Controle de Equações Dispersivas para as Ondas de Superfície
Contrôle d'equations Dispersives pour les Ondes de Surface

Thèse présentée par

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Résumé

Dans cette thèse, nous prouvons des résultats concernant le contrôle et la stabilisation d'équations dispersives étudiées sur un intervalle borné. Pour commencer, nous étudions la stabilisation interne du système de Gear-Grimshaw, qui est un système de deux équations de Korteweg-de Vries (KdV) couplées. Nous obtenons une décroissance exponentielle de l'énergie totale associée au modèle en introduisant une fonction de Lyapunov convenable. Ensuite, nous montrons des résultats de contrôlabilité à zéro et exacte pour l'équation de Korteweg-de Vries avec un contrôle distribué sur un sous-ensemble du domaine. Lorsque la région de contrôle est un sous-intervalle arbitraire, nous prouvons la contrôlabilité à zéro au moyen d'une nouvelle inégalité de Carleman. Nous en déduisons un résultat de contrôlabilité régionale, avec la fonction d'é tat commandée sur la partie gauche du complément de la région de contrôle. Par ailleurs, lorsque la région de contrôle est un voisinage de l'extrémité droite du domaine, nous obtenons également un résultat de contrôlabilité exacte dans un espace L^2 à poids. Enfin, dans la lignée du résultat de contrôlabilité au bord obtenu par L. Rosier pour KdV, nous prouvons que le système linéaire de Boussinesq de type KdV–KdV est exactement contrôlable lorsqu'au plus deux contrôles sont appliqués au bord. Notre méthode repose sur l'utilisation de multiplicateurs et l'approche de la dualité mentionnée ci-dessus. Lorsqu'un mécanisme d'amortissement est introduit au bord, nous montrons que le système non linéaire est aussi exactement contrôlable et que l'énergie associée au modèle décroit exponentiellement.

Mots-clés: Stabilisation, fonction de Lyapunov, contrôlabilité exacte, contrôlabilité à zéro, inégalité de Carleman, système de Gear-Grimshaw, équation de Korteweg-de Vries, système de Boussinesq.

Abstract

This work is devoted to prove a series of results concerning the control and stabilization properties of dispersive models posed on a bounded interval. Initially, we study the internal stabilization of a coupled system of two Korteweg-de Vries equations (KdV), the so-called Gear–Grimshaw system. Defining a convenient Lyapunov function we obtain the exponential decay of the total energy associated to model. Next, we prove results of null and exact controllability for the Korteweg-de Vries equation with a control acting internally on a subset of the domain. When the control region is an arbitrary subinterval, we prove the null controllability by mean of a new Carleman inequality. As a consequence, we obtain a regional controllability result, the state function being controlled on the left part of the complement of the control region. Moreover, when the control region is a neighborhood of the right endpoint, an exact controllability result in a weighted L^2-space is also established. Finally, in view of the result of the boundary controllability obtained by L. Rosier for the KdV equation, we prove that the linear system Boussinesq of KdV–KdV type is exactly controllable when the controls act in the boundary conditions. Our analysis is performed using multipliers and the duality approach mentioned above. Adding a damping mechanism in the boundary, it is proved that the nonlinear system is also exactly controllable and the energy associated to the model decays exponentially.

Key-Words: Stabilization, Lyapunov function, Exact controllability, Null controllablity, Carleman inequality, Gear–Grimshaw system, Korteweg-de Vries equation, Boussinesq system.

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Contents

1 Introduction Générale **1**

1.0.1 Histoire des "ondes progressives" . . . 1
1.0.2 Contrôlabilité et stabilisation d'EDP - Méthodes Utilisées . . . 4
1.0.2.1 Problème du contrôle interne . . . 4
1.0.2.2 Problème du contrôle au bord . . . 4
1.0.3 Contrôlabilité et observabilité . . . 5
1.0.3.1 Notions de contrôlabilité . . . 5
1.0.3.2 Opérateurs adjoints . . . 6
1.0.3.3 Tests de Contrôlabilité . . . 6
1.0.4 Stabilisabilité . . . 7
1.0.5 Problèmes et résultats principaux . . . 8
1.0.5.1 Stabilisation du système de Gear–Grimshaw . . . 8
1.0.5.2 Contrôlabilité de l'équation de Korteweg-de Vries . . . 11
1.0.5.3 Contrôlabilité du système de Boussinesq de type KdV-KdV . . . 15

2 General Introduction **22**

2.0.6 Historical review of the wave of translation . . . 22
2.0.7 Controllability and stabilization for PDEs - Methods Used . . . 25
2.0.7.1 Internal control problem . . . 25
2.0.7.2 Boundary control problem . . . 25
2.0.8 Controllability and observability . . . 26
2.0.8.1 Concepts of Controllability . . . 26
2.0.8.2 Adjoint operators . . . 27
2.0.8.3 Controllability tests . . . 27
2.0.9 Stabilizability . . . 28
2.0.10 Problems and Main Results . . . 29
2.0.10.1 Stabilization of the Gear–Grimshaw system . . . 29
2.0.10.2 Controllability for the Korteweg-de Vries equation . . . 32
2.0.10.3 Controllability of Boussinesq Equation KdV-KdV type 35

3 Stabilization of the Gear–Grimshaw system on a periodic domain **43**

3.1 Introduction . . . 43
3.2 Some technical lemmas . . . 45

3.3 Proof of Proposition 3.1 for $n = 0$ 49
3.4 Proof of Proposition 3.1 for $n = 1$ 51
3.5 Proof of Proposition 3.1 for $n = 2$ 56
3.6 Proof of the proposition for $n \geq 3$ 62

4 Internal Controllability for the Korteweg-de Vries Equation on a Bounded Domain 64
4.1 Introduction 64
4.2 Linear estimates 68
4.2.1 The linearized KdV equation 68
4.2.2 The modified KdV equation 68
4.2.2.1 Energy Estimates 69
4.3 Null controllability results 70
4.3.1 Null controllability of a linearized equation 70
4.3.1.1 Carleman inequality with internal observation 71
4.3.1.2 Internal observation 76
4.3.2 Null controllability of the nonlinear equation 81
4.3.2.1 Proof of Theorem 4.1 81
4.3.3 Proof of Theorem 4.3. 84
4.4 Exact controllability results 85
4.4.1 The linear system 86
4.4.2 Well-posedness in L^2_{xdx} 86
4.4.3 Well-posedness in $L^2_{(L-x)^{-1}dx}$ 89
4.4.4 Non-homogeneous system 92
4.4.5 Controllability of the linearized system 94
4.4.6 Exact controllability of the nonlinear system 97
4.5 Final Comments 99

5 Controllability of Boussinesq Equation KdV-KdV type on a Bounded Domain 102
5.1 Introduction 102
5.2 Well-Posedness 108
5.2.1 Linear homogeneous system 108
5.2.2 Adjoint System 110
5.2.3 Linear non-homogeneous system 111
5.3 Exact Boundary Controllability For The Linear System: Neumann boundary condition 113
5.3.1 Double control 114
5.3.2 Single control 122
5.4 Exact Boundary Controllability For The Linear System: Dirichlet boundary condition 129
5.4.1 Double control in L 130
5.4.2 Single control in L 139
5.4.3 Double mixed control of Dirichlet type 151

5.5 Exact Boundary Controllability For The Linear System: Mixed boundary condition 157
5.5.1 Double control 157
5.6 Nonlinear Problem 164
5.6.1 Well-Posedness in X_0. 165
5.6.2 Adjoint System 170
5.6.3 The nonhomogeneous system 171
5.7 Exact Boundary Controllability Results: The Linear System With Boundary Damping 172
5.7.1 Single Control of Neumann type 173
5.7.2 Double Control Mixed Type 175
5.8 Boundary Controllability Result: The Nonlinear System 180
5.9 Stabilization of Boussinesq System 188
5.9.1 The Linear Problem 189
5.9.2 Well-posedness and Exponential Stability 193

6 Conclusion and perspectives **195**
6.1 Conclusion 195
6.2 Perspectives 196
6.2.1 New damping mechanism 196
6.2.2 Internal Controllability for the Korteweg-de Vries Equation . . . 197
6.2.3 Controllability for the non linear Boussinesq System in critical cases 197

Chapter 1

Introduction Générale

Cette thèse est consacrée à la contrôlabilité et à la stabilisation de systèmes d'équations aux dérivées partielles dispersives. Le problème de la contrôlabilité consiste à voir si on peut amener la solution d'un système d'un état initial à un état final donnés par un choix approprié d'un contrôle interne ou d'un contrôle au bord. En ce qui concerne la stabilisation, nous étudions le comportement asymptotique des solutions. La question est : peut-on garantir que les trajectoires du système sont asymptotiquement stables lorsque $t \to +\infty$? Si c'est le cas, nous cherchons à déterminer le taux de décroissance de ces solutions.

Les modèles étudiés ici sont l'équation de Korteweg-de Vries (KdV) sur un domaine borné, un système de Boussinesq de type KdV–KdV sur un domaine borné et le système de Gear–Grimshaw dans un domaine périodique. L'étude de ces systèmes était motivée par les résultats obtenus pour l'équation de KdV. Par conséquent, avant de donner une description mathématique de ces problèmes, nous rappelons des faits historiques importants liés à cette équation.

1.0.1 Histoire des "ondes progressives"

En 1834, John Scott Russell, un ingénieur naval Écossais, observait l'*Union Canal* en Écosse lorsqu'il assista à un phénomène physique très spécial qu'il appela une *onde progressive* [68]. Il vit une onde particulière voyageant le long du canal sans perte de forme ou de vitesse, et il fut tellement captivé par cet évènement qu'il consacra son attention à ces ondes pendant plusieurs années, et il demanda à la communauté mathématique de trouver un modèle mathématique spécifique pour les décrire. Plus précisément, ses mots furent :

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of a vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original

figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation...."

Russell était à ce point fasciné par cette découverte que non seulement il construisit des réservoirs pour étudier les ondes de surface chez lui, mais il fit des recherches expérimentales et théoriques sur ce type d'ondes. Ses expériences, connues sous le nom de *"The wave line system of hull construction"*, consistaient à constituer une réserve de fluide derrière un obstacle, puis à ôter l'obstacle de sorte qu'une onde longue en forme de cloche se propageât le long du canal. Ses contributions révolutionnèrent l'architecture navale du 19ème siècle, et il reçut la médaille d'or de la *Royal Society of Edinburgh* pour ses travaux en 1837. Les expériences de Russell contredisaient les conjectures physiques telles que la théorie des ondes de surface d'Airy [3], dans laquelle une onde solitaire ne peut exister vu que sa forme ou sa vitesse changent au cours du temps, ou la théorie de G.G. Stokes [76], où les ondes de faible amplitude et qui ne changent pas de forme sont possibles, mais seulement en eau profonde et en régime périodique. Cependant, Stokes était informé de l'état d'inachèvement de la théorie de Russell:

"It is the opinion of Mr. Russell that the solitary wave is a phenomenon sui generis, in nowise deriving its character from the circumstances of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discovered."

Par conséquent, afin de convaincre la communauté physique, Scott Russell mit la communauté mathématique au défi de prouver théoriquement l'existence du phénomène auquel il avait assisté:

"Having ascertained that no one had succeeded in predicting the phenomenon which I have ventured to call the wave of translation,... it was not to be supposed that after its existence had been discovered and its phenomena determined, endeavors would not be made... to show how it ought to have been predicted from the known general equations of fluid motion. In other words, it now remained to the mathematician to predict the discovery after it had happened, i.e. to give a priori demonstration a posteriori."

Un certain nombre de chercheurs relevèrent le défi de Russell. Le premier mathématicien à le faire fut Joseph Boussinesq, un Français à la fois mathématicien et physicien, qui obtint des résultats importants [10] en 1871. En 1876, le physicien Anglais Lord Rayleigh obtint un résultat différent [60], et en 1895, le mathématicien néerlandais D.J. Korteweg et son étudiant G. de Vries donnèrent le dernier résultat significatif du 19ème siècle. En fait, Boussinesq considérait un modèle d'ondes longues, incompressibles et sans rotation dans un canal peu profond avec une section rectangulaire en négligeant la friction le long du bord. Il obtint l'équation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left(\frac{2h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right), \tag{1.0.1}$$

dans laquelle (t, x) sont les coordonnées d'une particule de fluide au temps t, h est l'amplitude de l'onde, H est la hauteur de l'eau en équilibre et g est la constante de gravitation.

De manière indépendante, Rayleigh considéra le même phénomène et ajouta l'hypothèse d'une onde stationnaire s'annulant à l'infini. Il considéra seulement une dépendance spatiale et captura le comportement désiré dans l'équation

$$\left(\frac{\partial h}{\partial x}\right)^2 + \frac{3}{H^3}h^2\left(h - h_0\right) = 0, \tag{1.0.2}$$

h_0 étant la hauteur de crête de l'onde et les autres paramètres étant comme ci-dessus. Cette équation a une solution explicite donnée par

$$h\left(x\right) = h_0 \operatorname{sech}^2\left(\sqrt{\frac{3h_0}{4H^3}}x\right).$$

En 1876, Rayleigh écrivit dans son article [60]:

"I have lately seen a memoir by M. Boussinesq, Comptes Rendus, Vol. LXXII, in which is contained a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to Boussinesq J."

La dernière preuve de l'existence *d'ondes solitaires (solitons)* fut donnée par Diederik Johannes Korteweg et Gustav de Vries. Ils construirent une équation aux dérivées partielles non linéaire qui présente une solution reproduisant le phénomène découvert par Russell. Ils donnèrent ainsi à l'équation de Korteweg-de Vries son nom, souvent abrégé en KdV. En 1895, ils publièrent un article aboutissant à l'équation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial_x}\left(\frac{1}{2}\eta^2 + \frac{3}{2}\alpha\eta + \frac{1}{3}\beta\frac{\partial^2\eta}{\partial x^2}\right), \tag{1.0.3}$$

dans laquelle η est l'élévation de la surface libre au dessus de la position d'équilibre, l est une constante arbitrairement petite reliée au mouvement du liquide, g est la constante de gravitation, et $\beta = \frac{l^3}{3} - \frac{Tl}{\rho g}$, où T est la tension de surface capillaire et ρ la densité. Éliminant les constantes physiques par le changement de variables

$$t \to \frac{1}{2}\sqrt{\frac{g}{l\beta}}t, \;\; x \to -\frac{x}{\beta} \;\; \text{et} \;\; u \to -\left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$$

on obtient l'équation de Korteweg-de Vries standard

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.0.4}$$

qui est un modèle décrivant la propagation d'ondes de faible amplitude et de grande longueur d'onde pour l'interface air-eau dans un canal de section rectangulaire. Les solutions stationnaires périodiques sont appelées *ondes cnoïdales.*

C.S. Gardner et G.K. Morikawa [30] trouvèrent une nouvelle application de ce modèle dans l'étude des ondes hydromagnétiques sans collision dans la perspective de décrire la propagation unidirectionnelle d'ondes de faible amplitude dans un milieu dispersif non linéaire. Par ailleurs, M. Kruskal et N. Zabusky [81] montrèrent que l'équation de KdV peut servir de modèle pour le problème de Fermi-Pasta-Ulam, qui décrit des ondes longitudinales se propageant dans un réseau unidimensionnel de masses identiques couplées par des cordes non-linéaires. D'autres applications ont été trouvées et sont étudiées actuellement.

1.0.2 Contrôlabilité et stabilisation d'EDP - Méthodes Utilisées

Nous souhaitons obtenir des résultats de contrôlabilité et de stabilisation pour des systèmes gouvernés par des EDP dispersives. Nous allons considérer deux problèmes fondamentaux concernant cette théorie : le contrôle interne et le contrôle au bord.

Les divers concepts de contrôlabilité, qui coïncident en dimension finie mais pas en général pour une EDP, vont être introduits puis caractérisés grâce à l'approche classique de la dualité (voir [28, 42]). Par exemple, on peut montrer que la contrôlabilité exacte d'un système est équivalente à l'observabilité du système adjoint. La preuve est basée sur la méthode H.U.M. (pour Hilbert Uniqueness Method) de J.-L. Lions (pour plus de détails voir [42, 43, 44]). Les tests de contrôlabilité donnés ici peuvent être vus comme des extensions naturelles du critère du rang de Kalman, cf. [19].

Une attention spéciale est donnée aux EDP avec un générateur infinitésimal anti-adjoint, pour lesquelles les notions de contrôlabilité et de stabilisabilité considérées ici coïncident. On regarde les deux concepts seulement en dimension infinie, en suivant de près [61].

Avant de formuler les problèmes de contrôle, nous introduisons quelques notations. Soit $\mathcal{P}(\mathcal{D})$ un opérateur différentiel, avec $\mathcal{P} \in \mathbb{C}[\tau, \xi_1, \ldots, \xi_n]$ et

$$\mathcal{D} = (-i\partial_t, -i\partial_{x_1}, \ldots, -i\partial_{x_n}).$$

Par exemple $\mathcal{P} = -\tau^2 + |\xi|^2$ donne l'opérateur des ondes $\mathcal{P}(\mathcal{D}) = \partial_t^2 - \Delta$. Soit $\Omega \subset \mathbb{R}^n$ un ensemble ouvert borné et suffisamment régulier, dont la frontière $\partial\Omega$ est notée Γ.

1.0.2.1 Problème du contrôle interne

Étant donnés un ensemble ouvert $\omega \subset \Omega$ avec une frontière régulière Γ, et un ensemble de conditions au bord écrites simplement sous la forme $\mathcal{B}(\mathcal{D}) z = 0$, on considère le problème de contrôle

$$\begin{cases} \mathcal{P}(\mathcal{D}) z = \mathcal{X}_\omega f & t > 0, x \in \Omega, \\ \mathcal{B}(\mathcal{D}) z = 0 & t > 0, x \in \Gamma, \\ z(0,x) = z_0(x) & x \in \Omega. \end{cases} \tag{1.0.5}$$

Ici, $f = f(t,x)$ est le contrôle interne, $z = z(t,x)$ est la fonction inconnue. Pour le problème de la contrôlabilité, étant donnés z_0 et z_1 dans un espace fonctionnel H, nous cherchons un contrôle $f \in L^2(0,T;U)$ (U étant un autre espace fonctionnel) tel que la solution z du système (1.0.5) satisfait $z(T,x) = z_1(x)$.

1.0.2.2 Problème du contrôle au bord

Étant donnés un ensemble ouvert $\gamma \subset \Gamma$ et deux ensembles de conditions au bord $\mathcal{B}_1(\mathcal{D}) z = \mathcal{X}_\gamma f$, $\mathcal{B}_2(\mathcal{D}) z = 0$, on considère le problème de contrôle

$$\begin{cases} \mathcal{P}(\mathcal{D}) z = 0 & t > 0, x \in \Omega, \\ \mathcal{B}_1(\mathcal{D}) z = \mathcal{X}_\gamma f & t > 0, x \in \Gamma, \\ \mathcal{B}_2(\mathcal{D}) z = 0 & t > 0, x \in \Gamma, \\ z(0,x) = z_0(x) & x \in \Omega. \end{cases} \tag{1.0.6}$$

Ici, $f = f(t, x)$ est le contrôle au bord. En général ω (resp. γ) est un sous-ensemble strict de Ω (resp. Γ).

Combinant un argument d'extension de domaine avec des résultats classiques de trace au bord, on peut souvent obtenir des résultats de contrôle frontière à partir de résultats de contrôle interne.

1.0.3 Contrôlabilité et observabilité

1.0.3.1 Notions de contrôlabilité

Étant donnés $z_0 \in H$ et $u \in L^2(0, T; U)$, on considère la solution $z : [0, T] \to H$ du problème de Cauchy

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0. \end{cases} \tag{1.0.7}$$

Rappelons que pour tout $z_0 \in \mathcal{D}(A)$ et tout $u \in W^{1,1}(0, T; U)$, le problème de Cauchy (1.0.7) admet une unique solution classique $z \in C([0, T]; \mathcal{D}(A)) \cap C^1([0, T]; H)$ donnée par la formule de Duhamel

$$z(t) = S(t) z_0 + \int_0^t S(t-s) Bu(s)\, ds, \; \forall t \in [0, T],$$

où $\{S(t)\}_{t \geq 0}$ est un semi-groupe engendré par l'opérateur A. Pour $z_0 \in H$ et $u \in L^1(0, T; U)$, la formule ci-dessus a encore un sens et définit la *solution faible* $z \in C([0, T], H)$ de (1.0.7).

Définition 1.0.1. *Le système (1.0.7) est exactement contrôlable en temps T si pour tout $z_0, z_T \in H$, il existe $u \in L^2(0, T; U)$ tel que la solution z de (1.0.7) vérifie $z(T) = z_T$;*

Définition 1.0.2. *Le système (1.0.7) est contrôlable à zéro en temps T si pour tout $z_0 \in H$, il existe $u \in L^2(0, T; U)$ tel que la solution z de (1.0.7) vérifie $z(T) = 0$.*

Introduisons l'opérateur $\mathcal{L}_T : L^2(0, T; U) \longrightarrow H$ défini par

$$\mathcal{L}_T u = \int_0^T S(T-t) Bu(s)\, ds.$$

Alors,

$$\text{Contrôlabilité exacte en temps } T \Leftrightarrow \operatorname{Im} \mathcal{L}_T = H; \tag{1.0.8}$$

$$\text{Contrôlabilité à zéro en temps } T \Leftrightarrow S(T) H \subset \operatorname{Im} \mathcal{L}_T. \tag{1.0.9}$$

En dimension finie, i.e. lorsque $A \in \mathbb{R}^{n \times n}$ et $B \in \mathbb{R}^{n \times m}$, les trois notions sont équivalentes, et équivalentes à une condition purement algébrique, la célèbre condition de rang de Kalman : $rank\,(B, AB, \ldots, A^{n-1}B) = n$. Notons que le temps T ne joue aucun rôle (pour plus de détails voir, par exemple, [19, 80]).

La situation est plus délicate pour les EDP :

– il n'y a pas de test algébrique pour la contrôlabilité ;

– le temps de contrôle joue un rôle crucial pour les EDP hyperboliques ;
– la réciproque de

$$\text{Contrôlabilité exacte} \Rightarrow \text{Contrôlabilité à zéro}$$

n'est pas vraie en général.

1.0.3.2 Opérateurs adjoints

Comme cela a été mentionné avant, les problèmes de contrôlabilité se ramènent à des preuves d'inégalités d'observabilité pour la solution du système adjoint. Rappelons quelques définitions :

L'*adjoint* d'un opérateur borné $B \in \mathcal{L}(U,H)$ est l'opérateur $B^* \in \mathcal{L}(H,U)$ défini par $(B^*z,u)_U = (z,Bu)_H$ pour tout $z \in H$ et tout $u \in U$. Par ailleurs, l'adjoint de l'opérateur (non borné) A est l'opérateur (non borné) A^* de domaine

$$\mathcal{D}(A^*) = \left\{ z \in H : \exists C \in \mathbb{R}^+, |(Ay,z)_H| \leq C \|y\|_H, \forall y \in \mathcal{D}(A) \right\}$$

et défini par

$$(Ay,z)_H = (y,A^*z)_H, \forall y \in \mathcal{D}(A), \forall z \in \mathcal{D}(A^*).$$

Rappelons que A^* engendre également un semi-groupe continu $\left(e^{tA^*}\right)_{t\geq 0}$ vérifiant $e^{tA^*} = S^*(t)$, $\forall t \geq 0$. Si $A^* = A$ (resp. $A^* = -A$) l'opérateur A est dit *auto-adjoint* (resp. *anti-adjoint*). Rappelons qu'un opérateur anti-adjoint engendre un groupe continu d'isométries (voir e.g. [54]).

1.0.3.3 Tests de Contrôlabilité

Les preuves des résultats cités ici sont classiques, et peuvent être trouvées, par exemple, dans [19, 42, 80, 87]. Ces tests sont basés sur la méthode HUM introduite par J.-L. Lions [42]. Un premier résultat assurant la contrôlabilité est donné par le théorème suivant.

Théorème A: Le système (1.0.7) est *exactement contrôlable en temps* $T > 0$ si, et seulement si, il existe une constante $c > 0$ telle que

$$\int_0^T \|B^*S^*(t)\,y_0\|_U^2\,dt \geq c\,\|y_0\|_H^2, \forall y_0 \in H. \tag{1.0.10}$$

(1.0.10) est appelée *inégalité d'observabilité*. Une telle inégalité signifie que l'application

$$\Upsilon : y_0 \longmapsto B^*S^*(\cdot)\,y_0,$$

est "inversible", au sens où il est possible de retrouver une information complète concernant la donnée initiale y_0 à partir d'une mesure sur $[0,T]$ de la sortie $B^*[S^*(t)\,y_0]$ (*propriété d'observabilité*).

Théorème B : Le système (1.0.7) est *contrôlable à zéro en temps* $T > 0$ si, et seulement si, il existe une constante $c > 0$ telle que

$$\int_0^T \|B^*S^*(t)\,y_0\|_U^2\,dt \geq c\,\|S^*(T)\,y_0\|_H^2, \forall y_0 \in H. \tag{1.0.11}$$

(1.0.11) est une inégalité d'observabilité faible, au sens où seulement $S^*(T)\, y_0$ peut être retrouvé, pas y_0.

La Méthode HUM On associe au problème à données initiale et frontière

$$\Sigma \qquad \begin{cases} \dot{z} = Az + Bu, \\ z(0) = 0, \end{cases}$$

son problème adjoint, obtenu en prenant l'adjoint (au sens des distributions) de l'opérateur $\partial_t - A$, à savoir $-\partial_t - A^*$:

$$\Sigma^* \qquad \begin{cases} \dot{y} = -A^* y, \\ y(T) = y_T. \end{cases}$$

Notons que le système Σ^* est sans contrôle et rétrograde. Pour tout $y_T \in H$, la solution y de Σ^* est donnée par $y(t) = S^*(T-t)\, y_T$.

On peut établir *l'identité clé*:

$$(z(T), y_T)_H = \int_0^T (u, B^* y)_U \, dt$$

qui permet d'assurer l'équivalence entre l'inégalité d'observabilité et la contrôlabilité du système Σ. Par ailleurs, nous signalons que :

– l'équation d'évolution dans le *problème adjoint* $\dot{y} = -A^* y$ diffère de celle associée à l'*opérateur adjoint* $\dot{y} = A^* y$ par un signe moins. Les solutions de la seconde donnent celles de la première simplement en changeant t en $\to T - t$ dedans ;

– la méthode HUM assure l'existence d'un opérateur borné $\Lambda : z_T \longmapsto u$ donnant le contrôle ;

– en général, il n'est pas nécessaire d'expliciter B et B^*. Les ingrédients importants dans HUM sont l'*identité clé* et l'*inégalité d'observabilité*.

1.0.4 Stabilisabilité

Dans cette dernière section, on s'intéresse à la stabilisabilité du système de contrôle (1.0.7). Pour ce faire, on considère $K \in \mathcal{L}(H, U)$, et on introduit l'opérateur A_K défini par $A_K z = Az + BKz$ pour $z \in \mathcal{D}(A_K) = \mathcal{D}(A)$, et on note $(S_K(t))_{t \geq 0}$ le semi-groupe engendré par A_K.

Le système (1.0.7) est dit *stabilisable exponentiellement* s'il existe un retour d'état $K \in \mathcal{L}(H, U)$ tel que l'opérateur A_K soit exponentiellement stable, i.e. il existe des constantes $C > 0$ et $\mu > 0$ telles que l'on ait

$$\|S_K(t)\| \leq C e^{-\mu t}, \forall t \geq 0.$$

Par ailleurs, le système (1.0.7) est dit *complètement stabilisable* s'il est exponentiellement stabilisable avec un taux de décroissance exponentielle arbitraire ; en d'autres termes, pour tout $\mu \in \mathbb{R}$, il existe une loi de retour $K \in \mathcal{L}(H, U)$ et une constante $C > 0$ telle que

$$\|S_K(t)\| \leq C e^{-\mu t}, \forall t \geq 0.$$

La stabilisation du système (1.0.7) est fortement reliée à la contrôlabilité du même système. Le premier résultat dans cette direction fut donné par Datko en 1972 (voir [22]).

Théorème C: Si le système (1.0.7) est contrôlable à zéro, alors il est stabilisable exponentiellement.

Le résultat qui suit donne une version en dimension infinie du théorème de Wonham.

Théorème D: Supposons que A engendre un groupe $(S(t))_{t\geq 0}$ d'opérateurs. Alors les propriétés suivantes sont équivalentes :

i) Le système (1.0.7) est exactement contrôlable en un certain temps $T > 0$;

ii) Le système (1.0.7) est contrôlable à zéro en un certain temps $T > 0$;

iii) Le système (1.0.7) est complètement stabilisable.

L'implication i) $\Rightarrow$ iii) est due à Slemrod [75], tandis que l'implication iii) $\Rightarrow$ i) est due à Megan [47] (voir aussi Zabczyk [80, Theorem 3.4 p. 229]). i) $\Rightarrow$ ii) est évident.

Le résultat qui précède s'applique en particulier à un opérateur anti-adjoint A, qui engendre un groupe d'isométries sur H. En fait, on peut dire un peu plus. D'abord, on a le principe de "contrôlabilité via la stabilisabilité" dû à Russell [69, 37]. D'autre part, des lois de retour stabilisant exponentiellement peuvent être données explicitement (par exemple, [45, Theorem 2.3]).

Théorème E: Supposons que A soit anti-adjoint. Alors les propriétés suivantes sont équivalentes :

– Le système (1.0.7) est exactement contrôlable pour un certain temps $T > 0$;

– Le système (1.0.7) est contrôlable à zéro pour un certain temps $T > 0$;

– Le système (1.0.7) est complètement stabilisable ;

– Le système (1.0.7) est stabilisable exponentiellement ;

– Pour tout opérateur auto-adjoint défini positif $S \in \mathcal{L}(U)$, l'opérateur $A - BSB^*$ engendre un semi-groupe exponentiellement stable sur H.

1.0.5 Problèmes et résultats principaux

Nous retournons aux problèmes considérés dans cette Thèse, à savoir l'étude des propriétés de contrôlabilité et de stabilisation de modèles qui décrivent la propagation d'ondes de surface. Nous commençons notre étude par le *système de Gear–Grimshaw sur un domaine périodique.*

1.0.5.1 Stabilisation du système de Gear–Grimshaw

Le premier travail de cette Thèse, effectué en collaboration avec V. Komornik et A. Pazoto [13], est centré sur les propriétés de décroissance des solutions du système

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x + k(u - [u]) = 0, \\ b_1 v' + rv_x + vv_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 uu_x \\ \qquad\qquad + b_2 a_1 (uv)_x + k(v - [v]) = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{1.0.12}$$

avec des conditions au bord périodiques

$$\partial_x^k u(0) = \partial_x^k u(1),\ k = 0, 1, 2, \ldots$$

Dans (1.0.12), $r, a_1, a_2, a_3, b_1, b_2, k$ sont des constantes réelles données vérifiant $b_1, b_2, k > 0$, $u(t,x), v(t,x)$ sont des fonctions réelles des variables $t \geq 0$ (le temps) et $0 \leq x \leq 1$ (l'espace), ∂_x et $'$ désignent les dérivées partielles par rapport à x et t, respectivement, et $[f]$ désigne la valeur moyenne de f définie par

$$[f] := \int_0^1 f(x)\ dx.$$

Lorsque $k = 0$, le système est celui proposé par Gear et Grimshaw [31] comme modèle pour décrire les interactions fortes de deux ondes longues internes de gravité dans un fluide stratifié où les deux ondes sont supposées correspondre à différents modes des équations du mouvement linéarisées. Il a la structure d'une paire d'équations de KdV avec des termes de couplage linéaires et non-linéaires, et a fait l'objet d'une recherche intensive ces dernières années. En ce qui concerne le problème de la stabilisation, la plupart des travaux se sont focalisés sur un intervalle borné avec un amortissement interne localisé en espace (voir, par exemple, [55] et les références dedans). En particulier, on renvoie le lecteur à [8] pour une analyse de la signification physique du système, et à [23, 24, 25, 26, 27] pour les résultats utilisés dans notre travail.

On peut vérifier (formellement) que l'énergie totale

$$E = \frac{1}{2}\int_0^1 b_2 u^2 + b_1 v^2\ dx$$

associée au modèle satisfait l'inégalité

$$E' = -k\int_0^1 b_2(u - [u])^2 + (v - [v])^2\ dx \leq 0$$

dans $(0, \infty)$, de sorte que l'énergie est décroissante. On peut donc se poser naturellement les question suivantes : est-ce que l'origine est asymptotiquement stable pour le système? Et si c'est le cas, peut-on donner un taux de décroissance des solutions ? Le but de ce travail est de répondre à ces questions.

Plus précisément, nous prouvons que pour tout entier fixe $s \geq 3$, les solutions sont exponentiellement stables dans les espaces de Sobolev

$$H_p^s(0,1) := \{u \in H^s(0,1)\ :\ \partial_x^n u(0) = \partial_x^n u(1), \quad n = 0, \ldots, s\}$$

avec conditions au bord périodiques. Ceci généralise un ancien résultat de Dávila dans [26] pour $s \leq 2$.

Avant d'énoncer le résultat de stabilisation mentionné plus haut, nous devons nous assurer du caractère bien posé du système. Ce point a été établi par Dávila dans [23] (voir aussi [24]) sous les hypothèses suivantes concernant les coefficients :

$$\begin{cases} a_3^2 b_2 < 1 \text{ et } r = 0 \\ b_2 a_1 a_3 - b_1 a_3 + b_1 a_2 - a_2 = 0 \\ b_1 a_1 - a_1 - b_1 a_2 a_3 + a_3 = 0 \\ b_1 a_2^2 + b_2 a_1^2 - b_1 a_1 - a_2 = 0. \end{cases} \tag{1.0.13}$$

En effet, sous les conditions (1.0.13), Dávila et Chaves [27] ont obtenu des invariants pour les solutions de (1.0.12). Ces invariants, combinés à l'approche introduite dans [9, 73], permettent d'établir le caractère bien posé globalement dans $H_p^s(0,1)$, pour tout $s \geq 0$. De plus, les auteurs donnent une dérivation plus simple des invariants découverts par Gear et Grimshaw, et Bona *et al.* [8]. Nous notons aussi que ces quantités conservées furent obtenues en utilisant les techniques développées pour l'équation de KdV seule [52] ; voir aussi [51].

Le résultat qui exprime le caractère bien posé du système s'énonce ainsi :

Théorème 1.0.1. *Supposons que la condition* (1.0.13) *soit satisfaite. Si* $\phi, \psi \in H_p^s(0,1)$ *sont données pour un entier* $s \geq 3$, *alors le système* (1.0.12) *a une unique solution vérifiant*

$$u, v \in C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1)).$$

De plus, l'application $(\phi,\psi) \mapsto (u,v)$ *est continue de* $\left(H_p^s(0,1)\right)^2$ *dans*

$$\left(C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1))\right)^2.$$

Pour $k = 0$, le résultat analogue sur la droite réelle $-\infty < x < \infty$ a été prouvé par Bona *et al.* [8], pour tout $s \geq 1$.

Avec le résultat d'existence globale à notre disposition, nous pouvons nous focaliser sur le problème de la stabilisation. Pour simplifier les notations, on ne considère que le cas

$$b_1 = b_2 = 1. \tag{1.0.14}$$

Alors les conditions (1.0.13) prennent la forme

$$\begin{array}{c} r = 0, \quad a_1^2 + a_2^2 = a_1 + a_2 \\ |a_3| < 1 \\ (a_1 - 1)a_3 = (a_2 - 1)a_3 = 0. \end{array} \tag{1.0.15}$$

Par conséquent, soit $a_3 = 0$ et $a_1^2 + a_2^2 = a_1 + a_2$, soit $0 < |a_3| < 1$ et $a_1 = a_2 = 1$.

Nous prouvons le résultat suivant :

Théorème 1.0.2. *Supposons* (1.0.14) *et* (1.0.15). *Si* $\phi, \psi \in H_p^s(0,1)$ *sont données pour un entier* $s \geq 3$, *alors la solution de* (1.0.12) *vérifie*

$$\|u(t) - [u(t)]\|_{H_p^s(0,1)} + \|v(t) - [v(t)]\|_{H_p^s(0,1)} = o\left(e^{-k't}\right), \quad t \to \infty,$$

pour chaque $k' < k$.

Un résultat similaire fut prouvé dans [38] pour l'équation de KdV classique en utilisant le famille (infinie) d'invariants de cette équation. Ces invariants permettent de construire une fonction de Lyapunov convenable qui permet d'établir la décroissance exponentielle des solutions. Ici, nous suivons la même approche en utilisant les résultats établis par Dávila et Chavez [27]. Ils prouvèrent que sous les hypothèses (1.0.13), le système (1.0.12) a également une suite infinie de quantités conservées (invariants), et ils conjecturèrent le résultat ci-dessus dans ce cas. A ce stade, on observe que les calculs sont simplifiés si l'on change u, v, ϕ et ψ en $u-[u]$, $v-[v]$, $\phi-[\phi]$ et $\psi-[\psi]$. Alors les nouvelles fonctions inconnues u et v satisfont le même système (1.0.12) avec ku et kv au lieu de $k(u-[u])$ et $k(v-[v])$. Dans la suite, nous considérons les solutions du système simplifié

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x + ku = 0, \\ v' + vv_x + v_{xxx} + a_3 u_{xxx} + a_2 uu_x + a_1(uv)_x + kv = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{1.0.16}$$

avec conditions au bord périodiques, correspondant aux données initiales ϕ, ψ à valeurs moyennes nulles.

Afin d'obtenir le résultat, nous prouvons un certain nombre d'identités et d'estimées pour les solutions de (1.0.12). D'après le Théorème 1.0.1, il suffit d'établir ces estimées pour des *solutions régulières*, i.e. pour des solutions correspondant à des données initiales ϕ, ψ C^∞ et périodiques. Pour de telles solutions, toutes les manipulations formelles qui suivent sont justifiées.

Enfin, il faut signaler qu'un résultat similaire a été obtenu dans [40] pour l'équation de KdV scalaire dans un domaine périodique. Les auteurs ont étudié le modèle d'un point de vue "contrôle" en insérant dans l'équation un terme force f à support dans un ouvert donné $\omega \subset \mathbb{T}$. Ils montrent que le système est *globalement* contrôlable et *globalement* stabilisable exponentiellement. La stabilisation est établie avec l'aide de certaines propriétés de propagation de compacité et de régularité dans des espaces de Bourgain pour les solutions du système linéaire correspondant. Nous renvoyons à [40] pour une bibliographie exhaustive sur le sujet.

1.0.5.2 Contrôlabilité de l'équation de Korteweg-de Vries

Le second travail de cette Thèse, en collaboration avec L. Rosier et A. Pazoto [14], est dévolu à l'étude de la contrôlabilité (interne) de l'équation de Korteweg-de Vries (KdV). L'équation de KdV a été introduite pour la première fois dans [39] comme un modèle pour la propagation d'ondes à la surface d'un liquide dans un canal. A présent, on sait que l'équation de KdV n'est pas seulement un bon modèle pour les ondes à la surface de l'eau, mais aussi un modèle approché dans des problèmes physiques ou l'on souhaite incorporer des effets dispersifs entrant en compétition avec une non-linéarité faible. En particulier, l'équation est communément acceptée comme un modèle mathématique pour la propagation unidirectionnelle d'ondes longues de faible amplitude dans des milieux dispersifs non linéaires. Dans l'étude mathématique de cette équation, on considère principalement la question du caractère bien posé sur la droite réelle. Cependant, l'utilisation pratique de KdV et des équations de la même famille ne concerne pas toujours le problème de Cauchy sur $\mathbb{R}$. Le problème de Cauchy avec données initiales et au bord apparaît naturellement, notamment lorsque l'on fait une étude numérique. Nous allons adopter ce point de vue, en nous focalisant sur le problème du contrôle pour l'équation de KdV posée sur un intervalle borné $(0, L)$. Ainsi, nous considérons le système

$$u_t + u_x + u_{xxx} + uu_x = 0, \quad \text{où } x \in [0, L] \text{ et } t \geq 0.$$

Afin d'étudier les propriétés de contrôlabilité, nous introduisons un système dynamique (système contrôlé) sur lequel nous agissons à l'aide d'un terme de contrôle afin de réaliser un objectif. Ici, on considère un système de contrôle où l'état, à chaque instant, est donné par la solution de l'équation de KdV avec un contrôle distribué dans un ouvert du domaine $(0, L)$. Plus précisément, nous étudions le système suivant avec contrôle interne :

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = f & \text{dans } (0,T) \times (0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{dans } (0,T), \\ u(0,x) = u_0(x) & \text{dans } (0,L), \end{cases} \tag{1.0.17}$$

où

• $\xi = \xi(t,x)$ est un coefficient de transport (constant dans le problème principal) et

• f, le terme force (contrôle distribué), est à support dans un ouvert donné $\omega \subset (0, L)$.

Notons que l'équation de KdV classique correspond à $\xi = 1 + \frac{u}{2}$.

Notre objectif principal est de voir si l'on peut contraindre les solutions de (1.0.17) à avoir certaines propriétés en choisissant un contrôle approprié f. Ainsi, on étudie les questions fondamentales de la théorie du contrôle pour les équations aux dérivées partielles:

Étant donnés un état initial $u_0(x)$ et un état terminal $u_1(x)$ dans un certain espace, peut-on trouver un contrôle approprié f de sorte que l'équation (1.0.17) admette une solution u qui soit égale à u_0 au temps $t = 0$ et à u_1 au temps $t = T$?

Si l'on peut toujours trouver un contrôle f qui amène le système décrit par (1.0.17) de n'importe quel état initial u_0 à n'importe quel état final u_1, on dit que le système (1.0.17) est **exactement contrôlable**. Si le système peut être amené de n'importe quel état à zéro à l'aide d'un contrôle f convenable (i.e. $u_1 \equiv 0$), on dit que le système est **contrôlable à zéro**.

L'étude du contrôle et de la stabilisation de l'équation de KdV a commencé avec les travaux de Russell [70] et Zhang [83] pour un système avec conditions au bord périodiques et contrôle interne. Depuis, les deux problèmes ont été étudiés intensivement, en particulier la contrôlabilité exacte avec contrôle au bord sur un domaine borné [16, 17, 20, 32, 33, 62, 64, 83, 85]. La plupart de ces travaux concernent le système

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{dans } (0,T) \times (0,L), \\ u(t,0) = g_1(t),\ u(t,L) = g_2(t),\ u_x(t,L) = g_3(t) & \text{dans } (0,T). \end{cases} \tag{1.0.18}$$

Le système (1.0.18) a d'abord été étudié par Rosier [62] en ne prenant que g_3 comme contrôle. Il montra que la contrôlabilité exacte dans $L^2(0, L)$ n'a pas lieu si L appartient à l'ensemble (dénombrable) suivant de longueurs critiques

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \ : k,\ l\ \in \mathbb{N}^* \right\}. \tag{1.0.19}$$

L'analyse développée dans [62] montre que le problème linéaire n'est pas contrôlable en raison de l'existence d'un espace de dimension finie d'états inatteignables. Il faut également signaler le résultat de contrôlabilité à zéro de Rosier [64] et de Glass et Guerrero [32]. Ils ont prouvé que le système (1.0.18) était contrôlable à zéro avec un contrôle appliqué à gauche.

De manière contrastée, la théorie mathématique du contrôle interne est moins développée. A notre connaissance, la contrôlabilité à zéro du système (1.0.17) a été étudiée seulement dans [32] lorsque le contrôle agit au voisinage du point $x = 0$. En ce qui concerne la contrôlabilité exacte, seulement des conditions au bord périodiques ont été considérées (voir [40] pour une bibliographie assez complète).

Les résultats établis dans cette Thèse donnent des réponses positives aux problèmes de contrôle posés plus haut. En ce qui concerne la contrôlabilité à zéro, notre résultat principal s'énonce ainsi :

Théorème 1.0.3. *Soient* $T > 0$, $\xi \in L^2(0,T;H^2(0,L)) \cap C^0([0,T];H^1(0,L))$ *et* $u_0 \in L^2(0,L)$. *Alors pour tout ouvert* $\omega \subset (0,L)$, *il existe une fonction* $f = v(t,x) = v \in L^2((0,T)\times\omega)$ *telle que la solution* $u \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ *de (1.0.17) vérifie* $u(T,\cdot) = 0$.

Suivant l'approche classique basée sur la dualité (voir [28, 42]), la propriété de contrôlabilité donnée dans le Théorème 1.0.3 sa ramène à une inégalité d'observabilité (à prouver) pour les solutions du système adjoint. Ici, l'inégalité d'observabilité est prouvée en combinant une inégalité de Carleman et des arguments classiques d'interpolation. Notre approche utilise certaines idées présentées dans [32], où les auteurs obtiennent un résultat similaire lorsque le contrôle agit sur un voisinage de l'extrémité gauche. Ici, nous établissons une nouvelle inégalité de Carleman qui permet de surmonter certaines difficultés techniques et d'étendre leur résultat à n'importe quel sous-intervalle ouvert de $(0,L)$. Cette inégalité ressemble un peu à celle établie par Rosier [62] pour l'étude des propriétés de contrôlabilité de (1.0.18).

Comme conséquence naturelle de la contrôlabilité à zéro du système linéarisé, nous obtenons la contrôlabilité à zéro du système non linéaire. En effet, en combinant le Théorème 1.0.3 et le théorème du point fixe de Kakutani (voir par exemple [82, Theorem 9.B]), on obtient le résultat suivant :

Théorème 1.0.4. *Pour* $\bar{u}_0 \in L^2(0,L)$, *on considère la solution* $\bar{u} \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H_0^1(0,L))$ *de*

$$\begin{cases} \bar{u}_t + \bar{u}_x + \bar{u}\bar{u}_x + \bar{u}_{xxx} = 0 & \textit{dans } (0,T)\times(0,L), \\ \bar{u}(t,0) = \bar{u}(t,L) = \bar{u}_x(t,L) = 0 & \textit{dans } (0,T), \\ \bar{u}(0,x) = \bar{u}_0(x) & \textit{dans } (0,L). \end{cases} \tag{1.0.20}$$

Alors, il existe $\delta > 0$ *tel que pour tout* $u_0 \in L^2(0,L)$ *vérifiant* $\|u_0 - \bar{u}_0\|_{L^2(0,L)} \leq \delta$, *il existe* $v \in (L^2(0,T)\times\omega)$ *tel que la solution* $u \in L^2(0,T;H^1(0,L)) \cap C^0([0,T];L^2(0,L))$ *de*

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 1_\omega v(t,x) & \textit{dans } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \textit{dans } (0,T), \\ u(0,x) = u_0(x) & \textit{dans } (0,L), \end{cases} \tag{1.0.21}$$

vérifie $u(T,\cdot)=\bar{u}(T,\cdot)$ *dans* $(0,L)$.

Le second problème que nous étudions ici concerne la contrôlabilité interne du système (1.0.17). A notre connaissance, le même problème a été étudié seulement dans [40] et [72] dans un domaine périodique $\mathbb{T}$ et avec un terme source de la forme

$$f(x,t)=(Gh)(x,t):=g(x)\left(h(x,t)-\int_{\mathbb{T}}g(y)h(y,t)dy\right).$$

La fonction h est considérée comme un nouveau contrôle, et $g(x)$ est une fonction donnée régulière, positive ou nulle, et telle que $\{g>0\}=\omega$, où ω désigne n'importe quel sous-intervalle ouvert. Dans notre cas, il n'est pas nécessaire de supposer que le domaine est périodique, mais on suppose que le terme source f est de la forme

$$f(x,t)=1_{\omega}h(t,x),$$

où ω est un voisinage ouvert du point $x=L$. En fait, lorsque le contrôle agit dans un voisinage de $x=L$, notre résultat de contrôlabilité exacte est obtenu dans l'espace $L^2_{\left(\frac{1}{L-x}\right)dx}$, où

$$L^2_{\left(\frac{1}{L-x}\right)dx}:=\left\{u:(0,L)\to\mathbb{R}:\int_0^L\frac{|u(x)|^2}{(L-x)}dx<\infty\right\}$$

est muni de son produit scalaire naturel. Plus précisément, nous allons prouver le résultat suivant :

Théorème 1.0.5. *Soient* $T>0$, $\omega=(l_1,l_2)=(L-\nu,L)$, *où* $0<\nu<L$. *Alors, il existe un nombre* $\delta>0$ *tel que pour tout* u_0, $u_1\in L^2_{\left(\frac{1}{L-x}\right)dx}$ *vérifiant*

$$\|u_0\|_{L^2_{\left(\frac{1}{L-x}\right)dx}}\le\delta,\quad \|u_1\|_{L^2_{\left(\frac{1}{L-x}\right)dx}}\le\delta$$

on peut trouver un contrôle $f\in L^2(0,T;H^{-1}(0,L))$ *tel que la solution*

$$u\in L^2(0,T;H^1(0,L))\cap C^0([0,T];L^2(0,L)$$

de

$$\begin{cases} u_t+u_x+uu_x+u_{xxx}=f & \textit{en } (0,T)\times(0,L),\\ u(t,0)=u(t,L)=u_x(t,L)=0 & \textit{en } (0,T),\\ u(0,x)=u_0(x) & \textit{en } (0,L). \end{cases}\tag{1.0.22}$$

vérifie

$$u(0,x)=u_0,\quad u(T,x)=u_1(x)\ \textit{in } (0,L)$$

et $u\in C^0([0,T],L^2_{\frac{1}{L-x}dx})$. *En outre,* $f\in L^2_{(T-t)dt}(0,T,L^2(0,L))$.

En fait , nous aurons à étudier le bien posé de la linéarisation de (1.0.22) dans l'espace $L^2_{\frac{1}{L-x}dx}$ et le bien posé de la (arriére) système adjoint dans le “espace dual” $L^2_{(L-x)dx}$. Pour ce faire, nous allons suivre quelques idées empruntées à [34], où le bien posé a été étudiée dans la moyenne pondérée espace $L^2_{\frac{x}{L-x}dx}$. L'inégalité d'observabilité nécessaire est obtenue par l'argument de compacité-unicité standard et une propriété de continuation unique. La contrôlabilité exacte est prolongée vers le système non linéaire en utilisant le principe de contraction.

Lorsque la commande agit loin du point de terminaison $x = L$, c'est à dire dans un intervalle $\omega = (l_1, l_2)$ avec $0 < l_1 < l_2 < L$, alors il n'y a aucune chance de contrôler exactement la fonction de l'etat sur (l_2, L) (voir par exemple [64]). Cependant, il est possible de contrôler la fonction d'état sur $(0, l_1)$, de sorte qu'un "contrôlabilité régionale" peut être établie:

Theorem 1.1. *Soit $T > 0$ et $\omega = (l_1, l_2)$ avec $0 < l_1 < l_2 < L$. Choisissez n'importe quel nombre $l_1' \in (l_1, l_2)$. Alors, il existe un certain nombre $\delta > 0$ tel que pour tout $u_0, u_1 \in L^2(0, L)$ satisfaisant*

$$||u_0||_{L^2(0,L)} \leq \delta, \qquad ||u_1||_{L^2(0,L)} \leq \delta,$$

on peut trouver un contrôle $f \in L^2(0, T, H^{-1}(0, L))$ avec supp$(f) \subset (0, T) \times \omega$ telle que la solution $u \in C^0([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ de (1.0.22) satisfait

$$u(T, x) = \begin{cases} u_1(x) & si \;\; x \in (0, l_1'); \\ 0 & si \;\; x \in (l_2, L). \end{cases} \tag{1.0.23}$$

La preuve du théorème 1.1 combine théorème 1.0.4, un résultat de contrôlabilité limite de [62], et l'utilisation d'une fonction de cut–off. Notez que la question de savoir si u peut aussi être contrôlé dans l'intervalle (l_1', l_2) est ouvert.

1.0.5.3 Contrôlabilité du système de Boussinesq de type KdV-KdV

Le troisième et dernier travail de cette Thèse, effectué en collaboration avec L. Rosier et A. Pazoto [15], est dévolu à l'étude des propriétés de contrôlabilité et du comportement asymptotique pour le système de Boussinesq de type KdV-KdV.

Les systèmes de Boussinesq ont été dégagés par Boussinesq dans [11] pour décrire la propagation bidirectionnelle d'ondes de gravité de faible amplitude et de grande longueur d'onde à la surface d'un canal. Ces systèmes et leurs généralisations d'ordre plus élevé apparaissent aussi comme modèles de propagation d'ondes de surface à haute crête pour les grands lacs ou les océans et dans d'autres contextes. Dans [6], les auteurs obtiennent une famille à quatre paramètres de systèmes de Boussinesq permettant de décrire le mouvement d'ondes longues de faible amplitude à la surface d'un fluide idéal sous l'action de la force de gravité et dans des situations où le mouvement est sensiblement bidimensionnel. Plus précisément, ils ont étudié la famille de systèmes de la forme :

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b \eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c \eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{1.0.24}$$

Dans (1.0.24), η est l'écart avec la position d'équilibre, et $w = w_\theta$ est la vitesse horizontale du fluide au niveau θh, où h est la profondeur du liquide au repos. Les paramètres

a, b, c, d, que l'on peut choisir dans une situation de modélisation donnée, doivent vérifier les relations

$$a+b=\frac{1}{2}\left(\theta^2-\frac{1}{3}\right), \quad c+d=\frac{1}{2}\left(1-\theta^2\right)\geq , \quad \theta\in[0,1], \tag{1.0.25}$$

où $\theta\in[0,1]$ spécifie quelle vitesse horizontale la variable w représente (cf. [6]).

Récemment, dans [50], les auteurs ont proposé un tableau assez complet des propriétés de contrôle de (1.0.24) sur un domaine périodique avec un terme force à support localisé. Selon les valeurs des quatre paramètres a, b, c, d, le système linéarisé peut être contrôlable en tout temps positif, ou seulement en temps grand, ou peut ne pas être contrôlable du tout. Ces résultats sont également étendus dans [50] au système non linéaire (1.0.24) dans le cas générique, i.e. lorsque tous les paramètres sont différents de 0.

En domaine borné, il n'existe à notre connaissance qu'un seul résultat dans la littérature. Rosier et Pazoto étudient dans [58] le comportement asymptotique du système (1.0.24) dans le cas spécial désigné par *système de Boussinesq de type KdV-KdV*. Ils ont considéré les paramètres (1.0.25) choisis ainsi : $a=c=1$ et $b=d=0$. Les auteurs ont donc considéré le système suivant

$$\begin{cases} \eta_t+w_x+(\eta w)_x+w_{xxx}=0 & \text{dans } (0,T)\times(0,L), \\ w_t+\eta_x+ww_x+\eta_{xxx}=0 & \text{dans } (0,T)\times(0,L), \end{cases} \tag{1.0.26}$$

satisfaisant les conditions au bord

$$\begin{cases} w(t,0)=w_{xx}(t,0)=0 & \text{dans } (0,T), \\ w_x(t,0)=\alpha_0\eta_x(t,0) & \text{dans } (0,T), \\ w(t,L)=\alpha_2\eta(t,L) & \text{dans } (0,T), \\ w_x(t,L)=-\alpha_1\eta_x(t,L) & \text{dans } (0,T), \\ w_{xx}(t,L)=-\alpha_2\eta_{xx}(t,L) & \text{dans } (0,T), \end{cases} \tag{1.0.27}$$

et les conditions initiales

$$\eta(0,x)=\eta_0(x), \;\; w(0,x)=w_0(x) \qquad \text{dans } (0,L). \tag{1.0.28}$$

Dans (1.0.26), α_0, α_1 et α_2 désignent des constantes réelles positive ou nulles. Le système de KdV–KdV possède probablement des solutions globales sur $\mathbb{R}$, ainsi que de bonnes propriétés de contrôle sur le tore [50].

Sous les conditions au bord ci-dessus, les auteurs ont observé que la dérivée de l'énergie associée au système (1.0.26), avec les conditions au bord (1.0.27)-(1.0.28), vérifie

$$\begin{aligned} \frac{dE}{dt} &= -\alpha_2\left|\eta(L,t)\right|^2-\alpha_1\left|\eta_x(L,t)\right|^2-\alpha_0\left|\eta_x(0,t)\right|^2 \\ &\quad -\frac{1}{3}w^3(L,t)-\int_0^L(\eta w)_x\,\eta dx \end{aligned}$$

où

$$E(t)=\frac{1}{2}\int_0^L\left(\eta^2+w^2\right)dx.$$

Cela indique que les conditions au bord jouent le rôle d'un amortissement en boucle fermée au moins pour le système linéarisé. Les questions qui suivent se posent naturellement :

- Est-ce que $E(t) \to 0$, lorsque $t \to +\infty$?

- Si c'est le cas, peut-on donner le taux de décroissance?

Le problème peut être facile à résoudre lorsque le modèle sous-jacent a une nature intrinsèquement dissipative. De plus, pour les systèmes couplés, pour obtenir la propriété de décroissance exponentielle désirée, le mécanisme d'amortissement doit être élaboré de manière appropriée afin de pouvoir stabiliser toutes les composantes du système.

Le résultat principal de Rosier et Pazoto fournit une réponse positive à ces questions.

Théorème 1.0.6. [58]*Supposons que $\alpha_0 \geq 0$, $\alpha_1 > 0$ et $\alpha_2 = 1$. Alors il existe des nombres $\rho > 0$, $C > 0$ et $\mu > 0$ tels que pour tout $(\eta_0, w_0) \in \left(L^2(I)\right)^2$ avec*

$$\|(\eta_0, w_0)\|_{(L^2(I))^2} \leq \rho,$$

le système (1.0.26)-(1.0.28) admette une unique solution

$$(\eta, w) \in C\left(\mathbb{R}^+; \left(L^2(I)\right)^2\right) \cap C\left(\mathbb{R}^{+*}; \left(H^1(I)\right)^2\right) \cap L^2\left((0,1); \left(H^1(I)\right)^2\right),$$

qui vérifie

$$\|(\eta, w)(t)\|_{(L^2(I))^2} \leq Ce^{-\mu t} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t \geq 0,$$

$$\|(\eta, w)(t)\|_{(H^1(I))^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t > 0.$$

Avec ce résultat à l'esprit, le troisième travail de cette Thèse se focalise sur le contrôle et la stabilisation du système (1.0.26) comme dans Micu *et al* [50]. Plus précisément, on considère le problème suivant :

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{dans } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{dans } (0,T) \times (0,L), \end{cases} \tag{1.0.29}$$

avec les conditions au bord

$$\begin{cases} \eta(t,0) = h_0(t), \ \eta(t,L) = h_1(t) & \text{sur } (0,T), \\ w(t,0) = g_0(t), \ w(t,L) = g_1(t) & \text{sur } (0,T), \\ \eta_x(t,0) = h_2(t), \ w_x(t,L) = g_2(t) & \text{sur } (0,T) \end{cases} \tag{1.0.30}$$

et les conditions initiales

$$\eta(0,x) = \eta_0(x), \ \ w(0,x) = w_0(x) \qquad \text{dans } (0,L). \tag{1.0.31}$$

Plus précisément, on considère d'abord des contrôles frontière de type Neumann en s'attendant à observer un phénomène similaire à celui observé dans [62] pour KdV, à savoir, l'existence d'un ensemble de longueurs critiques comme (1.0.19) pour le système linéaire associé à (1.0.29). De même, considérant un contrôle de type Dirichlet, on peut s'attendre à l'existence d'un ensemble de longueurs critiques du type

$$\begin{aligned} \mathcal{N}^* = \big\{ L \in \mathbb{R}_+^* : (a,b) \in \mathbb{C}^2 \text{ s.t. } ae^a = be^b = -(a+b) \\ \text{et } L = -\left(a^2 + ab + b^2\right) \big\} \end{aligned} \tag{1.0.32}$$

comme dans [33]. Les résultats de contrôlabilité pour le système linéaire (1.0.29) sont présentés de manière synthétique dans le tableau suivant:

	Contrôles							Propriétés	
Cas	h_0	h_1	h_2	g_0	g_1	g_2	Contrôles	Espace des états	critiques
1	0	0	$\star$	0	0	$\star$	$h_2, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in \left[H^{-1}(0,L)\right]^2$	$\mathcal{N}$
2	0	0	$\star$	0	0	0	$h_2 \in L^2(0,T)$	$(\eta_0, w_0) \in \left[H^{-1}(0,L)\right]^2$	$\mathcal{N}$
3	0	$\star$	0	0	$\star$	0	$h_1, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\emptyset$
4	0	0	0	0	$\star$	0	$g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N} \cup \mathcal{R}$
5	$\star$	0	0	0	$\star$	0	$h_0, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N} \cup \mathcal{R}$
6	0	0	$\star$	0	$\star$	0	$h_2, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
7	0	$\star$	0	0	0	$\star$	$h_1, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
8	$\star$	$\star$	0	0	0	0	$h_0, h_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
9	0	$\star$	$\star$	0	0	0	$h_1, h_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\emptyset$
10	0	0	$\star$	$\star$	0	0	$h_2, g_0 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$

Tableau 1. Résultats de contrôlabilité pour le système linéaire

où $\mathcal{N}$ est défini dans (1.0.19) et

$$\mathcal{R} = \left\{ \pi \sqrt{\left(\frac{1}{2} + 2k\right)^2 + \left(\frac{1}{2} + 2l\right)^2 + \left(\frac{1}{2} + 2k\right)\left(\frac{1}{2} + 2l\right)} : k, l \in \mathbb{N}^* \right\}. \tag{1.0.33}$$

Le résultat principal pour le système linéaire associé à (1.0.29) s'énonce de la manière suivante :

Théorème 1.0.7. [15]*Soient $\mathcal{N}$ et $\mathcal{R}$ défini par (1.0.19) et (1.0.33), respectivement.*

i) Pour tous $T > 0$, $L \in (0, +\infty) \setminus \mathcal{N}$ et (η_0, w_0), $(\eta_T, w_T) \in \left(H^{-1}(0, L)\right)^2$, il existe des contrôles $h_2, g_2 \in L^2(0,T)$ tels que la solution (η, w) associée au système linéarisé de (1.0.29)-(1.0.31), avec $h_i = 0$ et $g_i = 0$ pour $i = 0, 1$, dans la classe

$$(\eta, w) \in C^0\left(0, T; H^{-1}(0, L)\right)$$

vérifie $\eta(T, \cdot) = \eta_T$ et $w(T, \cdot) = w_T$ dans $(0, L)$;

ii) Pour tous $T > 0$, $L \in (0, +\infty) \setminus \mathcal{N}$ et (η_0, w_0), $(\eta_T, w_T) \in \left(H^{-1}(0, L)\right)^2$, il existe un contrôle $h_2 \in L^2(0,T)$ tel que la solution (η, w) associée au système linéarisé de (1.0.29)-(1.0.31), avec $h_i = 0$ et $g_i = 0$ for $i = 0, 1$ et $g_2 = 0$, dans la classe

$$(\eta, w) \in C^0\left(0, T; H^{-1}(0, L)\right)$$

vérifie $\eta(T, \cdot) = \eta_T$ et $w(T, \cdot) = w_T$ dans $(0, L)$;

iii) Pour tous $T > 0$, $L > 0$ et (η_0, w_0), $(\eta_T, w_T) \in X'$, il existe des contrôles $h_1, g_1 \in L^2(0,T)$ tels que la solution (η, w) associée au système linéarisé de (1.0.29)-(1.0.31), avec $h_i = 0$ et $g_i = 0$ pour $i = 0, 2$, dans la classe

$$(\eta, w) \in C^0\left(0, T; X'\right)$$

vérifie $\eta(T,\cdot) = \eta_T$ *et* $w(T,\cdot) = w_T$ *dans* $(0,L)$*, où* X' *désigne le dual de l'espace suivant*

$$X = \left\{(\eta, w) \in \left[H^2(0,L) \cap H_0^1(0,L)\right]^2 : \eta_x(0) = w_x(L) = 0\right\};$$

iv) Pour tous $T > 0$*,* $L \in (0,+\infty) \setminus (\mathcal{N} \cup \mathcal{R})$ *et* (η_0, w_0)*,* $(\eta_T, w_T) \in X'$*, il existe un contrôle* $g_1 \in L^2(0,T)$ *tel que la solution* (η, w) *associée au système linéarisé de (1.0.29)-(1.0.31), avec* $h_i = 0$ *et* $g_i = 0$ *pour* $i = 0,2$ *et* $g_1 = 0$*, dans la classe*

$$(\eta, w) \in C^0(0,T; X')$$

vérifie $\eta(T,\cdot) = \eta_T$ *et* $w(T,\cdot) = w_T$ *in* $(0,L)$*.*

Pour prouver ce théorème, nous utilisons la méthode H.U.M. (Hilbert Uniqueness Method) introduite par J.-L. Lions [42]. Plus précisément, utilisant des techniques de multiplicateurs et un argument de compacité-unicité dû à E. Zuazua (voir une annexe de [42]), on montre les inégalités d'observabilité requises pour appliquer la méthode.

