s)\tilde{L}(dz,ds).$$

Théorème. Soit f une fonction dans $C^2(\mathbb{R})$. Soit $L^K(t)$, $t \in [0,T]$, un processus de Lévy filtré avec $K(t,\cdot)$ satisfaisant la condition de la remarque. Alors la formule d'Itô a la forme suivante:

$$\begin{aligned} f\big(\tilde{L}^K(t)\big) &= f(\tilde{L}^K(0)) + \int_0^t \int_{\mathbb{R}_0} \Big\{\left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\right] \\ &+ D_{s^+,z}\left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\right]\Big\}\tilde{L}(\delta s,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \Big\{\left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big) - K(s,s)zf'(\tilde{L}^K(s-))\right] \\ &+ D_{s^+,z}\left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\right]\Big\}\nu(ds,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f'\big(\tilde{L}^K(r)\big)\frac{\partial}{\partial t}K(r,s)dr + f''\big(\tilde{L}^K(r)\big)zK(r,s)\frac{\partial}{\partial t}K(r,s)dr\right)\tilde{L}(\delta s,dz) \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}_0} f''\big(\tilde{L}^K(r)\big)zd\left(\int_0^r (K(r,s)^2)\nu(ds,dz)\right). \end{aligned} \tag{0.0.8}$$

Processus de Lévy doublement stochastique

Définition. Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité et m une mesure aléatoire (positive) sur $\mathbb{R}_+$ et à moments finis sur des ensembles bornés dans $\mathcal{B}(\mathbb{R}_+)$. Soit $\mathcal{F}^m \subset \mathcal{F}$ la tribu engendrée par

m. Soit $(\mathcal{F}_t, t \geqslant 0) \subset \mathcal{F}$ une filtration complète continue à droite. Une mesure aléatoire N est *un $\mathcal{F}_t$ processus de Poisson doublement stochastique* si

(i) $\mathbb{P}(N(\Delta) = k \mid \mathcal{F}^m) = \dfrac{m(\Delta)^k}{k!} e^{-m(\Delta)}$ pour tout ensemble borélien borné $\Delta \subset \mathbb{R}_+$, et $k \in \mathbb{N} \cup \{0\}$,

(ii) $N_0 := 0$, $N_t := N((0,t])$ est $\mathcal{F}_t$-measurable pour tout $t > 0$,

(iii) $\sigma(N(\Delta))$ et $\mathcal{F}_t$ sont conditionnellement indépendants sachant $\mathcal{F}^m$, pour tout $\Delta \subset]t, \infty[$.

Notons que le processus de Poisson doublement stochstique n'est pas en général homogène. On peut définir un processus de Lévy doublement stochastique L^Λ purement à sauts, en demandant que, conditionnellement à Λ, L^Λ est un processus de Lévy . Cependant ceci exige que Λ soit constante trajectoire par trajectoire. De manière plus générale on peut demander que L^Λ est, conditionnellement à Λ est un processus additif. On peut définir le processus de comptage des sauts suivant [82] à l'aide des processus de Poisson doublement stochastiques. Un calcul au sens de Malliavin (avec développement en chaos) semble plus difficile à obtenir (pour les processus additifs, voir [92]).

La formule de Clark-Ocone

En général, les processus de Lévy ne possèdent pas la propriété que des fonctionnelles de ces processus peuvent être représentées par une constante plus une integrale par rapport au processus (la représentation pévisible). Ceci est vrai dans le cas du mouvement Brownien et du processus de Poisson. Pour cela, afin d'obtenir une formule de Clark-Ocone, nous nous restreindrons au processus de Poisson doublement stochastique.

Soit $(\Lambda(t))_{t\in[0,T]}$ un processus de Poisson et $\mathcal{F}_t^\Lambda$ sa filtration naturelle. Notons par $\mathcal{F}^{N^\Lambda}$ la filtration engendrée par N^Λ.

Dans la suite nous discuterons deux représentations possibles, afin d'obtenir une décomposition en chaos adéquate.

Représentation 1

Soit

$$X(t) = \int_0^t \int_{\mathbb{R}_0} \theta(s,z) \tilde{N}^\Lambda(ds,dz) = \int_0^t \int_{\mathbb{R}_0} \theta(s,z) N^\Lambda(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \theta(s,z) \Lambda(ds,dz),$$

où $\Lambda(ds,dz) = \Lambda(ds)\nu(dz)$.

Pour $f \in C^2(\mathbb{R})$ on a:

$$\begin{aligned} f\big(X(t)\big) - f\big(X(0)\big) &= \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) + \theta(s,z)\big) - f\big(X(s-)\big) \right] \tilde{N}^\Lambda(ds,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) - \theta(s,z)\big) + f\big(X(s-) + \theta(s,z)\big) - 2f\big(X(s-)\big) \right] \tilde{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) - \theta(s,z)\big) + f\big(X(s-) + \theta(s,z)\big) - 2f\big(X(s-)\big) \right] \nu(ds,dz), \qquad (0.0.9) \end{aligned}$$

où $\nu(ds, dz) = \lambda ds \nu(dz)$ et λ est l'intensité de Λ.

En appliquant la formule d'Itô à

$$Y(t) = \exp\left\{\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{N}^\Lambda(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \left(e^{\theta(s,z)} + e^{-\theta(s,z)} - 2\right)\nu(ds,dz)\right\}$$

on obtient

$$dY(t) = Y(t^-)\left[\int_{\mathbb{R}_0} (e^{\theta(t,z)} - 1)\tilde{N}^\Lambda(dt,dz) + \int_{\mathbb{R}_0} \left(e^{\theta(t,z)} + e^{-\theta(t,z)} - 2\right)\tilde{\Lambda}(dt,dz)\right].$$

Afin d'obtenir un développement en chaos, il faut procéder par itération. Mais comme on a deux variables aléatoires N^Λ et Λ dont l'une dépend de l'autre, le développement n'est pas possible dans ce cas. Ce problème a été discuté dans [78] et les auteurs ont proposé une restriction sur la filtration. Dans notre cas, comme on prend un choix particulier de l'intensité (semimartingale), il est possible d'aller plus loin.

Représentation 2

Prenons maintenant $X(t)$ de la forme

$$X(t) = \int_0^t \int_{\mathbb{R}_0} \theta(s,z)N^\Lambda(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \theta(s,z)\nu(ds,dz).$$

On obtient la formule d'Itô suivante:

$$\begin{aligned} f(X_t) - f(X_0) = &\int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) + \theta(s,z)\big) - f\big(X(s-)\big)\right]\big(N^\Lambda(ds,dz) - \nu(ds,dz)\big) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) + \theta(s,z)\big) - f\big(X(s-)\big) - f'\big(X(s-)\big)\theta(s,z)\right]\nu(ds,dz). \quad (0.0.10)\end{aligned}$$

En appliquant cette formule au processus Y donné par:

$$Y(t) = \exp\left\{\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\big(N^\Lambda(ds,dz) - \nu(ds,dz)\big) - \int_0^t \int_{\mathbb{R}_0} \big(e^{\theta(s,z)} - 1 - \theta(s,z)\big)\nu(ds,dz)\right\},$$

il vient,

$$dY(t) = Y(t-)\int_{\mathbb{R}_0} \big(e^{\theta(s,z)} - 1\big)\big(N^\Lambda(ds,dz) - \nu(ds,dz)\big). \quad (0.0.11)$$

Posons $\hat{N}_t^\Lambda = N_t^\Lambda - \int_0^t \int_{\mathbb{R}_0} \nu(dz,ds)$. Alors,

$$dY(t) = Y(t-)\int_{\mathbb{R}_0} \big(e^{\theta(s,z)} - 1\big)\hat{N}^\Lambda(ds,dz). \quad (0.0.12)$$

Théorème de représentation d'Itô. *Soit $F \in L^2(\Omega, \mathcal{F}^N \vee \mathcal{F}^\Lambda)$. Alors il existe un unique processus prévisible $\psi \in L^2(\Omega, \mathcal{F}^N \vee \mathcal{F}^\Lambda)$ tel que:*

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \psi^2(t,z)\nu(dt,dz)\right] < \infty,$$

pour lequel

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \psi(t,z)\hat{N}^{\Lambda}(dt,dz)$$
$$= \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \psi(t,z)\tilde{N}^{\Lambda}(dt,dz) + \int_0^T \int_{\mathbb{R}_0} \psi(t,z)\tilde{\Lambda}(dt,dz).$$

Il en résulte que pour $(t_1, z_1) \in [0,T] \times \mathbb{R}_0$, il existe un processus prévisible $\psi_2(t_1, z_1, t_2, z_2)$, $(t_2, z_2) \in [0,T] \times \mathbb{R}_0$ tel que

$$\psi_1(t_1, z_1) = \mathbb{E}[\psi_1(t_1, z_1)] + \int_0^T \int_{\mathbb{R}_0} \psi_2(t_1, z_1, t_2, z_2)\hat{N}^{\Lambda}(dt_2, dz_2).$$

En itérant, nous obtenons

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{0.0.13}$$

pour des fonctions symétriques $f_n \in \tilde{L}^2(([0,T]\times\mathbb{R}_0)^n)$. Nous montrons alors le théorème suivant:

Théorème (Clark-Ocone). *Let* $F \in \mathbb{D}^{1,2}_{\hat{N}^{\Lambda}}$. *Then*

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}\big[D_{t,z}F \big| \mathcal{F}^{\hat{N}^{\Lambda}}\big]\hat{N}^{\Lambda}(dt,dz). \tag{0.0.14}$$

Processus de Poisson filtré doublement stochastique

L'idée derrière la généralisation des processus de Poisson et des processus de Lévy pour le cas doublement stochastique, est d'introduire la "fractionnalité" dans l'intensité du processus. Ainsi, on appelle $N_t^{\Lambda^K}$ un processus de Poisson composé filtré doublement stochastique si son intensité cumulative $\Lambda^K(t) = N^K(t)$, où $N^K(t)$ est un processus de Poisson filtré. Pour ce processus, il est possible de montrer la formule d'Itô suivante:

Soit

$$X(t) = \int_0^t \int_{\mathbb{R}_0} \theta(s,z)\hat{N}^{\Lambda^K}(d^-s, dz),$$

où θ est tel que, $\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} |\theta(s,z)| + \theta(s,z)^2 \nu^K(ds,dz)\right] < \infty$. Soit $Y(t) = f(X(t))$, $t \in [0,T]$, et f est une fonction $C^2(\mathbb{R})$ tel que $f(X(t-) + \theta(t,z)) - f(X(t-))$ appartient au domaine de l'opérateur de divergence $\hat{N}^K(\delta s, dz)$. Alors:

$$\begin{aligned} d^-Y(t) = &\int_{\mathbb{R}_0} \big(f(X(t-) + \theta(t,z)) - f(X(t-)) - f'(X(t-))\theta(t,z)\big)\, \nu^K(dt,dz) \\ &+ \int_{\mathbb{R}_0} \big(f(X(t^-) + \theta(t,z)) - f(X(t-))\big)\, \hat{N}^{\Lambda^K}(d^-t, dz) \\ = &\int_{\mathbb{R}_0} \big(f(X(t-) + \theta(t,z)) - f(X(t-)) - f'(X(t-))\theta(t,z)\big)\, \nu^K(dt,dz) \\ &+ \int_{\mathbb{R}_0} [(f(X(t-) + \theta(t,z)) - f(X(t-))) + D_{t,z}\left(f(X(t-) + \theta(t,z)) - f(X(t-))\right)]\, \hat{N}^{\Lambda^K}(\delta t, dz) \\ &+ 2\int_{\mathbb{R}_0} D_{t,z}\left(f(X(t-) + \theta(t,z)) - f(X(t-))\right) \nu^K(dt,dz). \end{aligned} \tag{0.0.15}$$

Chapitre 3 – Modèle de Black-Scholes fractionnaire sans arbitrage

Dans ce Chapitre nous traitons le problème d'arbitrage en lien avec les modèles de Black-Scholes fractionnaires. On montre que le MBF *ne doit pas être considéré comme diffusion mais plûtot comme un drift stochastique. Il s'ensuit qu'il est plus raisonnable d'ajouter un mouvement Brownien* (MB) *au modèle pour jouer le rôle de diffusion. On distingue les cas où la longue mémoire résulte de volatilité alternée ou des trends déterministes. On ajoute aussi des sauts au modèle et on calcule les prix des options européennes.*

Le problème d'évaluation des options et les opportunités d'arbitrage avec le modèle de Black-Scholes fractionnaire a été largement discuté au cours des dernières décennies. La motivation derrière l'utilisation de ce modèle vient de la présence de mémoire dans la dynamique des prix des actions. Afin de construire un modèle adéquat, une analyse des résultats empiriques sera d'une grande importance.

En effet, il est montré dans [26] que des études empiriques [4, 27, 73, 22, 63, 39] soutiennent la présence d'une dépendance à de grands horizons de temps entre la valeur absolue des rendements et pas entre les rendements eux même. Ce qui entraine qu'il est plus avantageux d'introduire la "fractionnalité" dans la volatilité et pas dans les rendements eux-mêmes. Ces résultats empiriques sont en cohérence avec l'hypothèse de l'efficience des marchés financiers (EMH) généralement connue comme la théorie de marché aléatoire definie par Fama [37] puis par Samuelson [86], qui postule que les prix sur le marché sont des martingales.

D'autre part, (Peter, 1994) [79] introduit l'hypothèse des marchés fractals (FMH) et met l'accent sur l'impact de l'information et des horizons d'investissement sur le comportement des investisseurs. Si toutes les informations ont le même impact sur les investisseurs, il n'y aura pas de liquidité parce que tous les investisseurs exécuteront la même transaction. Si la liquidité cesse, le marché devient instable et les mouvements extrêmes se produisent. Souvent, on associe la (FMH) à la présence de fractal et de longue mémoire dans les rendements des actifs financiers.

Dans la suite nous introduisons un modèle mixte avec un mouvement brownien et un processus de Volterra à mémoire. De cette manière le modèle pourra s'adapter aux différentes situations. En premier, en approximant le MBF par la plus proche martingale, nous montrons que le MBF peut être interprété comme drift stochastique. Nous terminons le chapitre avec un modèle mixte en ajoutant aussi des sauts avec mémoire.

Approximation du mouvement Brownien fractionnaire

Nous donnons une aproximation dans L^2 de $\left(B^H(t)\,,\ t \in [0,T]\right)$ par des semimartingales. Soit $\alpha = H - \frac{1}{2}$. Pour tout $\epsilon > 0$ définissons

$$B_H^\epsilon(t) = \int_0^t K_H^\epsilon(t,s)\, dW_s,$$

où

$$K_H^{\epsilon}(t,s) = C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s+\epsilon)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s+\epsilon)^{H-\frac{1}{2}} du \right].$$

Proposition. $B_H^{\epsilon}(t)$ converge vers $B^H(t)$ dans $L^2(\Omega)$ quand ϵ tends vers 0 et $H > \frac{1}{2}$. La convergence est uniforme par rapport à $t \in [0,T]$.

Nous répresentons $B_H^{\epsilon}(t)$ comme

$$B_H^{\epsilon}(t) = C_H \left(H - \frac{1}{2}\right) \int_0^t \int_0^u \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s+\epsilon)^{H-\frac{3}{2}} dW_s du \ + C_H \epsilon^{\alpha} W_t.$$

Cela montre que le mouvement Brownien fractionnaire peut être interprété comme un drift stochastique. On peut montrer aussi que la longue mémoire n'en résulte pas de la singularité à la diagonale du noyau.

Un modèle fractionnaire avec sauts

Dans cette section nous proposons un modèle fractionnaire avec sauts, avec des noyaux Volterra semimartingale et un processus de Poisson composé filtré doublement stochastique. Soit $(W(t))_{t\in[0,T]}$, $(B(t))_{t\in[0,T]}$ deux mouvements Browniens indépendants et $(N_P(t))_{t\in[0,T]}$ un processus de Poisson indépendent de W et B. Considérons le modèle suivant

$$\begin{aligned} S(t) = S(0) &+ \int_0^t \mu(s) S(s-) ds + \int_0^t \sigma_1(s) S(s-) dW(s) \\ &+ \int_0^t \sigma_2(s) S(s-) dB^K(s) + \int_0^t \int_{\mathbb{R}_0} S(s-) y N^{\Lambda^{\phi}}(dy, ds), \end{aligned} \tag{0.0.16}$$

où $B^K(t)$ est un processus de Volterra:

$$B^K(t) = \int_0^t K(t-s) dB(s),$$

et $N^{\Lambda^{\phi}}(dy,dt)$ est un processus de Poisson doublement stochastique filtré, avec intensité cumulative donnée par un processus de Poisson filtré indépendent de N,

$$\begin{aligned} \Lambda^{\phi}(t) &= \int_0^t \phi(t-s) N_P(ds) \\ &= \phi(0) N_P(t) + \int_0^t \int_0^u \phi'(u-s) N_P(ds) du. \end{aligned}$$

Soit $(X_t)_{t\in[0,T]}$ un processus Lévy Itô avec forme différentielle

$$dX_t = G(t)dt + F(t)dB(t) + \int_{\mathbb{R}_0} H(y,u) N_P(dy,du),$$

où $G \in L^1(\mathbb{P} \times \lambda)$, $F \in L^2(\mathbb{P} \times \lambda)$ adapté et $H \in L^2(\mathbb{P} \times \nu)$ adapté tel que

$$\int_0^T \int_{\mathbb{R}_0} e^{2H(y,t)} \nu(dy,dt) < +\infty.$$

Définissons la mesure de probabilité $\mathbb{Q}$ comme $\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = e^{X_t}$ pour $t \in [0,T]$.

Théorème. Il existe une mesure de probabilité risque neutre $\mathbb{Q}$ équivalente à $\mathbb{P}$ sous laquelle les prix actualisés $\tilde{S}_t = e^{-rt}S_t$, $t \in [0,T]$, sont des martingales, si les conditions suivantes sont satisfaites pour tout $t \in [0,T]$:

(1) $C(t) := (\mu(t) - r) + \int_0^t K'(t-s)\sigma_2(t))dB(s) + \int_0^t \int_{\mathbb{R}_0} \phi'(t-s)\ln(1+y)N_P(dy,ds)$

$$+K(0)\sigma_2(t)F(t) + \phi(0)\int_{\mathbb{R}_0} y\nu^{\mathbb{Q}}(dy) = 0$$

(2) $G(t) + \frac{1}{2}F(t)^2 + \int_{\mathbb{R}_0}\left(e^{H(t,y)} - 1\right)\nu(dy) = 0.$

Remarque. La condition $C(t) = 0$ donne une infinité de solutions possibles (F,G,H). Le marché est donc pas complet.

Chapitre 4 – Longue mémoire et structure à terme des taux d'intérêt

Dans ce chapitre, on étend les résultats du chapitre 3 aux modèles de taux d'intérêt. Nous montrons comment évaluer des contrats sur taux à court terme. Nous étudions la structure des modèles de taux d'intérêt à court terme comme le modèle de Vasicek et Cox-Ingersoll-Ross. Nous terminons le chapitre par l'étude du modèle de Heath-Jarrow-Morton (HJM) et nous donnons les conditions nécessaires et suffisantes pour l'évaluation des obligations en absence d'opportunité d'arbitrage.

Evaluation des contrats sur taux d'intérêt

Considérons que le taux à court terme suit le modèle géneral suivant:

$$dr(t) = \mu_r(t)dt + \sigma_{1r}(t)\,dW(t) + \sigma_{2r}(t)\,dB^K(t), \tag{0.0.17}$$

où $B^K(t) = \int_0^t K(t,s)dB(s)$, K un noyau semimartingale et W et B sont indépendants. Il s'ensuit que $B^K(t)$ admet la décomposition suivante:

$$\begin{aligned} B^K(t) &= \int_0^t K(s,s)dB(s) + \int_0^t \int_0^u K'(u,s)dB(s)du \\ &= \int_0^t K(s,s)dB(s) + A(t). \end{aligned} \tag{0.0.18}$$

Pour un contrat sur taux $V(t,T,r(t))$ avec T la date de maturité, l'application de la formule d'Itô donne:

$$dV(t,T,r(t)) = \left[\frac{\partial V}{\partial t} + \mu_r(t)\frac{\partial V}{\partial r} + \sigma_{2r}(t)A'(t)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}\sigma_{2r}^2(t)\frac{\partial^2 V}{\partial^2 r}\right]dt$$

$$+\left[\frac{\partial V}{\partial r}\sigma_{1r}(t)\right]dW(t)+\left[\frac{\partial V}{\partial r}\sigma_{2r}(t)K(t,t)\right]dB(t).$$

Ainsi par un peu de calculs, on peut montrer que $V(t)$ est solution de l'équation aux dérivées partielles suivante:

$$\frac{\partial V}{\partial t}(t)+\Bigg(\mu_r(t)-\lambda(t,r(t))\sqrt{\sigma_{1r}^2(t)+K(t,t)^2\sigma_{2r}^2(t)}$$
$$+\sigma_{2r}(t)\bigg(\int_0^t K'(t,s)dB_s\bigg)\Bigg)\frac{\partial V}{\partial r}(t)+\frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial r^2}(t)-r(t)V(t)=0,$$

où $\lambda(t,r(t))$ est la prime de risque.

Modèle de taux à court terme

On peut supposer que $r(t)$ suit un modèle de taux comme Vasicek ou Cox-Ingersoll-Ross avec W et B^K comme processus sous-jacent. Dans le cas du modèle de Vasicek, avec un choix particulier de la prime de risque, on peut se ramener à la forme suivante:

$$dr(t)=k(\theta-r(t))dt+\sigma_1 dW(t)+\sigma_2 K(t,t)\,dB^{\mathbb{Q}}(t).$$

Dans le cas de Cox-Ingersoll-Ross:

$$dr(t)=k(\theta-r(t))\,dt+\sigma_1\sqrt{r(t)}\,dW(t)+\sigma_2 K(t,t)\sqrt{r(t)}dB^{\mathbb{Q}}(t).$$

Ici $B^{\mathbb{Q}}$ est un mouvement Brownien sous la mesure risque neutre $\mathbb{Q}$.

Modèle de taux forward

Pour modéliser les taux forward, on propose un modèle de Heath-Jarrow-Morton avec deux sources de bruits W et B^K.

$$\begin{cases} df(t,T)=\mu_f(t,T)\,dt+\sigma_{f,1}(t,T)\,dW(t)+\sigma_{f,2}(t,T)\,dB^K(t),\\ f(0,T)=f^M(0,T), \end{cases} \tag{0.0.19}$$

$f^M(0,T)$ est le taux forward de maturité T observé sur le marché.

La dynamique des zéro coupons de maturité T est donnée par

$$P(t,T)=\exp\left(-\int_t^T f(t,u)du\right).$$

En appliquant la formule d'Itô pour les processus gaussiens, on montre la proposition suivante:

Proposition. Le prix d'une option zero coupon $P(t,T)$ de maturité T est solution de l'équation suivante:

$$dP(t,T)=P(t,T)\left[r_t-\mu_f^*(t,T)+\frac{1}{2}\sigma_{f,1}^*(t,T)^2+\frac{1}{2}\frac{\partial}{\partial t}\left(\int_t^T\left(K^*\sigma_{f,2}^*(t,s,T)\right)^2ds\right)\right]dt$$
$$-P(t,T)\sigma_{f,1}^*(t,T)dW_t-P(t,T)\int_t^T K(t,t)\sigma_{f,2}(t,u)dudB(t), \tag{0.0.20}$$

où

$$\mu_f^*(t,T) = \int_t^T \mu_f(t,u)du + \int_t^T \int_0^t K'(t,s)\sigma_{f,2}(t,u)dudB_s$$
$$\sigma_{f,1}^*(t,T) = \int_t^T \sigma_{f,1}(t,u)du$$
$$\sigma_{f,2}^*(t,s,T) = \int_t^T \sigma_{f,2}(t,s,u)du.$$

La condition de non arbitrage est donnée par la proposition suivante:

Proposition. Le modèle HJM est sans opportunités d'arbitrage, s'il existe un processus adapté $\{\lambda(t) : t \in [0,T]\}$ tel que:

$$-K(t,t)\sigma_{f,2}(t,T)\lambda(t) = -\mu_f(t,T) - \int_0^t K'(t,s)\sigma_{f,2}(t,T)dB_s$$
$$+ \sigma_{1,f}(t,T)\sigma_{1,f}^*(t,T) + \frac{1}{2}\frac{\partial}{\partial T}\frac{\partial}{\partial t}\left(\int_0^t \left(K^* \int_t^T \sigma_{f,2}(s,T)\right)^2 ds\right) dt.$$

Chapitre 5 – Longue mémoire et les dérivés de crédit

Dans ce chapitre nous traitons les modèles de risque de crédit. Ces modèles sont souvent classifiés comme structurels et à intensité réduite.

Les modèles structurels

Les modèles structurels supposent que l'évènement de défaut se produit pour une entreprise quand ses actifs atteignent un niveau suffisamment faible D par rapport à ses passifs. Supposons que l'actif de la firme A suit le modèle suivant:

$$dA(t) = \mu A(t)dt + \sigma_1 A(t)dW(t) + \sigma_2 A(t)dB^K(t), \quad A_0 > 0.$$

La solution de ce modèle est la suivante:

$$A(t) = A_0 \exp\left(\mu t + \sigma_1 W(t) + \sigma_2 B^K(t) - \frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 \int_0^t K(t,s)^2 ds\right).$$

La probabilité de défaut historique est alors:

$$\mathbb{P}\left(A(T) < D\right) = \mathcal{N}\left(\frac{\ln(\frac{D}{A_0}) - \mu T + \frac{1}{2}\sigma_1^2 T + \frac{1}{2}\sigma_2^2 \int_0^T K(t,s)^2 ds}{\sqrt{\sigma_1^2 T + \sigma_2^2 \int_0^T K(T,s)^2 ds}}\right)$$

où

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{1}{2}u^2\right) du.$$

La probabilité de défaut risque neutre est alors:

$$\mathbb{Q}\left(A(T) < D\right) = \mathcal{N}\left(\frac{\ln(\frac{D}{A_0}) - rT + \frac{1}{2}\sigma_1^2 T + \frac{1}{2}\sigma_2^2 K(T,T)^2}{\sqrt{T(\sigma_1^2 + \sigma_2^2 K(T,T)^2)}}\right).$$

Les modèles à intensité réduite

Les modèles à intensité réduite considèrent que le temps de défaut est déterminé comme le premier instant de saut d'un processus à saut exogène. On définit un temps d'arrêt τ adapté par rapport à une filtration $\mathcal{H}$. Le processus $H(t) = \mathbf{1}_{\{\tau \leqslant t\}}$ est un processus càdlàg adapté à $\mathcal{H}$ et il existe un processus progressivement mesurable $(\lambda(t))_{t \geqslant 0}$ tel que

$$H(t) - \int_0^{t \wedge \tau} \lambda(s) ds$$

est une martingale par rapport à la filtration $\mathcal{F}_t := \mathcal{H}_t \vee \mathcal{G}_t$, où $(\mathcal{G}_t)_{t \geqslant 0}$ est la filtration du marché contenant toute l'information autre que défaut et survie.

Pour modéliser $\lambda(t)$ on propose le modèle CIR fractionnaire

$$d\lambda(t) = \theta(\alpha - \lambda(t))dt + \sigma_1 \sqrt{\lambda(t)} dW(t) + \sigma_2 \sqrt{\lambda(t)} dB^K(t),$$

ou le modèle Vasicek exponentiel fractionnaire

$$d\lambda(t) = \lambda(t)(\theta - \kappa \ln \lambda(t))dt + \sigma_1 \lambda(t) dW(t) + \sigma_2 \lambda(t) dB^K(t).$$

Chapter 1

Lévy Processes and Gaussian semimartingales

1.1 Lévy processes: definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 1.1.1 *A Lévy process is a càdlàg stochastic process $(L_t)_{t\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ and $L_0 = 0$ such that:*

1. *Independent increments: for every increasing sequence of times $t_0, \dots, t_n$, the random variables L_{t_0}, $L_{t_1} - L_{t_0}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent.*
2. *Stationary increments: the law of $L_{t+h} - L_t$ does not depend on t.*
3. *Stochastic continuity: for every $\varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|L_{t+h} - L_t| > \varepsilon) = 0$.*

Simple examples of Lévy processes are compound Poisson processes.

Definition 1.1.2 *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where the jump size Y_i are i.i.d. with distribution f and (N_t) is a Poisson process with intensity λ, independent of $(Y_i)_{i\geqslant 1}$.

From the definition we can deduce the following properties:

1. The sample paths of X are càdlàg piecewise constant functions.
2. The jump times $(T_i)_{i\geqslant 1}$ have the same law as the jump times of the Poisson process N_t; they can be expressed as partial sums of independent exponential random variables with parameter λ.
3. The jump sizes $(Y_i)_{i\geqslant 1}$ are independent and identically distributed with law f.

1.1.1 The Lévy measure

Consider first a compound Poisson process $(X_t)_{t\geqslant 0}$ on $\mathbb{R}$ and denote by $\Delta X_t = X_t - X_{t\text{-}}$ the jump of X at time t. Rather than exploring ΔX_t itself further, we will find it more profitable to count jumps of specified size. Associate to X a random measure J_X on $[0,\infty[\times\mathbb{R}$ describing the jumps of X:
For every measurable set $A\subset\mathcal{B}(\mathbb{R}\setminus\{0\})$, $J_X\left([t_1,t_2]\times A\right)$ counts the number of jump times of X between t_1 and t_2 such that their jump sizes are in A. Then J_X is a Poisson random measure on $[0,\infty[\times\mathbb{R}$, with intensity measure $\nu\left(dx\right)dt$, defined such that

$$\nu\left(A\right)=\mathbb{E}\left[J_X\left([0,1]\times A\right)\right]=\mathbb{E}\left[\#\left\{t\in[0,1],\Delta X_t\in A\right\}\right]=\frac{1}{T}\mathbb{E}\left[\#\left\{t\in[0,T],\Delta X_t\in A\right\}\right],$$

where
$\nu\left(A\right)$ is the expected number, per unit time, of jumps whose size belongs to A.

The Lévy measure describes the expected number of jumps of a certain height in a time interval of length 1. In general, the Lévy measure ν is not a finite measure. If

$$\int_{\mathbb{R}}\nu\left(dx\right)<\infty,$$

we can define the probability measure F as the distribution of the jump sizes

$$F\left(dx\right)=\frac{\nu\left(dx\right)}{\lambda}.$$

We say that the jumps are of *finite activity.* In this case, a compound Poisson process can describe the jumps:

$$X_t=\sum_{s\in[0,t]}\Delta X_s=\int_{[0,t]\times\mathbb{R}}xJ_X\left(ds\times dx\right)=\sum_{i=1}^{N_t(\lambda)}Y_i,$$

where Y_i are *i.i.d* and N_t a Poisson process.

If $\nu(\mathbb{R})=\infty$, then an infinite number of small jumps is expected. General Lévy processes should be considered in this case.

Proposition 1.1.3 *Let X be a Lévy process without Gaussian component, of (dimension 1), as defined above.*

1. *If $\nu(\mathbb{R})<\infty$ then almost all paths of X have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.*
2. *If $\nu(\mathbb{R})=\infty$ then almost all paths of X have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.*

If $\int_{|x|>0}|x|^2\nu(dx)<\infty$, then X has variance

$$\text{var}\left(L_t\right)=t\ \text{var}\left(L_1\right)=t\int_{\mathbb{R}}x^2\nu\left(dx\right).$$

In the *finite activity case* we have a compound Poisson process and we can define a Stratonovich integral with respect to X. Let $(X_t)_{t\geqslant 0}$ have intensity measure λ and jump size distribution F. Its jump measure J_X is a Poisson random measure on $\mathbb{R}\times[0,\infty[$ with intensity measure

$$\nu(dx\times dt)=\nu(dx)dt=\lambda F(dx)dt.$$

In the *infinite activity case* we have to suppose the hypothesis of finite quadratic variations and then to use Itô calculus for semi martingales. This will be clarified later.

1.1.2 Lévy-Itô decomposition

Theorem 1.1.4 (Lévy-Itô decomposition) *Consider a triplet (μ,σ,ν) where $\mu\in\mathbb{R},\sigma\in\mathbb{R}_+$ and ν is a measure on $\mathbb{R}\setminus\{0\}$ that verifies*

$$\int_{|x|\leqslant 1}|x|^2\nu\left(dx\right)<\infty\ ;\qquad\int_{|x|>1}\nu\left(dx\right)<\infty.$$

Then there exists a probability space $(\Omega,\ \mathcal{F},\mathbb{P})$ on which four independent Lévy processes exist:

1. *$L_t^1=\mu t$, a constant drift,*
2. *L_t^2 is a Brownian motion W_t with variance σ^2,*
3. *$L_t^3=\int\limits_{|x|\geqslant 1;\,s\in[0,t]}xJ_L\left(ds\times dx\right)$ is a compound Poisson process,*
4. *$\tilde{L}_t^4=\int\limits_{\varepsilon\leqslant|x|<1;\,s\in[0,t]}x\tilde{J}_L\left(ds\times dx\right)=\int\limits_{\varepsilon\leqslant|x|<1;\,s\in[0,t]}x\{J_L\left(ds\times dx\right)-\nu(dx)ds\}$ a square integrable pure jump martingale with an a.s. countable number of jumps on each finite time interval of magnitude less than 1. Here J_L is a Poisson random measure on $[0,\infty[\times\mathbb{R}$ with intensity measure $\nu(dx)dt$.*

We take

$$\begin{aligned}L&=L^1+L^2+L^3+\lim_{\varepsilon\to 0}\tilde{L}^{4,\varepsilon}\\&=\mu t+W_t+\int\limits_{|x|\geqslant 1;\,s\in[0,t]}xJ_L\left(ds\times dx\right)+\lim_{\varepsilon\to 0}\int\limits_{\varepsilon\leqslant|x|<1;\,s\in[0,t]}x\{J_L\left(ds\times dx\right)-\nu(dx)ds\}.\end{aligned}$$

There exists a probability space on which a Lévy process $L=(L_t)_{t=0}$ with characteristic exponent

$$\psi(u)=iu\mu-\frac{u^2\sigma^2}{2}+\int_{\mathbb{R}}\left(e^{iux}-1-iux1_{\{|x|<1\}}\right)\nu\left(dx\right),\quad u\in\mathbb{R},$$

is defined. The convergence in the last term is a.s. and uniform in t on $[0,T]$.

The last two terms are discontinuous processes incorporating the jumps of L_t and are described by the Lévy measure ν.

The condition $\int_{|x|>1}\nu\left(dx\right)<\infty$ means that L has a finite number of jumps of size larger than 1. Then the sum $L_t^3=\sum_{0\leqslant s\leqslant t}^{|\Delta L_s|>1}\Delta L_s$ contains almost surely a finite number of terms and is a compound Poisson process.

For any $\varepsilon > 0$, the sum of jumps with amplitude between ε and 1

$$L_t^{4,\,\varepsilon} = \sum_{0\leqslant s\leqslant t}^{\varepsilon\leqslant|\Delta L_s|<1} \Delta L_s = \int\limits_{\varepsilon\leqslant|x|<1;\, s\in[0,t]} x J_L\,(ds\times dx)$$

is a compound Poisson process as long as $\varepsilon > 0$. The problem in that case is that ν may have infinitely many small jumps of size close to 0 and their sum does not converge. In order to let $\varepsilon \to 0$, we have to center the integral, i.e. to replace the jump integral by its compensated version

$$\tilde{L}_t^4 = \int\limits_{\varepsilon\leqslant|x|<1;\, s\in[0,t]} x\, \tilde{J}_L\,(ds\times dx) = \int\limits_{\varepsilon\leqslant|x|<1;\, s\in[0,t]} x\, \{J_L\,(ds\times dx) - \nu\,(dx)\,ds\}$$

which is a martingale. The condition $\int_{|x|\leqslant 1} |x|^2 \nu\,(dx) < \infty$ guarantees that $\tilde{L}_t^4$ is a square integrable martingale.