En ce qui concerne le problème non linéaire, avec le choix de conditions au bord du type (1.0.30), l'absence de l'effet régularisant de Kato dans le cas homogène nous a conduit à considérer des conditions au bord particulières pour que cet effet soit présent. Néanmoins, la question de la contrôlabilité pour des systèmes non linéaires avec des conditions au bord du type (1.0.30) reste un problème ouvert.

Ainsi, on considère le système

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{dans } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{dans } (0,T) \times (0,L), \end{cases} \tag{1.0.34}$$

vérifiant les conditions au bord

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0 & \text{dans } (0,T), \\ w(t,0) = w(t,L) = 0 & \text{dans } (0,T), \\ w_x(t,L) + \alpha_1 \eta_x(t,L) = g_2(t) & \text{dans } (0,T), \end{cases} \tag{1.0.35}$$

ou les conditions au bord

$$\begin{cases} \eta(t,L) = \eta_x(t,0) = 0 & \text{dans } (0,T), \\ w(t,0) = w(t,L) = 0 & \text{dans } (0,T), \\ \alpha_2 \eta(t,0) + \alpha_3 \eta_x(t,L) + w_{xx}(t,0) = h_0(t) & \text{dans } (0,T), \\ w_x(t,L) - \alpha_3 (\eta(t,0) - \eta_x(t,L)) = g_2(t) & \text{dans } (0,T), \end{cases} \tag{1.0.36}$$

où α_i est une constante positive pour $i = 1,2,3$, et en prenant les conditions initiales

$$\eta(0,x) = \eta_0(x), \quad w(0,x) = w_0(x) \qquad \text{dans } (0,L). \tag{1.0.37}$$

Avec les conditions au bord (1.0.35) (resp. (1.0.36)), on peut montrer que l'effet régularisant de Kato a bien lieu, et également montrer un résultat très similaire au Théorème 1.0.7 pour les systèmes ci-dessus. Les résultats qui suivent concernent le caractère bien posé et les propriétés de contrôlabilité des systèmes ci-dessus.

Théorème 1.0.8. *[15] Soient $X_0 = \left(L^2(0,L)\right)^2$, $T > 0$ et $L \in (0,+\infty) \backslash \mathcal{N}$, où $\mathcal{N}$ est défini par (1.0.19). Alors il existe une constante $\delta > 0$ telle que pour toute donnée initiale et toute donnée finale $\eta_0, w_0, \eta_T, w_T \in L^2(0,L)$ avec*

$$\|(\eta_0, w_0)\|_{X_0} \leq \delta \text{ et } \|(\eta_T, w_T)\|_{X_0} \leq \delta,$$

il existe un contrôle $g_2 \in L^2(0,T)$ tel que la solution

$$(\eta, w) \in C([0,T], X_0) \cap L^2\left(0,T; \left(H^1(0,L)\right)^2\right) \cap H^1\left(0,L; \left(H^{-2}(0,L)\right)^2\right),$$

du système (1.0.34) avec condition au bord (1.0.35)-(1.0.37) vérifie $\eta(T,\cdot) = \eta_T$ et $w(T,\cdot) = w_T$ dans $(0,L)$.

Théorème 1.0.9. *[15]Soient $T > 0$ et $L \in (0,+\infty) \backslash \mathcal{N}$. Alors il existe une constante $\delta > 0$ telle que pour toute donnée initiale et toute donnée finale $\eta_0, w_0, \eta_T, w_T \in L^2(0,L)$ avec*

$$\left\|\left(\eta^0, w^0\right)\right\|_{X_0} \leq \delta \text{ et } \left\|\left(\eta^1, w^1\right)\right\|_{X_0} \leq \delta,$$

il existe deux contrôles $(h_0(t), g_2(t)) \in \left(L^2(0,T)\right)^2$ tels que la solution

$$(\eta, w) \in C([0,T], X_0) \cap L^2\left(0,T; \left(H^1(0,L)\right)^2\right) \cap H^1\left(0,L; \left(H^{-2}(0,L)\right)^2\right),$$

du système (1.0.34) avec condition au bord (1.0.36)-(1.0.37) vérifie $\eta(T,\cdot) = \eta_T$ et $w(T,\cdot) = w_T$ dans $(0,L)$.

Enfin, on observe que si l'on choisit $g_2 = 0$ dans (1.0.35), on a une perte de stabilisation dans le résultat de Rosier et Pazoto [58]. Précisément, on prouve qu'avec seulement une dissipation, le système (1.0.26)-(1.0.28) est exponentiellement stable si $L \notin \mathcal{N}$. De plus, nous montrons aussi une décroissance exponentielle du système (1.0.34) avec les conditions au bord (1.0.35) ou (1.0.36), puisque la dérivée de l'énergie de ces systèmes satisfait, respectivement, les relations suivantes

$$\frac{d}{dt}E = -\alpha_1 |\eta_x(t,L)|^2 - \int_0^L (\eta w)_x \eta dx$$

et

$$\frac{d}{dt}E = -\alpha_2 |\eta(t,0)|^2 - \alpha_3 |\eta_x(t,L)|^2 - \int_0^L (\eta w)_x \eta dx.$$

Ainsi, comme dans [58], on obtient le résultat suivant :

Théorème 1.0.10. *[15]Supposons que $\alpha_1, \alpha_2, \alpha_3 > 0$ et $L \in (0,+\infty) \backslash \mathcal{N}$. Alors il existe des nombres $\rho > 0$, $C > 0$ et $\mu > 0$ tels que pour tout $(\eta_0, w_0) \in \left(L^2(I)\right)^2$ avec*

$$\|(\eta_0, w_0)\|_{(L^2(I))^2} \leq \rho,$$

le système (1.0.34) avec les conditions au bord (1.0.35) (resp. (1.0.36)) et les conditions initiales (1.0.37) admet une unique solution

$$(\eta, w) \in C\left(\mathbb{R}^+; \left(L^2(I)\right)^2\right) \cap C\left(\mathbb{R}^{+*}; \left(H^1(I)\right)^2\right) \cap L^2\left((0,1); \left(H^1(I)\right)^2\right),$$

qui vérifie

$$\left\|(\eta, w)(t)\right\|_{(L^2(I))^2} \leq Ce^{-\mu t}\left\|(\eta_0, w_0)\right\|_{(L^2(I))^2}, \ \forall t \geq 0,$$

$$\left\|(\eta, w)(t)\right\|_{(H^1(I))^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}}\left\|(\eta_0, w_0)\right\|_{(L^2(I))^2}, \ \forall t > 0.$$

En résumé, cette Thèse contient quatre chapitres (outre l'introduction) que l'on peut décrire de la façon suivante : le premier chapitre rassemble les résultats présentés dans [13], obtenus avec Vilmos Komornik et Ademir F. Pazoto et achevés lors d'une visite de l'Université de Strasbourg en Novembre 2012. Les second et troisième chapitres reproduisent les articles [14] et [15] en cours de soumission, obtenus avec Lionel Rosier et Ademir Pazoto, et achevés au cours de l'année de Doctorat passée à l'Université de Lorraine (Nancy). Enfin le dernier chapitre est une conclusion dans laquelle nous proposons une liste de problèmes ouverts concernant le contrôle d'équations aux dérivées partielles dispersives.

Chapter 2

General Introduction

This thesis deals with the controllability and stabilization of dispersive systems governed by partial differential equations. The controllability problem consists in analyzing whether the solution can be driven from a given initial state to a given terminal state by using an accurate control input or controls acting through the boundary conditions. In what concerns the stabilization, we study the asymptotic behavior of solutions, i.e., through an initial analysis of the signal energy associated with the model, the initial question is: Is it possible to ensure that the solutions are asymptotically stable for arbitrarily large $t \to +\infty$? Having positive response to this question, we are interested in seeking the rate of decay of these solutions.

The models studied here are the Korteweg-de Vries (KdV) equation on a bounded domain, a Boussinesq system of KdV–KdV type on a bounded domain and the Gear–Grimshaw system on a periodic domain. The study of both systems were motivated by the results obtained for the KdV equation. Therefore, before introducing the mathematical description of the problems, we recall some important historical facts related to this equation.

2.0.6 Historical review of the wave of translation

In 1834 John Scott Russell, a Scottish naval engineer, was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon which he called a wave of translation [68]. He saw a particular wave traveling through this channel without losing its shape or velocity, and was so captivated by this event that he focused his attention on these waves for several years and asked the mathematical community to find a specific mathematical model describing them. More precisely, his words were:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of a vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it

still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation...."

Russell was fascinated with his discovery to the point that he not only built water wave tanks at his home, but also did practical and theoretical research into these types of waves. His experiments, well-known as *"The wave line system of hull construction"*, consisted of raising an area of fluid behind an obstacle, then removing the obstacle so that a long, heap-shaped wave propagated down the channel. His developments revolutionized naval architecture in nineteenth century, and he was awarded the gold medal of the Royal Society of Edinburgh for his work in 1837. Russell's experiments contradicted physical conjectures such as G.B. Airy's water wave theory [3], in which the traveling wave could not exist because it eventually changed its speed or its shape, or G.G. Stokes' theory [76], where waves of finite amplitude and fixed form were possible, but only in deep water and only in periodic form. However, Stokes was aware of the unfinished state of Russell's theory:

"It is the opinion of Mr. Russell that the solitary wave is a phenomenon sui generis, in nowise deriving its character from the circumstances of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discovered."

Consequently, in order to convince the physics community, Scott Russell challenged the mathematical community to prove theoretically the existence of the phenomenon that he witnessed:

"Having ascertained that no one had succeeded in predicting the phenomenon which I have ventured to call the wave of translation,... it was not to be supposed that after its existence had been discovered and its phenomena determined, endeavors would not be made... to show how it ought to have been predicted from the known general equations of fluid motion. In other words, it now remained to the mathematician to predict the discovery after it had happened, i.e. to give a priori demonstration a posteriori."

A number of researchers took up Russell's challenge. The first mathematician to respond was Joseph Boussinesq, a French mathematician and physicist who got important results [10] in 1871. In 1876, the English physicist Lord Rayleigh obtained a different result [60], and in 1895 the Dutch mathematicians D.J. Korteweg and his student G. de Vries gave the last significant result of the 19th century [39]. In fact, Boussinesq considered a model of long, incompressible and rotation-free waves in a shallow channel with rectangular cross section neglecting the friction along the boundaries, and he obtained the equation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left(\frac{2h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right), \tag{2.0.1}$$

where (t, x) are the coordinates of a fluid particle at time t, h is the amplitude of the wave, H is the height of the water in equilibrium and g is the gravitational constant.

Rayleigh independently considered the same phenomenon and added the assumption of the existence of a stationary wave vanishing at infinity. He considered only spatial dependence and captured the desired behavior in the equation

$$\left(\frac{\partial h}{\partial x}\right)^2 + \frac{3}{H^3}h^2\left(h - h_0\right) = 0, \tag{2.0.2}$$

with h_0 being the crest of the wave and the other parameters defined as before. This equation has an explicit solution given by

$$h\left(x\right) = h_0 \operatorname{sech}^2\left(\sqrt{\frac{3h_0}{4H^3}}x\right).$$

In 1876, Rayleigh wrote in his article [60]:

"I have lately seen a memoir by M. Boussinesq, Comptes Rendus, Vol. LXXII, in which is contained a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to Boussinesq J."

The last proof of the existence of *"translation waves"* was given by Diederik Johannes Korteweg and Gustav de Vries. They constructed a nonlinear partial differential equation which has a solution describing the phenomenon discovered by Russell, thus giving the Korteweg-de Vries equation its name, often abbreviated as the KdV equation. In 1895, they published an article deriving the equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial_x}\left(\frac{1}{2}\eta^2 + \frac{3}{2}\alpha\eta + \frac{1}{3}\beta\frac{\partial^2 \eta}{\partial x^2}\right), \tag{2.0.3}$$

in which η is the surface elevation above the equilibrium level l, is a small arbitrary constant related to the motion of the liquid, g is the gravitational constant, and $=$ $\beta = \frac{l^3}{3} - \frac{Tl}{\rho g}$ with surface capillary tension T and density ρ. Eliminating the physical constants by the change of variables

$$t \to \frac{1}{2}\sqrt{\frac{g}{l\beta}}t, \quad x \to -\frac{x}{\beta} \quad \text{and} \quad u \to -\left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$$

one obtains the standard Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.0.4}$$

which is a model describing the propagation of small amplitude, long wavelength waves on an air-sea interface in a canal of rectangular cross section. The steady periodic wave-train solution is called the *cnoidal wave*.

C.S. Gardner and G.K. Morikawa [30] found a new application of this model in the study of collision-free hydro-magnetic waves in hopes of describing the unidirectional propagation of small but finite amplitude waves in a nonlinear dispersive medium. Also, M. Kruskal and N. Zabusky [81] showed that the KdV equation models the Fermi-Pasta-Ulam problem, as it describes longitudinal waves propagating in a one-dimensional lattice of equal masses coupled by nonlinear springs. Other applications have been found and are currently studied.

2.0.7 Controllability and stabilization for PDEs - Methods Used

We are interested in obtaining controllability and stabilization results for systems governed by dispersive PDEs. Therefore, we first deal with two major problems concerning this theory: the internal controllability and controllability at the border.

The various concepts of controllability, which agree in finite dimension but not in general for a PDE, are introduced and next characterized thanks to the classical duality approach (see [28, 42]). For instance, the exact controllability of a system is shown to be equivalent to the observability of the adjoint system. The proof given here is based on the Hilbert Uniqueness Method (HUM) due to J.-L. Lions (for more details see [42, 43, 44]). The controllability tests given here may be seen as natural extensions of Kalman rank criterion as shown in [19].

Special attention is given to PDEs with skew-adjoint infinitesimal generator, for whose the controllability and stabilizability concepts considered here agree. We only deal with both concepts in infinite dimension.

Before addressing the control problems, let us introduce some notations. Let $\mathcal{P}(\mathcal{D})$ denote a differential operator, with $\mathcal{P} \in \mathbb{C}[\tau, \xi_1, \ldots, \xi_n]$ and

$$\mathcal{D} = (-i\partial_t, -i\partial_{x_1}, \ldots, -i\partial_{x_n}).$$

E.g. $\mathcal{P} = -\tau^2 + |\xi|^2$ gives the wave operator $\mathcal{P}(\mathcal{D}) = \partial_t^2 - \Delta$. Let $\Omega \subset \mathbb{R}^n$ be a bounded (sufficiently smooth) open set, whose boundary $\partial\Omega$ is denoted by Γ.

2.0.7.1 Internal control problem

Given some open set $\omega \subset \Omega$ with a smooth boundary Γ, and a set of boundary conditions, merely written $\mathcal{B}(\mathcal{D})z = 0$, we consider the control problem

$$\begin{cases} \mathcal{P}(\mathcal{D})z = \mathcal{X}_\omega f & t > 0, x \in \Omega, \\ \mathcal{B}(\mathcal{D})z = 0 & t > 0, x \in \Gamma, \\ z(0,x) = z_0(x) & x \in \Omega. \end{cases} \tag{2.0.5}$$

Here, $f = f(t,x)$ is the internal control, $z = z(t,x)$ is the unknown function. For the controllability problem, given z_0 and z_1 in some functional space H, we seek for a control $f \in L^2(0,T;U)$ (U being another functional space) such that the solution z of the system (2.0.5) satisfies $z(T,x) = z_1(x)$.

2.0.7.2 Boundary control problem

Given some open set $\gamma \subset \Gamma$, and two sets of boundary conditions $\mathcal{B}_1(\mathcal{D})z = \mathcal{X}_\omega f$, $\mathcal{B}_2(\mathcal{D})z = 0$, we consider the control problem

$$\begin{cases} \mathcal{P}(\mathcal{D})z = 0 & t > 0, x \in \Omega, \\ \mathcal{B}_1(\mathcal{D})z = \mathcal{X}_\omega f & t > 0, x \in \Gamma, \\ \mathcal{B}_2(\mathcal{D})z = 0 & x \in \Omega, \\ z(0,x) = z_0(x) & x \in \Omega. \end{cases} \tag{2.0.6}$$

Here$f = f(t,x)$ is the boundary control. In general ω (resp. γ) is a strict subset of Ω (resp. Γ).

Using a domain extension together with classical trace results, one may often derive boundary control results from internal control results.

2.0.8 Controllability and observability

2.0.8.1 Concepts of Controllability

For given $z_0 \in H$, $u \in L^2(0,T;U)$, we consider the solution $z : [0,T] \to H$ of the Cauchy problem

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0. \end{cases} \tag{2.0.7}$$

Recall that for any $z_0 \in \mathcal{D}(A)$ and $u \in W^{1,1}(0,T;U)$, the Cauchy problem (2.0.7) admits a unique classical solution $z \in C([0,T];\mathcal{D}(A)) \cap C^1(0,T;H)$ given by Duhamel formula

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,ds,\ \forall t \in [0,T],$$

where $\{S(t)\}_{t\geq 0}$ is a semigroup generated by the operator A. For $z_0 \in H$ and $u \in L^1(0,T;U)$, the above formula is still meaningful and defines the *mild solution* of (2.0.7).

Definition 2.1. *System (2.0.7) is exactly controllable in time T if for any $z_0, z_T \in H$, there exists $u \in L^2(0,T;U)$ such that the solution z of (2.0.7) fulfills $z(T) = z_T$;*

Definition 2.2. *System (2.0.7) is null controllable in time T if for any $z_0 \in H$, there exists $u \in L^2(0,T;U)$ such that the solution z of (2.0.7) fulfills $z(T) = 0$.*

Let us introduce the operator $\mathcal{L}_T : L^2(0,T;U) \longrightarrow H$ defined by

$$\mathcal{L}_T u = \int_0^T S(T-t)Bu(s)\,ds.$$

Then,

$$\text{Exact controllability in time } T \Leftrightarrow \operatorname{Im}\mathcal{L}_T = H; \tag{2.0.8}$$

$$\text{Zero controllability in time } T \Leftrightarrow S(T)H \subset \operatorname{Im}\mathcal{L}_T. \tag{2.0.9}$$

In finite dimension, i. e., when $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, the three concepts are equivalent, and equivalent to a purely algebraic condition, the famous Kalman rank condition: $rank\,(B, AB, \ldots, A^{n-1}B) = n$. As a consequence, the time T plays no role (for more details see, for example, [19, 80]).

The situation is more tricky for PDE:

– there is no algebraic test for the controllability;

– the control time plays a role for hyperbolic PDE;

– the converses of

$$\text{Exact Controllability } \Rightarrow \text{ Null Controllability}$$

is not true in general.

2.0.8.2 Adjoint operators

As mentioned before, the controllability problems requires the proof an observability inequality for the solution of the adjoint system. Therefore, the following definition will be needed:

The *adjoint* of the bounded operator $B \in \mathcal{L}(U,H)$ is the operator $B^* \in \mathcal{L}(H,U)$ defined by $(B^*z,u)_U = (z,Bu)_H$ for all $z \in H$ and $u \in U$. Thus, the adjoint of the (unbounded) operator A is the unbounded operator A^* with domain

$$\mathcal{D}(A^*) = \left\{z \in H : \exists C \in \mathbb{R}^+, |(Ay,z)_H| \leq C \|y\|_H, \forall y \in \mathcal{D}(A)\right\}$$

and defined by

$$(Ay,z)_H = (y,A^*z)_H, \forall y \in \mathcal{D}(A), \forall z \in \mathcal{D}(A^*).$$

Therefore, A^* also generates a continuous semigroup $\left(e^{tA}\right)_{t\geq 0}$ fulfilling $e^{tA^*} = S^*(t)$, $\forall t \geq 0$. If $A^* = A$ (resp. $A^* = -A$) the operator A is said to be self-adjoint (resp. skew-adjoint). Recall that a skew-adjoint operator generates a continuous group of isometries (see e.g. [54]).

2.0.8.3 Controllability tests

The proofs of the results cited here are classic, and they can be found, for example in [19, 42, 80, 87]. These tests are based on the HUM method due to J.-L. Lions [42]. A first result ensures that the controllability can be given as follows:
Theorem A: The system (2.0.7) is *exactly controllable in time* $T > 0$ if and only if there exists a constant $c > 0$ such that

$$\int_0^T \|B^*S^*(t)\,y_0\|_U^2\,dt \geq c\,\|y_0\|_H^2, \forall y_0 \in H. \tag{2.0.10}$$

(2.0.10) is called an *observability inequality.* Such inequality means that the map

$$\Upsilon : y_0 \longmapsto B^*S^*(\cdot)\,y_0,$$

is boundedly invertible; i.e., it is possible to recover a complete information about the initial state y_0 from a measure on $[0,T]$ of the output $B^*[S^*(t)\,y_0]\,t$ (*observability property*).
Theorem B: The system (2.0.7) is *null controllable in time* $T > 0$ if and only if there exists a constant $c > 0$ such that

$$\int_0^T \|B^*S^*(t)\,y_0\|_U^2\,dt \geq c\,\|S^*(T)\,y_0\|_H^2, \forall y_0 \in H. \tag{2.0.11}$$

(2.0.11) is a weak observability inequality, i. e., only $S^*(T)\,y_0$ may be recovered, not y_0.

The Hilbert Uniqueness Method. We associate to the boundary-initial value problem

$$\Sigma \qquad \begin{cases} \dot{z} = Az + Bu, \\ z(0) = 0, \end{cases}$$

its adjoint problem, obtained by taking the distributional adjoint of the operator $\partial_t - A$, namely $-\partial_t - A^*$:

$$\Sigma^* \qquad \begin{cases} \dot{y} = -A^* y, \\ y(T) = y_T. \end{cases}$$

Note that Σ^* is without control and backwards in time. For any $y_T \in H$, the solution y of Σ^* is given by $y(t) = S^*(T-t)\, y_T$.

We assume the following *key identity*:

$$(z(t), y_T)_H = \int_0^T (u, B^* y)_U \, dt$$

to ensure the equivalence between observability inequality and controllability of the system Σ. In addition, we can conclude that:

– The evolution equation in the *adjoint problem* $\dot{y} = -A^* y$ differs from the one for the *adjoint operator* $\dot{y} = A^* y$ by a sign minus. Solutions of the second one give solutions of the first one just by changing $t \to T - t$ inside;

– HUM provides a bounded operator $\Lambda : z_T \longmapsto u$ giving the control;

– In general we don't need to explicit B and B^*. The important ingredients in HUM are the *key identity* and the *observability inequality*.

2.0.9 Stabilizability

In this last section we address the stabilizability of the control system (2.0.7). In order to do that, we consider $K \in \mathcal{L}(H, U)$, let A_K the operator $A_K z = Az + BKz$ with domain $\mathcal{D}(A_K) = \mathcal{D}(A)$ and by $(S_K(t))_{t \geq 0}$ the semigroup generated by A_K.

The system (2.0.7) is said to be *exponentially stabilizable* if there exists a feedback $K \in \mathcal{L}(H, U)$ such that the operator A_K is exponentially stable; i.e., for some constants $C > 0$ and $\mu > 0$,

$$\|S_K(t)\| \leq Ce^{-\mu t}, \forall t \geq 0.$$

On the other hand side, the system (2.0.7) is said to be *completely stabilizable* if it is exponentially stabilizable with an arbitrary exponential decay rate; i.e., for arbitrary $\mu \in \mathbb{R}$, there exists a feedback $K \in \mathcal{L}(H, U)$ and a constant $C > 0$ such that

$$\|S_K(t)\| \leq Ce^{-\mu t}, \forall t \geq 0.$$

Stabilization of the system (2.0.7) is is strongly related to controllability. The first result in this direction was given by Datko in 1972 (see [22]).

Theorem C: If the system (2.0.7) is null controllable, then it is exponentially stabilizable.

The next result gives an infinite dimensional version of Wonham's theorem.

Theorem D: Assume that A generates a group $(S(t))_{t\geq 0}$ of operators. Then the following properties are equivalent.

i) The system (2.0.7) is exactly controllable in some time $T > 0$;

ii) The system (2.0.7) is null controllable in some time $T > 0$;

iii) The system (2.0.7) is completely stabilizable.

The implication i) $\Rightarrow$ iii) is due to Slemrod [75], whereas the implication iii) $\Rightarrow$ i) is due to Megan [47] (see also Zabczyk [80, Theorem 3.4 p. 229]). i) $\Rightarrow$ ii) is obvious.

The previous result applies in particular to a skew-adjoint operator A, which generates a group of isometries on H. In fact more can be said. First, we have the "controllability via stabilizability" principle due to Russell [69, 37]. On the other hand, explicit exponentially stabilizing feedback laws may be given (for example, [45, Theorem 2.3]).
Theorem E: Assume that A is skew-adjoint. Then the following properties are equivalent.

– The system (2.0.7) is exactly controllable in some time $T > 0$;

– The system (2.0.7) is null controllable in some time $T > 0$;

– The system (2.0.7) is completely stabilizable;

– The system (2.0.7) is exponentially stabilizable;

–For every positive definite self-adjoint operator $S \in \mathcal{L}(U)$, the operator $A - BSB^*$ generates an exponentially stable semigroup on H.

2.0.10 Problems and Main Results

We now return to the main problem of this thesis, i.e., the study of controllability and stabilization properties of models that describe the propagation of a surface water wave. We begin our study with the *Gear–Grimshaw system on a periodic domain.*

2.0.10.1 Stabilization of the Gear–Grimshaw system

The first work of this thesis, in collaboration with V. Komornik and A. Pazoto [13], is to investigate the decay properties of the initial-value problem

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x + k(u - [u]) = 0, \\ b_1 v' + r v_x + v v_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 u u_x \\ \qquad\qquad\qquad\qquad + b_2 a_1 (uv)_x + k(v - [v]) = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{2.0.12}$$

with periodic boundary conditions

$$\partial_x^k u(0) = \partial_x^k u(1),\ k = 0, 1, 2, \ldots$$

In (2.0.12), $r, a_1, a_2, a_3, b_1, b_2, k$ are given real constants with $b_1, b_2, k > 0$, $u(t,x), v(t,x)$ are real-valued functions of the time and space variables $t \geq 0$ and $0 \leq x \leq 1$, the subscript x and the prime indicate the partial differentiation with respect to x and t, respectively, and $[f]$ denotes the mean value of f defined by

$$[f] := \int_0^1 f(x)\ dx.$$

When $k = 0$, system was proposed by Gear and Grimshaw [31] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been object of intensive research in recent years. In what concerns the stabilization problems, most of the works have been focused on a bounded interval with a localized internal damping (see, for instance, [55] and the references therein). In particular, we also refer to [8] for an extensive discussion on the physical relevance of the system and to [23, 24, 25, 26, 27] for the results used in this chapter.

We can (formally) check that the total energy

$$E = \frac{1}{2}\int_0^1 b_2u^2 + b_1v^2\ dx$$

associated with the model satisfies the inequality

$$E' = -k\int_0^1 b_2(u-[u])^2 + (v-[v])^2\ dx \le 0$$

in $(0,\infty)$, so that the energy in nonincreasing. Therefore, the following basic questions arise: are the solutions asymptotically stable for t sufficiently large? And if yes, is it possible to find a rate of decay? The aim of this work is to answer these questions.

More precisely, we prove that for any fixed integer $s \ge 3$, the solutions are exponentially stable in the Sobolev spaces

$$H_p^s(0,1) := \{u \in H^s(0,1)\ :\ \partial_x^n u(0) = \partial_x^n u(1), \quad n = 0,\dots,s\}$$

with periodic boundary conditions. This extends an earlier theorem of Dávila in [26] for $s \le 2$.

Before stating the stabilization result mentioned above, we first need to ensure the well posedness of the system. This was addressed by Dávila in [23] (see also [24]) under the following conditions on the coefficients:

$$\begin{array}{c} a_3^2b_2 < 1 \text{ and } r = 0 \\ b_2a_1a_3 - b_1a_3 + b_1a_2 - a_2 = 0 \\ b_1a_1 - a_1 - b_1a_2a_3 + a_3 = 0 \\ b_1a_2^2 + b_2a_1^2 - b_1a_1 - a_2 = 0. \end{array} \tag{2.0.13}$$

Indeed, under conditions (2.0.13), Dávila and Chaves [27] derived some conservation laws for the solutions of (2.0.12). Combined with an approach introduced in [9, 73], these conservation laws allow them to establish the global well-posedness in $H_p^s(0,1)$, for any $s \ge 0$. Moreover, the authors also give a simpler derivation of the conservation laws discovered by Gear and Grimshaw, and Bona et al [8]. We also observe that these conservation properties were obtained employing the techniques developed in [52] for the single KdV equation; see also [51].

The well-posedness result reads as follows:

Theorem 2.1. *Assume that condition* (2.0.13) *holds. If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \geq 3$*, then the system* (2.0.12) *has a unique solution satisfying*

$$u, v \in C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1)).$$

Moreover, the map $(\phi, \psi) \mapsto (u, v)$ *is continuous from* $\left(H_p^s(0,1)\right)^2$ *into*

$$\left(C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1))\right)^2.$$

For $k = 0$, the analogous theorem on the whole real line $-\infty < x < \infty$ was proved Bona et al. [8], for all $s \geq 1$.

With the global well-posedness result in hand, we can focus on the stabilization problem. For simplicity of notation we consider only the case

$$b_1 = b_2 = 1. \tag{2.0.14}$$

Then the conditions (2.0.13) take the simplified form

$$\begin{gathered} r = 0, \quad a_1^2 + a_2^2 = a_1 + a_2 \\ |a_3| < 1 \\ (a_1 - 1)a_3 = (a_2 - 1)a_3 = 0. \end{gathered} \tag{2.0.15}$$

Hence either $a_3 = 0$ and $a_1^2 + a_2^2 = a_1 + a_2$, or $0 < |a_3| < 1$ and $a_1 = a_2 = 1$.

We prove the following theorem:

Theorem 2.2. *Assume* (2.0.14) *and* (2.0.15)*. If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \geq 3$*, then the solution of* (2.0.12) *satisfies the estimate*

$$\|u(t) - [u(t)]\|_{H_p^s(0,1)} + \|v(t) - [v(t)]\|_{H_p^s(0,1)} = o\left(e^{-k't}\right), \quad t \to \infty,$$

for each $k' < k$.

An analogous theorem was proved in [38] for the usual KdV equation by using the infinite family of conservation laws for this equation. Such conservations lead to the construction of a suitable Lyapunov function that gives the exponential decay of the solutions. Here, we follow the same approach making use of the results established by Dávila and Chavez [27]. They proved that under the assumptions (2.0.13) system (2.0.12) also has an infinite family of conservation laws, and they conjectured the above theorem for this case. At this point we observe that some computations are simplified if we change u, v, ϕ and ψ to $u - [u]$, $v - [v]$, $\phi - [\phi]$ and $\psi - [\psi]$. Then, the new unknown functions u and v satisfy the same system (2.0.12) with ku and kv instead of $k(u - [u])$ and $k(v - [v])$. Hence we consider the solutions of the simplified system

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x + ku = 0, \\ v' + vv_x + v_{xxx} + a_3 u_{xxx} + a_2 u u_x + a_1 (uv)_x + kv = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{2.0.16}$$

with periodic boundary conditions, corresponding to initial data ϕ, ψ with zero mean values.

In order to obtain the result, we prove a number of identities and estimates for the solutions of (2.0.12). In view of Theorem 2.1 it suffices to establish these estimates for *smooth solutions*, i.e., to solutions corresponding to C^∞ initial data ϕ, ψ with periodic boundary conditions. For such solutions all formal manipulations in the sequel will be justified.

Finally, we also observe that a similar result was obtained in [40] for the scalar KdV equation on a periodic domain. The authors study the model from a control point of view with a forcing term f supported in a given open set of the domain. It is shown that the system is globally exactly controllable and globally exponentially stable. The stabilization is established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces for the solutions of the corresponding linear system. We also refer to [40] for a quite complete review on the subject.

2.0.10.2 Controllability for the Korteweg-de Vries equation

The second work of this thesis, in collaboration with L. Rosier and A. Pazoto [14], has the interest to investigate the properties of controllability for the equation of Korteweg-de Vries (KdV). The Korteweg–de Vries (KdV) equation can be written

$$u_t + u_{xxx} + u_x + uu_x = 0,$$

where $u = u(t,x)$ is real-valued function of two real variables t and x, and $u_t = \partial u/\partial t$, etc. The equation was first derived by Boussinesq [10] and Korteweg-de Vries [39] as a model for the propagation of water waves along a channel. The equation furnishes also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems.

The KdV equation has been intensively studied from various aspects of mathematics, including the well-posedness, the existence and stability of solitary waves, the integrability, the long-time behavior, etc. (see e.g. [35, 51]) The practical use of the KdV equation does not always involve the pure initial value problem. In numerical studies, one is often interested in using a finite interval (instead of the whole line) with three boundary conditions.

Here, we shall be concerned with the control properties of KdV, the control acting through a forcing term f incorporated in the equation:

$$u_t + u_{xxx} + u_x + uu_x = f, \quad t \in [0,T], \; x \in [0,L], \qquad + \; \text{b.c.} \tag{2.0.17}$$

Our main purpose is to see whether one can force the solutions of (2.0.17) to have certain desired properties by choosing an appropriate control input f. The focus here is on the *controllability* issue:

Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the equation (2.0.17) admits a solution u which equals u_0 at time $t = 0$ and u_1 at time $t = T$?

If one can always find a control input f to guide the system described by (2.0.17) from any given initial state u_0 to any given terminal state u_1, then the system (2.0.17) is said to be *exactly controllable*. If the system can be driven, by means of a control f, from any state to the origin (i.e. $u_1 \equiv 0$), then one says that system (2.0.17) is *null controllable*.

The study of the controllability and stabilization of the KdV equation started with the works of Russell and Zhang [72] for a system with periodic boundary conditions and an internal control. Since then, both the controllability and the stabilization have been intensively studied. (We refer the reader to [65] for a survey of the results up to 2009.) In particular, the exact boundary controllability of KdV on a finite domain was investigated in e.g. [16, 17, 20, 32, 33, 62, 64, 85]. Most of those works were concerned with the following system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{in } (0,T) \times (0,L), \\ u(t,0) = g_1(t),\ u(t,L) = g_2(t),\ u_x(t,L) = g_3(t) & \text{in } (0,T) \end{cases} \tag{2.0.18}$$

in which the boundary data g_1, g_2, g_3 can be chosen as control inputs. System (2.0.18) was first studied by Rosier [62] considering only the control input g_3 (i.e. $g_1 = g_2 = 0$). It was shown in [62] that the exact controllability of the linearized system holds in $L^2(0,L)$ if, and only if, L does not belong to the following countable set of *critical lengths*

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \ : k,\ l\ \in \mathbb{N}^* \right\}. \tag{2.0.19}$$

The analysis developed in [62] shows that when the linearized system is controllable, the same is true for the nonlinear one. Note that the converse is false, for it was proved in [16, 17, 20] that the (nonlinear) KdV equation is controllable even when L is a critical length. It is also worth mentioning some results due to Rosier [64] and Glass and Guerrero [32] with g_1 as control input (i.e. $g_2 = g_3 = 0$). They proved that system (2.0.18) is then null controllable, but not exactly controllable, because of the strong smoothing effect.

By contrast, the mathematical theory pertaining to the study of the internal controllability in a bounded domain is considerably less advanced. As far as we know, the null controllability problem for system (2.0.17) was only addressed in [32] when the control acts in a neighborhood of the endpoint $x = 0$. On the other hand, the exact controllability results in [40, 72] were obtained in a periodic domain.

The aim of this chapter is to address the controllability problem with a distributed control on a bounded domain. As far as the null controllability is concerned, our main result reads as follows:

Theorem 2.3. *Let $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < L$, and let $T > 0$. For $\bar{u}_0 \in L^2(0,L)$, let $\bar{u} \in C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$ denote the solution of*

$$\begin{cases} \bar{u}_t + \bar{u}_x + \bar{u}\,\bar{u}_x + \bar{u}_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \bar{u}(t,0) = \bar{u}(t,L) = \bar{u}_x(t,L) = 0 & \text{in } (0,T), \\ \bar{u}(0,x) = \bar{u}_0(x) & \text{in } (0,L). \end{cases} \tag{2.0.20}$$

Then there exists $\delta > 0$ *such that for any* $u_0 \in L^2(0,L)$ *satisfying* $\|u_0 - \bar{u}_0\|_{L^2(0,L)} \leq \delta$, *there exists* $f \in L^2((0,T)\times\omega)$ *such that the solution* $u \in C^0([0,T];L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ *of*

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{2.0.21}$$

satisfies $u(T,\cdot) = \bar{u}(T,\cdot)$ *in* $(0,L)$.

The null controllability is first established for a linearized system

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = 1_\omega f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{2.0.22}$$

by following the classical duality approach (see [28, 43]), which reduces the null controllability of (2.0.22) to an observability inequality for the solutions of the adjoint system. To prove the observability inequality, we derive a new Carleman estimate with an internal observation in $(0,T)\times(l_1,l_2)$ and use some interpolation arguments inspired by those in [32], where the authors derived a similar result when the control acts on a neighborhood on the left endpoint (that is, $l_1 = 0$). The null controllability is extended to the nonlinear system by applying Kakutani fixed-point theorem.

The second problem we address is related to the exact internal controllability of system (2.0.17). As far as we know, the same problem was studied only in [40, 72] in a periodic domain $\mathbb{T}$ with a distributed control of the form

$$f(x,t) = (Gh)(x,t) := g(x)(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)dy),$$

where $g \in C^\infty(\mathbb{T})$ was such that $\{g > 0\} = \omega$ and $\int_{\mathbb{T}} g(x)dx = 1$, and the function h was considered as a new control input. Here, we shall consider the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L). \end{cases} \tag{2.0.23}$$

As the smoothing effect is different from those in a periodic domain, the results in this chapter turn out to be very different from those in [40, 72]. First, for a controllability result in $L^2(0,L)$, the control f has to be taken in the space $L^2(0,T,H^{-1}(0,L))$. Actually, with any control $f \in L^2(0,T,L^2(0,L))$, the solution of (2.0.23) starting from $u_0 = 0$ at $t = 0$ would remain in $H^1_0(0,L)$ (see [32]). On the other hand, as for the boundary control, the localization of the distributed control plays a role in the results.

When the control acts in a neighborhood of $x = L$, we obtain the exact controllability in the weighted Sobolev space $L^2_{\frac{1}{L-x}dx}$ defined as

$$L^2_{\frac{1}{L-x}dx} := \{u \in L^1_{loc}(0,L); \int_0^L \frac{|u(x)|^2}{L-x}dx < \infty\}.$$

More precisely, we shall obtain the following result:

Theorem 2.4. *Let $T > 0$, $\omega = (l_1, l_2) = (L - \nu, L)$ where $0 < \nu < L$. Then, there exists $\delta > 0$ such that for any u_0, $u_1 \in L^2_{\frac{1}{L-x}dx}$ with*

$$\|u_0\|_{L^2_{\frac{1}{L-x}dx}} \leq \delta \quad \text{and} \quad \|u_1\|_{L^2_{\frac{1}{L-x}dx}} \leq \delta,$$

one can find a control input $f \in L^2(0, T; H^{-1}(0, L))$ such that the solution

$$u \in C^0([0, L], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$$

of (2.0.23) satisfies $u(T, x) = u_1(x)$ in $(0, L)$ and $u \in C^0([0, T], L^2_{\frac{1}{L-x}dx})$. Furthermore, $f \in L^2_{(T-t)dt}(0, T, L^2(0, L))$.

Actually, we shall have to investigate the well-posedness of the linearization of (2.0.23) in the space $L^2_{\frac{1}{L-x}dx}$ and the well-posedness of the (backward) adjoint system in the "dual space" $L^2_{(L-x)dx}$. To do this, we shall follow some ideas borrowed from [34], where the well-posedness was investigated in the weighted space $L^2_{\frac{x}{L-x}dx}$. The needed observability inequality is obtained by the standard compactness-uniqueness argument and some unique continuation property. The exact controllability is extended to the nonlinear system by using the contraction principle.

When the control is acting far from the endpoint $x = L$, i.e. in some interval $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < L$, then there is no chance to control exactly the state function on (l_2, L) (see e.g. [64]). However, it is possible to control the state function on $(0, l_1)$, so that a "regional controllability" can be established:

Theorem 2.5. *Let $T > 0$ and $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < L$. Pick any number $l_1' \in (l_1, l_2)$. Then there exists a number $\delta > 0$ such that for any $u_0, u_1 \in L^2(0, L)$ satisfying*

$$||u_0||_{L^2(0,L)} \leq \delta, \qquad ||u_1||_{L^2(0,L)} \leq \delta,$$

one can find a control $f \in L^2(0, T, H^{-1}(0, L))$ with $supp(f) \subset (0, T) \times \omega$ such that the solution $u \in C^0([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ of (2.0.23) satisfies

$$u(T, x) = \begin{cases} u_1(x) & \text{if } x \in (0, l_1'); \\ 0 & \text{if } x \in (l_2, L). \end{cases} \tag{2.0.24}$$

The proof of Theorem 2.5 combines Theorem 2.3, a boundary controllability result from [62], and the use of a cut-off function. Note that the issue whether u may also be controlled in the interval (l_1', l_2) is open.

2.0.10.3 Controllability of Boussinesq Equation KdV-KdV type

The third and last work of this thesis, in collaboration with L. Rosier and A. Pazoto [15], has the interest to investigate the properties of controllability and asymptotic behavior for the system of Boussinesq KdV-KdV type.

The classical Boussinesq systems were first derived by Boussinesq, in [11], to describe the two-way propagation of small amplitude, long wave length gravity waves on the surface of water in a canal. These systems and their higher-order generalizations also arise when modeling the propagation of long-crested waves on large lakes or on the ocean and in other contexts. In [6], the authors derived a four-parameter family of Boussinesq systems to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. More precisely, they studied a family of systems of the form

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{2.0.25}$$

In (2.0.25), η is the elevation from the equilibrium position, and $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid. The parameters a, b, c, d, that one might choose in a given modeling situation, are required to fulfill the relations

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad \theta \in [0,1]\,. \tag{2.0.26}$$

where $\theta \in [0,1]$ specifies which horizontal velocity the variable w represents (cf. [6]). Consequently,

$$a + b + c + d = \frac{1}{3}$$

As it has been proved in [6], the initial value problem for the linear system associated with (2.0.25) is well posed on $\mathbb{R}$ if either C_1 or C_2 is satisfied, where

$$\begin{aligned} &(C_1)\ b, d \geq 0,\ \ a \leq 0,\ \ c \leq 0; \\ &(C_2)\ b, d \geq 0,\ \ a = c > 0. \end{aligned}$$

In mathematical studies, considerations have been mainly given to pure initial value problems and well-posedness results [7]. However, the practical use of the above system and its relatives does not always involve the pure initial value problem. Instead, the initial boundary value problem often comes to the fore.

Recently, in [50], a rather complete picture of the control properties of (2.0.25) on a periodic domain with a locally supported forcing term was given. According to the values of the four parameters a, b, c, d, the linearized system may be controllable in any positive time, or only in large time, or may not be controllable at all. These results were also extended in [50] to the generic nonlinear system (2.0.25), i.e., when all the parameters are different from 0.

When $b = d = 0$ and (C_2) is satisfied, then necessarily $a = c = 1/6$. Nevertheless, the scaling $x \to x/\sqrt{6}$, $t \to t/\sqrt{6}$ gives an system equivalent to (2.0.25) for which $a = c = 1$, namely

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0, \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0. \end{cases} \tag{2.0.27}$$

When the model is posed on a bounded interval, Rosier and Pazoto, in [58], investigated the asymptotic behavior of the solutions assuming that $b = d = 0$ and $a = c = 1$.

More precisely, the authors studied the following Boussinesq system of KdV-KdV type

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + ww_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{2.0.28}$$

satisfying the boundary conditions

$$\begin{cases} w(t,0) = w_{xx}(t,0) = 0 & \text{in } (0,T), \\ w_x(t,0) = \alpha_0 \eta_x(t,0) & \text{in } (0,T), \\ w(t,L) = \alpha_2 \eta(t,L) & \text{in } (0,T), \\ w_x(t,L) = -\alpha_1 \eta_x(t,L) & \text{in } (0,T), \\ w_{xx}(t,L) = -\alpha_2 \eta_{xx}(t,L) & \text{in } (0,T), \end{cases} \tag{2.0.29}$$

and initial conditions

$$\eta(0,x) = \eta_0(x), \;\; w(0,x) = w_0(x) \qquad \text{in } (0,L). \tag{2.0.30}$$

In (2.0.28), α_0, α_1 and α_2 denote some nonnegative real constants. The KdV–KdV system is expected to admit global solutions on $\mathbb{R}$, and it also possesses good control properties on the torus [50].

Under the above boundary conditions, the authors observed that the derivative of the energy associated with the system (2.0.28), with boundary conditions (2.0.29)-(2.0.30) satisfies

$$\frac{dE}{dt} = -\alpha_2 \left|\eta(L,t)\right|^2 - \alpha_1 \left|\eta_x(L,t)\right|^2 - \alpha_0 \left|\eta_x(0,t)\right|^2 - \frac{1}{3} w^3(L,t) - \int_0^L (\eta w)_x \eta dx$$

where

$$E(t) = \frac{1}{2}\int_0^L (\eta^2 + w^2) dx.$$

This indicates that the boundary conditions play the role of a feedback damping mechanism, at least for the linearized system. Therefore, the following questions arise:

(i) Does $E(t) \to 0$, as $t \to +\infty$?

(ii) If it is the case, can we give the decay rate?

The problem might be easy to solve when the underlying model has a intrinsic dissipative nature. Moreover, in the context of coupled systems, in order to achieve the desired decay property, the damping mechanism has to be designed in an appropriate way in order to capture all the components of the system. The main result of Rosier and Pazoto provides a positive answer to those questions.

Theorem 2.6. *([58]) Assume that $\alpha_0 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 = 1$. Then there exist some numbers $\rho > 0$, $C > 0$ and $\mu > 0$ such that for any $(\eta_0, w_0) \in (L^2(I))^2$ with*

$$\|(\eta_0, w_0)\|_{(L^2(I))^2} \leq \rho,$$

the system (2.0.28)-(2.0.30) admits a unique solution

$$(\eta, w) \in C(\mathbb{R}^+; (L^2(I))^2) \cap C(\mathbb{R}^{+*}; (H^1(I))^2) \cap L^2((0,1); (H^1(I))^2),$$

which fulfills

$$\|(\eta,w)(t)\|_{(L^2(I))^2} \leq Ce^{-\mu t}\|(\eta_0,w_0)\|_{(L^2(I))^2}, \ \forall t \geq 0,$$

$$\|(\eta,w)(t)\|_{(H^1(I))^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}}\|(\eta_0,w_0)\|_{(L^2(I))^2}, \ \forall t > 0.$$

To our knowledge, the boundary control of the Boussinesq system of KdV-KdV type is completely open. The aim of this chapter is to investigate the control properties of the following system

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + ww_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{2.0.31}$$

with the boundary conditions

$$\begin{cases} \eta(t,0) = h_0(t),\ \eta(t,L) = h_1(t) & \text{in } (0,T), \\ w(t,0) = g_0(t),\ w(t,L) = g_1(t) & \text{in } (0,T), \\ \eta_x(t,0) = h_2(t),\ w_x(t,L) = g_2(t) & \text{in } (0,T) \end{cases} \tag{2.0.32}$$

and the initial conditions

$$\eta(0,x) = \eta_0(x), \ \ w(0,x) = w_0(x) \qquad \text{in } (0,L). \tag{2.0.33}$$

A similar problem was studied by Rosier [62] in the case of the KdV equation considering only one control,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = 0,\ u_x(t,L) = g_3(t) & \text{in } (0,T). \end{cases} \tag{2.0.34}$$

It was shown that the exact controllability of the linearized KdV equation holds in $L^2(0,L)$ if, and only if, L does not belong to the following (discrete) set of critical lengths

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2} \ : k,\ l\ \in \mathbb{N}^* \right\}. \tag{2.0.35}$$

To begin with, we consider the linearized Boussinesq system

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{2.0.36}$$

together with the boundary conditions (2.0.32) and the initial data (2.0.33).

The results established in this chapter show that, depending on the combination of the controls g_i and h_i, two sets of critical lengths appear; namely $\mathcal{N}$ and the (new) set

$$\mathcal{R} := \left\{ \pi\sqrt{(\frac{1}{2}+2k)^2 + (\frac{1}{2}+2l)^2 + (\frac{1}{2}+2k)(\frac{1}{2}+2l)} : k,l \in \mathbb{N}^* \right\}. \tag{2.0.37}$$

Introduce the space

$$X = \left\{ (\eta,w) \in \left[H^2(0,L)\cap H_0^1(0,L)\right]^2 : \eta_x(0) = w_x(L) = 0 \right\}; \tag{2.0.38}$$

and let X' denote the dual of X with respect to the pivot space $L^2(0,L)^2$. Some of the main results in this chapter are stated in the following theorem.

Theorem 2.7. *Let $\mathcal{N}$, $\mathcal{R}$, and X be defined by (2.0.35), (2.0.37), and (2.0.38), respectively. Then the following holds.*

(i) For any $T > 0$, $L \in (0,+\infty)\backslash\mathcal{N}$, $(\eta_0, w_0) \in (H^{-1}(0,L))^2$ and $(\eta_T, w_T) \in (H^{-1}(0,L))^2$ there exist some controls $h_2, g_2 \in L^2(0,T)$ such that the solution $(\eta, w) \in C^0([0,T],(H^{-1}(0,L))^2)$ of (2.0.36) and (2.0.32)-(2.0.33), with $h_i = 0$ and $g_i = 0$ for $i = 0,1$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$;

(ii) For any $T > 0$, $L \in (0,+\infty)\backslash\mathcal{N}$, $(\eta_0, w_0) \in (H^{-1}(0,L))^2$ and $(\eta_T, w_T) \in (H^{-1}(0,L))^2$, there exists a control $h_2 \in L^2(0,T)$ such that the solution $(\eta, w) \in C^0([0,T],(H^{-1}(0,L))^2)$ of (2.0.36) and (2.0.32)-(2.0.33), with $h_i = 0$ and $g_i = 0$ for $i = 0,1$ and $g_2 = 0$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$;

(iii) For any $T > 0$, $L > 0$, $(\eta_0, w_0) \in X'$ and $(\eta_T, w_T) \in X'$, there exist some controls $h_1, g_1 \in L^2(0,T)$ such that, the solution $(\eta, w) \in C([0,T],X'$ of (2.0.36) and (2.0.32)-(2.0.33), with $h_i = 0$ and $g_i = 0$ for $i = 0,2$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$;

(iv) For any $T > 0$, $L \in (0,+\infty)\backslash(\mathcal{N}\cup\mathcal{R})$, $(\eta_0, w_0) \in X'$ and $(\eta_T, w_T) \in X'$, there exists a control $g_1 \in L^2(0,T)$ such that the solution $(\eta, w) \in C^0([0,T],X')$ of (2.0.36) and (2.0.32)-(2.0.33), with $h_i = 0$ and $g_i = 0$ for $i = 0,2$ and $h_1 = 0$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$.

Actually, a more complete picture of the control results obtained in this chapter are presented in following table.

	Controls							Properties	
Case	h_0	h_1	h_2	g_0	g_1	g_2	Controls	State Space	Lengths
1	0	0	$\star$	0	0	$\star$	$h_2, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in \left[H^{-1}(0,L)\right]^2$	$\mathcal{N}$
2	0	0	$\star$	0	0	0	$h_2 \in L^2(0,T)$	$(\eta_0, w_0) \in \left[H^{-1}(0,L)\right]^2$	$\mathcal{N}$
3	0	$\star$	0	0	$\star$	0	$h_1, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\emptyset$
4	0	0	0	0	$\star$	0	$g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}\cup\mathcal{R}$
5	$\star$	0	0	0	$\star$	0	$h_0, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}\cup\mathcal{R}$
6	0	0	$\star$	0	$\star$	0	$h_2, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
7	0	$\star$	0	0	0	$\star$	$h_1, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
8	$\star$	$\star$	0	0	0	0	$h_0, h_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
9	0	$\star$	$\star$	0	0	0	$h_1, h_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\emptyset$
10	0	0	$\star$	$\star$	0	0	$h_2, g_0 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$

Table1. Controllability results for the linear system

To prove our control results, we use the classical duality approach based upon the Hilbert Uniqueness Method (H.U.M.) due to J.-L. Lions [42], which reduces our control properties to some observability inequalities for the adjoint systems. Next, to establish the observability inequalities, we use the compactness-uniqueness argument due to E. Zuazua (see the appendix in [42]) and some multipliers to reduce the problem to a spectral problem. The spectral problem is finally solved by using a method introduced in [62] and based on Fourier analysis and complex analysis.

Boussinesq system is more convenient than KdV as a model for the propagation of water waves, as it is adapted to the wave propagation in the two directions, and it is still valid after bounces of waves at the boundary. The initial value problem for Boussinesq system is less developed than for KdV, probably because of the complexity of the system. Nevertheless, it is striking that the control properties of Boussinesq system are better understood than for KdV: indeed, the critical lengths for Boussinesq system are explicitly given for any set of boundary controls, which is not the case for KdV (e.g. the critical lengths are not explicitly known with a Dirichlet control at the right point $x = L$, see [33]). This is probably due to the fact that $x = 0$ and $x = L$ (resp. w and η) play a symmetric role for the linearized Boussinesq system. The price to be paid is the lack of the Kato smoothing effect in general, which makes the extension of the control results to the nonlinear Boussinesq system delicate.

In what concerns the nonlinear problem, due to technical difficulties that come from the lack of regularity of solutions, special boundary conditions are used. The issue of the controllability of the nonlinear system (2.0.31) with the boundary conditions (2.0.32) will be investigated elsewhere.

Thus, we consider the system

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L). \end{cases} \tag{2.0.39}$$

satisfying either the boundary conditions

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0 & \text{in } (0,T), \\ w(t,0) = w(t,L) = 0 & \text{in } (0,T), \\ w_x(t,L) + \alpha_1 \eta_x(t,L) = g_2(t) & \text{in } (0,T), \end{cases} \tag{2.0.40}$$

or the boundary conditions

$$\begin{cases} \eta(t,L) = \eta_x(t,0) = 0 & \text{in } (0,T), \\ w(t,0) = w(t,L) = 0 & \text{in } (0,T), \\ \alpha_2 \eta(t,0) + \alpha_3 \eta_x(t,L) + w_{xx}(t,0) = h_0(t) & \text{in } (0,T), \\ w_x(t,L) - \alpha_3(\eta(t,0) - \eta_x(t,L)) = g_2(t) & \text{in } (0,T), \end{cases} \tag{2.0.41}$$

where α_i are positive constant for $i = 1, 2, 3$, and the initial conditions

$$\eta(0,x) = \eta_0(x), \;\; w(0,x) = w_0(x) \qquad \text{in } (0,L). \tag{2.0.42}$$

With (2.0.40) or (2.0.41), a global Kato smoothing effect similar to those for KdV can be derived. As a consequence, a result similar to Theorem 2.7 can be established for the system above. More precisely, the following results concerning the well-posedness and the exact controllability of the above systems will be established:

Theorem 2.8. *Let $X_0 = (L^2(0,L))^2$, $T > 0$ and $L \in (0,+\infty)\backslash\mathcal{N}$, where $\mathcal{N}$ is defined by (2.0.35). Then, there exists a constant $\delta > 0$ such that for any initial data $(\eta_0, w_0) \in X_0$ and and final data $(\eta_T, w_T) \in X_0$ satisfying*

$$\|(\eta_0, w_0)\|_{X_0} \leq \delta \;\; \text{and} \;\; \|(\eta_T, w_T)\|_{X_0} \leq \delta,$$

there exists a control $g_2 \in L^2(0,T)$ *such that the solution*

$$(\eta, w) \in C([0,T], X_0) \cap L^2(0,T; (H^1(0,L))^2) \cap H^1(0,L; (H^{-2}(0,L))^2),$$

of (2.0.39) *with* (2.0.42) *and the boundary conditions* (2.0.40) *satisfies* $\eta(T,\cdot) = \eta_T$ *and* $w(T,\cdot) = w_T$ *in* $(0,L)$.

Theorem 2.9. *Let* $T > 0$ *and* $L \in (0,+\infty)\backslash\mathcal{N}$. *Then, there exists a constant* $\delta > 0$ *such that for any initial data* $(\eta_0, w_0) \in X_0$ *and and final data* $(\eta_T, w_T) \in X_0$ *satisfying*

$$\left\|(\eta^0, w^0)\right\|_{X_0} \leq \delta \text{ and } \left\|(\eta^1, w^1)\right\|_{X_0} \leq \delta,$$

there exist two controls $(h_0, g_2) \in (L^2(0,T))^2$ *such that the solution*

$$(\eta, w) \in C([0,T], X_0) \cap L^2(0,T; (H^1(0,L))^2) \cap H^1(0,L; (H^{-2}(0,L))^2),$$

of (2.0.39) *with* (2.0.42) *and the boundary condition* (2.0.41) *satisfies* $\eta(T,\cdot) = \eta_T$ *and* $w(T,\cdot) = w_T$ *in* $(0,L)$.

The second part of the work is devoted to the study of the exponential decay of $E(t)$ when $g_2 = h_2 = 0$. In this case, the energy associated with (2.0.39) with boundary conditions (2.0.40) (resp. (2.0.41)) satisfies

$$\frac{d}{dt}E = -\alpha_1 \left|\eta_x(t,L)\right|^2 - \int_0^L (\eta w)_x \eta dx$$

(resp.

$$\frac{d}{dt}E = -\alpha_2 \left|\eta(t,0)\right|^2 - \alpha_3 \left|\eta_x(t,L)\right|^2 - \int_0^L (\eta w)_x \eta dx).$$

Thus, as in [58], we obtain the following result:

Theorem 2.10. *Assume that* $\alpha_1, \alpha_2, \alpha_3 > 0$ *and* $L \in (0,+\infty)\backslash\mathcal{N}$. *Then, there exist some numbers* $\rho > 0$, $C > 0$ *and* $\mu > 0$ *such that for any* $(\eta_0, w_0) \in (L^2(I))^2$ *with*

$$\|(\eta_0, w_0)\|_{(L^2(I))^2} \leq \rho,$$

the system (2.0.39) *with boundary conditions* (2.0.40) *(or* (2.0.41)*) and initial condition* (2.0.42) *admits a unique solution*

$$(\eta, w) \in C(\mathbb{R}^+; (L^2(I))^2) \cap C(\mathbb{R}^{+*}; (H^1(I))^2) \cap L^2((0,1); (H^1(I))^2),$$

which fulfills

$$\|(\eta, w)(t)\|_{(L^2(I))^2} \leq Ce^{-\mu t} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t \geq 0,$$

$$\|(\eta, w)(t)\|_{(H^1(I))^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t > 0.$$

In summary, this thesis consists of four chapters from the introduction divided as follows: The first chapter deals with results submitted and concentrated in [13], in collaboration with prof. Vilmos Komornik and Prof. Ademir F. Pazoto, finalized in a visit to the University of Strasbourg in November 2012. The second and third chapters are concentrated in the work [14] and [15] in process of submission in collaboration with Prof. Lionel Rosier and Prof. Ademir Pazoto, finalized during a sandwich doctor at the University of Lorraine - Nancy. Finally, the fourth chapter deals with conclusions on the studied systems and future open problems to start a search for good level in dispersive equations governed by partial differential equations.

Chapter 3

Stabilization of the Gear–Grimshaw system on a periodic domain

3.1 Introduction

The goal of this chapter is to investigate the decay properties of the initial-value problem

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x + k(u - [u]) = 0, \\ b_1 v' + r v_x + v v_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 u u_x \\ \qquad\qquad\qquad\qquad + b_2 a_1 (uv)_x + k(v - [v]) = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{3.1.1}$$

with periodic boundary conditions. In (3.1.1), $r, a_1, a_2, a_3, b_1, b_2, k$ are given real constants with $b_1, b_2, k > 0$, $u(t,x), v(t,x)$ are real-valued functions of the time and space variables $t \geq 0$ and $0 \leq x \leq 1$, the subscript x and the prime indicate the partial differentiation with respect to x and t, respectively, and $[f]$ denotes the mean value of f defined by

$$[f] := \int_0^1 f(x)\ dx.$$

When $k = 0$, system was proposed by Gear and Grimshaw [31] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been object of intensive research in recent years. In what concerns the stabilization problems, most of the works have been focused on a bounded interval with a localized internal damping (see, for instance, [55] and the references therein). In particular, we also refer to [8] for an extensive discussion on the physical relevance of the system and to [23, 24, 25, 26, 27] for the results used in this chapter.

We can (formally) check that the total energy

$$E = \frac{1}{2} \int_0^1 b_2 u^2 + b_1 v^2\ dx$$

associated with the model satisfies the inequality

$$E' = -k \int_0^1 b_2(u-[u])^2 + (v-[v])^2 \, dx \leq 0$$

in $(0,\infty)$, so that the energy in nonincreasing. Therefore, the following basic questions arise: are the solutions asymptotically stable for t sufficiently large? And if yes, is it possible to find a rate of decay? The aim of this chapter is to answer these questions.

More precisely, we prove that for any fixed integer $s \geq 3$, the solutions are exponentially stable in the Sobolev spaces

$$H_p^s(0,1) := \{u \in H^s(0,1) \ : \ \partial_x^n u(0) = \partial_x^n u(1), \quad n = 0,\dots,s\}$$

with periodic boundary conditions. This extends an earlier theorem of Dávila in [26] for $s \leq 2$.

Before stating the stabilization result mentioned above, we first need to ensure the well posedness of the system. This was addressed by Dávila in [23] (see also [24]) under the following conditions on the coefficients:

$$\begin{gathered} a_3^2 b_2 < 1 \text{ and } r = 0 \\ b_2 a_1 a_3 - b_1 a_3 + b_1 a_2 - a_2 = 0 \\ b_1 a_1 - a_1 - b_1 a_2 a_3 + a_3 = 0 \\ b_1 a_2^2 + b_2 a_1^2 - b_1 a_1 - a_2 = 0. \end{gathered} \tag{3.1.2}$$

Indeed, under conditions (3.1.2), Dávila and Chaves [27] derived some conservation laws for the solutions of (3.1.1). Combined with an approach introduced in [9, 73], these conservation laws allow them to establish the global well-posedness in $H_p^s(0,1)$, for any $s \geq 0$. Moreover, the authors also give a simpler derivation of the conservation laws discovered by Gear and Grimshaw, and Bona et al [8]. We also observe that these conservation properties were obtained employing the techniques developed in [52] for the single KdV equation; see also [51].

The well-posedness result reads as follows:

Theorem 3.1. *Assume that condition (3.1.2) holds. If $\phi, \psi \in H_p^s(0,1)$ for some integer $s \geq 3$, then the system (3.1.1) has a unique solution satisfying*

$$u, v \in C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1)).$$

Moreover, the map $(\phi,\psi) \mapsto (u,v)$ is continuous from $\left(H_p^s(0,1)\right)^2$ into

$$\left(C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1))\right)^2.$$

For $k = 0$, the analogous theorem on the whole real line $-\infty < x < \infty$ was proved Bona et al. [8], for all $s \geq 1$.

With the global well-posedness result in hand, we can focus on the stabilization problem. For simplicity of notation we consider only the case

$$b_1 = b_2 = 1. \tag{3.1.3}$$

Then the conditions (3.1.2) take the simplified form

$$r = 0, \quad a_1^2 + a_2^2 = a_1 + a_2, \quad |a_3| < 1, \quad \text{and} \quad (a_1 - 1)a_3 = (a_2 - 1)a_3 = 0. \qquad (3.1.4)$$

Hence either $a_3 = 0$ and $a_1^2 + a_2^2 = a_1 + a_2$, or $0 < |a_3| < 1$ and $a_1 = a_2 = 1$.

We prove the following theorem:

Theorem 3.2. *Assume* (3.1.3) *and* (3.1.4). *If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \geq 3$, *then the solution of* (3.1.1) *satisfies the estimate*

$$\|u(t) - [u(t)]\|_{H_p^s(0,1)} + \|v(t) - [v(t)]\|_{H_p^s(0,1)} = o\left(e^{-k't}\right), \quad t \to \infty$$

for each $k' < k$.

An analogous theorem was proved in [38] for the usual KdV equation by using the infinite family of conservation laws for this equation. Such conservations lead to the construction of a suitable Lyapunov function that gives the exponential decay of the solutions. Here, we follow the same approach making use of the results established by Dávila and Chavez [27]. They proved that under the assumptions (3.1.2) system (3.1.1) also has an infinite family of conservation laws, and they conjectured the above theorem for this case.

In order to obtain the result, we prove a number of identities and estimates for the solutions of (3.1.1). In view of Theorem 3.1 it suffices to establish these estimates for *smooth solutions*, i.e., to solutions corresponding to C^∞ initial data ϕ, ψ with periodic boundary conditions. For such solutions all formal manipulations in the sequel will be justified.

Finally, we also observe that a similar result was obtained in [40] for the scalar KdV equation on a periodic domain. The authors study the model from a control point of view with a forcing term f supported in a given open set of the domain. It is shown that the system is globally exactly controllable and globally exponentially stable. The stabilization is established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces for the solutions of the corresponding linear system. We also refer to [40] for a quite complete review on the subject.

The chapter is organized as follows. In Section 4.2 introduce the basic notations and we prove some technical lemmas. Sections 4.3 to 4.6 are devoted to the proof of the exponential decay in H_p^s, for $s = 0, 1, 2$ and $s \geq 3$, respectively.

3.2 Some technical lemmas

In the sequel all integrals are taken over the interval $(0, 1)$ so we omit the integration limits.

As explained in the introduction, all integrations by parts will be done for smooth periodic functions. Therefore, we will regularly use the simplified formulas

$$\int f_x g \, dx = -\int f g_x \, dx \quad \text{and} \quad \int f^n f_x \, dx = 0 \quad (n = 0, 1, \ldots)$$

without further explanation, and we will also use the simplified notation

$$f_n := \frac{d^n f}{dx^n}, \quad n = 1, 2, \ldots$$

As an example of the application of these rules we show that the mean-values of of the solutions are conserved:

Lemma 3.1. *The mean-values* $[u]$ *and* $[v]$ *of the solutions of* (3.1.1) *do not depend on* t*.*

Proof. We have

$$\begin{aligned}
[u]' &= -\int uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2 (uv)_x + k(u - [u]) \, dx \\
&= -\int \left(\frac{u^2}{2} + u_{xx} + a_3 v_{xx} + a_1 \frac{v^2}{2} + a_2 uv \right)_x + k(u - [u]) \, dx \\
&= -k \int (u - [u]) \, dx \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[v]' &= -\int vv_x + v_{xxx} + a_3 u_{xxx} + a_2 uu_x + a_1 (uv)_x + k(v - [v]) \, dx \\
&= -\int \left(\frac{v^2}{2} + v_{xx} + a_3 u_{xx} + a_2 \frac{u^2}{2} + a_1 uv \right)_x + k(v - [v]) \, dx \\
&= -k \int (v - [v]) \, dx \\
&= 0
\end{aligned}$$

by a straightforward computation. □

Motivated by this result we set $M = [\varphi]$, $N = [\psi]$ and we rewrite (3.1.1) by changing u, v, φ, ψ to $u - [u] = u - M$, $v - [v] = v - N$, $\varphi - [\varphi] = \varphi - M$ and $\psi - [\psi] = \psi - N$, respectively. Under our assumptions $r = 0$ and $b_1 = b_2 = 1$ we obtain the equivalent system

$$\begin{cases}
u' + (u + M)u_x + u_{xxx} + a_3 v_{xxx} + a_1 (v + N) v_x \\
\qquad\qquad + a_2((u + M)(v + N))_x + ku = 0, \\
v' + (v + N)v_x + v_{xxx} + a_3 u_{xxx} + a_2 (u + M) u_x \\
\qquad\qquad + a_1((u + M)(v + N))_x + kv = 0, \\
u(0, x) = \phi(x), \\
v(0, x) = \psi(x)
\end{cases} \tag{3.2.1}$$

with periodic boundary conditions, corresponding to initial data ϕ, ψ with zero mean values. Theorem 3.2 will thus follow from the following proposition:

Proposition 3.1. *Under the assumptions of Theorem 3.2 the smooth solutions of* (3.2.1) *satisfy the identity*

$$\int u(t)^2 + v(t)^2 \; dx = e^{-2kt} \int \phi^2 + \psi^2 \; dx, \quad t \geq 0, \tag{3.2.2}$$

and the estimates

$$e^{2k't} \int (\partial_x^n u(t))^2 + (\partial_x^n v(t))^2 \, dx \to 0 \quad \text{as} \quad t \to \infty$$

for all positive integers n and for all $k' < k$.

Remark 3.1. *For $n = 1$ the proposition and its proof remain valid under the weaker assumption that $|a_3| < 1$. We can also add the term rv_x to the equation by changing g to $g - rv^2$ in Lemma 3.6.*

Proposition 3.1 is proved by using the Lyapunov method. More precisely, we shall use the following lemma:

Lemma 3.2. *Let $f : (0, \infty) \to \mathbb{R}$ be a nonnegative function, and write $h_1 \approx h_2$ if $h_1 - h_2 = o(f)$ as $t \to \infty$.*

If there exists a function $g : (0, \infty) \to \mathbb{R}$ such that $g \approx 0$, $f + g$ is continuously differentiable, and $(f + g)' \approx -2kf$ for some positive number k, then

$$e^{2k't} f(t) \to 0 \quad \text{as} \quad t \to \infty$$

for each $k' < k$.

Proof. Fix $k'' > 0$ such that $k' < k'' < k$, and then fix $\epsilon > 0$ such that

$$\frac{1 - \epsilon}{1 + \epsilon} = \frac{k''}{k}.$$

Finally, choose a sufficiently large $t' > 0$ such that

$$(1 - \epsilon) f(t) \leq (f + g)(t) \leq (1 + \epsilon) f(t)$$

and

$$2k(1 - \epsilon) f(t) \leq -(f + g)'(t) \leq 2k(1 + \epsilon) f(t)$$

for all $t \geq t'$. Then for $t \geq t'$ we have

$$-(f + g)'(t) \geq 2k(1 - \epsilon) f(t) \geq 2k \frac{1 - \epsilon}{1 + \epsilon} (f + g)(t) = 2k''(f + g)(t),$$

whence

$$\frac{d}{dt} \left(e^{2k''t} (f + g)(t) \right) \leq 0.$$

It follows that

$$e^{2k''t} (f + g)(t) \leq e^{2k''t'} (f + g)(t')$$

for all $t \geq t'$, and hence

$$0 \leq e^{2k't} f(t) \leq \frac{e^{2k''t'} (f + g)(t')}{1 - \epsilon} e^{-2(k'' - k')t}$$

for all $t \geq t'$. We conclude by observing that $e^{-2(k'' - k')t} \to 0$ as $t \to \infty$. □

For the proof of the next result, we shall use the Hölder and Poincaré–Wirtinger inequalities in the following form. The second estimate will be used only for functions with mean value zero: $[u] = 0$.

Lemma 3.3. *If $p, q \in [0, \infty)$, then*

$$\|u\|_p \le \|u\|_q \quad \textit{for all} \quad u \in L^q(0,1) \quad \textit{and} \quad 1 \le p \le q \le \infty; \tag{3.2.3}$$

$$\|u - [u]\|_p \le \|u - [u]\|_q \quad \textit{for all} \quad u \in H^1(0,1) \quad \textit{and} \quad 1 \le p, q \le \infty. \tag{3.2.4}$$

We shall frequently use Lemma 3.2 together with the following result:

Lemma 3.4. *Let $n \ge 1$ and let α_m, β_m, $m = 0, \dots, n$ be nonnegative integers satisfying the two conditions*

$$2(\alpha_n + \beta_n) + \alpha_{n-1} + \beta_{n-1} \le 4$$

and

$$d := \sum_{m=0}^{n} (\alpha_m + \beta_m) \ge 2.$$

Then

$$\left| \int \prod_{m=0}^{n} u_m^{\alpha_m} v_m^{\beta_m} \; dx \right| \le \left(\int u_n^2 + v_n^2 \; dx \right) \left(\int u_{n-1}^2 + v_{n-1}^2 \; dx \right)^{\frac{d-2}{2}}.$$

If, moreover, $d \ge 3$ and

$$\int u_{n-1}^2 + v_{n-1}^2 \; dx \to 0,$$

then it follows that

$$\int \prod_{m=0}^{n} u_m^{\alpha_m} v_m^{\beta_m} \; dx = o\left(\int u_n^2 + v_n^2 \; dx \right)$$

as $t \to \infty$.

Proof. Setting

$$z_m := \sqrt{u_m^2 + v_m^2} \quad \text{and} \quad \gamma_m := \alpha_m + \beta_m, \quad m = 0, \dots, n$$

we have

$$\left| \int \prod_{m=0}^{n} u_m^{\alpha_m} v_m^{\beta_m} \; dx \right| \le \int \prod_{m=0}^{n} z_m^{\gamma_m} \; dx.$$

We are going to majorize the right side by using the Hölder and Poincaré–Wirtinger inequalities (3.2.3)–(3.2.4). We distinguish five cases according to the value of $\gamma_n + \gamma_{n-1}$: since $2\gamma_n + \gamma_{n-1} \le 4$ by our assumption, $\gamma_n + \gamma_{n-1} \le 4$.

If $\gamma_n + \gamma_{n-1} = 0$, then we have

$$\left| \int \prod_{m=0}^{n} z_m^{\gamma_m} \; dx \right| \le \prod_{m=0}^{n-2} \|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2 \, \|z_{n-1}\|_2^{d-2}.$$

If $\gamma_n + \gamma_{n-1} = 1$, then

$$\left|\int \prod_{m=0}^{n} z_m^{\gamma_m}\ dx\right| \le \|z_n\|_1 \prod_{m=0}^{n-2} \|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2 \|z_{n-1}\|_2^{d-2}.$$

If $\gamma_n + \gamma_{n-1} = 2$, then

$$\left|\int \prod_{m=0}^{n} z_m^{\gamma_m}\ dx\right| \le \|z_n\|_2^2 \prod_{m=0}^{n-2} \|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2 \|z_{n-1}\|_2^{d-2}.$$

If $\gamma_n + \gamma_{n-1} = 3$, then we have necessarily $\gamma_n = 1$ and $\gamma_{n-1} = 2$, so that

$$\left|\int \prod_{m=0}^{n} z_m^{\gamma_m}\ dx\right| \le \|z_n\|_2 \|z_{n-1}\|_\infty \|z_{n-1}\|_2 \prod_{m=0}^{n-2} \|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2 \|z_{n-1}\|_2^{d-2}.$$

Finally, if $\gamma_n + \gamma_{n-1} = 4$, then we have necessarily $\gamma_n = 0$ and $\gamma_{n-1} = 4$, so that

$$\left|\int \prod_{m=0}^{n} z_m^{\gamma_m}\ dx\right| \le \|z_{n-1}\|_\infty^2 \|z_{n-1}\|_2^2 \prod_{m=0}^{n-2} \|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2 \|z_{n-1}\|_2^{d-2}.$$

□

3.3 Proof of Proposition 3.1 for $n = 0$

Our proof is based on the following identity:

Lemma 3.5. *The solutions of* (3.2.1) *satisfy the following identity for all* $n = 0, 1, \dots$:

$$\begin{aligned}\left(\int u_n^2 + v_n^2\ dx\right)' = &-2k \int u_n^2 + v_n^2\ dx \\ &- 2\int u_n (u_1 u)_n + v_n (v_1 v)_n\ dx \\ &- 2a_1 \int u_n (v v_1)_n + v_n (uv)_{n+1}\ dx \\ &- 2a_2 \int v_n (u u_1)_n + u_n (uv)_{n+1}\ dx.\end{aligned} \tag{3.3.1}$$

Proof. We have

$$\begin{aligned}\left(\int u_n^2 + v_n^2\ dx\right)' &= \int 2u_n u_n' + 2v_n v_n'\ dx \\ &= \int -2u_n((u+M)u_1 + u_3 + a_3 v_3 + a_1(v+N)v_1 \\ &\qquad + a_2((u+M)(v+N))_1 + ku)_n\ dx \\ &\quad + \int -2v_n((v+N)v_1 + v_3 + a_3 u_3 + a_2(u+M)u_1 \\ &\qquad + a_1((u+M)(v+N))_1 + kv)_n\ dx.\end{aligned}$$

This yields the stated identity because

$$\begin{aligned}\int -2u_n u_{n+3} - 2v_n v_{n+3}\ dx &= \int 2u_{n+1}u_{n+2} + 2v_{n+1}v_{n+2}\ dx \\ &= \int (u_{n+1}^2)_1 + (v_{n+1}^2)_1\ dx = 0,\end{aligned}$$

$$a_3 \int -2u_n v_{n+3} - 2v_n u_{n+3}\ dx = a_3 \int -2u_n v_{n+3} + 2v_{n+3}u_n\ dx = 0,$$

$$\begin{aligned}-\,2M \int & u_n u_{n+1} + a_2 u_n v_{n+1} + a_2 v_n u_{n+1} + a_1 v_n v_{n+1}\ dx \\ &= -M \int \left(u_n^2 + 2a_2 u_n v_n + a_1 v_n^2\right)_1\ dx = 0,\end{aligned}$$

$$\begin{aligned}-\,2N \int & a_1 u_n v_{n+1} + a_2 u_n u_{n+1} + v_n v_{n+1} + a_1 v_n u_{n+1}\ dx \\ &= -N \int \left(2a_1 u_n v_n + a_2 u_n^2 + v_n^2\right)_1\ dx = 0\end{aligned}$$

and $(MN)_1 = 0$. □

Proof of the proposition for $n = 0$. In this case the last three integrals of the identity (3.3.1) vanish because

$$\begin{aligned}\int uu_1u + vv_1v\ dx &= \frac{1}{3}\int (u^3 + v^3)_1\ dx = 0, \\ \int uvv_1 + v(uv)_1\ dx &= \int (uvv)_1\ dx = 0\end{aligned}$$

and

$$\int vuu_1 + u(uv)_1\ dx = \int (vuu)_1\ dx = 0.$$

□

Proceeding by induction on n, let $n \geq 1$ and assume that the estimates

$$\int u_m^2 + v_m^2\ dx = o\left(e^{-2k't}\right) \quad \text{as} \quad t \to \infty \tag{3.3.2}$$

hold for all integers $m = 0, \ldots, n-1$ and for all $k' < k$. For $n = 1$ this follows from the stronger identity (3.2.2).

3.4 Proof of Proposition 3.1 for $n = 1$

For the proof of the case $n = 1$ we shall use an identity suggested by a conservation law discovered by Bona et al. [8].

Lemma 3.6. *Setting*

$$f := \int u_1^2 + v_1^2 + 2a_3 u_1 v_1 \; dx$$

and

$$g := -\frac{1}{3}\int (u^3 + v^3) + 3(a_1 u v^2 + a_2 u^2 v) \; dx,$$

we have the following identity:

$$(f + g)' = -2kf - 3kg. \tag{3.4.1}$$

Proof. The equality (3.4.1) will follow by combining the following four identities:

$$\begin{aligned} \left(\int u_1^2 + v_1^2 \; dx\right)' = & -2k \int u_1^2 + v_1^2 \; dx \\ & - \int u_1^3 + v_1^3 \; dx \\ & - 3a_1 \int u_1 v_1^2 \; dx \\ & - 3a_2 \int u_1^2 v_1 \; dx; \end{aligned} \tag{3.4.2}$$

$$\begin{aligned} \left(\int u_1 v_1 \; dx\right)' = & -2k \int u_1 v_1 \; dx + \int u u_1 v_2 + v v_1 u_2 \; dx \\ & - \frac{a_1}{2} \int 2v_2 u_1 u + 3v_1 u_1^2 + v_1^3 \; dx \\ & - \frac{a_2}{2} \int 2u_2 v_1 v + 3u_1 v_1^2 + u_1^3 \; dx; \end{aligned} \tag{3.4.3}$$

$$\begin{aligned} \left(\int u^3 + v^3 \; dx\right)' = & -3k \int u^3 + v^3 \; dx - 3 \int u_1^3 + v_1^3 \; dx \\ & - a_1 \int 3u^2 v v_1 + 2v^3 u_1 \; dx \\ & - a_2 \int 3v^2 u u_1 + 2u^3 v_1 \; dx. \\ & + 6a_3 \int u u_1 v_2 + v v_1 u_2 \; dx; \end{aligned} \tag{3.4.4}$$

$$\left(\int a_1uv^2 + a_2u^2v\ dx\right)' = -3k\int a_1uv^2 + a_2u^2v\ dx \tag{3.4.5}$$
$$+ a_1\int \frac{2}{3}v^3u_1 + u^2vv_1 - 3v_1^2u_1\ dx$$
$$+ a_2\int \frac{2}{3}u^3v_1 + v^2uu_1 - 3u_1^2v_1\ dx$$
$$- a_1a_3\int 2v_2u_1u + 3v_1u_1^2 + v_1^3\ dx$$
$$- a_2a_3\int 2u_2v_1v + 3u_1v_1^2 + u_1^3\ dx.$$

Proof of (3.4.2). We transform the identity (3.3.1) for $n = 1$ as follows. We have

$$\begin{aligned}\int u_1(u_1u)_1 + v_1(v_1v)_1\ dx &= \int u_2u_1u + u_1^3 + v_2v_1v + v_1^3\ dx\\ &= \int u_1^3 + v_1^3 + \frac{1}{2}(u_1^2)_1u + \frac{1}{2}(v_1^2)_1v\ dx\\ &= \frac{1}{2}\int u_1^3 + v_1^3\ dx,\end{aligned}$$

$$\begin{aligned}\int u_1(vv_1)_1 + v_1(uv)_2\ dx &= \int u_1v_1^2 + u_1vv_2 - v_2(uv)_1\ dx\\ &= \int u_1v_1^2 - v_2uv_1\ dx\\ &= \int u_1v_1^2 - \frac{1}{2}u(v_1^2)_1\ dx\\ &= \frac{3}{2}\int u_1v_1^2\ dx,\end{aligned}$$

and by symmetry

$$\int v_1(uu_1)_1 + u_1(uv)_2\ dx = \frac{3}{2}\int u_1^2v_1\ dx.$$

Using them (3.3.1) implies (3.4.2).

Proof of (3.4.3). We have

$$\begin{aligned}
\left(\int u_1 v_1\ dx\right)' &= \int u_1' v_1 + u_1 v_1'\ dx \\
&= \int -(uu_1 + u_3 + a_3 v_3 + a_1 v v_1 + a_2 (uv)_1 + ku)_1 v_1\ dx \\
&\qquad + \int -u_1 (v v_1 + v_3 + a_3 u_3 + a_2 u u_1 + a_1 (uv)_1 + kv)_1\ dx \\
&= -2k \int u_1 v_1\ dx + \int (uu_1 + u_3) v_2 + (v v_1 + v_3) u_2\ dx \\
&\qquad - a_1 \int (v v_1)_1 v_1 + u_1 (uv)_2\ dx \\
&\qquad - a_2 \int (uv)_2 v_1 + u_1 (u u_1)_1\ dx \\
&\qquad - a_3 \int v_4 v_1 + u_4 u_1\ dx \\
&= -2k \int u_1 v_1\ dx + \int u u_1 v_2 + v v_1 u_2\ dx \\
&\qquad + a_1 \int v v_1 v_2 + u_2 (uv)_1\ dx \\
&\qquad + a_2 \int (uv)_1 v_2 + u_2 u u_1\ dx
\end{aligned}$$

because

$$\int u_3 v_2 + v_3 u_2\ dx = \int u_3 v_2 - v_2 u_3\ dx = 0$$

and

$$\int v_4 v_1 + u_4 u_1\ dx = -\int v_3 v_2 + u_3 u_2\ dx = -\frac{1}{2} \int (v_2^2 + u_2^2)_1\ dx = 0.$$

Since

$$\begin{aligned}
\int v v_1 v_2 + u_2 (uv)_1\ dx &= \int \frac{1}{2} v (v_1^2)_1 + \frac{1}{2} (u_1^2)_1 v + u_2 u v_1\ dx \\
&= \int -\frac{1}{2} v_1^3 - \frac{1}{2} u_1^2 v_1 - u_1^2 v_1 - u_1 u v_2\ dx \\
&= -\frac{1}{2} \int 2 v_2 u_1 u + 3 v_1 u_1^2 + v_1^3\ dx
\end{aligned}$$

and by symmetry

$$\int u u_1 u_2 + v_2 (uv)_1\ dx = -\frac{1}{2} \int 2 u_2 v_1 v + 3 u_1 v_1^2 + u_1^3\ dx,$$

(3.4.3) follows from the previous identity.

Proof of (3.4.4). We have

$$\begin{aligned}\left(\int u^3\ dx\right)' &= \int 3u^2u'\ dx \\ &= \int -3u^2(uu_1+u_3+a_3v_3+a_1vv_1+a_2(uv)_1+ku)\ dx \\ &= \int -\frac{3}{4}\left(u^4\right)_1+3u\left(u_1^2\right)_1-3ku^3\ dx-3a_3\int u^2v_3\ dx \\ &\qquad -3a_1\int u^2vv_1\ dx-3a_2\int u^3v_1+\frac{1}{3}(u^3)_1v\ dx \\ &= -3\int u_1^3+ku^3\ dx-3a_1\int u^2vv_1\ dx-2a_2\int u^3v_1\ dx \\ &\qquad +6a_3\int uu_1v_2\ dx.\end{aligned}$$

We have an analogous identity for $\int v^3\ dx$ by symmetry; adding the we get (3.4.4).

Proof of (3.4.5). We have

$$\begin{aligned}\left(\int u^2v\ dx\right)' &= \int u'(2uv)+u^2v'\ dx \\ &= \int -2uv(uu_1+u_3+a_3v_3+a_1vv_1+a_2(uv)_1+ku)\ dx \\ &\qquad +\int -u^2(vv_1+v_3+a_3u_3+a_2uu_1+a_1(uv)_1+kv)\ dx \\ &= \int -2u^2u_1v+2u_2(uv)_1-u^2vv_1+2v_2uu_1\ dx-3k\int u^2v\ dx \\ &\qquad -a_1\int 2uvvv_1+u^2(uv)_1\ dx \\ &\qquad -a_2\int 2uv(uv)_1+u^3u_1\ dx \\ &\qquad -a_3\int 2uvv_3+u^2u_3\ dx.\end{aligned}$$

Here

$$\int -2u^2u_1v\ dx = -\frac{2}{3}\int (u^3)_1v\ dx = \frac{2}{3}\int u^3v_1,$$

$$\int -u^2vv_1\ dx = -\frac{1}{2}\int u^2(v^2)_1\ dx = \frac{1}{2}\int (u^2)_1v^2\ dx = \int v^2uu_1\ dx,$$

$$
\begin{aligned}
\int 2u_2(uv)_1 + 2v_2uu_1\ dx &= \int (2u_2u_1v + 2u_2uv_1) - (2v_1u_1^2 + 2v_1uu_2)\ dx \\
&= \int (u_1^2)_1v - 2v_1u_1^2\ dx \\
&= -3\int u_1^2v_1\ dx,
\end{aligned}
$$

$$
\int 2uvvv_1 + u^2(uv)_1\ dx = \int \frac{2}{3}u(v^3)_1 + u^3v_1 + \frac{1}{3}(u^3)_1v\ dx = \frac{2}{3}\int u^3v_1 - v^3u_1\ dx,
$$

$$
\int 2uv(uv)_1 + u^3u_1\ dx = \int \left((uv)^2 + \frac{1}{4}u^4\right)_1\ dx = 0,
$$

and

$$
\begin{aligned}
\int 2uvv_3 + u^2u_3\ dx &= \int -2(u_1v + uv_1)v_2 - 2uu_1u_2\ dx \\
&= \int 2(u_2v + u_1v_1)v_1 - u(v_1^2)_1 - u(u_1^2)_1\ dx \\
&= \int 2(u_2v + u_1v_1)v_1 + u_1v_1^2 + u_1^3\ dx \\
&= \int 2u_2v_1v + 3u_1v_1^2 + u_1^3\ dx,
\end{aligned}
$$

so that

$$
\begin{aligned}
\left(\int u^2v\ dx\right)' = \int \frac{2}{3}u^3v_1 + v^2uu_1 - 3u_1^2v_1\ dx - 3k\int u^2v\ dx \\
- \frac{2}{3}a_1\int u^3v_1 - v^3u_1\ dx - a_3\int 2u_2v_1v + 3u_1v_1^2 + u_1^3\ dx.
\end{aligned}
$$

By symmetry, we also have

$$
\begin{aligned}
\left(\int v^2u\ dx\right)' = \int \frac{2}{3}v^3u_1 + u^2vv_1 - 3v_1^2u_1\ dx - 3k\int v^2u\ dx \\
- \frac{2}{3}a_2\int v^3u_1 - u^3v_1\ dx - a_3\int 2v_2u_1u + 3v_1u_1^2 + v_1^3\ dx.
\end{aligned}
$$

Combining the last two identities (3.4.5) follows (some terms annihilate each other). □

Proof of the proposition for $n = 1$. It suffices to show that the functions f and g of Lemma 3.6 satisfy the conditions of Lemma 3.2. Since $|a_3| < 1$, we have $f \geq 0$. The other conditions follow from the already proven case $n = 0$ and from the second part of Lemma 3.4. We conclude by applying the lemma and then by observing that

$$
\int u_1^2 + v_1^2\ dx \leq \frac{1}{1 - |a_3|}\int u_1^2 + v_1^2 + 2a_3u_1v_1\ dx.
$$

□

3.5 Proof of Proposition 3.1 for $n = 2$

Lemma 3.7. *Setting*

$$f := \int u_2^2 + v_2^2 + 2a_3 u_2 v_2 \, dx,$$
$$g := -\frac{5}{3} \int (u_1^2 u + v_1^2 v) + a_1(2u_1 v_1 v + v_1^2 u) + a_2(2u_1 v_1 u + u_1^2 v) \, dx$$

and

$$h := \frac{2}{3} a_3 \int (1 - a_1)(2u_3 v_2 u + u_2 v_2 u_1) + (1 - a_2)(2v_3 u_2 v + u_2 v_2 v_1) \, dx,$$

we have

$$(f + g)' \approx -2kf + h. \tag{3.5.1}$$

Proof. The relationship (3.5.1) will follow by combining the following relations:

$$\begin{aligned} \left(\int u_2^2 + v_2^2 \, dx \right)' = -2k \int u_2^2 + v_2^2 \, dx \\ - 5 \int u_2^2 u_1 + v_2^2 v_1 \, dx \\ - 5a_1 \int 2u_2 v_2 v_1 + v_2^2 u_1 \, dx \\ - 5a_2 \int 2u_2 v_2 u_1 + u_2^2 v_1 \, dx; \end{aligned} \tag{3.5.2}$$

$$\begin{aligned} \left(\int u_2 v_2 \, dx \right)' &= -2k \int u_2 v_2 \, dx \\ &- \int u_3 v_2 u + v_3 u_2 v + 3u_2 v_2 (u_1 + v_1) \, dx \\ &- a_1 \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 2u_2 v_2 u_1 - u_3 v_2 u \, dx \\ &- a_2 \int \frac{5}{2}(u_2^2 + v_2^2) u_1 + 2u_2 v_2 v_1 - v_3 u_2 v \, dx; \end{aligned} \tag{3.5.3}$$

$$\begin{aligned} \left(\int u_1^2 u + v_1^2 v \, dx \right)' \approx -3 \int u_2^2 u_1 + v_2^2 v_1 \, dx \\ - 2a_3 \int u_3 v_2 u + v_3 u_2 v + 2u_2 v_2 (u_1 + v_1) \, dx; \end{aligned} \tag{3.5.4}$$

$$\left(\int 2u_1v_1v + v_1^2u \; dx\right)' \approx -3\int 2u_2v_2v_1 + v_2^2u_1 \; dx \qquad (3.5.5)$$
$$+ a_3\int -3(u_2^2+v_2^2)v_1 + 2u_3v_2u - 2u_2v_2u_1 \; dx;$$

$$\left(\int 2u_1v_1u + u_1^2v \; dx\right)' \approx -3\int 2u_2v_2u_1 + u_2^2v_1 \; dx \qquad (3.5.6)$$
$$+ a_3\int -3(u_2^2+v_2^2)u_1 + 2v_3u_2v - 2u_2v_2v_1 \; dx.$$

Proof of (3.5.2). We transform the last three integrals of the identity (3.3.1) in the following way:

$$\begin{aligned}
-2\int u_2(u_1u)_2 + v_2(v_1v)_2 \; dx &= -2\int 3u_2^2u_1 + u_2u_3u + 3v_2^2v_1 + v_2v_3v \; dx \\
&= -2\int 3u_2^2u_1 + \frac{1}{2}(u_2^2)_1u + 3v_2^2v_1 + \frac{1}{2}(v_2^2)_1v \; dx \\
&= -5\int u_2^2u_1 + v_2^2v_1 \; dx,
\end{aligned}$$

$$\begin{aligned}
-2a_1\int u_2(vv_1)_2 + v_2(uv)_3 \; dx &= -2a_1\int 3u_2v_1v_2 + u_2vv_3 - v_3(uv)_2 \; dx \\
&= -2a_1\int 3u_2v_1v_2 - 2v_3u_1v_1 - v_3uv_2 \; dx \\
&= -2a_1\int 3u_2v_1v_2 + 2v_2(u_1v_1)_1 - \frac{1}{2}u(v_2^2)_1 \; dx \\
&= -2a_1\int 5u_2v_1v_2 + \frac{5}{2}u_1v_2^2 \; dx \\
&= -5a_1\int 2u_2v_2v_1 + v_2^2u_1 \; dx,
\end{aligned}$$

and by symmetry

$$-2a_2\int v_2(uu_1)_2 + u_2(uv)_3 \; dx = -5a_2\int 2u_2v_2u_1 + u_2^2v_1 \; dx.$$

Combining these identities with (3.3.1) we obtain (3.5.2).

Proof of (3.5.3). We have

$$\begin{aligned}
\left(\int u_2 v_2\ dx\right)' &= \int u_2' v_2 + u_2 v_2'\ dx \\
&= -\int (u_1 u + u_3 + ku + a_3 v_3 + a_1 v_1 v + a_2 (uv)_1)_2 v_2\ dx \\
&\qquad - \int u_2 (v_1 v + v_3 + kv + a_3 u_3 + a_2 u_1 u + a_1 (uv)_1)_2\ dx \\
&= -2k \int u_2 v_2\ dx \\
&\qquad - a_3 \int v_5 v_2 + u_2 u_5\ dx - \int u_5 v_2 + u_2 v_5\ dx \\
&\qquad - \int (uu_1)_2 v_2 + u_2 (vv_1)_2\ dx \\
&\qquad - a_1 \int (vv_1)_2 v_2 + u_2 (uv)_3\ dx \\
&\qquad - a_2 \int (uv)_3 v_2 + u_2 (uu_1)_2\ dx.
\end{aligned}$$

Here

$$\int v_5 v_2 + u_2 u_5\ dx = -\int v_4 v_3 + u_3 u_4\ dx = -\frac{1}{2}\int (v_3^2 + u_3^2)_1\ dx = 0,$$

$$\int u_5 v_2 + u_2 v_5\ dx = \int u_5 v_2 - u_5 v_2\ dx = 0,$$

$$\begin{aligned}
&\int (uu_1)_2 v_2 + u_2 (vv_1)_2\ dx \\
&\quad = \int 3u_1 u_2 v_2 + u v_2 u_3 + v u_2 v_3 + 3 v_1 v_2 u_2\ dx,
\end{aligned}$$

$$\begin{aligned}
&\int (vv_1)_2 v_2 + u_2 (uv)_3\ dx \\
&= \int 3 v_2^2 v_1 + v_3 v_2 v + u_3 u_2 v + 3 u_2^2 v_1 + 3 u_2 v_2 u_1 + v_3 u_2 u\ dx \\
&= \int 3 v_2^2 v_1 + \frac{1}{2}(v_2^2)_1 v + \frac{1}{2}(u_2^2)_1 v + 3 u_2^2 v_1 + 3 u_2 v_2 u_1 + v_3 u_2 u\ dx \\
&= \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 3 u_2 v_2 u_1 + v_3 u_2 u\ dx \\
&= \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 3 u_2 v_2 u_1 - v_2 u_3 u - v_2 u_2 u_1\ dx \\
&= \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 2 u_2 v_2 u_1 - u_3 v_2 u\ dx.
\end{aligned}$$

By symmetry, we also have

$$\int (uu_1)_2 u_2 + v_2(uv)_3 \; dx = \int \frac{5}{2}(u_2^2 + v_2^2)u_1 + 2u_2v_2v_1 - v_3u_2v \; dx.$$

This proves (3.5.3).

Henceforth in all computations we integrate by parts and we apply Lemma 3.4 several times.

Proof of (3.5.4). We have

$$\begin{aligned}
\left(\int u_1^2 u \; dx\right)' &= \int 2u_1u_1'u + u_1^2u' \; dx \\
&= \int -u'(2u_2u + u_1^2) \; dx \\
&= \int (2u_2u + u_1^2)(u_1u + u_3 + ku + a_1v_1v + a_2(uv)_1 + a_3v_3) \; dx \\
&= k\int 2u_2u^2 + u_1^2u \; dx \\
&\qquad + \int u_1u(2u_2u + u_1^2) \; dx \\
&\qquad + \int u_3(2u_2u + u_1^2) \; dx \\
&\qquad + a_1\int v_1v(2u_2u + u_1^2) \; dx \\
&\qquad + a_2\int (uv)_1(2u_2u + u_1^2) \; dx \\
&\qquad + a_3\int v_3(2u_2u + u_1^2) \; dx.
\end{aligned}$$

Here all integrals are equivalent to zero by Lemma 3.4, except those containing u_3 or v_3. Since

$$\int u_3(2u_2u + u_1^2) \; dx = \int (u_2^2)_1u + u_3u_1^2 \; dx = -\int u_2^2u_1 + 2u_2^2u_1 \; dx = -3\int u_2^2u_1 \; dx$$

and

$$\begin{aligned}
\int v_3(2u_2u + u_1^2) \; dx &= 2\int v_3u_2u - v_2u_2u_1 \; dx \\
&= 2\int -v_2u_3u - v_2u_2u_1 - v_2u_2u_1 \; dx \\
&= -2\int u_3v_2u + 2u_2v_2u_1 \; dx,
\end{aligned}$$

we conclude that

$$\left(\int u_1^2u \; dx\right)' \approx -3\int u_2^2u_1 \; dx - 2a_3\int u_3v_2u + 2u_2v_2u_1 \; dx.$$

Adding this to the analogous relationship for $\int v_1^2 v\ dx$ we get (3.5.4).

Proof of (3.5.5) *and* (3.5.6). We have

$$\begin{aligned}\left(\int u_1v_1v\ dx\right)' &= \int u_1'v_1v + u_1v_1'v + u_1v_1v'\ dx\\ &= \int -u'(v_2v+v_1^2) - v'u_2v\ dx\\ &= \int (v_2v+v_1^2)(u_1u+u_3+ku+a_1v_1v+a_2(uv)_1+a_3v_3)\ dx\\ &\qquad + \int u_2v(v_1v+v_3+kv+a_2u_1u+a_1(uv)_1+a_3u_3)\ dx\\ &\approx \int v_2vu_3+v_1^2u_3+u_2vv_3\ dx + a_3\int (v_2v+v_1^2)v_3+u_2vu_3\ dx\\ &= \int (u_2v_2)_1v - u_2(v_1^2)_1\ dx + a_3\int (v_2v+v_1^2)v_3+u_2vu_3\ dx\\ &= -3\int u_2v_2v_1\ dx + a_3\int (v_2v+v_1^2)v_3+u_2vu_3\ dx.\end{aligned}$$

Since

$$\begin{aligned}\int (v_2v+v_1^2)v_3+u_2vu_3\ dx &= \int \frac{1}{2}(v_2^2)_1v - 2v_2^2v_1 + \frac{1}{2}v(u_2^2)_1\ dx\\ &= \int -\frac{1}{2}v_2^2v_1 - 2v_2^2v_1 - \frac{1}{2}u_2^2v_1\ dx\\ &= \int -\frac{5}{2}v_2^2v_1 - \frac{1}{2}u_2^2v_1\ dx,\end{aligned}$$

it follows that

$$\left(\int 2u_1v_1v\ dx\right)' \approx -6\int u_2v_2v_1\ dx - a_3\int (5v_2^2+u_2^2)v_1\ dx,$$

and then by symmetry

$$\left(\int 2u_1v_1u\ dx\right)' \approx -6\int u_2v_2u_1\ dx - a_3\int (5u_2^2+v_2^2)u_1\ dx.$$

Next we have

$$\begin{aligned}
\left(\int u_1^2 v\ dx\right)' &= \int 2u_1 u_1' v + u_1^2 v'\ dx \\
&= \int -(2u_2 v + 2u_1 v_1)u' + u_1^2 v'\ dx \\
&= \int (2u_2 v + 2u_1 v_1)(u_1 u + u_3 + ku + a_1 v_1 v + a_2 (uv)_1 + a_3 v_3)\ dx \\
&\qquad + \int -u_1^2 (v_1 v + v_3 + kv + a_2 u_1 u + a_1 (uv)_1 + a_3 u_3)\ dx \\
&\approx \int 2u_3 u_2 v + 2u_1 v_1 u_3 - u_1^2 v_3\ dx \\
&\qquad + a_3 \int (2u_2 v + 2u_1 v_1) v_3 - u_1^2 u_3\ dx \\
&= \int -u_2^2 v_1 - 2u_2 (u_1 v_1)_1 + 2u_1 u_2 v_2\ dx \\
&\qquad + a_3 \int (2u_2 v + 2u_1 v_1) v_3 - u_1^2 u_3\ dx \\
&= -3 \int u_2^2 v_1\ dx + a_3 \int (2u_2 v + 2u_1 v_1) v_3 - u_1^2 u_3\ dx.
\end{aligned}$$

Since

$$\begin{aligned}
\int (2u_2 v + 2u_1 v_1) v_3 - u_1^2 u_3\ dx &= \int -2v_2 (u_3 v + 2u_2 v_1 + u_1 v_2) + 2u_2^2 u_1\ dx \\
&= \int -2u_3 v_2 v - 4u_2 v_2 v_1 - 2v_2^2 u_1 + 2u_2^2 u_1\ dx \\
&= 2 \int v_3 u_2 v - u_2 v_2 v_1 + (u_2^2 - v_2^2) u_1\ dx,
\end{aligned}$$

it follows that

$$\left(\int u_1^2 v\ dx\right)' = -3 \int u_2^2 v_1\ dx + 2a_3 \int v_3 u_2 v - u_2 v_2 v_1 + (u_2^2 - v_2^2) u_1\ dx,$$

and then by symmetry

$$\left(\int v_1^2 u\ dx\right)' = -3 \int v_2^2 u_1\ dx + 2a_3 \int u_3 v_2 u - u_2 v_2 u_1 + (v_2^2 - u_2^2) v_1\ dx.$$

Combining the four relations we get (3.5.5) and (3.5.6). □

Proof of the proposition for $n = 2$. We consider the functions f, g, h of Lemma 3.7. If $a_3 = 0$ or if $a_1 = a_2 = 1$, then $h = 0$. If $|a_3| < 1$, then

$$\int u_n^2 + v_n^2\ dx \le \frac{1}{1 - |a_3|} \int u_n^2 + v_n^2 + 2a_3 u_n v_n\ dx.$$

Since by Lemma 3.4 and the induction hypothesis f and g satisfy the assumptions of Lemma 3.2, we may conclude as in case $n = 1$ above. □

3.6 Proof of the proposition for $n \geq 3$

We proceed by induction on n, so we assume that the proposition holds for smaller values of n.

By Lemma 3.5 we have

$$\begin{aligned}\left(\int u_n^2 + v_n^2\, dx\right)' = -2k \int u_n^2 + v_n^2\, dx \\ - 2\int u_n(u_1u)_n + v_n(v_1v)_n\, dx \\ - 2a_1 \int u_n(vv_1)_n + v_n(uv)_{n+1}\, dx \\ - 2a_2 \int v_n(uu_1)_n + u_n(uv)_{n+1}\, dx.\end{aligned} \tag{3.6.1}$$

If we differentiate the products in the last three integrals by using Leibniz's rule and the binomial formula, we obtain a sum of three-term products. Using the inequality $n \geq 3$, it follows from Lemma 3.4 that all terms are equivalent to zero, except those containing the factor u_{n+1} or v_{n+1}.

Indeed, the orders of differentiation of the three factors are n, j and $n+1-j$ with $1 \leq j \leq n$. Since the sum $2n+1$ of the differentiations satisfies the inequality $2n+1 < 2n+(n-1)$, we have

$$2(\alpha_n + \beta_n) + (\alpha_{n_1} + \beta_{n-1}) \leq 4,$$

and Lemma 3.4 applies.

Using again that $1 \leq n-2$, it follows that

$$\begin{aligned}\int u_n(u_1u)_n + v_n(v_1v)_n\, dx &\approx \int u_nu_{n+1}u + v_nv_{n+1}v\, dx \\ &= \frac{1}{2}\int (u_n^2)_1u + (v_n^2)_1v\, dx \\ &= -\frac{1}{2}\int u_n^2u_1 + v_n^2v_1\, dx \\ &\approx 0,\end{aligned}$$

$$\begin{aligned}\int u_n(vv_1)_n + v_n(uv)_{n+1}\, dx &\approx \int u_nvv_{n+1} + v_nu_{n+1}v + v_nuv_{n+1}\, dx \\ &= \int u_nvv_{n+1} - u_n(v_nv)_1 + \frac{1}{2}u(v_n^2)_1\, dx \\ &= \int -u_nv_nv_1 - \frac{1}{2}u_1v_n^2\, dx \\ &\approx 0,\end{aligned}$$

and by symmetry

$$\int v_n(uu_1)_n + u_n(uv)_{n+1}\ dx \approx 0.$$

Using these relations we infer from (3.6.1) that

$$\left(\int u_n^2 + v_n^2\ dx\right)' \approx -2k \int u_n^2 + v_n^2\ dx,$$

and we conclude as usual.

Chapter 4

Internal Controllability for the Korteweg-de Vries Equation on a Bounded Domain

4.1 Introduction

The Korteweg–de Vries (KdV) equation can be written

$$u_t + u_{xxx} + u_x + uu_x = 0,$$

where $u = u(t,x)$ is a real-valued function of two real variables t and x, and $u_t = \partial u/\partial t$, etc. The equation was first derived by Boussinesq [10] and Korteweg-de Vries [39] as a model for the propagation of water waves along a channel. The equation furnishes also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems.

The KdV equation has been intensively studied from various aspects of mathematics, including the well-posedness, the existence and stability of solitary waves, the integrability, the long-time behavior, etc. (see e.g. [35, 51]). The practical use of the KdV equation does not always involve the pure initial value problem. In numerical studies, one is often interested in using a finite interval (instead of the whole line) with three boundary conditions.

Here, we shall be concerned with the control properties of KdV, the control acting through a forcing term f incorporated in the equation:

$$u_t + u_x + u_{xxx} + uu_x = f, \quad t \in [0,T],\ x \in [0,L], \qquad + \ \text{b.c.} \tag{4.1.1}$$

Our main purpose is to see whether one can force the solutions of (4.1.1) to have certain desired properties by choosing an appropriate control input f. The focus here is on the *controllability* issue:

Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the equation (4.1.1) admits a solution u which equals

u_0 at time $t = 0$ and u_1 at time $t = T$?

If one can always find a control input f to guide the system described by (4.1.1) from any given initial state u_0 to any given terminal state u_1, then the system (4.1.1) is said to be *exactly controllable.* If the system can be driven, by means of a control f, from any state to the origin (i.e. $u_1 \equiv 0$), then one says that system (4.1.1) is *null controllable.*

The study of the controllability and stabilization of the KdV equation started with the works of Russell and Zhang [72] for a system with periodic boundary conditions and an internal control. Since then, both the controllability and the stabilization have been intensively studied. (We refer the reader to [65] for a survey of the results up to 2009.) In particular, the exact boundary controllability of KdV on a finite domain was investigated in e.g. [16, 17, 20, 32, 33, 62, 64, 85]. Most of those works were concerned with the following system

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{in } (0,T) \times (0,L), \\ u(t,0) = g_1(t),\ u(t,L) = g_2(t),\ u_x(t,L) = g_3(t) & \text{in } (0,T) \end{cases} \tag{4.1.2}$$

in which the boundary data g_1, g_2, g_3 can be chosen as control inputs. System (4.1.2) was first studied by Rosier [62] considering only the control input g_3 (i.e. $g_1 = g_2 = 0$). It was shown in [62] that the exact controllability of the linearized system holds in $L^2(0,L)$ if, and only if, L does not belong to the following countable set of *critical lengths*

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \ : k,\ l \in \mathbb{N}^* \right\}. \tag{4.1.3}$$

The analysis developed in [62] shows that when the linearized system is controllable, the same is true for the nonlinear one. Note that the converse is false, as it was proved in [16, 17, 20] that the (nonlinear) KdV equation is controllable even when L is a critical length. The existence of a discrete set of critical lengths for which the exact controllability of the linearized equation fails was also noticed by Glass and Guerrero in [33] when g_2 is taken as control input (i.e. $g_1 = g_3 = 0$). Finally, it is worth mentioning the result by Rosier [64] and Glass and Guerrero [32] for which g_1 is taken as control input (i.e. $g_2 = g_3 = 0$). They proved that system (4.1.2) is then null controllable, but not exactly controllable, because of the strong smoothing effect.

By contrast, the mathematical theory pertaining to the study of the internal controllability in a bounded domain is considerably less advanced. As far as we know, the null controllability problem for system (4.1.1) was only addressed in [32] when the control acts in a neighborhood of the left endpoint. On the other hand, the exact controllability results in [40, 72] were obtained on a periodic domain.

The aim of this chapter is to address the controllability issue for the KdV equation on a bounded domain with a distributed control. Our first main result is a null controllability result valid for any localization of the control region. Actually, a controllability to the trajectories is established:

Theorem 4.1. *Let $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < L$, and let $T > 0$. For $\bar{u}_0 \in L^2(0,L)$,* *let* $\bar{u} \in C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ *denote the solution of*

$$\begin{cases} \bar{u}_t + \bar{u}_x + \bar{u}\,\bar{u}_x + \bar{u}_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \bar{u}(t,0) = \bar{u}(t,L) = \bar{u}_x(t,L) = 0 & \text{in } (0,T), \\ \bar{u}(0,x) = \bar{u}_0(x) & \text{in } (0,L). \end{cases} \tag{4.1.4}$$

Then there exists $\delta > 0$ *such that for any* $u_0 \in L^2(0,L)$ *satisfying* $\|u_0 - \bar{u}_0\|_{L^2(0,L)} \leq \delta$, *there exists* $f \in L^2((0,T)\times\omega)$ *such that the solution* $u \in C^0([0,T];L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ *of*

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{4.1.5}$$

satisfies $u(T,\cdot) = \bar{u}(T,\cdot)$ *in* $(0,L)$.

The null controllability is first established for a linearized system

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = 1_\omega f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{4.1.6}$$

by following the classical duality approach (see [28, 42]), which reduces the null controllability of (4.1.6) to an observability inequality for the solutions of the adjoint system. To prove the observability inequality, we derive a new Carleman estimate with an internal observation in $(0,T)\times(l_1,l_2)$ and use some interpolation arguments inspired by those in [32], where the authors derived a similar result when the control acts on a neighborhood on the left endpoint (that is, $l_1 = 0$). The null controllability is extended to the nonlinear system by applying Kakutani fixed-point theorem.

The second problem we address is related to the exact internal controllability of system (4.1.1). As far as we know, the same problem was studied only in [40, 72] in a periodic domain $\mathbb{T}$ with a distributed control of the form

$$f(x,t) = (Gh)(x,t) := g(x)(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)dy),$$

where $g \in C^\infty(\mathbb{T})$ was such that $\{g > 0\} = \omega$ and $\int_{\mathbb{T}} g(x)dx = 1$, and the function h was considered as a new control input. Here, we shall consider the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L). \end{cases} \tag{4.1.7}$$

As the smoothing effect is different from those in a periodic domain, the results in this chapter turn out to be very different from those in [40, 72]. First, for a controllability result in $L^2(0,L)$, the control f has to be taken in the space $L^2(0,T,H^{-1}(0,L))$. Actually, with any control $f \in L^2(0,T,L^2(0,L))$, the solution of (4.1.7) starting from $u_0 = 0$ at $t = 0$ would remain in $H_0^1(0,L)$ (see [32]). On the other hand, as for the boundary control, the localization of the distributed control plays a role in the results.

When the control acts in a neighborhood of $x = L$, we obtain the exact controllability in the weighted Sobolev space $L^2_{\frac{1}{L-x}dx}$ defined as

$$L^2_{\frac{1}{L-x}dx} := \{u \in L^1_{loc}(0,L); \int_0^L \frac{|u(x)|^2}{L-x}dx < \infty\}.$$

More precisely, we shall obtain the following result:

Theorem 4.2. *Let $T > 0$, $\omega = (l_1, l_2) = (L-\nu, L)$ where $0 < \nu < L$. Then, there exists $\delta > 0$ such that for any $u_0,\ u_1 \in L^2_{\frac{1}{L-x}dx}$ with*

$$\|u_0\|_{L^2_{\frac{1}{L-x}dx}} \le \delta \quad \text{and} \quad \|u_1\|_{L^2_{\frac{1}{L-x}dx}} \le \delta,$$

one can find a control input $f \in L^2(0,T;H^{-1}(0,L))$ with supp$(f) \subset (0,T)\times\omega$ such that the solution $u \in C^0([0,L], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ of (4.1.7) satisfies $u(T,.) = u_1$ in $(0,L)$ and $u \in C^0([0,T], L^2_{\frac{1}{L-x}dx})$. Furthermore, $f \in L^2_{(T-t)dt}(0,T,L^2(0,L))$.

Actually, we shall have to investigate the well-posedness of the linearization of (4.1.7) in the space $L^2_{\frac{1}{L-x}dx}$ and the well-posedness of the (backward) adjoint system in the "dual space" $L^2_{(L-x)dx}$. To do this, we shall follow some ideas borrowed from [34], where the well-posedness was investigated in the weighted space $L^2_{\frac{x}{L-x}dx}$. The needed observability inequality is obtained by the standard compactness-uniqueness argument and some unique continuation property. The exact controllability is extended to the nonlinear system by using the contraction mapping principle.

When the control is acting far from the endpoint $x = L$, i.e. in some interval $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < L$, then there is no chance to control exactly the state function on (l_2, L) (see e.g. [64]). However, it is possible to control the state function on $(0, l_1)$, so that a "regional controllability" can be established:

Theorem 4.3. *Let $T > 0$ and $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < L$. Pick any number $l_1' \in (l_1, l_2)$. Then there exists a number $\delta > 0$ such that for any $u_0, u_1 \in L^2(0,L)$ satisfying*

$$||u_0||_{L^2(0,L)} \le \delta, \qquad ||u_1||_{L^2(0,L)} \le \delta,$$

one can find a control $f \in L^2(0,T,H^{-1}(0,L))$ with supp$(f) \subset (0,T)\times\omega$ such that the solution $u \in C^0([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ of (4.1.7) satisfies

$$u(T,x) = \begin{cases} u_1(x) & \text{if } x \in (0, l_1'); \\ 0 & \text{if } x \in (l_2, L). \end{cases} \tag{4.1.8}$$

The proof of Theorem 4.3 combines Theorem 4.1, a boundary controllability result from [62], and the use of a cutt-off function. Note that the issue whether u may also be controlled in the interval (l_1', l_2) is open.

The chapter is outlined as follows. In Section 2, we review some linear estimates from [32, 62] that will be used thereafter. Section 3 is devoted to the proof of Theorems 4.1 and 4.3. It contains the proof of a new Carleman estimate for the KdV equation with some internal observation (Proposition 4.5). In Section 4 we prove the well-posedness of KdV in the weighted spaces L^2_{xdx} and $L^2_{\frac{1}{L-x}dx}$ by using semigroup theory, and derive Theorem 4.2.

4.2 Linear estimates

We review a series of estimates for the system

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = f(t,x) & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L) \end{cases} \tag{4.2.1}$$

and its adjoint system. Here $f = f(t,x)$ is a function which stands for the control of the system, and $\xi = \xi(t,x)$ is a given function.

4.2.1 The linearized KdV equation

It was noticed in [62] that the operator $A = -\frac{\partial^3}{\partial x^3} - \frac{\partial}{\partial x}$ with domain

$$\mathcal{D}(A) = \left\{ w \in H^3(0,L);\ w(0) = w(L) = w_x(L) = 0 \right\} \subseteq L^2(0,L)$$

is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(0,L)$. More precisely, the following result was established in [62].

Proposition 4.1. *Let $u_0 \in L^2(0,L)$, $\xi \equiv 1$ and $f \equiv 0$. There exists a unique (mild) solution u of (4.2.1) with*

$$u \in C([0,T];L^2(0,L)) \cap L^2(0,T,H^1_0(0,L)). \tag{4.2.2}$$

Moreover, there exist positive constants c_1 and c_2 such that for all $u_0 \in L^2(0,L)$

$$\|u\|_{L^2(0,T;H^1(0,L))} + \|u_x(.,0)\|_{L^2(0,T)} \leq c_1 \|u_0\|_{L^2(0,L)}, \tag{4.2.3}$$

$$\|u_0\|^2_{L^2(0,L)} \leq \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + c_2 \|u_x(.,0)\|^2_{L^2(0,T)}. \tag{4.2.4}$$

If in addition $u_0 \in D(A)$, then (4.2.1) has a unique (classical) solution u in the class

$$u \in C([0,T];D(A)) \cap C^1([0,T];L^2(0,L)). \tag{4.2.5}$$

4.2.2 The modified KdV equation

We introduce a system related to the adjoint system to (4.2.1), namely

$$\begin{cases} -v_t - \xi v_x - v_{xxx} = f & \text{in } (0,T)\times(0,L), \\ v(t,0) = v(t,L) = v_x(t,0) = 0 & \text{in } (0,T), \\ v(T,x) = 0 & \text{in } (0,L), \end{cases} \tag{4.2.6}$$

for which we review some estimates borrowed from [32].

4.2.2.1 Energy Estimates

We introduce the following spaces

$$\begin{array}{ll} X_0 := L^2(0,T;H^{-2}(0,L)), & X_1 := L^2(0,T;H_0^2(0,L)), \\ \tilde{X}_0 := L^1(0,T;H^{-1}(0,L)), & \tilde{X}_1 := L^1(0,T;(H^3\cap H_0^2)(0,L)), \end{array} \tag{4.2.7}$$

and

$$\begin{array}{l} Y_0 := L^2((0,T)\times(0,L))\cap C^0([0,T];H^{-1}(0,L)), \\ Y_1 := L^2(0,T;H^4(0,L))\cap C^0([0,T];H^3(0,L)). \end{array} \tag{4.2.8}$$

The spaces $X_0, X_1, \tilde{X}_0, \tilde{X}_1, Y_0,$ and Y_1 are equipped with their natural norms. For instance, the spaces Y_0 and Y_1 are equipped with the norms

$$\|w\|_{Y_0} := \|w\|_{L^2((0,T)\times(0,L))} + \|w\|_{L^\infty(0,T;H^{-1}(0,L))}$$

and

$$\|w\|_{Y_1} := \|w\|_{L^2(0,T;H^4(0,L))} + \|w\|_{L^\infty(0,T;H^3(0,L))}.$$

For $\theta\in[0,1]$, we define the complex interpolation spaces (see [5] and [44])

$$X_\theta = (X_0,X_1)_{[\theta]},\ \tilde{X}_\theta = (\tilde{X}_0,\tilde{X}_1)_{[\theta]} \text{ and } Y_\theta = (Y_0,Y_1)_{[\theta]}.$$

Then,

$$X_{1/4} = L^2(0,T;H^{-1}(0,L)), \quad \tilde{X}_{1/4} = L^1(0,T;L^2(0,L)) \tag{4.2.9}$$

and

$$Y_{1/4} = L^2(0,T;H^1(0,L))\cap C^0([0,T];L^2(0,L)). \tag{4.2.10}$$

Furthermore,

$$X_{1/2} = L^2((0,T)\times(0,L)), \quad \tilde{X}_{1/2} = L^1(0,T;H_0^1(0,L)) \tag{4.2.11}$$

and

$$Y_{1/2} = L^2(0,T;H^2(0,L))\cap C^0([0,T];H^1(0,L)). \tag{4.2.12}$$

Proposition 4.2. *([32, Section 2.2.2]) Let $\xi\in Y_{\frac14}$ and*

$$f\in X_{\frac14}\cup\tilde{X}_{\frac14} = L^2(0,T;H^{-1}(0,L))\cup L^1(0,T;L^2(0,L)).$$

Then the solution v of (4.2.6) belongs to $Y_{\frac14}$, and there exists some constant $C = C(||\xi||_{Y_{\frac14}})>0$ such that

$$\|v\|_{L^\infty(0,T,L^2(0,L))} + \|v\|_{L^2(0,T;H^1(0,L))} + \|v_x(\cdot,L)\|_{L^2(0,T)} \le C(\|\xi\|_{Y_{1/4}})\,\|f\|_{L^2(0,T;H^{-1}(0,L))} \tag{4.2.13}$$

and

$$\|v\|_{L^\infty(0,T,L^2(0,L))} + \|v\|_{L^2(0,T;H^1(0,L))} + \|v_x(\cdot,L)\|_{L^2(0,T)} \le C(\|\xi\|_{Y_{1/4}})\,\|f\|_{L^1(0,T;L^2(0,L))}. \tag{4.2.14}$$

More can be said when $\xi \equiv 0$. Consider the following system

$$\begin{cases} -v_t - v_{xxx} = g & \text{in } (0,T)\times(0,L), \\ v(t,0) = v(t,L) = v_x(t,0) = 0 & \text{in } (0,T), \\ v(T,x) = 0 & \text{in } (0,L). \end{cases} \tag{4.2.15}$$

Proposition 4.3. *([32, Section 2.3.1]. If $g \in X_1 \cup \tilde{X}_1$, then $v \in Y_1$ and there exists a constant $C > 0$ such that*

$$\|v\|_{Y_1} + \|v_x(\cdot, L)\|_{H^1(0,T)} \le C\,\|g\|_{X_1} \tag{4.2.16}$$

and

$$\|v\|_{Y_1} + \|v_x(\cdot, L)\|_{H^1(0,T)} \le C\,\|g\|_{\tilde{X}_1}\,. \tag{4.2.17}$$

Proposition 4.4. *([32, Section 2.3.2]. If $g \in X_{1/2} \cup \tilde{X}_{1/2}$, then $v \in Y_{1/2}$, and there exists some constant $C > 0$ such that*

$$\|v\|_{Y_{1/2}} + \|v_x(\cdot, L)\|_{H^{1/3}(0,T)} + \|v_{xx}(\cdot, 0)\|_{L^2(0,T)} + \|v_{xx}(\cdot, L)\|_{L^2(0,T)} \le C\,\|g\|_{X_{1/2}} \tag{4.2.18}$$

and

$$\|v\|_{Y_{1/2}} + \|v_x(\cdot, L)\|_{H^{1/3}(0,T)} + \|v_{xx}(\cdot, 0)\|_{L^2(0,T)} + \|v_{xx}(\cdot, L)\|_{L^2(0,T)} \le C\,\|g\|_{\tilde{X}_{1/2}}\,. \tag{4.2.19}$$

4.3 Null controllability results

This section is devoted to the proof of Theorems 4.1 and 4.3.

4.3.1 Null controllability of a linearized equation

We first consider the system

$$\begin{cases} u_t + (\xi u)_x + u_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{4.3.1}$$

where $\xi = \xi(t,x)$ is a given function in $Y_{\frac{1}{4}}$, and $\omega = (l_1, l_2) \subset (0,L)$. Our aim is to prove the null controllability of (4.3.1). To this end, we shall establish an observability inequality for the corresponding adjoint system

$$\begin{cases} -v_t - \xi(t,x)v_x - v_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ v(t,0) = v(t,L) = v_x(t,0) = 0 & \text{in } (0,T), \\ v(T,x) = v_T(x) & \text{in } (0,L) \end{cases} \tag{4.3.2}$$

by using some Carleman inequality.

4.3.1.1 Carleman inequality with internal observation

Assume that $\omega = (l_1, l_2)$ with
$$0 < l_1 < l_2 < L.$$
Pick any function $\psi \in C^3([0,L])$ with

$$\psi > 0 \text{ in } [0,L]; \tag{4.3.3}$$
$$|\psi'| > 0,\ \ \psi'' < 0, \quad \text{and } \psi'\psi''' < 0 \text{ in } [0,L]\setminus\omega; \tag{4.3.4}$$
$$\psi'(0) < 0 \text{ and } \psi'(L) > 0; \tag{4.3.5}$$
$$\begin{aligned} &\min_{x\in[l_1,l_2]} \psi(x) = \psi(l_3) < \max_{x\in[l_1,l_2]} \psi(x) = \psi(l_1) = \psi(l_2), \\ &\max_{x\in[0,L]} \psi(x) = \psi(0) = \psi(L), \end{aligned} \tag{4.3.6}$$
$$\psi(0) < \frac{4}{3}\psi(l_3), \tag{4.3.7}$$

for some $l_3 \in (l_1, l_2)$. A convenient function ψ is defined on $[0,L]\setminus\omega$ as

$$\psi(x) = \begin{cases} \varepsilon x^3 - x^2 - x + c_1 & \text{if } x \in [0,l_1], \\ -\varepsilon x^3 + ax + c_2 & \text{if } x \in [l_2, L] \end{cases}$$

with $\varepsilon, a, c_1, c_2 > 0$ conveniently chosen. Note first that $\psi(l_1) = \psi(l_2)$ and $\psi(0) = \psi(L)$ if, and only if,

$$a = (L-l_2)^{-1}(l_1^2 + l_1 - \varepsilon l_2^3 - \varepsilon l_1^3 + \varepsilon L^3), \qquad c_1 = c_2 - \varepsilon L^3 + aL.$$

Then $a > 0$, $c_1 - c_2 > 0$ and (4.3.4)-(4.3.5) hold provided that $0 < \varepsilon \ll 1$. (4.3.3) and (4.3.7) hold for $c_2 \gg 1$. (4.3.6) is easy to satisfy.

Set
$$\varphi(t,x) = \frac{\psi(x)}{t(T-t)}. \tag{4.3.8}$$

For $f \in L^2(0,T;L^2(0,L))$ and $q_0 \in L^2(0,L)$, let q denote the solution of the system

$$q_t + q_{xxx} = f, \qquad t \in (0,T),\ x \in (0,L), \tag{4.3.9}$$
$$q(t,0) = q(t,L) = q_x(t,L) = 0, \qquad t \in (0,T), \tag{4.3.10}$$
$$q(0,x) = q_0(x), \qquad x \in (0,L). \tag{4.3.11}$$

Then the following Carleman inequality holds.