1.1.3 Lévy-Itô processes and Itô formula

Consider a Lévy process on $[0,T]$ with characteristic triplet (μ,σ,ν), and $(\gamma_t(x))_{t\in[0,T]}$ a random adapted function, left continuous in t, measurable in x, such that

$$\mathbb{E}\left[\int_0^T\int_{|x|>1}|\gamma_t\,(x)\,|\nu\,(dx)\,dt\right]<\infty;\qquad \mathbb{E}\left[\int_0^T\int_{|x|\leqslant 1}\gamma_t^2\,(x)\,\nu\,(dx)\,dt\right]<\infty.$$

A Lévy-Itô process has the form

$$X_t = \mu t + \sigma W_t + \int_0^t\int_{|x|>1}\gamma_s\,(x)\,J_X\,(ds\times dx) + \int_0^t\int_{|x|\leqslant 1}\gamma_s\,(x)\,\tilde{J}_X\,(ds\times dx).$$

We may also allow stochastic coefficients μ_s and σ_s: $[0,T]\times\Omega\to\mathbb{R}$ which are $\mathcal{F}_s$-adapted such that $\mathbb{E}\left[\int_0^T\{\mu_s^2+\sigma_s^2\}ds\right]<\infty$. Then we can write

$$X_t = \int_0^t\mu_s ds + \int_0^t\sigma_s dW_s + \int_0^t\int_{|x|>1}\gamma_s\,(x)\,J_X\,(ds\times dx) + \int_0^t\int_{|x|\leqslant 1}\gamma_s\,(x)\,\tilde{J}_X\,(ds\times dx).$$

We see that X_t is a semimartingale with quadratic variation

$$\langle X\rangle_t = \int_0^t\sigma_s^2 ds + \int_0^t\int_{\mathbb{R}}\gamma_s^2\,(x)\,\nu\,(dx)\,ds,$$

where we assume that the last integral is finite.

Proposition 1.1.5 (Itô formula) *Let X a Lévy-Itô process and $f \in C^2(\mathbb{R}_+, \mathbb{R})$. Then $f(t, X_t)$ is a Lévy-Itô process and the following formula holds:*

$$
\begin{aligned}
f(t, X_t) - f(0, X_0) = & \int_0^t \frac{\partial}{\partial t} f(s, X_s) ds + \int_0^t \frac{\partial}{\partial x} f(s, X_s) \left[\mu_s ds + \sigma_s dW_s\right] + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(s, X_s) \sigma_s^2 ds \\
& + \int_0^t \int_{|x|<1} \left\{ f(s, X_{s-} + \gamma_s(z)) - f(s, X_{s-}) - \frac{\partial}{\partial x} f(s, X_{s-}) \gamma_s(x) \right\} \nu(dx) ds \\
& + \int_0^t \int_{|x|<1} \left\{ f(s, X_{s-} + \gamma_s(z)) - f(s, X_{s-}) \right\} \tilde{J}_X(ds \times dx) \\
& + \int_0^t \int_{|x|\geqslant 1} \left\{ (f(s, X_{s-} + \gamma_s(z)) - f(s, X_{s-}) \right\} J_X(ds \times dx).
\end{aligned}
\tag{1.1.1}
$$

Example 1.1.6 *We have*

$$
\begin{aligned}
\exp(X_t) = 1 & + \int_0^t \exp(X_s) \left[\mu_s ds + \sigma_s dW_s\right] + \frac{1}{2} \int_0^t \exp(X_s) \sigma_s^2 ds \\
& + \int_0^t \int_{|x|<1} \left\{ \exp(X_{s-} + \gamma_s(x)) - \exp(X_{s-}) - \exp(X_{s-}) \gamma_s(x) \right\} \nu(dx) ds \\
& + \int_0^t \int_{|x|<1} (\exp(X_{s-} + \gamma_s(x)) - \exp(X_{s-})) \tilde{J}_X(ds \times dx) \\
& + \int_0^t \int_{|x|\geqslant 1} \exp(X_{s-} + \gamma_s(x)) - \exp(X_{s-}) J_X(ds \times dx).
\end{aligned}
$$

Corollary 1.1.7 *The process* $\exp(X)$ *is a local martingale if and only if:*

$$
\mu_s + \frac{1}{2}\sigma_s^2 + \int_{|x|<1} \left\{ \exp(\gamma_s(x)) - 1 - \gamma_s(x) \right\} \ \nu(dx) + \int_{|x|\geqslant 1} (\exp(\gamma_s(x)) - 1) \nu(dx) = 0 \quad a.s.,
$$

$$
\text{or} \quad \mu_s + \frac{1}{2}\sigma_s^2 + \int_{\mathbb{R}} \left\{ \exp(\gamma_s(x)) - 1 - \gamma_s(x) 1_{|x|<1} \right\} \ \nu(dx) = 0 \quad a.s.
$$

1.1.4 Stochastic integration for Lévy processes

Definition 1.1.8 *A stochastic process $(F_t)_{t\in[0,T]}$ is called simple predictable if it can be represented as:*

$$
F_t = F_0 1_{t=0} + \sum_{i=1}^{n} F_i 1_{]T_i, T_{i+1}]}(t),
$$

where $T_0 = 0 < T_1 < \ldots < T_n < T_{n+1} = T$ are non-anticipating random times, $n \in \mathbb{N}$ and each F_i is a bounded random variable whose value is revealed at T_i ($\mathcal{F}_{T_i}$measurable).

Definition 1.1.9 *We define the space $S(T, \nu)$ as the space of simple predictable functions F : $\Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$,*

$$
F(t, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} F_{ij} 1_{]T_i, T_{i+1}]}(t) 1_{A_j}(y),
$$

where $(F_{ij})_{j=1\dots m}$ are bounded $\mathcal{F}_{T_i}$-measurable random variables and $(A_j)_{j=1\dots m}$ disjoint subsets of $\mathbb{R}$ with $\nu([0,T]\times A_j)<\infty$.

The stochastic integral $\int_{[0,T]\times\mathbb{R}} F(s,y)L(ds,dy)$ is then defined as the random variable

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} F(s,y)L(ds,dy) &= \sum_{i=1}^n\sum_{j=1}^m F_{ij}L(]T_i,T_{i+1}]\times A_j)\\ &= \sum_{i=1}^n\sum_{j=1}^m F_{ij}\left[L_{T_{i+1}}(A_j)-L_{T_i}(A_j)\right].\end{aligned}$$

Analogously to the stochastic integral $X_T=\int_0^T\int_{\mathbb{R}} F(s,y)L(ds,dy)$ we may define the compensated integral $\tilde{X}_T=\int_{[0,T]\times\mathbb{R}} F(s,y)\tilde{L}(ds,dy)$ as

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} F(s,y)\tilde{L}(ds,dy) &= \sum_{i=1}^n\sum_{j=1}^m \tilde{L}(]T_i,T_{i+1}]\times A_j)\\ &= \sum_{i=1}^n\sum_{j=1}^m F_{ij}\left[L(]T_i,T_{i+1}]\times A_j)-\nu(]T_i,T_{i+1}]\times A_j)\right].\end{aligned}$$

For $t\in\,]0,T]$ we multiply $F(s,y)$ by $1_{[0,t]}(s)$ and define in the same way the integrals

$$X_t=\int_0^t\int_{\mathbb{R}} F(s,y)L(ds,dy)\quad\text{and}\quad \tilde{X}_t=\int_{[0,t]\times\mathbb{R}} F(s,y)\tilde{L}(ds,dy).$$

Suppose that $\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} F(s,y)^2\nu(ds,dy)\right]<\infty$. Then $\tilde{X}_t$, $0\leqslant t\leqslant T$, is a square integrable martingale verifying $\mathbb{E}[\tilde{X}_t]=0$ and the isometry formula:

$$\mathbb{E}[\tilde{X}_t^2]=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} F(s,y)^2\nu(ds,dy)\right].$$

Definition 1.1.10 *Define the Hilbert space $L^2(T,\nu)$ as the closure of $S(T,\nu)$ with respect to the scalar product:*

$$\langle F,G\rangle_{L^2(T,\nu)}=\int_0^T\int_{\mathbb{R}}\mathbb{E}\left[F(s,y)G(s,y)\right]\nu(ds,dy).$$

As in the Brownian motion case, we can prove that $S(T,\nu)$ is dense in $L^2(T,\nu)$ and for $F\in L^2(T,\nu)$, the process $\tilde{X}_t=\int_{[0,T]\times\mathbb{R}} F(s,y)\tilde{L}(ds,dy)$ is a square integrable martingale with $\langle F\rangle_{L^2(T,\nu)}<\infty$.

1.1.5 Girsanov type formula

Let $\mathbb{Q}$ be a probability on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\ \mathbb{P})$. We denote by $\mathbb{Q}_t$ the restrictions of $\mathbb{Q}$ with respect to the filtration $\mathcal{F}_t$.

Let $(X_t)_{t\in[0,T]}$ be a Lévy process satisfying the condition of the Corollary 1.1.7. Suppose that $\exp(X_t), (t \in [0,T])$ is a martingale. Define $\mathbb{Q}_t$ by

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \exp(X_t).$$

Then $\mathbb{Q}_t$ is a probability measure.

Let K a predictable process such that $\int_0^T \int_{\mathbb{R}} \mathbb{E}[|K_t(x)|^2]\, \nu(dx)dt < +\infty$. Define

$$M_t = \int_0^t \int_{|x|>0} K_s\,(x)\, \tilde{N}(ds, dx), \qquad U(s,x) = e^{\gamma_s(x)} - 1,$$

and suppose that

$$\int_0^T \int_{\mathbb{R}} \left(e^{\gamma_s(x)} - 1\right)^2 \nu\,(dx)\, dt < +\infty.$$

Finally set

$$B_t = W_t - \int_0^t \sigma\,(s)ds.$$

It follows

$$\begin{aligned} \tilde{N}_t^{\mathbb{Q}} &= M_t - \int_0^t \int_{|x|>0} K_s\,(x)\, U\,(s,\, x)\, \nu\,(dx)\, ds \\ &= \int_0^t \int_{|x|>0} K_t\,(x)\, N\,(dx, ds) - \int_0^t \int_{|x|>0} K_t\,(x)\, e^{\gamma_s(x)} \nu\,(dx)\, ds. \end{aligned}$$

Then, under $\mathbb{Q}$, the process $B = (B_t;\ 0 \leqslant t \leqslant T)$ is a Brownian motion, $\tilde{N}_t^{\mathbb{Q}} = \left(\tilde{N}_t^{\mathbb{Q}};\ 0 \leqslant t \leqslant T\right)$ is a $\mathbb{Q}$-martingale, $\nu^{\mathbb{Q}} = e^{\gamma}\nu$ is the $\mathbb{Q}$-compensator of N.

Proof. Applying Itô's formula to $\exp(X_t)$ with the martingale condition as in the corollary 1.1.7 gives

$$\exp(X_t) = 1 + \int_0^t \exp(X_s)\sigma_s dW_s + \int_0^t \int_{|x|<1} (\exp(X_{s-} + \gamma_s(x)) - \exp(X_{s-}))\, \tilde{N}\,(ds, dx)$$

$$+ \int_0^t \int_{|x|=1} \exp(X_{s-} + \gamma_s(x)) - \exp(X_{s-})\, \tilde{N}\,(ds, dx).$$

Similarly

$$B_t e^{X_t} = \int_0^t (1 + \sigma_s B_s) \exp(X_{s-}) dW_s + \int_0^t \int_{\mathbb{R}} \exp(X_{s-}) \left(e^{\gamma_s(x)} - 1\right) \tilde{N}\,(ds, dx)\,.$$

$$\tilde{N}_t^{\mathbb{Q}} \exp(X_t) = \int_0 \tilde{N}_{s-}^{\mathbb{Q}} \exp(X_{s-})\sigma_s dW_s + \int_0^t \int_{|x|>0} \exp(X_{s-}) \left(\tilde{N}_{s-}^{\mathbb{Q}} U\,(s,x) + K\,(s,x)\right) \tilde{N}\,(ds, dx)\,.$$

Hence $\tilde{N}_t^{\mathbb{Q}} \exp(X_t)$ and $B_t \exp(X_t)$ are martingales under $\mathbb{P}$. Now let $Z_t = B_t^2 - t$; applying Itô's formula, we show that Z_t is a martingale under $\mathbb{P}$. It follows from Lévy's Theorem that B_t is a Brownian motion. ■

1.2 Generalization of Poisson processes

1.2.1 Counting processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be our probability space. Consider a sequence of stopping time $(T_i;\ i \in \mathbb{N})$. Define the counting process $(N_t;\ t \in \mathbb{R}_+)$ associated to $(T_i;\ i \in \mathbb{N})$ as

$$N_t(\omega) = \sum_{i>0} 1_{\{T_i(\omega) \leqslant t\}}.$$

Then N_t takes positive integer values, and for $0 < s < t$, the random variable $N(t) - N(s)$ counts the number of events occurring in $(s, t]$.

Definition 1.2.1 *A random process $\{N_t;\ t \in \mathbb{R}_+\}$ is a **counting process** if it satisfies the following two conditions:*

1. *The trajectories of N are right continuous and piecewise constant with probability 1;*
2. *the process starts at 0, and for each t*

$$\Delta N_t = 0, \quad \text{or} \quad \Delta N_t = 1$$

   with probability one.

Here $\Delta N_t = N_t - N_{t^-}$ denotes the jumps of N at time t.

1.2.2 Poisson processes

Definition 1.2.2 *A counting process (point process) is an $(\mathcal{F}_t)$-Poisson process with intensity $\lambda > 0$ if and only if:*

1. *$N_t - N_s$ is independent of $\mathcal{F}_s$ for $0 \leqslant s < t < \infty$;*
2. *N_t is with stationary increments, i.e. $N_{t+h} - N_{s+h} \sim N_t - N_s$ for $(0 \leqslant s < t;\ h > 0)$;*
3. *$\mathbb{P}(N_{t+h} - N_t \geqslant 1) = \lambda h + o(h)$ and $\mathbb{P}(N_{t+h} - N_t \geqslant 2) = o(h)$.*

A special case is when the inter-arrival times $U_{n+1} = T_{n+1} - T_n$ are a sequence of i.i.d. random variables; in this case the sequence $\{T_n\}$ is called a renewal process.

Exponential random variables

Definition 1.2.3 *A positive random variable Y is said to follow an exponential distribution $\mathcal{E}(\lambda)$ with parameter $\lambda > 0$ if it has a probability density function of the form*

$$\lambda e^{-\lambda y} 1_{\{y \geqslant 0\}}.$$

The distribution function of Y is then given by

$$\forall y \in [0, \infty[\,; \qquad F_Y(y) = \mathbb{P}(Y \leqslant y) = 1 - \exp(-\lambda y).$$

The exponential distribution has the memoryless property, which means that if u is an exponential random variable, then

$$\forall t, s > 0, \qquad \mathbb{P}(T > t + s | T > t) = \frac{\int_{t+s}^{\infty} \lambda e^{-\lambda y} dy}{\int_{t}^{\infty} \lambda e^{-\lambda y} dy} = \mathbb{P}(T > s).$$

The distribution of $T - t$ knowing $T > t$ is the same as the distribution of T itself. We say that we have "absence of memory" which means that the distribution "forgets" the past.

Proposition 1.2.4 *Let $T > 0$ be a random variable such that*

$$\forall t, s > 0, \qquad \mathbb{P}(T > t + s | T > t) = \mathbb{P}(T > s);$$

then T has an exponential distribution.

Proof. Let $g(t) = \mathbb{P}(T > t)$. Then, for $t, s > 0$, we have

$$g(t + s) = \mathbb{P}(T > t + s | T > t)\mathbb{P}(T > t) = g(s)g(t).$$

We deduce as solution to this equation $g(t) = \exp(-\lambda t)$ for some $\lambda > 0$. Hence $1 - g$ is the distribution function of an exponential random variable with parameter λ.

■

The lack of memory of the exponential distribution is reflected in the fact that the instantaneous arrival rate for the exponential distribution is constant as a function of time.

The Poisson process is a continuous-time Markov chain with birth intensity λ. The Poisson model predicts exponential probability distribution of inter-arrival times, i.e., $U_{n+1} = T_{n+1} - T_n$ are independent random variables $\sim \mathcal{E}(\lambda)$ where λ is the intensity parameter. The time of the nth event T_n is a Gamma variable $\Gamma(n, \lambda)$ with density

$$f_{T_n}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t} 1_{\{t>0\}}.$$

The vector $(U_1, U_2, \ldots, U_n)$ has a density f_n defined as

$$f_n(t_1, t_2, \ldots, t_n) = \begin{cases} \lambda^n \exp(-\lambda t_n) & \text{if } \ 0 < t_1 < t_2 < \ldots < t_n; \\ 0 & \text{if } \ \text{not}. \end{cases}$$

The probability of an arrival at a certain fixed point in time is 0. From property (3) of Definition 1.2.2 we deduce that, as the length of an interval $[0, t]$ shrinks to zero,

$$\lim_{t \to 0} \frac{\mathbb{P}(N_t = 1)}{t} = \lambda \quad \text{and} \quad \lim_{t \to 0} \frac{\mathbb{P}(N_t \geqslant 2)}{t} = \lim_{t \to 0} \frac{o(t)}{t} \longrightarrow 0.$$

On the other hand, for $0 \leqslant s < t$,

$$\mathbb{P}(N_t = 0) = \mathbb{P}(N_s = 0, N_t - N_s = 0) = \mathbb{P}(N_s = 0)\mathbb{P}(N_{t-s} = 0). \tag{1.2.1}$$

Set $\alpha(t) = \mathbb{P}(N_t = 0)$. The solution to the equation $\alpha(t) = \alpha(s)\alpha(t - s)$ gives that

$$\mathbb{P}(N_t = 0) = e^{-\lambda t}.$$

In general, we can prove that

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0.$$

In particular,

$$\mathbb{E}[N_t] = \mathrm{Var}[N_t] = \lambda t \ ; \qquad \mathbb{E}[e^{iuN_t}] = \exp(\lambda t(e^{iu} - 1)),$$

and $\frac{N_t}{t} \to \lambda$ *p.s.* as $t \to \infty$.

Quadratic Variation

Let $(N_t)_{t\geqslant 0}$ be a Poisson process with intensity $\lambda > 0$. Then the quadratic variation of N is

$$[N, N]_t = \sum_{0\leqslant s\leqslant t} (\Delta N_t)^2.$$

Since the jumps of the Poisson process all have size 1, it follows that $[N, N]_t = N_t$ for all $t \geqslant 0$.

The *angle-bracket* process associated with an adapted process $(X_t)_{t\geqslant 0}$, denoted $(\langle X, X\rangle_t)_{t\geqslant 0}$ is defined as the predictable process such that

$$Z_t = [X, X]_t - \langle X, X\rangle_t$$

defines a process $(Z_t)_{t\geqslant 0}$ which is a martingale. In the case when $(X_t)_{t\geqslant 0}$ is an Itô processes (and in particular in the case of a Brownian motion) it follows that $\langle X, X\rangle = [X, X]_t$ for all $t \in [0, T]$, whereas in the case of the Poisson process $(N_t)_{t\in[0,T]}$ we have $\langle N, N\rangle_t = \lambda t$ for all $t \in [0, T]$.

Simulation of a Poisson process

To simulate a Poisson process with rate λ we use the property that the inter-arrival times u_i are independent $\mathcal{E}(\lambda)$ random variables.

First we simulate the inter-arrival times using the inversion method

$$T_i - T_{i-1} = u_i = -\frac{1}{\lambda}\ln(U_i)$$

where U_i are independent $U(0, 1)$ random variables, $i = 1, \ldots, n$. The event times, denoted $T_1, \ldots, T_n$, are then obtained by summing up the inter-arrival times

$$T_i = \sum_{j=1}^{i} u_i, \quad i = 1, \ldots, n.$$

This allows to generate the Poisson random variables.

1.2.3 The non-homogeneous Poisson process

As we have seen, Poisson processes assume constant rate λ, *i.e.* the distribution of the number of events in an interval depends only on the length of the interval and not on its position. Nevertheless statistical analysis shows that the constraints of *stationarity* and *independence* of increments are rather restrictive. The *non-homogeneous Poisson process* as its name indicates relaxes this stationarity assumption. Denoting by $\mathcal{F}_t$ the information up to time t, the $\mathcal{F}_t$-intensity function is defined by

$$\lambda(t, \mathcal{F}_t) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}\left[(N_{t+h} - N_t) > 0 \middle| \mathcal{F}_t\right] \tag{1.2.2}$$

where $\lambda(t, \mathcal{F}_t)$ is assumed to have sample paths that are left continuous with right-hand limits. This rate function defines the instantaneous probability of observing a jump at a given time point.

Definition 1.2.5 *Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a the probability space and $\{N_t,\, t \in [0,T]\}$ a stochastic process. Let $\mathcal{F}_t = \sigma\{N_s, s < t\}$ be the σ-field generated by the process, then $(N_t)_{t\in[0,T]}$ is a non-homogeneous Poisson process with intensity function $\lambda(t) : \mathbb{R}_+ \to \mathbb{R}_+$ if*

(1) $N_0 = 0$;

(2) *the number of events in disjoint intervals are independent random variables:* $\forall\ 0 \leqslant s \leqslant t$ *the increments $N_t - N_s$ are independent of the σ-field $\mathcal{F}_s$;*

(3) $\lim_{h\to 0} \frac{\mathbb{P}(N_{t+h}-N_t=1)}{h} = \lambda(t)$;

(4) $\lim_{h\to 0} \frac{\mathbb{P}(N_{t+h}-N_t\geqslant 2)}{h} = 0$.

Moreover assume that $\int_s^t \lambda(r)dr < \infty$ for all $s < t$, which is equivalent to saying that λ is locally integrable. Moreover we assume that the increments $N_t - N_s$ for the interval $(s, t[$ have a Poisson distribution:

$$\mathbb{P}(N_t - N_s = n) = e^{-\int_s^t \lambda(r)dr}\ \frac{(\int_s^t \lambda(r)dr)^n}{n!}$$

for all positive integers n. Finally, by condition (2) we can assume that the process has independent increments, *i.e.* for all disjoint intervals $(s_1, t_1], \ldots, (s_k, t_k]$, the random variables

$$\{N_{t_i} - N_{s_i} | i = 1, \ldots, k\}$$

are mutually independent. The function $\Lambda(t) = \int_0^t \lambda(u)du$ is called *cumulative intensity function.*

Example 1.2.6 (Piecewise constant intensity) *Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of the time interval $[0,T]$, and assume that the intensity is a piecewise constant function:*

$$\lambda(t) = \sum_{i=0}^{n-1} \lambda_i\, \mathbf{1}_{\{t_i < t \leqslant t_{i+1}\}}.$$

Change of time and non-homogeneous Poisson process

Let N_t^1 be a standard Poisson process with intensity $\lambda = 1$. Introduce an operational time scale function, given by the cumulative intensity, $\Lambda(t)$. By the following time change operation, the process:

$$N_{\Lambda(t)} = N_t \circ \Lambda(t)$$

is an non homogeneous Poisson process with intensity $\Lambda(t)$.

It is also possible to consider random intensity functions $\Lambda(t,\omega)$. Let $0 \leqslant t_1 < t_2 < t_3 < t_4$ then the covariance between the increments is:

$$\text{Cov}\left[(N_{t_2} - N_{t_1}), (N_{t_4} - N_{t_3})\right] = \text{Cov}\left[(\Lambda(t_2) - \Lambda(t_1)), (\Lambda(t_4) - \Lambda(t_3))\right].$$

Example 1.2.7 *Polya process*
Assume that $\Lambda(\omega)$ *follows a gamma distribution* $\Gamma(\alpha,\beta)$, *i.e. the probability density function is* $f_\Lambda(y) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, y^{\alpha-1}\, e^{-\beta y}$. *Then*

$$\begin{aligned}
\mathbb{P}(N_t = n) &= \int_0^\infty \mathbb{P}(N_t = n|\Lambda = \lambda) f_\Lambda(\lambda) d\lambda \\
&= \int_0^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda \\
&= \frac{t^n}{n!} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+n)}{(\beta+t)^{\alpha+n}} \int_0^\infty \frac{(\beta+t)^{\alpha+n}}{\Gamma(\alpha+n)} e^{-\lambda(\beta+t)} \lambda^{n+\alpha-1} d\lambda \\
&= \frac{(\alpha+n-1)!}{(\alpha-1)!n!} \left(\frac{t}{\beta+t}\right)^n \left(\frac{\beta}{\beta+t}\right)^\alpha
\end{aligned}$$

which is the distribution of a negative binomial random variable $N(\alpha,p)$ *with* $p = \frac{\beta}{\beta+t}$.

1.2.4 Doubly stochastic Poisson process

Definition 1.2.8 *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and* m *a (positive) random measure on* $\mathbb{R}_+$, *non-atomic and with finite moments on bounded sets in* $\mathcal{B}(\mathbb{R}_+)$. *Let* $\mathcal{F}^m \subset \mathcal{F}$ *denote the* σ*-field generated by* m. *Let* $(\mathcal{F}_t, t \geqslant 0) \subset \mathcal{F}$ *be a right-continuous and complete filtration. A random measure* N *is a doubly stochastic* $\mathcal{F}_t$*-Poisson process* (DSPP) *if*

(i) $\mathbb{P}(N(\Delta) = k \mid \mathcal{F}^m) = \frac{m(\Delta)^k}{k!} e^{-m(\Delta)}$ *for all bounded Borel sets* $\Delta \subset \mathbb{R}_+$,

(ii) $N_0 := 0$, $N_t := N((0,t])$ *is* $\mathcal{F}_t$*-measurable for all* $t > 0$,

(iii) $\sigma(N(\Delta))$ *and* $\mathcal{F}_t$ *are conditionally independent given* $\mathcal{F}^m$, *whenever* $\Delta \subset]t,\infty[$.

Intuitively, condition (iii) says that if a particular sample path of the (cumulative) intensity process m is known, the process N has exactly the same properties as the Poisson process with respect to $\mathcal{F}_t \vee \mathcal{F}^m$ with the deterministic hazard function $m(\omega)$. In particular, we have

$$\mathbb{P}\{N_t - N_s = k | \mathcal{F}_t \vee \mathcal{F}^m\} = \mathbb{P}\{N_t - N_s = k | \mathcal{F}^m\}, \tag{1.2.3}$$

i.e. conditionally on the σ-field $\mathcal{F}^m$ the increment $N_t - N_s$ is independent of the σ-field $\mathcal{F}_s$. When the intensity process $\lambda_t \equiv \lambda(t) = \{\lambda(t, x(t)) : t \geqslant 0\}$ given the information process $\{x(t) : t \geqslant 0\}$ exists, the measure $m(\Omega)$ is continuous with respect to the Lebesgue measure and the probability that the number of points occurring in an interval $[t_1, t_2)$ is k is given by:

$$\begin{aligned}
\mathbb{P}\left[N_{t_2} - N_{t_1} = k\right] &= \mathbb{E}\left\{\mathbb{P}\left[N_{t_2} - N_{t_1} = k \mid x(s) : t_1 < s < t_2\right]\right\} \\
&= \mathbb{E}\left[\frac{(m_{t_2} - m_{t_1})^k}{k!} e^{-(m_{t_2} - m_{t_1})}\right] \\
&= \mathbb{E}\left[\frac{1}{k!}\left(\int_{t_1}^{t_2} \lambda_s ds\right)^k e^{-\int_{t_1}^{t_2} \lambda_s ds}\right]
\end{aligned}$$

where

$$m(]0,t[) = \int_0^t \lambda(u)du.$$

Definition 1.2.9 *The centered doubly stochastic Poisson process* (CDSPP) *is defined as*

$$\tilde{N}(\Delta) := N(\Delta) - m(\Delta), \quad \Delta \in \mathcal{B}(\mathbb{R}_+). \tag{1.2.4}$$

Denote $\mathcal{F}^{\tilde{N}}$ the filtration generated by $\tilde{N}$ and by $\mathcal{F}^N$ the filtration generated by N . From [78] Theorem 2.8, we have that:

$$\mathcal{F}^{\tilde{N}} = \mathcal{F}^N \vee \mathcal{F}^m.$$

Moreover, we can verify that $\tilde{N}$ has the martingale property with respect to $\mathcal{F}_t$ and $\mathcal{F}_t \vee \mathcal{F}^m$.

Remark 1.2.10

a) $\mathcal{F}_t$ *is understood as "the history" up to time* t*. It contains the* σ*-field* $\sigma(N_s,\ 0 \leqq s \leqq t)$ *generated by* N*, but possibly also* σ*-fields generated by other processes, eventually independent of* N*. If* $\mathcal{F}_t = \sigma(N_s,\ 0 \leqq s \leqq t)$*, then* $\mathcal{F}_0$ *is trivial and* $\mathcal{F}^m$ *is contained in* $\mathcal{F}_0$.

b) *Condition* (iii) *implies that*

$$\mathbb{E}[f(N(\Delta)) \mid \mathcal{F}_t \vee \mathcal{F}^m] = \mathbb{E}[f(N(\Delta)) \mid \mathcal{F}^m]$$

for all $f : \mathbb{R}_+ \to \mathbb{R}$ *such that the conditional expectation above is defined and for all bounded Borel sets* $\Delta \subset]t, \infty[$.

c) *Condition* (i) *implies*

$$\mathbb{E}(f(N(\Delta)) \mid \mathcal{F}^m) = \sum_{k=0}^{\infty} f(k) \frac{m(\Delta)^k}{k!} e^{-m(\Delta)}.$$

In particular $\mathbb{E}(N(\Delta)) = \mathbb{E}(m(\Delta))$.

d) *By calculating the conditional variance one gets*

$$\mathrm{Var}(N(\Delta)) = \mathbb{E}(m(\Delta)) + \mathrm{Var}(m(\Delta)), \quad \mathbb{E}(N(\Delta)^2) = \mathbb{E}(m(\Delta)) + \mathbb{E}(m(\Delta)^2).$$

The higher order moments can be calculated as in [78].

e) *The conditional moments of the* CDSPP (1.2.4) *are*

$$\mathbb{E}\left[\tilde{N}(\Delta) \middle| \mathcal{F}^m\right] = 0, \tag{1.2.5}$$

$$\mathbb{E}\left[\tilde{N}(\Delta)^2 \middle| \mathcal{F}^m\right] = m(\Delta). \tag{1.2.6}$$

This yields

$$\mathbb{E}\left[\tilde{N}(\Delta)^2\right] = \mathrm{Var}\left(\tilde{N}(\Delta)\right) = \mathbb{E}\left[m(\Delta)\right]. \tag{1.2.7}$$

f) *The conditional characteristic function of* $N(\Delta)$ *is given by*

$$\mathbb{E}(e^{iuN(\Delta)} \mid \mathcal{F}^m) = \exp(m(\Delta)(e^{iu} - 1)), \quad u \in \mathbb{R}.$$

g) *We notice that the stochastic nature of the intensity causes the variance of the process to be greater than the variance of a homogeneous Poisson process with the same expected intensity measure.*

1.3 Gaussian semimartingales

Definition 1.3.1 *Let* ϕ *absolutely continuous on* $\mathbb{R}_+$ *with a square integrable derivative* ϕ'. *Set*

$$B_t = \int_0^t \phi(t-s)\, dW_s.$$

Proposition 1.3.2

1. $(B_t)_{t\in[0,T]}$ *is a Gaussian semimartingale and admits the following decomposition*

$$B_t = \phi(0)W_t + \int_0^t \int_0^u \phi'(u-s) dW_s du.$$

2. *The autocorrelation function*

$$\begin{aligned} R(t,s) = \mathbb{E}[B_t B_s] &= \int_0^{t\wedge s} \phi(t-u)\,\phi(s-u) du \\ &= \phi^2(0)\,(t\wedge s) \\ &\quad + \phi(0) \int_0^t \int_0^{r\wedge s} \phi'(r-u)\, dudr \\ &\quad + \phi(0) \int_0^s \int_0^{r\wedge t} \phi'(r-u)\, dudr \\ &\quad + \int_0^t \int_0^s \int_0^{r\wedge v} \phi'(r-u)\,\phi'(v-u)\, dudrdv. \end{aligned}$$

Proof. 1) By the stochastic Fubini theorem [82], we get

$$\begin{aligned} B_t &= \int_0^t \phi(t-s)\, dW_s \\ &= \phi(0)\, W_t + \int_0^t \int_s^t \phi'(u-s)\, dudW_s \\ &= \phi(0)\, W_t + \int_0^t \int_0^u \phi'(u-s) dW_s du. \end{aligned}$$

2) We have

$$\begin{aligned} R(t,s) &= \mathbb{E}[B_t B_s] \\ &= \mathbb{E}\left[\left(\phi(0)\, W_t + \int_0^t \int_u^t \phi'(r-u)\, drdW_u\right)\left(\phi(0)\, W_s + \int_0^s \int_u^s \phi'(r-u)\, drdW_u\right)\right]. \end{aligned}$$

■

Remark 1.3.3

(1) *As consequence we deduce that* $\langle B\rangle_t = \phi^2(0)t$.

(2) *Let* $A_t = \int_0^t \int_0^u \phi'(u-s)dW_s du$. *Then*

$$\begin{aligned}\mathbb{E}\left[A_t A_s\right] &= \mathbb{E}\left[\left(B_t - \phi(0)W_t\right)\left(B_s - \phi(0)W_s\right)\right] \\ &= \int_0^{t\wedge s} \phi\left(t-u\right)\phi\left(s-u\right)du - \phi\left(0\right)\int_0^{t\wedge s}\phi\left(t-u\right)du \\ &\quad - \phi\left(0\right)\int_0^{t\wedge s}\phi\left(s-u\right)du + \phi^2\left(0\right)\,\left(t\wedge s\right).\end{aligned}$$

(3) $\mathbb{E}\left[A_t B_t\right] = E\left[B_t^2\right] - \phi\left(0\right)\int_0^t \phi\left(t-u\right)du$.

(4) $\mathbb{E}\left[A_t W_t\right] = \int_0^t \phi\left(t-u\right)du - \phi\left(0\right)t$.

1.3.1 Stochastic integration

We briefly recall some basic elements of the stochastic calculus of variations with respect to Gaussian processes. In this review we restrict ourselves to Gaussian semimartingales as it is our concern. A more general presentation can be found in [1]. In the following we consider B_t as in the definition, with the same conditions on ϕ.

Fix an interval $[0,T]$. We suppose that B is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is generated by B. Denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]}\right\rangle_{\mathcal{H}} = R\left(t,s\right).$$

The mapping $1_{[0,t]} \to B_t$ provides an isometry between $\mathcal{H}$ and the first Wiener chaos H_1, that is the closed subspace of $L^2\left(\Omega\right)$ generated by B. The variable $B\left(\varphi\right)$ denotes the image in H_1 of an element $\varphi \in \mathcal{H}$.

Let $\mathscr{S}$ be the set of smooth cylindrical random variables of the form $F = f\left(B\left(\varphi_1\right), \ldots, B\left(\varphi_n\right)\right)$ where $n \geq 1$, $f \in C_b^\infty\left(\mathbb{R}^n\right)$ (f and all its derivatives are bounded) and $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$. The derivative of F is the element in $L^2\left(\Omega, \mathcal{H}\right)$ defined as

$$D^B F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(B\left(\varphi_1\right), \ldots, B\left(\varphi_n\right)\right)\varphi_j.$$

As usual, $\mathbb{D}_B^{1,2}$ is the closure of the set of smooth cylindrical random variables F with respect to the norm

$$\|F\|_{1,2}^2 = \mathbb{E}\left[|F|^2\right] + \mathbb{E}\left[\left\|D^B F\right\|_{\mathcal{H}}^2\right].$$

The divergence operator δ^B is defined as the adjoint of the derivative operator. If a random variable $u \in L^2\left(\Omega, \mathcal{H}\right)$ belongs to the domain of the divergence operator, then $\delta\left(u\right)$ is given by the duality relationship

$$\mathbb{E}\left[F\delta^B\left(u\right)\right] = \mathbb{E}\left[\left\langle D^B F, u\right\rangle_{\mathcal{H}}\right]$$

for every $F \in \mathbb{D}_B^{1,2}$.