Proposition 4.5. *Pick any $T > 0$. There exist two constants $C > 0$ and $s_0 > 0$ such that any $f \in L^2(0,T;L^2(0,L))$, any $q_0 \in L^2(0,L)$ and any $s \geq s_0$, the solution q of*

(4.3.9)-(4.3.11) *fulfills*

$$\int_0^T\int_0^L [s\varphi|q_{xx}|^2 + (s\varphi)^3|q_x|^2 + (s\varphi)^5|q|^2]e^{-2s\varphi}dxdt \\ + \int_0^T [(s\varphi|q_{xx}|^2 + (s\varphi)^3|q_x|^2)e^{-2s\varphi}]_{|x=0} + [s\varphi|q_{xx}|^2e^{-2s\varphi}]_{|x=L}dt \\ \le C\left(\int_0^T\int_0^L |f|^2e^{-2s\varphi}dxdt + \int_0^T\int_\omega [s\varphi|q_{xx}|^2 + (s\varphi)^3|q_x|^2 + (s\varphi)^5|q|^2]e^{-2s\varphi}dxdt\right) \tag{4.3.12}$$

Actually, we shall need a Carleman estimate for (4.3.2) with the potential $\xi \in Y_{\frac{1}{4}}$. Let

$$\tilde{\varphi}(t,x) = \varphi(t, L-x).$$

Corollary 4.1. *Let $\xi \in Y_{\frac{1}{4}}$. Then there exist some positive constants $\tilde{s}_0 = \tilde{s}_0(T, ||\xi||_{Y_{\frac{1}{4}}})$ and $C = C(T, ||\xi||_{Y_{\frac{1}{4}}})$ such that for all $s \ge \tilde{s}_0$ and all $v_T \in L^2(0,L)$, the solution v of* (4.3.2) *fulfills*

$$\int_0^T\int_0^L [s\tilde{\varphi}|v_{xx}|^2 + (s\tilde{\varphi})^3|v_x|^2 + (s\tilde{\varphi})^5|v|^2]e^{-2s\tilde{\varphi}}dxdt \\ \le C\int_0^T\int_\omega [s\tilde{\varphi}|v_{xx}|^2 + (s\tilde{\varphi})^3|v_x|^2 + (s\tilde{\varphi})^5|v|^2]e^{-2s\tilde{\varphi}}dxdt. \tag{4.3.13}$$

Proof of Proposition 4.5. We first assume that $q_0 \in D(A)$ and that $f \in C([0,T]; D(A))$, so that $q \in C([0,T]; D(A)) \cap C^1([0,T]; L^2(0,L))$. This will be sufficient to legitimate the following computations. The general case ($q_0 \in L^2(0,L)$ and $f \in L^2(0,T; L^2(0,L))$) follows by density. Indeed, if we set

$$p(t,x) := \sqrt{\varphi(t,l_3)}e^{-s\varphi(t,l_3)}q(t,x)$$

then p solves (4.3.9)-(4.3.11) with q_0 replaced by 0, and f replaced by

$$\tilde{f} = \sqrt{\varphi(t,l_3)}e^{-s\varphi(t,l_3)}f + \left(\frac{1}{2}\varphi_t(t,l_3)\varphi^{-\frac{1}{2}}(t,l_3) - s\varphi_t(t,l_3)\sqrt{\varphi(t,l_3)}\right)e^{-s\varphi(t,l_3)}q,$$

so that (with different constants C)

$$\begin{aligned}\int_0^T\int_0^L \varphi|q_{xx}|^2e^{-2s\varphi}dxdt &\le C||p||^2_{L^2(0,T,H^2(0,L))} \\ &\le C||\tilde{f}||^2_{L^2(0,T,L^2(0,L))} \\ &\le C\big(||f||^2_{L^2(0,T,L^2(0,L))} + ||q_0||^2_{L^2(0,L)}\big).\end{aligned}$$

Since

$$||q||^2_{L^2(0,T,H^1(0,L))} \le C\big(||f||^2_{L^2(0,T,L^2(0,L))} + ||q_0||^2_{L^2(0,L)}\big)$$

we conclude that we can pass to the limit in each term in (4.3.12), if we take a sequence $\{(q_0^n, f^n)\}_{n\geq 0}$ in $\mathcal{D}(A)\times C([0,T],\mathcal{D}(A))$ such that $q_0^n \to q_0$ in $L^2(0,L)$ and $f^n \to f$ in $L^2(0,T,L^2(0,L))$.

Assume from now on that $q_0 \in \mathcal{D}(A)$ and that $f \in C([0,T];\mathcal{D}(A))$. Let q denote the solution of (4.3.9)-(4.3.11), and let $u = e^{-s\varphi}q$, $w = e^{-s\varphi}L(e^{s\varphi}u)$, where

$$L = \partial_t + \partial_x^3. \tag{4.3.14}$$

Straightforward computations show that

$$w = Mu := u_t+u_{xxx}+3s\varphi_x u_{xx}+(3s^2\varphi_x^2+3s\varphi_{xx})u_x+(s^3\varphi_x^3+3s^2\varphi_x\varphi_{xx}+s(\varphi_t+\varphi_{xxx}))u. \tag{4.3.15}$$

Let M_1 and M_2 denote the (formal) self-adjoint and skew-adjoint parts of the operator M. We readily obtain that

$$M_1 u \ := \ 3s(\varphi_x u_{xx} + \varphi_{xx}u_x) + [s(\varphi_t + \varphi_{xxx}) + s^3\varphi_x^3]u, \tag{4.3.16}$$

$$M_2 u \ := \ u_t + u_{xxx} + 3s^2(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u). \tag{4.3.17}$$

On the other hand

$$||w||^2 = ||M_1u||^2 + ||M_2u||^2 + 2(M_1u, M_2u) \tag{4.3.18}$$

where $(u,v) = \int_0^T\!\int_0^L uv\,dx\,dt$ and $||w||^2 = (w,w)$. From now on, for the sake of simplicity, we write $\iint u$ (resp. $\int u\big|_0^L$) instead of $\int_0^T\!\int_0^L u(t,x)dxdt$ (resp. $\int_0^T u(t,x)\big|_{x=0}^L dt$). The proof of the Carleman inequality follows the same pattern as in [48, 66]. The first step provides an exact computation of the scalar product (M_1u, M_2u), whereas the second step gives the estimates obtained thanks to the (pseudoconvexity) conditions (4.3.3)-(4.3.7).

Step 1. Exact computation of the scalar product in (4.3.18).
Write

$$2(M_1u, M_2u) = 2\iint [s(\varphi_t+\varphi_{xxx})+s^3\varphi_x^3]uM_2u+2\iint 3s(\varphi_x u_{xx}+\varphi_{xx}u_x)M_2u =: I_1+I_2.$$

Let

$$\alpha := s(\varphi_t + \varphi_{xxx}) + s^3\varphi_x^3. \tag{4.3.19}$$

Using (4.3.17), we decompose I_1 into

$$I_1 = \iint 2\alpha u u_t + \iint 2\alpha u u_{xxx} + 3s^2 \iint 2\alpha u(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u).$$

Integrating by parts with respect to t or x, noticing that $u_{|x=0} = u_{|x=L} = u_{x|x=L} = 0$, and that $u_{|t=0} = u_{|t=T} = 0$ by (4.3.3), we obtain that

$$\begin{aligned} I_1 &= -\iint \alpha_t u^2 + (3\iint \alpha_x u_x^2 - \iint \alpha_{xxx}u^2 - \int \alpha u_x^2\big|_0^L) - 3s^2\iint \varphi_x^2\alpha_x u^2 \\ &= -\iint (\alpha_t + \alpha_{xxx} + 3s^2\varphi_x^2\alpha_x)u^2 + 3\iint \alpha_x u_x^2 - \int \alpha u_x^2\big|_0^L. \end{aligned} \tag{4.3.20}$$

Next, we compute

$$I_2 = 2\iint 3s(\varphi_x u_{xx} + \varphi_{xx}u_x)(u_t + u_{xxx} + 3s^2(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u)).$$

Performing integrations by parts, we obtain successively

$$\begin{aligned} 2\iint(\varphi_x u_{xx} + \varphi_{xx}u_x)u_t &= \iint \varphi_{xt}u_x^2, \\ 2\iint(\varphi_x u_{xx} + \varphi_{xx}u_x)u_{xxx} &= -3\iint \varphi_{xx}u_{xx}^2 + \iint \varphi_{4x}u_x^2 \\ &+ \int(\varphi_x u_{xx}^2 - \varphi_{3x}u_x^2 + 2\varphi_{xx}u_{xx}u_x)\big|_0^L, \end{aligned}$$

and

$$\begin{aligned} 2\iint(\varphi_x u_{xx} + \varphi_{xx}u_x)(\varphi_x^2 u_x + \varphi_x\varphi_{xx}u) = -3\iint \varphi_x^2\varphi_{xx}u_x^2 + \iint[(\varphi_x^2\varphi_{xx})_{xx} - (\varphi_x\varphi_{xx}^2)_x]u^2 \\ + \int \varphi_x^3 u_x^2\big|_0^L. \end{aligned}$$

Thus

$$\begin{aligned} I_2 = -9s\iint \varphi_{xx}u_{xx}^2 + \iint[-27s^3\varphi_x^2\varphi_{xx} + 3s(\varphi_{xt} + \varphi_{4x})]u_x^2 \\ + \iint 9s^3[(\varphi_x^2\varphi_{xx})_{xx} - (\varphi_x\varphi_{xx}^2)_x]u^2 + \int[3s(\varphi_x u_{xx}^2 - \varphi_{3x}u_x^2 + 2\varphi_{xx}u_x u_{xx}) + 9s^3\varphi_x^3 u_x^2]\big|_0^L \end{aligned} \tag{4.3.21}$$

Gathering together (4.3.20)-(4.3.21), we infer that

$$\begin{aligned} 2(M_1u, M_2u) \;=\; & \iint[-(\alpha_t + \alpha_{xxx} + 3s^2\varphi_x^2\alpha_x) + 9s^3((\varphi_x^2\varphi_{xx})_{xx} - (\varphi_x\varphi_{xx}^2)_x)]u^2 \\ & + \iint[3\alpha_x - 27s^3\varphi_x^2\varphi_{xx} + 3s(\varphi_{xt} + \varphi_{4x})]u_x^2 - 9s\iint \varphi_{xx}u_{xx}^2 \\ & + \int[3s\varphi_x u_{xx}^2 + (9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha)u_x^2 + 2\varphi_{xx}u_x u_{xx}]\big|_0^L \end{aligned} \tag{4.3.22}$$

Step 2. Estimation of each term in (4.3.22).
The estimates are given in a series of claims.
Claim 1. There exist some constants $s_1 > 0$ and $C_1 > 1$ such that for all $s \geq s_1$, we have

$$\iint[-(\alpha_t + \alpha_{xxx} + 3s^2\varphi_x^2\alpha_x) + 9s^3((\varphi_x^2\varphi_{xx})_{xx} - (\varphi_x\varphi_{xx}^2)_x)]u^2 \geq C_1^{-1}\iint(s\varphi)^5u^2 - C_1\int_0^T\int_\omega(s\varphi)^5u^2.$$

From (4.3.19), we see that the term in s^5 in the brackets reads

$$-3s^5\varphi_x^2(\varphi_x^3)_x = -9s^5\varphi_x^4\varphi_{xx} = -9s^5\frac{(\psi')^4\psi''}{t^5(T-t)^5}.$$

We infer from (4.3.4) that for some $\kappa_1 > 0$ and all $s > 0$

$$-9s^5\varphi_x^4\varphi_{xx} \geq \kappa_1(s\varphi)^5 \qquad (t,x) \in (0,T) \times ([0,L] \setminus \omega).$$

On the other hand, we have for some $\kappa_2 > 0$ and all $s > 0$

$$\begin{aligned} |\alpha_t| + |\alpha_{xxx}| + |9s^3((\varphi_x^2\varphi_{xx})_{xx} - (\varphi_x\varphi_{xx}^2)_x)| &\leq \kappa_2 s^3\varphi^4 \quad (t,x) \in (0,T) \times (0,L), \\ |3s^2\varphi_x^2\alpha_x| &\leq \kappa_2(s\varphi)^5 \quad (t,x) \in (0,T) \times \omega. \end{aligned}$$

Claim 1 follows then for all $s > s_1$ with s_1 large enough and some $C_1 > 1$.
CLAIM 2. There exist some constants $s_2 > 0$ and $C_2 > 1$ such that for all $s \geq s_2$, we have

$$\iint [3\alpha_x - 27s^3\varphi_x^2\varphi_{xx} + 3s(\varphi_{xt} + \varphi_{4x})]u_x^2 \geq C_2^{-1} \iint (s\varphi)^3 u_x^2 - C_2 \int_0^T\!\!\int_\omega (s\varphi)^3 u_x^2. \quad (4.3.23)$$

Indeed, the term in s^3 in the brackets is found to be

$$-18s^3\varphi_x^2\varphi_{xx} \geq \kappa_3(s\varphi)^3 \quad (t,x) \in (0,T) \times ([0,L] \setminus \omega)$$

for some $\kappa_3 > 0$ and all $s > 0$, by (4.3.4). On the other hand, we have for some $\kappa_4 > 0$ and all $s > 0$

$$\begin{aligned} |6s(\varphi_{tx} + \varphi_{4x})| &\leq \kappa_4 s\varphi^2 \quad (t,x) \in (0,T) \times (0,L), \\ |18s^3\varphi_x^2\varphi_{xx}| &\leq \kappa_4(s\varphi)^3 \quad (t,x) \in (0,T) \times \omega. \end{aligned}$$

Claim 2 follows for all $s \geq s_2$ with s_2 large enough and some $C_2 > 1$.
CLAIM 3. There exist some constants $s_3 > 0$ and $C_3 > 1$ such that for all $s \geq s_3$, we have

$$-9s \iint \varphi_{xx}u_{xx}^2 \geq C_3^{-1} \iint s\varphi u_{xx}^2 - C_3 \int_0^T\!\!\int_\omega s\varphi u_{xx}^2. \quad (4.3.24)$$

Claim 3 is clear, for $\psi'' < 0$ on $[0,L] \setminus \omega$.
CLAIM 4. There exist some constants $s_4 > 0$ and $C_4 > 1$ such that for all $s \geq s_4$, we have

$$\begin{aligned} \int [3s\varphi_x u_{xx}^2 + (9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha)u_x^2 + 2\varphi_{xx}u_x u_{xx}]\Big|_0^L \\ \geq C_4^{-1} \int_0^T [(s\varphi u_{xx}^2)_{|x=0} + (s\varphi u_{xx}^2)_{|x=L} + (s^3\varphi^3 u_x^2)_{|x=0}]dt. \end{aligned}$$

Since $u_{x|x=L} = 0$ and

$$[(9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha)u_x^2]_{|x=0} = [(8s^3\varphi_x^3 - s(\varphi_t + 4\varphi_{xxx}))u_x^2]_{|x=0},$$

we obtain with (4.3.5) for $s \geq s_4$ with s_4 large enough,

$$[(9s^3\varphi_x^3 - 3s\varphi_{xxx} - \alpha)u_x^2]\Big|_0^L \geq \kappa_5[(s\varphi)^3 u_x^2]_{|x=0}$$

and

$$3s\varphi_x u_{xx}^2|_0^L \geq \kappa_6([s\varphi u_{xx}^2]_{|x=0} + [s\varphi u_{xx}^2]_{|x=L})$$

for some constant $\kappa_5, \kappa_6 > 0$. Finally

$$|[2s\varphi_{xx}u_x u_{xx}]_{x=0}| \le \frac{\kappa_6}{2}[s\varphi u_{xx}^2]_{|x=0} + \kappa_7[s\varphi u_x^2]_{|x=0}$$

for some constant $\kappa_7 > 0$. Since $s\varphi(t,0) \ll (s\varphi)^3(t,0)$ for $s \gg 1$, Claim 4 follows.

We infer from Claims 1, 2, 3, and 4 that for some positive constants s_0, C and all $s \ge s_0$

$$\iint [(s\varphi)^5|u|^2+(s\varphi)^3|u_x|^2+s\varphi|u_{xx}|^2]+\int_0^T [(s\varphi u_{xx}^2)_{|x=0}+(s\varphi u_{xx}^2)_{|x=L}+(s^3\varphi^3u_x^2)_{|x=0}]dt$$
$$\le C(\iint |w|^2 + \int_0^T\int_\omega [(s\varphi)^5|u|^2 + (s\varphi)^3|u_x|^2 + s\varphi|u_{xx}|^2]\ ). \quad (4.3.25)$$

Replacing u by $e^{-s\varphi}q$ yields (4.3.12). □

Proof of Corollary 4.1. Note first that for $\xi \in Y_{\frac14}$ and $v_T \in L^2(0,L)$, one can prove that (4.3.2) has a unique solution $v \in Y_{\frac14}$, by using the contraction mapping principle for the integral equation. Corollary 4.1 follows from Proposition 4.5 by taking $q_0(x) = v_T(L-x)$, $q(t,x) = v(T-t, L-x)$, and $f(t,x) = -\xi(T-t, L-x)q_x(t,x)$, assuming first that $\xi \in Y_{\frac14} \cap L^\infty(Q)$ (so that $f \in L^2(Q)$). Indeed, with $u = e^{-s\varphi}q$,

$$w = e^{-s\varphi}L(e^{s\varphi}u) = -\xi(T-t, L-x)(u_x + s\varphi_x u),$$

so that

$$\begin{aligned}\iint |w|^2 dxdt &\le C\int_0^T\int_0^L |\xi(T-t,L-x)|^2(|u_x|^2 + |s\varphi_x u|^2)dxdt \\ &\le C\int_0^T ||\xi(T-t)||^2_{L^2(0,L)}\big(||u_x||^2_{L^\infty(0,L)} + ||s\varphi_x u||^2_{L^\infty(0,L)}\big)dt \quad (4.3.26)\\ &\le C||\xi||^2_{L^\infty(0,T,L^2(0,L))}\int_0^T\int_0^L [u_x^2 + u_{xx}^2 + \frac{s^2}{t^2(T-t)^2}(u^2+u_x^2)]dx.\end{aligned}$$

Combining (4.3.25) with (4.3.26), picking $s \gg 1$, and replacing again u by $e^{-s\varphi}v(T-t, L-x)$ yields (4.3.13). The result for $\xi \in Y_{\frac14}$ follows by density. □

4.3.1.2 Internal observation

We go back to the adjoint system (4.3.2). Our next goal is to remove the terms v_{xx} and v_x from the r.h.s. of (4.3.13). In addition to the weight $\tilde\varphi(t,x) = \frac{1}{t(T-t)}\psi(L-x)$, we introduce the functions

$$\hat\varphi(t) = \frac{1}{t(T-t)}\max_{x\in[0,L]}\psi(x) = \frac{\psi(0)}{t(T-t)} \text{ and } \check\varphi(t) = \frac{1}{t(T-t)}\min_{x\in[0,L]}\psi(x) = \frac{\psi(l_3)}{t(T-t)}, \quad (4.3.27)$$

where we used (4.3.6). By (4.3.7), we have

$$\hat\varphi(t) < \frac43\check\varphi(t), \quad t\in(0,T). \quad (4.3.28)$$

Lemma 4.1. *Let $0 < l_1 < l_2 < L$, $\xi \in Y_{\frac{1}{4}}$, and $\tilde{s}_0$ be as in Corollary 4.1. Then there exists a constant $C = C(T, ||\xi||_{Y_{\frac{1}{4}}}) > 0$ such that for any $s \geq \tilde{s}_0$ and any $v_T \in L^2(0, L)$, the solution v of (4.3.2) satisfies*

$$\int_Q \{(s\check{\varphi})^5|v|^2 + (s\check{\varphi})^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\} e^{-2s\hat{\varphi}} dxdt \quad \leq \quad C_1 s^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|^2_{L^2(\omega)} dt, \tag{4.3.29}$$

where $Q = (0, T) \times (0, L)$ and $\omega = (l_1, l_2) \subset (0, L)$.

Proof. We follow the same approach as in [32]. From (4.3.13) and (4.3.27)-(4.3.28), we first obtain

$$\begin{aligned}&\int_Q \left\{s^5\check{\varphi}^5|v|^2 + s^3\check{\varphi}^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\right\} e^{-2s\check{\varphi}} dxdt \\ &\leq C \int_0^T\int_\omega \left\{s^5\check{\varphi}^5 |v|^2 + s^3\check{\varphi}^3|v_x|^2 + s\check{\varphi}|v_{xx}|^2\right\} e^{-2s\check{\varphi}} dxdt =: C(I_0 + I_1 + I_2).\end{aligned} \tag{4.3.30}$$

Since $\check{\varphi}$ and $\hat{\varphi}$ do not depend on x, we clearly have that

$$I_1 \leq s^3 \int_0^T \check{\varphi}^3 e^{-2s\check{\varphi}} \|v(t,\cdot)\|^2_{H^1(\omega)} dt \tag{4.3.31}$$

and

$$I_2 \leq s \int_0^T \check{\varphi} e^{-2s\check{\varphi}} \|v(t,\cdot)\|^2_{H^2(\omega)} dt. \tag{4.3.32}$$

The following interpolation result will be used several times.

Proposition 4.6. *[2, Theorem 4.17] Let $p \in [1, \infty]$ and $m \in \mathbb{N}^*$. Then there exists a constant $K = K(m, p)$ such that for $0 \leq j \leq m$ and $u \in W^{m,p}(\omega)$ we have*

$$\|u\|_{j,p} \leq K \|u\|^{j/m}_{m,p} \|u\|^{(m-j)/m}_{0,p},$$

where $\|\cdot\|_{j,p}$ denotes the norm in the Sobolev space $W^{j,p}(\omega)$.

Using Proposition 4.6 with $j = 1$, $p = 2$ and $m = 8/3$ (resp. with $j = p = 2$ and $m = 8/3$) yields

$$\|v(t,\cdot)\|_{H^1(\omega)} \leq K_1 \|v(t,\cdot)\|^{3/8}_{H^{8/3}(\omega)} \|v(t,\cdot)\|^{5/8}_{L^2(\omega)} \tag{4.3.33}$$

and

$$\|v(t,\cdot)\|_{H^2(\omega)} \leq K_2 \|v(t,\cdot)\|^{3/4}_{H^{8/3}(\omega)} \|v(t,\cdot)\|^{1/4}_{L^2(\omega)}. \tag{4.3.34}$$

Replacing (4.3.33) and (4.3.34) in (4.3.31) and (4.3.32), respectively, yields

$$I_1 \leq Cs^3 \int_0^T \check{\varphi}^3 e^{-2s\check{\varphi}} \|v(t,\cdot)\|^{3/4}_{H^{8/3}(\omega)} \|v(t,\cdot)\|^{5/4}_{L^2(\omega)} dt \tag{4.3.35}$$

and

$$I_2 \leq Cs \int_0^T \check{\varphi} e^{-2s\check{\varphi}} \|v(t,\cdot)\|^{3/2}_{H^{8/3}(\omega)} \|v(t,\cdot)\|^{1/2}_{L^2(\omega)} dt. \tag{4.3.36}$$

Next, an application of Young inequality in (4.3.35) and (4.3.36) gives

$$I_1 \le Cs^3 \int_0^T \check{\varphi}^3 e^{-2s\check{\varphi}} e^{-\frac{3}{4}s\hat{\varphi}} e^{\frac{3}{4}s\hat{\varphi}} \check{\varphi}^{-\frac{27}{8}} \check{\varphi}^{\frac{27}{8}} \|v(t,\cdot)\|_{H^{8/3}(\omega)}^{3/4} \|v(t,\cdot)\|_{L^2(\omega)}^{5/4} dt$$
$$\le C_\epsilon s^6 \int_0^T e^{s(\frac{6}{5}\hat{\varphi}-\frac{16}{5}\check{\varphi})} \check{\varphi}^{51/5} \|v(t,\cdot)\|_{L^2(\omega)}^2 dt + \epsilon s^{-2} \int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|_{H^{8/3}(\omega)}^2 dt \quad (4.3.37)$$

and

$$I_2 \le Cs \int_0^T e^{-2s\check{\varphi}} e^{-\frac{3}{2}s\hat{\varphi}} e^{\frac{3}{2}s\hat{\varphi}} \check{\varphi}^{-\frac{27}{4}} \check{\varphi}^{\frac{31}{4}} \|v(t,\cdot)\|_{H^{8/3}(\omega)}^{3/2} \|v(t,\cdot)\|_{L^2(\omega)}^{1/2} dt$$
$$\le C_\epsilon s^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|_{L^2(\omega)}^2 dt + \epsilon s^{-2} \int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|_{H^{8/3}(\omega)}^2 dt, \quad (4.3.38)$$

for any $\epsilon > 0$. Note that

$$I_0 + s^6 \int_0^T e^{s(\frac{6}{5}\hat{\varphi}-\frac{16}{5}\check{\varphi})} \check{\varphi}^{51/5} \|v(t,\cdot)\|_{L^2(\omega)}^2 dt \le Cs^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|_{L^2(\omega)}^2 dt. \quad (4.3.39)$$

Gathering together (4.3.30) and (4.3.37)-(4.3.39), we obtain

$$\int_Q \left\{ s^5 \check{\varphi}^5 |v|^2 + s^3 \check{\varphi}^3 |v_x|^2 + s\check{\varphi} |v_{xx}|^2 \right\} e^{-2s\check{\varphi}} dxdt$$
$$\le Cs^{10} \int_0^T e^{s(6\hat{\varphi}-8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|_{L^2(\omega)}^2 dt + 2\epsilon s^{-2} \int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|_{H^{8/3}(\omega)}^2 dt. \quad (4.3.40)$$

It remains to estimate the integral term

$$\int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|_{H^{8/3}(\omega)}^2 dt.$$

Let $v_1(t,x) := \theta_1(t) v(t,x)$ with

$$\theta_1(t) = \exp(-s\hat{\varphi}) \check{\varphi}^{-\frac{1}{2}}.$$

Then v_1 satisfies the system

$$\begin{cases} -v_{1t} - v_{1xxx} = f_1 := \xi \theta_1 v_x - \theta_{1t} v & \text{in } (0,T) \times (0,L), \\ v_1(t,0) = v_1(t,L) = v_{1x}(t,0) = 0 & \text{in } (0,T), \\ v_1(T,x) = 0 & \text{in } (0,L). \end{cases} \quad (4.3.41)$$

Now, observe that, since $v_x(t,0) = 0$, $\xi \in L^\infty(0,T,L^2(0,L))$ and $|\theta_{1t}| \le Cs\check{\varphi}^{\frac{3}{2}} \exp(-s\hat{\varphi})$, we have

$$\|f_1\|_{L^2((0,T)\times(0,L))}^2 \le C ||\xi||_{L^\infty(0,T,L^2(0,L))}^2 \int_0^T e^{-2s\hat{\varphi}} ||v_x||_{L^\infty(0,L)}^2 dt + C \int_Q e^{-2s\hat{\varphi}} s^2 \check{\varphi}^3 |v|^2 dxdt$$
$$\le C \int_Q \left\{ s^2 \check{\varphi}^3 |v|^2 + s|v_x|^2 + s^{-1} |v_{xx}|^2 \right\} e^{-2s\check{\varphi}} dxdt \quad (4.3.42)$$

for some constant $C > 0$ and all $s \geq s_0$. Moreover, by Proposition 4.4, $v_1 \in Y_{1/2}$. Then, interpolating between $L^2(0,T;H^2(0,L))$ and $L^\infty(0,T;H^1(0,L))$, we infer that $v_1 \in L^4(0,T;H^{3/2}(0,L))$ and

$$\|v_1\|_{L^4(0,T;H^{3/2}(0,L))} \leq C \|f_1\|_{L^2((0,T)\times(0,L))}. \tag{4.3.43}$$

Let $v_2(t,x) := \theta_2(t)v(t,x)$ with

$$\theta_2 = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{5}{2}}.$$

Then v_2 satisfies system (4.3.41) with f_1 replaced by

$$f_2 := \xi\theta_2\theta_1^{-1}v_{1x} - \theta_{2t}\theta_1^{-1}v_1.$$

Observe that

$$\left|\theta_2\theta_1^{-1}\right| + \left|\theta_{2t}\theta_1^{-1}\right| \leq Cs.$$

On the other hand, since $\xi \in L^4(0,T;H^{\frac{1}{2}}(0,L))$ and $v_{1x} \in L^4(0,T;H^{\frac{1}{2}}(0,L))$ by (4.3.43), we infer that $\xi v_{1x} \in L^2(0,T;H^{1/3}(0,L))$ (the product of two functions in $H^{\frac{1}{2}}(0,L)$ being in $H^{\frac{1}{3}}(0,L)$). Thus, we obtain

$$\|f_2\|_{L^2(0,T;H^{1/3}(0,L))} \leq Cs \|v_1\|_{L^4(0,T;H^{3/2}(0,L))}. \tag{4.3.44}$$

Interpolating between (4.2.16) and (4.2.18), we have that $v_2 \in L^2(0,T;H^{7/3}(0,L)) \cap L^\infty(0,T;H^{4/3}(0,L))$ with

$$\|v_2\|_{L^2(0,T;H^{7/3}(0,L))\cap L^\infty(0,T;H^{4/3}(0,L))} \leq C \|f_2\|_{L^2(0,T;H^{1/3}(0,L))}. \tag{4.3.45}$$

Finally, let $v_3 := \theta_3(t)v(t,x)$ with

$$\theta_3(t) = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{9}{2}}.$$

Then v_3 satisfies system (4.3.41) with f_1 replaced by

$$f_3 := \xi\theta_3\theta_2^{-1}v_{2x} - \theta_{3t}\theta_2^{-1}v_2.$$

Again

$$\left|\theta_3\theta_2^{-1}\right| + \left|\theta_{3t}\theta_2^{-1}\right| \leq Cs.$$

Interpolating again between (4.2.16) and (4.2.18), we have that

$$\|v_3\|_{L^2(0,T;H^{8/3}(0,L))\cap L^\infty(0,T;H^{5/3}(0,L))} \leq C \|f_3\|_{L^2(0,T;H^{2/3}(0,L))}. \tag{4.3.46}$$

Since $\xi \in Y_{\frac{1}{4}}$, we have that $\xi \in L^3(0,T;H^{\frac{2}{3}}(0,L))$. On the other hand, by (4.3.45),

$$v_{2x} \in L^2(0,T;H^{4/3}(0,L)) \cap L^\infty(0,T;H^{1/3}(0,L)).$$

It follows that $v_{2x} \in L^6(0,T,H^{\frac{2}{3}}(0,L))$. Since $H^{\frac{2}{3}}(0,L)$ is an algebra, we conclude that $\xi v_{2x} \in L^2(0,T,H^{\frac{2}{3}}(0,L))$. Therefore

$$\|f_3\|_{L^2(0,T;H^{2/3}(0,L))} \leq Cs \|v_2\|_{L^2(0,T;H^{7/3}(0,L))\cap L^\infty(0,T;H^{4/3}(0,L))}. \tag{4.3.47}$$

Thus we infer from (4.3.42)-(4.3.47) that for some constants $C_1, C_2 > 0$ and all $s \geq s_0$

$$\begin{aligned} \|v_3\|^2_{L^2(0,T;H^{8/3}(0,L))} &\leq C_1 s^4 ||f_1||^2_{L^2((0,T)\times(0,L))} \\ &\leq C_2 \int_Q \left\{ s^6 \check{\varphi}^3 |v|^2 + s^5 |v_x|^2 + s^3 |v_{xx}|^2 \right\} e^{-2s\check{\varphi}} dx dt. \end{aligned} \quad (4.3.48)$$

Hence, replacing $v_3 = \exp(-s\hat{\varphi})\check{\varphi}^{-\frac{9}{2}} v$ in (4.3.48) yields for some constant $C_3 > 0$

$$\int_0^T e^{-2s\hat{\varphi}} \check{\varphi}^{-9} \|v(t,\cdot)\|^2_{H^{8/3}(\omega)} dt \leq C_3 s^2 \int_Q \left\{ (s\check{\varphi})^5 |v|^2 + (s\check{\varphi})^3 |v_x|^2 + s\check{\varphi} |v_{xx}|^2 \right\} e^{-2s\check{\varphi}} dx dt. \quad (4.3.49)$$

Then, picking $\epsilon = 1/(4C_3)$ in (4.3.40) results in

$$\int_Q s\check{\varphi} e^{-2s\hat{\varphi}} \left\{ s^4 \check{\varphi}^4 |v|^2 + s^2 \check{\varphi}^2 |v_x|^2 + |v_{xx}|^2 \right\} dx dt \quad \leq \quad C_4 s^{10} \int_0^T e^{s(6\hat{\varphi} - 8\check{\varphi})} \check{\varphi}^{31} \|v(t,\cdot)\|^2_{L^2(\omega)} dt$$

for all $s \geq \tilde{s}_0$ and some positive constant $C_4 = C_4(T, ||\xi||_{Y_{\frac{1}{4}}})$. □

We are in a position to prove the null controllability of system (4.3.1).

Theorem 4.4. *Let $T > 0$. Then there exists $\delta > 0$ such that for any $\xi \in Y_{1/4}$ with $||\xi||_{L^2(0,T,H^1(0,L))} \leq \delta$ and any $u_0 \in L^2(0, L)$, one may find a control $f \in L^2((0, T) \times \omega)$ such that the solution u of (4.3.1) fulfills $u(T, \cdot) = 0$.*

Proof. Scaling in (4.3.2) by v and $(L - x)v$, we obtain after some computations the estimate

$$||v||^2_{L^\infty(0,T,L^2(0,L))} + 2||v_x||^2_{L^2(0,T,L^2(0,L))} \leq C(L) \left(||v_T||^2_{L^2(0,L)} + ||\xi||^2_{L^2(0,T,H^1(0,L))} ||v_x||^2_{L^2(0,T,L^2(0,L))} \right)$$

for some constant $C(L) > 0$. It follows that if $||\xi||_{L^2(0,T,H^1(0,L))} \leq \delta := 1/\sqrt{C(L)}$, then we have

$$\max_{t\in[0,T]} ||v(t)||^2_{L^2(0,L)} + ||v_x||^2_{L^2(0,T,L^2(0,L))} \leq C(L) ||v_T||^2_{L^2(0,L)}. \quad (4.3.50)$$

Replacing $v(t)$ by $v(0)$ and v_T by $v(\tau)$ for $T/3 < \tau < 2T/3$ in (4.3.50), and integrating over $\tau \in (T/3, 2T/3)$, we obtain that

$$||v(0)||^2_{L^2(0,L)} \leq \frac{3C(L)}{T} \int_{\frac{T}{3}}^{\frac{2T}{3}} ||v(\tau)||^2_{L^2(0,L)} d\tau. \quad (4.3.51)$$

Combining (4.3.51) with Lemma 4.1 for a fixed value of $s \geq \tilde{s}_0$, we derive the following observability inequality

$$\int_0^L |v(0,x)|^2 dx \leq C_* \int_0^T \|v(t,\cdot)\|^2_{L^2(\omega)} dt \quad (4.3.52)$$

where $C_* = C_*(T, ||\xi||_{Y_{1/4}}) > 0$. Using (4.3.52), we can deduce the existence of a function $v \in L^2((0, T) \times \omega)$ as in Theorem 4.4 proceeding as follows.

On $L^2(0,L)$, we define the norm

$$\|v_T\|_B := \|v\|_{L^2((0,T)\times\omega)},$$

where v is the solution of (4.3.2) associated with v_T. The fact that $||\cdot||_B$ is a norm comes from (4.3.52) applied on (t,T) for $0<t<T$.

Let B denote the completion of $L^2(0,L)$ with respect to the above norm. We define a functional J on B by

$$J(v_T) := \frac{1}{2}\|v_T\|_B^2 + \int_0^L v(0,x)u_0(x)dx.$$

From (4.3.52) we infer that J is well defined and continuous on B. As it is strictly convex and coercive, it admits a unique minimum v_T^*, characterized by the Euler-Lagrange equation

$$\int_0^T\int_\omega v^* w dxdt + \int_0^L w(0,x)u_0(x)dx = 0, \qquad \forall w_T \in B, \tag{4.3.53}$$

where w (resp. v^*) denotes the solution of (4.3.2) associated with $w_T \in B$ (resp. $v_T^* \in B$). Define $f \in L^2((0,T)\times\omega)$ by

$$f := 1_\omega v^*, \tag{4.3.54}$$

and let u denote the solution of (4.3.1) associated with u_0 and f. Multiplying (4.3.1) by $w(t,x)$ and integrating by parts, we obtain for all $w_T \in L^2(0,L)$

$$\int_0^L u(T,x)w_T dx = \int_0^L u_0(x)w(0,x)dx + \int_0^T\int_\omega v^* w dxdt = 0, \tag{4.3.55}$$

where the second equality follows from (4.3.53). Therefore $u(T,\cdot)=0$. Finally, letting $w_T = v_T^*$ in (4.3.53) and using (4.3.52), we obtain

$$\int_0^T\int_\omega |f|^2 dxdt \le C_* \int_0^L |u_0(x)|^2 dx. \tag{4.3.56}$$

□

4.3.2 Null controllability of the nonlinear equation

In this section we prove Theorem 4.1. This is done by using a fixed-point argument.

4.3.2.1 Proof of Theorem 4.1

Consider u and $\bar{u}$ fulfilling system (4.1.5) and (4.1.4), respectively. Then $q = u - \bar{u}$ satisfies

$$\begin{cases} q_t + q_x + (\frac{q^2}{2} + \bar{u}q)_x + q_{xxx} = 1_\omega f(t,x) & \text{in } (0,T)\times(0,L), \\ q(t,0) = q(t,L) = q_x(t,L) = 0 & \text{in } (0,T), \\ q(0,x) = q_0(x) := u_0(x) - \bar{u}_0(x) & \text{in } (0,L). \end{cases} \tag{4.3.57}$$

The objective is to find f such that the solution q of (4.3.57) satisfies

$$q(T,\cdot)=0.$$

Given $\xi \in Y_{\frac{1}{4}}$ and $q_0 := u_0 - \bar{u}_0 \in L^2(0,L)$, we consider the control problem

$$q_t + q_x + (\xi q)_x + q_{xxx} = 1_\omega f(t,x) \qquad \text{in } (0,T)\times(0,L), \tag{4.3.58}$$
$$q(t,0) = q(t,L) = q_x(t,L) = 0 \qquad \text{in } (0,T), \tag{4.3.59}$$
$$q(0,x) = q_0(x) \qquad \text{in } (0,L). \tag{4.3.60}$$

We can prove the following estimate

$$\begin{aligned} ||q||^2_{L^\infty(0,T,L^2(0,L))} + 2||q_x||^2_{L^2(0,T,L^2(0,L))} &\le \tilde{C}(L)\big(||q_0||^2_{L^2(0,L)} \\ &+ ||\xi||^2_{L^2(0,T,H^1(0,L))}||q_x||^2_{L^2(0,T,L^2(0,L))} + ||f||^2_{L^2((0,T)\times\omega)}\big) \end{aligned} \tag{4.3.61}$$

Let $\tilde{\delta} = \min(\delta, 1/\sqrt{\tilde{C}(L)})$. We introduce the space

$$E := C^0([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L))$$

endowed with its natural norm

$$\|z\|_E := ||z||_{Y_{1/4}} + ||z||_{H^1(0,T,H^{-2}(0,L))}.$$

We consider in $L^2((0,T)\times(0,L))$ the following set

$$B := \left\{ z \in E;\ \|z\|_E \le 1 \ \text{ and } \ ||z||_{L^2(0,T,H^1(0,L))} \le \tilde{\delta} \right\}.$$

B is compact in $L^2((0,T)\times(0,L))$, by Aubin-Lions' lemma. We will limit ourselves to controls f fulfilling the condition

$$||f||^2_{L^2((0,T)\times\omega)} \le C_* ||q_0||^2_{L^2(0,L)} \tag{4.3.62}$$

where $C_* := C_*(T, ||\bar{u}||_{Y_{1/4}} + \frac{1}{2})$. We associate with any $z \in B$ the set

$$\begin{aligned} T(z) := \{q \in B;\ \ \exists f \in L^2((0,T)\times\omega) \text{ such that } f \text{ satisfies (4.3.62) and} \\ q \text{ solves (4.3.58)-(4.3.60) with } \xi = \bar{u} + \tfrac{z}{2} \text{ and } q(T,\cdot) = 0\big\}. \end{aligned}$$

Note that $||\bar{u}||_{L^2(0,T,H^1(0,L))} < \tilde{\delta}/2$ for $T \ll 1$. By Theorem 4.4 and (4.3.61), we see that if $\|q_0\|_{L^2(0,L)}$ and T are sufficiently small, then $T(z)$ is nonempty for all $z \in B$. We shall use the following version of Kakutani fixed point theorem (see e.g. [82, Theorem 9.B]):

Theorem 4.5. *Let F be a locally convex space, let $B \subset F$ and let $T : B \longrightarrow 2^B$. Assume that*

1. *B is a nonempty, compact, convex set;*
2. *$T(z)$ is a nonempty, closed, convex set for all $z \in B$;*

3. *The set-valued map $T : B \longrightarrow 2^B$ is upper-semicontinuous; i.e., for every closed subset A of F, $T^{-1}(A) = \{z \in B;\ T(z) \cap A \neq \varnothing\}$ is closed.*

Then T has a fixed point, i.e., there exists $z \in B$ such that $z \in T(z)$.

Let us check that Theorem 4.5 can be applied to T and

$$F = L^2((0,T) \times (0,L)).$$

The convexity of B and $T(z)$ for all $z \in B$ is clear. Thus (1) is satisfied. For (2), it remains to check that $T(z)$ is closed in F for all $z \in B$. Pick any $z \in B$ and a sequence $\left\{q^k\right\}_{k\in\mathbb{N}}$ in $T(z)$ which converges in F towards some function $q \in B$. For each k, we can pick some control function $f^k \in L^2((0,T) \times \omega)$ fulfilling (4.3.62) such that (4.3.58)-(4.3.60) are satisfied with $\xi = \bar{u} + \frac{z}{2}$ and $q^k(T,\cdot) = 0$. Extracting subsequences if needed, we may assume that as $k \to \infty$

$$f^k \to f \quad \text{in } L^2((0,T) \times \omega) \text{ weakly,} \tag{4.3.63}$$

$$q^k \to q \quad \text{in } L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L)) \text{ weakly,} \tag{4.3.64}$$

By (4.3.64), the boundedness of $||q^k||_{L^\infty(0,T,L^2(0,L))}$ and Aubin-Lions' lemma, $\{q^k\}_{k\in\mathbb{N}}$ is relatively compact in $C^0([0,T], H^{-1}(0,L))$. Extracting a subsequence if needed, we may assume that

$$q^k \to q \text{ strongly in } C^0([0,T], H^{-1}(0,L)).$$

In particular, $q(0,x) = q_0(x)$ and $q(T,x) = 0$. On the other hand, we infer from (4.3.64) that

$$\xi q^k \to \xi q \text{ in } L^2((0,T) \times (0,L)) \text{ weakly.}$$

Therefore, $(\xi q^k)_x \to (\xi q)_x$ in $\mathcal{D}'((0,T) \times (0,L))$. Finally, it is clear that

$$||f||^2_{L^2((0,T)\times\omega)} \leq C_* ||q_0||^2_{L^2(0,L)}$$

and that q satisfies (4.3.58) with $\xi = \bar{u} + \frac{z}{2}$ and $q(T,\cdot) = 0$. Thus $q \in T(z)$ and $T(z)$ is closed. Now, let us check (3). To prove that T is upper-semicontinuous, consider any closed subset A of F and any sequence $\left\{z^k\right\}_{k\in\mathbb{N}}$ in B such that

$$z^k \in T^{-1}(A), \quad \forall k \geq 0, \tag{4.3.65}$$

and

$$z^k \to z \ \text{ in } \ F \tag{4.3.66}$$

for some $z \in B$. We aim to prove that $z \in T^{-1}(A)$. By (4.3.65), we can pick a sequence $\left\{q^k\right\}_{k\in\mathbb{N}}$ in B with $q^k \in T(z^k) \cap A$ for all k, and a sequence $\left\{f^k\right\}_{k\in\mathbb{N}}$ in $L^2((0,T) \times \omega)$ such that

$$\begin{cases} q^k_t + q^k_x + ((\bar{u} + \dfrac{z^k}{2})q^k)_x + q^k_{xxx} = 1_\omega f^k(t,x) & \text{in } (0,T) \times (0,L), \\ q^k(t,0) = q^k(t,L) = q^k_x(t,L) = 0 & \text{in } (0,T), \\ q^k(0,x) = q_0(x) & \text{in } (0,L), \end{cases} \tag{4.3.67}$$

$$q^k(T,x) = 0, \qquad \text{in } (0,L), \tag{4.3.68}$$

and

$$\left\|f^k\right\|^2_{L^2((0,T)\times\omega)} \le C_* \left\|q_0\right\|^2_{L^2(0,L)}. \tag{4.3.69}$$

From (4.3.69) and the fact that z^k, $q^k \in B$, extracting subsequences if needed, we may assume that as $k \to \infty$,

$$\begin{array}{ll} f^k \to f & \text{in } L^2((0,T)\times\omega) \text{ weakly,} \\ q^k \to q & \text{in } L^2(0,T;H^1(0,L)) \cap H^1(0,T;H^{-2}(0,L)) \text{ weakly,} \\ q^k \to q & \text{in } C^0([0,T],H^{-1}(0,L)) \text{ strongly,} \\ q^k \to q & \text{in } F \text{ strongly,} \\ z^k \to z & \text{in } F \text{ strongly,} \end{array}$$

where $f \in L^2((0,T)\times\omega)$ and $q \in B$. Again, $q(0,x) = q_0(x)$ and $q(T,x) = 0$. We also see that (4.3.59) and (4.3.62) are satisfied. It remains to check that

$$q_t + q_x + ((\bar{u} + \frac{z}{2})q)_x + q_{xxx} = 1_\omega f(t,x). \tag{4.3.70}$$

Observe that the only nontrivial convergence in (4.3.67) is those of the nonlinear term $(z^k q^k)_x$. Note first that

$$||z^k q^k||_{L^2(0,T,L^2(0,L))} \le ||z^k||_{L^\infty(0,T,L^2(0,L))} ||q^k||_{L^2(0,T,L^\infty(0,L))} \le C,$$

so that, extracting a subsequence, one can assume that $z^k q^k \to f$ weakly in $L^2((0,T)\times(0,L))$. To prove that $f = zq$, it is sufficient to observe that for any $\varphi \in \mathcal{D}(Q)$,

$$\int_0^T\int_0^L z^k q^k \varphi dx dt \to \int_0^T\int_0^L zq\varphi dx dt,$$

for $z^k \to z$ and $q^k\varphi \to q\varphi$ in F. Thus

$$z^k q^k \to zq \qquad \text{in } L^2((0,T)\times(0,L)) \text{ weakly.}$$

It follows that $(z^k q^k)_x \to (zq)_x$ in $\mathcal{D}'((0,T)\times(0,L))$. Therefore, (4.3.70) holds and $q \in T(z)$. On the other hand, $q \in A$, since $q^k \to q$ in F and A is closed. We conclude that $z \in T^{-1}(A)$, and hence $T^{-1}(A)$ is closed.

Il follows from Theorem 4.5 that there exists $q \in B$ with $q \in T(q)$, i.e. we have found a control $f \in L^2((0,T)\times\omega)$ such that the solution of (4.3.57) satisfies $q(T,\cdot) = 0$ in $(0,L)$. The proof of Theorem 4.1 is complete. □

With Theorem 4.1 at hand, one can prove Theorem 4.3 about the regional controllability.

4.3.3 Proof of Theorem 4.3.

By Theorem 4.1, if δ is small enough one can find a control input $f \in L^2(0,T/2,L^2(0,L))$ with $\text{supp}(f) \subset (0,T)\times\omega$ such that the solution of (4.1.7) satisfies $u(T/2,.) \equiv 0$ in

$(0, L)$. Pick any number $l_2' \in (l_1', l_2)$ with $l_2' \notin \mathcal{N}$. (This is possible, the set $\mathcal{N}$ being discrete.) By [62, Theorem 1.3], if δ is small enough one can pick a function $h \in L^2(T/2, T)$ such that the solution $y \in C^0([T/2, T], L^2(0, l_2')) \cap L^2(T/2, T, H^1(0, l_2'))$ of the system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (T/2, T) \times (0, l_2'), \\ y(t, 0) = y(t, l_2') = 0, \quad y_x(t, l_2') = h(t) & \text{in } (T/2, T), \\ y(T/2, x) = 0 & \text{in } (0, l_2') \end{cases}$$

satisfies $y(T, x) = u_1(x)$ for $0 < x < l_2'$. We pick a function $\mu \in C^\infty([0, L])$ such that

$$\mu(x) = \begin{cases} 1 & \text{if } x < l_1', \\ 0 & \text{if } x > \frac{l_1' + l_2'}{2} \end{cases}$$

and set for $T/2 < t \le T$

$$u(t, x) = \begin{cases} \mu(x) y(t, x) & \text{if } x < l_2', \\ 0 & \text{if } x > l_2'. \end{cases}$$

Note that, for $T/2 < t < T$, $u_t + u_{xxx} + u_x + uu_x = f$ with

$$f = \mu(\mu - 1) yy_x + (\mu_{xxx} y + 3\mu_{xx} y_x + 3\mu_x y_{xx} + \mu_x y) + \mu\mu_x y^2.$$

Since $||y||^4_{L^4(0,T,L^4(0,l_2'))} \le C||y||^2_{L^\infty(0,T,L^2(0,L))} ||y||^2_{L^2(0,T,H^1(0,L))}$, it is clear that

$$f \in L^2(0, T, H^{-1}(0, L))$$

with $\text{supp}(f) \subset (0, T) \times (l_1, l_2)$. Furthermore, $u \in C([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ solves (4.1.7) and satisfies (4.1.8). □

4.4 Exact controllability results

Pick any function $\rho \in C^\infty(0, L)$ with

$$\rho(x) = \begin{cases} 0 & \text{if } \; 0 < x < L - \nu, \\ 1 & \text{if } \; L - \frac{\nu}{2} < x < L, \end{cases} \tag{4.4.1}$$

for some $\nu \in (0, L)$.

This section is devoted to the investigation of the exact controllability of the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f = (\rho(x) h)_x & \text{in } (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = u_x(t, L) = 0 & \text{in } (0, T), \\ u(0, x) = u_0(x) & \text{in } (0, L). \end{cases} \tag{4.4.2}$$

More precisely, we aim to find a control input $h \in L^2(0, T; L^2(0, L))$ (actually, with $(\rho(x) h(t, x))_x$ in some space of functions) to guide the system described by (4.4.2) in the time interval $[0, T]$ from any (small) given initial state u_0 in $L^2_{\frac{1}{L-x}dx}$ to any (small) given terminal state u_T in the same space. We first consider the linearized system, and next proceed to the nonlinear one. The results involve some weighted Sobolev spaces.

4.4.1 The linear system

For any measurable function $w : (0, L) \to (0, +\infty)$ (not necessarily in $L^1(0, L)$), we introduce the weighted L^2-space

$$L^2_{w(x)dx} = \{u \in L^1_{loc}(0, L);\ \int_0^L u(x)^2 w(x)dx < \infty\}.$$

It is a Hilbert space when endowed with the scalar product

$$(u, v)_{L^2_{w(x)dx}} = \int_0^L u(x)v(x)w(x)dx.$$

We first prove the well-posedness of the linear system associated with (4.4.2), namely

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = u_x(t, L) = 0 & \text{in } (0, T), \\ u(0, x) = u_0(x) & \text{in } (0, L), \end{cases} \tag{4.4.3}$$

in both the spaces L^2_{xdx} and $L^2_{\frac{1}{L-x}dx}$, following [34] where the well-posedness was established in $L^2_{\frac{x}{L-x}dx}$. We need the following result.

Theorem 4.6. *(see [34]) Let $W \subset V \subset H$ be three Hilbert spaces with continuous and dense embeddings. Let $a(v, w)$ be a bilinear form defined on $V \times W$ that satisfies the following properties:*
*(i) (**Continuity**)*

$$a(v, w) \le M||v||_V||w||_W, \quad \forall v \in V,\ \forall w \in W; \tag{4.4.4}$$

*(ii) (**Coercivity**)*

$$a(w, w) \ge m||w||^2_V, \quad \forall w \in W; \tag{4.4.5}$$

Then for all $f \in V'$ (the dual space of V), there exists $v \in V$ such that

$$a(v, w) = f(w), \qquad \forall w \in W. \tag{4.4.6}$$

If, in addition to (i) and (ii), $a(v, w)$ satisfies
*(iii) (**Regularity**) for all $g \in H$, any solution $v \in V$ of (4.4.6) with $f(w) := (g, w)_H$ belongs to W,*
then (4.4.6) has a unique solution $v \in W$. Let $D(A)$ denote the set of those $v \in W$ when g ranges over H, and set $Av = -g$. Then A is a maximal dissipative operator, and hence it generates a continuous semigroup of contractions $(e^{tA})_{t\ge0}$ in H.

4.4.2 Well-posedness in L^2_{xdx}

Theorem 4.7. *Let $A_1 u = -u_{xxx} - u_x$ with domain*

$$D(A_1) = \{u \in H^2(0, L) \cap H^1_0(0, L);\ u_{xxx} \in L^2_{xdx},\ u_x(L) = 0\} \subset L^2_{xdx}.$$

Then A_1 generates a strongly continuous semigroup in L^2_{xdx}.

Proof. Let

$$H = L^2_{xdx}, \quad V = H^1_0(0,L), \quad W = \{w \in H^1_0(0,L),\ w_{xx} \in L^2_{x^2dx}\},$$

be endowed with the respective norms

$$||u||_H := ||\sqrt{x}u||_{L^2(0,L)}, \quad ||v||_V := ||v_x||_{L^2(0,L)}, \quad ||w||_W := ||xw_{xx}||_{L^2(0,L)}.$$

Clearly, $V \subset H$ with a continuous (dense) embedding between two Hilbert spaces. On the other hand, we have that

$$||w_x||_{L^2} \le C||xw_{xx}||_{L^2} \qquad \forall w \in W. \tag{4.4.7}$$

First, we note that we have for $w \in \mathcal{T} := C^\infty([0,L]) \cap H^1_0(0,L)$ and $p \in \mathbb{R}$

$$\begin{aligned} 0 &\le \int_0^L (xw_{xx} + pw_x)^2 dx \\ &= \int_0^L (x^2 w_{xx}^2 + 2pxw_x w_{xx} + p^2 w_x^2) dx \\ &= \int_0^L x^2 w_{xx}^2 dx + (p^2 - p) \int_0^L w_x^2 dx + pLw_x^2(L). \end{aligned}$$

Taking $p = 1/2$ results in

$$\int_0^L w_x^2 dx \le 4 \int_0^L x^2 w_{xx}^2 dx + 2L|w_x(L)|^2. \tag{4.4.8}$$

The estimate (4.4.8) is also true for any $w \in W$, since $\mathcal{T}$ is dense in W. Let us prove (4.4.7) by contradiction. If (4.4.7) is false, then there exists a sequence $\{w^n\}_{n \ge 0}$ in W such that

$$1 = ||w_x^n||_{L^2} \ge n||xw_{xx}^n||_{L^2} \qquad \forall n \ge 0.$$

Extracting subsequences, we may assume that

$$\begin{aligned} w^n &\to w \quad \text{in } H^1_0(0,L) \text{ weakly} \\ xw^n_{xx} &\to 0 \quad \text{in } L^2(0,L) \text{ strongly} \end{aligned}$$

and hence $xw_{xx} = 0$, which gives $w(x) = c_1 x + c_2$. Since $w \in H^1_0(0,L)$, we infer that $w \equiv 0$. Since w^n is bounded in $H^2(L/2, L)$, extracting subsequences we may also assume that $w_x^n(L)$ converges in $\mathbb{R}$. We infer then from (4.4.8) that w^n is a Cauchy sequence in $H^1_0(0,L)$, so that

$$w^n \to w \quad \text{in } H^1_0(0,L) \text{ strongly},$$

and hence $||w_x||_{L^2} = \lim_{n\to\infty} ||w_x^n||_{L^2} = 1$. This contradicts the fact that $w \equiv 0$. The proof of (4.4.7) is achieved.

Thus $||\cdot||_W$ is a norm in W, which is clearly a Hilbert space, and $W \subset V$ with continuous (dense) embedding. Let

$$a(v,w) = \int_0^L v_x[(xw)_{xx} + xw]dx, \qquad v \in V,\ w \in W.$$

Let us check that (i), (ii), and (iii) in Theorem 4.6 hold. For $v \in V$ and $w \in W$,

$$\begin{aligned} |a(v,w)| &\le ||v_x||_{L^2}||xw_{xx}+2w_x+xw||_{L^2} \\ &\le ||v_x||_{L^2}\big(||xw_{xx}||_{L^2}+C||w_x||_{L^2}\big) \\ &\le C||v||_V||w||_W \end{aligned}$$

where we used Poincaré inequality and (4.4.7). This proves that the bilinear form a is well defined and continuous on $V \times W$. For (ii), we first pick any $w \in \mathcal{T}$ to obtain

$$\begin{aligned} a(w,w) &= \int_0^L w_x(xw_{xx}+2w_x+xw)dx \\ &= \frac{3}{2}\int_0^L w_x^2dx + [x\frac{w_x^2}{2}]|_0^L - \frac{1}{2}\int_0^L w^2dx \\ &\ge \frac{3}{2}\int_0^L w_x^2dx - \frac{1}{2}\int_0^L w^2dx. \end{aligned}$$

By Poincaré inequality

$$\int_0^L w^2(x)dx \le (\frac{L}{\pi})^2\int_0^L w_x^2(x)dx,$$

and hence

$$a(w,w) \ge (\frac{3}{2}-\frac{L^2}{2\pi^2})\int_0^L w_x^2dx.$$

This shows the coercivity when $L < \pi\sqrt{3}$. When $L \ge \pi\sqrt{3}$, we have to consider, instead of a, the bilinear form $a_\lambda(v,w) := a(v,w)+\lambda(v,w)_H$ for $\lambda \gg 1$. Indeed, we have by Cauchy-Schwarz inequality and Hardy inequality

$$\begin{aligned} ||w||_{L^2}^2 &\le ||x^{\frac{1}{2}}w||_{L^2}||x^{-\frac{1}{2}}w||_{L^2} \\ &\le \sqrt{L}||w||_H||x^{-1}w||_{L^2} \\ &\le \varepsilon||w_x||_{L^2}^2 + C_\varepsilon||w||_H^2 \end{aligned}$$

and hence

$$a_\lambda(w,w) \ge (\frac{3}{2}-\frac{\varepsilon}{2})||w||_V^2 + (\lambda - \frac{C_\varepsilon}{2})||w||_H^2.$$

Therefore, if $\varepsilon < 3$ and $\lambda > C_\varepsilon/2$, then a_λ is a continuous bilinear form which is coercive.

Let us have a look at the regularity issue. For given $g \in H$, let $v \in V$ be such that

$$a_\lambda(v,w) = (g,w)_H \qquad \forall w \in W,$$

i.e.

$$\int_0^L v_x((xw)_{xx}+xw)dx + \lambda\int_0^L v(x)w(x)xdx = \int_0^L g(x)w(x)xdx. \tag{4.4.9}$$

Picking any $w \in \mathcal{D}(0,L)$ results in

$$\langle x(v_{xxx}+v_x+\lambda v), w\rangle_{\mathcal{D}',\mathcal{D}} = \langle xg, w\rangle_{\mathcal{D}',\mathcal{D}} \qquad \forall w \in \mathcal{D}(0,L), \tag{4.4.10}$$

and hence

$$v_{xxx} + v_x + \lambda v = g \qquad \text{in } \mathcal{D}'(0,L). \tag{4.4.11}$$

Since $v \in H_0^1(0,L)$ and $g \in L^2_{xdx}$, we have that $v \in H^3(\varepsilon, L)$ for all $\varepsilon \in (0,L)$ and $v_{xxx} \in L^2_{xdx}$. Picking any $w \in \mathcal{T}$ and $\varepsilon \in (0,L)$, and scaling in (4.4.11) by xw yields

$$\int_\varepsilon^L v_x((xw)_{xx} + xw)dx + [v_{xx}(xw) - v_x(xw)_x]|_\varepsilon^L = \int_\varepsilon^L (g - \lambda v)xwdx.$$

Letting $\varepsilon \to 0$ and comparing with (4.4.9), we obtain

$$-Lv_x(L)w_x(L) = \lim_{\varepsilon\to 0}\big(\varepsilon v_{xx}(\varepsilon)w(\varepsilon) - v_x(\varepsilon)(w(\varepsilon) + \varepsilon w_x(\varepsilon))\big). \tag{4.4.12}$$

Since $v_{xxx} \in L^2_{xdx}$, we obtain successively for some constant $C > 0$ and all $\varepsilon \in (0,L)$

$$|v_{xx}(\varepsilon) - v_{xx}(L)| \leq (\int_\varepsilon^L x|v_{xxx}|^2dx)^{\frac{1}{2}}(\int_\varepsilon^L x^{-1}dx)^{\frac{1}{2}} \leq C|\log\varepsilon| \tag{4.4.13}$$

$$|v_x(\varepsilon)| \leq C. \tag{4.4.14}$$

We infer from (4.4.13) that $v \in H^2(0,L)$, and hence $v \in W$. Furthermore, letting $\varepsilon \to 0$ in (4.4.12) and using (4.4.13)-(4.4.14) yields $v_x(L) = 0$, since $w_x(L)$ was arbitrary. We conclude that $v \in \mathcal{D}(A_1)$. Conversely, it is clear that the operator $A_1 - \lambda$ maps $\mathcal{D}(A_1)$ into H, and actually onto H from the above computations. Hence $A_1 - \lambda$ generates a strongly semigroup of contractions in H. □

4.4.3 Well-posedness in $L^2_{(L-x)^{-1}dx}$

Theorem 4.8. *Let $A_2 u = -u_{xxx} - u_x$ with domain*

$$\mathcal{D}(A_2) = \{u \in H^3(0,L) \cap H_0^1(0,L);\ u_{xxx} \in L^2_{\frac{1}{L-x}dx} \text{ and } u_x(L) = 0\} \subset L^2_{\frac{1}{L-x}dx}.$$

Then A_2 generates a strongly continuous semigroup in $L^2_{\frac{1}{L-x}dx}$.

Proof. We will use Hille-Yosida theorem, and (partially) Theorem 4.6. Let

$$H = L^2_{\frac{1}{L-x}dx}, \quad V = \{u \in H_0^1(0,L),\ u_x \in L^2_{\frac{1}{(L-x)^2}dx}\}, \quad W = H_0^2(0,L), \tag{4.4.15}$$

be endowed respectively with the norms

$$||u||_H = ||(L-x)^{-\frac{1}{2}}u||_{L^2}, \quad ||u||_V = ||(L-x)^{-1}u_x||_{L^2}, \quad ||u||_W = ||u_{xx}||_{L^2}. \tag{4.4.16}$$

From [34], we know that V endowed with $||\cdot||_V$ is a Hilbert space, and that

$$||(L-x)^{-2}u||_{L^2} \leq \frac{2}{3}||(L-x)^{-1}u_x||_{L^2} \qquad \forall u \in V, \tag{4.4.17}$$

and hence

$$||u||_H \leq (\int_0^L \frac{L^3}{(L-x)^4}u^2(x)dx)^{\frac{1}{2}} \leq \frac{2}{3}L^{\frac{3}{2}}||u||_V \qquad \forall u \in V. \tag{4.4.18}$$

Thus $V \subset H$ with continuous embedding. From Poincaré inequality, we have that $||\cdot||_W$ is a norm on W equivalent to the H^2-norm. On the other hand, from Hardy inequality

$$\int_0^L \frac{v^2}{(L-x)^2}dx \le C\int_0^L v_x^2 dx \quad \forall v \in H^1(0,L) \text{ with } v(L)=0, \tag{4.4.19}$$

we have that

$$||v||_V \le C||v||_W \qquad \forall v \in W. \tag{4.4.20}$$

Thus $W \subset V$ with continuous embedding. It is easily seen that $\mathcal{D}(0,L)$ is dense in H, V, and W. Let

$$a(v,w) = \int_0^L [v_x(\frac{w}{L-x})_{xx} + v_x\frac{w}{L-x}]dx \qquad (v,w) \in V\times W.$$

Then

$$\begin{aligned}|a(v,w)| &\le |\int_0^L v_x(\frac{w_{xx}}{L-x} + 2\frac{w_x}{(L-x)^2} + 2\frac{w}{(L-x)^3} + \frac{w}{L-x})dx| \\ &\le ||w_{xx}||_{L^2}||\frac{v_x}{L-x}||_{L^2} + 2||\frac{w_x}{L-x}||_{L^2}||\frac{v_x}{L-x}||_{L^2} \\ &+ ||\frac{v_x}{L-x}||_{L^2}\big(2||\frac{w}{(L-x)^2}||_{L^2} + ||w||_{L^2}\big) \\ &\le C||v||_V||w||_W\end{aligned}$$

by (4.4.17), (4.4.18), and (4.4.20). This shows that a is well defined and continuous. Let us look at the coercivity of a. Pick any $w \in \mathcal{D}(0,L)$. Then

$$\begin{aligned}a(w,w) &= \int_0^L w_x\big(\frac{w_{xx}}{L-x} + 2\frac{w_x}{(L-x)^2} + 2\frac{w}{(L-x)^3} + \frac{w}{L-x}\big)dx \\ &= \frac{3}{2}\int_0^L \frac{w_x^2}{(L-x)^2}dx - 3\int_0^L \frac{w^2}{(L-x)^4}dx - \frac{1}{2}\int_0^L \frac{w^2}{(L-x)^2}dx \\ &\ge \frac{1}{6}\int_0^L \frac{w_x^2}{(L-x)^2}dx - \frac{1}{2}\int_0^L \frac{w^2}{(L-x)^2}dx\end{aligned}$$

where we used (4.4.17) for the last line. Note that, using Cauchy-Schwarz inequality and (4.4.17), we have that

$$\begin{aligned}||\frac{w}{L-x}||_{L^2}^2 &\le ||(L-x)^{-\frac{1}{2}}w||_{L^2}||(L-x)^{-\frac{3}{2}}w||_{L^2} \\ &\le \frac{2\sqrt{L}}{3}||w||_H||w||_V \\ &\le \varepsilon||w||_V^2 + \frac{L}{9\varepsilon}||w||_H^2.\end{aligned} \tag{4.4.21}$$

If we pick $\varepsilon \in (0,1/3)$, we infer that for all $w \in \mathcal{D}(0,L)$

$$a(w,w) + \frac{L}{18\varepsilon}||w||_H^2 \ge \big(\frac{1}{6} - \frac{\varepsilon}{2}\big)||w||_V^2 \ge C||w||_V^2. \tag{4.4.22}$$

The result is also true for any $w \in W$, by density. This shows that the continuous bilinear form

$$a_\lambda(v,w) = a(v,w) + \lambda(v,w)_H$$

is coercive for $\lambda > L/6$. Let $g \in H$ be given. By Theorem 4.6, there is at least one solution $v \in V$ of

$$a_\lambda(v,w) = (g,w)_H \qquad \forall w \in W. \tag{4.4.23}$$

Pick such a solution $v \in V$, and let us prove that $v \in \mathcal{D}(A_2)$. Picking any $w \in \mathcal{D}(0,L)$ in (4.4.23) yields

$$v_{xxx} + v_x + \lambda v = g \qquad \text{in } \mathcal{D}'(0,L). \tag{4.4.24}$$

As $g \in L^2(0,L)$ and $v \in H^1(0,L)$, we have that $v_{xxx} \in L^2(0,L)$, and $v \in H^3(0,L)$. Pick finally w of the form $w(x) = x^2(L-x)^2\overline{w}(x)$, where $\overline{w} \in C^\infty([0,L])$ is arbitrary chosen. Note that $w \in W$ and that $w/(L-x) \in H_0^1(0,L) \cap C^\infty([0,L])$. Multiplying in (4.4.24) by $w/(L-x)$ and integrating over $(0,L)$, we obtain after comparing with (4.4.23)

$$0 = -v_x(\frac{w}{L-x})_x|_0^L = -v_x\big((2xL-3x^2)\overline{w} + x^2(L-x)\overline{w}_x\big)|_0^L = v_x(L)L^2\overline{w}(L).$$

As $\overline{w}(L)$ can be chosen arbitrarily, we conclude that $v_x(L) = 0$. Using (4.4.19) twice, we infer that $v_x + \lambda v \in H$, and hence $v_{xxx} = g - (v_x + \lambda v) \in H$. Therefore $v \in \mathcal{D}(A_2)$. Thus, for $\lambda > L/6$ we have that $A_2 - \lambda : \mathcal{D}(A_2) \to H$ is onto. Let us check that $A_2 - \lambda$ is also dissipative in H. Pick any $w \in \mathcal{D}(A_2)$. Then we obtain after some integrations by parts that

$$(A_2 w, w)_H = -\frac{3}{2}\int_0^L \frac{w_x^2}{(L-x)^2}dx + 3\int_0^L \frac{w^2}{(L-x)^4}dx + \frac{1}{2}\int_0^L \frac{w^2}{(L-x)^2}dx - \frac{w_x^2(0)}{2L}$$

and

$$(A_2 w - \lambda w, w)_H \le -(\frac{1}{6} - \frac{\varepsilon}{2})||w||_V^2 - \frac{w_x^2(0)}{2L} \le 0$$

for $\varepsilon < 1/3$ and $\lambda = L/(18\varepsilon)$. We conclude that $A_2 - \lambda$ is maximal dissipative for $\lambda > L/6$, and thus it generates a strongly continuous semigroup of contractions in H by Hille-Yosida theorem. □

A global Kato smoothing effect as in [34, 62] can as well be derived.

Proposition 4.7. *Let H and V be as in (4.4.15)-(4.4.16), and let $T > 0$ be given. Then there exists some constant $C = C(L,T)$ such that for any $u_0 \in H$, the solution $u(t) = e^{tA_2}u_0$ of (4.4.3) satisfies*

$$||u||_{L^\infty(0,T,H)} + ||u||_{L^2(0,T,V)} \le C||u_0||_H. \tag{4.4.25}$$

Proof. We proceed as in [34]. First, we notice that $\mathcal{D}(A_2)$ is dense in H, so that it is sufficient to prove the result when $u_0 \in \mathcal{D}(A_2)$. Note that the estimate $||u||_{L^\infty(0,T,H)} \le C||u_0||_H$ is a consequence of classical semigroup theory. Assume $u_0 \in \mathcal{D}(A_2)$, so that $u_t = A_2 u$ in the classical sense. Taking the inner product in H with u yields

$$(u_t, u)_H = -a(u,u) \le -C||u||_V^2 + \frac{L}{18\varepsilon}||u||_H^2$$

where we used (4.4.22). An integration over $(0,T)$ completes the proof of the estimate of $||u||_{L^2(0,T,V)}$. □

4.4.4 Non-homogeneous system

In this section we consider the nonhomogeneous system

$$u_t + u_x + u_{xxx} = f(t,x) \qquad \text{in } (0,T)\times(0,L), \tag{4.4.26}$$
$$u(t,0) = u(t,L) = u_x(t,L) = 0 \qquad \text{in } (0,T), \tag{4.4.27}$$
$$u(0,x) = u_0 \qquad \text{in } (0,L). \tag{4.4.28}$$

We need the prove the existence of a "reasonable" solution when solely

$$f \in L^2(0,T,H^{-1}(0,L)).$$

Proposition 4.8. *Let $u_0 \in L^2_{xdx}$ and $f \in L^2(0,T;H^{-1}(0,L))$. Then there exists a unique solution $u \in C([0,T],L^2_{xdx}) \cap L^2(0,T,H^1(0,L))$ to (4.4.26)-(4.4.28). Furthermore, there is some constant $C>0$ such that*

$$||u||_{L^\infty(0,T,L^2_{xdx})} + ||u||_{L^2(0,T,H^1(0,L))} \le C\big(||u_0||_{L^2_{xdx}} + ||f||_{L^2(0,T,H^{-1}(0,L))}\big). \tag{4.4.29}$$

Proof. Assume first that $u_0 \in \mathcal{D}(A_1)$ and $f \in C^0([0,T],\mathcal{D}(A_1))$ to legitimate the following computations. Multiplying each term in (4.4.26) by xu and integrating over $(0,\tau)\times(0,L)$ where $0<\tau<T$ yields

$$\int_0^\tau\int_0^L xuf\,dxdt = \frac{1}{2}\int_0^L x|u(\tau,x)|^2dx - \frac{1}{2}\int_0^L x|u_0(x)|^2dx$$
$$+\frac{3}{2}\int_0^\tau\int_0^L |u_x|^2dxdt - \frac{1}{2}\int_0^\tau\int_0^L |u|^2dxdt. \tag{4.4.30}$$

$\langle .,. \rangle_{H^{-1},H^1_0}$ denoting the duality pairing between $H^{-1}(0,L)$ and $H^1_0(0,L)$, we have that for all $\varepsilon>0$

$$\int_0^\tau\int_0^L xuf\,dxdt = \int_0^\tau \langle f, xu\rangle_{H^{-1},H^1_0} \le \frac{\varepsilon}{2}\int_0^\tau\int_0^L u_x^2dxdt + C_\varepsilon\int_0^\tau ||f||^2_{H^{-1}}dt. \tag{4.4.31}$$

The last term in the l.h.s. of (4.4.30) is decomposed as

$$\frac{1}{2}\int_0^\tau\int_0^L |u|^2dxdt = \frac{1}{2}\int_0^\tau\int_0^{\sqrt{\varepsilon}} |u|^2dxdt + \frac{1}{2}\int_0^\tau\int_{\sqrt{\varepsilon}}^L |u|^2dxdt =: I_1 + I_2.$$

We claim that

$$I_1 \le \frac{\varepsilon}{2}\int_0^\tau\int_0^L |u_x|^2dxdt, \tag{4.4.32}$$
$$I_2 \le \frac{1}{2\sqrt{\varepsilon}}\int_0^\tau\int_0^L x|u|^2dxdt. \tag{4.4.33}$$

For (4.4.32), since $u(0,t)=0$ we have that for $(t,x)\in(0,T)\times(0,\sqrt{\varepsilon})$

$$|u(x,t)| \le \int_0^{\sqrt{\varepsilon}} |u_x|dx \le \varepsilon^{\frac{1}{4}}\Big(\int_0^{\sqrt{\varepsilon}} |u_x|^2dx\Big)^{\frac{1}{2}}$$

and hence
$$\int_0^{\sqrt{\varepsilon}} |u|^2 dx \le \varepsilon \int_0^{\sqrt{\varepsilon}} |u_x|^2 dx$$
which gives (4.4.32) after integrating over $t \in (0,\tau)$. (4.4.33) is obvious.

Gathering together (4.4.30)-(4.4.33), we obtain
$$\frac{1}{2}\int_0^L x|u(\tau,x)|^2 dx + (\frac{3}{2}-\varepsilon)\int_0^\tau\int_0^L |u_x|^2 dxdt \le \frac{1}{2}\int_0^L x|u_0(x)|^2 dx + \frac{1}{2\sqrt{\varepsilon}}\int_0^\tau\int_0^L x|u|^2 dxdt + C_\varepsilon \int_0^\tau ||f||^2_{H^{-1}} dt.$$
Letting $\varepsilon = 1$ and applying Gronwall's lemma, we obtain
$$||u||^2_{L^\infty(0,T,L^2_{xdx})} + ||u_x||^2_{L^2(0,T,L^2(0,L))} \le C(T)\big(||u_0||^2_{L^2_{xdx}} + ||f||^2_{L^2(0,T,H^{-1}(0,L))}\big).$$
This gives (4.4.29) for $u_0 \in D(A_1)$ and $f \in C^0([0,T],D(A_1))$. A density argument allows us to construct a solution $u \in C([0,T],L^2_{xdx}) \cap L^2(0,T,H^1(0,L))$ of (4.4.26)-(4.4.28) satisfying (4.4.29) for $u_0 \in L^2_{xdx}$ and $f \in L^2(0,T,H^{-1}(0,L))$. The uniqueness follows from classical semigroup theory. □

Our goal now is to obtain a similar result in the spaces H and V introduced in (4.4.15)-(4.4.16). To do that, we limit ourselves to the situation when $f = (\rho(x)h)_x$ with $h \in L^2(0,T,L^2(0,L))$.

Proposition 4.9. *Let $u_0 \in H$ and $h \in L^2(0,T,L^2(0,L))$, and set $f := (\rho(x)h)_x$. Then there exists a unique solution $u \in C([0,T],H) \cap L^2(0,T,V)$ to (4.4.26)-(4.4.28). Furthermore, there is some constant $C > 0$ such that*
$$||u||_{L^\infty(0,T,H)} + ||u||_{L^2(0,T,V)} \le C\big(||u_0||_H + ||h||_{L^2(0,T,L^2(0,L))}\big). \tag{4.4.34}$$
Proof. Assume that $u_0 \in \mathcal{D}(A_2)$ and $h \in C_0^\infty((0,T)\times(0,L))$, so that $f \in C^1([0,T],H)$. Taking the inner product of $u_t - A_2 u - f = 0$ with u in H yields
$$(u_t,u)_H = -a(u,u) + (f,u)_H \le -C||u||^2_V + \frac{L}{18\varepsilon}||u||^2_H + (f,u)_H, \tag{4.4.35}$$
where we used (4.4.22). Then
$$\begin{aligned} |(f,u)_H| &= |\int_0^L (\rho(x)h)_x \frac{u}{L-x} dx| \\ &= |\int_0^L \rho(x)h\big(\frac{u_x}{L-x} + \frac{u}{(L-x)^2}\big)dx| \\ &\le C||h||_{L^2}(||\frac{u_x}{L-x}||_{L^2} + ||\frac{u}{(L-x)^2}||_{L^2}) \\ &\le C||h||_{L^2}||u||_V, \end{aligned}$$
where we used (4.4.17) in the last line. Thus, we have that
$$|(f,u)_H| \le \frac{C}{2}||u||^2_V + C'||h||^2_{L^2}$$

which, when combined with (4.4.35), gives after integration over $(0,\tau)$ for $0<\tau<T$

$$||u(\tau)||_H^2 + C\int_0^\tau ||u||_V^2 dt \le ||u_0||_H^2 + C''\Big(\int_0^\tau ||u||_H^2 dt + \int_0^\tau\int_0^L |h|^2 dxdt\Big).$$

An application of Gronwall's lemma yields (4.4.34) for $u_0 \in \mathcal{D}(A_2)$ and $h \in C_0^\infty((0,T)\times(0,L))$. A density argument allows us to construct a solution $u \in C([0,T],H)\cap L^2(0,T,V)$ of (4.4.26)-(4.4.28) satisfying (4.4.34) for $u_0 \in H$ and $h \in L^2(0,T,L^2(0,L))$. The uniqueness follows from classical semigroup theory. □

4.4.5 Controllability of the linearized system

We turn our attention to the control properties of the linear system

$$\begin{aligned} u_t + u_{xxx} + u_x &= f = (\rho(x)h)_x, && (4.4.36)\\ u(t,0) = u(t,L) = u_x(t,L) &= 0, && (4.4.37)\\ u(0,x) &= u_0(x). && (4.4.38)\end{aligned}$$

Theorem 4.9. *Let $T>0$, $\nu \in (0,L)$ and $\rho(x)$ as in (4.4.1). Then there exists a continuous linear operator $\Gamma : L^2_{\frac{1}{L-x}dx} \to L^2(0,T,L^2(0,L)) \cap L^2_{(T-t)dt}(0,T,H^1(0,L))$ such that for any $u_1 \in L^2_{\frac{1}{L-x}dx}$, the solution u of (4.4.36)-(4.4.38) with $u_0 = 0$ and $h = \Gamma(u_1)$ satisfies $u(T,x) = u_1(x)$ in $(0,L)$.*

Note that the forcing term $f = (\rho(x)h)_x$ is actually a *function* in $L^2_{(T-t)dt}(0,T,L^2(0,L))$ supported in $(0,T)\times(L-\nu,L)$.

Proof. We use the Hilbert Uniqueness Method (see e.g. [42]). Introduce the adjoint system

$$\begin{aligned} -v_t - v_{xxx} - v_x &= 0, && (4.4.39)\\ v(t,0) = v(t,L) = v_x(t,0) &= 0, && (4.4.40)\\ v(T,x) &= v_T(x). && (4.4.41)\end{aligned}$$

If $u_0 \equiv 0$, $v_T \in \mathcal{D}(0,L)$, and $h \in \mathcal{D}((0,T)\times(0,L))$, then multiplying in (4.4.36) by v and integrating over $(0,T)\times(0,L)$ gives

$$\int_0^L u(T,x)v_T(x)dx = \int_0^T\int_0^L (\rho(x)h)_x v dxdt = -\int_0^T\int_0^L \rho(x)hv_x dxdt.$$

The usual change of variables $x \to L-x$, $t\to T-t$, combined with Proposition 4.8, gives

$$||v||_{L^\infty(0,T,L^2_{(L-x)dx})} + ||v||_{L^2(0,T,H^1(0,L))} \le C||v_T||_{L^2_{(L-x)dx}}.$$

By a limiting argument, we obtain that for all $h \in L^2(0,T,L^2(0,L))$ and all $v_T \in L^2_{(L-x)dx}$,

$$\langle u(T,.), v_T\rangle_{L^2_{\frac{1}{L-x}dx}, L^2_{(L-x)dx}} = -\int_0^T (h, \rho(x)v_x)_{L^2} dt,$$

where u and v denote the solutions of (4.4.36)-(4.4.38) and (4.4.39)-(4.4.41), respectively, and $\langle \cdot, \cdot \rangle_{L^2_{\frac{1}{L-x}dx}, L^2_{(L-x)dx}}$ denotes the duality pairing between $L^2_{\frac{1}{L-x}dx}$ and $L^2_{(L-x)dx}$. We have to prove the following observability inequality

$$||v_T||^2_{L^2_{(L-x)dx}} \le C \int_0^T \int_0^L |\rho(x) v_x|^2 dxdt \tag{4.4.42}$$

or, equivalently, letting $w(t,x) = v(T-t, L-x)$,

$$||w_0||^2_{L^2_{xdx}} \le C \int_0^T \int_0^L |\rho(L-x) w_x|^2 dxdt \tag{4.4.43}$$

where w solves

$$\begin{cases} w_t + w_{xxx} + w_x = 0, \\ w(t,0) = w(t,L) = w_x(t,L) = 0, \\ w(0,x) = w_0(x). \end{cases} \tag{4.4.44}$$

From [62], we know that for any $q \in C^\infty([0,T] \times [0,L])$

$$-\int_0^T \int_0^L (q_t + q_{xxx} + q_x) \frac{w^2}{2} dxdt + \int_0^L (q\frac{w^2}{2})(T,x)dx - \int_0^L (q\frac{w^2}{2})(0,x)dx \\ + \frac{3}{2}\int_0^T \int_0^L q_x w_x^2 dxdt + \int_0^T (q\frac{w_x^2}{2})(t,0)dt = 0.$$

We pick $q(t,x) = (T-t)b(x)$, where $b \in C^\infty([0,L])$ is nondecreasing and satisfies

$$b(x) = \begin{cases} x & \text{if } \ 0 < x < \nu/4, \\ 1 & \text{if } \ \nu/2 < x < L. \end{cases}$$

This yields

$$\begin{aligned} ||w_0||^2_{L^2_{xdx}} &\le C(L,\nu) \int_0^L b(x) w_0^2(x) dx \\ &\le C(T,L,\nu)\big(\int_0^T \int_0^{\frac{\nu}{2}} w_x^2 dxdt + \int_0^T \int_0^L w^2 dxdt \big). \end{aligned} \tag{4.4.45}$$

If the estimate

$$||w_0||^2_{L^2_{xdx}} \le C \int_0^T \int_0^{\frac{\nu}{2}} w_x^2 dxdt \tag{4.4.46}$$

fails, then one can find a sequence $\{w_0^n\} \subset L^2_{xdx}$ such that

$$1 = ||w_0^n||^2_{L^2_{xdx}} > n \int_0^T \int_0^{\frac{\nu}{2}} |w_x^n|^2 dxdt, \tag{4.4.47}$$

where w^n denotes the solution of (4.4.44) with w_0 replaced by w_0^n. By (4.4.29) and (4.4.47), $\{w^n\}$ is bounded in $L^2(0,T,H^1(0,L))$, hence also in $H^1(0,T,H^{-2}(0,L))$ by (4.4.44). Extracting a subsequence, we have by Aubin-Lions' lemma that w^n converges strongly in $L^2(0,T,L^2(0,L))$. Thus, using (4.4.45) and (4.4.47), we see that w_0^n is a Cauchy sequence in L^2_{xdx}, and hence it converges strongly in this space. Let w_0 denote its limit in L^2_{xdx}, and let w denote the corresponding solution of (4.4.44). Then

$$||w_0||_{L^2_{xdx}} = 1,$$
$$w^n \to w \qquad \text{in } L^2(0,T,H^1(0,L)).$$

But $w_x^n \to 0$ in $L^2(0,T,L^2(0,\nu/2))$ by (4.4.47). Thus $w_x \equiv 0$ in $(0,T)\times(0,\nu/2)$, and hence $w(t,x) = g(t)$ (for some function g) in $(0,T)\times(0,\nu/2)$. Since w satisfies (4.4.44), we infer from $w(t,0) = 0$ that $w \equiv 0$ in $(0,T)\times(0,\nu/2)$, and also in $(0,T)\times(0,L)$ by Holmgren's theorem. This would imply that $w(0,x) = 0$, in contradiction with $||w_0||_{L^2_{xdx}} = 1$. Therefore (4.4.46) is proved, and (4.4.43) follows at once.

We are in a position to apply H.U.M. Let $\Lambda(v_T) = (L-x)^{-1}u(T,.) \in L^2_{(L-x)dx}$, where u solves (4.4.36)-(4.4.38) with $h = -\rho(x)v_x$. Then $\Lambda : L^2_{(L-x)dx} \to L^2_{(L-x)dx}$ is clearly continuous. On the other hand, from (4.4.42)

$$\big(\Lambda(v_T), v_T\big)_{L^2_{(L-x)dx}} = \langle u(T,.), v_T\rangle_{L^2_{\frac{1}{L-x}dx}, L^2_{(L-x)dx}} = \int_0^T ||\rho(x)v_x||^2_{L^2}dt \geq C||v_T||^2_{L^2_{(L-x)dx}},$$

and it follows that the map $v_T \to \Lambda(v_T)$ is invertible in $L^2_{(L-x)dx}$.

Define the map $\Gamma :\ L^2_{\frac{1}{L-x}dx} \to L^2(0,T,L^2(0,L))$ by $\Gamma(u_1) = h := -\rho(x)v_x$, where v is the solution of (4.4.39)-(4.4.41) with $v_T = \Lambda^{-1}((L-x)^{-1}u_1)$. Γ is continuous from $L^2_{\frac{1}{L-x}dx}$ to $L^2(0,T,L^2(0,L))$, and the solution u of (4.4.36)-(4.4.38) with $u_0 = 0$ and $h = \Gamma(u_1)$ satisfies $u(T,.) = u_1$. To prove that Γ is also continuous from $L^2_{\frac{1}{L-x}dx}$ into $L^2_{(T-t)dt}(0,T,H^1(0,L))$, it is sufficient to prove the following estimate

$$\int_0^T ||v(t)||^2_{H^2}(T-t)dt \leq C||v_T||^2_{L^2_{(L-x)dx}},$$

for the solutions of (4.4.39)-(4.4.41) or, alternatively, the estimate

$$\int_0^T ||w||^2_{H^2}\, tdt \leq C||w_0||^2_{L^2_{xdx}} \tag{4.4.48}$$

for the solutions of (4.4.44). By Proposition 4.8,

$$\int_0^T ||w||^2_{H_0^1(0,L)}dt \leq C||w_0||^2_{L^2_{xdx}}. \tag{4.4.49}$$

This yields for $w_0 \in L^2(0,L)$

$$\int_0^T ||w||^2_{H_0^1(0,L)}dt \leq C||w_0||^2_{L^2}. \tag{4.4.50}$$

Assume now that $w_0 \in \mathcal{D}(A)$, and let $u_0 = Aw_0 = -w_{0,xxx} - w_{0,x}$. Denote by w (resp. u) the solution of (4.4.44) issuing from w_0 (resp. u_0). Then

$$Aw = -w_{xxx} - w_x = u \in L^2(0,T,H_0^1(0,L)),$$

and we infer that $w \in L^2(0,T,H^4(0,L))$. By interpolation, this gives that $w \in L^2(0,T,H^2(0,L))$ if $w_0 \in H_0^1(0,L)$, with an estimate of the form

$$\int_0^T ||w||^2_{H^2(0,L)} dt \leq C||w_0||^2_{H_0^1(0,L)}. \tag{4.4.51}$$

The different constants C in (4.4.49)-(4.4.51) may be taken independent of T for $0 < T < T_0$. Thus, using Fubini's theorem, we obtain

$$\int_0^T s||w(s)||^2_{H^2} ds = \int_0^T (\int_t^T ||w(s)||^2_{H^2} ds) dt \leq C \int_0^T ||w(t)||^2_{H_0^1(0,L)} dt \leq C||w_0||^2_{L^2_{xdx}}.$$

This completes the proof of (4.4.48) and of Theorem 4.9. □

4.4.6 Exact controllability of the nonlinear system

Our aim is to prove the local exact controllability in $L^2_{\frac{1}{L-x}dx}$ of system (4.4.2). Note that the solutions of (4.4.2) can be written as

$$u = u_L + u_1 + u_2,$$

where u_L is the solution of (4.4.3) with initial data $u_0 \in L^2_{\frac{1}{L-x}dx}$, u_1 is solution of

$$\begin{cases} u_{1,t} + u_{1,x} + u_{1,xxx} = f = (\rho(x)h)_x & \text{in } (0,T) \times (0,L), \\ u_1(t,0) = u_1(t,L) = u_{1,x}(t,L) = 0 & \text{in } (0,T), \\ u_1(0,x) = 0 & \text{in } (0,L) \end{cases} \tag{4.4.52}$$

with $h = h(t,x) \in L^2(0,T;L^2(0,L))$, and u_2 is solution of

$$\begin{cases} u_{2,t} + u_{2,x} + u_{2,xxx} = g(t,x) & \text{in } (0,T) \times (0,L), \\ u_2(t,0) = u_2(t,L) = u_{2,x}(t,L) = 0 & \text{in } (0,T), \\ u_2(0,x) = 0 & \text{in } (0,L), \end{cases} \tag{4.4.53}$$

with $g = g(t,x) = -uu_x$.

The following result is concerned with the solutions of the non-homogeneous system (4.4.53).

Proposition 4.10. *(i) Let H and V be as in (4.4.15)-(4.4.16) If $u,v \in L^2(0,T;V)$, then $uv_x \in L^1(0,T;H)$. Furthermore, the map*

$$(u,v) \in L^2(0,T;V)^2 \to uv_x \in L^1(0,T;H)$$

is continuous and there exists a constant $c > 0$ such that

$$\|uv_x\|_{L^1(0,T;H)} \leq c\,\|u\|_{L^2(0,T;V)}\,\|v\|_{L^2(0,T;V)}\,. \tag{4.4.54}$$

(ii) For $g \in L^1(0,T;H)$, the mild solution u of (4.4.53) given by Duhamel formula satisfies

$$u_2 \in C([0,T]\,;H) \cap L^2(0,T;V) =: \mathcal{G}$$

and we have the estimate

$$||u_2||_{L^\infty(0,T,H)} + ||u_2||_{L^2(0,T,V)} \leq C||g||_{L^1(0,T,H)}. \tag{4.4.55}$$

Proof. For $u, v \in V$, we have

$$||uv_x||_{L^2_{\frac{1}{L-x}dx}} \leq ||u||_{L^\infty}||\frac{v_x}{\sqrt{L-x}}||_{L^2} \leq C||u||_V||v||_V.$$

This gives (i). For (ii), we first assume that $g \in C^1([0,T],H)$, so that

$$u_2 \in C^1([0,T],H) \cap C^0([0,T],\mathcal{D}(A_2)).$$

Taking the inner product of $u_{2,t} = A_2u_2 + g$ with u_2 in H yields

$$(u_{2,t}, u_2)_H \leq -C||u_2||_V^2 + C'||u_2||_H^2 + (g, u_2)_H \tag{4.4.56}$$

where C, C' denote some positive constants. Integrating over $(0,T)$ and using the classical estimate

$$||u_2||_{L^\infty(0,T,H)} \leq C||g||_{L^1(0,T,H)}$$

coming from semigroup theory, we obtain (ii) when $g \in C^1([0,T],H)$. The general case ($g \in L^1(0,T,H)$) follows by density. □

Let $\Theta_1(h) := u_1$ and $\Theta_2(g) := u_2$, where u_1 (resp. u_2) denotes the solution of (4.4.52) (resp. (4.4.53)). Then $\Theta_1 : L^2(0,T;L^2(0,L)) \to \mathcal{G}$ and $\Theta_2 : L^1(0,T;L^2_{\frac{1}{L-x}dx}) \to \mathcal{G}$ are well-defined continuous operators, by Propositions 4.9 and 4.10.

Using Proposition 4.10 and the contraction mapping principle, one can prove as in [34, 56, 62] the existence and uniqueness of a solution $u \in \mathcal{G}$ of (4.4.2) when the initial data u_0 and the forcing term h are small enough. As the proof is similar to those of Theorem 4.10, it will be omitted.

We are in a position to prove the main result of Section 4.4, namely the (local) exact controllability of system (4.4.2).

Theorem 4.10. *Let $T > 0$. Then there exists $\delta > 0$ such that for any u_0, $u_1 \in L^2_{\frac{1}{L-x}dx}$ satisfying $\|u_0\|_{L^2_{\frac{1}{L-x}dx}} \leq \delta$, $\|u_1\|_{L^2_{\frac{1}{L-x}dx}} \leq \delta$, one can find a control function $h \in L^2(0,T;L^2(0,L))$ such that the solution $u \in \mathcal{G}$ of (4.4.2) satisfies $u(T,\cdot) = u_1$ in $(0,L)$.*

As in the linear case, the forcing term $f = (\rho(x)h)_x$ is actually a *function* in $L^2_{(T-t)dt}(0,T,L^2(0,L))$ supported in $(0,T) \times (L-\nu, L)$.

Proof. To prove this result, we apply the contraction mapping principle, following closely [62]. Let $\mathcal{F}$ denote the nonlinear map

$$\mathcal{F} : L^2(0,T;V) \to \mathcal{G},$$

defined by

$$\mathcal{F}(u) = u_L + \Theta_1 \circ \Gamma(u_T - u_L(T,\cdot) + \Theta_2(uu_x)(T,\cdot)) - \Theta_2(uu_x),$$

where u_L is the solution of (4.4.3) with initial data $u_0 \in L^2_{\frac{1}{L-x}dx}$, Θ_1 and Θ_2 are defined as above, and Γ is as in Theorem 4.9.

Remark that if u is a fixed point of $\mathcal{F}$, then u is a solution of (4.4.2) with the control $h = \Gamma(u_T - u_L(T,\cdot) + \Theta_2(uu_x)(T,\cdot))$, and it satisfies

$$u(T,\cdot) = u_T,$$

as desired. In order to prove the existence of a fixed point of $\mathcal{F}$, we apply the Banach fixed-point theorem to the restriction of $\mathcal{F}$ to some closed ball $\overline{B}(0,R)$ in $L^2(0,T;V)$.
(i) *$\mathcal{F}$ is contractive.* Pick any $u, \tilde{u} \in \overline{B}(0,R)$. Using (4.4.34) and (4.4.54)-(4.4.55), we deduce that for some constant C, independent of u, $\tilde{u}$, and R, we have

$$\|\mathcal{F}(u) - \mathcal{F}(\tilde{u})\|_{L^2(0,T;V)} \leq 2CR\,\|u - \tilde{u}\|_{L^2(0,T;V)}\,. \tag{4.4.57}$$

Hence, $\mathcal{F}$ is contractive if R satisfies

$$R < \frac{1}{4C}, \tag{4.4.58}$$

where C is the constant in (4.4.57).
(ii) *$\mathcal{F}$ maps $\overline{B}(0,R)$ into itself.* Using Proposition 4.7 and the continuity of the operators Γ, Θ_1, and Θ_2, we infer the existence of a constant $C' > 0$ such that for any $u \in \overline{B}(0,R)$, we have

$$\|\mathcal{F}(u)\|_{L^2(0,T;V)} \leq C'(\|u_0\|_{L^2_{\frac{1}{L-x}dx}} + \|u_T\|_{L^2_{\frac{1}{L-x}dx}} + R^2).$$

Thus, taking R satisfying (4.4.58) and $R < 1/(2C')$ and assuming that $\|u_0\|_{L^2_{\frac{1}{L-x}dx}}$ and $\|u_T\|_{L^2_{\frac{1}{L-x}dx}}$ are small enough, we obtain that the operator $\mathcal{F}$ maps $\overline{B}(0,R)$ into itself. Therefore the map $\mathcal{F}$ has a fixed point in $\overline{B}(0,R)$ by the Banach fixed-point Theorem. The proof of Theorem 4.10 is complete. □

4.5 Final Comments

Let us consider the following system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f & \text{in } (0,T)\times(0,L), \\ u(t,0) = u(t,L) = u_x(t,L) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x) & \text{in } (0,L), \end{cases} \tag{4.5.1}$$

where $\mathrm{supp}(f) \subset (0,T)\times\omega$ with $\omega = (l_1,l_2) \subset (0,L)$. When ω is any open interval in $(0,L)$, system (4.5.1) is (locally) null controllable in $L^2(0,L)$ with control inputs

$f \in L^2((0,T) \times (0,L))$. The position of the support is represented in Figure 1.

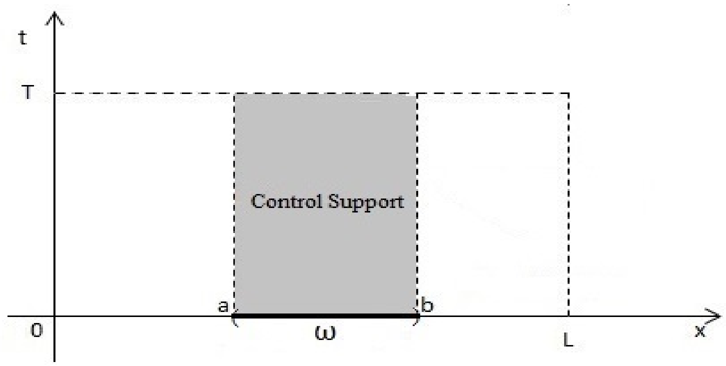

Fig. 1 - Null controllability: ω is any subinterval of $(0, L)$.

If $l_2 = L$ (see Figure 2) and f takes the form $f = (\rho(x)h)_x$ with $h \in L^2(0,T; L^2(0,L))$ and $\text{supp}(\rho) \subset \omega = (L - \nu, L)$, then system (4.5.1) is exactly controllable in time T in $L^2_{\frac{1}{L-x}dx}$. Note that the state space is a strict subspace of $L^2(0,L)$, and that the control inputs f are taken in $L^2(0,T,H^{-1}(0,L)) \cap L^2_{(T-t)dt}(0,T,L^2(0,L))$.

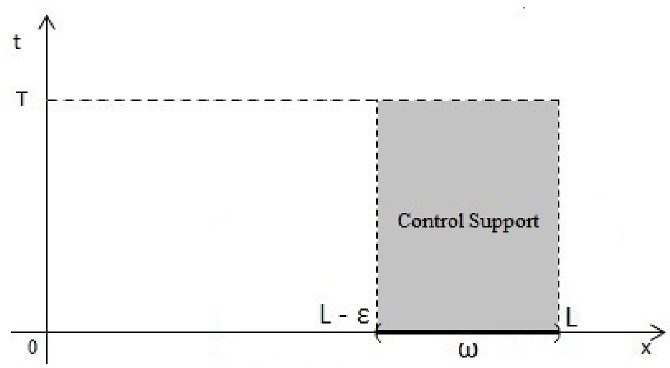

Fig. 2 - Exact controllability: ω is a neighborhood of $x = L$

Finally, taking again $\omega = (l_1, l_2) \subset (0, L)$, we derived a regional controllability in time T for system (4.5.1), in the sense that we had exact controllability on the interval $(0, l_1)$ and null controllability on the interval (l_2, L), taking control inputs $f \in L^2(0,T; H^{-1}(0,L))$ with $\text{supp}(f) \subset (0,T) \times \omega$, and initial data $u_0 \in L^2(0,L)$ (see Figure 3). The issue whether u can also be controlled on (l_1, l_2) is open.

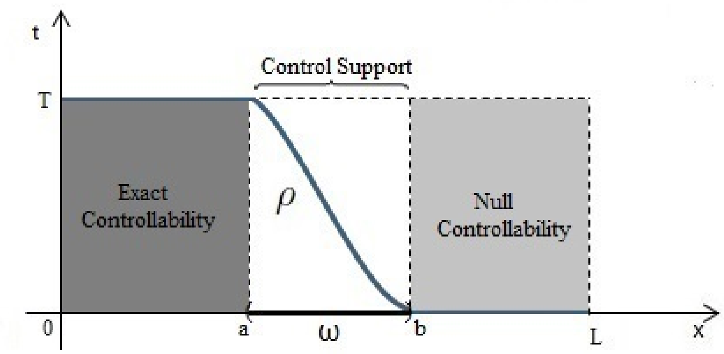

Fig. 3 - Regional controllability: ω is any subinterval of $(0, L)$

Chapter 5

Controllability of Boussinesq Equation KdV-KdV type on a Bounded Domain

5.1 Introduction

The classical Boussinesq systems were first derived by Boussinesq, in [11], to describe the two-way propagation of small amplitude, long wave length gravity waves on the surface of water in a canal. These systems and their higher-order generalizations also arise when modeling the propagation of long-crested waves on large lakes or on the ocean and in other contexts. In [6], the authors derived a four-parameter family of Boussinesq systems to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. More precisely, they studied a family of systems of the form

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{5.1.1}$$

In (5.1.1), η is the elevation from the equilibrium position, and $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid. The parameters a, b, c, d, that one might choose in a given modeling situation, are required to fulfill the relations

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad \theta \in [0, 1]. \tag{5.1.2}$$

where $\theta \in [0, 1]$ specifies which horizontal velocity the variable w represents (cf. [6]). Consequently,

$$a + b + c + d = \frac{1}{3}$$

As it has been proved in [6], the initial value problem for the linear system associated with (5.1.1) is well posed on $\mathbb{R}$ if either C_1 or C_2 is satisfied, where

$$\begin{aligned} &(C_1) \qquad b, d \geq 0,\ a \leq 0,\ c \leq 0; \\ &(C_2) \qquad b, d \geq 0,\ a = c > 0. \end{aligned}$$

In mathematical studies, considerations have been mainly given to pure initial value problems and well-posedness results [7]. However, the practical use of the above system and its relatives does not always involve the pure initial value problem. Instead, the initial boundary value problem often comes to the fore.

Recently, in [50], a rather complete picture of the control properties of (5.1.1) on a periodic domain with a locally supported forcing term was given. According to the values of the four parameters a, b, c, d, the linearized system may be controllable in any positive time, or only in large time, or may not be controllable at all. These results were also extended in [50] to the generic nonlinear system (5.1.1), i.e., when all the parameters are different from 0.

When $b = d = 0$ and (C_2) is satisfied, then necessarily $a = c = 1/6$. Nevertheless, the scaling $x \to x/\sqrt{6}$, $t \to t/\sqrt{6}$ gives an system equivalent to (5.1.1) for which $a = c = 1$, namely

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0, \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0. \end{cases} \tag{5.1.3}$$

When the model is posed on a bounded interval, Rosier and Pazoto, in [58], investigated the asymptotic behavior of the solutions assuming that $b = d = 0$ and $a = c = 1$. More precisely, the authors studied the following Boussinesq system of KdV-KdV type

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{5.1.4}$$

satisfying the boundary conditions

$$\begin{cases} w(t,0) = w_{xx}(t,0) = 0 & \text{in } (0,T), \\ w_x(t,0) = \alpha_0 \eta_x(t,0) & \text{in } (0,T), \\ w(t,L) = \alpha_2 \eta(t,L) & \text{in } (0,T), \\ w_x(t,L) = -\alpha_1 \eta_x(t,L) & \text{in } (0,T), \\ w_{xx}(t,L) = -\alpha_2 \eta_{xx}(t,L) & \text{in } (0,T), \end{cases} \tag{5.1.5}$$

and initial conditions

$$\eta(0,x) = \eta_0(x), \quad w(0,x) = w_0(x) \qquad \text{in } (0,L). \tag{5.1.6}$$

In (5.1.4), α_0, α_1 and α_2 denote some nonnegative real constants. The KdV–KdV system is expected to admit global solutions on $\mathbb{R}$, and it also possesses good control properties on the torus [50].

Under the above boundary conditions, the authors observed that the derivative of the energy associated with the system (5.1.4), with boundary conditions (5.1.5)-(5.1.6) satisfies

$$\frac{dE}{dt} = -\alpha_2 \left|\eta(L,t)\right|^2 - \alpha_1 \left|\eta_x(L,t)\right|^2 - \alpha_0 \left|\eta_x(0,t)\right|^2 - \frac{1}{3} w^3(L,t) - \int_0^L (\eta w)_x \eta dx$$

where

$$E(t) = \frac{1}{2} \int_0^L (\eta^2 + w^2) dx.$$

This indicates that the boundary conditions play the role of a feedback damping mechanism, at least for the linearized system. Therefore, the following questions arise:
(i) Does $E(t) \to 0$, as $t \to +\infty$?
(ii) If it is the case, can we give the decay rate?
The problem might be easy to solve when the underlying model has a intrinsic dissipative nature. Moreover, in the context of coupled systems, in order to achieve the desired decay property, the damping mechanism has to be designed in an appropriate way in order to capture all the components of the system. The main result of Rosier and Pazoto provides a positive answer to those questions.

Theorem 5.1. *([58]) Assume that $\alpha_0 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 = 1$. Then there exist some numbers $\rho > 0$, $C > 0$ and $\mu > 0$ such that for any $(\eta_0, w_0) \in (L^2(I))^2$ with*

$$\|(\eta_0, w_0)\|_{(L^2(I))^2} \leq \rho,$$

the system (5.1.4)-(5.1.6) admits a unique solution

$$(\eta, w) \in C(\mathbb{R}^+; (L^2(I))^2) \cap C(\mathbb{R}^{+*}; (H^1(I))^2) \cap L^2((0,1); (H^1(I))^2),$$

which fulfills

$$\|(\eta, w)(t)\|_{(L^2(I))^2} \leq Ce^{-\mu t} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t \geq 0,$$
$$\|(\eta, w)(t)\|_{(H^1(I))^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t > 0.$$

To our knowledge, the boundary control of the Boussinesq system of KdV-KdV type is completely open. The aim of this chapter is to investigate the control properties of the following system

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{5.1.7}$$

with the boundary conditions

$$\begin{cases} \eta(t,0) = h_0(t),\ \eta(t,L) = h_1(t) & \text{in } (0,T), \\ w(t,0) = g_0(t),\ w(t,L) = g_1(t) & \text{in } (0,T), \\ \eta_x(t,0) = h_2(t),\ w_x(t,L) = g_2(t) & \text{in } (0,T) \end{cases} \tag{5.1.8}$$

and the initial conditions

$$\eta(0,x) = \eta_0(x), \ \ w(0,x) = w_0(x) \qquad \text{in } (0,L). \tag{5.1.9}$$

A similar problem was studied by Rosier [62] in the case of the KdV equation considering only one control,

$$\begin{cases} u_t + u_x + u_{xxx} + u u_x = 0 & \text{in } (0,T) \times (0,L), \\ u(t,0) = u(t,L) = 0,\ u_x(t,L) = g_3(t) & \text{in } (0,T). \end{cases} \tag{5.1.10}$$

It was shown that the exact controllability of the linearized KdV equation holds in $L^2(0,L)$ if, and only if, L does not belong to the following (discrete) set of critical lengths

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \ : k,\ l \in \mathbb{N}^* \right\}. \tag{5.1.11}$$

To begin with, we consider the linearized Boussinesq system

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.1.12}$$

together with the boundary conditions (5.1.8) and the initial data (5.1.9).

The results established in this chapter show that, depending on the combination of the controls g_i and h_i, two sets of critical lengths appear; namely $\mathcal{N}$ and the (new) set

$$\mathcal{R} := \left\{ \pi\sqrt{(\frac{1}{2}+2k)^2 + (\frac{1}{2}+2l)^2 + (\frac{1}{2}+2k)(\frac{1}{2}+2l)} : k,l \in \mathbb{N}^* \right\}. \tag{5.1.13}$$

Introduce the space

$$X = \left\{ (\eta, w) \in \left[H^2(0,L)\cap H_0^1(0,L)\right]^2 : \eta_x(0) = w_x(L) = 0 \right\}; \tag{5.1.14}$$

and let X' denote the dual of X with respect to the pivot space $L^2(0,L)^2$. Some of the main results in this chapter are stated in the following theorem.

Theorem 5.2. *Let $\mathcal{N}$, $\mathcal{R}$, and X be defined by (5.1.11), (5.1.13), and (5.1.14), respectively. Then the following holds.*

(i) For any $T>0$, $L\in(0,+\infty)\backslash\mathcal{N}$, $(\eta_0, w_0)\in(H^{-1}(0,L))^2$ and $(\eta_T, w_T)\in(H^{-1}(0,L))^2$ there exist some controls $h_2, g_2 \in L^2(0,T)$ such that the solution $(\eta, w)\in C^0([0,T],(H^{-1}(0,L))^2)$ of (5.1.12) and (5.1.8)-(5.1.9), with $h_i = 0$ and $g_i = 0$ for $i = 0,1$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$;

(ii) For any $T>0$, $L\in(0,+\infty)\backslash\mathcal{N}$, $(\eta_0, w_0)\in(H^{-1}(0,L))^2$ and $(\eta_T, w_T)\in(H^{-1}(0,L))^2$, there exists a control $h_2\in L^2(0,T)$ such that the solution $(\eta,w)\in C^0([0,T],(H^{-1}(0,L))^2)$ of (5.1.12) and (5.1.8)-(5.1.9), with $h_i = 0$ and $g_i = 0$ for $i = 0,1$ and $g_2 = 0$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$;

(iii) For any $T>0$, $L>0$, $(\eta_0, w_0)\in X'$ and $(\eta_T, w_T)\in X'$, there exist some controls $h_1, g_1\in L^2(0,T)$ such that, the solution $(\eta,w)\in C([0,T],X'$ of (5.1.12) and (5.1.8)-(5.1.9), with $h_i = 0$ and $g_i = 0$ for $i = 0,2$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$;

(iv) For any $T>0$, $L\in(0,+\infty)\backslash(\mathcal{N}\cup\mathcal{R})$, $(\eta_0, w_0)\in X'$ and $(\eta_T, w_T)\in X'$, there exists a control $g_1\in L^2(0,T)$ such that the solution $(\eta,w)\in C^0([0,T],X')$ of (5.1.12) and (5.1.8)-(5.1.9), with $h_i = 0$ and $g_i = 0$ for $i = 0,2$ and $h_1 = 0$, fulfills $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$.

Actually, a more complete picture of the control results obtained in this chapter are presented in following table.

	Controls							Properties	
Case	h_0	h_1	h_2	g_0	g_1	g_2	Control Inputs	State	Lengths
1	0	0	$\star$	0	0	$\star$	$h_2, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in [H^{-1}(0,L)]^2$	$\mathcal{N}$
2	0	0	$\star$	0	0	0	$h_2 \in L^2(0,T)$	$(\eta_0, w_0) \in [H^{-1}(0,L)]^2$	$\mathcal{N}$
3	0	$\star$	0	0	$\star$	0	$h_1, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\emptyset$
4	0	0	0	0	$\star$	0	$g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N} \cup \mathcal{R}$
5	$\star$	0	0	0	$\star$	0	$h_0, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N} \cup \mathcal{R}$
6	0	0	$\star$	0	$\star$	0	$h_2, g_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
7	0	$\star$	0	0	0	$\star$	$h_1, g_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
8	$\star$	$\star$	0	0	0	0	$h_0, h_1 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$
9	0	$\star$	$\star$	0	0	0	$h_1, h_2 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\emptyset$
10	0	0	$\star$	$\star$	0	0	$h_2, g_0 \in L^2(0,T)$	$(\eta_0, w_0) \in X'$	$\mathcal{N}$

Table1. Controllability results for the linear system

To prove our control results, we use the classical duality approach based upon the Hilbert Uniqueness Method (H.U.M.) due to J.-L. Lions [42], which reduces our control properties to some observability inequalities for the adjoint systems. Next, to establish the observability inequalities, we use the compactness-uniqueness argument due to E. Zuazua (see the appendix in [42]) and some multipliers to reduce the problem to a spectral problem. The spectral problem is finally solved by using a method introduced in [62] and based on Fourier analysis and complex analysis.

Boussinesq system is more convenient than KdV as a model for the propagation of water waves, as it is adapted to the wave propagation in the two directions, and it is still valid after bounces of waves at the boundary. The initial value problem for Boussinesq system is less developed than for KdV, probably because of the complexity of the system. Nevertheless, it is striking that the control properties of Boussinesq system are better understood than for KdV: indeed, the critical lengths for Boussinesq system are explicitly given for any set of boundary controls, which is not the case for KdV (e.g. the critical lengths are not explicitly known with a Dirichlet control at the right point $x = L$, see [33]). This is probably due to the fact that $x = 0$ and $x = L$ (resp. w and η) play a symmetric role for the linearized Boussinesq system. The price to be paid is the lack of the Kato smoothing effect in general, which makes the extension of the control results to the nonlinear Boussinesq system delicate.

In what concerns the nonlinear problem, due to technical difficulties that come from the lack of regularity of solutions, special boundary conditions are used. The issue of the controllability of the nonlinear system (5.1.7) with the boundary conditions (5.1.8) will be investigated elsewhere.

Thus, we consider the system

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L). \end{cases} \tag{5.1.15}$$

satisfying either the boundary conditions

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0 & \text{in } (0,T), \\ w(t,0) = w(t,L) = 0 & \text{in } (0,T), \\ w_x(t,L) + \alpha_1 \eta_x(t,L) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.1.16}$$

or the boundary conditions

$$\begin{cases} \eta(t,L) = \eta_x(t,0) = 0 & \text{in } (0,T), \\ w(t,0) = w(t,L) = 0 & \text{in } (0,T), \\ \alpha_2 \eta(t,0) + \alpha_3 \eta_x(t,L) + w_{xx}(t,0) = h_0(t) & \text{in } (0,T), \\ w_x(t,L) - \alpha_3(\eta(t,0) - \eta_x(t,L)) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.1.17}$$

where α_i are positive constant for $i = 1, 2, 3$, and the initial conditions

$$\eta(0,x) = \eta_0(x), \;\; w(0,x) = w_0(x) \qquad \text{in } (0,L). \tag{5.1.18}$$

With (5.1.16) or (5.1.17), a global Kato smoothing effect similar to those for KdV can be derived. As a consequence, a result similar to Theorem 5.2 can be established for the system above. More precisely, the following results concerning the well-posedness and the exact controllability of the above systems will be established:

Theorem 5.3. *Let $X_0 = (L^2(0,L))^2$, $T > 0$ and $L \in (0,+\infty)\backslash\mathcal{N}$, where $\mathcal{N}$ is defined by (5.1.11). Then, there exists a constant $\delta > 0$ such that for any initial data $(\eta_0, w_0) \in X_0$ and and final data $(\eta_T, w_T) \in X_0$ satisfying*

$$\|(\eta_0, w_0)\|_{X_0} \leq \delta \;\; \text{and} \;\; \|(\eta_T, w_T)\|_{X_0} \leq \delta,$$

there exists a control $g_2 \in L^2(0,T)$ such that the solution

$$(\eta, w) \in C([0,T], X_0) \cap L^2(0,T; (H^1(0,L))^2) \cap H^1(0,L; (H^{-2}(0,L))^2),$$

of (5.1.15) with (5.1.18) and the boundary conditions (5.1.16) satisfies $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$.

Theorem 5.4. *Let $T > 0$ and $L \in (0,+\infty)\backslash\mathcal{N}$. Then, there exists a constant $\delta > 0$ such that for any initial data $(\eta_0, w_0) \in X_0$ and and final data $(\eta_T, w_T) \in X_0$ satisfying*

$$\left\|(\eta^0, w^0)\right\|_{X_0} \leq \delta \;\; \text{and} \;\; \left\|(\eta^1, w^1)\right\|_{X_0} \leq \delta,$$

there exist two controls $(h_0, g_2) \in (L^2(0,T))^2$ such that the solution

$$(\eta, w) \in C([0,T], X_0) \cap L^2(0,T; (H^1(0,L))^2) \cap H^1(0,L; (H^{-2}(0,L))^2),$$

of (5.1.15) with (5.1.18) and the boundary condition (5.1.17) satisfies $\eta(T,\cdot) = \eta_T$ and $w(T,\cdot) = w_T$ in $(0,L)$.

The second part of the work is devoted to the study of the exponential decay of $E(t)$ when $g_2 = h_2 = 0$. In this case, the energy associated with (5.1.15) with boundary conditions (5.1.16) (resp. (5.1.17)) satisfies

$$\frac{d}{dt}E = -\alpha_1 \left|\eta_x(t,L)\right|^2 - \int_0^L (\eta w)_x \eta dx$$

(resp.
$$\frac{d}{dt}E = -\alpha_2 \left|\eta(t,0)\right|^2 - \alpha_3 \left|\eta_x(t,L)\right|^2 - \int_0^L (\eta w)_x \eta dx).$$
Thus, as in [58], we obtain the following result:

Theorem 5.5. *Assume that $\alpha_1, \alpha_2, \alpha_3 > 0$ and $L \in (0,+\infty)\backslash\mathcal{N}$. Then, there exist some numbers $\rho > 0$, $C > 0$ and $\mu > 0$ such that for any $(\eta_0, w_0) \in (L^2(I))^2$ with*
$$\|(\eta_0, w_0)\|_{(L^2(I))^2} \leq \rho,$$
the system (5.1.15) with boundary conditions (5.1.16) (or (5.1.17)) and initial condition (5.1.18) admits a unique solution
$$(\eta, w) \in C(\mathbb{R}^+; (L^2(I))^2) \cap C(\mathbb{R}^{+*}; (H^1(I))^2) \cap L^2((0,1); (H^1(I))^2),$$
which fulfills
$$\|(\eta, w)(t)\|_{(L^2(I))^2} \leq Ce^{-\mu t} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t \geq 0,$$
$$\|(\eta, w)(t)\|_{(H^1(I))^2} \leq C\frac{e^{-\mu t}}{\sqrt{t}} \|(\eta_0, w_0)\|_{(L^2(I))^2}, \ \forall t > 0.$$

5.2 Well-Posedness

5.2.1 Linear homogeneous system

In this section we study the existence of solution of the linear system
$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{5.2.1}$$
satisfying the boundary conditions
$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0 & \text{in } (0,T) \\ w(t,0) = w(t,L) = w_x(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.2.2}$$
and the initial conditions
$$\eta(0,x) = \eta^0(x), \ \ w(0,x) = w^0(x) \quad \text{in } (0,L). \tag{5.2.3}$$
Let $X_0 = \left(L^2(0,L)\right)^2$ endowed with the usual inner product and consider the operator $A : \mathcal{D}(A) \subset X_0 \to X_0$, where
$$\mathcal{D}(A) = \left\{(\eta, w) \in \left(H^3(0,L)\right)^2; \eta(0) = w(0) = \eta(L) = w(L) = \eta_x(0) = w_x(L) = 0\right\},$$
and
$$A(\eta, w) = \begin{pmatrix} -w_x - w_{xxx} \\ -\eta_x - \eta_{xxx} \end{pmatrix}, \forall (\eta, w) \in \mathcal{D}(A). \tag{5.2.4}$$

With the notation introduce above, system (5.2.1) can be now written as an abstract Cauchy problem in X_0

$$\left\{ \begin{array}{l} (\eta, w)_t = A(\eta, w), \\ (\eta, w)(0) = (\eta^0, w^0). \end{array} \right. \tag{5.2.5}$$

On the other hand, the adjoint of the operator A (denoted by A^*) is give by

$$A^*(\varphi, \psi) = \begin{pmatrix} \psi_x + \psi_{xxx} \\ \varphi_x + \varphi_{xxx} \end{pmatrix}, \ \forall (\varphi, \psi) \in \mathcal{D}(A^*), \tag{5.2.6}$$

where $A^* : \mathcal{D}(A^*) \subset X_0 \to X_0$ with

$$\mathcal{D}(A^*) = \left\{ (\varphi, \psi) \in \left(H^3(0, L)\right)^2 ; \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(L) = 0 \right\}.$$

Proposition 5.1. *The operators A and A^* are dissipative in X_0.*

Proof. Consider $(\eta, w) \in \mathcal{D}(A)$. By multiplying the first equation of the system (4.2.1) by η, the second one by w and integrating by parts in $(0, L)$, we obtain

$$\int_0^L (w_x + w_{xxx})\eta dx = -\int_0^L w\eta_x dx - \int_0^L w_{xx}\eta_x dx$$

and

$$\int_0^L (\eta_x + \eta_{xxx}) w dx = -\int_0^L w\eta_x dx - \int_0^L \eta_{xx} w_x dx.$$

Therefore,

$$\langle A(\eta, w), (\eta, w) \rangle_{X_0} = -\int_0^L (\eta_x w_x)_x dx = 0.$$

Hence A is dissipative in X_0. Analogously, we deduce that

$$\langle A^*(\varphi, \psi), (\varphi, \psi) \rangle_{X_0} = 0, \ \forall (\varphi, \psi) \in \mathcal{D}(A^*),$$

i.e., A^* is dissipative in X_0. □

Since A and A^* are both dissipative, A is a closed operator and the respective domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense and compactly embedded in X_0 we conclude that A generates a C^0 semigroup of contractions in X_0 which will be denoted by $(S(t))_{t \geq 0}$. Then, classical existence results give us the global well-posedness for (5.2.1)-(5.2.3):

Theorem 5.6. *Let $(\eta^0, w^0) \in X_0$. Then, there exists a unique weak solution $(\eta, w) = S(\cdot)(\eta^0, w^0)$ of (5.2.1)-(5.2.3) such that*

$$(\eta, w) \in C([0, T]; X_0). \tag{5.2.7}$$

Moreover, if $(\eta^0, w^0) \in \mathcal{D}(A)$, then (5.2.1)-(5.2.3) has a unique (classical) solution (η, w) such that

$$(\eta, w) \in C([0, T]; \mathcal{D}(A)) \cap C^1(0, T; X_0). \tag{5.2.8}$$

Using the previous results and some interpolation argument, we derive the global well-posedness result in each space $[H^s(0, L)]^2$, for $s \in [0, 3]$.

Corollary 5.1. *For any* $s \in [0,3]$ *and any* $(\eta^0, w^0) \in [H^s(0,L)]^2$ *the solution* (η, w) *of (5.2.1)-(5.2.3) belongs to* $C([0,T];[H^s(0,L)]^2)$*.*

Remark 5.1. *Observe that due to the boundary conditions (5.2.2) we can not prove the so-called Kato smoothing effect. Therefore, for the analysis of the controllability properties, we consider more regular initial data.*

5.2.2 Adjoint System

In this subsection, we introduce the time-backward system associated to (5.2.1)-(5.2.2). First we multiply the first equation of (5.2.1) by φ and the second one by ψ and integrating in $(0,T)\times(0,L)$, i.e.,

$$\int_0^T\int_0^L (\eta_t + w_x + w_{xxx})\,\varphi dxdt = 0$$

and

$$\int_0^T\int_0^L (w_t + \eta_x + \eta_{xxx})\,\psi dxdt = 0.$$

Assuming that the functions η, w, φ, ψ are sufficiently regular, we obtain, after integration by parts,

$$\begin{aligned}
0 &= \int_0^T\int_0^L [\eta(-\varphi_t - \psi_x - \psi_{xxx}) + w\,(-\psi_t - \varphi_x - \varphi_{xxx})]\,dxdt + \int_0^L [\eta\varphi + w\psi]_0^T\,dx \\
&+\int_0^T (w\,(t,L)\,\varphi\,(t,L) - w\,(t,0)\,\varphi\,(t,0))\,dt + \int_0^T (w_{xx}\,(t,L)\,\varphi\,(t,L) - w_{xx}\,(t,0)\,\varphi\,(t,0))\,dt \\
&-\int_0^T (w_x\,(t,L)\,\varphi_x\,(t,L) - w_x\,(t,0)\,\varphi_x\,(t,0))\,dt + \int_0^T (w\,(t,L)\,\varphi_{xx}\,(t,L) - w\,(t,0)\,\varphi_{xx}\,(t,0))\,dt \\
&+\int_0^T (n\,(t,L)\,\psi\,(t,L) - \eta\,(t,0)\,\psi\,(t,0))\,dt + \int_0^T (\eta_{xx}\,(t,L)\,\psi\,(t,L) - \eta_{xx}\,(t,0)\,\psi\,(t,0))\,dt \\
&-\int_0^T (\eta_x\,(t,L)\,\psi_x\,(t,L) - \eta_x\,(t,0)\,\psi_x\,(t,0))\,dt + \int_0^T (\eta\,(t,L)\,\psi_{xx}\,(t,L) - \eta\,(t,0)\,\psi_{xx}\,(t,0))\,dt.
\end{aligned} \tag{5.2.9}$$

Having (5.2.9) in hands, we consider the following time-backward system

$$\begin{cases} \varphi_t + \psi_x + \psi_{xxx} = 0 & \text{in } (0,T)\times(0,L)\,, \\ \psi_t + \varphi_x + \varphi_{xxx} = 0 & \text{in } (0,T)\times(0,L)\,, \end{cases} \tag{5.2.10}$$

satisfying the boundary conditions

$$\begin{cases} \varphi\,(t,0) = \varphi\,(t,L) = \varphi_x\,(t,0) = 0 & \text{in } (0,T) \\ \psi\,(t,0) = \psi\,(t,L) = \psi_x\,(t,L) = 0 & \text{in } (0,T)\,, \end{cases} \tag{5.2.11}$$

and the initial conditions

$$\varphi\,(T,x) = \varphi^1\,(x)\,,\ \ \psi\,(T,x) = \psi^1\,(x) \qquad \text{in } (0,L)\,. \tag{5.2.12}$$

Remark that the change of variable $t \mapsto T-t$ reduces system (5.2.10)-(5.2.12) to

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.2.13}$$

satisfying the boundary conditions

$$\begin{cases} \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0 & \text{in } (0,T) \\ \psi(t,0) = \psi(t,L) = \psi_x(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.2.14}$$

and the initial conditions

$$\varphi(0,x) = \varphi^0(x), \quad \psi(0,x) = \psi^0(x) \qquad \text{in } (0,L). \tag{5.2.15}$$

Thus, (5.2.13)-(5.2.15) is equivalent to

$$\begin{cases} (\varphi,\psi)_t = A^*(\varphi,\psi); \\ (\varphi,\psi)(0) = (\varphi^0,\psi^0), \end{cases}$$

where A^* is given by (5.2.6). Observe that the properties of the solutions of (5.2.13)-(5.2.15) are similar to the ones deduced in Theorem 5.6 and Corollary 5.1. More precisely, we have

Theorem 5.7. *Let $(\varphi^0,\psi^0) \in X_0$. Then there exist a unique weak solution $(\varphi,\psi) = S(\cdot)(\varphi^0,\psi^0)$ of (5.2.13)-(5.2.15) such that*

$$(\varphi,\psi) \in C([0,T];X_0). \tag{5.2.16}$$

Moreover, if $(\varphi^0,\psi^0) \in \mathcal{D}(A^*)$, then (5.2.13)-(5.2.15) has a unique (classical) solution (φ,ψ) such that

$$(\varphi,\psi) \in C([0,T];\mathcal{D}(A^*)) \cap C^1(0,T;X_0). \tag{5.2.17}$$

Using the previous results and some interpolation argument, we derive the global well-posedness result in each space $[H^s(0,L)]^2$, for $s \in [0,3]$.