Proposition 1.3.2 yields for any pair of step functions φ and ψ in $\mathcal{E}$

$$\begin{aligned}\langle \varphi, \psi \rangle_{\mathcal{H}} &= \phi^2(0) \int_0^T \varphi(r)\psi(r)\, dr \\ &+ \phi\,(0) \int_0^T \int_0^r \varphi(r)\phi'\,(r-u)\, dudr \\ &+ \phi\,(0) \int_0^T \int_0^r \psi(r)\phi'\,(r-u)\, dudr \\ &+ \int_0^T \int_0^T \int_0^{r\wedge v} \varphi(r)\psi(v)\, \phi'\,(r-u)\, \phi'\,(v-u)\, dudrdv.\end{aligned}$$

Now set $Z(t,s) = \phi\,(t-s)$. Then Z defines an operator on $L^2([0,T])$ by

$$(Zh)(t) = \int_0^t Z(t,s)\, h(s)\, ds.$$

Consider the linear operator Z^* from $\mathcal{E}$ to $L^2\,([0,T])$ defined by

$$(Z^*\varphi)\,(s) = \phi\,(0)\,\varphi\,(s) + \int_s^T \varphi\,(u)\,\phi'\,(u-s)\, du.$$

The operator Z^* is an isometry from $\mathcal{E}$ to $L^2\,([0,T])$ which extends to the Hilbert space $\mathcal{H}$; thus we have $\mathcal{H} = (Z^*)^{-1}\left(L^2\,([0,T])\right)$. Analogously to $\mathbb{D}_B^{1,2}$, we may define the spaces $\mathbb{D}_B^{1,2}(\mathbb{H})$ of $\mathbb{H}$-valued random variables for an arbitrary separable Hilbert space $\mathbb{H}$. Then we obtain

$$\mathbb{D}_B^{1,2}\,(\mathcal{H}) = (Z^*)^{-1}\mathbb{D}_W^{1,2}\left(L^2\,([0,T])\right).$$

On the other hand, we have for smooth random variables F the following identity

$$\mathbb{E}\left[\langle u, D^B F\rangle_{\mathcal{H}}\right] = \mathbb{E}\left[\langle Z^* u, D^W F\rangle_{L^2([0,T])}\right], \quad u \in L^2(\Omega, \mathcal{H})$$

As a consequence, we obtain $\mathrm{Dom}\,\delta^B = (Z^*)^{-1}\mathrm{Dom}\,\delta^W$.

For $\varphi \in \mathcal{E}$ consider the seminorm

$$\|\varphi\|_Z^2 = \phi^2(0) \int_0^T \varphi^2(s)\, ds + \int_0^T \left(\int_s^T \varphi\,(u)\,\phi'\,(u-s)\, du \right)^2 ds.$$

Denote by $\mathcal{H}_Z$ the completion of $\mathcal{E}$ with respect to this seminorm. Since $\|\varphi\|_{\mathcal{H}} \leqslant \sqrt{2}\|\varphi\|_Z$, the space $\mathcal{H}_Z$ is continuously embedded into $\mathcal{H}$. In addition, the space $\mathbb{D}^{1,2}\,(\mathcal{H}_Z)$ is included in the domain of δ^B, and for any $u \in \mathbb{D}^{1,2}\,(\mathcal{H}_Z)$, we have $\delta^B(u) = \int_0^T Z^* u_s\, \delta W_s$, see [1] for details.

For processes $u \in \mathbb{D}^{1,2}\,(\mathcal{H}_Z)$ we use the notation $\delta^B(u) = \int_0^T u_s\, \delta B_s$, and therefore we have the formula

$$\int_0^T u_s\, \delta B_s = \int_0^T Z^* u_s\, \delta W_s.$$

1.3.2 Itô formula

We have seen that for Gaussian semimartingales one can define stochastic integrals by classical Itô theory or as divergence integrals. Obtaining an Itô formula is directly related to the definition of the stochastic integral. In the sequel we explain the connections between the different definitions of stochastic integration.

Let F be a twice continuously differentiable function $F \in C^2(\mathbb{R})$ satisfying the growth condition:

$$\max\left\{|F(x)|\,,\ |F'(x)|,\ |F''(x)|\right\} \leqslant ce^{\lambda|x|^2}.$$

According to [1], Theorem 2, we have that $F'(B_t) \in \mathbb{D}^{1,2}(\mathcal{H}_Z)$ and for each $t \in [0,T]$ the following formula holds:

$$F(B_t) = F(0) + \int_0^t \frac{\partial F}{\partial x}(B_s)\,\delta B_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial^2 x}(B_s)\,dR_s$$

with

$$\begin{aligned}\int_0^t \frac{\partial F}{\partial x}(B_s)\,\delta B_s &= \phi(0)\int_0^t \frac{\partial F}{\partial x}(B_s)\,\delta W_s + \int_0^t\int_s^t \frac{\partial F}{\partial x}(B_r)\,\phi'(r-s)\,dr\delta W_s \\ &= \phi(0)\int_0^t \frac{\partial F}{\partial x}(B_s)\,\delta W_s + \int_0^t\int_0^r \frac{\partial F}{\partial x}(B_r)\,\phi'(r-s)\,\delta W_s dr\end{aligned}$$

and

$$\int_0^t \frac{\partial^2 F}{\partial x^2}(B_s)\,dR_s = \int_0^t \frac{\partial^2 F}{\partial x^2}(B_r)\,d\left(\int_0^r (\phi(r-s))^2 ds\right).$$

On the other hand, we have

$$B_t = \phi(0)\,W_t + \int_0^t\int_0^u \phi'(u-s)dW_s du = \phi(0)\,W_t + A_t$$

where A_t is of bounded variations. The Itô formula for semimartingales yields

$$\begin{aligned}F(B_t) &= F(0) + \int_0^t \frac{\partial F}{\partial x}(B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(B_s)d\langle B\rangle_s \\ &= F(0) + \phi(0)\int_0^t \frac{\partial F}{\partial x}(B_s)dW_s + \int_0^t\int_0^u \frac{\partial F}{\partial x}(B_u)\,\phi'(u-s)dW_s du \\ &\quad + \frac{\phi^2(0)}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(B_s)\,ds.\end{aligned} \tag{1.3.1}$$

We deduce the following equality

$$\begin{aligned}&\int_0^t\int_0^u \frac{\partial F}{\partial x}(B_u)\,\phi'(u-s)dW_s du + \frac{\phi^2(0)}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(B_s)\,ds \\ &= \int_0^t\int_0^r \frac{\partial F}{\partial x}(B_r)\,\phi'(r-s)\,\delta W_s dr + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(B_r)\,d\left(\int_0^r (\phi(r-s))^2 ds\right).\end{aligned} \tag{1.3.2}$$

Example 1.3.4 *We have*

$$\begin{aligned}\exp\left(B\left(t\right)\right) &= 1+\phi\left(0\right)\int_0^t \exp\left(\phi\left(0\right)W_s\right)\exp\left(A_s\right)dW_s \\ &+\int_0^t\int_0^u \exp\left(\phi\left(0\right)W_u\right)\exp\left(A_u\right)\phi'\left(u-s\right)dW_s du \\ &+\frac{\phi^2(0)}{2}\int_0^t \exp\left(\phi\left(0\right)W_s\right)\exp\left(A_s\right)ds\end{aligned}$$

where $A_t=\int_0^t\int_0^u \phi'\left(u-s\right)dW_s du$.

We may write as well

$$\begin{aligned}\exp\left(B\left(t\right)\right) &= 1+\phi\left(0\right)\int_0^t \exp\left(B_s\right)\delta W_s+\int_0^t\int_s^t \exp\left(B_r\right)\phi'\left(r-s\right)dr\delta W_s \\ &+\frac{1}{2}\int_0^t \exp\left(B_r\right)d\left(\int_0^r\left(\phi\left(r-s\right)\right)^2 ds\right).\end{aligned}$$

1.4 Nonlinear SDE driven by fractional Brownian motion

In this section we consider some stochastic differential equations which we are going to use many times in the subsequent chapters. Let $B^H(t)$ be a fractional Brownian motion. Consider one-dimensional, nonlinear Itô SDE of the form

$$\begin{cases} dX_t = f(X_t,t)dt+g(X_t,t)dB^H(t) \\ X(0)=0, \quad t\geqslant 0,\ H\in(0,1)\end{cases} \tag{1.4.1}$$

where $f(x,t)$ and $g(x,t)$ are sufficiently smooth and the integration *w.r.t* the FBM is in the Wick sense as in [5]. In [97] the authors showed that under certain conditions, we can derive a solution to problem (1.4.1) by exact linearization. The following theorem establishes the linearization conditions for (1.4.1). In the following we will use the integral $\int$ without limits to design the primitive.

Theorem 1.4.1 (see [97]). *The Itô* SDE (1.4.1) *is linearizable to*

$$dY_t=\left(a(t)Y_t+b(t)\right)dt+\left(c(t)Y_t+e(t)\right)dB^H(t) \tag{1.4.2}$$

via an invertible transformation

$$y=h(x,t), \qquad \frac{\partial h(x,t)}{\partial x}\neq 0 \quad (\textit{at least locally}) \tag{1.4.3}$$

if and only if condition

$$\frac{\partial}{\partial x}\left(g(x,t)L\right)=0 \tag{1.4.4}$$

or

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial x}\left(g(x,t)\frac{\partial}{\partial x}\left(g(x,t)L\right)\right)}{\frac{\partial}{\partial x}\left(g(x,t)L\right)}\right)=0 \tag{1.4.5}$$

is satisfied. Here

$$L=\frac{\partial}{\partial t}\left(\frac{1}{g}\right)+\frac{\partial}{\partial x}\left(\frac{f}{g}-Ht^{2H-1}\frac{\partial g}{\partial x}\right). \tag{1.4.6}$$

This implies that finding a solution to problem (1.4.1) is reduced to find a solution to (1.4.2). We can distinguish the following cases depending on $c(t)=0$ or not.

1. If $c(t)\equiv 0$ the condition (1.4.4) is satisfied with

$$h(x,t)=\left(\int \alpha(t)dt\right)^{-1}\left(\int \frac{1}{g(x,t)}dx\right).$$

   We can write

$$dY_t=\beta(t)e(t)dt+e(t)dB^H(t)$$

   where $e(t)=\left(\int \alpha(t)dt\right)^{-1}$ and α,β verifying the equality

$$\alpha(t)y+\beta(t)=\int \frac{\partial}{\partial t}\left(\frac{1}{g}\right)dx+\left(\frac{f}{g}\right)-Ht^{2H-1}\frac{dg}{dx},$$

   and $y=h(x,t)$.

2. If $c(t)\not\equiv 0$, the condition (1.4.5) is satisfied for

$$h(x,t)=e^{-M(t,x)\int \frac{1}{g(x,t)}dx}$$

   where $M(t,x)=\dfrac{\frac{\partial}{\partial x}\left(g(x,t)\frac{\partial}{\partial x}\left(g(x,t)L\right)\right)}{\frac{\partial}{\partial x}\left(g(x,t)L\right)}$.

By applying Itô's formula, see [97], we can deduce for $H\in(0,1)$ the following linear solution to the problem (1.4.1)

$$Y_t=\frac{1}{F}\left(\int^t F(s)b(s)ds-2H\int^t s^{2H-1}F(s)c(s)e(s)ds+\int^t F(s)e(s)dB^H_s+Y_0\right)$$

where

$$F(t,B^H_t)=\exp\left(-\int^t c(s)dB^H(s)+\int^t s^{2H-1}c^2(s)ds-\int^t a(s)ds\right).$$

1.4.1 Ornstein-Uhlenbeck SDE

Consider the following form of the coefficients:

$$f(x,t)=p(t)x\left(q(t)-\ln(x)\right)$$
$$g(x,t)=r(t)x.$$

In a first step we compute

$$L=\frac{\partial}{\partial t}\left(\frac{1}{g}\right)+\frac{\partial}{\partial x}\left(\frac{f}{g}-Ht^{2H-1}\frac{\partial g}{\partial x}\right).$$

In fact

- $\dfrac{\partial}{\partial t}\left(\dfrac{1}{g}\right) = \dfrac{-\dot{r}(t)}{(r(t))^2}\dfrac{1}{x}$,
- $\dfrac{\partial}{\partial x}\left(\dfrac{f}{g}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{p(t)x\,(q(t)-\ln(x))}{r(t)x}\right) = -\dfrac{p(t)}{r(t)x}$,
- $\dfrac{\partial^2 g}{\partial x^2} = 0.$

It follows the expression of L:

$$L = \frac{-\dot{r}(t)}{(r(t))^2}\frac{1}{x} - \frac{p(t)}{r(t)x}$$

satisfying $\frac{\partial}{\partial x}g(x,t)L = 0$.

By the last theorem, the solution to the Ornstein-Uhlenbeck SDE is given by (see [97]),

$$\begin{aligned} X(t) = \exp\Bigg[\Bigg(&\int_0^t \left[p(s)q(s) - Hs^{2H-1}(r(s))^2\right] e^{\int_0^s p(r)dr}ds \\ &+ \int_0^t r(s)e^{\int_0^r p(r)dr}dB^H(s) + \ln(X(0))\Bigg) e^{-\int p(t)dt}\Bigg]. \end{aligned} \tag{1.4.7}$$

Example 1.4.2 *Set $p=1$, $H=1/2$, $r(t)=\sigma$ and $q(t)=q$. Then Equation* (1.4.7) *becomes*

$$\begin{aligned} X(t) &= \exp\left[\left(\int_0^t \left[q - \frac{1}{2}\sigma^2\right] e^{\int_0^s dr}ds + \int_0^t \sigma e^{\int_0^s dr}dW(s) + \ln(X(0))\right) e^{-\int_0^t ds}\right] \\ &= \exp\left[\left(\int_0^t \left[q - \frac{1}{2}\sigma^2\right] e^{s}ds + \int_0^t \sigma e^{\int_0^s dr}dW(s) + \ln(X(0))\right) e^{-t}\right] \\ &= \exp\left[\left(\int_0^t \left[q - \frac{1}{2}\sigma^2\right] e^{s}ds + \int_0^t \sigma e^{s}dW(s) + \ln(X(0))\right) e^{-t}\right] \\ &= \exp\left[\ln(X(0))e^{-t} + \left(\left[q - \frac{1}{2}\sigma^2\right](1-e^{-t}) + \sigma\int_0^t e^{-(t-s)}dW(s)\right)\right]. \end{aligned}$$

Using the Novikov criterion

$$\mathbb{E}\left[\exp\left(\sigma\int_0^t e^{-(t-s)}dW(s) - \sigma^2\left[\frac{1-e^{-2t}}{4}\right]\right)\right] = 1,$$

we deduce

$$\mathbb{E}[X(t)] = \exp\left[\ln(X(0))e^{-t} + \left(\left[q - \frac{1}{2}\sigma^2\right](1-e^{-t}) + \sigma^2\left[\frac{1-e^{-2t}}{4}\right]\right)\right]$$

and

$$\lim_{t\to\infty}\mathbb{E}[X(t)] = \exp\left(\left[q - \frac{1}{2}\sigma^2\right] + \frac{\sigma^2}{4}\right),$$

which is the long run mean.

Remark 1.4.3 *Let $Z \sim N(\mu, \sigma)$, then $X = e^Z$ follows a log-normal with mean μ_X and variance σ_X^2:*

$$\begin{cases} \mu_X = e^{\mu + \frac{\sigma^2}{2}} \\ \sigma_X^2 = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}. \end{cases}$$

We can recover the long run mean without the Novikov criterion. In fact

$$\mathbb{E}\left[\left(\int_0^t e^{-(t-s)} dW(s)\right)^2\right] = \int_0^t e^{-2(t-s)} ds = \frac{1}{2}\left[e^{-2(t-s)}\right]_0^t = \frac{1 - e^{-2t}}{2}.$$

1.4.2 Cox-Ingersoll-Ross SDE

Let

$$f(x,t) = l(t)(\theta(t)) - x)$$

$$g(x,t) = \sigma(t)\sqrt{x}.$$

We compute

- $\frac{\partial}{\partial t}\left(\frac{1}{g}\right) = -\frac{\dot{\sigma}(t)}{(\sigma(t))^2\sqrt{x}}$
- $\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = -\frac{1}{2}\left(\frac{l(t)\theta(t)}{\sigma(t)}\right)\left(\frac{1}{x\sqrt{x}}\right) - \frac{1}{2}\left(\frac{l(t)}{\sigma(t)}\right)\left(\frac{1}{\sqrt{x}}\right)$
- $\frac{\partial^2 g}{\partial x^2} = -\frac{1}{4}\frac{\sigma(t)}{x\sqrt{x}}.$

We deduce

$$L = \left(-\frac{\dot{\sigma}(t)}{(\sigma(t))^2} - \frac{1}{2}\frac{l(t)}{\sigma(t)}\right)\left(\frac{1}{\sqrt{x}}\right) + \left(-\frac{1}{2}\frac{l(t)\theta(t)}{\sigma(t)} + \frac{Ht^{2H-1}\sigma(t)}{4}\right)\left(\frac{1}{x\sqrt{x}}\right)$$

and

$$\frac{\partial}{\partial x}(g(x,t)L) = \left[\frac{l(t)\theta(t)}{2} - \frac{Ht^{2H-1}\sigma(t)^2}{4}\right]\frac{1}{x^2}.$$

The linearization condition $\frac{\partial}{\partial x}(g(x,t)L) = 0$ is fulfilled for $\sigma(t) = \sqrt{\frac{2l(t)\theta(t)}{Ht^{2H-1}}}$, which is a limitation for this model since this assumption is not practical. Usually obtaining a solution for this model is done by a discrete scheme.

If we take

$$\sigma(t) = \sqrt{\frac{2l(t)\theta(t)}{Ht^{2H-1}}},$$

we can deduce that, see [97],

$$x(t) = \left(\frac{1}{2}\exp\left(\exp\left(-\int_0^t \frac{1}{2}l(s)ds\right)\right) \times \left(\int_0^t e^{\left(\int_0^s l(r)dr\right)}\sigma(s)dB^H(s) + \sqrt{x(0)}\right)\right)^2.$$

Chapter 2

Stochastic Calculus for fractional Lévy Processes

In this chapter we discuss different approaches to fractional Lévy processes and propose a definition based on a fractional generalization of Cox processes by allowing the intensity to be fractional. We point out that there are some ambiguities concerning already existing definitions. Here we propose an anticipative stochastic calculus via Malliavin calculus and derive an Itô formula using the relation between pathwise integration and divergence integration.

2.1 Definitions

2.1.1 Fractional Brownian motion and fractional Lévy process

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a fixed filtered probability space. For $T > 0$ and $t \in [0, T]$ we have the following Molchan-Golosov representation of fractional Brownian motion with Hurst index $H \in]0, 1[$:

$$B^H(t) = \int_0^t Z_H(t,s)\, dW_s,$$

where

$$Z_H(t,s) = C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}}\, du \right] \quad \text{for } t > s,$$

and

$$C_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right]^{\frac{1}{2}}.$$

Here β denotes the classical beta function.

One can represent $Z_H(t,s)$ in terms of a fractional integral,

$$Z_H(t,s) = C_H s^{\frac{1}{2}-H} \left(I_{T^-}^{H-\frac{1}{2}} \left((\cdot)^{H-\frac{1}{2}} 1_{[0,t[}(\cdot) \right) \right)(s), \quad s, t \in]0, T],$$

where $I_{T^-}^{H-\frac{1}{2}}$ is the *right-sided Riemann-Liouville fractional operator* of order $H - \frac{1}{2}$ over $[0, T]$.

In order to extend this approach to general Lévy processes, we proceed as follows, see [95].

Definition 2.1.1 *Let $(L_t)_{t\geqslant 0}$ be a Lévy process without Gaussian component such that $\mathbb{E}L_1 = 0$ and $\mathbb{E}L_1^2 < \infty$. Let $H \in]0,1[$. We call the stochastic process*

$$Y_t = \int_0^t Z_H(t,s)dL_s \tag{2.1.1}$$

a fractional Lévy process by Molchan-Golosov transformation.

2.2 Trajectories, increments and simulation

In this section we compare different processes in order to understand the trajectorial behavior of the process defined in (2.1.1). The increments of a FBM are a moving average of the increments of a BM. We start by simulating the trajectories of a BM, then of a FBM. After we will consider a pure jump process, and then a fractional jump process as defined in (2.1.1). Our objective is to show that the transformation we apply to get a FBM from the BM is not necessarily adequate to obtain a FLP from a Lévy process. Lévy process in our aim is to modelize the jumps.

2.2.1 Brownian motion

Let us start by simulating a Brownian motion. Consider a Brownian motion $(W_t)_{t\in[0,T]}$ with volatility σ and drift b. Consider a discretization of n fixed times $(0 = t_0, t_1, t_2, \ldots, t_n = t)$. The simulation of W_t is the realization of the vector $(W_{t_1}, \ldots, W_{t_n})$. For this, we proceed in three steps:

1. Simulate n independent standard normal random variables $(N_1, N_2, \ldots, N_n)$.
2. Set $\Delta X_i = \sigma N_i\sqrt{t_i - t_{i-1}} + b(t_i - t_{i-1})$.
3. Put $X_{t_i} = \sum_{k=1}^{i} \Delta X_i$.

The formula is

$$W_{t_i} = W_{t_{i-1}} + \sigma N_i\sqrt{t_i - t_{i-1}} + b(t_i - t_{i-1}).$$

For a Brownian motion with $b = 0$ and $\sigma = 1$, we have

$$W_{t_i} = W_{t_{i-1}} + N_i\sqrt{t_i - t_{i-1}}.$$

Hence, the increments are:

$$W_{t_i} - W_{t_{i-1}} = N_i\sqrt{t_i - t_{i-1}}$$

which depend on the mesh size of the discretization $t_i - t_{i-1}$.

2.2.2 Fractional Brownian motion and fractional Lévy process

The process that we want to simulate is a FBM of Liouville type:

$$B_t^H = \int_0^t (t-s)^{H-\frac{1}{2}}dW_s, \quad 0 < H < 1.$$

Consider a discretization $(0 = t_0, t_1, t_2, \dots, t_n = t)$ of the interval $[0, t]$. We have the following approximation (in law):

$$\begin{aligned} B_t^n &\approx \sum_{i=0}^{n-1} (t - t_i)^{H-\frac{1}{2}} (W_{t_{i+1}} - W_{t_i}) \\ &\approx (t - t_0)^{H-\frac{1}{2}} (W_{t_1} - W_{t_0}) + (t - t_1)^{H-\frac{1}{2}} (W_{t_2} - W_{t_1}) + \dots \\ &\quad \dots + (t - t_{n-1})^{H-\frac{1}{2}} (W_{t_n} - W_{t_{n-1}}). \end{aligned}$$

For $H > 1/2$, we see that more weight is assigned to past increments of the Brownian motion W_t which gives the long memory behavior of the process.

For $H < 1/2$, we have the inverse situation. More weight is assigned to recent increments of the Brownian motion W_t which yields the short memory behavior.

In the case $H < 1/2$, we see for instance that $(t - t_{n-1})^{H-\frac{1}{2}} \to \infty$ as $t_{n-1} \to t$, whereas

$$(t - t_{n-1})^{H-\frac{1}{2}} (W_t - W_{t_{n-1}}) \sim (t - t_{n-1})^{H-\frac{1}{2}} \sqrt{t - t_{n-1}}\, N(0, 1)$$

stays finite. Hence the fact that $W_{t_n} - W_{t_{n-1}} \sim N(0, \sqrt{t_n - t_{n-1}})$ is a useful ingredient to control the stochastic integral when $t_{n-1} \to t_n = t$.

Let $0 < s < t$, $\alpha = H - 1/2$ with $H > 1/2$, and consider the increment

$$\begin{aligned} B_t^H - B_s^H &= \int_0^t (t - r)^{H-\frac{1}{2}} dW_r - \int_0^s (s - r)^{H-\frac{1}{2}} dW_r \\ &= \int_0^s \left[(t - r)^{H-\frac{1}{2}} - (s - r)^{H-\frac{1}{2}} \right] dW_r + \int_s^t (t - r)^{H-\frac{1}{2}} dW_r \\ &\approx \sum_{i:\, 0 \leqslant t_i \leqslant s} [(t - t_i)^\alpha - (s - t_i)^\alpha] \left(W_{t_{i+1}} - W_{t_i} \right) + \sum_{i:\, s < t_i < t} (t - t_i)^\alpha \left(W_{t_{i+1}} - W_{t_i} \right) \\ &\approx \sum_{i:\, t_i < t} \psi_{(t,s)}(t_i) \left(W_{t_{i+1}} - W_{t_i} \right), \end{aligned}$$

where

$$\psi_{(t,s)}(t_i) = \begin{cases} (t - t_i)^\alpha - (s - t_i)^\alpha & \text{if } \ 0 \leqslant t_i \leqslant s, \\ (t - t_i)^\alpha & \text{if } \ s < t_i < t. \end{cases}$$

Hence we can interpret the increment $B_t^H - B_s^H$ as a weighted sum of past increments of a Brownian motion. For $H > 1/2$, as $s \to t$, we see that $\psi_{(t,s)} \to 0$ and

$$\psi_{(t,s)}(t_i) \left(W_{t_{i+1}} - W_{t_i} \right) \longrightarrow 0,$$

hence

$$B_t^H - B_s^H \to 0 \quad \mathbb{P}\text{-a.s.}$$

If we replace B^H by L^H the behaviour of the paths of L^H is given by the proposition from [95]:

Proposition 2.2.1

1. *For $H > 1/2$, a FLP has almost surely Hölder continuous paths of any order strictly less than $H - 1/2$.*
2. *For $H < 1/2$, a FLP has discontinuous and unbounded sample paths with positive probability.*

2.3 Fractional white noise and fractional Lévy white noise

In this section we recall some results from [51] and [4] in order to make a discrete analysis of the FLP.

2.3.1 Fractional white noise and ARIMA process

In a formal sense the derivative of a BM is the continuous time white noise process with constant spectral density. Fractional Brownian motion $B^H(t)$ with Hurst index $H \in]0,1[$ is a generalization of a BM and in a formal sense the continuous time fractional noise process would correspond to $\frac{dB^H(t)}{dt}$.

The discrete time analogue of Brownian motion is a random walk, or an ARIMA$(0,1,0)$ process x_t defined by

$$\nabla x_t = (1-B)x_t = \epsilon_t,$$

where B is the backward shift operator defined by $Bx_t = x_{t-1}$ and where the ϵ_t are i.i.d. random variables. The first difference of x_t is the discrete time white noise process ϵ_t. We can write x_t as

$$x_t = (1-B)^{-1}\epsilon_t,$$

and using

$$(1-B)^{-1} = \lim_{j\to\infty}(1+B+B^2+\ldots+B^j),$$

we get

$$x_t = \sum_{j=0}^{\infty} \epsilon_{t-j}$$

(here the index j runs over all points of the interval $[0,t]$ and is not an integer).

We define fractionally differentiated white noise with parameter H to be the $(\frac{1}{2}-H)$th fractional difference of discrete time white noise. For $-1/2 < d < 1/2$, the fractional difference operator ∇^d is defined in a natural way by a binomial series as

$$\nabla^d = (1-B)^d = \sum_{k=0}^{\infty}\binom{d}{k}(-B)^k = 1 - dB - \frac{1}{2}d(1-d)B^2 - \frac{1}{6}(1-d)(2-d)B^3 - \ldots$$

Letting $d = H - \frac{1}{2}$, the discrete fractional noise is the process

$$\epsilon_t^H = \nabla^{-d}\epsilon_t, \quad \text{or} \quad \nabla^d \epsilon_t^H = \epsilon_t,$$

where ϵ_t is the white noise process. We call $\{x_t\}$ an ARIMA$(0,d,0)$ process.

Set $S_T^H = \sum_{t=1}^{T}\epsilon_t^H$. In [4] it is established that

$$\lim_{T\to\infty}\left[\frac{1}{T^H}\right] S_{[rT]}^H \longrightarrow B_r^H \quad \text{in law},$$

where $r \in [0,1]$ and $d = H - \frac{1}{2}$.

Theorem 2.3.1 (Theorem 1, Hosking [51]) *Let $\{x_t\}$ be an* ARIMA$(0,d,0)$ *process.*

1. *If $d < \frac{1}{2}$, the process $\{x_t\}$ is stationary and has the infinite moving average representation*

$$x_t = \psi(B)\epsilon_t := \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$$

where

$$\psi_k = \frac{d(1+d)\dots(k-1+d)}{k!} = \frac{(k+d-1)!}{k!(d-1)!}.$$

As $k \to \infty$, we have $\psi_k \sim \frac{k^{d-1}}{(d-1)!}$.

2. *If $d > -\frac{1}{2}$, the process $\{x_t\}$ is invertible and has the infinite autoregressive representation*

$$\pi(B)x_t := \sum_{k=0} \pi_k x_{t-k} = \epsilon_t,$$

where

$$\pi_k = \frac{-d(1-d)\dots(k-1-d)}{k!} = \frac{(k-d-1)!}{k!(-d-1)!}.$$

As $k \to \infty$, we have $\pi_k \sim \frac{k^{-d-1}}{(-d-1)!}$.

3. *The covariance function of $\{x_t\}$ is given by*

$$\gamma_k = \mathbb{E}[x_t x_{t-k}] = \frac{(-1)^k(-2d)!}{(k-d)!(-k-d)!}.$$

Remark 2.3.2 *The representation*

$$x_t = \psi(B)\epsilon_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}$$

is in terms of an infinite sum for a finite interval $[0,T]$. This is due to the fact that the process has infinite variations on every finite interval of time. The question whether such a representation can still hold for a process with finite variations, like a compound Poisson process.

2.3.2 The FBM of Riemann-Liouville type

As in [94] consider a time series of ARIMA type defined as

$$Y_s = (1-L)^{-d}\Phi(L)^{-1}\Theta(L)\epsilon_s, \quad s = 0,1,2,\dots,[T]$$

where (ϵ_s) is a sequence of centered and uncorrelated random variables of the same variance σ, furthermore L is the lag operator and Φ, Θ are polynomials of L with roots outside of the unit disk. Suppose that the difference order d is greater than $\frac{1}{2}$, so that Y is a non-stationary process. It is known that such an ARIMA process exhibits a long range dependence; it is a long memory process.

We shall establish a relation between Y and the stochastic fractional integral

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \quad 0 < H < 1.$$

First note that Y has a moving average representation as

$$Y_s = \sum_{k=1}^{s} h_{s-k}^{(d)} \epsilon_k,$$

where the moving average coefficients h can be approximated as

$$h_s^{(d)} \approx \frac{\Theta(1)}{\Phi(1)\Gamma(d)} s^{d-1}$$

for large s, Γ being the Gamma function.

Consider now a continuous-time process Z defined as

$$Z_r = \frac{1}{T^{d-\frac{1}{2}}} Y_{[Tr]}, \quad 0 \leqslant r \leqslant 1,$$

where $[x]$ stands for the integer part of x. By some calculations and using Donsker's theorem, see [42], we obtain

$$\begin{aligned} Z_r &= \frac{1}{T^{d-\frac{1}{2}}} \sum_{k=1}^{[Tr]} h_{[Tr]-k}^{(d)} \varepsilon_k \\ &\approx \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} \sum_{k=1}^{[Tr]} \left(r - \frac{k}{T}\right)^{d-1} \left(W_{\frac{k}{T}} - W_{\frac{k-1}{T}}\right), \end{aligned}$$

where W is a standard Brownian motion.

As $T \to \infty$, the last sum converges (in law) to

$$Z_r \approx \frac{\sigma\Theta(1)}{\Phi(1)\Gamma(d)} \int_0^r (r-s)^{d-1} dW_s, \quad 0 \leqslant r \leqslant 1, \ d > \frac{1}{2}.$$

Put $d - 1 = -\alpha$, then $\alpha < \frac{1}{2}$ and we can write

$$Z_r = C(\alpha) \int_0^t (t-u)^{-\alpha} dW_u = C(\alpha) B_t, \qquad 0 \leqslant t \leqslant T.$$

Hence we see that the process B_t is in fact a limiting case of an ARIMA process of long memory. The integral form of B_t is obtained by passing to the limit in the sum expression. In case of finite increments (as for example the case of a compound Poisson process) we do not have to pass to the limit and cannot obtain an integral form: the representation will stay a sum.

2.3.3 Fractional Lévy process in discrete time

Lévy processes without Gaussian component, are discrete processes. One of the differences between Brownian motion and Lévy process is that in the first case we have infinite number of increments (ϵ_{t-k}) while the increments of a Lévy process with size $\neq 0$ cannot be infinite. If we ignore the jumps of size $\cong 0$ *i.e.*, the case of a Compound Poisson process for example, we have a finite number of jumps (increments) and this number is given by $N(t)$ a Poisson process.

Lévy white noise

Let

$$L(t) = \sum_{k:\, 0 \leqslant T_k \leqslant t} Y_{T_k} = \int_{[0,t]\times\mathbb{R}} y\, J_L(dy, ds), \quad t \geqslant 0,$$

be a pure Lévy process where $\{Y_{T_k},\ k \geqslant 1\}$ is a collection of real-valued independent i.i.d. random variables. The first moment of L is

$$\mathbb{E}[L(t)] = \int_{[0,t]\times\mathbb{R}} y\, \nu(dy, ds)$$

and we suppose for the second moment

$$\mathbb{E}[L(t)^2] = \int_{[0,t]\times\mathbb{R}} y^2\, \nu(y, ds) < \infty.$$

Lévy processes and Brownian motion have both stationary and independent increments. Nevertheless, the processes differ in a significant way, for example, B is a Gaussian process but L not necessarily. Also, B has continuous sample paths a.s. while the sample paths of L are right continuous with discontinuities at the jump times of L. As for the Gaussian white noise, we define in a formal way the Lévy white noise processes as formal derivatives of L, that is,

$$\epsilon_L(t) = \frac{d}{dt} L(t), \quad t \geqslant 0.$$

The Lévy white noise process can be viewed as a sequence of i.i.d. pulses arriving at the jump times. The above definitions are formal because both Brownian motion and Lévy process are not differentiable.

Consider now the case of a compound Poisson process. Given the stopping times

$$\{T_1, T_2, \ldots, T_{N(t)}\}$$

where $N(t)$ is the Poisson process counting the number of jumps in the interval $[0, t]$, we see that for $0 \leqslant s \leqslant t$:

$$\epsilon_p(s) = \begin{cases} Y_{T_i} & \text{if } s \in \{T_1, T_2, \ldots, T_N\}, \\ 0 & \text{if not.} \end{cases}$$

We see from this construction that we cannot argue like in the Section 2.3.2. The integral form

$$\int_0^t (t-s)^{-\alpha} dL(s)$$

cannot be understood as the natural Lévy generalization of the fractional Brownian motion. It may be more appropriate to call it a convoluted Lévy process.

2.4 The Mittag-Leffler Poisson process

As we know, if the waiting times between the successive jumps are i.i.d. and exponential, we get a Poisson process which is Markovian. However, more general waiting time distributions are also relevant in applications. In this section we mention a non-Markovian renewal process with a waiting time distribution described by the Mittag-Leffler function. One may see [61, 41, 64] for this purpose.

The concept of renewal process has been developed as a stochastic model for the class of counting processes where the time between successive events are i.i.d. non-negative random variables, following a given probability law.

Consider a renewal process with waiting times $U_1, U_2, \ldots$ and let

$$T_0 = 0, \quad T_k = \sum_{j=1}^{k} U_j, \quad k \geqslant 1.$$

That means, $T_1 = U_1$ is the first renewal time, $T_2 = U_1 + U_2$ the time of the second renewal, and so on. The process is specified if the probability law of the waiting times is known. Let $\phi(t)$ be the probability density function and $\Phi(t)$ the cumulative distribution function,

$$\phi(t) = \frac{d}{dt}\Phi(t), \quad \Phi(t) = \mathbb{P}(U \leqslant t) = \int_0^t \phi(s)ds.$$

It is common to refer to $\Phi(t)$ as the *failure probability* and to

$$\Psi(t) = \mathbb{P}(U > t) = \int_t^\infty \phi(s)ds = 1 - \Phi(t),$$

as the *survival probability*.

Let $F_1(t) = \Phi(t)$, $f_1(t) = \phi(t)$, and in general

$$F_k(t) = \mathbb{P}(T_k = U_1 + \ldots + U_k \leqslant t), \qquad f_k(t) = \frac{d}{dt}F_k(t), \quad k \geqslant 1.$$

Hence $F_k(t)$ represents the probability that the sum of the first k waiting times does not exceed t and $f_k(t)$ is less or equal to t. In other words, $F_k(t)$ is the probability that the kth arrival happens before t.

Setting for reasons of consistency $F_0(t) \equiv 1$ and $f_0(t) = \delta(t)$ (the Dirac function at 0), we observe for $k \geqslant 0$ that

$$\mathbb{P}(N(t) = k) = \mathbb{P}(U_k < t,\ U_{k+1} > t) = \int_0^t f_k(s)\Psi(t-s)ds.$$

In the Poissonian case, $f_k(t)$ turns to be the k-folded convolution of $\phi(t)$ with itself,

$$f_k(t) = \left(\phi^{*k}\right)(t),$$

and hence

$$\mathbb{P}(N(t) = k) = \left(\phi^{*k} * \Psi\right)(t).$$

Our objective in the sequel is to relax the hypothesis of independency, so that $f_k(t) \neq \big(\phi^{*k}\big)(t)$ in general. Recall that the survival probability for the Poisson renewal process

$$\Psi(t) = \mathbb{P}(U > t) = e^{-\lambda t}, \quad t \geqslant 0,$$

obeys the ordinary differential equation

$$\frac{d}{dt}\Psi(t) = -\lambda\Psi(t), \quad t \geqslant 0, \quad \Psi(0^+) = 1,$$

in other words

$$\lambda = \frac{-\Psi'(t)}{\Psi(t)} = \frac{\phi(t)}{\Psi(t)} = \frac{\phi(t)}{1-\Phi(t)}.$$

Generalization to the Mittag-Leffler type

As in [61, 41, 64] a fractional generalization of the Poisson renewal process is simply obtained by replacing the first derivative in $\frac{d}{dt}\Psi(t)$ by the fractional derivative of order β in Caputo sense:

$$D^\beta\Psi(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\Psi'(s)}{(t-s)^\beta}ds = -\lambda\Psi(t), \quad t \geqslant 0, \quad 0 < \beta \leqslant 1, \quad \Psi(0^+) = 1. \quad (2.4.1)$$

The Mittag-Leffler function is a natural fractional generalization of the exponential function which characterizes the Poisson process.

The Mittag-Leffler function of parameter β is defined in the complex plane $\mathbb{C}$ by the power series

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+1)}, \quad \beta > 0, \quad z \in \mathbb{C}.$$

When $\beta = 1$ it reduces to $\exp(z)$. The solution to Equation (2.4.1) is known to be

$$\Psi(t) = E_\beta\big(-\lambda t^\beta\big), \quad t \geqslant 0, \quad 0 < \beta \leqslant 1.$$

Let us compare some properties of a fractional Poisson process and a Poisson process. Denote by $U_{n+1} = T_{n+1} - T_n$ the inter-arrival time.

	Poisson process ($\beta = 1$)	FPP ($\beta < 1$)
$\mathbb{P}(U_{n+1} > t)$	$e^{-\lambda t}$	$E_\beta[-\lambda t^\beta]$
$f_{U_n}(t)$	$\lambda e^{-\lambda t}$	$\lambda t^{\beta-1}E_\beta(-\lambda t^\beta)$
$\mathbb{P}(N(t) = n)$	$\frac{(\lambda t)^n}{n!}e^{-\lambda t}$	$\frac{(\lambda t^\beta)^n}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\lambda t^\beta)^k}{\Gamma(\beta(k+n)+1)}$
Mean	λt	$\frac{\lambda t^\beta}{\Gamma(\beta+1)}$

2.5 Additive processes

In this section we recall some elements of stochastic calculus of variations related to additive processes. For this topic one may consult [101, 78] or [88]; a shorter way can be found in [92] which we are following in the sequel. Additive processes are obtained from Lévy processes by relaxing the condition of stationarity of the increments.

Definition 2.5.1 (Additive process) *A stochastic process* $(X_t)_{t\geqslant 0}$ *on* $\mathbb{R}$ *is called an additive process if it is càdlàg, satisfies* $X_0 = 0$ *and the following properties:*

1. *Independent increments: for every increasing sequence of times* $t_0, \ldots, t_n$ *the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*

2. *Stochastic continuity:* $\forall \epsilon > 0$, $\mathbb{P}\left[|X_{t+h} - X_t| \geqslant \epsilon\right] \xrightarrow{h\to 0} 0$.

A consequence of relaxing the hypothesis of stationarity of the increments is that the intensity λ is no longer constant; it may be deterministic or even stochastic. The positions and sizes of $(X_t)_{t\in[0,T]}$ are described by a Poisson random measure on $[0,T]\times\mathbb{R}$:

$$J_X = \sum_{t\in[0,T]} \delta_{(t,\Delta X_t)}$$

with time inhomogeneous intensity given by $\nu_t(dx)dt$:

$$\mathbb{E}[J_X([t_1,t_2]\times A)] = \int_{t_1}^{t_2} \nu_s(A)ds.$$

Note that X_t is a semimartingale. The compensated version $\tilde{J}_X$ can be defined by $\tilde{J}_X = J_X(dt,dx) - \nu_t(dx)dt$. Due to the independence of their increments, additive processes are spatially but not temporally homogeneous Markov processes.

Theorem 2.5.2 (see Sato [87], Theorems 9.1–9.8)
If $\{X_t, t\geqslant 0\}$ *is an additive process on* $\mathbb{R}^d$*, then for every* t*, the distribution of* X_t *is infinitely divisible. The law of* X_t *is uniquely determined by its spot characteristics* $(A_t, \Gamma_t, \nu_t)_{t\geqslant 0}$*, and the characteristic function of* X_t *has a Lévy-Khinchin representation:*

$$\mathbb{E}[\exp(iuX_t)] = \exp[\psi_t(u)]$$

where $\psi_t(u) = iuA_t - \dfrac{1}{2}u\Gamma_t u + \displaystyle\int_{\mathbb{R}^d} (e^{iux - 1 - iux\mathbf{1}_{|x|\leqslant 1}})\nu_s(dx).$

The spot characteristics $(A_t, \Gamma_t, \nu_t)_{t\geqslant 0}$ *satisfy the following conditions:*

1. *For all* t, Γ_t *is a positive definite* $d\times d$ *matrix and* ν_t *a positive measure on* $\mathbb{R}^d$ *satisfying* $\nu_t(\{0\}) = 0$ *and* $\int_{\mathbb{R}^d}(|x|^2\wedge 1)\nu_t(dx) < \infty$.

2. *Positiveness:* $A_0 = 0$, $\Gamma_0 = 0$, $\nu_0 = 0$ *and for all* s,t *such that* $s\leqslant t$, $\Gamma_t - \Gamma_s$ *is a positive definite matrix and* $\nu_t(B)\geqslant \nu_s(B)$ *for all measurable sets* $B\in\mathcal{B}(\mathbb{R}^d)$.

3. *Continuity: if* $s\to t$ *then* $A_s\to A_t, \Gamma_s\to\Gamma_t$ *and* $\nu_s(B)\to\nu_t(B)$ *for all* $B\in\mathcal{B}(\mathbb{R}^d)$ *such that* $B\subset\{x : |x|\geqslant\epsilon\}$ *for some* $\epsilon > 0$.

The positions and sizes of jumps of $(X_t)_{t\in[0,T]}$ are described by a Poisson random measure on $[0,t]\times\mathbb{R}^d$,

$$J_X = \sum_{t\in[0,T]} \delta_{(t,\Delta X_t)}$$

with time inhomogeneous intensity given by $\nu_t(dx)dt$:

$$\mathbb{E}[J_X([t_1,t_2]\times A)] = \int_{t_1}^{t_2} \nu_s(A)ds.$$

2.5.1 Additive processes and chaotic representation

In this section we recall some results from [92] to construct a family of martingales generated by an additive process. Assume that X_t is centered and has a moment of all orders and take $d = 1$. It is well known that the law of X_t is infinitely divisible for all $t \geqslant 0$. As explained for additive processes the stationarity of increments is not necessary. Take $\sigma = 0$. Consider the variations of the process X (see Meyer [71]):

$$X_t^{(1)} = X_t,$$
$$X_t^{(2)} = [X, X]_t = \sum_{0<s\leqslant t} (\Delta X_s)^2,$$
$$X_t^{(n)} = \sum_{0<s\leqslant t} (\Delta X_s)^n, \quad n \geqslant 3.$$

Write, when the moments exist,

$$F_2(t) = \int_0^t \int_{\mathbb{R}_0} x^2 \nu_s(dx)ds \quad \textit{and} \quad F_n(t) = \int_0^t \int_{\mathbb{R}_0} x^n \nu_s(dx)ds, \quad n \geqslant 3.$$

Proposition 2.5.3 [92] *The functions $F_n(t), n \geqslant 2$, are continuous and have finite variation on finite intervals. Moreover, for n even, they are increasing.*

The Teugels martingales have been introduced by Nualart and Schoutens [77] for Lévy processes and are defined by

$$Y_t^{(1)} = X_t$$
$$Y_t^{(n)} = X^{(n)} - F_n(t), \quad n \geqslant 2.$$

The Teugels martingales has been extended to the case of additive processes by Solé and Utzet [92]. The authors show that they are square integrable martingales with optional quadratic covariation

$$[Y^{(n)}, Y^{(m)}] = X^{(n+m)},$$

and since $F_{2n}(t)$ is increasing, the predictable quadratic variation of $Y^{(n)}$ is

$$\langle Y^{(n)} \rangle_t = F_{2n}(t).$$

Recall the formula that relates the moments $\{\mu_n, n \geqslant 0\}$ of a random variable Z (with moment generating function in some open interval containing 0) and its cumulants $\{\kappa_n, n \geqslant 1\}$:

$$\exp\left\{\sum_{n=1}^{\infty} \kappa_n \frac{u^n}{n!}\right\} = \sum_{n=1}^{\infty} \mu_n \frac{u^n}{n!}.$$

Both series converge in a neighborhood of 0 and the last equality is the relationship between $\psi(u) = \mathbb{E}[e^{uZ}]$ and $\log(\psi(u))$. One easily verifies the first three relations:

$$\mu_1 = \kappa_1$$
$$\mu_2 = \kappa_1^2 + \kappa_2$$
$$\mu_3 = \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3.$$

In general, μ_n is a polynomial Γ_n of $\kappa_1, \ldots, \kappa_n$, called Kendall polynomial. Thus, we have $\mu_n = \Gamma_n(\kappa_1, \ldots, \kappa_n)$ with $\Gamma_0 = 1$.

Theorem 2.5.4 [92] *Let X be a centered additive process with finite moments of all orders. Then the process*

$$M_t^{(n)} = \Gamma\left(X_t, -F_2(t), \ldots, -F_n(t)\right)$$

is a martingale.

This result allows according to [92] to define a chaos expansion for additive processes.

2.6 Stochastic integration, the case of a deterministic intensity

In the previous section we have seen how to establish chaos expansions beyond classical Lévy processes, i.e. to the case of additive processes where we relax the stationarity and allow a non-constant intensity. In this section we propose to define a stochastic integral by means of an isometry, as we did in Section 1.3 for Gaussian processes. We will not repeat all steps in detail but we will just show how the method works. Later in Section 2.7 more details will be provided.

First of all, recall some elements of the classical chaos expansion with respect to a compensated Poisson measure following [34]. For an extension to the case of Lévy process one may see [32] and [33].

Let

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz), \quad t \geqslant 0,$$

be a compensated Poisson measure and $L^2(\nu^n)$ (λ is the Lebesgue measure on $\mathbb{R}_+$) be the space of deterministic real functions f such that

$$\|f\|_{L^2(\nu^n)} = \left(\int_{([0,T]\times\mathbb{R}_0)^n} f^2(t_1, z_1, \ldots, t_n, z_n)\nu_t(dz_1)dt_1 \ldots \nu_t(dz_n, dt_n)\right)^{1/2} < \infty$$

where

$$\int_0^t \int_C \nu_s(dz, ds) = \mathbb{E}\left[\int_0^t \int_C z\ N(ds, dz)\right] < \infty, \quad t \geqslant 0 \text{ and } C \in \mathcal{B}(\mathbb{R}_0).$$

We denote the space of symmetric functions in $L^2(\nu^n_\cdot)$ by $\tilde{L}^2(\nu^n)$. Then, for $f \in \tilde{L}^2(\nu^n_\cdot)$, we define

$$I_n(f) = \int_{([0,T]\times\mathbb{R}_0)^n} f(t_1, z_1, \ldots, t_n, z_n)\, \tilde{N}^{\otimes n}(dt, dz).$$

For $f, g \in \tilde{L}^2(\nu^n)$ we have the following relations

$$\mathbb{E}[I_m(f)I_n(g)] = \begin{cases} 0 & \text{if } n \neq m \\ (g, f)_{L^2(\nu^n)} & \text{if } n = m \end{cases}$$

where

$$(g, f)_{L^2(\nu^n)} = \int_{([0,T]\times\mathbb{R}_0)^n} g(t_1, z_1, \ldots, t_n, z_n) f(t_1, z_1, \ldots, t_n, z_n)\nu(dz_1, dt_1) \ldots \nu(dz_n, dt_n).$$

In this context N has a constant intensity. Using a Theorem of Cinlar [24] we will construct an isometry allowing to define stochastic integrals with respect to non homogeneous Poisson processes (NHPP).

Theorem 2.6.1 ([24] Cinlar*, 1975)
Let $\Lambda(t), t \geqslant 0$ be a positive-valued, continuous, strictly increasing function, and let $X(t)$ be a NHPP *with continuous cumulative intensity $\Lambda(t)$. Then the random variables $T_1, T_2, \ldots$ are event times corresponding to a non-homogeneous Poisson process with expectation function $\Lambda(t)$ if and only if $\Lambda(T_1), \Lambda(T_2), \ldots$ are the event times corresponding to a homogeneous Poisson process $N(t) = X(\Lambda^{-1}(t))$ with intensity rate* 1.

The generalization to the case of compound Poisson processes can be obtained by a similar proof as in the theorem. Let $X(t)$ be a compound Poisson process with Lévy measure $\nu(dz, dt) = \nu(dz)\Lambda(dt)$. Set $N(t) = X(\Lambda^{-1}(t))$. We have to show that $N(t)$ has intensity equal to 1. In fact,

$$\begin{aligned}\mathbb{E}\left[N(t+s) - N(t)\right] &= \mathbb{E}\left[X(\Lambda^{-1}(t+s)) - X(\Lambda^{-1}(t))\right] \\ &= \int_{\Lambda^{-1}(t)}^{\Lambda^{-1}(t+s)} \int_{\mathbb{R}_0} z\nu(dz)d\Lambda(r) \\ &= \left[\Lambda(r)\right]_{\Lambda^{-1}(t)}^{\Lambda^{-1}(t+s)} \int_{\mathbb{R}_0} z\nu(dz) \\ &= s\int_{\mathbb{R}_0} z\nu(dz).\end{aligned}$$

Consider two compound Poisson processes N_t^1 with intensity 1 and N_t^Λ with cumulative intensity $\Lambda(t)$. For $f \in L^2(\lambda \times \nu^\Lambda)$ consider the process

$$Y(t) = \int_0^t \int_{\mathbb{R}_0} f(s,z)\tilde{N}^\Lambda(ds, dz),$$

where f is such that

$$\int_0^t \int_{\mathbb{R}_0} f^2(s,z)\,\nu(dz)\Lambda(ds) < \infty,$$

Fix an interval $[0, T]$. We define the following transformation

$$\begin{aligned}Y(t) &= \int_0^t \int_{\mathbb{R}_0} f(s,z)\tilde{N}^\Lambda(ds, dz) \\ &= \int_0^T \int_{\mathbb{R}_0} 1_{[0,t]}(s) f(s,z)\tilde{N}^\Lambda(ds, dz) \\ &= \int_0^T \int_{\mathbb{R}_0} 1_{[0,\Lambda(t)]}(s) f(\Lambda^{-1}(s), z)\tilde{N}^1(ds, dz).\end{aligned}$$

divergence integral† as follows:

$$\delta^{\tilde{N}^\Lambda}(f) = \int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{[0,\Lambda(t)]}(s) f(\Lambda^{-1}(s), z)\tilde{N}^1(dz, ds).$$

Example 2.6.2 *One may consider cumulative intensities of the form*

$$\Lambda_t^H = \int_0^t K^H(t,s)d\lambda_s$$

where λ_t is a deterministic intensity.

*Theorem 7.4
†More details on the definition of the divergence integral are given in Section 2.8

2.7 Filtered Lévy process

We start in a first step by establishing the notion of a filtered compound Poisson process N_t^K. We show by the means of the characteristic function that the increments are dependent. After that we generalize to a filtered Lévy process (FLP). We propose a stochastic calculus in terms of divergence integral and derive an Itô formula.

2.7.1 Filtered compound Poisson process

Filtered Poisson processes (FPP) provide models for a wide variety of random phenomena. First introduced in the Physics of noise as a model for shot noise and in radioactivity, this process was later generalized and applied in many areas like *telecom* [99] to model the number of busy servers, or in *insurance* to model the number of claims. One can find also some applications in financial modeling. Our motivation for introducing this class of processes is the dependence of the increments of this processes. More precisely, we will use it to model the intensity of a point process.

Definition 2.7.1 (Filtered compound Poisson process) *Let K be a kernel which is absolutely continuous with respect to t on $\mathbb{R}_+$ and square integrable,*

$$\int_0^t \int_{\mathbb{R}_0} K(t,s,z)^2 \nu(dz,ds) < \infty.$$

Consider, when it exists, the following process:

$$N_t^K = \int_0^t \int_{\mathbb{R}_0} K(t,s,z) N(ds,dz) = \sum_{i=0}^{N(t)} K(t,\tau_i,Z_i), \tag{2.7.1}$$

where $\{(\tau_i, Z_i):\ i \in \mathbb{N}\}$ denote impact times and corresponding marks of N. We call it a filtered Poisson processes.

2.7.2 Characteristic function and properties

The characteristic function of $\{N_t^K:\ 0 \leqslant t_0 \leqslant t \leqslant T\}$ is defined as

$$\Phi_{N_t^K}(\alpha) = \mathbb{E}\left\{\exp\left[i\alpha N_t^K\right]\right\}. \tag{2.7.2}$$

From the book of Snyder [91] (Theorem 5.1) we obtain:

$$\begin{aligned}
\Phi_{N_t^K}(\alpha) &= \exp\left\{\int_{t_0}^t \lambda(s)\mathbb{E}_Z\left[e^{i\alpha K(t,s,Z)} - 1\right] ds\right\} \\
&= \exp\left\{\int_{t_0}^t \lambda(s)\left(\int_{\mathbb{R}_0} (e^{i\alpha K(t,s,z)} - 1)\nu(dz)\right) ds\right\} \\
&= \exp\left\{\int_{t_0}^t \int_{\mathbb{R}_0} (e^{i\alpha K(t,s,z)} - 1)\lambda(s)\nu(dz)ds\right\}.
\end{aligned} \tag{2.7.3}$$

Taking the first derivative with respect to α and evaluating it at $\alpha = 0$, the expectation of N_t^K is

$$\mathbb{E}[N_t^K] = \int_{t_0}^{t} \int_{\mathbb{R}_0} K(t,s,z)\lambda(s)\,\nu(dz)ds. \tag{2.7.4}$$

Taking the second derivative with respect to α and evaluating it at $\alpha = 0$, the variance of N_t^K is seen to be

$$\mathrm{Var}[N_t^K] = \int_{t_0}^{t} \int_{\mathbb{R}_0} K(t,s,z)^2\lambda(s)\nu(dz)ds. \tag{2.7.5}$$

Characteristic function of the increments

Let $0 = t_0 \leqslant t_1 < t_2 < t_3 < t_4 \leqslant T$. The characteristic function of the increments is defined as

$$\begin{aligned}
\Phi_{N_{t_2}^K - N_{t_1}^K, N_{t_4}^K - N_{t_3}^K}(\alpha_1,\alpha_2) &= \mathbb{E}\left[e^{i\alpha_1\left(N_{t_2}^K - N_{t_1}^K\right) + i\alpha_2\left(N_{t_4}^K - N_{t_3}^K\right)}\right] \\
= \mathbb{E}\Bigg[\exp\Bigg\{\int_{t_0}^{t_1} &\left(e^{i\alpha_1 K(t_2,s,Z) - i\alpha_1 K(t_1,s,Z) + i\alpha_2 K(t_4,s,Z) - i\alpha_2 K(t_3,s,Z)} - 1\right)\lambda(s)ds \\
&+ \int_{t_1}^{t_2} \left(e^{i\alpha_1 K(t_2,s,Z) + i\alpha_2 K(t_4,s,Z) - i\alpha_2 K(t_3,s,Z)} - 1\right)\lambda(s)ds \\
&+ \int_{t_2}^{t_3} \left(e^{i\alpha_2 K(t_4,s,Z) - i\alpha_2 K(t_3,s,Z)} - 1\right)\lambda(s)ds \\
&+ \int_{t_3}^{t_4} \left(e^{i\alpha_2 K(t_4,s,Z)} - 1\right)\lambda(s)ds\Bigg\}\Bigg].
\end{aligned} \tag{2.7.6}$$

We see that the increments are not independent. If $K(t,s,z) = z$, we have

$$\begin{aligned}
\Phi_{N_{t_2} - N_{t_1}, N_{t_4} - N_{t_3}}(\alpha_1,\alpha_2) &= \mathbb{E}\left[\exp\left\{\int_{t_1}^{t_2} \left(e^{i\alpha_1 Z} - 1\right)\lambda(s)ds + \int_{t_3}^{t_4} \left(e^{i\alpha_2 Z} - 1\right)\lambda(s)ds\right\}\right] \\
&= \Phi_{N_{t_2} - N_{t_1}}(\alpha_1)\Phi_{N_{t_4} - N_{t_3}}(\alpha_2).
\end{aligned}$$

In this case the increments are independent.

Remark 2.7.2 *If we consider the case of a filtered Poisson process then one should take* $Z = 1$ *and consequently* $\nu(dz) = \delta_1$.

2.7.3 Filtered Lévy process, stochastic calculus and Itô formula

A filtered (pure)[‡] Lévy process is obtained from a compound Poisson process by replacing the Poisson measure in Definition 2.7.1 by a pure Lévy measure.

$$L^K(t) = \int_0^t \int_{\mathbb{R}_0} K(t,s)zL(ds,dz),$$

[‡]without Brownian component

and

$$\begin{aligned}\tilde{L}^K(t) &= \int_0^t \int_{\mathbb{R}_0} K(t,s)z\tilde{L}(ds,dz) \\ &= \int_0^t \int_{\mathbb{R}_0} K(t,s)zL(ds,dz) - \int_0^t \int_{\mathbb{R}_0} K(t,s)z\nu(ds,dz) \\ &= \int_0^t \int_{\mathbb{R}_0} K(t,s)zL(ds,dz) - \int_0^t \int_{\mathbb{R}_0} z\nu^K(ds,dz). \end{aligned} \tag{2.7.7}$$

In order to define a stochastic integration with respect to filtered Lévy processes, we use the isometry proposed by [1] for Gaussian processes and adapt it to the case of filtered Lévy processes.

We consider in this section regular kernels of the form $K(t,s,z) = K(t,s)z$ defined on $[0,T] \times \mathbb{R}_0$. As already seen, the increments of Lévy processes have different properties than those of Gaussian processes. For Brownian motion, the increments tends to 0 which can help if we consider singular kernels. But this is not the case for Lévy processes. For this reason we will only consider the case of regular kernels to avoid the singularities.

In the sequel we impose the following condition on $K(t,\cdot)$.

(C) For all $s \in [0,T), K(\cdot,s)$ has bounded variation on the interval $(s,T]$ and

$$\int_0^T |K|\left((s,T),s\right)^2 ds < \infty. \tag{2.7.8}$$

Fix an interval $[0,T]$. We suppose that $\tilde{L}^K$ is defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ where $\mathcal{F}$ is generated by $\tilde{L}^K$. Denote by $\mathcal{E}$ the set of simple functions on $[0,T]\times\mathbb{R}_0$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle z1_{[0,t]}, z1_{[0,s]}\rangle_{\mathcal{H}} = R(t,s) = \int_0^{t\wedge s}\int_{\mathbb{R}_0} \big(K(t,s)z\big)^2 \nu(dz,ds).$$

The mapping $1_{[0,T]} \to \tilde{L}^K$ provides an isometry between $\mathcal{H}$ and the first chaos H_1, that is the closed subspace of $L^2(\Omega)$ generated by $\tilde{L}^K$. The variable $\tilde{L}^K(\varphi)$ denotes the image in H_1 of an element $\varphi \in \mathcal{H}$.

Let $\mathcal{S}$ be the set of smooth cylindrical random variables of the form

$$F = f\big(\tilde{L}^K(\varphi_1),\ldots,\tilde{L}^K(\varphi_n)\big)$$

where $n \geqslant 1$, $f \in C_b^\infty\big(\mathbb{R}^n\big)$ (f and all its derivatives are bounded), and $\varphi_1,\ldots,\varphi_n \in \mathcal{H}$. The derivative of F is defined as the element in $L^2(\Omega,\mathcal{H})$ defined as

$$D^{\tilde{L}^K} F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\big(\tilde{L}^K(\varphi_1),\ldots,\tilde{L}^K(\varphi_n)\big)\varphi_j.$$

As usual, $\mathbb{D}^{1,2}_{\tilde{L}^K}$ denotes the closure of the set $\mathcal{S}$ of smooth cylindrical random variables F with respect to the norm

$$\|F\|_{1,2}^2 = \mathbb{E}\left[|F|^2\right] + \mathbb{E}\left[\|D^{\tilde{L}^K}F\|_{\mathcal{H}}^2\right].$$

The divergence operator $\delta^{\tilde{L}^K}$ is defined as the adjoint of the derivative operator.

Consider the linear operator K^* from $\mathcal{E}$ to $L^2\big([0,T]\times\mathbb{R}_0\big)$ defined by

$$\big(K^*\phi\big)(s,z) = K(s,s)\phi(s,z) + \int_s^T \phi(t,z)K(dt,s). \tag{2.7.9}$$

The operator K^* is an isometry from $\mathcal{E}$ to $L^2([0,T],\mathbb{R}_0)$ which extends to the Hilbert space $\mathcal{H}$; thus we have $\mathcal{H} = (K^*)^{-1}\left(L^2([0,T]\times\mathbb{R}_0)\right)$. Analogously to $\mathbb{D}^{1,2}_{\tilde{L}^K}$, we may consider the space $\mathbb{D}^{1,2}_{\tilde{L}^K}(\mathbb{H})$ of $\mathbb{H}$-valued random variables for an arbitrary separable Hilbert space $\mathbb{H}$. Then we obtain

$$\mathbb{D}^{1,2}_{\tilde{L}^K}(\mathcal{H}) = (K^*)^{-1}\mathbb{D}^{1,2}_{\tilde{L}}L^2([0,T]\times\mathbb{R}_0).$$

As a consequence, we obtain $\operatorname{Dom}\delta^{\tilde{L}^K} = (K^*)^{-1}\ \operatorname{Dom}\delta^{\tilde{L}}$.

For $\varphi\in\mathcal{E}$ consider the seminorm

$$\|\phi\|_K^2 = \int_0^T\int_{\mathbb{R}_0}\varphi(s)^2K(s,s)^2z^2\ \nu(dz)ds + \int_0^T\int_{\mathbb{R}_0}\left(\int_s^T|\varphi(r)||K|(dr,s)\right)^2 z^2\ \nu(dz,ds).$$

Denote by $\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to this seminorm. Since $\|\varphi\|_{\mathcal{H}}\leqslant\sqrt{2}\|\varphi\|_K$, the space $\mathcal{H}_K$ is continuously embedded into $\mathcal{H}$. In addition, the space $\mathbb{D}^{1,2}_{\tilde{L}^K}(\mathcal{H}_{\mathcal{K}})$ is included in the domain of $\delta^{\tilde{L}^K}$ and for any $\theta\in\mathbb{D}^{1,2}_{\tilde{L}^K}(\mathcal{H}_K)$, we have $\delta^{\tilde{L}^K}(\theta) = \int_0^T\theta(s,z)\tilde{L}^K(ds,dz)$ and therefore we have the formula

$$\int_0^T\theta(s,z)\tilde{L}^K(\delta s,dz) = \int_0^T K^*\theta(s,z)\tilde{L}(\delta s,dz).$$

Remark 2.7.3 *If we impose*[§] *the stronger condition of square integrability for the derivative*

$$\int_0^T\int_{\mathbb{R}_0}\left(\frac{\partial}{\partial t}K(t,s)z\right)^2\nu(dz,ds)<\infty, \tag{2.7.10}$$

then the last isometry is valid and we have

$$\begin{aligned} L_t^K &= \int_0^t\int_{\mathbb{R}_0}K(t,s)zL(ds,dz) \qquad (2.7.11)\\ &= \int_0^t\int_{\mathbb{R}_0}K(s,s)zL(ds,dz) + \int_0^t\int_{\mathbb{R}_0}\int_s^t K'(r,s)zdrL(ds,dz)\\ &= \int_0^t\int_{\mathbb{R}_0}K(s,s)zL(ds,dz) + \int_0^t\int_{\mathbb{R}_0}\int_0^r K'(r,s)zL(ds,dz)dr. \end{aligned}$$

The stochastic calculus of Itô is valid in this case.

[§] see Chapter 1, Proposition 1.3.2

Following [34], let $\mathcal{M}^K$ denote the set of stochastic functions $\theta(t,z),\ t \in [0,T],\ z \in \mathbb{R}_0$, such that

1. $\theta(t,z) \equiv \theta(\omega,t,z)$ is adapted such that

$$\begin{aligned}\|\theta\|^2_{L^{2,K}(\Omega\times\lambda\times\nu)} &:= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(s,z)(K(s,s))^2 \nu(dz)ds\right] \\ &+\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_s^T \theta(s,z)K(dr,s)\right)^2 \nu(dz)ds\right] < \infty.\end{aligned}$$

2. $D_{t^+,z}\theta(t,z) := \lim_{s\to t^+} D_{s,z}\theta(t,z)$ exists in $L^{2,K}(\Omega\times\lambda\times\nu)$.

3. $\theta(t,z) + D_{t^+,z}\theta(t,z)$ is Skorohod integrable.

Let $\mathbb{M}^K_{1,2}$ be the closure of the linear span $\mathcal{M}^K$ with respect to the norm given by

$$\|\theta\|^2_{\mathbb{M}^K_{1,2}} := \|\theta\|^2_{L^{2,K}(\Omega\times\lambda\times\nu)} + \|D_{t^+,z}\theta(t,z)\|^2_{L^{2,K}(\Omega\times\lambda\times\nu)}. \tag{2.7.12}$$

Definition 2.7.4 *The forward integral $X(t)$, $t \in [0,T]$ of a stochastic function $\theta(t,z), t \in [0,T], z \in \mathbb{R}_0$ with respect to $\tilde{L}^K$ is defined as*

$$\begin{aligned}X(t) &= \int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{L}^K(d^-s,dz) \\ &= \lim_{m\to\infty} \sum_{0\leqslant s_i\leqslant t} \Delta X_s 1_{U_m} = \lim_{m\to\infty} \sum_{0\leqslant s_i\leqslant t} 1_{U_m}\theta(s_i,z)\Delta\tilde{L}^K(d^-s_i,dz),\end{aligned} \tag{2.7.13}$$

if the limit exists in $L^2(\Omega)$. Here, U_m, $m = 1,2,\ldots$, is an increasing sequence of compact sets $U_m \subseteq \mathbb{R}_0$ with $\nu(U_m) < \infty$ exhausting $\mathbb{R}_0$.

Lemma 2.7.5 *If $\theta \in \mathbb{M}^K_{1,2}$, then its forward integral exists and*

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{L}^K(d^-s,dz) &= \int_0^t \int_{\mathbb{R}_0} D_{s^+,z}\theta(s,z)\nu^K(ds,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[\theta(s,z) + D_{s^+,z}\theta(s,z)\right]\tilde{L}^K(\delta s,dz).\end{aligned} \tag{2.7.14}$$

Proof. We have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{L}^K(d^-s,dz) &= \int_0^t \int_{\mathbb{R}_0} \theta(s,z)K(s,s)\tilde{L}(d^-s,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t \theta(r,z)K(dr,s)\right)\tilde{L}(d^-s,dz).\end{aligned} \tag{2.7.15}$$

By Lemma 15.5 of [34]

$$\begin{aligned}&= \int_0^t \int_{\mathbb{R}_0} D_{s^+,z}\theta(s,z)K(s,s)\nu(dz)ds + \int_0^t \int_{\mathbb{R}_0} K(s,s)\left[\theta(s,z) + D_{s^+,z}\theta(s,z)\right]\tilde{L}(\delta s,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t D_{r,z}\theta(r,z)K(dr,s)\right)\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t \big(\theta(r,z) + D_{r,z}\theta(r,z)\big)K(dr,s)dr\right)\tilde{L}(\delta s,dz).\end{aligned} \tag{2.7.16}$$

By arranging the last sum we get the proof of the lemma. ■

Let $f \in C^2(\mathbb{R})$ satisfying the growth conditions

$$\max\left\{|f(x)|,\ |f'(x)|,\ |f''(x)|\right\} \leqslant ce^{ax^2}, \tag{2.7.17}$$

where a and c are positive constants such that $a < \frac{1}{4}(\sup_{0\leqslant t\leqslant T} R(t,t))^{-1}$ where $R(t,t)$ is the autocorrelation function.

Lemma 2.7.6 *Let f be a function of class $C^2(\mathbb{R})$. Let $L^K(t), t \in [0,T]$, be a filtered Lévy process where $K(t,.)$ satisfies the condition of Remark* 2.7.3. *Then the forward Itô formula has the following form*

$$\begin{aligned} f\left(\tilde{L}^K(t)\right) &= f\left(\tilde{L}^K(0)\right) + \int_0^t \int_{\mathbb{R}_0} \left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\right]\tilde{L}(d^-s, dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big) - f'\left(\tilde{L}^K(s-)\right)\right]\nu(ds,dz) \\ &+ \int_0^t f'\big(\tilde{L}^K(s-)\big)d\tilde{A}(s) \end{aligned} \tag{2.7.18}$$

where

$$\tilde{A}(t) = \tilde{L}^K(t) - \int_0^t \int_{\mathbb{R}_0} K(s,s)\tilde{L}(dz,ds).$$

Proof. Under the condition of the Remark 2.7.3, we can write

$$\begin{aligned} \tilde{L}^K(t) &= \int_0^t \int_{\mathbb{R}_0} K(s,s)z\tilde{L}(ds,dz) + \int_0^t \int_{\mathbb{R}_0} \int_0^r \frac{\partial}{\partial t} K(r,s)z\tilde{L}(ds,dz)dr \\ &= \int_0^t \int_{\mathbb{R}_0} K(s,s)z\tilde{L}(ds,dz) + \tilde{A}(t), \end{aligned} \tag{2.7.19}$$

where $\tilde{A}(t)$ is with bounded variation. We get the proof by applying Itô's formula (Proposition 1.1.5). ■

Theorem 2.7.7 *Let f be a function in $C^2(\mathbb{R})$ satisfying the growth condition* (2.7.17). *Let $L^K(t)$, $t \in [0,T]$, be a filtered Lévy process where $K(t,\cdot)$ satisfies the condition of Remark* 2.7.3.