Corollary 5.2. *For any $s \in [0,3]$ and any $(\varphi^0,\psi^0) \in [H^s(0,L)]^2$ the solution (φ,ψ) of (5.2.13)-(5.2.15) belongs to $C([0,T];[H^s(0,L)]^2)$.*

5.2.3 Linear non-homogeneous system

Now we use the adjoint system to define our solution by transposition. Consider the nonhomogeneous system given by

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.2.18}$$

satisfying the boundary conditions

$$\begin{cases} \eta(t,0) = h_0(t), \ \ \eta(t,L) = h_1(t), \ \ \eta_x(t,0) = h_2(t) & \text{in } (0,T) \\ w(t,0) = g_0(t), \ \ w(t,L) = g_1(t), \ \ w_x(t,L) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.2.19}$$

and the initial conditions

$$\eta(0,x)=\eta^0(x)\,,\quad w(0,x)=w^0(x)\qquad \text{in }\ (0,L)\,. \tag{5.2.20}$$

From (5.2.9), (5.2.11) and (5.2.19), we have that

$$\begin{aligned}0=\int_0^L[\eta\varphi+w\psi]_0^T\,dx-\int_0^T g_2(t)\,\varphi_x(t,L)\,dt+\int_0^T\left(g_1(t)\,\varphi_{xx}(t,L)-g_0(t)\,\varphi_{xx}(t,0)\right)dt\\+\int_0^T h_2(t)\,\psi_x(t,0)\,dt+\int_0^T\left(h_1(t)\,\psi_{xx}(t,L)-h_0(t)\,\psi_{xx}(t,0)\right)dt.\end{aligned} \tag{5.2.21}$$

We introduce the Hilbert spaces

$$H=\left[H_0^1(0,L)\right]^2 \tag{5.2.22}$$

endowed with the usual inner product and

$$X=\left\{(\eta,w)\in\left[H^2(0,L)\cap H_0^1(0,L)\right]^2:\eta_x(0)=w_x(L)=0\right\}\subset\left[H^2(0,L)\right]^2, \tag{5.2.23}$$

endowed with the inner product of $[H^2(0,L)]^2$. Observe that the dual of the space H is $H':=[H^{-1}(0,L)]^2$ and the duality pairing $\langle\cdot,\cdot\rangle_{H'\times H}$ is defined by

$$\left\langle\left(\eta^0,w^0\right),\left(\varphi^0,\psi^0\right)\right\rangle_{H'\times H}=\left\langle\eta^0,\varphi^0\right\rangle_{H^{-1}(0,L)\times H_0^1(0,L)}+\left\langle w^0,\psi^0\right\rangle_{H^{-1}(0,L)\times H_0^1(0,L)}.$$

For the dual of the space X we denote

$$X':=\left(\left\{(\eta,w)\in\left[H^2(0,L)\cap H_0^1(0,L)\right]^2:\eta_x(L)=w_x(0)=0\right\}\right)'$$

and the duality pairing $\langle\cdot,\cdot\rangle_{X'\times X}$ is defined by

$$\left\langle\left(\eta^0,w^0\right),\left(\varphi^0,\psi^0\right)\right\rangle_{X'\times X}=\left\langle\eta^0,\varphi^0\right\rangle_{X'\times X}+\left\langle w^0,\psi^0\right\rangle_{X'\times X}.$$

Replacing T by t in (5.2.21) and considering $h_i(t)=0$, $g_i(t)=0$, for $i=0,1$ it follows that

$$\begin{aligned}\langle(\eta(t),w(t)),(\varphi(t),\psi(t))\rangle_{H'\times H}=-\int_0^t h_2(s)\,\psi_x(s,0)\,ds+\int_0^t g_2(s)\,\varphi_x(s,L)\,dt\\+\langle(\eta^0,w^0),(\varphi^0,\psi^0)\rangle_{H'\times H}.\end{aligned} \tag{5.2.24}$$

Now, considering $h_2=g_2=0$ in (5.2.21) and replacing T by t, we obtain

$$\begin{aligned}\langle(\eta(t),w(t)),(\varphi(t),\psi(t))\rangle_{X'\times X}=\int_0^t h_0(s)\,\psi_{xx}(s,0)\,ds-\int_0^t h_1(s)\,\psi_{xx}(s,L)\,dt\\\int_0^t g_0(s)\,\varphi_{xx}(s,0)\,ds-\int_0^t g_1(s)\,\varphi_{xx}(s,L)\,dt+\langle(\eta^0,w^0),(\varphi^0,\psi^0)\rangle_{X'\times X}.\end{aligned} \tag{5.2.25}$$

Definition 5.1. *i) Given* $T > 0$, $(\eta^0, w^0) \in H'$ *and* $(h_2, g_2) \in [L^2(0,T)]^2$*, we call a solution by transposition of (5.2.18)-(5.2.20) with* $h_i = g_i = 0$*, for* $i = 0, 1$*, a function*

$$(\eta, w) \in C([0,T], H'), \tag{5.2.26}$$

satisfying (5.2.24), where (φ, ψ) *is a solution of*

$$\begin{cases} \varphi_t + \psi_x + \psi_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \psi_t + \varphi_x + \varphi_{xxx} = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{5.2.27}$$

satisfying the boundary conditions

$$\begin{cases} \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0 & \text{in } (0,T) \\ \psi(t,0) = \psi(t,L) = \psi_x(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.2.28}$$

and the initial conditions

$$\varphi(T,x) = 0, \quad \psi(T,x) = 0 \qquad \text{in } (0,L). \tag{5.2.29}$$

ii) Given $T > 0$, $(\eta^0, w^0) \in X'$ *and* $(h_0, g_0, h_1, g_1) \in [L^2(0,T)]^4$*, we call a solution by transposition of (5.2.18)-(5.2.20) with* $h_2 = g_2 = 0$*, a function*

$$(\eta, w) \in C([0,T], X'), \tag{5.2.30}$$

satisfying (5.2.25), where (φ, ψ) *is a solution of (5.2.27)-(5.2.29).*

Remark 5.2. *Observe that the maps*

$$\Xi_0 : (\varphi^1, \psi^1) \in H \longmapsto (\varphi(t), \psi(t)) \in H$$

and

$$\Xi_1 : (\varphi^1, \psi^1) \in X \longmapsto (\varphi(t), \psi(t)) \in H,$$

are isomorphisms of Hilbert spaces for any $t \in \mathbb{R}$*, therefore (5.2.24) and (5.2.25) define (5.2.26) and (5.2.30), respectively, in a unique way.*

5.3 Exact Boundary Controllability For The Linear System: Neumann boundary condition

This section is devoted to the analysis of the exact controllability property of the linear system corresponding to (5.1.7) with boundary controls of Neumann type. More precisely, given $T > 0$ and (η^0, w^0), $(\eta^T, w^T) := (\eta^1, w^1) \in \Sigma$, we study the existence of controls $(h_2, g_2) \in \Sigma_1$ such that the solution (η, w) of the system

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x) & \text{in } (0,L), \end{cases} \tag{5.3.1}$$

satisfying the boundary conditions

$$\begin{cases} \eta(t,0) = 0, \quad \eta(t,L) = 0, \quad \eta_x(t,0) = h_2(t) & \text{in } (0,T) \\ w(t,0) = 0, \quad w(t,L) = 0, \quad w_x(t,L) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.3.2}$$

satisfies

$$\eta(T,\cdot)=\eta^1, \quad w(T,\cdot)=w^1 \text{ in } \Sigma. \tag{5.3.3}$$

The spaces Σ and Σ_1 will be defined later.

Definition 5.2. *Let $T>0$. System (5.3.1) is exact controllable in time T if for any initial and final data (η^0,w^0), $(\eta^1,w^1)\in\Sigma$, there exist control functions $(h_2,g_2)\in\Sigma_1$ such that the solution of (5.3.1)-(5.3.2) satisfies (5.3.3).*

Remark 5.3. *Without loss of generality, we may study only the exact controllability property for the case $\eta^0=w^0=0$. Indeed, let (η^0,w^0), (η^1,w^1) be arbitrarily in Σ and let $(h_2,g_2)\in\Sigma_1$ be controls which lead the solution (φ,ψ) of (5.3.1) from the zero initial data to the final state $(\eta^1,w^1)-S(T)(\eta^0,w^0)$ (we recall that $(S(t))_{t\geq 0}$ is the semigroup generated by the differential operator A corresponding to (5.3.1)). It follows immediately that these controls also lead to the solution $(\eta,w)+S(\cdot)(\eta^0,w^0)$ of (5.3.1) from (η^0,w^0) to the final state (η^1,w^1).*

From now on, we shall consider only the case $\eta^0=w^0=0$. For the analysis of the controllability we will consider several cases regarding the amount of controls on (5.3.2).

5.3.1 Double control

In this section we study the exact controllability, in time T, for the system (5.3.1)-(5.3.2). We first give an equivalent condition for the exact controllability property:

Lemma 5.1. *Let $(\eta^1,w^1)\in\Sigma:=H'$. Then, there exist two control $(h_2(t),g_2(t))\in\Sigma_1:=\left[L^2(0,L)\right]^2$, such that the solution (η,w) of (5.3.1)-(5.3.2) satisfies (5.3.3) if and only if*

$$\left\langle\left(\eta^1,w^1\right),\left(\varphi^1,\psi^1\right)\right\rangle_{H'\times H}=\int_0^T\left(h_2(t)\,\psi_x(t,0)-g_2(t)\,\varphi_x(t,L)\right)dt, \tag{5.3.4}$$

for any $(\varphi^1,\psi^1)\in H$, (φ,ψ) being the solution of the backward system (5.2.10)-(5.2.12).

Proof. The relation (5.3.4) is obtained multiplying the equations in (5.3.1) by the solution (φ,ψ) of (5.2.10)-(5.2.12) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result

Theorem 5.8. *Let $\mathcal{N}=\left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}}:k,l\in\mathbb{N}^*\right\}$. Then, $\forall\ L\in(0,+\infty)\setminus\mathcal{N}$ and $\forall$ $T>0$, $\exists\ C(T,L)>0$ such that*

$$\left\|\left(\varphi^1,\psi^1\right)\right\|_H^2\leq C\int_0^T\left(\left|\varphi_x(t,L)\right|^2+\left|\psi_x(t,0)\right|^2\right)dt, \tag{5.3.5}$$

holds for any $(\varphi^1,\psi^1)\in H$, where (φ,ψ) is the solution of (5.2.10)-(5.2.12) with initial data (φ^1,ψ^1).

In order to prove Theorem 5.8, we need some basic estimates for the solution of adjoint system (5.2.10)-(5.2.12). Therefore, the following result will be needed:

Lemma 5.2. *For any (φ,ψ) solution of (5.2.10)-(5.2.11) with initial data $(\varphi^1,\psi^1)\in H$, there exists $C>0$ such that*

$$\|(\varphi,\psi)\|_{L^2(0,T;H)} \leq C\left\|\left(\varphi^1,\psi^1\right)\right\|_H. \tag{5.3.6}$$

Moreover there exist $c_1>0$, $c_2>0$ and $c_3>0$, such that

$$\left\|\left(\varphi^1,\psi^1\right)\right\|_H \leq c_1\|(\varphi,\psi)\|_{L^2(0,T;H)}, \tag{5.3.7}$$

$$\|(\varphi,\psi)\|^2_{L^2(0,T;H)} \leq c_2\left\{\left\|\left(\varphi^1,\psi^1\right)\right\|^2_{(L^2(0,L))^2}+\int_0^T\left|\varphi_x(t,L)\right|^2dt\right\} \tag{5.3.8}$$

and

$$\|(\varphi,\psi)\|^2_{L^2(0,T;H)} \leq c_3\left\{\left\|\left(\varphi^1,\psi^1\right)\right\|^2_{(L^2(0,L))^2}+\int_0^T\left|\psi_x(t,0)\right|^2dt\right\}. \tag{5.3.9}$$

Proof. Observe that the estimate (5.3.6) holds directly from Theorem 5.7 and Corollary 5.2 and, since

$$(\varphi,\psi)\in C^0([0,T];H),$$

it follows that $(\varphi(t),\psi(t))\in H$, for all $t\in(0,T)$.

Now, pick any $U_1=(\varphi^1,\psi^1)\in\left(H^3(0,L)\right)^2$ and write $U(t)=(\eta(t),w(t))=S(t)U_1$. Let $V(t)=U_t(t)=A^*U(t)$. Then V is a mild solution of the system

$$\begin{cases} V_t=A^*V\\ V(T)=A^*U_1\in X_0,\end{cases}$$

and therefore $\|V(t)\|_{X_0}=\|V_1\|_{X_0}$. Since $V(t)=A^*U(t)$, $V_1=A^*U_1$, and the norms $\|U\|_{X_0}+\|AU\|_{X_0}$ and $\|U\|_{(H^3(0,L))^2}$ are equivalent in $\left(H^3(0,L)\right)^2$, we conclude that $\|U(t)\|_{(H^3(0,L))^2}=\|U_1\|_{(H^3(0,L))^2}$. The fact that (5.3.7) is still valid for H^s, $0<s<3$, follows from a standard interpolation argument, since $H^s=\left[H^0,H^3\right]_{s/3}$.

Now, we prove (5.3.8). We multiply the first equation of (5.2.10) by $x\psi$, the second one by $x\varphi$ and integrate in $(0,T)\times(0,L)$, to obtain

$$\int_0^L\int_0^T(x\psi)(\varphi_t+\psi_x+\psi_{xxx})\,dxdt=0$$

and

$$\int_0^L\int_0^T(x\varphi)(\psi_t+\varphi_x+\varphi_{xxx})\,dxdt=0.$$

By integration by parts we have

$$\int_0^L\int_0^T x\varphi_t\psi dxdt+\frac{3}{2}\int_0^L\int_0^T|\psi_x|^2dxdt-\frac{1}{2}\int\int|\psi|^2dxdt=0 \tag{5.3.10}$$

and

$$\int_0^L\int_0^T x\psi_t\varphi dxdt+\frac{3}{2}\int_0^L\int_0^T|\varphi_x|^2dxdt-\frac{1}{2}\int_0^L\int_0^T|\varphi|^2dxdt-\frac{L}{2}\int_0^T|\varphi_x(L)|^2dt=0. \tag{5.3.11}$$

Adding (5.3.10) and (5.3.11) we infer that

$$\begin{aligned}\frac{3}{2}\int_0^L\int_0^T\left(|\varphi_x|^2+|\psi_x|^2\right)dxdt=&-\int_0^L\int_0^T(x\varphi\psi)_t\,dxdt+\frac{1}{2}\int_0^L\int_0^T\left(|\varphi|^2+|\psi|^2\right)dxdt\\&+\frac{L}{2}\int_0^T|\varphi_x(L)|^2dt.\end{aligned} \tag{5.3.12}$$

On the other hand, from the energy identity it follows that

$$\begin{aligned}-\int_0^L\int_0^T(x\varphi\psi)_t\,dxdt&=-\int_0^L x\varphi^1(x)\,\psi^1(x)\,dx+\int_0^L x\varphi^0(x)\,\psi^0(x)\,dx\\&\leq L\int_0^L\left(\left|\varphi^1\right|^2+\left|\psi^1\right|^2\right)dx\end{aligned} \tag{5.3.13}$$

and

$$\int_0^L\int_0^T\left(|\varphi|^2+|\psi|^2\right)=\left\|\left(\varphi^1,\psi^1\right)\right\|_{(L^2(0,L))^2}, \tag{5.3.14}$$

we can combine (5.3.12), (5.3.13) and (5.3.14) to obtain (5.3.8).

In order to prove (5.3.9), we multiply the first equation of (5.2.10) by $(x-L)\,\psi$, the second one by $(x-L)\,\varphi$ and integrate by parts in $(0,T)\times(0,L)$, to obtain

$$\begin{aligned}\frac{3}{2}\int_0^L\int_0^T\left(|\varphi_x|^2+|\psi_x|^2\right)dxdt=&-\int_0^L\int_0^T((x-L)\,\varphi\psi)_t\,dxdt+\frac{1}{2}\int_0^L\int_0^T\left(|\varphi|^2+|\psi|^2\right)dxdt\\&-\frac{L}{2}\int_0^T|\psi_x(0)|^2dt.\end{aligned}$$

Now, using the arguments used in the proof of (5.3.8) we obtain the result. □

With Lemma 5.2 in hands, we can prove Theorem 5.8.

Proof of Theorem 5.8. The change of variable $t=T-t$ transforms (5.2.10)-(5.2.12) into (5.2.13)-(5.2.15). Hence, inequality (5.3.5) is equivalent to

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_H^2\leq C\int_0^T\left(|\varphi_x(t,L)|^2+|\psi_x(t,0)|^2\right)dt, \tag{5.3.15}$$

for any $(\varphi^0,\psi^0)\in H$, where (φ,ψ) is the solution of (5.2.13)-(5.2.15) with initial data (φ^0,ψ^0). First, observe that adding (5.3.8) and (5.3.9), we obtain

$$\|(\varphi,\psi)\|^2_{L^2(0,T;H)}\leq c\left\{\left\|\left(\varphi^0,\psi^0\right)\right\|^2_{(L^2(0,L))^2}+\int_0^T\left(|\varphi_x(t,L)|^2+|\psi_x(t,0)|^2\right)dt\right\},$$

where $c := \max\{c_2, c_3\}$. Then, from (5.3.7), it follows that

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_H^2 \le c_1 c\left\{\left\|\left(\varphi^0,\psi^0\right)\right\|_{(L^2(0,L))^2}^2 + \int_0^T \left(\left|\varphi_x(t,L)\right|^2 + \left|\psi_x(t,0)\right|^2\right) dt\right\}. \quad (5.3.16)$$

Now we prove the observability (5.3.15). We proceed as in [62, Proposition 3.3]. Let us suppose that (5.3.15) does not hold. In this case, it follows that there exists a sequence $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}}$ in H such that

$$\left\|\left(\varphi_n^0, \psi_n^0\right)\right\|_H = 1, \quad (5.3.17)$$

$$\int_0^T \left|\varphi_{n,x}(t,L)\right|^2 dt \to 0 \text{ in } L^2(0,T) \quad (5.3.18)$$

and

$$\int_0^T \left|\psi_{n,x}(t,0)\right|^2 dt \to 0 \text{ in } L^2(0,T), \quad (5.3.19)$$

where (φ_n, ψ_n) is the solution of (5.2.13)-(5.2.15) with initial data (φ_n^0, ψ_n^0). From (5.3.6) and (5.3.17), we obtain that $\{(\varphi_n, \psi_n)\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H)$ and from (5.2.13) we have that $\{((\varphi_n)_t, (\psi_n)_t)\}_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;(H^{-2}(0,L))^2\right)$. Since

$$H \hookrightarrow_{cc} X_0 := \left[L^2(0,L)\right]^2 \hookrightarrow H^{-2}(0,L),$$

being the first embedding compact, it follows that $\{(\varphi_n, \psi_n)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;X_0)$. Therefore, there exists a subsequence, still denoted by the same index, such that

$$(\varphi_n, \psi_n) \longrightarrow (\varphi, \psi) \text{ in } L^2(0,T;X_0).$$

Moreover, since $\{(\varphi_n, \psi_n)\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H) \cap H^1\left(0,T;(H^{-2}(0,L))^2\right)$ from Corollary 4 in [74] we obtain a subsequence satisfying

$$(\varphi_n, \psi_n) \longrightarrow (\varphi, \psi) \text{ in } C\left([0,T];\left(H^{-1}(0,L)\right)^2\right), \text{ for any } T > 0. \quad (5.3.20)$$

On the other hand, (5.3.16), (5.3.18), (5.3.19) and (5.3.17) together with compact embedding $H \hookrightarrow (L^2(0,L))^2$ allow us to conclude that $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H. Therefore, at least for a subsequence,

$$\left(\varphi_n^0, \psi_n^0\right) \longrightarrow \left(\varphi^0, \psi^0\right) \text{ in } H. \quad (5.3.21)$$

In particular, from (5.3.20)

$$(\varphi, \psi)(0) = \lim_{n\to+\infty} (\varphi_n, \psi_n)(0) = \lim_{n\to\infty} \left(\varphi_n^0, \psi_n^0\right) = \left(\varphi^0, \psi^0\right).$$

As $(\varphi, \psi) \in L^\infty(0,T;H) \cap C\left([0,T], (H^{-1}(0,L))^2\right)$, from Lemma 1.4 in [77] we deduce that

$$(\varphi, \psi) \in C_\omega([0,T];H),$$

where C_ω represent the space of weakly continuous functions form $[0,T]$ into H, and $(\varphi,\psi) = S(\cdot)(\varphi^0,\psi^0)$. Furthermore, from (5.3.17) and (5.3.21),

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_H = 1 \tag{5.3.22}$$

and from (5.3.18) and (5.3.19) , we have that

$$\begin{aligned} 0 &= \lim_{n\to+\infty}\inf\left\{\int_0^T \left(|\varphi_{n,x}(t,L)|^2 + |\psi_{n,x}(t,0)|^2\right)dt\right\} \\ &\geq \int_0^T \left(|\varphi_x(t,L)|^2 + |\psi_x(t,0)|^2\right)dt. \end{aligned}$$

Then,

$$\varphi_x(\cdot,L) = 0 \text{ and } \psi_x(\cdot,0) = 0. \tag{5.3.23}$$

Hence, (φ,ψ) is a solution of

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \varphi(0,x) = \varphi^0(x), \;\; \psi(0,x) = \psi^0(x) & \text{in } (0,L), \end{cases} \tag{5.3.24}$$

satisfying the boundary conditions

$$\begin{cases} \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0 & \text{in } (0,T) \\ \psi(t,0) = \psi(t,L) = \psi_x(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.3.25}$$

and, in addition,

$$\varphi_x(\cdot,L) = \psi_x(\cdot,0) = 0. \tag{5.3.26}$$

Remark that (5.3.22) implies that the solutions of (5.3.24)-(5.3.26) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following result:

Lemma 5.3. *For any $T>0$, let N_T denote the space of the initial states $(\varphi^0,\psi^0)\in H$ such that the solution $(\varphi,\psi) = S(\cdot)(\varphi^0,\psi^0)$ of (5.3.24)-(5.3.25) satisfies (5.3.26). Then, for $L\in(0,+\infty)\setminus\mathcal{N}$, $N_T = \{0\}$, $\forall T>0$.*

Proof. Let A be the operator

$$A(\varphi,\psi) = \begin{pmatrix} \psi_x + \psi_{xxx} \\ \varphi_x + \varphi_{xxx} \end{pmatrix}, \forall(\varphi,\psi)\in\mathcal{D}(A),$$

with

$$\mathcal{D}(A) = \left\{(\varphi,\psi)\in\left(H^3(0,L)\right)^2; \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(L) = 0\right\}.$$

We prove that

1. $\dim(N_T) < +\infty$;
2. $N_T \subset D(A)$;
3. $A(N_T)\subset N_T$.

Assertion 1 follows from the fact that $\overline{B_1(0)}$ is a compact subset of N_T. Indeed, if $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}}$ is a sequence in the unit ball $\{(\varphi^0, \psi^0) \in N_T : \|(\varphi^0, \psi^0)\|_H \leq 1\}$, we can use the same argument used in the proof of Theorem 5.8 above to conclude that $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}}$ has a subsequence converging in H, i.e., $\overline{B_1(0)}$ is a compact subset of N_T. Hence, by Riesz' Theorem N_T is finite-dimensional.

In order to prove 2, we first observe that

$$T_1 < T_2 \Rightarrow N_{T_2} \subset N_{T_1} \Rightarrow \dim(N_{T_2}) \leq \dim(N_{T_1}).$$

Indeed, if $(\varphi^0, \psi^0) \in N_{T_2}$, then the solution of (5.3.24)-(5.3.25) satisfies (5.3.26) in $L^2(0, T_2)$. In particular, as $T_1 < T_2$, the solution of (5.3.24)-(5.3.25) also satisfies (5.3.26) in $L^2(0, T_1)$, then $(\varphi^0, \psi^0) \in N_{T_1}$. Thus, the map $T \longmapsto \dim(N_T)$ defined in $\mathbb{R}^+$, with values in $\mathbb{N}$, is nonincreasing, which allows us to conclude that there exist $T, \epsilon > 0$, such that

$$\dim(N_t) = \dim(N_T), \ \forall t \in [T, T+\epsilon].$$

We prove that $N_T \subset D(A)$. Let $(\varphi^0, \psi^0) \in N_T$, $(\varphi, \psi) = S(\cdot)(\varphi^0, \psi^0)$ and $0 < t < \epsilon$. Since

$$S(\tau)\left(S(t)\left(\varphi^0, \psi^0\right)\right) = S(\tau + t)\left(\varphi^0, \psi^0\right),$$

for $0 \leq \tau \leq T$ and $(\varphi^0, \psi^0) \in N_{T+\epsilon} = N_T$, we obtain that $S(t)(\varphi^0, \psi^0) \in N_T$, and the (corresponding) solution $\left(\tilde{\varphi}, \tilde{\psi}\right) := S(\cdot)(S(t)(\varphi^0, \psi^0))$ satisfies

$$\tilde{\varphi}_x(\cdot, L) = \tilde{\psi}_x(\cdot, 0) = 0 \text{ in } L^2(0, L).$$

Then,

$$\frac{S(\tau)(\varphi^0, \psi^0) - (\varphi^0, \psi^0)}{\tau} \in N_T, \tag{5.3.27}$$

for τ small enough. Indeed,

$$\frac{S(\tau)(\varphi^0, \psi^0) - (\varphi^0, \psi^0)}{\tau} \in H,$$

and

$$\begin{aligned}\left(\bar{\varphi}, \bar{\psi}\right) &:= S(\cdot)\left(\frac{S(\tau)(\varphi^0, \psi^0) - (\varphi^0, \psi^0)}{\tau}\right) = \left(\frac{S(\cdot)\,S(\tau)(\varphi^0, \psi^0) - S(\cdot)(\varphi^0, \psi^0)}{\tau}\right) \\ &= \frac{\left(\tilde{\varphi}, \tilde{\psi}\right) - (\varphi, \psi)}{\tau},\end{aligned}$$

is a solution of the problem and satisfies

$$\bar{\varphi}_x(\cdot, L) = \bar{\psi}_x(\cdot, 0) = 0 \text{ in } L^2(0, L).$$

Moreover, note that

$$\left(\varphi^0, \psi^0\right) \in \mathcal{D}(A) \Leftrightarrow \lim_{t\to 0^+} \frac{S(t)(\varphi^0, \psi^0) - (\varphi^0, \psi^0)}{t} \text{ exist in } H.$$

Now, we prove the existence of this limit in H. Set

$$M_T := \left\{ \left(\tilde{\varphi}, \tilde{\psi}\right) = S(\tau)\left(\tilde{\varphi}^0, \tilde{\psi}^0\right) : 0 \leq \tau \leq T : \left(\tilde{\varphi}^0, \tilde{\psi}^0\right) \in N_T \right\}.$$

Observe that

$$M_T \subset C([0,T]; H)$$

and if $(\varphi, \psi) \in M_T$,

$$(\varphi, \psi) \in H^1\left(0, T+\epsilon; \left(H^{-2}(0,L)\right)^2\right).$$

Therefore,

$$\lim_{t \to 0^+} \frac{(\varphi(t+\cdot), \psi(t+\cdot)) - (\varphi, \psi)}{t} = (\varphi', \psi')(\cdot) \in L^2\left(0, T; \left(H^{-2}(0,L)\right)^2\right).$$

On the other hand, by (5.3.27)

$$\frac{(\varphi(t+\cdot), \psi(t+\cdot)) - (\varphi, \psi)}{t} \in M_T,$$

for $0 < t < \epsilon$. Moreover, note that $\dim(M_T) < +\infty$, by the same arguments used to prove that $\dim(N_T)$ is finite. So, M_T is a subspace of $L^2\left(0, T; (H^{-2}(0,L))^2\right)$ which has finite dimension. Consequently, M_T is closed in $L^2\left(0, T; (H^{-2}(0,L))^2\right)$ and $(\varphi', \psi') \in M_T \subset C([0,T]; H)$, i.e.,

$$(\varphi, \psi) \in C^1([0,T]; H).$$

Thus,

$$(\varphi', \psi')(0) = \lim_{t \to 0^+} \frac{(\varphi(t), \psi(t)) - (\varphi^0, \psi^0)}{t} = \lim_{t \to 0^+} \frac{S(t)(\varphi^0, \psi^0) - (\varphi^0, \psi^0)}{t}$$

exist in H. Then, it follows that $(\varphi^0, \psi^0) \in \mathcal{D}(A)$.

Finally, we prove 3. As $\dim(N_T) < +\infty$ and N_T is a subspace of H, it follows that N_T is closed in H. Then, if $(\varphi^0, \psi^0) \in N_T$,

$$A\left((\varphi^0, \psi^0)\right) = \lim_{t \to +\infty} \frac{S(t)(\varphi^0, \psi^0) - (\varphi^0, \psi^0)}{t} \in N_T,$$

therefore,

$$A(N_T) \subset N_T,$$

which concludes the proof of 1, 2, and 3.

If $N_T \neq \{0\}$, the map $(\varphi^0, \psi^0) \in \mathbb{C}N_T \longrightarrow A((\varphi^0, \psi^0)) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue. Hence, there exists $\lambda \in \mathbb{C}$ and $(\varphi^0, \psi^0) \in (H^3(0,L))^2 \setminus \{(0,0)\}$ such that

$$\begin{cases} \lambda\varphi^0 = (\psi^0)' + (\psi^0)''', \\ \lambda\psi^0 = (\varphi^0)' + (\varphi^0)''', \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = (\psi^0(0))' = 0. \end{cases} \tag{5.3.28}$$

To conclude the proof of the Lemma 5.3, we prove that this does not hold if $L \notin \mathcal{N}$.

Lemma 5.4. *Let $L>0$. Consider the assertion*

$$(\mathcal{F}) \qquad \exists\lambda\in\mathbb{C},\ \exists\left(\varphi^0,\psi^0\right)\in\left(H^3(0,L)\right)^2\setminus\{(0,0)\} \text{ such that}$$

$$\begin{cases} \lambda\varphi^0=\left(\psi^0\right)'+\left(\psi^0\right)''', \\ \lambda\psi^0=\left(\varphi^0\right)'+\left(\varphi^0\right)''', \\ \varphi^0(0)=\varphi^0(L)=\left(\varphi^0(0)\right)'=\left(\varphi^0(L)\right)'=0, \\ \psi^0(0)=\psi^0(L)=\left(\psi^0(L)\right)'=\left(\psi^0(0)\right)'=0. \end{cases}$$

Then, $(\mathcal{F})$ holds if and only if $L\in\mathcal{N}$.

Proof. Observe that setting $v^0:=\varphi^0+\psi^0$ and $u^0=\varphi^0-\psi^0$, it follows that (v^0,u^0) satisfies

$$\begin{cases} \lambda v^0=\left[\left(v^0\right)'+\left(v^0\right)'''\right], \\ \lambda u^0=-\left[\left(u^0\right)'+\left(u^0\right)'''\right], \\ v^0(0)=v^0(L)=\left(v^0(0)\right)'=\left(v^0(L)\right)'=0, \\ u^0(0)=u^0(L)=\left(u^0(L)\right)'=\left(u^0(0)\right)'=0. \end{cases}$$

Therefore, using the same argument of [62, Lema 3.5] the proof of Lemma 5.4 holds. This completes the proof of Lemma 5.3 and also the proof of Theorem 5.8. □

The following theorem gives a positive answer for the control problem (5.3.1)-(5.3.2):

Theorem 5.9. *Let $T>0$ and $L\in(0,+\infty)\setminus\mathcal{N}$. Then, the system (5.3.1)-(5.3.2) is exactly controllable in time T.*

Proof. Let us define the following functional

$$\Lambda\left(\varphi^1,\psi^1\right)=\frac{1}{2}\left(\left\|\varphi_x(\cdot,L)\right\|^2_{L^2(0,T)}+\left\|\psi_x(\cdot,0)\right\|^2_{L^2(0,T)}\right)-\int_0^L\left(\eta^1(x)\varphi^1(x)+w^1(x)\psi^1(x)\right)dx \tag{5.3.29}$$

where $(\varphi^1,\psi^1)\in H$ and (φ,ψ) is the solution of the backward system (5.2.10)-(5.2.11) with initial data (φ^1,ψ^1).

Let $\left(\hat{\varphi}^1,\hat{\psi}^1\right)\in H$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.3.4) is satisfied with $h_2(t)=\psi_x(t,0)\in L^2(0,T)$ and $g_2=\varphi_x(t,L)\in L^2(0,T)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.3.4) and (5.3.5), it follows that

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right)&=\frac{1}{2}\left(\left\|\varphi_x(\cdot,L)\right\|^2_{L^2(0,T)}+\left\|\psi_x(\cdot,0)\right\|^2_{L^2(0,T)}\right)-\int_0^L\left(\eta^1(x)\varphi^1(x)+w^1(x)\psi^1(x)\right)dx\\ &\geq C^{-1}\left\|\left(\varphi^1,\psi^1\right)\right\|_H.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.9 is complete. □

Remark 5.4. *When $(\eta^0,w^0)=0$, H.U.M yields a (linear) continuous selection of the control, namely*

$$\Gamma:\left(\eta^1,w^1\right)\in\left(H'\right)^2\longrightarrow\left(\varphi_x(\cdot,L),\psi_x(\cdot,L)\right)\in\left(L^2(0,T)\right)^2,$$

where (φ,ψ) denotes the solution of (5.2.10)-(5.2.11) associated to with $(\varphi^1,\psi^1)=\Lambda^{-1}(\eta^1,w^1)$.

5.3.2 Single control

In this section we study the exact controllability, in time T, for the system (5.3.1)-(5.3.2) with $h_2 = 0$. We first give an equivalent condition for the exact controllability property:

Lemma 5.5. *Let $(\eta^1, w^1) \in H'$. Then, there exists a control $g_2(t) \in L^2(0, L)$, such that the solution (η, w) of (5.3.1)-(5.3.2), with $h_2 = 0$, satisfies (5.3.3) if and only if*

$$\left\langle \left(\eta^1, w^1\right), \left(\varphi^1, \psi^1\right) \right\rangle_{H' \times H} = -\int_0^T g_2(t)\, \varphi_x(t, L)\, dt \tag{5.3.30}$$

for any $(\varphi^1, \psi^1) \in H$, (φ, ψ) being the solution of the backward system (5.2.10)-(5.2.12).

Proof. The relation (5.3.30) is obtained multiplying the equation in (5.3.1) by the solution (φ, ψ) of (5.2.10)-(5.2.12) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result:

Theorem 5.10. *Let $\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2+kl+l^2}{3}} : k, l \in \mathbb{N}^* \right\}$. Then $\forall\ L \in (0, +\infty) \setminus \mathcal{N}$ and $\forall$ $T > 0$, $\exists\ C(T, L) > 0$ such that*

$$\left\| \left(\varphi^1, \psi^1\right) \right\|_H^2 \leq C \int_0^T \left|\varphi_x(t, L)\right|^2 dt, \tag{5.3.31}$$

holds for any $(\varphi^1, \psi^1) \in H$, where (φ, ψ) is the solution of (5.2.10)-(5.2.12) with initial data (φ^1, ψ^1).

Proof. The change of variable $t = T - t$ transforms (5.2.10)-(5.2.12) into (5.2.13)-(5.2.15). Hence, inequality (5.3.31) is equivalent to

$$\left\| \left(\varphi^0, \psi^0\right) \right\|_H^2 \leq C \int_0^T \left|\varphi_x(t, L)\right|^2 dt, \tag{5.3.32}$$

for any $(\varphi^0, \psi^0) \in H$, where (φ, ψ) is the solution of (5.2.13)-(5.2.15) with initial data (φ^0, ψ^0). First, observe that by (5.3.7) and (5.3.8), we obtain

$$\left\| \left(\varphi^0, \psi^0\right) \right\|_H^2 \leq c_1 c_2 \left\{ \left\| \left(\varphi^0, \psi^0\right) \right\|_{(L^2(0,L))^2}^2 + \int_0^T \left|\varphi_x(t, L)\right|^2 dt \right\}.$$

Now, we proceed by contradiction and argue as in the proof of Theorem 5.8. In this case we obtain a solution (φ, ψ) of

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ \varphi(0, x) = \varphi^0(x), \ \ \psi(0, x) = \psi^0(x) & \text{in } (0, L), \end{cases} \tag{5.3.33}$$

satisfying the boundary conditions

$$\left\{\begin{array}{ll}\varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0 & \text{in } (0,T),\\ \psi(t,0)=\psi(t,L)=\psi_x(t,L)=0 & \text{in } (0,T),\end{array}\right. \tag{5.3.34}$$

and, in addition,

$$\varphi_x(\cdot,L)=0 \tag{5.3.35}$$

and

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_H=1. \tag{5.3.36}$$

Remark that (5.3.36) implies that the solutions of (5.3.33)-(5.3.35) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following result:

Lemma 5.6. *For any $T>0$ let N_T denote the space of the initial states $(\varphi^0,\psi^0)\in H$ such that the solution $(\varphi,\psi)=S(\cdot)(\varphi^0,\psi^0)$ of (5.3.33)-(5.3.34) satisfies (5.3.35). Then, for $L\in(0,+\infty)\setminus\mathcal{N}$, $N_T=\{0\}$, $\forall T>0$.*

Proof. Let A be the operator

$$A(\varphi,\psi)=\begin{pmatrix}\psi_x+\psi_{xxx}\\ \varphi_x+\varphi_{xxx}\end{pmatrix}, \forall(\varphi,\psi)\in\mathcal{D}(A),$$

with

$$\mathcal{D}(A)=\left\{(\varphi,\psi)\in\left(H^3(0,L)\right)^2;\varphi(0)=\psi(0)=\varphi(L)=\psi(L)=\varphi_x(0)=\psi_x(L)=0\right\}.$$

Arguing as in the proof of Lemma 5.3, we conclude that N_T verifies

1. $\dim(N_T)<+\infty$;
2. $N_T\subset D(A)$;
3. $A(N_T)\subset N_T$.

If $N_T\neq\{0\}$, the map $(\varphi^0,\psi^0)\in\mathbb{C}N_T\longrightarrow A((\varphi^0,\psi^0))\in\mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at last) one eigenvalue. Then, there exist $\lambda\in\mathbb{C}$ and $(\varphi^0,\psi^0)\in(H^3(0,L))^2\setminus\{(0,0)\}$ satisfying

$$\left\{\begin{array}{l}\lambda\varphi^0=\left[(\psi^0)'+(\psi^0)'''\right],\\ \lambda\psi^0=\left[(\varphi^0)'+(\varphi^0)'''\right],\\ \varphi^0(0)=\varphi^0(L)=(\varphi^0(0))'=(\varphi^0(L))'=0,\\ \psi^0(0)=\psi^0(L)=(\psi^0(L))'=0.\end{array}\right. \tag{5.3.37}$$

To conclude the proof of the Lemma 5.6, we prove in the following lemma that (5.3.37) does not hold if $L\notin\mathcal{N}$.

Lemma 5.7. *Let $L>0$. Consider the assertion*

$(\mathcal{F}_1)$ $\quad\exists\lambda\in\mathbb{C}$, $\exists\left(\varphi^0,\psi^0\right)\in\left(H^3(0,L)\right)^2\setminus\{(0,0)\}$ *such that*

$$\left\{\begin{array}{l}\lambda\varphi^0=\left[(\psi^0)'+(\psi^0)'''\right],\\ \lambda\psi^0=\left[(\varphi^0)'+(\varphi^0)'''\right],\\ \varphi^0(0)=\varphi^0(L)=(\varphi^0(0))'=(\varphi^0(L))'=0,\\ \psi^0(0)=\psi^0(L)=(\psi^0(L))'=0.\end{array}\right.$$

Then, $(\mathcal{F}_1)$ holds if and only if $L\in\mathcal{N}$.

Proof. We use an argument which is similar to the one used in [62, Lema 3.5]. Assume that (φ^0, ψ^0) satisfies $(\mathcal{F}_1)$ and let us denote by $(\varphi, \psi) \in (H^2(\mathbb{R}))^2$ their prolongation by 0 to $\mathbb{R}$. Then,

$$\begin{cases} -\lambda\varphi + \psi' + \psi''' = \psi''(0)\,\delta_0 + \psi'(0)\,(\delta_0)' - \psi''(L)\,\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda\psi + \varphi' + \varphi''' = \varphi''(0)\,\delta_0 - \varphi''(L)\,\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \end{cases} \tag{5.3.38}$$

where δ_{x_0} and $(\delta_{x_0})'$ denote the Dirac measure at x_0. Note that the $(\mathcal{F}_1)$ is equivalent to the existence of complex numbers α, α', β, γ, γ', λ with $(\alpha, \alpha', \beta, \gamma, \gamma') \neq (0,0,0,0,0)$ and $(\varphi, \psi) \in (H^2(\mathbb{R}))^2$ with compact support in $[-L, L]$ such that

$$\begin{cases} -\lambda(\varphi + \psi) + (\psi + \varphi)' + (\psi + \varphi)''' = \alpha\delta_0 + \beta(\delta_0)' + \gamma\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda(\varphi - \psi) + (\varphi - \psi)' + (\varphi - \psi)''' = \alpha'\delta_0 - \beta(\delta_0)' + \gamma'\delta_L & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases} \tag{5.3.39}$$

Let us introduce the notation $\hat{\varphi}(\xi) = \int_0^L \varphi(\xi)\, e^{-ix\xi}dx$ and $\hat{\psi}(\xi) = \int_0^L \psi(\xi)\, e^{-ix\xi}dx$. Then, taking the Fourier transform in (5.3.38) we obtain

$$-\lambda\hat{\varphi}(\xi) + (i\xi)\,\hat{\psi}(\xi) + (i\xi)^3\,\hat{\psi}(\xi) + \psi''(L)\,e^{-iL\xi} - \psi''(0) - (i\xi)\,\psi'(0) = 0 \tag{5.3.40}$$

and

$$-\lambda\hat{\psi}(\xi) + (i\xi)\,\hat{\varphi}(\xi) + (i\xi)^3\,\hat{\varphi}(\xi) - \varphi''(0) + \varphi''(L)\,e^{-iL\xi} = 0. \tag{5.3.41}$$

Then, adding (5.3.40) and (5.3.41) the follow identity holds

$$\left(-\lambda + (i\xi) + (i\xi)^3\right)\left(\hat{\varphi}(\xi) + \hat{\psi}(\xi)\right) = (\varphi''(0) + \psi''(0)) + (i\xi)\,\psi'(0) + (-\psi''(L) - \varphi''(L))\,e^{-iL\xi}.$$

We denote

$$\hat{u}(\xi) := \left(\hat{\varphi}(\xi) + \hat{\psi}(\xi)\right) = \frac{(\varphi''(0) + \psi''(0)) + (i\xi)\,\psi'(0) + (-\psi''(L) - \varphi''(L))\,e^{-iL\xi}}{-\lambda + (i\xi) + (i\xi)^3}. \tag{5.3.42}$$

We also take the difference between (5.3.40) and (5.3.41) to obtain

$$\left(-\lambda - (i\xi) - (i\xi)^3\right)\left(\hat{\varphi}(\xi) - \hat{\psi}(\xi)\right) = (-\varphi''(0) + \psi''(0)) + (i\xi)\,\psi'(0) + (-\psi''(L) + \varphi''(L))\,e^{-iL\xi}.$$

Here, we denote

$$\hat{v}(\xi) := \left(\hat{\varphi}(\xi) - \hat{\psi}(\xi)\right) = \frac{(-\varphi''(0) + \psi''(0)) + (i\xi)\,\psi'(0) + (-\psi''(L) + \varphi''(L))\,e^{-iL\xi}}{-\lambda - (i\xi) - (i\xi)^3}.$$

Introducing the change of variable $\xi \longmapsto -\xi$, we have that

$$\hat{v}(-\xi) = \frac{(-\varphi''(0) + \psi''(0)) - (i\xi)\,\psi'(0) + (-\psi''(L) + \varphi''(L))\,e^{iL\xi}}{-\lambda + (i\xi) + (i\xi)^3}. \tag{5.3.43}$$

Setting $\lambda = ip$, we write (5.3.42) and (5.3.43) as

$$\hat{u}(\xi) = i\frac{\alpha + (i\xi)\,\beta + \gamma e^{-iL\xi}}{\xi^3 - \xi + p} \tag{5.3.44}$$

and

$$\hat{v}(-\xi) = i\frac{\alpha' - (i\xi)\beta + \gamma' e^{iL\xi}}{\xi^3 - \xi + p}. \tag{5.3.45}$$

Using Paley-Wiener theorem (see [79, Section 4, page 161]) and the usual characterization of $H^2(\mathbb{R})$ by means of the Fourier transform we see that $(\mathcal{F}_1)$ is equivalent to the existence of $p \in \mathbb{C}$ and

$$(\alpha, \alpha', \beta, \gamma, \gamma') \in \mathbb{C}^5 \backslash (0,0,0,0,0),$$

such that

$$f(\xi) := \frac{\alpha + (i\xi)\beta + \gamma e^{-iL\xi}}{\xi^3 - \xi + p}$$

and

$$g(\xi) := \frac{\alpha' - (i\xi)\beta + \gamma' e^{iL\xi}}{\xi^3 - \xi + p}$$

satisfies

a) f and g are entire function in $\mathbb{C}$;

b) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2 \left(1 + |\xi|^2\right)^2 d\xi < \infty$ and $\displaystyle\int_{\mathbb{R}} |g(\xi)|^2 \left(1 + |\xi|^2\right)^2 d\xi < \infty$;

c) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c(1 + |\xi|)^k \exp(l \operatorname{Im} \xi)$ for some positive constants c and k.

Remark that f and g are entire if and only if the roots μ_0, μ_1 and μ_2 of $Q(\xi) := \xi^3 - \xi + p$ are roots of

$$r_1(\xi) := \alpha + (i\xi)\beta + \gamma e^{-iL\xi} \tag{5.3.46}$$

and

$$r_2(\xi) := \alpha' - (i\xi)\beta + \gamma' e^{iL\xi}. \tag{5.3.47}$$

Furthermore, these roots are also roots of the sum of (5.3.46) and (5.3.47), i.e.,

$$s(\xi) := (\alpha + \alpha') + \gamma e^{-iL\xi} + \gamma' e^{iL\xi}. \tag{5.3.48}$$

Observe that if the roots of (5.3.48) are simple, a) holds if the roots of $Q(\xi)$ are simples and also roots of (5.3.48). Consequently, if a) is true, then b) and c) are satisfied.

The next steps are devoted to find the roots of (5.3.48) and to prove that they are simple. Indeed, consider the equation $s(\xi) = 0$. Multiplying this equality by $e^{iL\xi}$, we obtain that

$$\gamma' \left(e^{iL\xi}\right)^2 + (\alpha + \alpha') e^{iL\xi} + \gamma = 0, \tag{5.3.49}$$

i.e.,

$$P\left(e^{iL\xi}\right) = 0,$$

where P denotes a polynomial function in $e^{iL\xi}$. Letting $x = e^{iL\xi}$, we have

$$P(x) = \gamma' x^2 + (\alpha + \alpha') x + \gamma. \tag{5.3.50}$$

The roots of $P(x)$ are

$$x_1 = \frac{-(\alpha + \alpha') + \left((\alpha + \alpha')^2 - 4\gamma\gamma'\right)^{1/2}}{2\gamma'} \tag{5.3.51}$$

and

$$x_2 = \frac{-(\alpha+\alpha') - \left((\alpha+\alpha')^2 - 4\gamma\gamma'\right)^{1/2}}{2\gamma'}. \tag{5.3.52}$$

Therefore,

$$P\left(e^{iL\xi}\right) = 0 \Leftrightarrow e^{iL\xi} = x_1 \text{ or } e^{iL\xi} = x_2. \tag{5.3.53}$$

Then, we deduce that the roots of $s(\xi)$ lies in

$$\Upsilon := \left\{\xi_1 + \frac{2k_1\pi}{L} : k_1 \in \mathbb{N}^*\right\} \cup \left\{\xi_2 + \frac{2k_2\pi}{L} : k_2 \in \mathbb{N}^*\right\}.$$

Now, to conclude the analysis for f and g, we consider three cases:
Case 1: Suppose that $Q(\xi)$ has three simple roots μ_0, μ_1 and μ_2 in Υ. Then, f and g are entire, i.e., they have the same simple roots, if

$$\beta = \psi'(0) = 0. \tag{5.3.54}$$

Thus, from (5.3.37), (5.3.54) and of Lemma 5.4 we conclude that $(\mathcal{F}_1)$ holds if and only if $L \in \mathcal{N}$.
Case 2: Suppose that $Q(\xi)$ has a root of order three, namely, μ_0. In this case,

$$Q(\mu_0) = Q'(\mu_0) = Q''(\mu_0) = 0. \tag{5.3.55}$$

Then, $\mu_0 = 0$ and

$$Q(\mu_0) = 0 \Leftrightarrow p = 0.$$

If $p = 0$, we obtain that

$$Q(\xi) = \xi^3 - \xi = \xi(\xi+1)(\xi-1), \tag{5.3.56}$$

which is a contradiction.
Case 3: Suppose that $Q(\xi)$ has one double root μ_0 and a simple root μ_2. In this case,

$$Q(\mu_0) = Q'(\mu_0) = 0 \text{ and } Q(\mu_2) = 0. \tag{5.3.57}$$

Then, from (5.3.57) we obtain

$$\mu_0 = \frac{1}{\sqrt{3}} \text{ or } \mu_0 = -\frac{1}{\sqrt{3}}.$$

If $\mu_0 = \frac{1}{\sqrt{3}}$,

$$p = -\left(\mu_0^3 - \mu_0\right) = -\left(\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}, \tag{5.3.58}$$

and, therefore, we obtain

$$Q(\xi) = \xi^3 - \xi + \frac{2}{3\sqrt{3}} = \left(\xi - \frac{1}{\sqrt{3}}\right)^2 \left(\xi + \frac{2}{\sqrt{3}}\right). \tag{5.3.59}$$

Consequently, we deduce that

$$\mu_2 = -\frac{2}{\sqrt{3}}. \tag{5.3.60}$$

The case $\mu_0 = -\frac{1}{\sqrt{3}}$ is analogous and it will be omitted. Since μ_0 and μ_2 are roots of $Q(\xi)$, they μ_0 is also roots of r_1, r_1', r_2 and r_2', furthermore, μ_2 is also roots of r_1 and r_2, where r_1 and r_2 are defined in (5.3.46) and (5.3.47), respectively. Here $'$ denotes the derivative with respect to ξ. Thus,

$$\begin{cases} \alpha + i\frac{1}{\sqrt{3}}\beta + \gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right) = 0 \\ i\beta - (iL)\,\gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right) = 0, \end{cases} \tag{5.3.61}$$

$$\begin{cases} \alpha' - i\frac{1}{\sqrt{3}}\beta + \gamma' \exp\left(iL\frac{1}{\sqrt{3}}\right) = 0 \\ -i\beta + (iL)\,\gamma' \exp\left(iL\frac{1}{\sqrt{3}}\right) = 0 \end{cases} \tag{5.3.62}$$

and

$$\begin{cases} \alpha + i\left(-\frac{2}{\sqrt{3}}\right)\beta + \gamma \exp\left(iL\frac{2}{\sqrt{3}}\right) = 0 \\ \alpha' - i\left(-\frac{2}{\sqrt{3}}\right)\beta + \gamma' \exp\left(-iL\frac{2}{\sqrt{3}}\right) = 0. \end{cases} \tag{5.3.63}$$

Finally, we find L such that (5.3.61)-(5.3.63) are satisfied. Indeed, from the second equations of (5.3.61) and (5.3.62), we get

$$\beta = \gamma L \exp\left(-iL\frac{1}{\sqrt{3}}\right) = \gamma' L \exp\left(iL\frac{1}{\sqrt{3}}\right), \tag{5.3.64}$$

which give us that

$$\gamma' = \gamma \exp\left(-2iL\frac{1}{\sqrt{3}}\right). \tag{5.3.65}$$

Replacing (5.3.64) and (5.3.65) in the first equations of (5.3.61) and (5.3.62), respectively, we have

$$\alpha + i\frac{1}{\sqrt{3}}\gamma L \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right) = 0 \tag{5.3.66}$$

and

$$\alpha' - i\frac{1}{\sqrt{3}}\gamma L \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma \exp\left(-2iL\frac{1}{\sqrt{3}}\right) \exp\left(iL\frac{1}{\sqrt{3}}\right) = 0. \tag{5.3.67}$$

Now, replacing (5.3.64) and (5.3.65) in (5.3.63) it follows that

$$\begin{cases} \alpha + i\left(-\frac{2}{\sqrt{3}}\right)\gamma L \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma \exp\left(iL\frac{2}{\sqrt{3}}\right) = 0 \\ \alpha' - i\left(-\frac{2}{\sqrt{3}}\right)\gamma L \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma \exp\left(-2iL\frac{1}{\sqrt{3}}\right)\exp\left(-iL\frac{2}{\sqrt{3}}\right) = 0. \end{cases} \tag{5.3.68}$$

From (5.3.66) and (5.3.67), we obtain α and α', i.e.,

$$\alpha = -\gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right)\left(\frac{iL}{\sqrt{3}} + 1\right) \tag{5.3.69}$$

and

$$\alpha' = \gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right)\left(\frac{iL}{\sqrt{3}} - 1\right). \tag{5.3.70}$$

Finally, replacing (5.3.69) and (5.3.70) in (5.3.68), it follows that

$$\gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right)\left(\frac{-3iL}{\sqrt{3}} - 1\right) + \gamma \exp\left(iL\frac{2}{\sqrt{3}}\right) = 0 \tag{5.3.71}$$

and

$$\gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right)\left(\frac{3iL}{\sqrt{3}} - 1\right) + \gamma \exp\left(-2iL\frac{2}{\sqrt{3}}\right) = 0. \tag{5.3.72}$$

Thus, we have that $\gamma = 0$ or $\gamma \neq 0$. If $\gamma = 0$, the solution of (5.3.61)-(5.3.63) is the trivial one, i.e., $\gamma = \gamma' = \beta = \alpha = 0$. If $\gamma \neq 0$ we can add (5.3.71) and (5.3.72) to obtain

$$-2\exp\left(-iL\frac{1}{\sqrt{3}}\right) + \exp\left(2iL\frac{1}{\sqrt{3}}\right) + \exp\left(-4iL\frac{1}{\sqrt{3}}\right) = 0.$$

Multiplying this equation by $\exp\left(4iL\frac{1}{\sqrt{3}}\right)$ and denoting $\nu := \exp\left(3iL\frac{1}{\sqrt{3}}\right)$, we have

$$\nu^2 - 2\nu + 1 = 0 \Leftrightarrow (\nu - 1)^2 = 0.$$

Since $\nu = 1$ is a root of $P(v) = \nu^2 - 2\nu + 1$, it follows that

$$1 = \exp\left(3iL\frac{1}{\sqrt{3}}\right).$$

Taking the modulus, it follows that $L = 0$.

Therefore, from the cases 1, 2 and 3 we deduce that $(\mathcal{F}_1)$ holds if and only if $L \in \mathcal{N}$. This completes the proof of Lemmas 5.6, 5.7 and Theorem 5.10. □

The following theorem gives a positive answer for the control problem (5.3.1)-(5.3.2) with $h_2 = 0$:

Theorem 5.11. *Let $T > 0$ and $L \in (0, +\infty) \setminus \mathcal{N}$. Then the system (5.3.1)-(5.3.2) is exactly controllable, with $h_2 = 0$, in time T.*

Proof. Let us define the following functional

$$\Lambda\left(\varphi^1, \psi^1\right) = \frac{1}{2}\left\|\varphi_x(\cdot, L)\right\|^2_{L^2(0,T)} - \int_0^L \left(\eta^1(x)\,\varphi^1(x) + w^1(x)\,\psi^1(x)\right) dx \tag{5.3.73}$$

where $(\varphi^1, \psi^1) \in H$ and (φ, ψ) is the solution of the backward system (5.2.10)-(5.2.11) with initial data (φ^1, ψ^1).

Let $\left(\hat{\varphi}^1, \hat{\psi}^1\right) \in H$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.3.30) is satisfied with $g_2 = \varphi_x(t, L) \in L^2(0, T)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.3.30) and (5.3.31), holds that

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right) &= \frac{1}{2}\left\|\varphi_x\left(\cdot,L\right)\right\|^2_{L^2(0,T)} - \int_0^L \left(\eta^1\left(x\right)\varphi^1\left(x\right) + w^1\left(x\right)\psi^1\left(x\right)\right)dx \\ &\geq C^{-1}\left\|\left(\varphi^1,\psi^1\right)\right\|_H.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.11 is complete. □

Remark 5.5. *An important question is whether system (5.3.1)-(5.3.2) is exactly controllable in time $T>0$, where $g_2=0$. Observe that, in this case it would be necessary to prove an observability inequality of the type*

$$\left\|(\varphi,\psi)\right\|^2_H \leq C\int_0^T \left|\psi_x\left(t,0\right)\right|^2 dt,$$

for any $(\varphi^1,\psi^1)\in H$, where (φ,ψ) is the solution of (5.2.10)-(5.2.11) with initial data (φ^1,ψ^1). Note that it can be done by using Lemma 5.2 together with the contradiction argument used in the proof of the Theorem 5.10. Necessarily, we would have an important difference in the proof of Lemma 5.7 since we would have the following assertion:

$$(\mathcal{F}_2) \qquad \exists\lambda\in\mathbb{C},\ \exists\left(\varphi^0,\psi^0\right)\in\left(H^3\left(0,L\right)\right)^2\backslash\{(0,0)\}\ \text{such that}$$

$$\begin{cases}\lambda\varphi^0 = \left[\left(\psi^0\right)' + \left(\psi^0\right)'''\right], \\ \lambda\psi^0 = \left[\left(\varphi^0\right)' + \left(\varphi^0\right)'''\right], \\ \varphi^0\left(0\right) = \varphi^0\left(L\right) = \left(\varphi^0\left(0\right)\right)' = 0, \\ \psi^0\left(0\right) = \psi^0\left(L\right) = \left(\psi^0\left(L\right)\right)' = \left(\psi^0\left(0\right)\right)' = 0.\end{cases}$$

Then $(\mathcal{F}_2)$ holds if and only if $L\in\mathcal{N}$. In order to prove this result we can use the ideas introduced to prove Lemma 5.7. Thus, the next result about the exact controllability of the system (5.3.1)-(5.3.2) with $g_2=0$ is also holds.

Theorem 5.12. *Let $T>0$ and $L\in(0,+\infty)/\mathcal{N}$. Then the system (5.3.1)-(5.3.2) is exactly controllable, with $g_2=0$, in time T.*

5.4 Exact Boundary Controllability For The Linear System: Dirichlet boundary condition

This section is devoted to the analysis of the exact controllability property of the linear system corresponding to (5.1.7) with boundary controls of Dirichlet type. More precisely, given $T>0$ and (η^0,w^0), $\left(\eta^T,w^T\right):=(\eta^1,w^1)\in\bar{\Sigma}$, we study the existence of the controls $(h_0,g_0,h_1,g_1)\in\bar{\Sigma}_1$ such that the solution (η,w) of the system

$$\begin{cases}\eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \eta\left(0,x\right) = \eta^0\left(x\right),\ \ w\left(0,x\right) = w^0\left(x\right) & \text{in } (0,L),\end{cases} \tag{5.4.1}$$

satisfying the boundary conditions

$$\begin{cases} \eta(t,0) = h_0(t), \ \ \eta(t,L) = h_1(t), \ \ \eta_x(t,0) = 0 & \text{in } (0,T) \\ w(t,0) = g_0(t), \ \ w(t,L) = g_1(t), \ \ w_x(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.4.2}$$

satisfies

$$\eta(T,\cdot) = \eta^1, \quad w(T,\cdot) = w^1 \text{ in } \bar{\Sigma}. \tag{5.4.3}$$

The spaces $\bar{\Sigma}$ and $\bar{\Sigma}_1$ will be defined later.

Definition 5.3. *Let $T > 0$. System (5.4.1) is exact controllable in time T if for any initial and final data (η^0, w^0), $(\eta^1, w^1) \in \bar{\Sigma}$, there exist control functions $(h_0, g_0, h_1, g_1) \in \bar{\Sigma}_1$ such that the solution of (5.4.1)-(5.4.2) satisfies (5.4.3).*

We consider several cases regarding the amount of controls on (5.4.2).

5.4.1 Double control in L

In this section we consider (5.4.2) with $h_0 = g_0 = 0$. We first give an equivalent condition for the exact controllability property:

Lemma 5.8. *Let $(\eta^1, w^1) \in \bar{\Sigma} := X'$. Then, there exist two control $(h_1(t), g_1(t)) \in \bar{\Sigma}_1 := [L^2(0,L)]^2$, such that the solution (η, w) of (5.4.1)-(5.4.2) with $h_0 = g_0 = 0$, satisfies (5.4.3) if and only if*

$$\left\langle (\eta^1, w^1), (\varphi^1, \psi^1) \right\rangle_{X' \times X} = -\int_0^T \left(h_1(t)\psi_{xx}(t,L) + g_1(t)\varphi_{xx}(t,L)\right) dt, \tag{5.4.4}$$

for any $(\varphi^1, \psi^1) \in X$, (φ, ψ) being the solution of the backward system (5.2.10)-(5.2.12).

Proof. The relation (5.4.4) is obtained multiplying the equations in (5.4.1) by the solution (φ, ψ) of (5.2.10)-(5.2.12) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result:

Theorem 5.13. *For any $T > 0$ and $L > 0$ there exist $C = C(T,L) > 0$ such that the inequality*

$$\left\|(\varphi^1, \psi^1)\right\|_X^2 \le C \int_0^T \left(|\varphi_{xx}(t,L)|^2 + |\psi_{xx}(t,L)|^2\right) dt, \tag{5.4.5}$$

holds for any $(\varphi^1, \psi^1) \in X$, where (φ, ψ) is the solution of (5.2.10)-(5.2.11) with initial data (φ^1, ψ^1).

For the proof of Theorem 5.13, we need some basic estimates for the solution of the adjoint system (5.2.10)-(5.2.12). Therefore, we prove the following lemma:

Lemma 5.9. *For any (φ, ψ) solution of (5.2.10)-(5.2.11) with initial data $(\varphi^1, \psi^1) \in X$, there exist $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$, such that*

$$\left\|(\varphi^1, \psi^1)\right\|_X \le c_1 \left\|(\varphi, \psi)\right\|_{L^2(0,T;X)}, \tag{5.4.6}$$

$$\|\varphi_{xx}(t,L)\|^2_{L^2(0,T)}+\|\psi_{xx}(t,L)\|^2_{L^2(0,T)}\le c_2\left\|\left(\varphi^1,\psi^1\right)\right\|^2_X \tag{5.4.7}$$

and

$$\|\varphi_{xx}(t,0)\|^2_{L^2(0,T)}+\|\psi_{xx}(t,0)\|^2_{L^2(0,T)}\le c_3\left\|\left(\varphi^1,\psi^1\right)\right\|^2_X. \tag{5.4.8}$$

Proof. Observe that (5.4.6) follows from the proof of (5.3.7).