Then the Itô formula takes the following form

$$\begin{aligned}
f\big(\tilde{L}^K(t)\big) &= f(\tilde{L}^K(0)) + \int_0^t \int_{\mathbb{R}_0} \Big\{ \big[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\big] \\
&\quad + D_{s+,z}\Big[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\Big] \Big\} \tilde{L}(\delta s, dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \Big\{ \big[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big) - K(s,s)zf'(\tilde{L}^K(s-))\big] \\
&\quad + D_{s+,z}\Big[f\big(\tilde{L}^K(s-) + K(s,s)z\big) - f\big(\tilde{L}^K(s-)\big)\Big] \Big\} \nu(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s)dr + f''\big(\tilde{L}^K(r)\big) z K(r,s) \frac{\partial}{\partial t} K(r,s)dr \right) \tilde{L}(\delta s, dz) \\
&\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}_0} f''\big(\tilde{L}^K(r)\big) z d \left(\int_0^r (K(r,s)^2) \nu(ds,z) \right). \qquad (2.7.20)
\end{aligned}$$

Proof. In order to prove the last formula, we will have to apply Lemma 2.7.6 in a first step, then Lemma 2.7.5 to the jump semimartingale part of $\tilde{L}^K$ and for $\tilde{A}_t$. It is easy to derive the result for the first part. We just have to apply Lemma 2.7.5. For $\tilde{A}(t)$, we notice that

$$\int_0^t \int_{\mathbb{R}_0} D_{s,z} f(\tilde{L}^K(s)) \tilde{L}^K(ds,dz) = \int_0^t \int_{\mathbb{R}_0} K(t,s) z f'(\tilde{L}^K(s)) \tilde{L}^K(ds,dz)$$

and perform the following calculation:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_0} f'\big(\tilde{L}^K(s)\big) d\tilde{A}(d^-s, dz) = \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s) dr \right) \tilde{L}(d^-s, dz) \\
&= \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t D_{s,z} f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s)dr \right) \nu(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s)dr + D_{s,z} f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s)dr \right) \tilde{L}(\delta s, dz) \\
&= \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f''\big(\tilde{L}^K(r)\big) z K(r,s) \frac{\partial}{\partial t} K(r,s)dr \right) \nu(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s)dr + f''\big(\tilde{L}^K(r)\big) z K(r,s) \frac{\partial}{\partial t} K(r,s)dr \right) \tilde{L}(\delta s, dz) \\
&= \frac{1}{2} \int_0^t \int_{\mathbb{R}_0} f''\big(\tilde{L}^K(r)\big) z d \left(\int_0^r (K(r,s)^2) \nu(ds,z) \right) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left(\int_s^t f'\big(\tilde{L}^K(r)\big) \frac{\partial}{\partial t} K(r,s)dr + f''\big(\tilde{L}^K(r)\big) z K(r,s) \frac{\partial}{\partial t} K(r,s)dr \right) \tilde{L}(\delta s, dz).
\end{aligned}$$

■

2.8 Itô's formula for doubly stochastic Lévy Process

A possible definition of a doubly stochastic Lévy Process is to develop the notion of the doubly stochastic Poisson process defined in section 1.2.4 and to follow the construction of Lévy processes in [82]. We consider a pure jump process and suppose that the counting processes N^I of the jumps of size contained in intervals $I \subset \mathbb{R} \setminus \{0\}$ are doubly stochastic Poisson processes with associated measure Λ^I. As in [82] one can show that $\tilde{N}^I := N^I - \Lambda^I$ is a martingale with respect to underlying filtration of the pure jump process. Moreover, for two disjoint intervals $I, I' \subset \mathbb{R} \setminus \{0\}$, $\tilde{N}^I$ and $\tilde{N}^{I'}$ are orthogonal martingales. Therefore we define a doubly stochastic Lévy process (DSLP) L^Λ by the sum of its jumps, separating large and small jumps. Let $I_1 = (-\infty, -1) \cup (1, +\infty)$ and $I_n = \left[(-\frac{1}{n}, -\frac{1}{n+1}) \cup (\frac{1}{n+1}, \frac{1}{n})\right], n \geqslant 1$ and let

$$\begin{aligned} L_t^\Lambda &= \sum_{0 \leqslant s \leqslant t} \Delta L_s^\Lambda 1_{I_1}(\Delta L_s^\Lambda) + \sum_{0 \leqslant s \leqslant t} \Delta L_s^\Lambda 1_{\cup_n I_n}(\Delta L_s^\Lambda) \\ &= \int_{I_1} x N_t(dz) + \int_{\cup_n I_n} x \tilde{N}_t(dz) + \int_{\cup_n I_n} x \Lambda(t) \nu(dz), \end{aligned}$$

where $\nu_t(I) = \mathbb{E}\left[N_t^I\right]$. The second integral is well defined if $\int_{\cup_n I_n} min\{1, z^2\} \nu_t(dz) < \infty$ for all $t > 0$. Notice that the processes constructed in this way are not necessarily Lévy processes and in general even not with independent increments. For the case where L is an additive process (see section 2.5), we refer to [92]. Here L^Λ is an additive process if Λ has independent increments and is independent of L.

Let us mention the following particular choices for Λ.

a) Λ is a homogeneous Poisson process. In this case L^Λ is a process with independent and stationary increments if it is a doubly stochastic Poisson process. However, it is not a Poisson process since the probability of the jump size to be strictly bigger than 1 is positive.

b) $\Lambda(t) = \int_0^t \frac{N(s)}{s} ds$ $(t \geqslant 0)$, where N is a homogeneous Poisson process with intensity $\lambda > 0$. We have $\mathbb{E}[\Lambda(t)] = \int_0^t \frac{1}{s} \mathbb{E}[N(s)] ds = \lambda t = \mathbb{E}[N(t)]$. However, contrary to the case (a), Λ is continuous.

c) $\Lambda^K(t) = \int_0^t K(t,s) dN(s)$, where K is a kernel. This type of processes has been studied in section 2.7 and will be considered again in section 2.10.

For the rest of this section we consider processes X given by

$$\begin{aligned} dX(t) &= \int_{|z|<1} K(t,z) \tilde{L}^\Lambda(dt, dz) + \int_{|z|>1} H(t,z) L^\Lambda(dt, dz) \qquad (2.8.1) \\ &= \int_{|z|<1} K(t,z) L^\Lambda(dt, dz) - \int_{|z|<1} K(t,z) \Lambda(dt) \nu(dz) + \int_{|z|>1} H(t,z) L^\Lambda(dt, dz), \end{aligned}$$

where Λ is as in case (a).

$$\mathbb{E}\left[\int_0^t \int_D z L^\Lambda(ds, dz)\right] = \int_0^t \int_D z \lambda ds \nu(dz), \quad D \in \mathcal{B}(\mathbb{R}_0),$$

and $H(t,z)$, $K(t,z)$ are adapted random functions left-continuous in t, measurable in z, such that

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} \{K(s,z)^2 + H(s,z)^2\} ds \nu(dz)\right] < \infty. \qquad (2.8.2)$$

Proposition 2.8.1 *Let $f \in C^2(\mathbb{R})$ and $X(t)$ be as defined above. Then the following Itô formula holds:*

$$\begin{aligned} f\big(X(t)\big) &= f\big(X(0)\big) + \int_0^t \int_{|z|>1} \big[f\big(X(s-)+H(s,z)\big) - f\big(X(s-)\big)\big]\, L^{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{|z|<1} \big[f\big(X(s-)+K(s,z)\big) - f\big(X(s-)\big)\big]\, \tilde{L}^{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{|z|<1} \big[f\big(X(s-)-K(s,z)\big) + f\big(X(s-)+K(s,z)\big) - 2f\big(X(s-)\big)\big]\, \Lambda(ds,dz), \end{aligned}$$

where $\Lambda(ds,dz) = \Lambda(ds)\nu(dz)$.

Proof. By the same arguments as for the classical Itô formula for Lévy processes, we have

$$\begin{aligned} f\big(X(t)\big) - f\big(X(0)\big) &= \int_0^t \int_{|z|>1} \big[f\big(X(s-)+H(s,z)\big) - f\big(X(s-)\big)\big]\, L^{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{|z|<1} \big[f\big(X(s-)+K(s,z)\big) - f\big(X(s-)\big)\big]\, L^{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{|z|<1} \big[f\big(X(s-)-K(s,z)\big) - f\big(X(s-)\big)\big]\, \Lambda(ds,dz) \\ &= f\big(X(0)\big) + \int_0^t \int_{|z|>1} \big[f\big(X(s-)+H(s,z)\big) - f\big(X(s-)\big)\big]\, L^{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{|z|<1} \big[f\big(X(s-)+K(s,z)\big) - f\big(X(s-)\big)\big]\, \tilde{L}^{\Lambda}(ds,dz) \\ &+ \int_0^t \int_{|z|<1} \big[f\big(X(s-)-K(s,z)\big) + f\big(X(s-)+K(s,z)\big) - 2f\big(X(s-)\big)\big]\, \Lambda(ds,dz). \end{aligned}$$

To prove the last formula, we should prove that the integral

$$\int_0^t \int_{|z|<1} \big[f\big(X(s-)-K(s,z)\big) + f\big(X(s-)+K(s,z)\big) - 2f\big(X(s-)\big)\big]\, \Lambda(ds,dz) \tag{2.8.3}$$

is well defined.

More precisely, we have to find under which conditions on f the integral exists. In fact,

$$\begin{aligned}
&\int_0^t \int_{|z|<1} \left[f\big(X(s-)-K(s,z)\big)+f\big(X(s-)+K(s,z)\big)-2f\big(X(s-)\big)\right]\Lambda(ds,dz)\\
&\quad=\int_0^t \int_{|z|<1} \left[f\big(X(s-)-K(s,z)\big)+f\big(X(s-)+K(s,z)\big)-2f\big(X(s-)\big)\right]\tilde{\Lambda}(ds,dz)\\
&\quad+\int_0^t \int_{|z|<1} \left[f\big(X(s-)-K(s,z)\big)+f\big(X(s-)+K(s,z)\big)-2f\big(X(s-)\big)\right]\nu(ds,dz)\\
&\quad=\int_0^t \int_{|z|<1} \left[f\big(X(s-)-K(s,z)\big)-f\big(X(s-)+K(s,z)\big)-2f\big(X(s-)\big)\right]\tilde{\Lambda}(ds,dz)\\
&\quad+\int_0^t \int_{|z|<1} \left[f\big(X(s-)+K(s,z)\big)-f\big(X(s-)\big)-K(s,z)f'\big(X(s-)\big)\right]\nu(ds,dz)\\
&\quad-\int_0^t \int_{|z|<1} \left[f\big(X(s-)\big)-f\big(X(s-)-K(s,z)\big)-K(s,z)f'\big(X(s-)\big)\right]\nu(ds,dz)
\end{aligned}$$

where $\nu(ds,dz)=\lambda ds\nu(dz)$. We can easily prove for $f\in C^2(\mathbb{R})$ the existence of the last integrals in the equality because it correspond to the same proof of the classical Itô formula, see [28] for example. ■

2.9 The Clark-Ocone formula

It is well known that Lévy processes in general do not have the property that functionals of the process can be represented as a constant plus an integral with respect to the process itself (the predictable representation property). In fact the only Lévy processes possessing this property are the Brownian motion and the Poisson process. In this section we discuss two Itô representations formulas for two processes obtained from N^Λ where Λ is a homogeneous Poisson process and N^Λ a doubly stochastic compound Poisson process with Lévy measure ν and N independent of Λ. Due to the dependence structure between N^Λ and Λ, the chaos expansion and also divergence and derivative operators are not easy to get. The authors in [78] avoided this difficulty by considering a restriction on the filtration. For the case we are studying, i.e., the cumulative intensity Λ is a Poisson process independent of N, we can avoid this difficulty.

2.9.1 The Itô representation formula

Let $(\Lambda(t))_{t\in[0,T]}$ be a Poisson process and let $\mathcal{F}_t^\Lambda$ be its natural filtration. Denote by $\mathcal{F}^{N^\Lambda}$ the filtration generated by N^Λ. We have in general $\mathcal{F}_t^{\tilde{N}^\Lambda}\subset\mathcal{F}_t^{N^\Lambda}\vee\mathcal{F}_t^\Lambda$ and $\mathcal{F}_t^{N^\Lambda}\subset\mathcal{F}_t^{N}\vee\mathcal{F}_t^\Lambda$.

Representation 1

Let

$$X(t)=\int_0^t\int_{\mathbb{R}_0}\theta(s,z)\tilde{N}^\Lambda(ds,dz)=\int_0^t\int_{\mathbb{R}_0}\theta(s,z)N^\Lambda(ds,dz)-\int_0^t\int_{\mathbb{R}_0}\theta(s,z)\Lambda(ds,dz),\quad(2.9.1)$$

where $\Lambda(ds,dz)=\Lambda(ds)\nu(dz)$.

For $f \in C^2(\mathbb{R})$ the Itô formula yields

$$f\big(X(t)\big) - f\big(X(0)\big) = \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) + \theta(s,z)\big) - f\big(X(s-)\big)\right] \tilde{N}^\Lambda(ds,dz)$$
$$+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) - \theta(s,z)\big) + f\big(X(s-) + \theta(s,z)\big) - 2f\big(X(s-)\big)\right] \tilde{\Lambda}(ds,dz)$$
$$+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) - \theta(s,z)\big) + f\big(X(s-) + \theta(s,z)\big) - 2f\big(X(s-)\big)\right] \nu(ds,dz), \qquad (2.9.2)$$

where $\nu(ds,dz) = \lambda ds \nu(dz)$ and λ is the intensity of Λ.

Applying the Itô formula of Proposition 2.8.1 to

$$Y(t) = \exp\left\{\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{N}^\Lambda(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \left(e^{\theta(s,z)} + e^{-\theta(s,z)} - 2\right) \nu(ds,dz)\right\} \qquad (2.9.3)$$

yields

$$dY(t) = Y(t^-)\left[\int_{\mathbb{R}_0} (e^{\theta(t,z)} - 1)\tilde{N}^\Lambda(dt,dz) + \int_{\mathbb{R}_0} \left(e^{\theta(t,z)} + e^{-\theta(t,z)} - 2\right) \tilde{\Lambda}(dt,dz)\right]. \qquad (2.9.4)$$

In order to get a chaos expansion one should proceed by iteration. But the presence of two random variables N^Λ and Λ does not allow to get a chaos decomposition.

Remark 2.9.1 *If $\Lambda(t)$ is a semimartingale with compensator $\langle\Lambda\rangle_t$, then $N^\Lambda(t) - \langle\Lambda\rangle_t$ is an $\mathcal{F}^N \vee \mathcal{F}^\Lambda$-martingale. In fact,*

$$N^\Lambda(t) - \langle\Lambda\rangle_t = \big(N^\Lambda(t) - \Lambda(t)\big) + (\Lambda(t) - \langle\Lambda\rangle_t) = \tilde{N}^\Lambda(t) + \tilde{\Lambda}(t).$$

Representation 2

Take now:

$$X(t) = \int_0^t \int_{\mathbb{R}_0} \theta(s,z)N^\Lambda(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \theta(s,z)\nu(ds,dz)$$

The Itô formula yields

$$f(X_t) - f(X_0) = \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) + \theta(s,z)\big) - f\big(X(s-)\big)\right] \big(N^\Lambda(ds,dz) - \nu(ds,dz)\big)$$
$$+ \int_0^t \int_{\mathbb{R}_0} \left[f\big(X(s-) + \theta(s,z)\big) - f\big(X(s-)\big) - f'\big(X(s-)\big)\theta(s,z)\right] \nu(ds,dz). \qquad (2.9.5)$$

If we apply the Itô formula (1.1.1)

$$Y(t) = \exp\left\{\int_0^t \int_{\mathbb{R}_0} \theta(s,z)\big(N^\Lambda(ds,dz) - \nu(ds,dz)\big) - \int_0^t \int_{\mathbb{R}_0} \big(e^{\theta(s,z)} - 1 - \theta(s,z)\big)\nu(ds,dz)\right\},$$

we get

$$dY(t) = Y(t-)\int_{\mathbb{R}_0} \big(e^{\theta(s,z)} - 1\big)\big(N^\Lambda(ds,dz) - \nu(ds,dz)\big). \qquad (2.9.6)$$

Set $\hat{N}_t^\Lambda = N_t^\Lambda - \int_0^t \int_{\mathbb{R}_0} \nu(dz, ds)$. Then we have

$$dY(t) = Y(t-) \int_{\mathbb{R}_0} \left(e^{\theta(s,z)} - 1\right) \hat{N}^\Lambda(ds, dz). \tag{2.9.7}$$

For $F \in L^2\left(\Omega, \mathcal{F}^N \vee \mathcal{F}^\Lambda\right)$, by taking a sequence F_n of linear combinations of *Wick-Doléans-Dade* exponentials 2.9.6 such that $F_n \to F$, we have the following theorem:

Theorem 2.9.2 (The Itô representation Theorem) *Let $F \in L^2(\Omega, \mathcal{F}^N \vee \mathcal{F}^\Lambda)$. Then there exists a unique predictable process $\psi \in L^2(\Omega, \mathcal{F}^N \vee \mathcal{F}^\Lambda)$ verifying*

$$\int_0^T \int_{\mathbb{R}_0} \mathbb{E}\left[\psi^2(t,z)\right] \nu(dt, dz) < \infty, \tag{2.9.8}$$

for which we have

$$\begin{aligned} F &= \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \psi(t,z) \hat{N}^\Lambda(dt, dz) \\ &= \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \psi(t,z) \tilde{N}^\Lambda(dt, dz) + \int_0^T \int_{\mathbb{R}_0} \psi(t,z) \tilde{\Lambda}(dt, dz). \end{aligned} \tag{2.9.9}$$

The proof of this theorem is similar to the proof of Theorem 9.10 [34].

Hence, for all $(t_1, z_1) \in [0,T] \times \mathbb{R}_0$, there exists a predictable process $\psi_2(t_1, z_1, t_2, z_2)$, $(t_2, z_2) \in [0,T] \times \mathbb{R}_0$ such that

$$\psi_1(t_1, z_1) = \mathbb{E}[\psi_1(t_1, z_1)] + \int_0^T \int_{\mathbb{R}_0} \psi_2(t_1, z_1, t_2, z_2) \hat{N}^\Lambda(dt_2, dz_2).$$

This gives

$$\begin{aligned} F = \mathbb{E}[F] &+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\psi_1(t_1, z_1)] \hat{N}^\Lambda(dt_1, dz_1) \\ &+ \int_0^T \int_{\mathbb{R}_0} \int_0^{t_1^-} \int_{\mathbb{R}_0} \psi_2(t_1, z_1, t_2, z_2) \hat{N}^\Lambda(dt_1, dz_1) \hat{N}^\Lambda(dt_2, dz_2). \end{aligned}$$

By iteration, we can get

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.9.10}$$

where $f_n \in \tilde{L}^2(([0,T] \times \mathbb{R}_0)^n)$ are symmetric functions. This allows us to prove the following theorem.

Theorem 2.9.3 *Let $F \in \mathbb{D}^{1,2}_{\hat{N}^\Lambda}$. Then*

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}\left[D_{t,z} F \middle| \mathcal{F}^{\hat{N}^\Lambda}\right] \hat{N}^\Lambda(dt, dz). \tag{2.9.11}$$

For the complete proof, see Theorem 12.16 [34].

Remark 2.9.4 *Let $\Lambda(t)$ a Poisson process with intensity λ. Then*

$$\begin{aligned}\mathbb{E}\left[(\hat{N}_t^\Lambda)^2\right] &= \mathbb{E}\left[\left(N^\Lambda - \lambda t\right)^2\right] \\ &= \mathbb{E}\left[\left(N^\Lambda\right)^2\right] - 2\mathbb{E}\left[N^\Lambda\right]\left(\lambda t\right) + \left(\lambda t\right)^2 \\ &= \mathbb{E}\left[\Lambda_t\right] + \mathbb{E}\left[\Lambda_t^2\right] - 2\mathbb{E}\left[N^\Lambda\right]\left(\lambda t\right) + \left(\lambda t\right)^2 \\ &= \lambda t + \lambda t + \left(\lambda t\right)^2 - 2\left(\lambda t\right)^2 + \left(\lambda t\right)^2 \\ &= 2\lambda t. \end{aligned} \tag{2.9.12}$$

For $X(t,z)$ Skorohod integrable and $F \in \mathbb{D}_{\hat{N}}^{1,2}$, the duality formula of Di Nunno et al. [34] (Theorem 12.10) yields

$$2\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} X(s,z) D_{s,z} F \nu(dz) ds\right] = \mathbb{E}\left[F \int_0^t \int_{\mathbb{R}_0} X(s,z) \hat{N}^\Lambda(\delta t, dz)\right]. \tag{2.9.13}$$

Moreover if $X(t,z) \cdot (F + D_{t,z}F)$, $t \in [0,T]$, $z \in \mathbb{R}$, is Skorohod integrable, the integration by part formula becomes

$$\begin{aligned} &F \int_0^t \int_{\mathbb{R}_0} X(s,z) \hat{N}^\Lambda(\delta t, dz) \\ &\quad = \int_0^t \int_{\mathbb{R}_0} X(s,z)(F + D_{s,z}F) \hat{N}^\Lambda(\delta t, dz) + 2\int_0^t \int_{\mathbb{R}_0} X(s,z) D_{s,z} F \nu(dz) \lambda ds. \end{aligned} \tag{2.9.14}$$

2.10 Doubly stochastic filtered compound Poisson processes (DS-FCPP)

Let N^{Λ^K} a doubly stochastic filtered compound Poisson process (DSPP) with cumulative intensity Λ^K, where Λ^K is a filtered Poisson process with kernel K (see the beginning of the section 2.7).

Let us write

$$\Lambda^K(t) = \int_0^t K(t,s) dM(s),$$

where M is a homogeneous Poisson process with intensity λ. If K is the Molchan-Golosov kernel with $H > \frac{1}{2}$, Y. Mishura and K. Tikanmäki [95] have shown that Λ^K is absolutely continuous. Notice that the increments of Λ^K are neither independend nor stationnary.

Denote $\mathbb{D}^{1,2}_{\hat{N}_t^{\Lambda^K}}$ by $\mathbb{D}^{1,2}_K$. For $\theta \in \mathbb{D}^{1,2}_K$, following the proof of Lemma 2.7.5 and by Equation (2.9.14), we get

$$\begin{aligned} \int_0^t \int_{\mathbb{R}_0} \theta(s,z) \hat{N}^{\Lambda^K}(d^- s, dz) = 2\int_0^t \int_{\mathbb{R}_0} D_{s,z}\theta(s,z) \nu^K(dz) ds \\ + \int_0^t \int_{\mathbb{R}_0} \big(\theta(s,z) + D_{s,z}\theta(s,z)\big) \hat{N}^{\Lambda^K}(\delta s, dz). \end{aligned} \tag{2.10.1}$$

Characteristic function

The method of conditioning as in [91] allows to obtain:

$$\begin{aligned}\Phi_{N_t^{\Lambda^K}}(\alpha) &= M_{\Lambda^K(t)}\left(\int_{\mathbb{R}_0}(e^{i\alpha z}-1)\nu(dz)\right)\\ &= \exp\left\{\int_0^t\left[e^{\left(\int_{\mathbb{R}_0}(e^{i\alpha z}-1)\nu(dz)\right)K(t,s)}-1\right]\lambda(s)ds\right\},\end{aligned} \tag{2.10.2}$$

where $M_{\Lambda(t)}$ is the moment generating function[¶] of $\Lambda(t)$.

From the characteristic function, we obtain the expectation and the variance:

$$\mathbb{E}[N_t^{\Lambda^K}] = \mathbb{E}[\Lambda^K(t)]\left[\int_{\mathbb{R}_0} z\nu(dz)\right],$$

$$\mathrm{Var}[N_t^{\Lambda^K}] = \mathbb{E}[\Lambda^K(t)]\left[\int_{\mathbb{R}_0} z^2\nu(dz)\right] + \mathrm{Var}[\Lambda^K(t)]\left[\int_{\mathbb{R}_0} z\nu(dz)\right]^2.$$

2.10.1 Itô formula for DSFCPP

Let

$$X(t) = \int_0^t\int_{\mathbb{R}_0}\theta(s,z)\hat{N}^{\Lambda^K}(d^-s,dz), \tag{2.10.3}$$

where $\hat{N}^{\Lambda^K}$ is a filtered doubly stochastic compound Poisson process and θ is such that

$$\mathbb{E}\left[\int_0^t\int_{\mathbb{R}_0}|\theta(s,z)|+\theta(s,z)^2\nu^K(ds,dz)\right]<\infty.$$

Let $Y(t) = f(X(t))$, $t\in[0,T]$, where f is a function in $C^2(\mathbb{R})$ such that $f(X(t-)+\theta(t,z)) - f(X(t-))$ belongs to the domain of the divergence operator $\hat{N}^K(\delta s,dz)$. Then we have the following formula by (2.10.1) :

$$\begin{aligned}d^-Y(t) &= \int_{\mathbb{R}_0}\left(f(X(t-)+\theta(t,z)) - f(X(t-)) - f'(X(t^-))\theta(t,z)\right)\nu^K(dt,dz)\\ &\quad+\int_{\mathbb{R}_0}\left(f(X(t^-)+\theta(t,z)) - f(X(t-))\right)\hat{N}^{\Lambda^K}(d^-t,dz)\\ &= \int_{\mathbb{R}_0}\left(f(X(t-)+\theta(t,z)) - f(X(t-)) - f'(X(t-))\theta(t,z)\right)\nu^K(dt,dz)\\ &\quad+\int_{\mathbb{R}_0}\left[\left(f(X(t-)+\theta(t,z)) - f(X(t-))\right) + D_{t,z}\left(f(X(t-)+\theta(t,z)) - f(X(t-))\right)\right]\hat{N}^{\Lambda^K}(\delta t,dz)\\ &\quad+2\int_{\mathbb{R}_0}D_{t,z}\left(f(X(t-)+\theta(t,z)) - f(X(t-))\right)\nu^K(dt,dz).\end{aligned} \tag{2.10.4}$$

[¶] $M_{\Lambda(t)}(\alpha) = \mathbb{E}\{e^{\alpha\Lambda(t)}\}$

Chapter 3

An arbitrage free fractional Black-Scholes model

In this chapter we discuss the arbitrage problem in connection with fractional Brownian motion (FBM) in the Black-Scholes model. We show that FBM should not be considered as a volatility but as a stochastic trend. Therefore, it is reasonable to add a Brownian motion as diffusion part. We distinguish the cases whether long memory is the result of volatility clustering or the result of deterministic trends. In addition, we suggest to add in the model fractional Lévy motion as defined in Chapter 2, and we derive the price of European options in this case.

3.1 Introduction

The problem of option pricing and arbitrage opportunities in the fractional Black-Scholes model has been largely discussed during the last decades, see for example [15, 67, 66, 74, 102, 93]. For instance, [84, 90] prove that this model allows arbitrage opportunities and construct an example where such situations can be seen clearly. Later in [20], using advanced techniques from stochastic calculus, the author proves that if $H > 3/4$ then the mixed fractional Brownian motion defined as sum of a FBM and a BM is equivalent to a martingale under a certain probability measure. A remark here is that the Hurst index is unlikely to be $> 3/4$.

The motivation for using a fractional Black-Scholes model comes from the presence of long (or short) range dependence in the dynamics of assets prices in some real-world cases. The problem that this model is however not free of arbitrage represents a big challenge for pricing options. Nevertheless, people agree that in practice the presence of long or short memory in the financial data of asset returns should not permit traders to realize arbitrage opportunities on the market. In principle the question is why the model allows arbitrage opportunities despite the fact that the fractional model fits better to the data. We shall show in the sequel that the structure of the model as introduced already presents some failures.

First of all, we make precise that long (short) memory may result from volatility clustering or from drift shift. We will see later that the distinction between these two cases is crucial and that it has not been adequately taken into account in previous papers on this topic.

The main idea of our work is to show that arbitrage opportunities come from the structure of the model and not really from the fractional Brownian motion or the long memory. We will show that the fractional Brownian motion plays the role of a stochastic trend. The model should be corrected by adding a diffusion term (random motion) which is usually a Brownian motion.

3.1.1 Efficient market hypothesis versus fractal market hypothesis

Efficient market hypothesis

The efficient market hypothesis (EMH), commonly known through the random walk theory defined by (Fama, 1965) [37], postulates that market prices fully reflect all available information. The market immediately achieves equilibrium with the arrival of new information. Fama (1970) describes the efficient market as a market where all available information is already reflected in the assets prices. Later in [38], he extends the definition by stating that prices reflect available information to the point where marginal gain from using the information equals marginal cost of obtaining it.

Samuelson [86] formulates in 1965 independently the EMH in terms of martingales. Both approaches lead to Brownian motion, a process with i.i.d. normally distributed increments. The practical implication of this theory is that investors should be rational and homogeneous, that standard deviation is a meaningful measure of risk, and that there is a trade-off between risk and return, whereas future returns are unpredictable.

The most stringent requirement of the random walk theory is that observations have to be independent in the sense that the current change in prices can not be inferred from previous changes. In other words, the process modeling asset prices should be a martingale or fair game.

When modeling asset prices, the EMH facilitates the mathematical issues. Nevertheless, there is more and more evidence that this hypothesis is a pure theoretical assumption which is frequently violated in practice. In [25], the authors discuss two typical cases where the EMH does not hold. The first is when investors do not react to new information quickly and in unbiased manner. The second is when investors follow a “value strategy”, i.e. buy stocks that have low prices relative to their accounting book values, dividends, or historical prices, in order to outperform the market.

Fractal market hypothesis

The fractal market hypothesis (FMH) was proposed by (Peter, 1994) [79] and emphasizes the impact of information and investment horizons on the behavior of the investors. If all information had the same impact on all investors there would be no liquidity because all investors would be executing the same trading. If liquidity ceases, market becomes unstable and extreme movements occur.

Usually, one connect the FMH with the presence of fractality and long range dependence in the return processes of financial assets.

While Brownian motion succeeds in modeling the random walk movements (martingales) in assets prices, it fails in the other situations, e.g. when the movements perform perfectly a random walk. Therefore there is need to correction by adding an other source of noise, which is fractional in our study.

However, according to [26], empirical studies [4, 27, 73, 22, 63, 39] indicate positive dependence over large time horizon in absolute returns but not in the returns themselves, showing that it is more interesting to use fractional processes as models of volatility rather than for modeling prices directly. We propose a mixed fractional model instead of a pure fractional one. In this way we do not contradict [26] because we include a Brownian motion, a process with i.i.d. increments to the model, but we add a fractional noise to correct for the cases where there is evidence of long memory and fractality.

3.1.2 Fractional Brownian motion

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a filtered probability space. We define a Brownian motion $(W_t)_{0\leqslant t\leqslant T}$ with respect to this space.

Definition 3.1.1 *A centered Gaussian process $B^H = \{B_t^H, t \geqslant 0\}$ is called fractional Brownian motion* (FBM) *of Hurst parameter $H \in (0,1)$ if it has the covariance function:*

$$R_H(t,s) = \mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right]. \tag{3.1.1}$$

The FBM is a self similar process; for $H = \frac{1}{2}$ the FBM is just standard Brownian motion. However if $H \neq \frac{1}{2}$, then B^H is neither a semimartingale nor a Markov process.

The increments of B^H exhibit long range dependence when $H > \frac{1}{2}$ and short range dependence when $H < \frac{1}{2}$. Motivated by empirical studies on financial data of assets returns, many authors suggested to use the FBM instead of the classical BM to capture the long range dependence.

Proposition 3.1.2 *Let $\{B_t^H, t \in [0,1]\}$ be a fractional Brownian motion with Hurst index H. Consider the p-variations*

$$V_{n,p} = \sum_{i=1}^{n}\left|B^H\left(\frac{i}{n}\right) - B^H\left(\frac{i-1}{n}\right)\right|^p \xrightarrow{\text{in probability}} \begin{cases} 0 & \text{if } pH > 1, \\ \infty & \text{if } pH < 1, \end{cases} \tag{3.1.2}$$

as $n \longrightarrow \infty$.

Representation of FBM

For $T > 0$, $t \in [0,T]$ and $H \in (0,1)$, we have the following Molchan-Golosov representation for fractional Brownian motion:

$$B^H(t) = \int_0^t K_H(t,s)\, dW_s$$

where

$$K_H(t,s) = C_H\left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\, du\right]$$

and $C_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}}$ for $t > s$.

The Riemann-Liouville representation is defined as

$$B_{RL}^H(t) = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}} dW(s). \tag{3.1.3}$$

Itô formula

First, as B^H is not a semimartingale, we choose to use the divergence integration to define the stochastic integral with respect to B^H. Following [76] (Theorem 5.2.2), we have for $F \in C^2(\mathbb{R})$ satisfying the growth condition and for all $t \in [0,T]$, that the process $\{F'(B_s^H)1_{[0,T]}(s)\}$ belongs to $\operatorname{Dom}\delta$ and the following Itô formula holds:

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H)\delta B_s^H + H\int_0^t F''(B^H(s))s^{2H-1}ds. \tag{3.1.4}$$

Long range dependence in time series

Long range dependence in time series means that observations far away from each other are still strongly correlated. In general, it is stated and accepted that a Hurst exponent $H \in]\frac{1}{2}, 1[$ yields long range dependence in a time series. In this case the correlation function of the increments $\rho(k)$ decays slowly with a hyperbolic rate, i.e. for some positive constant c_ρ, we have

$$\lim_{k\to\infty} \frac{\rho(k)}{c_\rho k^{2H-2}} = 1. \tag{3.1.5}$$

We observe that $\rho(k) \to 0$ as $k \to \infty$.

Mixed fractional Brownian motion

Definition 3.1.3 *We define a mixed fractional Brownian motion with parameter α, β and H as the linear combination of standard Brownian motion and fractional Brownian motion, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by:*

$$Y_t = \alpha W_t + \beta B_t^H, \tag{3.1.6}$$

where W_t is a Brownian motion, B_t^H is an independent fractional Brownian motion of Hurst parameter $H \in (0,1)$ and α, β are two real positive constants.

3.1.3 Classical Black Scholes theory, comments and remarks

The objective of this section is to comment on some basic features of the Black-Scholes (B-S) option pricing theory. A clear understanding of these concepts is helpful for being able to generalize to a fractional B-S framework. In fact, in various situations related to this topic, certain shortcomings in the literature are due to an inadequate interpretation of those notions.

Fix an interval $[0, T]$. Consider the classical Black-Scholes model driven by a Brownian motion B_t adapted to the filtration $\mathcal{F}_t$, with parameters μ and σ such that $\mathbb{E}[\int_0^T \{\mu_s^2 + \sigma_s^2\} ds]$:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t, \quad \sigma_t > 0. \tag{3.1.7}$$

The parameter μ_t is known as the trend or the drift. "*Trends*" are defined as the general direction of the price of an asset. Usually, we consider μ as a constant. But in general it can be deterministic or even stochastic as long as it still predictable or adapted:

$$\mu_{t_i} = \mathbb{E}\left[\ln\left(\frac{S(t_{i+1})}{S(t_i)}\right) \middle| \mathcal{F}_{t_i}\right]. \tag{3.1.8}$$

The parameter σ_t is the volatility. "*Volatility*" is a statistical measure of the dispersion of returns for a given asset. It does not measure the direction of the price changes, it should be considered as a measurement of risk. In the Gaussian statistics it is the standard deviation.