In order to prove (5.4.7) we introduce $\rho=\rho(x)\in C^3([0,L])$ satisfying

$$\rho(x)=\begin{cases}0, & \text{if } x\in[0,L/2]\\ 1 & \text{if } x\in[3L/4,L].\end{cases} \tag{5.4.9}$$

Let us consider the functions

$$\tilde{\varphi}(t,x):=\rho(x)\varphi(t,x)\ \text{ and }\ \tilde{\psi}(t,x):=\rho(x)\psi(t,x),$$

which fulfill

$$\begin{cases}\tilde{\varphi}_t+\tilde{\psi}_x+\tilde{\psi}_{xxx}=\rho_x\psi+\rho_{xxx}\psi+3\rho_{xx}\psi_x+3\rho_x\psi_{xx}\\ \tilde{\psi}_t+\tilde{\varphi}_x+\tilde{\varphi}_{xxx}=\rho_x\varphi+\rho_{xxx}\varphi+3\rho_{xx}\varphi_x+3\rho_x\varphi_{xx}.\end{cases} \tag{5.4.10}$$

Now, multiplying the first equation of (5.4.10) by $\tilde{\psi}_{xx}$, the second one by $\tilde{\varphi}_{xx}$ and integrating in $(0,L)$, we obtain

$$\begin{aligned}&\int_0^L\tilde{\varphi}_t(t,x)\tilde{\psi}_{xx}(t,x)\,dx+\int_0^L\tilde{\psi}_x(t,x)\tilde{\psi}_{xx}(t,x)\,dx+\int_0^L\tilde{\psi}_{xxx}(t,x)\tilde{\psi}_{xx}(t,x)\,dx\\ &=\int_0^L\rho_x(x)\psi(t,x)\tilde{\psi}_{xx}(t,x)\,dx+\int_0^L\rho_{xxx}(x)\psi(t,x)\tilde{\psi}_{xx}(t,x)\,dx\\ &+3\int_0^L\Big(\rho_{xx}(x)\psi_x(t,x)\tilde{\psi}_{xx}(t,x)+\rho_x(x)\psi_{xx}(t,x)\tilde{\psi}_{xx}(t,x)\Big)\,dx\end{aligned} \tag{5.4.11}$$

and

$$\begin{aligned}&\int_0^L\tilde{\psi}_t(t,x)\tilde{\varphi}_{xx}(t,x)\,dx+\int_0^L\tilde{\varphi}_x(t,x)\tilde{\varphi}_{xx}(t,x)\,dx+\int_0^L\tilde{\varphi}_{xxx}(t,x)\tilde{\varphi}_{xx}(t,x)\,dx\\ &=\int_0^L\rho_x(x)\varphi(t,x)\tilde{\varphi}_{xx}(t,x)\,dx+\int_0^L\rho_{xxx}(x)\varphi(t,x)\tilde{\varphi}_{xx}(t,x)\,dx\\ &+3\int_0^L\left(\rho_{xx}(x)\varphi_x(t,x)\tilde{\varphi}_{xx}(t,x)+\rho_x(x)\varphi_{xx}(t,x)\tilde{\varphi}_{xx}(t,x)\right)dx.\end{aligned} \tag{5.4.12}$$

We first analyze the terms on the left hand side of (5.4.11). From the boundary condition and (5.4.9), we obtain, after integration by parts, (here we omit (t,x))

$$\int_0^L\tilde{\varphi}_t\tilde{\psi}_{xx}=\tilde{\varphi}_t\tilde{\psi}_x(t,x)\Big|_0^L-\int_0^L\tilde{\varphi}_{tx}\tilde{\psi}_x=-\int_0^L\tilde{\varphi}_{tx}\tilde{\psi}_x, \tag{5.4.13}$$

$$\int_0^L\tilde{\psi}_x\tilde{\psi}_{xx}=\int_0^L\left(\frac{\tilde{\psi}_x^2}{2}\right)_x=0 \tag{5.4.14}$$

and

$$\int_0^L \tilde{\psi}_{xxx}\tilde{\psi}_{xx} = \int_0^L \left(\frac{\tilde{\psi}_{xx}^2}{2}\right)_x = \frac{1}{2}\left|\tilde{\psi}_{xx}\Big|_{x=L}\right|^2. \tag{5.4.15}$$

A similar analysis can be done for the terms on the left hand side of (5.4.12):

$$\int_0^L \tilde{\psi}_t\tilde{\varphi}_{xx} = -\int_0^L \tilde{\psi}_{tx}\tilde{\varphi}_x, \tag{5.4.16}$$

$$\int_0^L \tilde{\varphi}_x\tilde{\varphi}_{xx} = \int_0^L \left(\frac{\tilde{\varphi}_x^2}{2}\right)_x = \frac{1}{2}\left|\tilde{\varphi}_x|_{x=L}\right|^2 \tag{5.4.17}$$

and

$$\int_0^L \tilde{\varphi}_{xxx}\tilde{\varphi}_{xx} = \frac{1}{2}\left|\tilde{\varphi}_{xx}|_{x=L}\right|^2. \tag{5.4.18}$$

Then, from (5.4.13)-(5.4.18) the following identity holds

$$\begin{aligned}\int_0^L \left(\tilde{\varphi}_x\tilde{\psi}_x\right)_t &+ \frac{1}{2}\left|\tilde{\varphi}_{xx}|_{x=L}\right|^2 + \frac{1}{2}\left|\tilde{\psi}_{xx}\Big|_{x=L}\right|^2 + \frac{1}{2}\left|\tilde{\varphi}_x|_{x=L}\right|^2 \\ &= \int_0^L \left\{\rho_x\psi + \rho_{xxx}\psi + 3\left(\rho_{xx}\psi_x + \rho_x\psi_{xx}\right)\right\}\tilde{\psi}_{xx} \\ &+ \int_0^L \left\{\rho_x\varphi + \rho_{xxx}\varphi + 3\left(\rho_{xx}\varphi_x + \rho_x\varphi_{xx}\right)\right\}\tilde{\varphi}_{xx}.\end{aligned} \tag{5.4.19}$$

Integrating in the t variable and estimating the right hand side terms, we deduce that

$$\begin{aligned}\int_0^T &\left(\left|\varphi_{xx}|_{x=L}\right|^2 + \left|\psi_{xx}|_{x=L}\right|^2\right) dt \\ &\leq C_1\left(\|\psi\|^2_{L^2(0,T;H^2(0,L))} + \|\psi\|^2_{C^0(0,T;H^1(0,L))} + \|(\varphi^1,\psi^1)\|^2_H\right) \\ &+ C_2\left(\|\varphi\|^2_{L^2(0,T;H^2(0,L))} + \|\varphi\|^2_{C^0(0,T;H^1(0,L))} + \|(\varphi^1,\psi^1)\|^2_H\right).\end{aligned} \tag{5.4.20}$$

Note that, (φ,ψ) is the solution of (5.2.10)-(5.2.11) with initial data $(\varphi^1,\psi^1)\in X$, in particular,

$$(\varphi,\psi)\in L^2(0,T;X),$$

therefore,

$$\int_0^T \left(\left|\varphi_{xx}|_{x=L}\right|^2 + \left|\psi_{xx}|_{x=L}\right|^2 + \left|\varphi_x|_{x=L}\right|^2\right) dt \leq C\left\|\left(\varphi^1,\psi^1\right)\right\|^2_X.$$

Thus, (5.4.7) holds.

Finally, observe that modifying the function $\rho=\rho(x)\in C^3([0,L])$ by

$$\rho(x) = \begin{cases} 1, & \text{if } x\in[0,L/2] \\ 0 & \text{if } x\in[3L/4,L].\end{cases} \tag{5.4.21}$$

the same calculations above ensures that (5.4.8) holds and the Lemma is proved. □

With Lemma 5.9 in hands, we can prove Theorem 5.13.

Proof of Theorem 5.13. The change of variable $t = T - t$ transforms (5.2.10)-(5.2.12) into (5.2.13)-(5.2.15). Hence, inequality (5.4.5) is equivalent to

$$\left\|\left(\varphi^0, \psi^0\right)\right\|_X^2 \leq C \int_0^T \left(\left|\varphi_{xx}(t, L)\right|^2 + \left|\psi_{xx}(t, L)\right|^2\right) dt, \tag{5.4.22}$$

for any $(\varphi^0, \psi^0) \in X$, where (φ, ψ) is the solution of (5.2.13)-(5.2.15) with initial data (φ^0, ψ^0).

We assume that (5.4.22) is not true. Then, there exists a sequence $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}} \in X$ such that

$$\left\|\left(\varphi_n^0, \psi_n^0\right)\right\|_X = 1, \tag{5.4.23}$$

$$\left\|\varphi_{n,xx}(\cdot, L)\right\|_{L^2(0,T)}^2 \longrightarrow 0, \text{ as } n \to \infty \tag{5.4.24}$$

and

$$\left\|\psi_{n,xx}(\cdot, L)\right\|_{L^2(0,T)}^2 \longrightarrow 0, \text{ as } n \to \infty \tag{5.4.25}$$

where $(\varphi_n, \psi_n) \in Z := L^2(0, T; X)$ is the solution of (5.2.13)-(5.2.15) with initial data (φ_n^0, ψ_n^0). Let us denote

$$Y := L^2\left(0, T; \left(H^{7/4}(0, L)\right)^2\right) \cap C^0\left([0, T]; \left(H^1(0, L)\right)^2\right).$$

We show that there exist some positive constant c_1 such that, $\forall (\varphi, \psi) \in Z$ solution of (5.2.13)-(5.2.15), one has

$$\begin{aligned}\|(\varphi, \psi)\|_Z^2 \leq\ & c_1 \left(\left\|\varphi_{xx}(\cdot, L)\right\|_{L^2(0,T)}^2 + \left\|\psi_{xx}(\cdot, L)\right\|_{L^2(0,T)}^2\right) \\ & + c_1 \left(\|(\varphi, \psi)\|_Y^2 + \left\|\left(\varphi^0, \psi^0\right)\right\|_{(H^1(0,L))^2}^2\right).\end{aligned} \tag{5.4.26}$$

For that purpose, we differentiate the equations in (5.2.13) with respect to x to obtain

$$\begin{cases} \varphi_{tx} - \psi_{xx} - \psi_{xxxx} = 0 & \text{in } (0, T) \times (0, L), \\ \psi_{tx} - \varphi_{xx} - \varphi_{xxxx} = 0 & \text{in } (0, T) \times (0, L). \end{cases} \tag{5.4.27}$$

Now, multiplying the first equation of (5.4.27) by $x\psi_x$, the second one by $x\varphi_x$ and integrating in $(0, T) \times (0, L)$, we obtain, after integration by parts,

$$\begin{aligned} 0 =\ & \int_0^T \int_0^L x\psi_x\varphi_{tx} + \frac{1}{2}\int_0^T \int_0^L \psi_x^2 - \int_0^T \psi_x(0)\,\psi_{xx}(0) \\ & + \frac{L}{2}\int_0^T \psi_{xx}^2(L) - \frac{3}{2}\int_0^T \int_0^L \psi_{xx}^2 \end{aligned} \tag{5.4.28}$$

and

$$\begin{aligned} 0 =\ & \int_0^T \int_0^L x\psi_{tx}\varphi_x + \frac{1}{2}\int_0^T \int_0^L \varphi_x^2 + \frac{L}{2}\int_0^T \varphi_x^2(L) \\ & + \int_0^T \varphi_x(L)\,\varphi_{xx}(L) + \frac{L}{2}\int_0^T \varphi_{xx}^2(L) - \frac{3}{2}\int_0^T \int_0^L \varphi_{xx}^2. \end{aligned} \tag{5.4.29}$$

Therefore, adding (5.4.28) and (5.4.29) and using Young inequality, we get

$$
\begin{aligned}
\frac{3}{2}\int_0^T\int_0^L\left(\varphi_{xx}^2+\psi_{xx}^2\right) \leq & \int_0^T\int_0^L\left(x\varphi_x\psi_x\right)_t+\frac{1}{2}\int_0^T\int_0^L\left(\varphi_x^2+\psi_x^2\right)\\
&+\frac{L}{2}\int_0^T\left(\left|\varphi_{xx}(L)\right|^2+\left|\psi_{xx}(L)\right|^2\right)+\frac{L}{2}\int_0^T\left|\varphi_x(L)\right|^2\\
&+\frac{1}{2}\int_0^T\left(\left|\varphi_{xx}(L)\right|^2+\left|\varphi_x(L)\right|^2\right) \qquad (5.4.30)\\
&+\int_0^T\left(\frac{\epsilon}{2}\left|\psi_{xx}(0)\right|^2+\frac{1}{2\epsilon}\left|\psi_x(0)\right|^2\right).
\end{aligned}
$$

From (5.4.8), (5.4.6) and the embedding $\left(H^{7/4}(0,L)\right)^2 \hookrightarrow \left(H^1(0,L)\right)^2$, we can bound the last term in the inequality above, as follows

$$
\int_0^T\frac{\epsilon}{2}\left|\psi_{xx}(0)\right|^2 dt \leq \epsilon c_3\left\|\left(\varphi^0,\psi^0\right)\right\|_X^2 \leq \epsilon c_3 c_1\left\|(\varphi,\psi)\right\|_{L^2(0,T;X)}^2,
$$

and

$$
\int_0^T\left(\frac{1}{2\epsilon}\left|\psi_x(0)\right|^2+\frac{1}{2}\left|\varphi_x(L)\right|^2\right)dt \leq C\left\|(\varphi,\psi)\right\|_Y^2,
$$

for some $C>0$. Therefore, from (5.4.30), we get

$$
\begin{aligned}
\int_0^T\int_0^L\left(\varphi_{xx}^2+\psi_{xx}^2\right) \leq & C\left(\left\|\varphi_{xx}(\cdot,L)\right\|_{L^2(0,T)}^2+\left\|\psi_{xx}(\cdot,L)\right\|_{L^2(0,T)}^2\right)\\
&+C\left(\left\|(\varphi,\psi)\right\|_Y^2+\left\|\left(\varphi^0,\psi^0\right)\right\|_{(H^1(0,L))^2}^2\right), \qquad (5.4.31)
\end{aligned}
$$

for some $C>0$. Thus, (5.4.26) follows.

Now, since X is compactly embedded in $\left(H^{7/4}(0,L)\right)^2 \cap \left(H^1(0,L)\right)^2$ one may extract a subsequence of $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ converging strongly to (φ,ψ) in Y. Moreover since the embedding $X \hookrightarrow \left(H^1(0,L)\right)^2$ is compact, from (5.4.23) we deduce that $\{(\varphi_n^0,\psi_n^0)\}_{n\in\mathbb{N}}$ converges to (φ^0,ψ^0) in $\left(H^1(0,L)\right)^2$, at least for a subsequence. Denote such subsequences by the same index. From Lemma 5.9 and (5.4.24)-(5.4.26), we deduce that the sequence $\{(\varphi_n^0,\psi_n^0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Hence, it converges to $(\varphi^0,\psi^0)\in X$ and (see the proof of Theorem 5.8) $(\varphi,\psi)\in C^0([0,T];X)$ is a weak solution of (5.2.13)-(5.2.15) satisfying

$$
\varphi(0,\cdot)=\varphi^0 \text{ and } \psi(0,\cdot)=\psi^0 \qquad (5.4.32)
$$

and

$$
\left\|\left(\varphi^0,\psi^0\right)\right\|_X=1. \qquad (5.4.33)
$$

On the other hand, (5.4.7) allow us to conclude that

$$
\left(\varphi_n\right)_{xx}(\cdot,L) \longrightarrow \varphi_{xx}(\cdot,L) \text{ and } \left(\psi_n\right)_{xx}(\cdot,L) \longrightarrow \psi_{xx}(\cdot,L),
$$

as $n\to\infty$. Then, from (5.4.24) and (5.4.25) we obtain $\varphi_{xx}(\cdot,L)=\psi_{xx}(\cdot,L)=0$.

Hence, (φ,ψ) is a solution of

$$\begin{cases} \varphi_t-\psi_x-\psi_{xxx}=0 & \text{in } (0,T)\times(0,L)\,,\\ \psi_t-\varphi_x-\varphi_{xxx}=0 & \text{in } (0,T)\times(0,L)\,,\\ \varphi(0,x)=\varphi^0(x)\,,\ \ \psi(0,x)=\psi^0(x) & \text{in } (0,L)\,, \end{cases} \tag{5.4.34}$$

satisfying the boundary conditions

$$\begin{cases} \varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0 & \text{in } (0,T)\\ \psi(t,0)=\psi(t,L)=\psi_x(t,L)=0 & \text{in } (0,T)\,, \end{cases} \tag{5.4.35}$$

and, in addition,

$$\varphi_{xx}(\cdot,L)=\psi_{xx}(\cdot,L)=0. \tag{5.4.36}$$

Remark that (5.4.33) implies that the solutions of (5.4.34)-(5.4.35) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following unique continuation result:

Lemma 5.10. *Let $(\varphi^0,\psi^0)\in X$. Then if*

$$\left.\begin{array}{r} (\varphi,\psi)\ \ \textit{is a solution of (5.4.34)-(5.4.35)}\\ (\varphi,\psi)\ \ \textit{satisfies (5.4.36)} \end{array}\right\}\Rightarrow \varphi^0=\psi^0=0. \tag{5.4.37}$$

Proof. Let N_T be the space of initial data $(\varphi^0,\psi^0)\in X$ such that the corresponding solution of (5.4.34)-(5.4.35) satisfies $\varphi_{xx}(\cdot,L)=\psi_{xx}(\cdot,L)=0$ in $L^2(0,L)$. The space N_T has the following properties:

1. $\dim(N_T)<+\infty$;
2. $N_T\subset D(A)$;
3. $A(N_T)\subset N_T$, where A is the operator

$$A(\varphi,\psi)=\begin{pmatrix}\psi_x+\psi_{xxx}\\ \varphi_x+\varphi_{xxx}\end{pmatrix},\forall(\varphi,\psi)\in\mathcal{D}(A)\,,$$

with

$$\mathcal{D}(A)=\left\{(\varphi,\psi)\in\left(H^3(0,L)\right)^2;\varphi(0)=\psi(0)=\varphi(L)=\psi(L)=\varphi_x(0)=\psi_x(L)=0\right\}.$$

The proof of 1,2 and 3 is very similar the proof of Lemma 5.3, therefore it will be omitted. Thus, the unique continuation principle (5.4.37) does not hold if and only if $N_T\neq\{0\}$ or, equivalently,

$$\text{there exists } \lambda\in\mathbb{C} \text{ and } (\varphi,\psi)\in N_T, \text{ such that } (\varphi,\psi)\neq 0 \text{ and } A(\varphi,\psi)=\lambda(\varphi,\psi)\,. \tag{5.4.38}$$

Note that (5.4.38) means that there exists a nontrivial solution (φ,ψ) of the system

$$\begin{cases} -\lambda\varphi+\psi'+\psi'''=0 & \text{in } (0,T)\times(0,L)\,,\\ -\lambda\psi+\varphi'+\varphi'''=0 & \text{in } (0,T)\times(0,L)\,, \end{cases} \tag{5.4.39}$$

satisfying the boundary conditions

$$\begin{cases} \varphi(0)=\varphi(L)=\varphi'(0)=\varphi''(L)=0 & \text{in } (0,T)\\ \psi(0)=\psi(L)=\psi'(L)=\psi''(L)=0 & \text{in } (0,T)\,. \end{cases} \tag{5.4.40}$$

Thus, to conclude the proof of Lemma 5.10, the following result is needed.

Lemma 5.11. *If* (λ, φ, ψ) *is solution of (5.4.39)-(5.4.40), then*

$$\varphi = \psi = 0. \tag{5.4.41}$$

Proof. Let us remark that $\lambda = 0$ is not a solution of (5.4.39)-(5.4.40). Indeed, $\lambda = 0$ implies that (φ, ψ) is the solution of

$$\begin{cases} \psi' + \psi''' = 0 & \text{in } (0,T) \times (0,L), \\ \varphi' + \varphi''' = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{5.4.42}$$

satisfying the boundary conditions (5.4.40). But, the unique solution of this problem is $\varphi = \psi = 0$, hence $\lambda = 0$ is not a solution of (5.4.38).

Now, we introduce the Fourier transforms of φ and ψ,

$$\hat{\varphi}(\xi) = \int_0^L \varphi(x) e^{-ix\xi} dx \text{ and } \hat{\psi}(\xi) = \int_0^L \psi(x) e^{-ix\xi} dx.$$

Thus, $\hat{\varphi}$ and $\hat{\psi}$ are entire functions.

By multiplying the first and the second equations in (5.4.39) by $e^{-ix\xi}$ and integrating by parts, we obtain

$$\begin{cases} -\lambda\hat{\varphi} + \left(i\xi + (i\xi)^3\right)\hat{\psi} = (i\xi)\psi'(0) + \psi''(0), \\ -\lambda\hat{\psi} + \left(i\xi + (i\xi)^3\right)\hat{\varphi} = -(i\xi)\varphi'(L)e^{-i\xi L} + \varphi''(0). \end{cases}$$

Denote $\hat{u}(\xi) := \hat{\varphi}(\xi) + \hat{\psi}(\xi)$ and $\hat{v}(\xi) := \hat{\varphi}(\xi) - \hat{\psi}(\xi)$. Thus, setting $\lambda = ip$

$$\hat{u}(\xi) = \frac{i}{\xi^3 - \xi + p}\left((i\xi)\psi'(0) + \psi''(0) - (i\xi)\varphi'(L)e^{-i\xi L} + \varphi''(0)\right) \tag{5.4.43}$$

and

$$\hat{v}(\xi) = \frac{1}{ip - i\xi - (i\xi)^3}\left((i\xi)\psi'(0) + \psi''(0) + (i\xi)\varphi'(L)e^{-i\xi L} - \varphi''(0)\right).$$

Now, we consider the change of variable $\xi \mapsto -\xi$, to obtain that

$$\hat{v}(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(-(i\xi)\psi'(0) + \psi''(0) - (i\xi)\varphi'(L)e^{i\xi L} - \varphi''(0)\right). \tag{5.4.44}$$

Denote $\alpha = (\varphi''(0) + \psi''(0))$, $\beta = \psi'(0)$, $\nu = -\varphi'(L)$ and $\alpha' = (-\varphi''(0) + \psi''(0))$. Adding (5.4.43) and (5.4.44), we get

$$\hat{u}(\xi) + \hat{v}(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha + \alpha' + 2\nu i\xi\cos(L\xi)\right) := \frac{r_1(\xi)}{Q(\xi)}$$

and

$$\hat{u}(\xi) - \hat{v}(-\xi) = \frac{i}{\xi^3 - \xi + p}\left(\alpha - \alpha' + 2\beta i\xi + \nu i\xi(-2i)\sin(L\xi)\right) := \frac{r_2(\xi)}{Q(\xi)},$$

where are have used that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and } \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

Remark that α, β, ν and α' determine uniquely $\hat{\varphi}$ and $\hat{\psi}$ and, consequently, φ and ψ. Moreover, $\hat{u}$ and $\hat{v}$ are entire if and only if $\hat{u}(\xi) + \hat{v}(-\xi)$ and $\hat{u}(\xi) - \hat{v}(-\xi)$ are entire.

Therefore, the next steps are devoted to the analyze the roots of r_1, r_2 and Q defined above. Necessarily, $\hat{u} + \hat{v}$ and $\hat{u} - \hat{v}$ are entire if the roots of r_1, r_2 are also roots of Q. First, observe that

$$\begin{cases} \alpha + \alpha' + 2\nu i\xi \cos(L\xi) = 0 \\ \alpha - \alpha' + 2\beta i\xi + \nu i\xi(-2i)\sin(L\xi) = 0 \end{cases} \Leftrightarrow \begin{cases} 2\nu i\xi \cos(L\xi) = -(\alpha + \alpha') \\ 2\nu i\xi \sin(L\xi) = \dfrac{\alpha - \alpha'}{i} + 2\beta\xi \end{cases} \tag{5.4.45}$$

or, equivalently,

$$\begin{cases} 4(\nu i\xi)^2 \cos^2(L\xi) = (\alpha + \alpha')^2 \\ 4(\nu i\xi)^2 \sin^2(L\xi) = -(\alpha - \alpha')^2 + 4\beta^2\xi^2 + \dfrac{4}{i}\beta\xi(\alpha - \alpha'). \end{cases} \tag{5.4.46}$$

Therefore, adding the identities in (5.4.46) and using the basic relation $\cos^2(L\xi) + \sin^2(L\xi) = 1$, we obtain that

$$\left(\beta^2 + \nu^2\right)\xi^2 + \frac{(\alpha - \alpha')}{i}\beta\xi + \alpha\alpha' = 0. \tag{5.4.47}$$

Taking (5.4.47) into account, we obtain a contradiction. Indeed, (5.4.47) allows us to conclude that r_1 and r_2 have, at least, two roots, unless

$$\begin{cases} \beta^2 + \nu^2 &= 0, \\ \dfrac{(\alpha - \alpha')}{i}\beta &= 0, \\ \alpha\alpha' &= 0. \end{cases} \tag{5.4.48}$$

From the first equation of (5.4.48) we have that $\beta = \pm i\nu$. We analyze the case $\beta = i\nu$, since the case $\beta = -i\nu$ is similar. If $\beta = i\nu$, by the second equation of (5.4.48), it follows that

$$(\alpha - \alpha')\nu = 0. \tag{5.4.49}$$

If $\nu = 0$, then $\beta = 0$ and, therefore, $\psi'(0) = \varphi'(L) = 0$. Thus, (φ, ψ) is a solution of the system

$$\begin{cases} -\lambda\varphi + \psi' + \psi''' = 0 & \text{in } (0,T)\times(0,L), \\ -\lambda\psi + \varphi' + \varphi''' = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.4.50}$$

satisfying the boundary conditions

$$\begin{cases} \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = \varphi''(L) = 0 & \text{in } (0,T) \\ \psi(0) = \psi(L) = \psi'(L) = \psi'(0) = \psi''(L) = 0 & \text{in } (0,T). \end{cases} \tag{5.4.51}$$

Then, $(\varphi, \psi) \equiv 0$, for all $L > 0$, which concludes the proof.

Now, we consider the case $\nu \neq 0$. Thus, from (5.4.49) we get $\alpha = \alpha'$ and from the third equation of (5.4.48) its follows that $\alpha = \alpha' = 0$. Returning to the system (5.4.45) the following identities holds

$$\begin{cases} \xi \cos(L\xi) = 0, \\ 2i^2\xi - 2i^2\xi \sin(L\xi) = 0 \end{cases} \Leftrightarrow \begin{cases} \xi \cos(L\xi) = 0, \\ \xi \left(1 - \sin(L\xi)\right) = 0. \end{cases} \tag{5.4.52}$$

The roots of (5.4.52) are $\xi = 0$ or $L\xi = \dfrac{\pi}{2} + 2k\pi$, $k \in \mathbb{N}^*$.

Recall that the idea we have in mind is to check if the roots of r_1 and r_2 are the roots of Q, i.e.,

$$\begin{cases} Q(0) = 0, \\ Q\left(\dfrac{\pi}{2L} + \dfrac{k\pi}{L}\right) = 0. \end{cases} \tag{5.4.53}$$

If $Q(0) = 0$, then, necessary $p = 0$ and, therefore $Q(\xi) = \xi(\xi + 1)(\xi - 1)$. Thus, the roots of $Q(\xi)$ are $\mu_0 = 0$, $\mu_1 = 1$ and $\mu_2 = -1$. We will show that these roots can not be simultaneously roots of (5.4.52). Indeed, replacing $\xi = \mu_1 = 1$ we have

$$\begin{cases} \cos(L) = 0, \\ \sin(L) = 1. \end{cases}$$

Then,

$$L = \frac{\pi}{2} + 2k_1\pi, \text{ for } k_1 \in \mathbb{N}^*. \tag{5.4.54}$$

If $\xi = \mu_2 = -1$, then

$$\begin{cases} \cos(-L) = 0, \\ \sin(-L) = 1, \end{cases}$$

i.e.,

$$-L = -\frac{\pi}{2} + 2k_2\pi, \text{ for } k_2 \in \mathbb{N}^*, \tag{5.4.55}$$

which contradicts (5.4.54).

Lastly, we consider $L\xi = \dfrac{\pi}{2} + k\pi$, $k \in \mathbb{N}^*$. Then,

$$L\mu_i = \frac{\pi}{2} + k_i\pi, k_i \in \mathbb{N}^*, \text{ for } i = 0, 1, 2, \tag{5.4.56}$$

where μ_i are the roots of $Q(\xi)$. Observing that such roots satisfy

$$\mu_0 + \mu_1 + \mu_2 = 0,$$

it follows that

$$\frac{3\pi}{2} + \pi(k_0 + k_1 + k_2) = 0,$$

which is not possible, since k_1,k_2,$k_3 \in \mathbb{N}^*$. Then, (5.4.56) not are root of $Q(\xi)$.

Thus, $\nu = 0$ and (5.4.50)-(5.4.51) holds, for all $L > 0$. This allow us to conclude that $(\varphi, \psi) = (0, 0)$. This complete the proofs of the Lemmas 5.10 and 5.11 and the Theorem 5.13.

□

The following theorem solves the control problem (5.4.1)-(5.4.2) with $h_0 = g_0 = 0$:

Theorem 5.14. *Let $T > 0$ and $L > 0$. Then, the system (5.4.1)-(5.4.2) is exactly controllable, with $h_0 = g_0 = 0$, in time T.*

Proof. Let us define the following functional

$$\Lambda\left(\varphi^1, \psi^1\right) = \frac{1}{2}\left(\|\varphi_{xx}(\cdot, L)\|^2_{L^2(0,T)} + \|\psi_{xx}(\cdot, L)\|^2_{L^2(0,T)}\right) - \int_0^L \left(\eta^1(x)\varphi^1(x) + w^1(x)\psi^1(x)\right)dx \tag{5.4.57}$$

where $(\varphi^1, \psi^1) \in X$ and (φ, ψ) is the solution of the backward system (5.2.10)-(5.2.11) with initial data (φ^1, ψ^1).

Let $\left(\hat{\varphi}^1, \hat{\psi}^1\right) \in X$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.4.4) is satisfied with $h_1 = \psi_{xx}(t, L) \in L^2(0, T)$ and $g_1 = \varphi_{xx}(t, L) \in L^2(0, T)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.4.4) and (5.4.5), holds that

$$\begin{aligned}\Lambda\left(\varphi^1, \psi^1\right) &= \frac{1}{2}\left(\|\varphi_{xx}(\cdot, L)\|^2_{L^2(0,T)} + \|\psi_{xx}(\cdot, L)\|^2_{L^2(0,T)}\right) \\ &\quad - \int_0^L \left(\eta^1(x)\varphi^1(x) + w^1(x)\psi^1(x)\right)dx \\ &\geq C^{-1}\left\|\left(\varphi^1, \psi^1\right)\right\|_X.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.14 is complete. □

5.4.2 Single control in L

In this section we consider (5.4.2) with $h_0 = g_0 = h_1 = 0$. The following lemma gives an equivalent condition for the exact controllability property:

Lemma 5.12. *Let $(\eta^1, w^1) \in X'$. Then, there exist one control $g_1(t) \in L^2(0, L)$, such that the solution (η, w) of (5.4.1)-(5.4.2) with $h_0 = g_0 = h_1 = 0$ satisfies (5.3.3) if and only if*

$$\left\langle\left(\eta^1, w^1\right), \left(\varphi^1, \psi^1\right)\right\rangle_{X' \times X} = -\int_0^T g_1(t)\varphi_{xx}(t, L)\,dt, \tag{5.4.58}$$

for any $(\varphi^1, \psi^1) \in X$, (φ, ψ) being the solution of the backward system (5.2.10)-(5.2.12).

Proof. The relation (5.4.58) is obtained multiplying (5.4.1) by the solution (φ, ψ) of (5.2.10)-(5.2.12) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result

Theorem 5.15. *Let*

$$\mathcal{N}=\left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}}:k,l\in\mathbb{N}^*\right\} \tag{5.4.59}$$

and

$$\mathcal{R}=\left\{\pi\sqrt{\left(\frac{1}{2}+2k\right)^2+\left(\frac{1}{2}+2l\right)^2+\left(\frac{1}{2}+2k\right)\left(\frac{1}{2}+2l\right)}:k,l\in\mathbb{N}^*\right\}. \tag{5.4.60}$$

Then, $\forall\ L\in(0,+\infty)\setminus(\mathcal{N}\cup\mathcal{R})$ and $\forall\ T>0$, $\exists C(T,L)>0$ such that

$$\left\|\left(\varphi^1,\psi^1\right)\right\|_X^2\leq C\int_0^T\left|\varphi_{xx}(t,L)\right|^2dt, \tag{5.4.61}$$

for any $(\varphi^1,\psi^1)\in X$, where (φ,ψ) is the solution of (5.2.10)-(5.2.12) with initial data (φ^1,ψ^1).

Proof. The change of variable $t=T-t$ transforms (5.2.10)-(5.2.12) into (5.2.13)-(5.2.15). Hence, inequality (5.4.61) is equivalent to

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_X^2\leq C\int_0^T\left|\varphi_{xx}(t,L)\right|^2dt, \tag{5.4.62}$$

for any $(\varphi^0,\psi^0)\in X$, where (φ,ψ) is the solution of (5.2.13)-(5.2.15) with initial data (φ^0,ψ^0).

We assume that (5.4.62) not is true. Then, there exists a sequence $\{(\varphi_n^0,\psi_n^0)\}_{n\in\mathbb{N}}\in X$ such that

$$\left\|\left(\varphi_n^0,\psi_n^0\right)\right\|_X=1 \tag{5.4.63}$$

and

$$\left\|\varphi_{n,xx}(\cdot,L)\right\|_{L^2(0,T)}^2\longrightarrow 0\text{ as }n\to\infty, \tag{5.4.64}$$

where $(\varphi_n,\psi_n)\in Z$ is solution of (5.2.13)-(5.2.15) with initial data (φ_n^0,ψ_n^0). Let us denote

$$Y:=L^2\left(0,T;\left(H^{7/4}(0,L)\right)^2\right)\cap C^0\left([0,T];\left(H^1(0,L)\right)^2\right).$$

We show that there exist a positive constant c_1 such that: $\forall(\varphi,\psi)\in Z$ solution of (5.2.13)-(5.2.15), one has

$$\|(\varphi,\psi)\|_Z^2\leq c_1\left(\|\varphi_{xx}(\cdot,L)\|_{L^2(0,T)}^2+\|(\varphi,\psi)\|_Y^2+\left\|\left(\varphi^0,\psi^0\right)\right\|_{(H^1(0,L))^2}\right). \tag{5.4.65}$$

For that purpose, we use the multiplier method. Multiplying the first equation of (5.2.13) by $(x-L)\psi_{xx}$ and the second one by $(x-L)\varphi_{xx}$ and integrating in $(0,T)\times(0,L)$, we obtain

$$\int_0^T\int_0^L(x-L)\psi_{xx}\left(\varphi_t-\psi_x-\psi_{xxx}\right)dxdt=0 \tag{5.4.66}$$

and

$$\int_0^T \int_0^L (x-L)\,\varphi_{xx}\,(\psi_t-\varphi_x-\varphi_{xxx})\,dxdt=0. \tag{5.4.67}$$

We first analyze the terms in (5.4.66):

$$\begin{aligned}\int_0^T \int_0^L (x-L)\,\psi_{xx}\varphi_t &= \int_0^T (x-L)\,\psi_x\varphi_t|_0^L - \int_0^T \int_0^L \psi_x\varphi_t - \int_0^L \int_0^T (x-L)\,\psi_x\varphi_{tx}\\ &= \int_0^T \int_0^L |\psi_x|^2 + \int_0^T \int_0^L \psi_x\psi_{xxx} - \int_0^L \int_0^T (x-L)\,\psi_x\varphi_{tx}\\ &= \int_0^T \int_0^L |\psi_x|^2 - \int_0^T \int_0^L |\psi_{xx}|^2 - \int_0^T \psi_x\,(0)\,\psi_{xx}\,(0) \qquad (5.4.68)\\ &- \int_0^L \int_0^T (x-L)\,\psi_x\varphi_{tx},\end{aligned}$$

$$\begin{aligned}-\int_0^T \int_0^L (x-L)\,\psi_{xx}\psi_x &= -\int_0^T \left(\frac{x-L}{2}\right)|\psi_x|^2\Big|_0^L + \frac{1}{2}\int_0^T \int_0^L |\psi_x|^2\\ &= -\frac{L}{2}\int_0^T \psi_x^2\,(0) + \frac{1}{2}\int_0^T \int_0^L |\psi_x|^2 \qquad (5.4.69)\end{aligned}$$

and

$$\begin{aligned}-\int_0^T \int_0^L (x-L)\,\psi_{xx}\psi_{xxx} &= -\int_0^T \left(\frac{x-L}{2}\right)|\psi_{xx}|^2\Big|_0^L + \frac{1}{2}\int_0^T \int_0^L |\psi_{xx}|^2\\ &= -\frac{L}{2}\int_0^T \psi_{xx}^2\,(0) + \frac{1}{2}\int_0^T \int_0^L |\psi_{xx}|^2. \qquad (5.4.70)\end{aligned}$$

Replacing (5.4.68)-(5.4.70) into (5.4.66) it follows that

$$\begin{aligned}\frac{1}{2}\int_0^T \int_0^L |\psi_{xx}|^2\,dxdt &= -\frac{L}{2}\int_0^T |\psi_{xx}\,(t,0)|^2\,dt - \frac{L}{2}\int_0^T |\psi_x\,(t,0)|^2\,dt\\ -\int_0^T \psi_x\,(t,0)\,\psi_{xx}\,(t,0)\,dt &+ \frac{3}{2}\int_0^T \int_0^L |\psi_x|^2\,dxdt - \int_0^T \int_0^L (x-L)\,\psi_x\varphi_{tx}dxdt.\end{aligned} \tag{5.4.71}$$

Now, we analyze (5.4.67):

$$\begin{aligned}\int_0^T \int_0^L (x-L)\,\varphi_{xx}\psi_t &= \int_0^T (x-L)\,\varphi_x\psi_t|_0^L - \int_0^T \int_0^L \varphi_x\psi_t - \int_0^L \int_0^T (x-L)\,\varphi_x\psi_{tx}\\ &= \int_0^T \int_0^L |\varphi_x|^2 - \int_0^T \int_0^L |\varphi_{xx}|^2 + \int_0^T \varphi_x\,(L)\,\varphi_{xx}\,(L) \qquad (5.4.72)\\ &- \int_0^L \int_0^T (x-L)\,\varphi_x\psi_{tx},\end{aligned}$$

$$-\int_0^T\int_0^L (x-L)\,\psi_{xx}\psi_x = -\int_0^T \left(\frac{x-L}{2}\right)|\varphi_x|^2\Big|_0^L + \frac{1}{2}\int_0^T\int_0^L |\varphi_x|^2$$
$$= \frac{1}{2}\int_0^T\int_0^L |\varphi_x|^2 \tag{5.4.73}$$

and

$$-\int_0^T\int_0^L (x-L)\,\varphi_{xx}\varphi_{xxx} = -\int_0^T \left(\frac{x-L}{2}\right)|\varphi_{xx}|^2\Big|_0^L + \frac{1}{2}\int_0^T\int_0^L |\varphi_{xx}|^2$$
$$= -\frac{L}{2}\int_0^T |\varphi_{xx}(0)|^2 + \frac{1}{2}\int_0^T\int_0^L |\varphi_{xx}|^2. \tag{5.4.74}$$

Replacing (5.4.72)-(5.4.74) into (5.4.67) we get

$$\begin{aligned} \frac{1}{2}\int_0^T\int_0^L |\varphi_{xx}|^2\,dxdt &= -\frac{L}{2}\int_0^T |\varphi_{xx}(t,0)|^2\,dt + \int_0^T \varphi_x(t,L)\,\varphi_{xx}(t,L)\,dt \\ &\quad + \frac{3}{2}\int_0^T\int_0^L |\varphi_x|^2\,dxdt - \int_0^T\int_0^L (x-L)\,\varphi_x\psi_{tx}dxdt. \end{aligned} \tag{5.4.75}$$

Therefore, adding (5.4.71) and (5.4.75) and using Young inequality, is follows that

$$\begin{aligned} \frac{1}{2}\int_0^T\int_0^L \left(|\varphi_{xx}|^2 + |\psi_{xx}|^2\right)dxdt &\le \int_0^T \left(\frac{1}{2\epsilon}\psi_x(t,0) + \frac{\epsilon}{2}\psi_{xx}(t,0)\right)dt \\ &\quad + \frac{1}{2}\int_0^T \left(|\varphi_x(t,L)|^2 + |\varphi_{xx}(t,L)|^2\right)dt + \frac{3}{2}\int_0^T\int_0^L |\varphi_x|^2\,dxdt \\ &\quad + \frac{3}{2}\int_0^T\int_0^L |\psi_x|^2\,dxdt - \int_0^T\int_0^L \left((x-L)\,\psi_x\varphi_x\right)_t dxdt. \end{aligned} \tag{5.4.76}$$

Arguing as in (5.4.30) and (5.4.31), (5.4.65) follows.

Hence, proceeding as in the proof of the Theorem 5.13 we obtain (φ,ψ) solution of

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \varphi(0,x) = \varphi^0(x), \ \ \psi(0,x) = \psi^0(x) & \text{in } (0,L), \end{cases} \tag{5.4.77}$$

satisfying

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_X = 1, \tag{5.4.78}$$

$$\begin{cases} \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0 & \text{in } (0,T), \\ \psi(t,0) = \psi(t,L) = \psi_x(t,L) = 0 & \text{in } (0,T) \end{cases} \tag{5.4.79}$$

and

$$\varphi_{xx}(\cdot,L) = 0. \tag{5.4.80}$$

Remark that (5.4.78) implies that the solutions of (5.4.77)-(5.4.80) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following result:

Lemma 5.13. *For any $T > 0$ let N_T denote the space of the initial states $(\varphi^0, \psi^0) \in X$ such that the solution $(\varphi, \psi) = S(\cdot)(\varphi^0, \psi^0)$ of (5.4.77)-(5.4.79) satisfies (5.4.80). Then, for $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$, $N_T = \{0\}$, $\forall T > 0$.*

Proof. Let A be the operator

$$A(\varphi, \psi) = \begin{pmatrix} \psi_x + \psi_{xxx} \\ \varphi_x + \varphi_{xxx} \end{pmatrix}, \forall (\varphi, \psi) \in \mathcal{D}(A),$$

with

$$\mathcal{D}(A) = \left\{ (\varphi, \psi) \in \left(H^3(0, L)\right)^2 ; \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(L) = 0 \right\}.$$

Using the same argument of the proof of Lemma 5.3, N_T verifies

1. $\dim(N_T) < +\infty$;
2. $N_T \subset D(A)$;
3. $A(N_T) \subset N_T$.

If $N_T \neq \{0\}$, the map $(\varphi^0, \psi^0) \in \mathbb{C}N_T \longrightarrow A((\varphi^0, \psi^0)) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue. Hence, there exist $\lambda \in \mathbb{C}$, $(\varphi^0, \psi^0) \in (H^3(0, L))^2 \setminus \{(0, 0)\}$ such that

$$\begin{cases} \lambda\varphi^0 = \left[(\psi^0)' + (\psi^0)'''\right], \\ \lambda\psi^0 = \left[(\varphi^0)' + (\varphi^0)'''\right], \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))'' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = 0. \end{cases} \tag{5.4.81}$$

To conclude the proof of Lemma 5.13, we prove that this does not hold if $L \notin \mathcal{N}$.

Lemma 5.14. *Let $L > 0$. Consider the assertion*

$$(\mathcal{F}_2) \qquad \exists \lambda \in \mathbb{C},\ \exists \left(\varphi^0, \psi^0\right) \in \left(H^3(0, L)\right)^2 \setminus \{(0, 0)\} \text{ such that}$$

$$\begin{cases} \lambda\varphi^0 = \left[(\psi^0)' + (\psi^0)'''\right], \\ \lambda\psi^0 = \left[(\varphi^0)' + (\varphi^0)'''\right], \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))'' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = 0. \end{cases}$$

Then, $(\mathcal{F}_2)$ holds if and only if $L \in \mathcal{N} \cup \mathcal{R}$.

Proof. We use an argument similar to the one used in [62, Lema 3.5]. Assume that (φ^0, ψ^0) satisfies $(\mathcal{F}_2)$ and let us denote by $(\varphi, \psi) \in (H^2(\mathbb{R}))^2$ its prolongation by 0 to $\mathbb{R}$. Then,

$$\begin{cases} -\lambda\varphi + \psi' + \psi''' = \psi''(0)\delta_0 + \psi'(0)(\delta_0)' - \psi''(L)\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda\psi + \varphi' + \varphi''' = \varphi''(0)\delta_0 - \varphi'(L)(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}), \end{cases} \tag{5.4.82}$$

where δ_{x_0} and $(\delta_{x_0})'$ denote the Dirac measure at x_0. Note that the $(\mathcal{F}_2)$ is equivalent to the existence of complex numbers α, α', β, γ, γ', λ with $(\alpha, \alpha', \beta, \gamma, \gamma') \neq (0, 0, 0, 0, 0)$

and $(\varphi,\psi)\in\left(H^2(\mathbb{R})\right)^2$ with compact support in $[-L,L]$ satisfying

$$\begin{cases} -\lambda(\varphi+\psi)+(\psi+\varphi)'+(\psi+\varphi)'''=\alpha\delta_0+\beta(\delta_0)'+\gamma\delta_L+\gamma'(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda(\varphi-\psi)+(\varphi-\psi)'+(\varphi-\psi)'''=\alpha'\delta_0-\beta(\delta_0)'+\gamma\delta_L+\gamma'(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases} \tag{5.4.83}$$

We introduce the notation $\hat{\varphi}(\xi)=\int_0^L\varphi(\xi)e^{-ix\xi}dx$ and $\hat{\psi}(\xi)=\int_0^L\psi(\xi)e^{-ix\xi}dx$. Then, taking the Fourier transform in (5.4.82) we obtain

$$-\lambda\hat{\varphi}(\xi)+(i\xi)\hat{\psi}(\xi)+(i\xi)^3\hat{\psi}(\xi)+\psi''(L)e^{-iL\xi}-\psi''(0)-(i\xi)\psi'(0)=0 \tag{5.4.84}$$

and

$$-\lambda\hat{\psi}(\xi)+(i\xi)\hat{\varphi}(\xi)+(i\xi)^3\hat{\varphi}(\xi)-\varphi''(0)+(i\xi)\varphi'(L)e^{-iL\xi}=0. \tag{5.4.85}$$

Adding (5.4.84) and (5.4.85) it follows that

$$\begin{aligned}\left(-\lambda+(i\xi)+(i\xi)^3\right)\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)=&\left(\varphi''(0)+\psi''(0)\right)+(i\xi)\psi'(0)\\ &-\psi''(L)e^{-iL\xi}-(i\xi)\varphi'(L)e^{-iL\xi}.\end{aligned}$$

Then, we denote

$$\hat{u}(\xi):=\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)=\frac{\left(\varphi''(0)+\psi''(0)\right)+(i\xi)\psi'(0)-\psi''(L)e^{-iL\xi}-(i\xi)\varphi'(L)e^{-iL\xi}}{-\lambda+(i\xi)+(i\xi)^3}. \tag{5.4.86}$$

Now, taking the difference between (5.4.84) and (5.4.85) the following identity holds

$$\begin{aligned}\left(-\lambda-(i\xi)-(i\xi)^3\right)\left(\hat{\varphi}(\xi)-\hat{\psi}(\xi)\right)=&\left(-\varphi''(0)+\psi''(0)\right)+(i\xi)\psi'(0)\\ &-\psi''(L)e^{-iL\xi}+(i\xi)\varphi'(L)e^{-iL\xi}.\end{aligned}$$

We denote

$$\hat{v}(\xi):=\left(\hat{\varphi}(\xi)-\hat{\psi}(\xi)\right)=\frac{\left(-\varphi''(0)+\psi''(0)\right)+(i\xi)\psi'(0)-\psi''(L)e^{-iL\xi}+(i\xi)\varphi'(L)e^{-iL\xi}}{-\lambda-(i\xi)-(i\xi)^3}.$$

In the identity above, consider the change of variable $\xi\longmapsto-\xi$ and the new function

$$\hat{v}(-\xi)=\frac{\left(-\varphi''(0)+\psi''(0)\right)-(i\xi)\psi'(0)-\psi''(L)e^{iL\xi}-(i\xi)\varphi'(L)e^{iL\xi}}{-\lambda+(i\xi)+(i\xi)^3}. \tag{5.4.87}$$

Setting $\lambda=ip$ it is possible to write (5.4.86) and (5.4.87) as

$$\hat{u}(\xi)=i\frac{\alpha+(i\xi)\beta+\gamma e^{-iL\xi}+\gamma'(i\xi)e^{-iL\xi}}{\xi^3-\xi+p} \tag{5.4.88}$$

and

$$\hat{v}(-\xi)=i\frac{\alpha'-(i\xi)\beta+\gamma e^{iL\xi}+\gamma'(i\xi)e^{iL\xi}}{\xi^3-\xi+p}. \tag{5.4.89}$$

Using Paley-Wiener theorem (see [79, Section 4, page 161]) and the usual characterization of $H^2(\mathbb{R})$ by means of their Fourier transforms we see that $(\mathcal{F}_2)$ is equivalent to the existence of $p \in \mathbb{C}$ and

$$(\alpha, \alpha', \beta, \gamma, \gamma') \in \mathbb{C}^5 \setminus (0,0,0,0,0)$$

such that

$$f(\xi) := \frac{\alpha + (i\xi)\beta + \gamma e^{-iL\xi} + \gamma'(i\xi) e^{-iL\xi}}{\xi^3 - \xi + p}$$

and

$$g(\xi) := \frac{\alpha' - (i\xi)\beta + \gamma e^{iL\xi} + \gamma'(i\xi) e^{iL\xi}}{\xi^3 - \xi + p}$$

satisfies

a) f and g are entire function in $\mathbb{C}$;

b) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2 \left(1+|\xi|^2\right)^2 d\xi < \infty$ and $\displaystyle\int_{\mathbb{R}} |g(\xi)|^2 \left(1+|\xi|^2\right)^2 d\xi < \infty$;

c) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c(1+|\xi|)^k \exp(l \operatorname{Im} \xi)$ for some positive constants c and k.

Remark that f and g are entire if only if the roots μ_0, μ_1 and μ_2 of $Q(\xi) := \xi^3 - \xi + p$ are roots of

$$r_1(\xi) := \alpha + (i\xi)\beta + \gamma e^{-iL\xi} + \gamma'(i\xi) e^{-iL\xi} \tag{5.4.90}$$

and

$$r_2(\xi) := \alpha' - (i\xi)\beta + \gamma e^{iL\xi} + \gamma'(i\xi) e^{iL\xi}. \tag{5.4.91}$$

In particular, f and g are entire if and only if $f+g$ and $f-g$ are entire, where

$$\begin{cases} f+g := (\alpha+\alpha') + 2(\gamma + \gamma' i\xi)\cos(L\xi) \\ f-g := (\alpha-\alpha') + 2(i\xi)\beta + (\gamma + \gamma'(i\xi))(-2i)\sin(L\xi). \end{cases} \tag{5.4.92}$$

Here we use that $\cos y = \dfrac{e^{iy}+e^{-iy}}{2}$ and $\sin y = \dfrac{e^{iy}-e^{-iy}}{2i}$.

If the roots of (5.4.92) are simple, a) holds if the roots of $\xi^3 - \xi + p$ are simples and also roots of (5.4.92). Observe that if a) is true, then b) and c) are satisfied. We find the roots of (5.4.92) and prove that they are simple:

$$\begin{cases} 2(\gamma + \gamma' i\xi)\cos(L\xi) = -(\alpha+\alpha') \\ 2(\gamma + \gamma' i\xi)\sin(L\xi) = \dfrac{(\alpha-\alpha')}{i} + 2\xi\beta \end{cases}$$

or, equivalently,

$$\begin{cases} 4(\gamma + \gamma' i\xi)^2\cos^2(L\xi) = (\alpha+\alpha')^2 \\ 4(\gamma + \gamma' i\xi)^2\sin^2(L\xi) = -(\alpha-\alpha')^2 + 4\xi^2\beta^2 + \dfrac{4}{i}\beta\xi(\alpha-\alpha'). \end{cases} \tag{5.4.93}$$

Therefore, adding the identities in (5.4.93) and using the basic relation $\cos^2(L\xi) + \sin^2(L\xi) = 1$, we have

$$\left(\beta^2 + (\gamma')^2\right)\xi^2 + \left(\frac{(\alpha-\alpha')}{i}\beta - 2\gamma\gamma' i\right)\xi + \left(\alpha\alpha' - \gamma^2\right) = 0. \tag{5.4.94}$$

Taking (5.4.94) into account, we obtain a contradiction. Indeed, (5.4.94) allow us to conclude that (5.4.92), at least, two roots, unless

$$\left\{ \begin{aligned} \beta^2 + (\gamma')^2 = 0, \\ \frac{(\alpha - \alpha')}{i}\beta - 2\gamma\gamma' i = 0, \\ \alpha\alpha' - \gamma^2 = 0. \end{aligned} \right. \tag{5.4.95}$$

From the first equation of (5.4.95), we obtain $\beta = i\gamma'$ and $\beta = -i\gamma'$.

We analyze the first case, since the second is analogous and it will be omitted. If $\beta = i\gamma'$, the second equation of (5.4.95) give us that

$$\frac{(\alpha - \alpha')}{i}\beta - 2\gamma\gamma' i = 0 \Rightarrow ((\alpha - \alpha') - 2\gamma i)\,\gamma' = 0. \tag{5.4.96}$$

Now, we consider two cases:
a) If $\gamma' = 0$, then $\beta = 0$. Thus, from (5.4.90) and (5.4.91) we have

$$f(\xi) = \frac{\alpha + \gamma e^{-iL\xi}}{\xi^3 - \xi + p}$$

and

$$g(\xi) = \frac{\alpha' + \gamma e^{iL\xi}}{\xi^3 - \xi + p}.$$

Then, [62, Lema 3.5] ensures that $\gamma' = \beta = 0$ and, $(\mathcal{F}_2)$ holds if and only if $L \in \mathcal{N}$.

b) If $\gamma' \neq 0$, the third equation of (5.4.95) and (5.4.96) give us that (observe that $\alpha, \alpha' \neq 0$)

$$\left\{ \begin{aligned} \alpha - \alpha' = 2\gamma i \\ \alpha\alpha' = \gamma^2 \end{aligned} \right. \Leftrightarrow \left\{ \begin{aligned} \alpha = \alpha' + 2\gamma i, \\ \alpha = \frac{\gamma^2}{\alpha'}. \end{aligned} \right.$$

Thus,

$$\frac{\gamma^2}{\alpha'} = \alpha' + 2\gamma i \Leftrightarrow (\alpha' + i\gamma)^2 = 0$$

and, therefore

$$\alpha' = -i\gamma \text{ and } \alpha = i\gamma. \tag{5.4.97}$$

Returning to (5.4.92) and replacing (5.4.97), $f + g$ can be written as

$$f + g = 2(\gamma + \gamma' i\xi)\cos(L\xi). \tag{5.4.98}$$

Let us find the roots of $f + g$. Note that if $2(\gamma + \gamma' i\xi)\cos(L\xi) = 0$, then

$$\xi = -\frac{\gamma}{i\gamma'} \tag{5.4.99}$$

or

$$L\xi = \frac{\pi}{2} + k\pi,\ k \in \mathbb{N}^*. \tag{5.4.100}$$

As previously mentioned we suppose that, the roots of $f+g$ are (also) roots of $Q(\xi)$. Then, we consider μ_0, μ_1 and μ_2 three simple roots of $Q(\xi)$ and introduce

$$\Delta := \{\mu_0, \mu_1, \mu_2 : \mu_i \text{ is root of } Q(\xi)\,, \text{ for } i = 0,1,2\}\,.$$

We consider two cases:

I) $-\dfrac{\gamma}{i\gamma'} \notin \Delta$;

II) $-\dfrac{\gamma}{i\gamma'} \in \Delta$.

The case I ensures that

$$L\mu_i = \frac{\pi}{2} + k_i\pi \in \Delta, \tag{5.4.101}$$

with $k_i \in \mathbb{N}^*$, for $i = 0,1,2$. Then, μ_0, μ_1 and μ_2 satisfies

$$L(\mu_1 - \mu_0) = (k_1 - k_0)\,\pi$$

and

$$L(\mu_2 - \mu_1) = (k_2 - k_1)\,\pi.$$

Denoting $l_1 = k_1 - k_0$ and $l_2 = k_2 - k_1$, from the relations above, we get

$$\mu_1 = \frac{l_1\pi}{L} + \mu_0,\ l_1 \in \mathbb{N}^* \tag{5.4.102}$$

and

$$\mu_2 = \frac{l_2\pi}{L} + \mu_1 = \frac{(l_1 + l_2)\,\pi}{L} + \mu_0,\ l_1, l_2 \in \mathbb{N}^*. \tag{5.4.103}$$

On the other hand, we know that

$$(\xi - \mu_0)(\xi - \mu_1)(\xi - \mu_2) = \xi^3 - \xi + p,$$

since we are assuming that μ_0, μ_1 and μ_2 are simple roots of Q. Thus,

$$\mu_0 + \mu_1 + \mu_2 = 0, \tag{5.4.104}$$

$$\mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2 = -1, \tag{5.4.105}$$

$$\mu_0\mu_1\mu_2 = p. \tag{5.4.106}$$

From (5.4.102), (5.4.103) and (5.4.104) it follows that

$$3\mu_0 + \left(\frac{l_2 + 2l_1}{L}\right)\pi = 0 \Rightarrow L\mu_0 = -\left(\frac{2l_1 + l_2}{3}\right)\pi, \text{ for } l_1, l_2 \in \mathbb{N}^*. \tag{5.4.107}$$

But if the roots of $Q(\xi)$ has the form (5.4.101), from (5.4.107) we obtain

$$-\left(\frac{2l_1 + l_2}{3}\right)\pi = \frac{\pi}{2} + k_0.$$

This equality does not hold since l_1 and l_2 are natural numbers different from zero. Therefore, we conclude that the roots of $Q(\xi)$ are not given by (5.4.100).

The case II ensures exactly the opposite situation, that is, $-\dfrac{\gamma}{i\gamma'} \in \Delta$. Remember that we are analyzing the case in which $\beta = i\gamma'$, thus (5.4.97) holds. Thus,

$$f + g = 2\left(\gamma + \gamma' i\xi\right)\cos\left(L\xi\right) \tag{5.4.108}$$

and

$$f - g = 2i\left(\gamma + \gamma' i\xi\right)\left(1 - \sin\left(L\xi\right)\right). \tag{5.4.109}$$

If we denote

$$\mu_0 = -\frac{\gamma}{i\gamma'}$$

and μ_0, μ_1 and μ_2 the simple roots of $Q\left(\xi\right)$, then

$$\mu_1 \neq \mu_2 \neq \mu_0. \tag{5.4.110}$$

On the other hand, μ_1 and μ_2 must be the roots of (5.4.108) and (5.4.109), i.e.

$$\begin{cases} 2\left(\gamma + \gamma' i\mu_1\right)\cos\left(L\mu_1\right) = 0, \\ 2i\left(\gamma + \gamma' i\mu_1\right)\left(1 - \sin\left(L\mu_1\right)\right) = 0, \end{cases}$$

and

$$\begin{cases} 2\left(\gamma + \gamma' i\mu_2\right)\cos\left(L\mu_2\right) = 0, \\ 2i\left(\gamma + \gamma' i\mu_2\right)\left(1 - \sin\left(L\mu_2\right)\right) = 0. \end{cases}$$

But from (5.4.110),

$$\begin{cases} \cos\left(L\mu_1\right) = 0, \\ \sin\left(L\mu_1\right) = 1, \end{cases}$$

and

$$\begin{cases} \cos\left(L\mu_2\right) = 0, \\ \sin\left(L\mu_2\right) = 1. \end{cases}$$

From these relations it follows that

$$L\mu_1 = \frac{\pi}{2} + 2k_1\pi,\ k_1 \in \mathbb{N}^* \Leftrightarrow \mu_1 = \frac{\pi}{L}\left(\frac{1}{2} + 2k_1\right),\ k_1 \in \mathbb{N}^*$$

and

$$L\mu_2 = \frac{\pi}{2} + 2k_2\pi,\ k_2 \in \mathbb{N}^* \Leftrightarrow \mu_2 = \frac{\pi}{L}\left(\frac{1}{2} + 2k_2\right),\ k_2 \in \mathbb{N}^*.$$

Note that (5.4.104) ensure that $\mu_0 = -\mu_1 - \mu_2$ and from (5.4.105) we obtain the relation

$$-\left(\mu_1 + \mu_2\right)^2 + \mu_1\mu_2 = -1 \Leftrightarrow \mu_1^2 + \mu_2^2 + \mu_1\mu_2 = 1. \tag{5.4.111}$$

Now, replacing μ_1 and μ_2 in (5.4.111) we obtain

$$L = \pi\sqrt{\left(\frac{1}{2} + 2k_1\right)^2 + \left(\frac{1}{2} + 2k_2\right)^2 + \left(\frac{1}{2} + 2k_1\right)\left(\frac{1}{2} + 2k_2\right)}$$

with $k_1, k_2 \in \mathbb{N}^*$ and $k_1 \neq k_2$.

Again using the relation (5.4.104), replacing μ_1 and μ_2, we have that

$$\mu_0 = -\frac{\pi}{L}\left(1 + 2\left(k_1 + k_2\right)\right),$$

with $\mu_0 \neq \mu_i$, for $i = 1, 2$. Therefore, chosen γ and γ' such that

$$-\frac{\gamma}{i\gamma'} = \mu_0,$$

$(\mathcal{F}_2)$ holds if and only if $L \in (\mathcal{N} \cup \mathcal{R})$. This completes the proof of the Lemmas 5.13 and 5.14 and Theorem 5.15. $\square$

The following theorem solves the control problem for (5.4.1)-(5.4.2) with $h_0 = g_0 = h_1 = 0$:

Theorem 5.16. *Let $T > 0$ and $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$. Then, the system (5.4.1)-(5.4.2) is exactly controllable, with $h_0 = g_0 = h_1 = 0$, in time T.*

Proof. Let us define the following functional

$$\Lambda\left(\varphi^1, \psi^1\right) = \frac{1}{2}\left\|\varphi_{xx}\left(\cdot, L\right)\right\|^2_{L^2(0,T)} - \int_0^L \left(\eta^1\left(x\right)\varphi^1\left(x\right) + w^1\left(x\right)\psi^1\left(x\right)\right) dx \qquad (5.4.112)$$

where $(\varphi^1, \psi^1) \in X$ and (φ, ψ) is the solution of the backward system (5.2.10)-(5.2.11) with initial data (φ^1, ψ^1).