We understand from the structure of SDE (3.1.7) that $\sigma_t S_t dB_t$ is a perturbation component with zero mean. As we know, the reason why one does not consider $\sigma_t dB_t$ rather than $\sigma_t S_t dB_t$ is that one wants perturbation proportional to S_t.

The solution to SDE (3.1.7) is

$$S_t = S_0 e^{\int_0^t (\mu_s - \frac{1}{2}\sigma_s{}^2) ds + \int_0^t \sigma_s dB_s}. \tag{3.1.9}$$

If r is the risk free rate and if $\tilde{S}_t = e^{-rt}S_t$ denotes the discounted prices, we have:

$$d\tilde{S}_t = (\mu_t - r)\,\tilde{S}_t dt + \sigma_t \tilde{S}_t dB_t. \tag{3.1.10}$$

Set $W_t = B_t - \frac{\mu_t - r}{\sigma_t} = B_t - \theta_t$. Then Equation (3.1.10) writes as

$$d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t.$$

The quantity θ_t is interpreted as "*risk premia*" or "*excess return*" per unit of risk and relates to the compensation that a risk averse investor expects to receive for bearing risks. Risk aversion is the reluctance of a person to accept a bargain with an uncertain payoff rather than another bargain with a more certain, but possibly lower expected payoff. The compensation for risk depends on the investor perception of the underlying risks and on the price they require per unit of risk.

Define a probability measure $\mathbb{Q}$ such that:

$$M_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \left(\frac{\mu_s - r}{\sigma_s}\right) dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu_s - r}{\sigma_s}\right)^2 ds\right).$$

Then M_t is a martingale with expectation $\mathbb{E}\left[M_t\right] = 1$. Under this measure W_t is a Brownian motion and $\tilde{S}_t$ is martingale. This result follows from the application of Girsanov's theorem.

When we come to pricing contingent claims, European options for example, an interesting result is that the prices of calls and puts does not depend on the drift μ. In fact, the Black-Scholes theory suppose that the market should renumerate only the risk of incertitude represented by σ. Any predictable drift in excess of the risk free rate is supposed to have no risk. Hence this excess drift or trend will be part of the risk premia and can be removed by a change of measure.

For instance recall the classical formulas for the value of a European call C at time t with maturity T and strike K:

$$C(t, S_t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \tag{3.1.11}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

We observe that μ does not enter in these equations. This result will have an important impact later when we consider fractional models.

3.2 Calibration of the fractional model

3.2.1 Maximum likelihood estimators

Statistical inference for fractional diffusion processes and mixed fractional diffusion processes have been investigated in a large number of references [72, 81, 23, 100, 35, 52, 100]. The estimation of the Hurst parameter H is usually obtained by the rescaled analysis method (R/S), see for example [9, 40, 45, 50].

Hu and Nualart (2011) [52] studied the problem of consistency and strong consistency of the maximum likelihood estimators of the mean and variance of the drifted FBM observed at discrete time. They considered the following model

$$Y_t = \mu t + \sigma B_t^H, \qquad t \geqslant 0,$$

where μ and σ are constants to be estimated from discrete observations of the process Y, and $(B_t^H, t \geqslant 0)$ is the fractional Brownian motion FBM of Hurst parameter $H \in]0,1[$. Assume that the process is observed at discrete time instants $(\Delta t, 2\Delta t, \ldots, N\Delta t)$.

Suppose that we are given N observations*

$$\mathbf{Y} = (Y_{\Delta t}, Y_{2\Delta t}, ..., Y_{N\Delta t})^T.$$

Letting $\mathbf{t} = (\Delta t, 2\Delta t, \ldots, N\Delta t)^T$, the joint probability density function of Y writes as

$$g(\mathbf{Y}) = (2\pi\sigma^2)^{\frac{-N}{2}} |\Gamma_H|^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{Y} - \mu\mathbf{t})^T \Gamma_H^{-1}(\mathbf{Y} - \mu\mathbf{t})\right)$$

where

$$\Gamma_H = \left(\mathrm{Cov}(B_{i\Delta t}^H, B_{j\Delta t}^H)\right)_{i,j=1,2,\ldots,N} = \frac{1}{2}(\Delta t)^{2H}\left(i^{2H} + j^{2H} + |i-j|^{2H}\right)_{i,j=1,2,\ldots,N}.$$

The maximum likelihood estimators of μ and σ^2 from the observation $\mathbf{Y}$ are given by

$$\begin{aligned} \hat{\mu} &= \frac{\mathbf{t}^T\Gamma_H^{-1}\mathbf{Y}}{\mathbf{t}^T\Gamma_H^{-1}\mathbf{t}}, \quad \text{and} \\ \hat{\sigma}^2 &= \frac{1}{N}\frac{(\mathbf{Y}^T\Gamma_H^{-1}\mathbf{Y})(\mathbf{Y}^T\Gamma_H^{-1}\mathbf{t}) - (\mathbf{t}^T\Gamma_H^{-1}\mathbf{Y})^2}{\mathbf{t}^T\Gamma_H^{-1}\mathbf{t}}. \end{aligned}$$

In [100] the authors considered statistical inference for the mixed fractional Brownian motion as defined in Definition 3.1.3. Set $\epsilon = \beta/\alpha$, then Equation (3.1.6) becomes:

$$Y_t = \alpha(W_t + \epsilon B_t^H). \tag{3.2.1}$$

Using Malliavin calculus, the authors were able to give an estimator for ϵ by maximizing the likelihood function with respect to ϵ^2, that is

$$L_N(\mathbf{Y}, \epsilon^2) = -\frac{N}{2}\ln(2\pi) + \frac{N}{2}\ln N - \frac{N}{2}\ln(\mathbf{Y}^T\Gamma^{-1}\mathbf{Y}) - \frac{1}{2}\ln|\Gamma| - \frac{N}{2}, \tag{3.2.2}$$

or equivalently, by minimizing the function,

$$M_N(\mathbf{Y}; \epsilon^2) = \frac{1}{2}\ln|\Gamma| + \frac{N}{2}\ln(\mathbf{Y}^T\Gamma^{-1}\mathbf{Y}). \tag{3.2.3}$$

*The log returns for example

3.2.2 The deterministic drift in fractional models

The aim of the review in the previous section is to apply these concepts to estimate the parameter of fractional SDEs used in financial modeling. For the classical Black-Scholes SDE one commonly works with the following estimations:

$$\hat{\mu}(\Delta t) = \frac{1}{N}\sum_{i=1}^{N} \log\left[\frac{S_{i\Delta t}}{S_{(i+1)\Delta t}}\right], \tag{3.2.4}$$

$$\hat{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N} \left(\log\left[\frac{S_{i\Delta t}}{S_{(i+1)\Delta t}}\right] - \hat{\mu}\Delta t\right)^2, \tag{3.2.5}$$

where $S_{i\Delta t}$ are the historical asset prices at times $i\Delta t$.

Suppose now that S_t solves the fractional Black-Scholes SDE obtained by replacing the Brownian motion by a fractional Brownian motion, so that

$$S_t = S_0 \exp\left(\mu t + \sigma B_t^H - \frac{1}{2}\sigma^2 t^{2H}\right). \tag{3.2.6}$$

Then we can write

$$\frac{S_{t_{i+1}}}{S_{t_i}} = \exp\left(\mu(t_{i+1} - t_i) + \sigma(B_{t_{i+1}}^H - B_{t_i}^H) - \left(\frac{1}{2}\sigma^2 t_{i+1}^{2H} - \frac{1}{2}\sigma^2 t_i^{2H}\right)\right) \tag{3.2.7}$$

and

$$\mathbb{E}\left[\frac{S_{t_{i+1}}}{S_{t_t}}\right] = \exp\left(\mu(t_{i+1} - t_i) + \left(\frac{1}{2}\sigma^2 (t_{i+1} - t_i)^{2H}\right) - \left(\frac{1}{2}\sigma^2 t_{i+1}^{2H} - \frac{1}{2}\sigma^2 t_i^{2H}\right)\right). \tag{3.2.8}$$

Remark 3.2.1 *For $H = \frac{1}{2}$,*

$$\mathbb{E}\left[\frac{S_{t_{i+1}}}{S_{t_t}}\right] = \exp\left(\mu(t_{i+1} - t_i)\right). \tag{3.2.9}$$

In this case, $\hat{\mu}$ is the estimator of μ.

But for $H \neq \frac{1}{2}$, we can see from (3.2.8) that the parameters for fractional SDE change over time and cannot still constant.

3.3 Exclusion of arbitrage

In this section we discuss two existing methods which were developed to exclude arbitrage in the FBM model. The first one has been introduced by Guasoni (2006) [44], the second one by Cheridito (2003) [21]. Both methods consider a pure fractional noise which in our opinion gives rise to a problem of economic interpretation because it does not contain a diffusion part.

On the other hand, for semimartingales, the fundamental theorem of asset pricing [31] gives that

no arbitrage $\iff$ *existence of a risk neutral measure.*

In other words, in order to price financial derivatives fairly, one should have existence of an equivalent martingale measure (EMM) under which derivatives can be evaluated. Preferably the EMM should exist in closed form and not just formally or theoretically. This is for instance not the case with the two previous methods were existence of the EMM can be deduced exist but no explicit expression can be determined. This is a clear limitation of these methods as far as the pricing problem is concerned.

3.3.1 Exclusion of arbitrage according to Guasoni

In a previous paper [43] the author discusses the problem of pricing without semimartingales. When transaction costs are taken into account exclusion of arbitrage is possible. Later in [44] the author applies this result to the FBM which is not a semimartingale. But even not being a semimartingale, the process nevertheless has many desirable properties that were not taken into account before, for instance, the predictability of the increments. In the sequel we will give a short review of the method of [44]. For related results see also [7].

Consider a market model with a riskless and a risky asset, based on a filtered probability space $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$. The riskless asset is used as numéraire, and hence is assumed to be identically equal 1. The risky asset $(X_t)_{t\in[0,\infty)}$ is a càdlàg process adapted to $\mathcal{F}_t$ and strictly positive a.s.

Let $(\theta_t)_{t\in[0,\infty)}$ be a strategy which represents the number of shares in the risky asset held at time t, and each unit of numéraire traded in the risky asset generates a transaction cost of k units, which is charged to the riskless asset account. Consider an elementary strategy θ, which requires a finite number of transactions at some stopping times $(\tau_i)_{i=1}^n$. We have that

$$\theta = \sum_{i=1}^{n-1} \theta^{(i)} 1_{]\tau_i,\tau_{i+1}]}$$

for some random variables $(\theta^{(i)})_{i=1}^n$ where $\theta^{(i)}$ is $\mathcal{F}_{\tau_i}$-measurable. We also conventionally set $\theta(0) = 0$. Then the liquidation value of a portfolio with zero initial capital is:

$$V_t(\theta) = \sum_{i:\,\tau_i<t} \theta^{(i)}\big(X_{\tau_{i+1}} - X_{\tau_i}\big) - k \sum_{i:\,\tau_i<t} X_{\tau_i}\,|\theta^{(i+1)} - \theta^{(i)}| - kX_t|\theta_t|. \tag{3.3.1}$$

where the first term accounts for the capital gain, the second for the cost incurred in the various transactions, and the third for the final cost of liquidation. In continuous time

$$V_t(\theta) = (\theta.X)_t - k\int_{[0,t]} X_s d\theta_s - kX_t|\theta_t|. \tag{3.3.2}$$

Proposition 3.3.1 (see [44]) *Let X be a càdlàg process adapted to $\mathcal{F}_t$ strictly positive a.s. and let k, $T > 0$. If for all stopping times τ such that $\mathbb{P}(\tau < T) > 0$ we have that*

$$\mathbb{P}\left(\sup_{t\in[\tau,T]}\left|\frac{X_\tau}{X_t} - 1\right| < k,\ \tau < T\right) > 0, \tag{3.3.3}$$

then X is arbitrage free with transaction costs k in the interval $[0, T]$.

The main idea of the above criterion is the following: in order to realize arbitrage, at some time τ one has to start trading. This decision immediately generates transaction costs that must be recovered at a later time, and this is possible only if the asset price moves enough in the future. Hence, if at all times there is a remote possibility of arbitrarily small price changes, then downside risk cannot be eliminated, and arbitrage is impossible.

3.3.2 Exclusion of arbitrage according to Cheridito (no high frequency trading)

Cheridito (2003) [21] shows that when postulating the existence of an arbitrary small minimal amount of time $h > 0$ that must lie between two consecutive transactions, all kinds of arbitrage opportunities can be excluded. The assertion is that traders can act arbitrarily fast but not as fast as the market. The Cheridito class of trading strategies is defined as follows:

Definition 3.3.2 *The set of simple predictable integrands with bounded support is given by*

$$S(F) = g_0 1_0 + \sum_{j=1}^{n-1} g_j 1_{]\tau_j, \tau_{j+1}]}, \quad n \geqslant 2,\ 0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_n,$$

where all τ_j are $(\mathcal{F}_t)$-stopping times, g_0 is a real number and the g_j are real $\mathcal{F}_{\tau_j}$-measurable random variables and moreover τ_n is bounded.

Definition 3.3.3 (Cheridito class of trading strategies)
For any $h > 0$, let

$$S^h(F) = \left\{ g_0 1_0 + \sum_{j=1}^{n-1} g_j 1_{]\tau_j, \tau_{j+1}]} \in S(F) :\ \forall j, \tau_{j+1} \geqslant \tau_j + h \right\}$$

and let $\pi(F) = \bigcup_{h>0} S^h(F)$ be the class of trading strategies.

The idea here is to exclude continuous trading and moreover to require a minimal fixed time between successive trades. The fixed time can be as small as we want, but once chosen, it cannot be changed.

A comment on this method is that when we consider discrete markets, we add a randomness between the trading dates. In fact, consider Stratonovich discretization of the FBM with a fixed step $\Delta > 0$ which correspond to the time interval between the transactions:

$$\begin{aligned} B^H_{s+\Delta} - B^H_s &\simeq \sqrt{2H} \int_0^{s+\Delta} (s + \Delta - r)^{H-\frac{1}{2}} dW_r - \sqrt{2H} \int_0^s (s - r)^{H-\frac{1}{2}} dW_r \\ &\simeq \sqrt{2H} \int_0^s \left[(s + \Delta - r)^{H-\frac{1}{2}} - (s - r)^{H-\frac{1}{2}} \right] dW_r + \sqrt{2H} \int_s^{s+\Delta} (s + \Delta - r)^{H-\frac{1}{2}} dW_r \\ &\simeq \sqrt{2H} \int_0^s \left[(s + \Delta - r)^{H-\frac{1}{2}} - (s - r)^{H-\frac{1}{2}} \right] dW_r + \sqrt{2H} \Delta^{H-\frac{1}{2}} (W_{s+\Delta} - W_s). \end{aligned} \tag{3.3.4}$$

We see that a martingale part $\sqrt{2H}\Delta^{H-\frac{1}{2}}(W_{s+\Delta} - W_s)$ appears in the increments. We deduce that the discrete FBM can be identified in a formal way with a semimartingale. Using this fact one is able exclude arbitrage.

3.4 Approximation of the fractional Brownian motion

In this Section we give an L^2 approximation for $\left(B^H(t),\ t \in [0,T]\right)$ by semimartingales. Let $\alpha = H - \frac{1}{2}$. For every $\epsilon > 0$ define

$$B_H^\epsilon(t) = \int_0^t Z_H^\epsilon(t,s) dW_s + C_H \epsilon^{H-\frac{1}{2}} W_t = \int_0^t K_H^\epsilon(t\, s)\, dW_s$$

where

$$\begin{aligned} Z_H^\epsilon(t,s) &= C_H \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s+\epsilon)^{H-\frac{3}{2}} du \\ &= C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s+\epsilon)^{H-\frac{1}{2}} - \epsilon^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s+\epsilon)^{H-\frac{1}{2}} du\right] \end{aligned}$$

and

$$\begin{aligned} K_H^\epsilon(t,s) &= Z_H^\epsilon(t,s) + C_H \varepsilon^{H-\frac{1}{2}} \\ &= C_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s+\epsilon)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s+\epsilon)^{H-\frac{1}{2}} du\right]. \end{aligned}$$

Proposition 3.4.1 *$B_H^\epsilon(t)$ converge to $B^H(t)$ in $L^2(\Omega)$ as ϵ tends to 0 and $H > \frac{1}{2}$. The convergence is uniform with respect to $t \in [0,T]$.*

Remark 3.4.2 *For more general result, we refer to [72].*

Proof. We recall that for all $a, b > 0$,

$$(a+b)^\alpha \leqslant (a)^\alpha + (b)^\alpha, \quad \alpha \in [0,1], \tag{3.4.1}$$

$$(a-b)^2 \leqslant \left|a^2 - b^2\right| \leqslant a^2 + b^2. \tag{3.4.2}$$

For $0 \leqslant s \leqslant t$ and $\alpha = H - \frac{1}{2}$ we have

$$\begin{aligned} & K_H^\epsilon(t,s) - Z_H^\epsilon(t,s) \\ &\quad = C_H \left[\left(\frac{t}{s}\right)^\alpha \left[(t-s+\epsilon)^\alpha - (t-s)^\alpha\right] - \alpha s^{-\alpha} \int_s^t u^{\alpha-1} \left[(u-s+\epsilon)^\alpha - (u-s)^\alpha\right] du\right]. \end{aligned}$$

The Itô isometry yields

$$\begin{aligned} \mathbb{E}\left|B_H^\epsilon(t) - B^H(t)\right|^2 &= \int_0^t |K_H^\epsilon(t,s) - K_H(t,s)|^2 ds \\ &= \int_0^t \left(C_H \left[\left(\frac{t}{s}\right)^\alpha \left[(t-s+\epsilon)^\alpha - (t-s)^\alpha\right]\right)^2 ds \\ &\quad + \int_0^t \left(C_H \alpha s^{-\alpha} \int_s^t u^{\alpha-1} \left[(u-s+\epsilon)^\alpha - (u-s)^\alpha\right] du\right)^2 ds. \end{aligned}$$

From (3.4.1) we deduce for $a = t - s, b = \epsilon$ that

$$
\begin{aligned}
\mathbb{E}\left|B_H^\epsilon(t) - B^H(t)\right|^2 &\leqslant \int_0^t \left(C_H\left[\left(\frac{t}{s}\right)^\alpha [\epsilon]^\alpha\right]\right)^2 ds + \int_0^t \left(\alpha C_H s^{-\alpha}\int_s^t u^{\alpha-1}[\epsilon]^\alpha du\right)^2 ds \\
&\leqslant (C_H)^2\epsilon^{2\alpha}\int_0^t \left(\frac{t}{s}\right)^{2\alpha} + \alpha^2 s^{-2\alpha}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^2 ds \\
&\leqslant (C_H)^2\epsilon^{2\alpha}\left[t^{2\alpha}\left(\frac{t^{-2\alpha+1}}{-2\alpha+1}\right) + \int_0^t \left[t^{2\alpha}s^{-2\alpha} - 2s^{-2\alpha}t^\alpha s^\alpha + 1\right]ds\right] \\
&\leqslant (C_H)^2\epsilon^{2\alpha}\left[\left(\frac{t}{-2\alpha+1}\right) + t^{2\alpha}\left(\frac{t^{-2\alpha+1}}{-2\alpha+1}\right) - 2t^\alpha\left(\frac{t^{-\alpha+1}}{-\alpha+1}\right) + t\right] \\
&\leqslant (C_H)^2\epsilon^{2\alpha}\left[\left(\frac{t}{-2\alpha+1}\right) + \left(\frac{t}{-2\alpha+1}\right) - 2\left(\frac{t}{-\alpha+1}\right) + t\right] \\
&\leqslant (C_H)^2\epsilon^{2\alpha}\left[\left(\frac{t}{-\alpha+1}\right) + \left(\frac{t}{-\alpha+1}\right) - 2\left(\frac{t}{-\alpha+1}\right) + t\right] \\
&\leqslant (C_H)^2\epsilon^{2\alpha}t \\
&\leqslant (C_H)^2\epsilon^{2\alpha}T
\end{aligned}
$$

for all $t \in [0, \mathrm{T}]$. ■

3.4.1 Martingale measures

By the stochastic Fubini theorem [82] we have

$$
\begin{aligned}
B_H^\epsilon(t) &= \int_0^t Z_H^\epsilon(t,s)dW_s + C_H\varepsilon^{H-\frac{1}{2}}\, W_t \\
&= \underbrace{C_H\left(H - \frac{1}{2}\right)\int_0^t\int_0^u \left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s+\epsilon)^{H-\frac{3}{2}}dW_s du}_{\text{process of bounded variation}} + \underbrace{C_H(\epsilon)^\alpha W_t}_{\text{martingale}}.
\end{aligned}
$$

Let

$$D_t^\epsilon = \mathcal{E}(M^\epsilon)_t = \exp\left(M_t^\epsilon - \frac{1}{2}\langle M^\varepsilon\rangle_t\right), \tag{3.4.3}$$

$$M_t^\epsilon = -\frac{(H-\frac{1}{2})}{\epsilon^\alpha}\int_0^t \left(\int_0^u \left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s+\epsilon)^{H-\frac{3}{2}}dW_s\right)dW_u. \tag{3.4.4}$$

Proposition 3.4.3
Le $T > 0$ and $t \in [0, T]$. There exists an equivalent probability measure $\mathbb{Q}^\epsilon$ such that $D_t^\epsilon = \frac{d\mathbb{Q}_t^\epsilon}{d\mathbb{P}_t}$ under which $\tilde{B}_H^\epsilon(t)$ is a $\mathbb{Q}^\varepsilon$-Brownian motion.

Proof. We have

$$
\begin{aligned}
W_t &= \frac{1}{C_H(\epsilon)^\alpha}\, B_H^\epsilon(t) + \left(-\frac{(H-\frac{1}{2})}{\epsilon^\alpha}\int_0^t\int_0^u \left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s+\epsilon)^{H-\frac{3}{2}}dW_s du\right) \\
&= \tilde{B}_H^\epsilon(t) + \left(-\frac{(H-\frac{1}{2})}{\epsilon^\alpha}\int_0^t\int_0^u \left(\frac{u}{s}\right)^{H-\frac{1}{2}}(u-s+\epsilon)^{H-\frac{3}{2}}dW_s du\right)
\end{aligned}
$$

By the Girsanov Theorem the problem is resumed to find a martingale M_t^ϵ such that

$$\tilde{B}_H^\epsilon(t) = W_t - \langle W_t,\ M_t^\epsilon \rangle .$$

The aim is to show that $\tilde{B}_H^\epsilon(t)$ is a $\mathbb{Q}^\epsilon$-Brownian motion where $d\mathbb{Q}^\epsilon = D_t^\epsilon d\mathbb{P}_t$ and D_t^ϵ is given by (3.4.3).

We deduce that M_t^ϵ solves the following equation:

$$\langle W_t, M_t^\epsilon \rangle = -\frac{\left(H-\frac{1}{2}\right)}{\epsilon^\alpha} \int_0^t \int_0^u \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s+\epsilon)^{H-\frac{3}{2}} dW_s du.$$

Therefore M_t^ϵ is given by (3.4.4).

Set $b(t,\omega) = \int_0^t \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s+\epsilon)^{H-\frac{3}{2}} dW_s$. It follows that

$$\begin{aligned} dB_H^\epsilon(t) &= C_H\left(H-\frac{1}{2}\right)\left(\int_0^t \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s+\epsilon)^{H-\frac{3}{2}} dW_s\right) dt\ + C_H(\epsilon)^\alpha dW_t \\ &= C_H\left(H-\frac{1}{2}\right) b(t,\omega)\, dt\ + C_H(\epsilon)^\alpha dW_t. \end{aligned}$$

In order to prove that $(D_t^\epsilon)_{t\in[0,T]}$ is a martingale we have to verify the Novikov condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |b(u,\omega)|^2 du\right)\right] < \infty.$$

This can be done as in Cheridito [19] page 58. ∎

Remark 3.4.4 *By a similar proof we can find an expression of the martingale measure for the process*

$$B_t = \int_0^t \phi(t-s) dW_s$$

where ϕ *verifies the hypothesis of Section* 1.3. *It is given by*

$$D_t = \mathcal{E}(M_t)$$

where

$$M_t = -\frac{1}{\phi(0)} \int_0^t \left(\int_0 \phi'(u-s) dW_s\right) dW_u.$$

3.4.2 Regular kernels and long range dependence

This section is to answer the question what happens to the long range dependence (l.r.d.) behavior when we switch from a singular kernel to a regular kernel. Intuitively, one could think that the l.r.d. is a consequence of the singularity in $t = s$. In the following we give a simple example derived from the *Riemann-Liouville* representation of the FBM and we show that for this case the l.r.d is not affected. In addition we show that the increments covariation of this process has the same asymptotic behavior as the FBM.

Definition 3.4.5 *For any $H \in (0,1)$, we define a centered Gaussian semimartingale*

$$\left\{B_t^H,\ t \in [0,T]\right\}$$

of Hurst index H as follows: $B_t^H = \sqrt{2H}\int_0^t (t-s+1)^{H-\frac{1}{2}} dW_s$ where W_t is a Wiener process. Letting $\alpha = H - \frac{1}{2}$ we have

(1) $\mathbb{E}\left[\left(B_t^H\right)^2\right] = (2\alpha+1)\int_0^t (t-s+1)^{2\alpha} ds = (t+1)^{2\alpha+1} - 1.$

(2) *The autocorrelation function*

$$\begin{aligned} R_H(t,s) = \mathbb{E}\left[B_t^H B_s^H\right] &= (2\alpha+1)\int_0^{t\wedge s} (t-u+1)^\alpha (s-u+1)^\alpha du \\ &= \frac{1}{2}\left[(t+1)^{2\alpha+1} + (s+1)^{2\alpha+1} - 2 - 2\mathbb{E}\left[\left(B_t^H - B_s^H\right)^2\right]\right]. \end{aligned}$$

For $0 < s < t$, we have

$$\begin{aligned} \mathbb{E}\left[\left(B_t^H - B_s^H\right)^2\right] &= 2H\int_0^s \{(t-u+1)^\alpha - (s-u+1)^\alpha\}^2 du + 2H\int_s^t (t-u+1)^{2\alpha} du \\ &= 2H\int_0^s \{(t-u+1)^\alpha - (s-u+1)^\alpha\}^2 du + \left[(t-s+1)^{2\alpha+1} - 1\right]. \end{aligned}$$

(3) $\left(B_t^H\right)_{t\in[0,T]}$ *is not self-similar and its increments are non-stationary.*

Set $Z(t,s) = \sqrt{2H}(t-s+1)^\alpha$ for $t \in [0,T]$ and $s \leqslant t$. Then $Z(t,s)$ is smooth in (t,s) and bounded on compact sets. In fact, we have for $H \in (0,1)$ and $0 < s < t$:

$$|Z(t,s)| = \max\left((t+1)^\alpha, 1\right).$$

It follows that

$$\begin{aligned} Z(t,s) &= Z(s,s) + \int_s^t \frac{\partial}{\partial u} Z(u,s) du \\ &= \sqrt{2H}\left(1 + \alpha\int_s^t (u-s+1)^{\alpha-1} du\right) \end{aligned}$$

and

$$B_t^H = \sqrt{2H}\ W_t + \alpha\sqrt{2H}\int_0^t \int_s^t (u-s+1)^{\alpha-1} du dW_s.$$

Since $\int_0^t \int_s^t (u-s+1)^{2\alpha-2} duds < \infty$, we have by the stochastic Fubini theorem

$$\int_0^t \int_s^t (u-s+1)^{\alpha-1} du dW_s = \int_0^t \int_0^u (u-s+1)^{\alpha-1} dW_s du.$$

We deduce the following representation

$$B_t^H = \sqrt{2H}\ W_t + \alpha\sqrt{2H}\int_0^t \int_0^u (u-s+1)^{\alpha-1} dW_s du$$

where $\int_0^t \int_0^u (u-s+1)^{\alpha-1} dW_s du$ is a process with bounded variations and $(W_t)_{t\geqslant 0}$ a Wiener process. The quadratic variation of B_t^H is calculated as

$$\left\langle B^H\right\rangle_t = 2H\langle W\rangle_t\ + 0 = 2Ht.$$

Proposition 3.4.6 *For $H > \frac{1}{2}$ the increments of $\left(B^H(t),\ t \in [0,T]\right)$ exhibit long range dependence, and for $0 \leqslant j < k$, such that $\frac{1}{k-j} \to 0$, we have*

$$\begin{aligned}&\operatorname{Cov}\left[\left(B^H(k+1) - B^H(k)\right), \left(B^H(j+1) - B^H(j)\right)\right]\\ &\quad = \frac{1}{2}\left[(k-j+1)^{2H} - 2(k-j)^{2H} + (k-j-1)^{2H}\right].\end{aligned}$$

Proof. Consider the sequence of increments $W_H(j) = B^H(j+1) - B^H(j)$. We have $\mathbb{E}\left[W_H(j)\right] = 0$ and

$$\begin{aligned}\operatorname{Cov}\left[W_H(k), W_H(j)\right] &= \operatorname{Cov}\left[\left(B^H(k+1) - B^H(k)\right), \left(B^H(j+1) - B^H(j)\right)\right]\\ &= \frac{1}{2}\Big\{\left[(k+2)^{2\alpha+1} + (j+2)^{2\alpha+1} - 2 - 2\mathbb{E}\left[\left(B^H_{k+1} - B^H_{j+1}\right)^2\right]\right]\\ &\quad - \left[(k+2)^{2\alpha+1} + (j+1)^{2\alpha+1} - 2 - 2\mathbb{E}\left[\left(B^H_{k+1} - B^H_{j}\right)^2\right]\right]\\ &\quad - \left[(k+1)^{2\alpha+1} + (j+2)^{2\alpha+1} - 2 - 2\mathbb{E}\left[\left(B^H_{k} - B^H_{j+1}\right)^2\right]\right]\\ &\quad + \left[(k+1)^{2\alpha+1} + (j+1)^{2\alpha+1} - 2 - 2\mathbb{E}\left[\left(B^H_{k} - B^H_{j}\right)^2\right]\right]\Big\}.\end{aligned}$$

Now for $0 < r_1 < r_2$, we have

$$\mathbb{E}\left[\left(B^H_{r_2} - B^H_{r_1}\right)^2\right] = 2H\int_0^{r_1}\{(r_2-u+1)^\alpha - (r_1-u+1)^\alpha\}^2du + 2H\int_{r_1}^{r_2}(r_2-u+1)^{2\alpha}du.$$

Set $v = r_1 - u + 1$ and $u = \frac{v}{r_2 - r_1}$, then

$$\begin{aligned}\mathbb{E}\left[\left(B^H_{r_2} - B^H_{r_1}\right)^2\right] = (r_2-r_1)^{2H}&\left[\left(1 + \frac{1}{r_2-r_1}\right)^{2\alpha+1} - \left(\frac{1}{r_2-r_1}\right)^{2\alpha+1}\right.\\ &\left. + 2H\int_{\frac{1}{r_2-r_1}}^{\frac{r_1+1}{r_2-r_1}}\{(1-u)^\alpha - (u)^\alpha\}^2du\right].\end{aligned}$$

Let

$$g(r_2,r_1) = \left(1 + \frac{1}{r_2-r_1}\right)^{2\alpha+1} - \left(\frac{1}{r_2-r_1}\right)^{2\alpha+1} + 2H\int_{\frac{1}{r_2-r_1}}^{\frac{r_1+1}{r_2-r_1}}\{(1-u)^\alpha - (u)^\alpha\}^2du.$$

It follows that

$$\begin{aligned}\operatorname{Cov}\left[W_H(k), W_H(j)\right] = \frac{1}{2}&\left[-(k-j)^{2H}g(k+1,j+1) + (k-j+1)^{2H}g(k+1,j)\right]\\ &+ \left[(k-j-1)^{2H}g(k,j+1) - (k-j)^{2H}g(k,j)\right].\end{aligned}$$

As $\frac{1}{r_2-r_1} \to 0$, we can check $g(r_2,r_1) \to 1$. Thus

$$\operatorname{Cov}\left[W_H(k), W_H(j)\right] \approx \frac{1}{2}\left((k-j+1)^{2H} - 2(k-j)^{2H} + (k-j-1)^{2H}\right)$$

which agrees with the increments covariation of standard fractional Brownian motion. Taking the last equality into account and the fact that

$$\left(1+\frac{1}{r_2-r_1}\right)^{2\alpha+1}-\left(\frac{1}{r_2-r_1}\right)^{2\alpha+1}\to 1,$$

we deduce the long range dependence which finishes the proof of the proposition. ∎

3.5 Fractional Black-Scholes model with jumps

3.5.1 Classical Merton model

In the Merton model the dynamics of the risky asset $S(t)$, the price of the risky asset at time t, jumps at the proportions $Y_1,\ldots,Y_n$ at the times $T_1,\ldots,T_n$. The process $(S(t)\,,\ t\geqslant 0)$ is an adapted, right-continuous process such that on the time intervals $[T_n,T_{n+1})$,

$$dS(t)=S(t)\left(\mu dt+\sigma dW(t)\right),\quad 0\leqslant t\leqslant T,$$

where $W(t)$ is a Brownian motion.

At time $t=T_n$, the jumps of $S(t)$ are given by $\Delta S_n=S(T_n)-S(T_{n-})=S(T_{n-})\,Y_n$. It follows that $S(T_n)=S(T_{n-})(1+Y_n)$, and for $t\in[0,T_1)$,

$$S(t)=S(0)\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W(t)\right).$$

Consequently, the left-hand limit at T_1 writes as

$$S(T_{1-})=\lim_{r\to T_1}S(r)=S(0)\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)T_1+\sigma W(T_1)\right),$$

whereas

$$S(T_1)=S(0)(1+Y_1)\exp\left(\left(\left(\mu-\frac{\sigma^2}{2}\right)\right)T_1+\sigma W(T_1)\right).$$

Repeating this scheme, we obtain

$$\begin{aligned}
S(t)&=S(0)\left[\prod_{n=1}^{N(t)}(1+Y_n)\right]\exp\left(\left(\left(\mu-\frac{\sigma^2}{2}\right)\right)t+\sigma W(t)\right)\\
&=S(0)\exp\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W(t)+\log\left(\prod_{n=1}^{N(t)}(1+Y_n)\right)\right]\\
&=S(0)\exp\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W(t)+\sum_{n=1}^{N(t)}\log(1+Y_n)\right]\\
&=S(0)\exp\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W(t)+\int_0^t\log(1+Y_n)\,dN(s)\right].
\end{aligned}$$

On the other hand, for $t \in]T_i, T_j[$ and $0 < i < j$, we have

$$S(t) = S(0) + \int_0^t S(s-)\left(\mu ds + \sigma(s)dW(s)\right) + \sum_{n=1}^{N(t)} S(T_{n-})Y_n$$

and in differential form:

$$\begin{aligned} dS(t) &= S(t-)\left(\mu dt + \sigma dW(t)\right) + S(t-)Y_t dN_t \\ &= S(t-)dX_t \end{aligned}$$

where X_t is a Lévy process with differential form

$$dX_t = \mu dt + \sigma dW(t) + \int_{\mathbb{R}\setminus\{0\}} y\ N(dy, dt).$$

Remark 3.5.1 *For $t \in]T_i,\, T_j[$ we may suppose that $S(t) = S(t-)$.*

3.5.2 A fractional model with jumps

In this Section we propose a fractional jump diffusion model with a Volterra semimartingale kernel and a fractional jump process which is a filtered doubly stochastic compound Poisson process. Let $(W(t))_{t\in[0,T]}$, $(B(t))_{t\in[0,T]}$ be two independent Brownian motions and $(N_P(t))_{t\in[0,T]}$ a Poisson process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ independent of W and B. Consider the following model

$$\begin{aligned} S(t) = S(0) &+ \int_0^t \mu(s)S(s-)ds + \int_0^t \sigma_1(s)S(s-)dW(s) \\ &+ \int_0^t \sigma_2(s)S(s-)dB^K(s) + \int_0^t \int_{\mathbb{R}_0} S(s-)yN^{\Lambda^\phi}(dy, ds), \end{aligned} \tag{3.5.1}$$

where $B^K(t)$ is a Volterra process with regular kernel as in Definition 1.3.1,

$$B^K(t) = \int_0^t K(t-s)dB(s),$$

and $N^{\Lambda^\phi}(dy, dt)$ is a filtered doubly stochastic Poisson process, with cumulative intensity given by a filtered Poisson process independent of N, that is

$$\begin{aligned} \Lambda^\phi(t) &= \int_0^t \phi(t-s)N_P(ds) \\ &= \phi(0)N_P(t) + \int_0^t \int_0^u \phi'(u-s)N_P(ds)du. \end{aligned} \tag{3.5.2}$$

Let $(X_t)_{t\geqslant 0}$ be a Itô Lévy process with differential form

$$dX_t = G(t)dt + F(t)dB(t) + \int_{\mathbb{R}_0} H(y,t)N_P(dy, dt), \tag{3.5.3}$$

where $G \in L^1(\mathbb{P})$, $F \in L^2(\mathbb{P})$ adapted and H predictable such that $\int_0^T \int_{\mathbb{R}_0} E[H(t,y)^2]\,\nu(dy, dt) < +\infty$. Suppose that e^{X_t} $(t \in [0,T])$ is a martingale. Define the probability measure $\mathbb{Q}$ through $\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = e^{X_t}$ for $t \in [0,T]$.

Theorem 3.5.2 *There exists a risk neutral probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *under which the discounted price* $\tilde{S}_t = e^{-rt}S_t$, $t \in [0,T]$, *is a martingale, if the following two conditions are satisfied:*

(1) $C(u) := (\mu(u) - r) + \int_0^u K'(u-s)\sigma_2(u))dB(s) + \int_0^u \int_{\mathbb{R}_0} \phi'(u-s)\ln(1+y)N_P(dy,ds)$

$$+K(0)\sigma_2(u)F(u) + \phi(0)\int_{\mathbb{R}_0} y\nu^{\mathbb{Q}}(dy) = 0$$

(2) $G(t) + \frac{1}{2}F(t)^2 + \int_{\mathbb{R}_0}\left(e^{H(t,y)} - 1\right)\nu(dy) = 0.$

Remark 3.5.3 *The condition* $C(u) = 0$ *allows an infinity of solutions* (F, G, H). *Hence there exists an infinity of risk neutral measures* $\mathbb{Q}$ *under which the discounted prices are martingales. In general market models with Lévy processes are known to be incomplete.*

The condition $C(u) = 0$ *yields*

$$F(u) = -\frac{1}{K(0)\sigma_2(u)}\Bigg[(\mu(u) - r) + \int_0^u K'(u-s)\sigma_2(u))dB(s) + \int_0^u \int_{\mathbb{R}_0} \phi'(u-s)\ln(1+y)N_P(dy,ds) + \phi(0)\int_{\mathbb{R}_0} y\nu^{\mathbb{Q}}(dy)\Bigg]$$

where $G(t)$ *is a function of* t, N_P, B, H, *given by*

$$G(t) = -\frac{1}{2}F(t)^2 - \int_{\mathbb{R}_0}\left(e^{H(y,t)} - 1\right)\nu(dy). \tag{3.5.4}$$

We now turn to the proof of Theorem 3.5.2.

Proof. We apply the Girsanov Theorem of Section 1.1.5 and condition (2) of the theorem, under the hypothesis that $\exp(X_t)$ is a martingale. Then $B^{\mathbb{Q}}(t) = B(t) - \int_0^t F(u)du$ is a Brownian motion under $\mathbb{Q}$ and $N_P(dy,dt)$ a compound Poisson process with $\mathbb{Q}$-intensity measure $\nu^{\mathbb{Q}}(dy,du) = e^{H(y,u)}\nu(dy,du)$.

On the other hand, Proposition 1.3.2 applied to B^K allows us to write

$$S(t) = S(0) + \int_0^t \mu(s)S(s-)ds + \int_0^t\int_0^u K'(u-s)\sigma_2(u)S(u-)dB(s)du + \int_0^t \sigma_1(u)S(u-)dW(u) + K(0)\int_0^t \sigma_2(u)S(u-)dB(u) + \int_0^t\int_{\mathbb{R}_0} S(u-)yN^{\Lambda^\phi}(dy,du). \tag{3.5.5}$$

Letting $\tilde{S}(t) = e^{-rt}S(t)$ and applying Itô's formula (2.8.1) and (1.3.1) to $\ln\tilde{S}(t)$, we get

$$\ln\big(\tilde{S}(t)\big) = \ln\big(\tilde{S}(0)\big) + \int_0^t \frac{1}{\tilde{S}(u-)}\left[(\mu(u) - r)\tilde{S}(u-) + \int_0^u K'(u-s)\sigma_2(u)\tilde{S}(u-)dB(s)\right]du + \int_0^t \sigma_1(u)dW(u) - \frac{1}{2}\int_0^t \sigma_1(u)^2du + K(0)\int_0^t \sigma_2(u)dB(u) - \frac{\big(K(0)\big)^2}{2}\int_0^t \sigma_2(u)^2du + \int_0^t\int_{\mathbb{R}_0}\ln\big(1+y\big)N^{\Lambda^\phi}(dy,du). \tag{3.5.6}$$

Set $B^{\mathbb{Q}}(t) = B(t) - \int_0^t F(u)du$. By means of Equation (3.5.2) we obtain

$$\begin{aligned}
\ln\big(\tilde{S}(t)\big) &= \ln\big(\tilde{S}(0)\big) + \int_0^t \Big[(\mu(u) - r) + \int_0^u K'(u-s)\sigma_2(u)dB(s) + K(0)\sigma(u)F(u) \\
&+ \int_0^u \int_{\mathbb{R}_0} \phi'(u-s)\ln(1+y)N_P(dy,ds) + \phi(0)\int_{\mathbb{R}_0} y\nu^{\mathbb{Q}}(dy)\Big] du \\
&+ \int_0^t \sigma_1(u)dW(u) - \frac{1}{2}\int_0^t \sigma_1(u)^2 du \\
&+ K(0)\int_0^t \sigma_2(u)dB^{\mathbb{Q}}(u) - \frac{\big(K(0)\big)^2}{2}\int_0^t \sigma_2(u)^2 du \\
&+ \int_0^t \int_{\mathbb{R}_0} \ln\big(1+y\big)N^{\Lambda^\phi}(dy,du) \\
&- \int_0^t \int_0^u \int_{\mathbb{R}_0} \phi'(u-s)\ln\big(1+y\big)N_P(dy,ds)du - \phi(0)\int_0^t \int_{\mathbb{R}_0} y\nu^{\mathbb{Q}}(dy)du. \qquad (3.5.7)
\end{aligned}$$

Set

$$\begin{aligned}
C(u) := \Big[&(\mu(u) - r) + \int_0^u K'(u-s)\sigma_2(u))dB(s) \\
&+ \int_0^u \phi'(u-s)\ln(1+y)N_P(dy,ds) + K(0)\sigma_2(u)F(u) + \phi(0)\int_{\mathbb{R}_0} y\nu^{\mathbb{Q}}(dy)\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{S}(t) = S(0)e^{\int_0^t C(u)du}\exp\Big[&\int_0^t \sigma_1(u)dW(u) - \frac{1}{2}\int_0^t \sigma_1(u)^2 du + K(0)\int_0^t \sigma_2(u)dB^{\mathbb{Q}}(u) \\
&- \frac{\big(K(0)\big)^2}{2}\int_0^t \sigma_2(u)^2 du + \int_0^t \ln\big(1+y\big)N^{\Lambda^\phi}(dy,du) \\
&- \int_0^t \int_0^u \phi'(u-s)\ln\big(1+y\big)N_P(dy,ds)du \\
&- \int_0^t \int_{\mathbb{R}} y\nu^{\mathbb{Q}}(dy)du\Big]. \qquad (3.5.8)
\end{aligned}$$

Putting $C(u) = 0$ which corresponds to condition (1) of the Theorem and applying again Itô's formula to the exponential function, we obtain

$$\begin{aligned}
\tilde{S}(t) = S(0) &+ \int_0^t \sigma_1(u)\tilde{S}(u)dW(u) \\
&+ K(0)\int_0^t \sigma_2(u)\tilde{S}(u)dB^{\mathbb{Q}}(u) + \int_0^t \tilde{S}(u)yN^{\Lambda^\phi}(dy,du) \\
&- \int_0^t \int_0^u \phi'(u-s)\tilde{S}(u)N_P(dy,ds)du - \int_{\mathbb{R}} \tilde{S}(u)y\nu^{\mathbb{Q}}(dy,du) \\
= S(0) &+ \int_0^t \sigma_1(u)\tilde{S}(u)dW(u) \\
+ K(0)&\int_0^t \tilde{S}(u)\sigma_2(u)dB^{\mathbb{Q}}(u) + \int_0^t \tilde{S}(u)y\tilde{N}^{\Lambda^\phi}(dy,du) \\
&+ \phi(0)\int_0^t \int_{\mathbb{R}_0} \tilde{S}(u)yN_P(dy,du) - \phi(0)\int_0^t \int_{\mathbb{R}} \tilde{S}(u)y\nu^{\mathbb{Q}}(dy,du) \\
= S(0) &+ \int_0^t \sigma_1(u)\tilde{S}(u)dW(u) \\
+ K(0)&\int_0^t \tilde{S}(u)\sigma_2(u)dB^{\mathbb{Q}}(u) + \int_0^t \tilde{S}(u)y\tilde{N}^{\Lambda^\phi}(dy,du) \\
&+ \phi(0)\int_0^t \int_{\mathbb{R}_0} \tilde{S}(u)y\tilde{N}_p(dy,du). \qquad (3.5.9)
\end{aligned}$$

■

Chapter 4

Long memory, interest rates and term structure modeling

4.1 Introduction

The valuation of fixed-income securities and interest rate derivatives, from the most simple structures to complex structures found in structured finance and interest rate derivative markets, depends on the interest rate model and term structure model used by the investor. The term structure of interest rates gives the relationship between the yield on an investment and the term to maturity of the investment. Understanding and modeling the term structure of interest rates represents one of the most challenging topics in quantitative finance.

Term structure theory seeks to identify elements or factors that may explain the dynamics of interest rates. These factors can be random or stochastic. Therefore, interest rate models use statistical processes to describe the stochastic properties of the factors.

There exist three major theories to explain the relation between the interest rates of various maturities: the expectation hypothesis, the liquidity preference and the preferred habitat theory. Some references add the market segmentation theory, see [83].

It is not of our concern in this chapter to go into the details of these theories; we just mention that the goal of these theories is to try to explain the shift between interest rates of different maturities. Given the short rate r at time $t \geqslant 0$ and a maturity $0 \leqslant t \leqslant T$, we suppose the term structure to follow the dynamics:

$$R(t,T) = \frac{1}{T-t}\left[\int_t^T \mathbb{E}_t(r(s))\,ds + \int_t^T L\,(s,T)\,ds\right],$$

where $L(t,T) \geqslant 0$ denotes the instantaneous term premium at time t for a bond maturing at time T and $\mathbb{E}_t$ the conditional expectation. The expectation hypothesis assumes $L(t,T) = 0$, while the liquidity preference theory [47] attributes the premium to the risk aversion of the investors, who prefer short term maturities and require a premium to engage in long term lending. The preferred habitat theory attributes the premium to the offer and demand.

Earlier term structure models explain interest rate behavior in terms of the dynamics of the short rate which refers to the interest rate for an infinitesimal period. An example for short rate is the zero coupon rate. One factor models assume the short rate to follow a statistical process and that all other interest rates are functions of the short rate.

Historical studies indicate that the changes in interest rates for different maturities are not perfectly correlated. Three main factors: parallel, twists and butterfly movements explain more than 95 % of the changes in the yield curve.

In modeling the short term interest rate, one of the challenges is the accommodation of all relevant features in a single model specification. Some of those features include persistence, fat tailed distributions, long run mean reversion, and the level dependence of volatility.

In this Chapter we discuss the long memory and persistence behavior of stochastic interest rates. Obviously, fractional Brownian motion is a natural candidate to model these phenomena. When $\frac{1}{2} < H < 1$, the FBM can model the persistence. Persistence in our context means that if the trend has been positive in the immediate past then there is a high probability that it will continue to rise. If the trend is negative, it will tend to continue to fall.

If $0 < H < \frac{1}{2}$, the FBM will however be anti-persistent. That is, if the trend has been positive in the immediate past there is a high probability that it will become negative. Conversely, if negative, it will tend to reverse to a positive direction.

4.1.1 Spot interest rates

The base of any interest rate model is the zero coupon bond rate, paying one unit of currency at the maturity T. Denote by $P(t,T)$ the price at date $t \in [0,T]$. We have $P(T,T) = 1$.

Definition 4.1.1

1. *We call instantaneous continuously compounded interest rate the quantity $R(t,T)$ defined as*
$$R(t,T) = \frac{-\ln(P(t,T))}{(T-t)}.$$
Sometimes $R(t,T)$ is also called zero coupon rate; it is the constant continuous rate over the period $[t,T]$ such that
$$P(t,T) = e^{-R(t,T)(T-t)}.$$

2. *We call simply compounded interest rate the quantity $L(t,T)$ defined by*
$$L(t,T) = \frac{1-P(t,T)}{P(t,T)(T-t)};$$
hence
$$P(t,T) = \frac{1}{1+L(t,T)(T-t)}.$$

4.1.2 Forward rates

Consider the present date t and two future dates S and T, $t < S < T$.

Definition 4.1.2

1. *The continuously compounded forward rate for the period $[S,T]$ is the rate $R_t(S,T)$ satisfying*
$$e^{R_t(S,T)(T-S)} P_t(T) = P_t(S) \qquad \forall t < S < T.$$

2. *The simply compounded forward rate for the period* $[S,T]$ *is the rate* $L_t(S,T)$ *satisfying*

$$\big(1 + L_t(S,T)(T-S)\big)P_t(T) = P_t(S) \qquad \forall t < S < T.$$

As the Libor rates are simply compounded, we deduce the forward Libor rate as

$$L(t,S,T) = \frac{P_t(S) - P_t(T)}{(T-S)P_t(T)}.$$

4.1.3 Default free bonds

Default free-bond is when the owner of the bond is assured at initiation to get the specified interest (coupon) and the principal when the bond expires. In contrast to corporate bonds, it is assumed that that bonds issued by sovereign governments of developed countries can be considered default free. They still nevertheless highly sensitive to fluctuations in the interest rates.

There are two types of bonds: Zero coupon bonds, also known as discount bonds or strips, and coupon bonds. A zero-coupon bond makes a single payment on its maturity date, while a coupon bond makes interest payments at regular dates up to and including the maturity date. A coupon bond may be regarded as a set of strips, with the payment at each coupon date and at maturity being equivalent to a zero-coupon bond maturing on that date.

Definition 4.1.3 *A zero coupon bond with maturity date* T *is a contract that guarantees to pay the bond holder* Z *units of currency at time* T*. The face value* Z *is usually substituted by* 1*. Let* $P(t,T)$ *denote the price at time* $t \leqslant T$ *of a default-free zero coupon bond with maturity date* T*.*

1. *For fixed* t*,* $P(t,T)$ *is a function of* T *which we assume to be differentiable with respect to* T*.*

2. *For fixed* T*,* $P(t,T)$ *turns to be a stochastic process.*

Under the assumption of differentiability of the bond prices with respect to the maturity date, the *instantaneous forward rate* $f_t(T)$ can be defined as

$$f(t,T) = -\frac{\partial \log[P(t,T)]}{\partial T}. \tag{4.1.1}$$

The *instantaneous short rate* $r(t)$ at time t is defined as

$$r(t) = f(t,t).$$

Some simple transformations of Equation (4.1.1) yield

$$P(t,T) = \exp\left[-\int_t^T f(t,s)ds\right]. \tag{4.1.2}$$

The instantaneous short rate $r(t)$ at time t is defined as

$$r(t) = f(t,t)$$

and gives the interest rate at which a person can borrow money for an infinitely small period of time starting from time t. It is reasonable to assume that $r(t)$ is stochastic. Then we define the stochastic discount factor

$$D(t,T) = \exp\left(-\int_t^T r_s ds\right).$$

Definition 4.1.4 (arbitrage-free bond) *A family $P(t,T)$, $t \leqslant T \leqslant T^*$, of adapted processes is called an arbitrage free family of bond prices relative to r if the following conditions hold:*

1. *$P(T,T)$= 1 for all $T \in [0,T^*]$*

2. *There exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbb{P}$ such that for all $t \in [0,T]$ the discounted bond price*
$$\tilde{P}(t,T) = D(0,t)P(t,T)$$
is a martingale under $\mathbb{P}^*$.

As $\tilde{P}(t,T)$ follows a martingale under $\mathcal{P}^*$, we have:

$$\tilde{P}(t,T) = \mathbb{E}^{\mathbb{P}^*}\left[\tilde{P}(T,T) \mid \mathcal{F}_t\right] \quad \text{for } t \leqslant T,$$

which leads to the following expression of the bond prices

$$P(t,T) = \mathbb{E}^{\mathbb{P}^*}\left[-\int_t^T r(s)ds \mid \mathcal{F}_t\right].$$

4.1.4 Swap rate

An interest rate swap is an agreement between two parties to exchange future interest rate cash flows at specific intervals, calculated on a notional principal amount, during the life of the contract (maturity). A plain Vanilla swap refers to an exchange of a fixed rate against a floating rate which is Libor in general. The value of an interest rate swap is the net difference between the discounted value of the cash flows the counterpart expects to make and the discounted value of the cash leg on which the flows are received.

At initiation a swap has a value equal to zero. When the fixed leg is paid and the floating leg is received, the interest rate swap is called "payer IRS" and in the other case "receiver IRS". Let $t = t_0 < t_1 < \ldots < t_n$ be the flow dates and $\delta_n = t_{n+1} - t_n$ adjusted to the basis convention*.

The present value (PV) at time t of borrowing one unit of cash at a fixed rate k with coupon paid at times $t_i, i = 1, \ldots, n$ and with $\delta_i = t_i - t_{i-1}$ is

$$PV(\textit{fixed leg}) = \sum_{i=1}^{n} P(t,t_i)\delta_i k + P(t,t_n).$$

and the present value at time $t = T_0$ of a stream of floating rate cash flows is

$$PV(\textit{floating leg}) = \sum_{i=1}^{n} P(t,t_i)\delta_i L(t_{i-1},t_i) + P(t,t_n).$$

Thus, the present value at time $t = t_0$ of a payer IRS is given by

$$PV(\textit{payer swap}) = \sum_{i=1}^{n} P(t_,t_i)\delta_i(L(t_{i-1},t_i) - k).$$

*The basis can be actually /360, 30/360 ...

Definition 4.1.5 (Swap rate) *The swap rate is the rate $S(t,t_i)$ that must be inserted for k in order to have*

$$PV(\textit{fixed leg}) = PV(\textit{floating leg}).$$

Taking into account that the present value of the floating rate is equal to 1, we can express the swap rate as

$$S(t,T_n) = \frac{1 - P(t,t_n)}{\sum_{i=1}^{n} \delta_i P(t,t_i)}.$$

4.2 Single factor short rate models

Original interest rate models describe the dynamics of the short rate. Later ones which model the whole term structure focus on the forward rate. Short rate models can be classified into equilibrium and no-arbitrage models. Equilibrium models start from a description of the economy and derive the term structure endogenously assuming that the market is at equilibrium. Arbitrage free models assume that there are no arbitrage opportunities on the market. The existence of a risk neutral measure $\mathbb{Q}$ implies that the arbitrage free price of a zero-coupon bond at time t with maturity T is given by the conditional expectation

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\right].$$

Vasicek (1977) [98] is the earliest and most famous general equilibrium short rate model. Two of the most important no-arbitrage short-rate models are the Hull-White model (1990) [54] and the Black-Karasinski (1991) model [14].

Single factor models assume that all the information about the term structure at any point in time can be summarized by one single specific factor. As a consequence only the short term interest rate $r(t)$ and the maturity T will be considered. Practically, when opting for a single factor model, we make the assumption that there is perfect correlation between movement in the rates at different maturities and that the shifts in the term structure are parallel.

In particular if $V(t)$ is the value at time t of an interest rate derivative with maturity T, we can write

$$V(t) \equiv V(t,T,r(t)).$$

For pricing interest rate contingent claims, the two known methodologies are the PDE approach and the martingale approach.

4.2.1 Partial differential equation for pricing interest rate derivatives

Consider a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ on which two independent Brownian motion $W(t)$ and $B(t)$ are defined. Assume that the dynamics of the short term rate is given by

$$dr(t) = \mu_r(t)dt + \sigma_{1r}(t)\,dW(t) + \sigma_{2r}(t)\,dB^K(t) \tag{4.2.1}$$

such that

- $B^K(t) = \int_0^t K(t,s)dB(s)$ where K (with respect to t) is absolutely continuous on $\mathbb{R}_+$ with a square integrable density K'. As a consequence we have the following decomposition for B^K,

$$B^K(t) = \int_0^t K(s,s)dB(s) + \int_0^t \int_0^u K'(u,s)dB(s)du$$
$$= \int_0^t K(s,s)dB(s) + A(t), \tag{4.2.2}$$

where we set $A(t) = \int_0^t \int_0^u K'(u,s)dB(s)du$. We observe that $A(t)$ is of bounded variations. Furthermore, the differential form of $A(t)$ is

$$dA(t) = A'(t)dt = \left(\int_0^t K'(t,s)dB(s) \right) dt.$$

- $\mu_r(t) = \mu_r(t, r(t))$ and $\sigma_{\cdot r}(t) = \sigma_{\cdot r}(t, r(t))$ are measurable Lipschitz continuous functions $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfying the growth conditions.

Equation (4.2.1) becomes

$$dr(t) = \mu_r(t)dt + \sigma_{1r}(t)\, dW(t) + \sigma_{2r}(t)K(t,t)dB(t) + \sigma_{2r}(t)dA(t), \tag{4.2.3}$$

where the integral $\int_0^t \sigma_{2r}(s)dA(s)$ is a Lebesgue type integral. The reason of taking the integral this way is that the application of Itô's formula is not obvious in the other cases. Of course, if $\sigma_{2r}(t)$ is deterministic and is not a function of $r(t)$, one could generalize the integral.

Itô's formula yields

$$dV(t,T,r(t)) = \frac{\partial V}{\partial r}dr(t) + \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial^2 r}dt + \frac{1}{2}\sigma_{2r}^2 K(t,t)^2 \frac{\partial^2 V}{\partial^2 r}dt$$
$$= \left[\frac{\partial V}{\partial t} + \mu_r(t)\frac{\partial V}{\partial r} + \sigma_{2r}(t)A'(t)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}\sigma_{2r}^2(t)\frac{\partial^2 V}{\partial^2 r} \right] dt$$
$$+ \left[\frac{\partial V}{\partial r}\sigma_{1r}(t) \right] dW(t) + \left[\frac{\partial V}{\partial r}\sigma_{2r}(t)K(t,t) \right] dB(t). \tag{4.2.4}$$

To obtain the instantaneous return on the contract V consider

$$\frac{dV(t)}{V(t)} = \frac{1}{V(t)} \left[\frac{\partial V}{\partial t}(t) + \mu_r(t)\frac{\partial V}{\partial r}(t) + \sigma_{2r}(t)A'(t)\frac{\partial V}{\partial r}(t) + \frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial r^2}(t) + \frac{1}{2}\sigma_{2r}^2(t)\frac{\partial^2 V}{\partial^2 r}(t) \right] dt$$
$$+ \frac{1}{V(t)} \left[\frac{\partial V}{\partial r}\sigma_{1r}(t)dW_t + \frac{\partial V}{\partial r}(t)\sigma_{2r}(t)K(t,t)dB_t \right]$$
$$= \mu_V(t)dt + \sigma_{1V}(t)dW_t + \sigma_{2V}(t)dB(t),$$

where we set

$$\mu_V(t) = \frac{1}{V(t)} \left[\frac{\partial V}{\partial t}(t) + \mu_r(t)\frac{\partial V}{\partial r}(t) + \sigma_{2r}(t)A'(t)\frac{\partial V}{\partial r}(t) + \frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial r^2}(t) + \frac{1}{2}\sigma_{2r}^2(t)\frac{\partial^2 V}{\partial^2 r}(t) \right],$$
$$\sigma_{1V}(t) = \frac{1}{V(t)}\sigma_{1r}\frac{\partial V}{\partial r}(t),$$
$$\sigma_{2V}(t) = \frac{1}{V(t)}\sigma_{2r}(t)\; K(t,t)\frac{\partial V}{\partial r}(t).$$

In order to avoid arbitrage opportunities, the expected return on the contract V must be equal to the risk free rate. This implies the existence of a market risk premium defined as the excess return over the risk free rate per unit of risk which is the volatility in this framework. Denote by $\lambda(t,r(t))$ this premium; it is defined such that

$$\lambda(t,r(t)) = \frac{\mu_V(t) - r(t)}{\sqrt{\sigma_{1V}^2(t) + \sigma_{2V}^2(t)}}. \tag{4.2.5}$$

The instantaneous return on the contract can be expressed as

$$\mu_V(t) = r(t) + \lambda(t,r(t))\sqrt{\sigma_{1V}^2(t) + \sigma_{2V}^2(t)}. \tag{4.2.6}$$

Replacing μ_{V_1}, σ_{1V} and σ_{2V} by their respective values leads to

$$\begin{aligned}
\mu_V(t) &= r(t) + \lambda(t,r(t))\sqrt{\sigma_{1V}^2(t) + \sigma_{2V}^2(t)} \\
&= r(t) + \lambda(t,r(t))\frac{1}{V}\frac{\partial V}{\partial r}\sqrt{\sigma_{1r}^2 + K(t,t)^2\sigma_{2r}^2} \\
&= \frac{1}{V}\left[\frac{\partial V}{\partial t} + \mu_r(t)\frac{\partial V}{\partial r} + \left(\int_0^t K'(t,s)dB_s\right)\sigma_{2r}\frac{\partial V}{\partial r} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2 V}{\partial r^2}\right].
\end{aligned}$$

The following PDE yields

$$\begin{aligned}
\frac{\partial V}{\partial t}(t) + \left(\mu_r(t) - \lambda(t,r(t))\sqrt{\sigma_{1r}^2 + K(t,t)^2\sigma_{2r}^2}(t) + \sigma_{2r}(t)\left(\int_0^t K'(t,s)dB_s\right)\right)\frac{\partial V}{\partial r}(t) \\
+ \frac{1}{2}\sigma_{1r}^2(t)\frac{\partial^2 V}{\partial r^2}(t) - r(t)V(t) = 0.
\end{aligned} \tag{4.2.7}$$

By conditioning on every path of B_t we can obtain a solution to this PDE via a finite difference scheme. The mean of these solutions will be the solution to (4.2.7).

4.2.2 Affine short rate models and the risk premium

An important class of short rate models is the class of affine term structure models where the continuously compounded zero rate $R(t,T)$ is an affine function of $r(t)$:

$$R(t,T) = a(t,T) + b(t,T)r(t),$$

and a, b are deterministic function of time. One can show that this is satisfied if the zero coupon bond price takes the form:

$$P(t,T) = D(t,T)e^{-A(t,T)r(t)}$$

where A and D deterministic functions. In general to specify completely an affine short rate model, one needs to choose a market price of risk. We will discuss some examples in the sequel.

The Vasicek Model

(Vasicek, 1977) [98] has been the first in the literature to propose a one factor affine short rate model. Here we propose to add a fractional noise to the original model. We assume that the dynamics under the historical probability $\mathbb{P}$ are such that

$$\begin{aligned} dr(t) &= \tilde{k}\left[\tilde{\theta} - r(t)\right]dt + \sigma_1 dW(t) + \sigma_2 dB^K(t) \\ &= \tilde{k}\left[(\tilde{\theta} + \frac{\sigma_2}{\tilde{k}}\int_0^t K'(t,s)dB_s) - r(t)\right]dt + \sigma_1 dW(t) + \sigma_2 K(t,t)dB(t), \end{aligned} \tag{4.2.8}$$

where W, B are independent Brownian motion and $\sigma_1, \sigma_2 > 0$ are constants. We see that the mean parameter $\tilde{\theta}$ which is constant in the classical case becomes stochastic in the mixed fractional case. Set

$$\tilde{\theta}^K(t) = \tilde{\theta} + \frac{\sigma_2}{\tilde{k}}\int_0^t K'(t,s)dB(s).$$

Then Equation (4.2.8) writes as

$$dr(t) = \tilde{k}\left[(\tilde{\theta}^K(t)) - r(t)\right]dt + \sigma_1 dW(t) + \sigma_2 K(t,t)dB(t). \tag{4.2.9}$$

The solution to Equation (4.2.8) is then

$$\begin{aligned} r(t) &= r(0)e^{-\tilde{k}t} + \tilde{k}\int_0^t \tilde{\theta}^K(s)e^{-\tilde{K}(t-s)}ds + \int_0^t \sigma_1 e^{-\tilde{k}(t-s)}dW(s) + \int_0^t \sigma_2 K(s,s)e^{-\tilde{k}(t-s)}dB(s) \\ &= r(0)e^{-\tilde{k}t} + \tilde{k}\int_0^t (\tilde{\theta} + \frac{1}{\tilde{k}}\int_0^s \sigma_2 K'(s,r)dB_r)e^{-\tilde{k}(t-s)}ds \\ &\quad + \int_0^t \sigma_1 e^{-\tilde{k}(t-s)}dW(s) + \int_0^t K(s,s)\sigma_2 e^{-\tilde{k}(t-s)}dB(s) \\ &= r(0)e^{-\tilde{k}t} + \tilde{\theta}\big(1 - e^{-\tilde{k}(t-s)}\big) + \int_0^t \sigma_1 e^{-\tilde{k}(t-s)}dW(s) + \int_0^t \sigma_2 e^{-\tilde{k}(t-s)}dB^K(s). \end{aligned}$$

Choose the market price of risk λ of the form

$$\lambda(t) = ar(t) - b - \sigma_2\int_0^t K'(t,s)dB(s),$$

and set

$$\int_0^t K(s,s)dB^{\mathbb{Q}}(s) = \int_0^t K(s,s)dB(s) - \int_0^t \lambda(s)ds.$$

Then (4.2.8) becomes

$$\begin{aligned} dr(t) &= \tilde{k}\left[\tilde{\theta} - r(t)\right]dt + \sigma_1 dW(t) + \sigma_2 K(t,t)dB^{\mathbb{Q}}(t) + ar(t)dt - bdt \\ &= (\tilde{k} - a)\left[\frac{\tilde{k}\tilde{\theta} - b}{\tilde{k} - a} - r(t)\right]dt + \sigma_1 dW(t) + \sigma_2 K(t,t)\,dB^{\mathbb{Q}}(t). \end{aligned}$$

Setting

$$k = \tilde{k} - a\sigma_1$$
$$\theta = \frac{\tilde{k}\tilde{\theta} - b}{\tilde{k} - a},$$

the model preserves its affine structure and has the following dynamics under the risk neutral measure $\mathbb{Q}$:

$$dr_t = k(\theta - r(t))\,dt + \sigma_1\,dW(t) + \sigma_2 K(t,t)dB^{\mathbb{Q}}(t). \tag{4.2.10}$$

The Cox-Ingersoll-Ross Model

The main drawback of the Vasicek model is that it does not exclude negative interest rates. (Cox-Ingersoll-Ross, 1985) [30] propose the following model under the historical measure $\mathbb{P}$:

$$dr(t) = \tilde{k}(\tilde{\theta} - r(t))dt + \sigma\sqrt{r(t)}dW(t), \tag{4.2.11}$$

with positive constants $\tilde{k}, \tilde{\theta}$ and σ positive. If we choose the market price of risk such that

$$\lambda(t) = a\sqrt{r_t} - \frac{b}{\sqrt{r(t)}},$$
$$W^{\mathbb{Q}}(t) = W(t) - \int_0^t \lambda(s)ds,$$
$$k = \tilde{k} + a\sigma,$$
$$\theta = \frac{\tilde{k}\tilde{\theta}}{\tilde{k} + a\sigma},$$

then the model preserves its affine structure and shows the following dynamics under the risk neutral measure $\mathbb{Q}$:

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW^{\mathbb{Q}}(t). \tag{4.2.12}$$

We propose to add an additional (Volterra) noise, so that the model then writes as

$$dr(t) = \tilde{k}(\tilde{\theta} - r(t))dt + \sigma_1\sqrt{r(t)}dW(t) + \sigma_2\sqrt{r(t)}dB^K(t). \tag{4.2.13}$$

Take

$$\lambda(t) = a\sqrt{r(t)} + \frac{b}{\sqrt{r(t)}} - \frac{\sigma_2}{\sqrt{r(t)}}\int_0^t K'(t,s)dB_s,$$

set

$$\int_0^t K(s,s)dB^{\mathbb{Q}}(s) = \int_0^t K(s,s)dB(s) - \int_0^t \lambda(s)ds,$$

and

$$k = \tilde{k} - a\sigma_1, \quad \theta = \frac{\tilde{k}\tilde{\theta} - b}{\tilde{k} - a}.$$

The dynamics of (4.2.13) under the measure $\mathbb{Q}$ becomes

$$dr(t) = k(\theta - r_t)\, dt + \sigma_1 \sqrt{r_t}\, dW(t) + \sigma_2 K(t,t) \sqrt{r(t)} dB^{\mathbb{Q}}(t). \tag{4.2.14}$$

Moreover, if $k\theta > 0$, the $\mathbb{Q}$-solution is strictly positive, see [55]. Taking into account the fact that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} > 0,$$

we can deduce that the $\mathbb{P}$-solution is also strictly positive.

4.3 Infinite dimensional stochastic interest rates, the HJM framework

Infinite dimensional interest rates consider the entire term structure of interest rates rather than a finite number of state variables. Ho and Lee [49] in 1986 were the first to model the whole term structure using a set of zero coupon bonds. Heath, Jarrow and Morton [46] in (1992) consider forward rates rather than bond prices. The model explains the whole term structure dynamics in an arbitrage free framework and is compatible with an equilibrium model. This model was later extended to deal with multiple factors.

For many reasons, it is desirable and sometimes necessary to use models which include more than one source of randomness. Simply by looking at historical interest rate data, we can see that changes in interest rates with different maturities are not perfectly correlated as one factor models predict. Moreover, for interest rate options which deal with more than one date spot rate, a one factor model would possibly overprice the contract.

The HJM multifactor generalization has been developed as

$$\begin{cases} df(t,T) = \mu_f(t,T)dt + \displaystyle\sum_{i=1}^{d} \sigma_{f,i}(t,T)dW_i(t), \\ f(0,T) = f^M(0,T), \end{cases} \tag{4.3.1}$$

where

- $\mu_f(t,T)$ is the drift of the forward rate with maturity T adapted to the filtration of $\{W_i,\ 1 \leqslant i \leqslant d\}$,
- $\sigma_{f_i}(t,T)$ are deterministic volatilities
- $W_i(t)$ are independent standard Brownian motions
- $f^M(0,T)$ is the forward rate with maturity T observed on the market.

In the sequel we propose a model with two sources of noise, one driven by a standard Brownian motion and the second by a Volterra process (with semimartingale kernel) independent of the Brownian motion:

$$\begin{cases} df(t,T) = \mu_f(t,T)\, dt + \sigma_{f,1}(t,T)\, dW(t) + \sigma_{f,2}(t,T)\, dB^K(t), \\ f(0,T) = f^M(0,T). \end{cases} \tag{4.3.2}$$

As explained above, part of the Volterra process can be assimilated to a stochastic trend, hence we do not need to attribute a risk premium to this part of the factor. It turns out that even if the model is driven by two source of noise, it can remain a one factor model if the martingale part of B^K is zero.

4.3.1 Absence of arbitrage opportunities

Recall that the dynamic of zero coupon bonds can be obtained from the forward rate as follows

$$P(t,T) = \exp\left(- \int_t^T f(t,u)du \right). \tag{4.3.3}$$

Proposition 4.3.1 *The price of a zero coupon bond $P(t,T)$ with maturity T solves the following SDE:*

$$\begin{aligned} dP(t,T) = P(t,T)\left[r(t) - \mu_f^*(t,T) + \frac{1}{2}\sigma_{f,1}^*(t,T)^2 + \frac{1}{2}\frac{\partial}{\partial t}\left(\int_t^T \left(K^* \sigma_{f,2}^*(t,s,T) \right)^2 ds \right) \right] dt \\ - P(t,T)\sigma_{f,1}^*(t,T)dW_t - P(t,T)\int_t^T K(t,t)\sigma_{f,2}(t,u)dudB(t) \end{aligned} \tag{4.3.4}$$

where

$$\begin{aligned} \mu_f^*(t,T) &= \int_t^T \mu_f(t,u)du + \int_t^T \int_0^t K'(t,s)\sigma_{f,2}(t,u)dudB_s \\ \sigma_{f,1}^*(t,T) &= \int_t^T \sigma_{f,1}(t,u)du \\ \sigma_{f,2}^*(t,s,T) &= \int_t^T \sigma_{f,2}(t,s,u)du. \end{aligned}$$

Proof. Set $F(t,T) = -\int_t^T f(t,u)du$. By the stochastic Fubini theorem (see [82] for example) and by differentiation, we can obtain:

$$\begin{aligned} dF(t,T) &= f(t,t)dt - \int_t^T \mu_f(t,u)dudt - \int_t^T \sigma_{f,1}(t,u)dudW(t) - \int_t^T \sigma_{f,2}(t,u)dudB^K(t) \\ &= r(t)dt - \int_t^T \mu_f(t,u)dudt - \int_t^T \sigma_{f,1}(t,u)dudW(t) - \int_t^T \sigma_{f,2}(t,u)dudB^K(t) \end{aligned}$$

The stochastic Fubini theorem then gives

$$\begin{aligned} dF(t,T) = r(t)dt - \int_t^T \mu_f(t,u)dudt - \int_t^T \int_0^t K'(t,s)\sigma_{f,2}(t,u)dudB_s dt \\ - \int_t^T \sigma_{f,1}(t,u)dudW(t) - \int_t^T K(t,t)\sigma_{f,2}(t,u)dudB(t). \end{aligned}$$

Applying the Itô's formula[†] to $P(t,T) = e^{F(t,T)}$ gives

$$\begin{aligned} dP(t,T) &= P(t,T)dF(t,T) + \frac{1}{2}P(t,T)\Big(\int_t^T \sigma_{1,f}(t,u)du\Big)^2 dt \\ &\quad + \frac{1}{2}P(t,T)\frac{\partial}{\partial t}\left(\int_0^t \left(K^* \int_t^T \sigma_{f,2}(s,u)du\right)^2 ds\right) dt \\ &= P(t,T)\left[r(t)dt - \int_t^T \mu_f(t,u)dudt - \int_t^T \int_0^t K'(t,s)\sigma_{f,2}(t,u)dudB_s dt\right. \\ &\quad \left. + \frac{1}{2}\Big(\int_t^T \sigma_{1,f}(t,u)du\Big)^2 dt + \frac{1}{2}\frac{\partial}{\partial t}\left(\int_0^t \left(K^* \int_t^T \sigma_{f,2}(s,u)du\right)^2 ds\right) dt\right] \\ &\quad - P(t,T)\left[\int_t^T \sigma_{f,1}(t,u)dudW(t) - \int_t^T K(t,t)\sigma_{f,2}(t,u)dudB(t)\right]. \qquad (4.3.5) \end{aligned}$$

■

Remark 4.3.2 *If $K(t,t)$ is constant then*

$$P(t,T)\int_t^T K(t,t)\sigma_{f,2}(t,u)dudB(t) = P(t,T)K(t,t)\sigma^*_{f,2}(t,T)dB(t).$$

On the other hand, since

$$df(t,T) = \mu_f(t,T)dt + \sigma_{f,1}(t,T)dW(t) + \sigma_{f,2}(t,T)dB^K(t),$$

we have

$$r(T) = \lim_{t\to T^-} f(t,T) = f(0,T) + \int_0^T \mu_f(s,T)ds \int_0^T \sigma_{f,1}(s,T)dW(s) + \int_0^T \sigma_{f,2}(s,T)dB^K(s).$$

Consider now a cash account $B(t)$ at time t with $B(0) = 1$, following the differential equation:

$$dB(t) = r(t)B(t)dt.$$

We obtain

$$\begin{aligned} B(t) &= \exp\left[\int_0^t r(u)du\right] \\ &= \exp\left[\int_0^t f(0,u)du + \int_0^t \int_s^t \mu_f(s,u)duds\right. \\ &\quad \left. + \int_0^t \int_s^t \sigma_{f,1}(s,u)dudW(s) + \int_0^t \int_s^t \sigma_{f,2}(s,u)dudB^K(s)\right]. \qquad (4.3.6) \end{aligned}$$

Proposition 4.3.3 *The HJM model is arbitrage free, if there exist an adapted process $\{\lambda(t) : t \in [0,T]\}$ such that:*

$$\begin{aligned} -K(t,t)\sigma_{f,2}(t,T)\lambda(t) &= -\mu_f(t,T) - \int_0^t K'(t,s)\sigma_{f,2}(t,T)dB_s \\ &\quad + \sigma_{1,f}(t,T)\sigma^*_{1,f}(t,T) + \frac{1}{2}\frac{\partial}{\partial T}\frac{\partial}{\partial t}\left(\int_0^t \left(K^* \int_t^T \sigma_{f,2}(s,T)\right)^2 ds\right) dt. \end{aligned}$$

[†]see [2] for the divergence integral with respect to u

Proof. From Equation (4.1.2) we have

$$
\begin{aligned}
P(t,T) &= \exp\left[-\int_t^T f(t,u)du\right] \\
&= \exp\left[-\int_t^T f(0,u)du - \int_0^t\int_t^T \mu_f(s,u)duds \right. \\
&\quad \left. -\int_0^t\left(\int_t^T \sigma_{f,1}(s,u)du\right)dW(s) - \int_0^t\left(\int_t^T \sigma_{f,2}(s,u)du\right)dB^K(s)\right].
\end{aligned}
$$

Define the discounted asset

$$
\begin{aligned}
Z(t) &= \frac{P(t,T)}{B(t)} \\
&= \exp\left[-\int_0^T f(0,u)du - \int_0^t\int_s^T \mu_f(s,u)duds \right. \\
&\quad \left. -\int_0^t\left(\int_s^T \sigma_{f,1}(s,u)du\right)dW(s) - \int_0^t\left(\int_s^T \sigma_{f,2}(s,u)du\right)dB^K(s)\right]. \qquad (4.3.7)
\end{aligned}
$$

Itô's formula yields

$$
\begin{aligned}
dZ(t) \;=\; Z(t)&\left[-\int_t^T \mu_f(t,u)dudt - \int_t^T\int_0^t K'(t,s)\sigma_{f,2}(t,u)dudB_sdt \right. \\
&+\frac{1}{2}\left(\int_t^T \sigma_{1,f}(t,u)du\right)^2 dt + \frac{1}{2}\frac{\partial}{\partial t}\left(\int_0^t\left(K^*\int_t^T \sigma_{f,2}(s,u)du\right)^2 ds\right)dt \\
&\left. -\int_t^T \sigma_{f,1}(t,u)dudW(t) - \int_t^T K(t,t)\sigma_{f,2}(t,u)dudB(t)\right]. \qquad (4.3.8)
\end{aligned}
$$

Let $\lambda(t)$ be such that

$$
\begin{aligned}
-\int_t^T K(t,t)\sigma_{f,2}(t,u)du\;[dB(t)+\lambda(t)dt] = &-\int_t^T \mu_f(t,u)dudt \\
&-\int_t^T\int_0^t K'(t,s)\sigma_{f,2}(t,u)dudB(s)dt \\
&+\frac{1}{2}\left(\int_t^T \sigma_{1,f}(t,u)du\right)^2 dt \\
&+\frac{1}{2}\frac{\partial}{\partial t}\left(\int_0^t\left(K^*\int_t^T \sigma_{f,2}(s,u)du\right)^2 ds\right)dt \\
&-\int_t^T K(t,t)\sigma_{f,2}(t,u)du\,dB(t). \qquad (4.3.9)
\end{aligned}
$$

By the Girsanov theorem, there exists a probability measure $\mathbb{Q}$ under which the process $B^{\mathbb{Q}}(t) = B(t)+\lambda(t)$ is a $\mathbb{Q}$-Brownian motion. Then we have that

$$
dZ(t) = Z(t)\left[-\int_t^T \sigma_{f,1}(t,u)dudW(t) - \int_t^T K(t,t)\sigma_{f,2}(t,u)dudB^{\mathbb{Q}}(t)\right] \qquad (4.3.10)
$$

is a martingale under $\mathbb{Q}$. From Equation (4.3.9) we can deduce by derivation with respect to T:

$$
\begin{aligned}
-K(t,t)\sigma_{f,2}(t,T)\lambda(t) = & -\mu_f(t,T) - \int_0^t K'(t,s)\sigma_{f,2}(t,T)dB_s \\
& + \sigma_{1,f}(t,T)\sigma_{1,f}^*(t,T) + \frac{1}{2}\frac{\partial}{\partial T}\frac{\partial}{\partial t}\left(\int_0^t \left(K^* \int_t^T \sigma_{f,2}(s,T)\right)^2 ds\right) dt
\end{aligned}
$$

which yields the proof of the proposition. ■

Chapter 5

Long memory with credit derivative pricing

5.1 Introduction

A credit derivative is a derivative security that has a payoff which is conditioned on the occurrence of a credit event. The credit event is defined with respect to a reference entity and the reference credit assets issued by the reference entity. If the credit event occurs, the default payment has to be made by one of the counterparties.

In another formulation, credit derivatives may be seen as derivative securities whose payoff depends on the credit quality of a certain issuer. This credit quality can be measured by the credit rating of the issuer or by the yield spread of his bonds over the yield of a comparable default-free bond.

Participants in the market for credit derivatives can be divided into five major groups. Banks form the largest group. The second largest group consists of insurances and re insurances. Other groups are hedge funds and investment funds as well as industrials of different branches.

Purposes for using credit derivatives are for credit risk management as hedging counterparty or country risk and as a funding opportunity for banks through the securitisation of loan portfolios or for portfolio optimization for bond and loan portfolio managers.

The quantification of default risk is one of the most challenging problems related to credit risk management and credit derivatives. Up to now, we don't have a tool at hand which is good enough to give a satisfying estimation of the default probability of the different actors on the market. However, investors rely on mathematical models developed during the past decades.

Credit risk models are often classified as structural or reduced form models. Structural approach (sometimes called the firm's value model) was first introduced by Black and Scholes (1973) [13] and uses the evolution of the firm's structural variables such as asset and debt values to determine the time of default. Defaults occur as soon as the firm's asset value falls below a certain threshold. In Black and Cox (1976) [12] this approach is extended to allow defaults before maturity of the debt if the value of the firm hits a lower boundary. The probability of the first passage time is useful in this approach.

Reduced form models consider the case that the default occurs in an unexpected way, i.e. by an inaccessible stopping time. The first models of this type were developed by Jarrow and Turnbull (1995) [57], Madan and Unal (1998) [96] and Due and Singleton (1997) [36]. Jarrow and Turnbull consider the simplest case where the default is driven by a Poisson process with constant intensity with known payoff at default. This is changed in the Madan and Unal model where the intensity of the default is driven by an underlying stochastic process that is interpreted as a firm's value process, and the payoff at default is a random variable drawn at default, it is not predictable before default. Madan and Unal estimate the parameters of their process using rates for certificates of deposit in the Savings and Loan Industry. Lando (1998)[60] developed the Cox process methodology with the iterated conditional expectations. His model has a default payoff in terms a certain number of default-free bonds and he applies his results to a Markov chain model of credit ratings transitions.

In the literature the reduced form framework has been split into two different approaches. In the "intensity based approach" a unique flow of information is considered and the credit event is a stopping time with respect to the filtration containing the whole information of the financial market. The former refer to "hazard process approach" and relies on the introduction of a reference filtration enlarged by the progressive knowledge of the credit event.

In the following we will introduce some credit sensitivities with values depending on the credit risk of the counterparties.

5.1.1 Defaultable bonds and credit spreads

For corporates and countries there is no guarantee that the bond-holders can get all their face value at the maturity date. Therefore, investors would be unwilling to pay as much as for undefaultable bonds, hence the price of a defaultable bonds is lower than a default-free bond. This difference is to compensate the investor for the credit risk of the issuer. Three main factors affecting the credit risk are the default rate, the recovery rate and the exposure at default.

The pricing of defaultable bonds is more complex because one has to take into account the possibility that the issuer may default. Assume that we are pricing under a risk neutral measure $\mathbb{Q}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \mathcal{F}_{t\in[0,T]})$. Consider the stopping time τ such that the event $\{\tau \leqslant t\}$ belongs to the σ-field $\mathcal{F}_t$, for every $t \geqslant 0$.

Assume that the time of default for a company is given by the stopping time τ. Consider further a defaultable zero-coupon bond with face value 1 unit of currency at maturity T. The structure of the payout is

$$\text{Payout}(T) = \begin{cases} 1 & \text{if } \tau > T \\ R & \text{if } \tau \leqslant T \end{cases} \tag{5.1.1}$$

where R is the recovery rate*, that is the amount that the investor (the bond holder) will receive if the company defaults before time T.

Under the non arbitrage conditions the price of the defaultable bond at time t with maturity T is

$$\begin{aligned} \bar{P}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) \text{Payout}(T)|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) [1_{\tau>T} + R 1_{t\leqslant\tau\leqslant T}]|\mathcal{F}_t\right] \\ &= P(t,T)\mathbb{E}^{\mathbb{Q}}\left[1_{\tau>T}|\mathcal{F}_t\right] + R\ P(t,T)\mathbb{E}^{\mathbb{Q}}\left[1_{0\leqslant\tau\leqslant T}|\mathcal{F}_t\right] \\ &= P(t,T)\mathbb{Q}\left[\tau > T\right] + R\ P(t,T)\mathbb{Q}\left[0 \leqslant \tau \leqslant T\right]. \end{aligned}$$

*typically R is between 0 and 40 percent

Define the "default risky short rate" $\bar{r}_s$ by

$$\bar{P}(t,T) = \exp\left(-\int_t^T \bar{r}(s)ds\right).$$

The *credit spread* is defined as the positive difference $\lambda = \bar{r} - r$, where r is the risk free rate. The yield spread is defined as

$$\mathrm{YS}(t,T) = \frac{1}{T-t}\int_t^T (\bar{r}_s - r_s)ds = \frac{1}{T-t}\ln\left(\frac{P(t,T)}{\bar{P}(t,T)}\right).$$

5.1.2 Credit Default Swap

In a standard Credit Default Swap (CDS) contract, a party A buys an insurance from B against default of a third party C. A default occurs if party C fails to make a required payment on his debt. The protection buyer A pays the protection seller B a fee (premium) $s^\dagger$ at a fixed intervals (quarterly) until maturity if no default happens, or until default. This premium is called the CDS spread. In the last situation, i.e. a default of the counterparty C, the counterparty B has to make a default payment, typically equal to $(1-R)$ times the notional value N of the contract, where R is the recovery rate which we assume constant in our study.

Denote by T the maturity of the contract and consider $\{t_1 < t_2 < \ldots < t_k\}$ the premium payment dates. Let $t = t_0$ be the actual date $(t_0 < t_1)$ and let $t_k = T$. Let τ represent the default time. Suppose that no fee is payed after default. The value of the fixed premium leg payed by A to B is:

$$L_{\text{prem}}(t) = \sum_{i=1}^{k} s \times N \times (t_i - t_{i-1}) \times D(t,t_i) \times 1_{\{\tau > t_i\}}, \tag{5.1.2}$$

where D is the discount factor. The value of the floating leg or protection leg payed by B in case of default is

$$L_{\text{pro}}(t) = N \times \mathrm{LGD} \times D(t,\tau) \times 1_{\{\tau \leqslant T\}}, \tag{5.1.3}$$

where LGD is the loss given default. To price this contract we should take the risk neutral expectation of the cash flows:

$$\begin{aligned} P(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[L_{\text{prem}}(t) - L_{\text{pro}}(t)\right] \\ &= \sum_{i=1}^{k} s \times N \times (t_i - t_{i-1}) \times D(t,t_i) \times \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau > t_i\}}\right] - N \times \mathrm{LGD} \times D(t,\tau) \times \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau \leqslant T\}}\right]. \end{aligned}$$

At initiation the value of the contract should equal zero. We can then deduce the value of the CDS spread as

$$s = \frac{\mathrm{LGD} \times D(t,\tau) \times \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau \leqslant T\}}\right]}{\sum_{i=1}^{k}(t_i - t_{i-1}) \times D(t,t_i) \times \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau > t_i\}}\right]}.$$

We see that the difficulty of pricing such a contract as well for all credit derivatives is to estimate the values $\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau \leqslant T\}}\right]$ and $\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau > t_i\}}\right]$. We will give an answer to that in the next sections.

† in basis point of the notional

5.1.3 Survival and default probability

Theoretically, the probability of survival of defaultable bond issuer could be implied from the market using the relationship,

$$S(t,T) = \frac{\bar{P}(t,T)}{P(t,T)},$$

where $S(t,T)$ is the survival probability, $\bar{P}$ is price of a defaultable bond and P the price of a default free bond.

The term structure of the survival probability verifies the following properties:

1. $S(t,T)$ is a function of T and could be estimated from defaultable bond spread on a continuum set of maturities.
2. $S(t,T)$ is a non-negative and decreasing function of T.
3. $S(t,T)$ is increasing in t.
4. We have $S(t,+\infty) = 0$.

From this we can derive that the probability of default over the time interval $[t,T]$ is

$$\mathrm{PD}(t,T) = 1 - S(t,T).$$

The hazard rate $\lambda(t)$ is the probability that the default occurs in a small interval dt given that the default did not occur before time t:

$$\lambda(t) = \lim_{h\to 0} \frac{1}{h} P(\tau \leqslant t+h|\tau > t). \tag{5.1.4}$$

5.2 Credit risk models

5.2.1 Structural models for credit risk, firm's value models

Structural models assume that a default event occurs for a firm when its assets reach a sufficient low level relatively to its liabilities. A popular example is the Merton model [70]. The Merton model assumes that the total value of the assets of the firm A_t follows a geometric Brownian motion,

$$dA(t) = \mu A(t)dt + \sigma A(t)dW(t), \quad A_0 > 0, \tag{5.2.1}$$

where

- μ is the mean rate of return on the assets
- σ is the volatility.

The model assumes that debt consists of a single outstanding bond with face value D and maturity T. The default event occurs for a firm at maturity if the total value of the assets is less than the value of the debt. If no default happens, then the shareholder will receive a cash flow at T equal to $A_T - D$, so that the equity E can be viewed as a European call option on the firm's asset value.

Define on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ two independent Brownian motions $(W_t)_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$. Consider a fractional Volterra process

$$B^K(t) = \int_0^t K(t,s)dB(s), \tag{5.2.2}$$

where K is a kernel verifying the conditions of Proposition 1.3.2. We propose to modify the model in order to include memory in the value of the assets firms. The model becomes

$$dA(t) = \mu A(t)dt + \sigma_1 A(t)dW(t) + \sigma_2 A(t)dB^K(t), \quad A_0 > 0. \tag{5.2.3}$$

The solution to this equation can be obtained by application of the Itô formula for Gaussian processes of Chapter 1, Section 1.3.2,

$$A(t) = A_0 \exp\left(\mu t + \sigma_1 W(t) + \sigma_2 B^K(t) - \frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 \int_0^t K(t,s)^2 ds\right). \tag{5.2.4}$$

Risk neutral and historical default probability

From (5.2.4) we deduce the historical default probability,

$$\begin{aligned}\mathbb{P}(A(T) < D) &= \mathbb{P}\left(A_0 \exp\left(\mu T + \sigma_1 W(T) + \sigma_2 B^K(T) - \frac{1}{2}\sigma_1^2 T - \frac{1}{2}\sigma_2^2 \int_0^T K(T,s)^2 ds\right) < D\right) \\ &= \mathbb{P}\left(\sigma_1 W(T) + \sigma_2 B^K(T) < \ln(\frac{D}{A_0}) - \mu T + \frac{1}{2}\sigma_1^2 T + \frac{1}{2}\sigma_2^2 \int_0^T K(t,s)^2 ds\right).\end{aligned}$$

Since $W(T)$, $B^K(T)$ are independent and since $\frac{\sigma_1 W(T) + \sigma_2 B^K(T)}{\sqrt{\sigma_1^2 T + \sigma_2^2 \int_0^T K(T,s)^2 ds}}$ is a standard normal random variable, we have

$$\mathbb{P}(A(T) < D) = \mathcal{N}\left(\frac{\ln(\frac{D}{A_0}) - \mu T + \frac{1}{2}\sigma_1^2 T + \frac{1}{2}\sigma_2^2 \int_0^T K(t,s)^2 ds}{\sqrt{\sigma_1^2 T + \sigma_2^2 \int_0^T K(T,s)^2 ds}}\right)$$

where

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{1}{2}u^2\right) du.$$

On the other hand, following the results of Chapter 3, the risk neutral default probability is given by

$$\mathbb{Q}(A(T) < D) = \mathcal{N}\left(\frac{\ln(\frac{D}{A_0}) - rT + \frac{1}{2}\sigma_1^2 T + \frac{1}{2}\sigma_2^2 K(T,T)^2}{\sqrt{T(\sigma_1^2 + \sigma_2^2 K(T,T)^2)}}\right). \tag{5.2.5}$$

The value of equity, debt and yield spread

- The value of equity can be obtained as a European call option on the firm assets,

$$\begin{aligned}E(t) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(A(T) - D)^+\right] = \text{BSCall}\left(A(t), D, r, \sqrt{\sigma_1^2 + K(t,t)^2\sigma_2^2}, T - t\right) \\ &= A(t)\mathcal{N}(d_1) - e^{-r(T-t)} D\mathcal{N}(d_2),\end{aligned} \tag{5.2.6}$$

where

$$d_1 = \frac{\ln\left(\frac{A(t)}{D}\right) + (r + \frac{\sigma_1^2 + K(t,t)^2\sigma_2^2}{2})(T-t)}{\sqrt{(\sigma_1^2 + K(t,t)^2\sigma_2^2)(T-t)}}, \qquad d_2 = \frac{\ln\left(\frac{A(t)}{D}\right) + (r - \frac{\sigma_1^2 + K(t,t)^2\sigma_2^2}{2})(T-t)}{\sqrt{(\sigma_1^2 + K(t,t)^2\sigma_2^2)(T-t)}}.$$

- Bond holders receive $\min(D, A(T)) = A(T) - (A(T) - D)^+ = D - (D - A(T))^+$. Therefore the value of the debt $D_t^{\mathbb{Q}}$ at time t is equal to the value of a zero coupon bond minus a European put option,

$$D^{\mathbb{Q}}(t) = e^{-r(T-t)}D - \left(-A(t)\mathcal{N}(-d_1) + De^{-r(T-t)}\mathcal{N}(-d_2)\right) \tag{5.2.7}$$

- A zero coupon defaultable bond with face value 1 and maturity T will have the price

$$\bar{P}(t,T) = \frac{D^{\mathbb{Q}}(t)}{D}, \tag{5.2.8}$$

and has a yield spread

$$\begin{aligned} \text{YS}(t,T) &= \frac{1}{T-t}\ln\left(\frac{P(t,T)}{\bar{P}(t,T)}\right) = -\frac{1}{T-t}\ln\left(\frac{De^{-r(T-t)}}{D^{\mathbb{Q}}(t)}\right) \\ &= -\frac{1}{T-t}\ln\left(e^{r(T-t)}\frac{A(t)}{D}(1-\mathcal{N}(d_1)) + \mathcal{N}(d_2)\right). \end{aligned}$$

5.2.2 Reduced form models

Reduced form models also known as the intensity based models or hazard rate models were studied extensively over the past years, see for example [36, 57, 56, 62, 11, 89, 60]. Reduced form models do not consider the relation between default and firm's value in an explicit manner as the structural models. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data. The firm goes into default whenever the exogenous random variable shifts unexpectedly.

The question of long memory in corporate credit spread was discussed in McCarthy et al. [58]. The authors investigate the long memory property of corporate bond yield spreads. Their results showed that there is strong evidence of persistence in corporate bonds yield spreads and in the spread between corporate bond yields and long term treasury bond yields. An explication for this fact is the persistence of risk premiums. Rating agencies do not frequently adjust bond ratings which results in a persistence of risk premiums. They conclude that, with long memory, time patterns should be accounted for investigating decisions and forecasting models.

In our approach we propose to model the hazard rate inferred from the yield spread of defaultable bonds by a fractional Cox process exhibiting long memory.

5.2.3 Filtration and information sets

Fix a filtered probability space $(\Omega, (\mathcal{H}_t)_{t\geqslant 0}, \mathbb{P})$ and denote by $(\mathcal{G}_t)_{t\geqslant 0}$ the information available to the investor at time t. A *stopping time* with respect to the filtration $(\mathcal{H}_t)_{t\geqslant 0}$ is a random variable τ valued in $[0,\infty]$ such that the event $\{\tau \leqslant t\}$ belongs to $\mathcal{H}_t$ for all $t \geqslant 0$. Then τ is called predictable if there is an increasing sequence of stopping time (T_n) such that $\tau > T_n$ and $\lim_{n\to\infty} T_n = \tau$ a.s. The stopping time τ is called *totally inaccessible* if $P(\tau = T < \infty) = 0$ for all predictable times T.

In intensity based models, the default time τ is a stopping time in a filtration $\mathcal{H}$. The process $(H(t) = \mathbf{1}_{\tau\leqslant t},\ t \geqslant 0)$ is a $\mathcal{H}$-adapted increasing càdlàg process, hence a $\mathcal{H}$-submartingale, and the Doob-Meyer decomposition theorem states that there exists a unique $\mathcal{H}$-predictable increasing process Λ called the compensator such that the process $M(t) = H(t) - \Lambda(t)$ is a $\mathcal{H}$-martingale. If Λ is absolutely continuous with respect to the Lebesgue measure, i.e. if there exists a bounded progressively measurable process $(\lambda(t))_{t\geqslant 0}$ such that

$$\Lambda(t) = \int_0^t \lambda(s)ds,$$

we say that τ admits the intensity λ. If the intensity is predictable, it is essentially unique, see Brémaud (1981) [16].

In the hazard process approach, we assume that there is a market filtration $(\mathcal{G}_t)_{t\geqslant 0}$ containing all information other than default or survival. We can interpret the filtration $\mathcal{G}_t$ as the default free information generated by default free assets. We suppose the existence of an $(\mathcal{H}_t)$ adapted process $\Lambda_t = \int_0^t \lambda(s)ds$ such that

$$H_t - \int_0^{t\wedge\tau} \lambda(s))ds$$

is a martingale under the full filtration $\mathcal{F}_t := \mathcal{H}_t \vee \mathcal{G}_t$.

5.2.4 Credit spread and default probability

Implied default probabilities can be inferred from the credit spread on corporate bonds or from the CDS spread. For the difference between credit spread from corporate bonds and CDS spread, one may see [53]. In general, the probability of default is calculated as:

$$\mathrm{PD} = \frac{\mathrm{s}}{\mathrm{LGD}} = \frac{\mathrm{s}}{1 - \mathrm{RR}}, \tag{5.2.9}$$

where

- s is the CDS spread
- PD is the probability of default
- LGD is the loss given default (%)
- RR is the recovery rate (%).

Implied default probabilities inferred from corporate bonds or from the CDS spread are risk neutral default probabilities. Default probability inferred from historical data[‡] are historical probabilities. In general, we explain any excess return on corporate bond as the difference between risk neutral probability and real probability.

[‡]Given for instance by rating agency such as Moody's or Standard and Poor's

Credit spreads require a model that prevents negative values. On the other hand, there is an empirical mean reversion in credit quality. This means that good credit quality firms tend to deteriorate and vice versa. A model that fits this requirements is:

$$d\lambda(t) = \theta(\alpha - \lambda(t))dt + \sigma(t)\sqrt{\lambda(t)}dW(t) \tag{5.2.10}$$

where $\lambda(t)$ is the intensity or hazard rate of default. Taking into account the long memory behavior of credit spreads as suggested by [58], we can propose fractional CIR model for the intensity

$$d\lambda(t) = \theta(\alpha - \lambda(t))dt + \sigma_1(t)\sqrt{\lambda(t)}dW(t) + \sigma_2(t)\sqrt{\lambda(t)}dB^K(t), \tag{5.2.11}$$

or the fractional exponential Vasicek model

$$d\lambda(t) = \lambda(t)(\theta - \kappa \ln \lambda(t))dt + \sigma_1(t)\lambda(t)dW(t) + \sigma_2(t)\lambda(t)dB^K(t), \tag{5.2.12}$$

where $\lambda(t)$ is the intensity of the default process H.

5.3 Pricing options with default risk

In Chapter 3 we have seen how to price options under the assumption that the asset prices $S(t)$ follow a fractional Black-Scholes model, but we didn't take into consideration the case that a firm may default. Previously we got the price of options by evaluation under the risk neutral measure. Then we could compute the risk premia or the excess return over the risk free rate by unit of volatility. Thus the assumption was to attribute excess return to volatility. But of course part of the excess return observed on the market is due to the probability of default of the firm. Investors will require a premium for bearing more risk which refers here to the risk of default. We will consider just a Brownian motion for the volatility part, the former case was already investigated in Chapter 3. To complete the model, we suppose the following

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) - S(t)(\text{LGD})dN_t, \tag{5.3.1}$$

where $N(t)$ is a Cox process with historical intensity process $\lambda^{\mathbb{P}}(t)$ as defined in Equation (5.2.11) for example. Suppose that there exists an $\mathcal{F}_t$-adapted process $\gamma(t)$ such that

$$e^{\gamma(t)} = \frac{\lambda^{\mathbb{P}}(t)}{\lambda^{\mathbb{Q}}(t)}.$$

Set

$$X(t) = \int_0^t (\mu(s) + \lambda^{\mathbb{Q}}(s) - r)ds + \int_0^t \gamma(s)dN(s) - \int_0^t F(s)ds$$

such that

$$\mu(s) + (\text{LGD})\lambda^{\mathbb{Q}}(s) - r + \frac{1}{2}(\text{LGD})(e^{\gamma(s)} - 1)\lambda^{\mathbb{Q}}(s) = 0.$$

Define $\left.\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right| = e^{X(t)}$. Then the Girsanov theorem of Section 1.1.5 yields that the dynamics of $S(t)$ under $\mathbb{Q}$ are as follows

$$S(t) = rS(t)\,dt + \sigma S(t)\,dW^{\mathbb{Q}}(t) - S(t)(\text{LGD})\,d\tilde{N}_t^{\mathbb{Q}}.$$

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Abstract. The purpose of this work is the analysis of financial models, especially for option pricing, interest rates and credit risk, with stochastic processes having memory and eventually discontinuities, characteristics which can be observed frequently in the statistical evaluation of financial data. Fractional Brownian motion seems to be a natural tool for modeling continuous phenomena with memory. However, contrary to the classical models which are usually formulated in terms of Brownian motion or Lévy processes and analysed with the Itô stochastic calculus, the models with fractional Brownian motion require a different approach and more advanced methods of analysis. Moreover, questions of pricing under non-arbitrage conditions with a fractional Brownian motion cannot be treated by the fundamental theorem of asset pricing which holds true only in the class of semimartingales. First, stochastic calculus for classes of jump processes with memory in the jump times is studied. Fractional (or filtered) Lévy processes are considered with different regularity assumptions on the kernel, especially with the assumption which implies that fractional Lévy processes are semimartingales. An Itô formula is proven and a chaos decomposition is considered. Filtered doubly stochastic Lévy processes are investigated as well. Then financial models formulated with these processes are considered. Risk neutral probability measures are studied for fractional Black-Scholes models with jumps, which are modeled by filtered doubly stochastic Poisson processes. Stochastic interest rates are modeled in terms of stochastic differential equations driven by (continuous) mixed processes, in particular the Vasicek, the Cox-Ingersoll-Ross and the Heath-Jarrow-Morton models. It is shown that arbitrage free pricing of interest rate derivatives can be carried out by solving random partial differential equations. Finally credit risk models are studied.

Key words: fractional Brownian motion, semimartingale kernel, fractional Lévy process, filtered doubly stochastic Lévy process, Itô formula, Clark-Ocone formula, fractional Black-Scholes model with jumps, interest rate models, Vasicek, Cox-Ingersoll-Ross, Heath-Jarrow-Morton, credit risk models

Résumé. Ce travail étudie des modèles financiers pour les prix d'options, les taux d'intérêts et le risque de crédit, dirigés par des processus stochastiques à mémoire et avec discontinuités, des propriétés qu'on observe fréquemment dans les données financières. Le mouvement brownien fractionnaire est un outil naturel pour modéliser la mémoire dans le cadre des processus continus, mais l'analyse stochastique ne peut en général pas se faire par le calcul d'Itô qui est appliqué aux modèles classiques formulés en termes du mouvement brownien ou des processus de Lévy; le mouvement brownien fractionnaire exige des méthodes d'analyse plus avancées. En plus, le théorème fondamental de la finance sur le pricing sous des conditions d'absence d'arbitrage est valable pour les semimartingales et ne s'applique donc pas au brownien fractionnaire. Dans ce travail nous étudions d'abord le calcul stochastique pour des classes de processus avec mémoire dans le temps des sauts. Nous considérons les processus de Lévy filtrés (ou fractionnaires) sous différentes hypothèses sur la régularité des noyaux, en particulier sous l'hypothèse qui implique que le processus de Lévy filtré est une semimartingale, et nous démontrons une formule d'Itô et une formule de Clark-Ocone. Les processus de Lévy filtrés doublement stochastique sont également étudiés. Dans la suite nous nous intéressons aux probabilités risque neutre pour le modèle de Black - Scholes avec sauts modélisés par un processus de Poisson filtré doublement stochastique. Nous étudions également des modèles de taux d'intérêts en termes d'équations différentielles stochastiques, en particulier les modèles de Vasicek, de Cox-Ingersoll-Ross et de Heath-Jarrow-Morton. Nous montrons que les prix des options sur des taux d'intérêt peuvent être déduits d'équations aux dérivées partielles aléatoires. Finalement nous étudions le risque de crédit.

Mots clés: mouvement Brownien fractionnaire, noyau de semimartingale, processus de Lévy fractionnaire, process de Lévy filtré doublement stochastique, formule d'Itô, formule de Clark-Ocone, modèle de Black-Scholes fractionnaire avec sauts, modèles de taux d'intérêt, Vasicek, Cox-Ingersoll-Ross, Heath-Jarrow-Morton, modèles du risque de crédit