Let $\left(\hat{\varphi}^1, \hat{\psi}^1\right) \in X$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.4.58) is satisfied with $g_1 = \varphi_{xx}\left(t, L\right) \in L^2\left(0, T\right)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.4.58) and (5.4.61), holds that

$$\begin{aligned}\Lambda\left(\varphi^1, \psi^1\right) &= \frac{1}{2}\left(\left\|\varphi_x\left(\cdot, L\right)\right\|^2_{L^2(0,T)} + \left\|\psi_x\left(\cdot, 0\right)\right\|^2_{L^2(0,T)}\right) \\ &\quad - \int_0^L \left(\eta^1\left(x\right)\varphi^1\left(x\right) + w^1\left(x\right)\psi^1\left(x\right)\right) dx \\ &\geq C^{-1}\left\|\left(\varphi^1, \psi^1\right)\right\|_X.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.16 is complete. $\square$

Remark 5.6. *An important is whether system (5.4.1)-(5.4.2) is exactly controllable in two situations: i) $h_1 = g_1 = 0$ and ii) $g_0 = h_1 = g_1 = 0$. Observe that, for the first case, it would be necessary to prove that, for any $T > 0$, there exist $C = C\left(T, L\right) > 0$, such that*

$$\left\|(\varphi, \psi)\right\|^2_X \leq C \int_0^T \left(\left|\varphi_{xx}\left(t, 0\right)\right|^2 + \left|\psi_{xx}\left(t, 0\right)\right|^2\right) dt,$$

holds for any $(\varphi^1, \psi^1) \in X$, where (φ, ψ) is the solution of (5.2.10)-(5.2.11) with initial data (φ^1, ψ^1). For the second case, it would be necessary to prove that

$$\left\|(\varphi, \psi)\right\|^2_X \leq C \int_0^T \left(\left|\psi_{xx}\left(t, 0\right)\right|^2\right) dt,$$

for any $(\varphi^1, \psi^1) \in X$*, where* (φ, ψ) *is the solution of (5.2.10)-(5.2.11) with initial data* (φ^1, ψ^1)*. Note that both inequalities can be obtained following the same steps of the proof of Theorems 5.13 and 5.15. Necessarily, we have to treat new spectral problems. However, considering the change of variable* $t \mapsto T - t$ *and* $x \mapsto L - x$*, we obtain the following problem*

$$\begin{cases} \tilde{\eta}_t + \tilde{w}_x + \tilde{w}_{xxx} = 0 & \text{in } (0,T) \times (0,L)\,, \\ \tilde{w}_t + \tilde{\eta}_x + \tilde{\eta}_{xxx} = 0 & \text{in } (0,T) \times (0,L)\,, \\ \tilde{\eta}(T,x) = \tilde{\eta}^T(x)\,, \ \ \tilde{w}(T,x) = \tilde{w}^T(x) & \text{in } (0,L)\,, \end{cases} \tag{5.4.113}$$

satisfying the boundary conditions

$$\begin{cases} \tilde{\eta}(t,0) = 0, \ \ \tilde{\eta}(t,L) = h_0(t)\,, \ \ \tilde{\eta}_x(t,L) = 0 & \text{in } (0,T) \\ \tilde{w}(t,0) = 0, \ \ \tilde{w}(t,L) = g_0(t)\,, \ \ \tilde{w}_x(t,0) = 0 & \text{in } (0,T)\,, \end{cases} \tag{5.4.114}$$

for the first case, and

$$\begin{cases} \tilde{\eta}(t,0) = 0, \ \ \tilde{\eta}(t,L) = h_0(t)\,, \ \ \tilde{\eta}_x(t,L) = 0 & \text{in } (0,T) \\ \tilde{w}(t,0) = 0, \ \ \tilde{w}(t,L) = 0, \ \ \tilde{w}_x(t,0) = 0 & \text{in } (0,T)\,, \end{cases} \tag{5.4.115}$$

for the second case. Thus, the exact controllability of (5.4.113)-(5.4.114) is obtained if we prove that

$$\left\| \left(\tilde{\varphi}, \tilde{\psi}\right) \right\|_X^2 \le C \int_0^T \left(\left|\tilde{\varphi}_{xx}(t,L)\right|^2 + \left|\tilde{\psi}_{xx}(t,L)\right|^2 \right) dt,$$

for any $\left(\tilde{\varphi}^0, \tilde{\psi}^0\right) \in X$*, where* $\left(\tilde{\varphi}, \tilde{\psi}\right)$ *is the solution of*

$$\begin{cases} \tilde{\varphi}_t + \tilde{\psi}_x + \tilde{\psi}_{xxx} = 0 & \text{in } (0,T) \times (0,L)\,, \\ \tilde{\psi}_t + \tilde{\varphi}_x + \tilde{\varphi}_{xxx} = 0 & \text{in } (0,T) \times (0,L)\,, \\ \tilde{\varphi}(0,x) = \tilde{\varphi}^0(x)\,, \ \ \tilde{w}(0,x) = \tilde{\psi}^0(x) & \text{in } (0,L)\,, \end{cases} \tag{5.4.116}$$

satisfying the boundary conditions

$$\begin{cases} \tilde{\varphi}(t,0) = \tilde{\varphi}(t,L) = \tilde{\varphi}_x(t,L) = 0 & \text{in } (0,T)\,, \\ \tilde{\psi}(t,0) = \tilde{\psi}(t,L) = \tilde{\psi}_x(t,0) = 0 & \text{in } (0,T)\,, \end{cases} \tag{5.4.117}$$

for $T > 0$ *and* $L > 0$*. In order to obtain the controllability result for (5.4.113)-(5.4.114) it is necessary to prove that*

$$\left\| \left(\tilde{\varphi}, \tilde{\psi}\right) \right\|_X^2 \le C \int_0^T \left|\tilde{\psi}_{xx}(t,L)\right|^2 dt,$$

for any $\left(\tilde{\varphi}^0, \tilde{\psi}^0\right) \in X$*, where* $\left(\tilde{\varphi}, \tilde{\psi}\right)$ *is the solution of (5.4.116)-(5.4.117), for* $T > 0$ *and* $L \in (0,\infty) \setminus (\mathcal{N} \cup \mathcal{R})$*. Therefore, we transfer the problems of the Dirichlet condition on zero for the Dirichlet condition on L. This is exactly what was done in Theorem 5.15 and Lemmas 5.13 and 5.14. Then, the following theorems holds:*

Theorem 5.17. *Let $T > 0$ and $L > 0$. Then, the system (5.4.1)-(5.4.2) is exactly controllable, with $h_1 = g_1 = 0$, in time $T = 0$.*

Theorem 5.18. *Let $T > 0$ and $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$. Then, the system (5.4.1)-(5.4.2) is exactly controllable, with $g_0 = h_1 = g_1 = 0$, in time $T = 0$.*

Remark 5.7. *Note that due to change of variable described in Remark 5.6, Theorems 5.17 and 5.18 show that the solution of the systems can be driven from any instant T to the zero initial data.*

5.4.3 Double mixed control of Dirichlet type

In this section we consider (5.4.2) with $g_0 = h_1 = 0$. We first give an equivalent condition for the exact controllability property:

Lemma 5.15. *Let $(\eta^1, w^1) \in X'$. Then, there exist two control $(h_0(t), g_1(t)) \in [L^2(0, L)]^2$, such that the solution (η, w) of (5.4.1)-(5.4.2) with $g_0 = h_1 = 0$, satisfies (5.4.3) if and only if*

$$\left\langle \left(\eta^1, w^1\right), \left(\varphi^1, \psi^1\right)\right\rangle_{X' \times X} = \int_0^T \left(h_0(t)\, \psi_{xx}(t, 0) - g_1(t)\, \varphi_{xx}(t, L)\right) dt \tag{5.4.118}$$

for any $(\varphi^1, \psi^1) \in X$, (φ, ψ) being the solution of the backward system (5.2.10)-(5.2.12).

Proof. The relation (5.4.118) is obtained multiplying the equations in (5.4.1) by the solution (φ, ψ) of (5.2.10)-(5.2.12) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result:

Theorem 5.19. *Set $\mathcal{N}$ and $\mathcal{R}$ defined as in Theorem 5.15. Then, for $T > 0$ and $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$, there exist $C = C(T, L) > 0$ such that the inequality*

$$\left\|\left(\varphi^1, \psi^1\right)\right\|_X^2 \leq C \int_0^T \left(\left|\varphi_{xx}(t, L)\right|^2 + \left|\psi_{xx}(t, 0)\right|^2\right) dt, \tag{5.4.119}$$

holds for any $(\varphi^1, \psi^1) \in X$, where (φ, ψ) is the solution of (5.2.10)-(5.2.11) with initial data (φ^1, ψ^1).

Proof. The proof follows closely the proof of Theorem 5.15, therefore we omitted the details. The change of variable $t = T - t$ transforms (5.2.10)-(5.2.11) into (5.2.13)-(5.2.15). Hence, the proof of inequality (5.4.119) is equivalent to the proof of the inequality

$$\left\|\left(\varphi^0, \psi^0\right)\right\|_X^2 \leq C \int_0^T \left(\left|\varphi_{xx}(t, L)\right|^2 + \left|\psi_{xx}(t, 0)\right|^2\right) dt, \tag{5.4.120}$$

for any $(\varphi^0, \psi^0) \in X$, where (φ, ψ) is the solution of (5.2.13)-(5.2.15) with initial data (φ^0, ψ^0).

We assume that (5.4.120) is not true. Then, there exist a sequence $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}} \in X$ such that

$$\left\| \left(\varphi_n^0, \psi_n^0\right) \right\|_X = 1, \tag{5.4.121}$$

$$\left\| \varphi_{n,xx} \left(\cdot, L\right) \right\|^2_{L^2(0,T)} \longrightarrow 0, \text{ as } n \to \infty \tag{5.4.122}$$

and

$$\left\| \psi_{n,xx} \left(\cdot, 0\right) \right\|^2_{L^2(0,T)} \longrightarrow 0, \text{ as } n \to \infty \tag{5.4.123}$$

where $(\varphi_n, \psi_n) \in Z$ is the solution of (5.2.13)-(5.2.15) with initial data (φ_n^0, ψ_n^0). Let us denote

$$Y := L^2 \left(0, T; \left(H^{7/4} \left(0, L\right)\right)^2\right) \cap C^0 \left(\left[0, T\right]; \left(H^1 \left(0, L\right)\right)^2\right).$$

We show that there exist some positive constant c_1 such that: $\forall \left(\varphi, \psi\right) \in Z$ solution of (5.2.13)-(5.2.15), one has

$$\begin{aligned} \left\| (\varphi, \psi) \right\|^2_Z \leq c_1 \left(\left\| \varphi_{xx} \left(\cdot, L\right) \right\|^2_{L^2(0,T)} + \left\| \psi_{xx} \left(\cdot, 0\right) \right\|^2_{L^2(0,T)} \right) \\ + c_1 \left(\left\| (\varphi, \psi) \right\|^2_Y + \left\| \left(\varphi^0, \psi^0\right) \right\|_{(H^1(0,L))^2} \right). \end{aligned} \tag{5.4.124}$$

For that purpose, we use the multiplier method. Multiplying the first equation of (5.2.13) by $(x - L)\, \psi_{xx}$, the second one by $(x - L)\, \varphi_{xx}$ and integrating in $(0, T) \times (0, L)$, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^T \int_0^L \left|\psi_{xx}\right|^2 dxdt = -\frac{L}{2} \int_0^T \left|\psi_{xx} \left(t, 0\right)\right|^2 dt - \frac{L}{2} \int_0^T \left|\psi_x \left(t, 0\right)\right|^2 dt \\ - \int_0^T \psi_x \left(t, 0\right) \psi_{xx} \left(t, 0\right) dt + \frac{3}{2} \int_0^T \int_0^L \left|\psi_x\right|^2 dxdt - \int_0^T \int_0^L \left(x - L\right) \psi_x \varphi_{tx} dxdt \end{aligned} \tag{5.4.125}$$

and

$$\begin{aligned} \frac{1}{2} \int_0^T \int_0^L \left|\varphi_{xx}\right|^2 dxdt = -\frac{L}{2} \int_0^T \left|\varphi_{xx} \left(t, 0\right)\right|^2 dt - \int_0^T \varphi_x \left(t, L\right) \varphi_{xx} \left(t, L\right) dt \\ + \frac{3}{2} \int_0^T \int_0^L \left|\varphi_x\right|^2 dxdt - \int_0^T \int_0^L \left(x - L\right) \varphi_x \psi_{tx} dxdt. \end{aligned} \tag{5.4.126}$$

Hence, proceeding as in the proof of the Theorem 5.13 we obtain (φ, ψ) solution of

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ \varphi\left(0, x\right) = \varphi^0\left(x\right), \ \ \psi\left(0, x\right) = \psi^0\left(x\right) & \text{in } (0, L), \end{cases} \tag{5.4.127}$$

satisfying

$$\left\| \left(\varphi^0, \psi^0\right) \right\|_X = 1, \tag{5.4.128}$$

$$\begin{cases} \varphi\left(t, 0\right) = \varphi\left(t, L\right) = \varphi_x\left(t, 0\right) = 0 & \text{in } (0, T) \\ \psi\left(t, 0\right) = \psi\left(t, L\right) = \psi_x\left(t, L\right) = 0 & \text{in } (0, T) \end{cases} \tag{5.4.129}$$

and

$$\varphi_{xx}(\cdot, L) = \psi_{xx}(\cdot, 0) = 0. \tag{5.4.130}$$

Remark that (5.4.128) implies that the solutions of (5.4.127)-(5.4.130) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following result:

Lemma 5.16. *For any $T>0$ let N_T denote the space of the initial states $(\varphi^0, \psi^0) \in X$ such that the solution $(\varphi, \psi) = S(\cdot)(\varphi^0, \psi^0)$ of (5.4.127)-(5.4.129) satisfies (5.4.130). Then, for $L \in (0, +\infty) \setminus (\mathcal{N} \cup \mathcal{R})$, $N_T = \{0\}$, $\forall T > 0$.*

Proof. Let A be the operator

$$A(\varphi, \psi) = \begin{pmatrix} \psi_x + \psi_{xxx} \\ \varphi_x + \varphi_{xxx} \end{pmatrix}, \forall (\varphi, \psi) \in \mathcal{D}(A),$$

with

$$\mathcal{D}(A) = \left\{ (\varphi, \psi) \in \left(H^3(0, L)\right)^2; \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(L) = 0 \right\}.$$

Using the same argument of the proof of Lemma 5.3, N_T verifies
1. $\dim(N_T) < +\infty$;
2. $N_T \subset D(A)$;
3. $A(N_T) \subset N_T$.

If $N_T \neq \{0\}$, the map $(\varphi^0, \psi^0) \in \mathbb{C}N_T \longrightarrow A((\varphi^0, \psi^0)) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at last) one eigenvalue, hence there exist $\lambda \in \mathbb{C}$, $(\varphi^0, \psi^0) \in (H^3(0, L))^2 \setminus \{(0, 0)\}$ such that

$$\left\{ \begin{array}{l} \lambda \varphi^0 = \left[(\psi^0)' + (\psi^0)''' \right], \\ \lambda \psi^0 = \left[(\varphi^0)' + (\varphi^0)''' \right], \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))'' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = (\psi^0(0))'' = 0. \end{array} \right. \tag{5.4.131}$$

To conclude the proof of the Lemma 5.16, we prove that this does not hold if $L \notin \mathcal{N}$.

Lemma 5.17. *Let $L > 0$. Consider the assertion*

$(\mathcal{F}_3)$ $\quad \exists \lambda \in \mathbb{C}$, $\exists \left(\varphi^0, \psi^0\right) \in \left(H^3(0, L)\right)^2 \setminus \{(0, 0)\}$ *such that*

$$\left\{ \begin{array}{l} \lambda \varphi^0 = \left[(\psi^0)' + (\psi^0)''' \right], \\ \lambda \psi^0 = \left[(\varphi^0)' + (\varphi^0)''' \right], \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))'' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = (\psi^0(0))'' = 0. \end{array} \right.$$

Then, $(\mathcal{F}_3)$ holds if and only if $L \in (\mathcal{N} \cup \mathcal{R})$.

Proof. We follows the argument used in [62, Lema 3.5]. Assume that (φ^0, ψ^0) satisfies $(\mathcal{F}_3)$ and let us denote by $(\varphi, \psi) \in (H^2(\mathbb{R}))^2$ its prolongation by 0 to $\mathbb{R}$. Then,

$$\left\{ \begin{array}{ll} -\lambda \varphi + \psi' + \psi''' = \psi'(0)(\delta_0)' - \psi''(L)\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda \psi + \varphi' + \varphi''' = \varphi''(0)\delta_0 - \varphi'(L)(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}), \end{array} \right. \tag{5.4.132}$$

where δ_{x_0} and $(\delta_{x_0})'$ denote the Dirac measure at x_0. Note that the $(\mathcal{F}_3)$ is equivalent to the existence of complex numbers α, β, γ, γ', λ with $(\alpha,\beta,\gamma,\gamma')\neq(0,0,0,0)$ and $(\varphi,\psi)\in\left(H^2(\mathbb{R})\right)^2$ with compact support in $[-L,L]$ satisfying

$$\begin{cases} -\lambda(\varphi+\psi)+(\psi+\varphi)'+(\psi+\varphi)'''=\alpha\delta_0+\beta(\delta_0)'+\gamma\delta_L+\gamma'(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda(\varphi-\psi)+(\varphi-\psi)'+(\varphi-\psi)'''=-\alpha\delta_0-\beta(\delta_0)'+\gamma\delta_L+\gamma'(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases} \tag{5.4.133}$$

We introduce the notation $\hat{\varphi}(\xi)=\int_0^L\varphi(\xi)e^{-ix\xi}dx$ and $\hat{\psi}(\xi)=\int_0^L\psi(\xi)e^{-ix\xi}dx$. Then, taking the Fourier transform in (5.4.132) we obtain that (see (5.4.86) and (5.4.87))

$$\hat{u}(\xi):=\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)=\frac{\varphi''(0)+(i\xi)\psi'(0)-\psi''(L)e^{-iL\xi}-(i\xi)\varphi'(L)e^{-iL\xi}}{-\lambda+(i\xi)+(i\xi)^3}. \tag{5.4.134}$$

and

$$\hat{v}(\xi):=\left(\hat{\varphi}(\xi)-\hat{\psi}(\xi)\right)=\frac{-\varphi''(0)+(i\xi)\psi'(0)-\psi''(L)e^{-iL\xi}+(i\xi)\varphi'(L)e^{-iL\xi}}{-\lambda-(i\xi)-(i\xi)^3}.$$

For $\hat{v}$ consider the change of variable $\xi\longmapsto-\xi$ and the new function

$$\hat{v}(-\xi)=\frac{-\varphi''(0)-(i\xi)\psi'(0)-\psi''(L)e^{iL\xi}-(i\xi)\varphi'(L)e^{iL\xi}}{-\lambda+(i\xi)+(i\xi)^3}. \tag{5.4.135}$$

Setting $\lambda=ip$ we can write (5.4.134) and (5.4.135) as

$$\hat{u}(\xi)=i\frac{\alpha+(i\xi)\beta+\gamma e^{-iL\xi}+\gamma'(i\xi)e^{-iL\xi}}{\xi^3-\xi+p} \tag{5.4.136}$$

and

$$\hat{v}(-\xi)=i\frac{-\alpha-(i\xi)\beta+\gamma e^{iL\xi}+\gamma'(i\xi)e^{iL\xi}}{\xi^3-\xi+p}. \tag{5.4.137}$$

Using Paley-Wiener theorem (see [79]) and the usual characterization of $H^2(\mathbb{R})$ by means of Fourier transforms we see that $(\mathcal{F}_3)$ is equivalent to the existence of $p\in\mathbb{C}$ and

$$(\alpha,\beta,\gamma,\gamma')\in\mathbb{C}^4\setminus(0,0,0,0)$$

such that

$$f(\xi):=\frac{\alpha+(i\xi)\beta+\gamma e^{-iL\xi}+\gamma'(i\xi)e^{-iL\xi}}{\xi^3-\xi+p}$$

and

$$g(\xi):=\frac{-\alpha-(i\xi)\beta+\gamma e^{iL\xi}+\gamma'(i\xi)e^{iL\xi}}{\xi^3-\xi+p}$$

satisfies

a) f and g are entire function in $\mathbb{C}$;

b) $\displaystyle\int_{\mathbb{R}}|f(\xi)|^2\left(1+|\xi|^2\right)^2d\xi<\infty$ and $\displaystyle\int_{\mathbb{R}}|g(\xi)|^2\left(1+|\xi|^2\right)^2d\xi<\infty$;

c) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c(1+|\xi|)^k \exp(l \operatorname{Im} \xi)$ for some positive constants c and k.

Remark that f and g are entire if and only if the roots μ_0, μ_1 and μ_2 of $Q(\xi) := \xi^3 - \xi + p$ are roots of

$$r_1(\xi) := \alpha + (i\xi)\beta + \gamma e^{-iL\xi} + \gamma'(i\xi)e^{-iL\xi} \tag{5.4.138}$$

and

$$r_2(\xi) := -\alpha - (i\xi)\beta + \gamma e^{iL\xi} + \gamma'(i\xi)e^{iL\xi}. \tag{5.4.139}$$

In particular, f and g are entire if and only if $f+g$ and $f-g$ are entire, where

$$\begin{cases} f+g := 2(\gamma + \gamma' i\xi)\cos(L\xi) \\ f-g := 2\alpha + 2(i\xi)\beta + (\gamma + \gamma'(i\xi))(-2i)\sin(L\xi). \end{cases} \tag{5.4.140}$$

Here we use that $\cos y = \dfrac{e^{iy}+e^{-iy}}{2}$ and $\sin y = \dfrac{e^{iy}-e^{-iy}}{2i}$.

If the roots of (5.4.140) are simple, a) holds if the roots of $Q(\xi)$ are simples and also roots of (5.4.140). Observe that if a) is true, then b) and c) are satisfied. We find the roots of (5.4.140) and prove that they are simple:

$$\begin{cases} 4(\gamma + \gamma' i\xi)^2 \cos^2(L\xi) = 0 \\ 4(\gamma + \gamma' i\xi)^2 \sin^2(L\xi) = -4\alpha^2 + 4\xi^2\beta^2 + \dfrac{8}{i}\beta\xi\alpha. \end{cases} \tag{5.4.141}$$

Therefore, adding the identities in (5.4.141) and using the basic relation $\cos^2(L\xi) + \sin^2(L\xi) = 1$, we have

$$\left(\beta^2 + (\gamma')^2\right)\xi^2 + \left(\frac{2\alpha\beta}{i} - 2\gamma\gamma' i\right)\xi - \left(\alpha^2 + \gamma^2\right) = 0. \tag{5.4.142}$$

Taking (5.4.142) into account, we obtain a contradiction. Indeed, (5.4.142) allows us to conclude that (5.4.142), at least, two roots, unless

$$\begin{cases} \beta^2 + (\gamma')^2 = 0, \\ \alpha\beta + \gamma\gamma' = 0, \\ \alpha^2 + \gamma^2 = 0. \end{cases} \tag{5.4.143}$$

From the first equation of (5.4.143) we obtain $\beta = i\gamma'$ and $\beta = -i\gamma'$.

We analyze the first case, since the second case is analogous and it will be omitted. If $\beta = i\gamma'$, the second equation of (5.4.143) give us that

$$\alpha\beta + \gamma\gamma' = 0 \Rightarrow (\alpha i + \gamma)\gamma' = 0. \tag{5.4.144}$$

Now we consider two cases:

a) If $\gamma' = 0$, then $\beta = 0$. Thus, from (5.4.138) and (5.4.139) we have

$$f(\xi) = \frac{\alpha + \gamma e^{-iL\xi}}{\xi^3 - \xi + p}$$

and

$$g(\xi)=\frac{-\alpha+\gamma e^{iL\xi}}{\xi^3-\xi+p}.$$

Then, [62, Lema 3.5] ensure that for $\gamma'=\beta=0$ and $(\mathcal{F}_3)$ holds if and only if $L\in\mathcal{N}$.

b) If $\gamma'\neq 0$, the second and the third equation of (5.4.143) give us that $\alpha=i\gamma$, since $\beta=i\gamma'$. Then from (5.4.140) we obtain that

$$f+g=2\left(\gamma+\gamma' i\xi\right)\cos\left(L\xi\right),$$

and

$$f-g=2i\left(\gamma+\gamma' i\xi\right)\left(1-\sin\left(L\xi\right)\right).$$

Therefore, the conclusion of the Lemma 5.17 follows exactly as the conclusion of Lemma 5.14. Thus, $(\mathcal{F}_3)$ holds if and only if $L\in(\mathcal{N}\cup\mathcal{R})$. This completes the proof of the Lemmas 5.16 and 5.17 and Theorem 5.19. □

The following theorem solves the control problem (5.4.1)-(5.4.2) with $g_0=h_1=0$:

Theorem 5.20. *Let $T>0$ and $L\in(0,+\infty)\setminus(\mathcal{N}\cup\mathcal{R})$. Then, the system (5.4.1)-(5.4.2) is exactly controllable, with $g_0=h_1=0$, in time T.*

Proof. Let us define the following functional

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right)=&\frac{1}{2}\left(\|\varphi_{xx}(\cdot,L)\|^2_{L^2(0,T)}+\|\psi_{xx}(\cdot,0)\|^2_{L^2(0,T)}\right)\\&-\int_0^L\left(\eta^1(x)\varphi^1(x)+w^1(x)\psi^1(x)\right)dx,\end{aligned}\tag{5.4.145}$$

where $(\varphi^1,\psi^1)\in X$ and (φ,ψ) is the solution of the backward system (5.2.10)-(5.2.11) with initial data (φ^1,ψ^1).

Let $\left(\hat{\varphi}^1,\hat{\psi}^1\right)\in X$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.4.118) is satisfied with $h_0=\psi_{xx}(t,0)\in L^2(0,T)$ and $g_1=\varphi_{xx}(t,L)\in L^2(0,T)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.4.118) and (5.4.119), holds that

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right)=&\frac{1}{2}\left(\|\varphi_{xx}(\cdot,L)\|^2_{L^2(0,T)}+\|\psi_{xx}(\cdot,0)\|^2_{L^2(0,T)}\right)\\&-\int_0^L\left(\eta^1(x)\varphi^1(x)+w^1(x)\psi^1(x)\right)dx\\\geq& C^{-1}\left\|\left(\varphi^1,\psi^1\right)\right\|_X.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.20 is complete. □

Remark 5.8. *In this section we prove the controllability of the system (5.4.1)-(5.4.2) with $g_0=h_1=0$. Observe that, if we consider $h_0=g_1=0$, the problem of controllability is reduced to prove the following observability inequality: For $T>0$ and*

$L\in(0,+\infty)\setminus(\mathcal{N}\cup\mathcal{R})$, *there exist* $C=C(T,L)>0$ *such that the inequality*

$$\left\|\left(\varphi^{1},\psi^{1}\right)\right\|_{X}^{2}\leq C\int_{0}^{T}\left(\left|\varphi_{xx}\left(t,0\right)\right|^{2}+\left|\psi_{xx}\left(t,L\right)\right|^{2}\right)dt, \tag{5.4.146}$$

holds for any $(\varphi^1,\psi^1)\in X$, *where* (φ,ψ) *is the solution of (5.2.10)-(5.2.11) with initial data* (φ^1,ψ^1). *The proof follows exactly the same steps of the previous one, so it will be omitted. Thus, we also have the following theorem:*

Theorem 5.21. *Let* $T>0$ *and* $L\in(0,+\infty)\setminus(\mathcal{N}\cup\mathcal{R})$. *Then, the system (5.4.1)-(5.4.2) is exactly controllable, with* $h_0=g_1=0$, *in time* T.

5.5 Exact Boundary Controllability For The Linear System: Mixed boundary condition

This section is devoted to the analysis of the exact controllability property of the linear system corresponding to (5.1.7) with mixed boundary controls. More precisely, given $T>0$ and (η^0,w^0), $\left(\eta^T,w^T\right):=(\eta^1,w^1)\in\bar{\Sigma}$, we study the existence of the controls $(h_1,g_1,h_2,g_2)\in\bar{\Sigma}_1$ such that the solution (η,w) of the system

$$\begin{cases}\eta_t+w_x+w_{xxx}=0 & \text{in } (0,T)\times(0,L),\\ w_t+\eta_x+\eta_{xxx}=0 & \text{in } (0,T)\times(0,L),\\ \eta(0,x)=\eta^0(x), \;\; w(0,x)=w^0(x) & \text{in } (0,L),\end{cases} \tag{5.5.1}$$

satisfying the boundary conditions

$$\begin{cases}\eta(t,0)=0,\;\; \eta(t,L)=h_1(t),\;\; \eta_x(t,0)=h_2(t) & \text{in } (0,T)\\ w(t,0)=0,\;\; w(t,L)=g_1(t),\;\; w_x(t,L)=g_2(t) & \text{in } (0,T),\end{cases} \tag{5.5.2}$$

satisfies

$$\eta(T,\cdot)=\eta^1,\quad w(T,\cdot)=w^1 \text{ in } \bar{\Sigma}. \tag{5.5.3}$$

The spaces $\bar{\Sigma}$ and $\bar{\Sigma}_1$ will be defined later.

Definition 5.4. *Let* $T>0$. *System (5.5.1) is exact controllable in time* T *if for any initial and final data* (η^0,w^0), $(\eta^1,w^1)\in\bar{\Sigma}$, *there exist control functions* $(h_1,g_1,h_2,g_2)\in\bar{\Sigma}_1$ *such that the solution of (5.5.1)-(5.5.2) satisfies (5.5.3).*

For the analysis of the controllability we will consider several cases regarding the amount of controls on (5.5.2).

5.5.1 Double control

In this section we consider (5.5.2) with $h_1=g_2=0$. We first give an equivalent condition for the exact controllability property:

Lemma 5.18. *Let* $(\eta^1, w^1) \in \bar{\Sigma} := X'$. *Then, there exist two control* $(g_1(t), h_2(t)) \in \bar{\Sigma}_1 := [L^2(0,L)]^2$, *such that the solution* (η, w) *of (5.5.1)-(5.5.2), with* $h_1 = g_2 = 0$, *satisfies (5.5.3) if and only if*

$$\left\langle \left(\eta^1, w^1\right), \left(\varphi^1, \psi^1\right)\right\rangle_{X' \times X} = -\int_0^T \left(h_2(t)\,\psi_x(t,0) + g_1(t)\,\varphi_{xx}(t,L)\right) dt \tag{5.5.4}$$

for any $(\varphi^1, \psi^1) \in X$, (φ, ψ) *being the solution of the backward system (5.2.10)-(5.2.12).*

Proof. The relation (5.5.4) is obtained multiplying the equations in (5.5.1) by the solution (φ, ψ) of (5.2.10)-(5.2.12) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result:

Theorem 5.22. *For any* $T > 0$ *and* $L \in \mathcal{N}$ *there exists* $C = C(T, L) > 0$ *such that the inequality*

$$\left\|\left(\varphi^1, \psi^1\right)\right\|_X^2 \le C \int_0^T \left(\left|\varphi_{xx}(t,L)\right|^2 + \left|\psi_x(t,0)\right|^2\right) dt, \tag{5.5.5}$$

holds for any $(\varphi^1, \psi^1) \in X$, *where* (φ, ψ) *is the solution of (5.2.10)-(5.2.11) with initial data* (φ^1, ψ^1).

Proof. The change of variable $t = T - t$ transforms (5.2.10)-(5.2.12) into (5.2.13)-(5.2.15). Hence, inequality (5.5.5) is equivalent to

$$\left\|\left(\varphi^0, \psi^0\right)\right\|_X^2 \le C \int_0^T \left(\left|\varphi_{xx}(t,L)\right|^2 + \left|\psi_x(t,0)\right|^2\right) dt, \tag{5.5.6}$$

for any $(\varphi^0, \psi^0) \in X$, where (φ, ψ) is the solution of (5.2.13)-(5.2.15) with initial data (φ^0, ψ^0).

We assume that (5.5.6) does not holds. Then, there exists a sequence $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}} \in X$ such that

$$\left\|\left(\varphi_n^0, \psi_n^0\right)\right\|_X = 1, \tag{5.5.7}$$

$$\left\|\varphi_{n,xx}(\cdot, L)\right\|_{L^2(0,T)}^2 \longrightarrow 0, \text{ as } n \to \infty \tag{5.5.8}$$

and

$$\left\|\psi_{n,x}(\cdot, 0)\right\|_{L^2(0,T)}^2 \longrightarrow 0, \text{ as } n \to \infty \tag{5.5.9}$$

where $(\varphi_n, \psi_n) \in Z$ is the solution of (5.2.13)-(5.2.15) with initial data (φ_n^0, ψ_n^0). Let us denote

$$Y := L^2\left(0, T; \left(H^{7/4}(0,L)\right)^2\right) \cap C^0\left([0,T]; \left(H^1(0,L)\right)^2\right).$$

We show that there exist some positive constant C such that: $\forall (\varphi, \psi) \in Z$ solution of (5.2.13)-(5.2.15), one has

$$\|(\varphi, \psi)\|_Z^2 \le C\left(\left\|\varphi_{xx}(\cdot, L)\right\|_{L^2(0,T)}^2 + \left\|\psi_x(\cdot, 0)\right\|_{L^2(0,T)}^2 + \|(\varphi, \psi)\|_Y^2 + \left\|\left(\varphi^0, \psi^0\right)\right\|_{(H^1(0,L))^2}\right). \tag{5.5.10}$$

For that purpose, we use the multiplier method. Multiplying the first equation of (5.2.13) by $(x-L)\,\psi_{xx}$, the second one by $(x-L)\,\varphi_{xx}$ and integrating in $(0,T)\times(0,L)$, we obtain

$$\frac{1}{2}\int_0^T\int_0^L |\psi_{xx}|^2\,dxdt = -\frac{L}{2}\int_0^T |\psi_{xx}(t,0)|^2\,dt - \frac{L}{2}\int_0^T |\psi_x(t,0)|^2\,dt$$
$$-\int_0^T \psi_x(t,0)\,\psi_{xx}(t,0)\,dt + \frac{3}{2}\int_0^T\int_0^L |\psi_x|^2\,dxdt - \int_0^T\int_0^L (x-L)\,\psi_x\varphi_{tx}dxdt \tag{5.5.11}$$

and

$$\begin{aligned}\frac{1}{2}\int_0^T\int_0^L |\varphi_{xx}|^2\,dxdt = -\frac{L}{2}\int_0^T |\varphi_{xx}(t,0)|^2\,dt - \int_0^T \varphi_x(t,L)\,\varphi_{xx}(t,L)\,dt \\ +\frac{3}{2}\int_0^T\int_0^L |\varphi_x|^2\,dxdt - \int_0^T\int_0^L (x-L)\,\varphi_x\psi_{tx}dxdt.\end{aligned} \tag{5.5.12}$$

Hence, proceeding as in the proof of the Theorem 5.13 we obtain (φ,ψ) solution of

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \varphi(0,x) = \varphi^0(x),\ \ \psi(0,x) = \psi^0(x) & \text{in } (0,L), \end{cases} \tag{5.5.13}$$

satisfying

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_X = 1, \tag{5.5.14}$$

$$\begin{cases} \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0 & \text{in } (0,T) \\ \psi(t,0) = \psi(t,L) = \psi_x(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.5.15}$$

and, in addition,

$$\varphi_{xx}(\cdot,L) = \psi_x(\cdot,0) = 0. \tag{5.5.16}$$

Remark that (5.5.14) implies that the solutions of (5.5.13)-(5.5.16) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following result:

Lemma 5.19. *For any $T>0$, let N_T denote the space of the initial states $(\varphi^0,\psi^0)\in X$ such that the solution $(\varphi,\psi) = S(\cdot)(\varphi^0,\psi^0)$ of (5.5.13)-(5.5.15) satisfies (5.5.16). Then, for $L\in(0,+\infty)\setminus\mathcal{N}$, where $\mathcal{N}$ is defined by (5.4.59), $N_T=\{0\}$, $\forall T>0$.*

Proof. Let A be the operator

$$A(\varphi,\psi) = \begin{pmatrix} \psi_x + \psi_{xxx} \\ \varphi_x + \varphi_{xxx} \end{pmatrix}, \forall(\varphi,\psi)\in\mathcal{D}(A),$$

with

$$\mathcal{D}(A) = \left\{(\varphi,\psi)\in\left(H^3(0,L)\right)^2; \varphi(0)=\psi(0)=\varphi(L)=\psi(L)=\varphi_x(0)=\psi_x(L)=0\right\}.$$

Using the same argument of the proof of Lemma 5.3, N_T verifies

1. $\dim(N_T) < +\infty$;
2. $N_T \subset D(A)$;
3. $A(N_T) \subset N_T$.

If $N_T \neq \{0\}$, the map $(\varphi^0, \psi^0) \in \mathbb{C}N_T \longrightarrow A((\varphi^0, \psi^0)) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at last) one eigenvalue, hence there exist $\lambda \in \mathbb{C}$, $(\varphi^0, \psi^0) \in (H^3(0,L))^2 \setminus \{(0,0)\}$ such that

$$\begin{cases} \lambda\varphi^0 = \left[(\psi^0)' + (\psi^0)'''\right], \\ \lambda\psi^0 = \left[(\varphi^0)' + (\varphi^0)'''\right], \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))'' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = (\psi^0(0))' = 0. \end{cases} \tag{5.5.17}$$

To conclude the proof of the Lemma 5.19, we prove that this does not hold if $L \notin \mathcal{N}$.

Lemma 5.20. *Let $L > 0$. Consider the assertion*

$$(\mathcal{F}_4) \qquad \exists\lambda \in \mathbb{C},\ \exists\left(\varphi^0, \psi^0\right) \in \left(H^3(0,L)\right)^2 \setminus \{(0,0)\} \textit{ such that}$$

$$\begin{cases} \lambda\varphi^0 = \left[(\psi^0)' + (\psi^0)'''\right], \\ \lambda\psi^0 = \left[(\varphi^0)' + (\varphi^0)'''\right], \\ \varphi^0(0) = \varphi^0(L) = (\varphi^0(0))' = (\varphi^0(L))'' = 0, \\ \psi^0(0) = \psi^0(L) = (\psi^0(L))' = (\psi^0(0))' = 0. \end{cases}$$

Then, $(\mathcal{F}_4)$ holds if and only if $L \in \mathcal{N}$.

Proof. We follow the argument used in [62, Lema 3.5]. Assume that (φ^0, ψ^0) satisfies $(\mathcal{F}_3)$ and let us denote by $(\varphi, \psi) \in (H^2(\mathbb{R}))^2$ its prolongation by 0 to $\mathbb{R}$. Then,

$$\begin{cases} -\lambda\varphi + \psi' + \psi''' = \psi''(0)\,\delta_0 - \psi''(L)\,\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda\psi + \varphi' + \varphi''' = \varphi''(0)\,\delta_0 - \varphi'(L)\,(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}), \end{cases} \tag{5.5.18}$$

where δ_{x_0} and $(\delta_{x_0})'$ denote the Dirac measure at x_0. Note that the $(\mathcal{F}_4)$ is equivalent to the existence of complex numbers α, α', γ, γ', λ with $(\alpha, \alpha', \gamma, \gamma') \neq (0,0,0,0)$ and $(\varphi, \psi) \in (H^2(\mathbb{R}))^2$ with compact support in $[-L, L]$ satisfying

$$\begin{cases} -\lambda(\varphi+\psi) + (\psi+\varphi)' + (\psi+\varphi)''' = \alpha\delta_0 + \gamma\delta_L + \gamma'(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda(\varphi-\psi) + (\varphi-\psi)' + (\varphi-\psi)''' = \alpha'\delta_0 + \gamma\delta_L + \gamma'(\delta_L)' & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases} \tag{5.5.19}$$

We introduce the notation $\hat{\varphi}(\xi) = \int_0^L \varphi(\xi)\, e^{-ix\xi} dx$ and $\hat{\psi}(\xi) = \int_0^L \psi(\xi)\, e^{-ix\xi} dx$. Then, taking the Fourier transform in (5.5.18) we obtain that

$$\hat{u}(\xi) := \left(\hat{\varphi}(\xi) + \hat{\psi}(\xi)\right) = \frac{(\varphi''(0) + \psi''(0)) - \psi''(L)\, e^{-iL\xi} - (i\xi)\,\varphi'(L)\, e^{-iL\xi}}{-\lambda + (i\xi) + (i\xi)^3}. \tag{5.5.20}$$

and

$$\hat{v}(\xi) := \left(\hat{\varphi}(\xi) - \hat{\psi}(\xi)\right) = \frac{(-\varphi''(0) + \psi''(0)) - \psi''(L)\, e^{-iL\xi} + (i\xi)\,\varphi'(L)\, e^{-iL\xi}}{-\lambda - (i\xi) - (i\xi)^3}.$$

For $\hat{v}$ consider the change of variable $\xi \longmapsto -\xi$ and the new function

$$\hat{v}(-\xi) = \frac{(-\varphi''(0) + \psi''(0)) - \psi''(L)\, e^{iL\xi} - (i\xi)\,\varphi'(L)\, e^{iL\xi}}{-\lambda + (i\xi) + (i\xi)^3}. \tag{5.5.21}$$

Setting $\lambda = ip$ we can write (5.5.20) and (5.5.21) as

$$\hat{u}(\xi) = i\frac{\alpha + \gamma e^{-iL\xi} + \gamma'(i\xi)\, e^{-iL\xi}}{\xi^3 - \xi + p} \tag{5.5.22}$$

and

$$\hat{v}(-\xi) = i\frac{\alpha' + \gamma e^{iL\xi} + \gamma'(i\xi)\, e^{iL\xi}}{\xi^3 - \xi + p}. \tag{5.5.23}$$

Using Paley-Wiener theorem (see [79]) and the usual characterization of $H^2(\mathbb{R})$ by means of their Fourier transforms we see that $(\mathcal{F}_4)$ is equivalent to the existence of $p \in \mathbb{C}$ and

$$(\alpha, \alpha', \gamma, \gamma') \in \mathbb{C}^4 \setminus (0,0,0,0)$$

such that

$$f(\xi) := \frac{\alpha + \gamma e^{-iL\xi} + \gamma'(i\xi)\, e^{-iL\xi}}{\xi^3 - \xi + p} \tag{5.5.24}$$

and

$$g(\xi) := \frac{\alpha' + \gamma e^{iL\xi} + \gamma'(i\xi)\, e^{iL\xi}}{\xi^3 - \xi + p} \tag{5.5.25}$$

satisfies

a) f and g are entire function in $\mathbb{C}$;

b) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2 \left(1 + |\xi|^2\right)^2 d\xi < \infty$ and $\displaystyle\int_{\mathbb{R}} |g(\xi)|^2 \left(1 + |\xi|^2\right)^2 d\xi < \infty$;

c) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c\,(1 + |\xi|)^k \exp(l \operatorname{Im} \xi)$ for some positive constants c and k.

Remark that f and g are entire if and only if the roots μ_0, μ_1 and μ_2 of $Q(\xi) := \xi^3 - \xi + p$ are roots of

$$r_1(\xi) := \alpha + \gamma e^{-iL\xi} + \gamma'(i\xi)\, e^{-iL\xi}$$

and

$$r_2(\xi) := \alpha' + \gamma e^{iL\xi} + \gamma'(i\xi)\, e^{iL\xi}.$$

In particular, f and g are entire if and only if $f + g$ and $f - g$ are entire, where

$$\begin{cases} f + g := (\alpha + \alpha') + 2(\gamma + \gamma' i\xi) \cos(L\xi) \\ f - g := (\alpha - \alpha') + (\gamma + \gamma' i\xi)(-2i) \sin(L\xi). \end{cases} \tag{5.5.26}$$

Here we use that $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$ and $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$.

If the roots of (5.5.26) are simple, a) holds if the roots of $Q(\xi)$ are simples and also roots of (5.5.26). Observe that if a) is true, then b) and c) are satisfied. We find the roots of (5.5.26) and prove that are simple. We analyze the following cases:

Case I: $Q(\xi)$ has a simple roots.

Let us consider the system

$$\begin{cases} (\alpha+\alpha')+2(\gamma+\gamma' i\xi)\cos(L\xi)=0 \\ (\alpha-\alpha')+(\gamma+\gamma' i\xi)(-2i)\sin(L\xi)=0 \end{cases}$$

or, equivalently,

$$\begin{cases} 4(\gamma+\gamma' i\xi)^2\cos^2(L\xi)=(\alpha+\alpha')^2 \\ 4(\gamma+\gamma' i\xi)^2\sin^2(L\xi)=-(\alpha-\alpha')^2. \end{cases} \tag{5.5.27}$$

Therefore, adding the identities in (5.5.27) and using the basic relation $\cos^2(L\xi)+\sin^2(L\xi)=1$, we have

$$-(\gamma')^2\xi^2+2\gamma\gamma' i\xi+\left(\gamma^2-\alpha\alpha'\right)=0. \tag{5.5.28}$$

Taking (5.5.28) into account, we obtain a contradiction. Indeed, (5.5.28) allow us to conclude that (5.5.26) has, at least, two roots, unless

$$\begin{cases} (\gamma')^2=0, \\ \gamma\gamma'=0, \\ (\gamma^2-\alpha\alpha')=0. \end{cases} \tag{5.5.29}$$

From the first equation of (5.5.29) we obtain $\gamma'=0$ and from (5.5.24) and (5.5.25) it follows that

$$f(\xi):=\frac{\alpha+\gamma e^{-iL\xi}}{\xi^3-\xi+p}$$

and

$$g(\xi):=\frac{\alpha'+\gamma e^{iL\xi}}{\xi^3-\xi+p}.$$

Then, [62, Lema 3.5] ensure that $(\mathcal{F}_4)$ holds if and only if $L\in\mathcal{N}$.

Case II: $Q(\xi)$ has a root of order three.

If $Q(\xi)$ has a root of order three, namely, μ_0, then

$$Q(\mu_0)=Q'(\mu_0)=Q''(\mu_0)=0,$$

and, therefore,

$$\mu_0=0\Rightarrow Q(0)=p=0.$$

Thus, $Q(\xi)=\xi(\xi+1)(\xi-1)$. This is a contradiction, because $Q(\xi)$ has a root of order three.

Case III: $Q(\xi)$ has a double root.

In this case, we consider $\mu_0=\mu_1$ and μ_2 roots of $Q(\xi)$. By previous computations (see Lemma 5.7), we obtain

$$\mu_0=\mu_1=\frac{1}{\sqrt{3}},\ \mu_2=-\frac{2}{\sqrt{3}}\text{ and }p=\frac{2}{3\sqrt{3}} \tag{5.5.30}$$

or

$$\mu_0=\mu_1=-\frac{1}{\sqrt{3}},\ \mu_2=\frac{2}{\sqrt{3}}\text{ and }p=-\frac{2}{3\sqrt{3}}. \tag{5.5.31}$$

We analyze the case wherein the roots are given by (5.5.30). In this case $\mu_0 = \mu_1$ and μ_2 should be roots of (5.5.24) and (5.5.25), that is

$$\begin{cases} \alpha + \gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma'\left(i\frac{1}{\sqrt{3}}\right)\exp\left(-iL\frac{1}{\sqrt{3}}\right) = 0, \\ -(iL)\,\gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma' i \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma'\left(\frac{L}{\sqrt{3}}\right)\exp\left(-iL\frac{1}{\sqrt{3}}\right) = 0, \end{cases} \tag{5.5.32}$$

$$\begin{cases} \alpha' + \gamma \exp\left(iL\frac{1}{\sqrt{3}}\right) + \gamma'\left(i\frac{1}{\sqrt{3}}\right)\exp\left(iL\frac{1}{\sqrt{3}}\right) = 0, \\ (iL)\,\gamma \exp\left(iL\frac{1}{\sqrt{3}}\right) + \gamma' i \exp\left(iL\frac{1}{\sqrt{3}}\right) - \gamma'\left(\frac{L}{\sqrt{3}}\right)\exp\left(iL\frac{1}{\sqrt{3}}\right) = 0, \end{cases} \tag{5.5.33}$$

and

$$\begin{cases} \alpha + \gamma \exp\left(iL\frac{2}{\sqrt{3}}\right) + \gamma'\left(-i\frac{2}{\sqrt{3}}\right)\exp\left(iL\frac{2}{\sqrt{3}}\right) = 0, \\ \alpha' + \gamma \exp\left(-iL\frac{2}{\sqrt{3}}\right) + \gamma'\left(-i\frac{2}{\sqrt{3}}\right)\exp\left(-iL\frac{2}{\sqrt{3}}\right) = 0. \end{cases} \tag{5.5.34}$$

Finally, we obtain that the solution of this system, i.e., find L such that (5.5.32)-(5.5.34) are satisfied. From the second equation of (5.5.32) and second equation of (5.5.33), we obtain the following system in function of γ and γ'

$$\begin{cases} -(iL)\,\gamma \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma' i \exp\left(-iL\frac{1}{\sqrt{3}}\right) + \gamma'\left(\frac{L}{\sqrt{3}}\right)\exp\left(-iL\frac{1}{\sqrt{3}}\right) = 0, \\ (iL)\,\gamma \exp\left(iL\frac{1}{\sqrt{3}}\right) + \gamma' i \exp\left(iL\frac{1}{\sqrt{3}}\right) - \gamma'\left(\frac{L}{\sqrt{3}}\right)\exp\left(iL\frac{1}{\sqrt{3}}\right) = 0, \end{cases}$$

or, equivalently,

$$\begin{cases} -(iL)\,\gamma + \gamma' i + \gamma'\left(\frac{L}{\sqrt{3}}\right) = 0 \\ (iL)\,\gamma + \gamma' i - \gamma'\left(\frac{L}{\sqrt{3}}\right) = 0 \end{cases} \Leftrightarrow 2\gamma' i = 0$$

Then, $\gamma = \gamma' = 0$, and using (for example) (5.5.34), we have that $\alpha = \alpha' = 0$. Thus, the system (5.5.32)-(5.5.34) has a trivial solution. The case wherein $\mu_0 = \mu_1$ and μ_2 are of the form (5.5.31) is analogous.

Thus, we conclude from Cases I, II and III that $(\mathcal{F}_4)$ holds if and only if $L \in \mathcal{N}$. This completes the proof of Lemmas 5.19, 5.20 and Theorem 5.22. □

The following theorem solves the control problem (5.5.1)-(5.5.2) with $h_1 = g_2 = 0$:

Theorem 5.23. *Let $T > 0$ and $L \in (0, +\infty) \setminus \mathcal{N}$. Then the system (5.5.1)-(5.5.2) is exactly controllable, with $h_1 = g_2 = 0$, in time T.*

Proof. Let us define the following functional

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right) &= \frac{1}{2}\left(\left\|\varphi_{xx}\left(\cdot,L\right)\right\|^2_{L^2(0,T)}+\left\|\psi_x\left(\cdot,0\right)\right\|^2_{L^2(0,T)}\right)\\ &\quad -\int_0^L\left(\eta^1\left(x\right)\varphi^1\left(x\right)+w^1\left(x\right)\psi^1\left(x\right)\right)dx \end{aligned} \tag{5.5.35}$$

where $(\varphi^1,\psi^1)\in X$ and (φ,ψ) is the solution of the backward system (5.2.10)-(5.2.11) with initial data (φ^1,ψ^1).

Let $\left(\hat{\varphi}^1,\hat{\psi}^1\right)\in X$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.5.4) is satisfied with $h_2=\psi_x\left(t,0\right)\in L^2\left(0,T\right)$ and $g_1=\varphi_{xx}\left(t,L\right)\in L^2\left(0,T\right)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.5.4) and (5.5.5), holds that

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right) &= \frac{1}{2}\left(\left\|\varphi_{xx}\left(\cdot,L\right)\right\|^2_{L^2(0,T)}+\left\|\psi_x\left(\cdot,0\right)\right\|^2_{L^2(0,T)}\right)\\ &\quad -\int_0^L\left(\eta^1\left(x\right)\varphi^1\left(x\right)+w^1\left(x\right)\psi^1\left(x\right)\right)dx\\ &\geq C^{-1}\left\|\left(\varphi^1,\psi^1\right)\right\|_X.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.23 is complete. □

Remark 5.9. *In this section we prove the controllability of the system (5.5.1)-(5.5.2) with $h_1=g_2=0$. Observe that, if we consider $g_1=h_2=0$, the problem of controllability is reduced to prove the following observability inequality: For $T>0$ and $L\in(0,+\infty)\setminus\mathcal{N}$, there exist $C=C\left(T,L\right)>0$, such that the inequality*

$$\left\|\left(\varphi^1,\psi^1\right)\right\|^2_X\leq C\int_0^T\left(\left|\varphi_x\left(t,L\right)\right|^2+\left|\psi_{xx}\left(t,L\right)\right|^2\right)dt, \tag{5.5.36}$$

holds for any $(\varphi^1,\psi^1)\in X$, where (φ,ψ) is the solution of (5.2.10)-(5.2.11) with initial data (φ^1,ψ^1). The proof follows exactly the same steps of the proof of Lemmas 5.3 and 5.4, so it will be omitted. Thus, the following theorem holds.

Theorem 5.24. *Let $T>0$ and $L\in(0,+\infty)\setminus\mathcal{N}$. Then the system (5.5.1)-(5.5.2) is exactly controllable, with $g_1=h_2=0$, in time T.*

5.6 Nonlinear Problem

This section is devoted to the study of the nonlinear problem (5.1.7), namely,

$$\begin{cases}\eta_t+w_x+\left(\eta w\right)_x+w_{xxx}=0 & \text{in } \left(0,T\right)\times\left(0,L\right),\\ w_t+\eta_x+ww_x+\eta_{xxx}=0 & \text{in } \left(0,T\right)\times\left(0,L\right).\end{cases} \tag{5.6.1}$$

Observe that with the boundary conditions (5.1.8), the Kato smoothing effect does not holds. Therefore, we consider the solution of the system (5.6.1) satisfying the following boundary conditions

$$\begin{cases} \eta(t,0)=0, \ \ \eta(t,L)=0, \ \ \eta_x(t,0)=0 & \text{in } (0,T) \\ w(t,0)=0, \ \ w(t,L)=0, \ \ w_x(t,L)+\alpha_1\eta_x(t,L)=g_2(t) & \text{in } (0,T), \end{cases} \tag{5.6.2}$$

and

$$\begin{cases} \eta(t,L)=0, \ \ \eta_x(t,0)=0, \ \ \alpha_2\eta(t,0)+\alpha_3\eta_x(t,L)+w_{xx}(t,0)=h_0(t) & \text{in } (0,T) \\ w(t,0)=0, \ \ w(t,L)=0, \ \ w_x(t,L)-\alpha_3(\eta(t,0)-\eta_x(t,L))=g_2(t) & \text{in } (0,T), \end{cases} \tag{5.6.3}$$

where α_i are positive constants for $i=1,2,3$, and the initial conditions

$$\eta(0,x)=\eta^0(x), \ \ w(0,x)=w^0(x) \quad \text{in } (0,L). \tag{5.6.4}$$

5.6.1 Well-Posedness in X_0.

We study the existence of solutions of the linear system corresponding to (5.6.1)

$$\begin{cases} \eta_t+w_x+w_{xxx}=0 & \text{in } (0,T)\times(0,L), \\ w_t+\eta_x+\eta_{xxx}=0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.6.5}$$

satisfying, initially, the following boundary conditions

$$\begin{cases} \eta(t,0)=0, \ \ \eta(t,L)=0, \ \ \eta_x(t,0)=0 & \text{in } (0,T) \\ w(t,0)=0, \ \ w(t,L)=0, \ \ w_x(t,L)+\alpha_1\eta_x(t,L)=0 & \text{in } (0,T), \end{cases} \tag{5.6.6}$$

and the initial conditions

$$\eta(0,x)=\eta^0(x), \ \ w(0,x)=w^0(x) \quad \text{in } (0,L). \tag{5.6.7}$$

Let $X_0=\left(L^2(0,L)\right)^2$ endowed with the usual inner product and consider the operator $A:\mathcal{D}(A)\subset X_0\to X_0$, where

$$\mathcal{D}(A)=\left\{ \begin{array}{c} (\eta,w)\in\left(H^3(0,L)\right)^2; \eta(0)=w(0)=\eta(L)=w(L)=\eta_x(0)=0 \\ \text{and } w_x(L)+\alpha_1\eta_x(L)=0 \end{array} \right\},$$

and

$$A(\eta,w)=\begin{pmatrix} -w_x-w_{xxx} \\ -\eta_x-\eta_{xxx} \end{pmatrix}, \forall(\eta,w)\in\mathcal{D}(A). \tag{5.6.8}$$

With the notation introduce above, system (5.6.5) can be now written as an abstract Cauchy problem in X_0

$$\begin{cases} (\eta,w)_t=A(\eta,w), \\ (\eta,w)(0)=(\eta^0,w^0). \end{cases} \tag{5.6.9}$$

On the other hand, the adjoint of the operator A (denoted by A^*) is give by

$$A^*(\varphi,\psi)=\begin{pmatrix} \psi_x+\psi_{xxx} \\ \varphi_x+\varphi_{xxx} \end{pmatrix}, \forall(\varphi,\psi)\in\mathcal{D}(A^*), \tag{5.6.10}$$

where $A^*:\mathcal{D}(A^*)\subset X_0\to X_0$ with

$$\mathcal{D}(A^*)=\left\{ \begin{array}{c} (\varphi,\psi)\in\left(H^3(0,L)\right)^2; \varphi(0)=\psi(0)=\varphi(L)=\psi(L)=\varphi_x(0)=0 \\ \text{and } \psi_x(L)-\alpha_1\varphi_x(L)=0 \end{array} \right\}.$$

Proposition 5.2. *The operators A and A^* are dissipative in X_0.*

Proof. Consider $(\eta, w) \in \mathcal{D}(A)$. By multiplying the first equation of the system (5.6.5) by η, the second one by w and integrating by parts in $(0, L)$, we obtain

$$\int_0^L (-w_x - w_{xxx})\, \eta dx = \int_0^L w\eta_x dx + \int_0^L w_{xx}\eta_x dx$$

and

$$\int_0^L (-\eta_x - \eta_{xxx})\, w dx = -\int_0^L w\eta_x dx + \int_0^L \eta_{xx} w_x dx.$$

Therefore,

$$\begin{aligned}\langle A(\eta, w), (\eta, w)\rangle_{X_0} &= \int_0^L (\eta_x w_x)_x\, dx = \eta_x(L)\, w_x(L) \\ &= -\alpha_1 |\eta_x(L)|^2 \leq 0.\end{aligned}$$

Hence A is dissipative in X_0. Analogously, we deduce that

$$\langle A^*(\varphi, \psi), (\varphi, \psi)\rangle_{X_0} = -\alpha_1 |\varphi_x(L)|^2 \leq 0,\ \forall (\varphi, \psi) \in \mathcal{D}(A^*),$$

i.e., A^* is dissipative in X_0. □

Since A and A^* are both dissipative, A is a closed operator and the respective domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense and compactly embedded in X_0 we conclude that A generates a C^0 semigroup of contractions in X_0 which will be denoted by $(S(t))_{t\geq 0}$. Then, classical existence results give us the global well-posedness for (5.6.5)-(5.6.7):

Theorem 5.25. *Let $(\eta^0, w^0) \in X_0$. Then, there exists a unique weak solution $(\eta, w) = S(\cdot)(\eta^0, w^0)$ of (5.6.5)-(5.6.7) such that*

$$(\eta, w) \in C([0, T]; X_0). \tag{5.6.11}$$

Moreover, if $(\eta^0, w^0) \in \mathcal{D}(A)$, then (5.6.5)-(5.6.7) has a unique (classical) solution (η, w) such that

$$(\eta, w) \in C([0, T]; \mathcal{D}(A)) \cap C^1(0, T; X_0). \tag{5.6.12}$$

Additional regularity results for the weak solutions of (5.6.5)-(5.6.7) are proven in the next theorem.

Theorem 5.26. *Let $(\eta^0, w^0) \in X_0$ and $(\eta, w) = S(\cdot)(\eta^0, w^0)$. Then, for any $T > 0$*

$$\begin{aligned}\int_0^L \left(|\eta^0(x)|^2 + |w^0(x)|^2\right) dx - \int_0^L \left(|\eta(T, x)|^2 + |w(T, x)|^2\right) dx \\ = 2\alpha_1 \int_0^T |\eta_x(t, L)|^2\, dt,\end{aligned} \tag{5.6.13}$$

and

$$\begin{aligned}\frac{T}{2}\int_0^L \left(|\eta^0(x)|^2 + |w^0(x)|^2\right) dx = \frac{1}{2}\int_0^T\int_0^L \left(|\eta|^2 + |w|^2\right) dxdt \\ + \alpha_1 \int_0^T (T - t)\, |\eta_x(t, L)|^2\, dt.\end{aligned} \tag{5.6.14}$$

Moreover, there exist a positive constant $C = C(T, L)$ *such that*

$$\|(\eta, w)\|_{L^2\left(0,T;(H^1(0,L))^2\right)} \leq C \left\|\left(\eta^0, w^0\right)\right\|_{X_0}. \tag{5.6.15}$$

Remark 5.10. *Observe that Theorem 5.26 reveals a Kato smoothing effect, this is possible due to damping in the boundary condition of Neumann type.*

Proof of Theorem 5.26. Let C denote a positive constant which may vary from line to line. Pick any $(\eta^0, w^0) \in \mathcal{D}(A)$. Multiplying the first equation of (5.6.5) by η, the second one by w, adding the two obtained equations and integrating over $(0,T)\times(0,L)$, we obtain after some integrations by parts (5.6.13). The identity may be extended to any initial state $(\eta^0, w^0) \in X_0$ by a density argument. Multiplying the first equation of (5.6.5) by $(T-t)\eta$, the second one by $(T-t)w$, and integrating over $(0,T)\times(0,L)$ we derive (5.6.14) in a similar way. Let us proceed to the proof of (5.6.15). Multiply the first equation of (5.6.5) by xw, the second by $x\eta$, integrate over $(0,T)\times(0,L)$. After some integrations by parts and using the boundary conditions (5.6.6), we obtain that

$$\begin{aligned}\frac{3}{2}\int_0^T\int_0^L \left(w_x^2+\eta_x^2\right) dxdt = -\int_0^T\int_0^L x\left(\eta w\right)_t + \frac{1}{2}\int_0^T\int_0^L \left(w^2+\eta^2\right) dxdt \\ +\frac{L\left(\alpha_1^2+1\right)}{2}\int_0^T \left|\eta_x(t,L)\right|^2 dt\end{aligned} \tag{5.6.16}$$

Observe that,

$$\int_0^T\int_0^L \left(w^2+\eta^2\right) dxdt \leq 2\int_0^T E(0)\,dt = 2TE(0), \tag{5.6.17}$$

where

$$E(t) = \frac{1}{2}\int_0^L \left(w^2+\eta^2\right) dx$$

is the energy associated of the system (5.6.5). Furthermore,

$$\begin{aligned}-\int_0^T\int_0^L x\left(\eta w\right)_t &= -\int_0^L x\eta(T,x)\,w(T,x)\,dx + \int_0^L x\eta^0(x)\,w^0(x)\,dx \\ &\leq L\int_0^L \left(\frac{|\eta(T,x)|^2+|w(T,x)|^2}{2}\right) dx + L\int_0^L \left(\frac{|\eta^0|^2+|w^0|^2}{2}\right) dx \\ &\leq LE(T) + L\int_0^L \left(\frac{|\eta^0|^2+|w^0|^2}{2}\right) dx \leq 2LE(0),\end{aligned} \tag{5.6.18}$$

since $E(t)$ is decreasing. Thus, from (5.6.16)-(5.6.18), we obtain that

$$\frac{3}{2}\int_0^T\int_0^L \left(w_x^2+\eta_x^2\right) dxdt \leq 2TE(0) + 2LE(0) + \left(\frac{L\left(\alpha_1^2+1\right)}{2\alpha_1}\right)\int_0^T \alpha_1 \left|\eta_x(t,L)\right|^2 dt. \tag{5.6.19}$$

Then, (5.6.15) follows from (5.6.19) and (5.6.13). □

Now, we prove the same properties for the system (5.6.5) with the boundary conditions

$$\begin{cases} \eta(t,L)=0, \;\; \eta_x(t,0)=0, \;\; \alpha_2\eta(t,0)+\alpha_3\eta_x(t,L)+w_{xx}(t,0)=0 & \text{in } (0,T) \\ w(t,0)=0, \;\; w(t,L)=0, \;\; w_x(t,L)-\alpha_3(\eta(t,0)-\eta_x(t,L))=0 & \text{in } (0,T), \end{cases} \tag{5.6.20}$$

and the initial conditions (5.6.7). Consider the operator $A:\mathcal{D}(A)\subset X_0\to X_0$, where

$$\mathcal{D}(A)=\left\{ \begin{array}{l} (\eta,w)\in\left(H^3(0,L)\right)^2; w(0)=\eta(L)=w(L)=\eta_x(0)=0, \\ \alpha_2\eta(0)+\alpha_3\eta_x(L)+w_{xx}(0)=0 \text{ and } w_x(L)-\alpha_3(\eta(0)-\eta_x(L))=0 \end{array}\right\}$$

and $A^*:\mathcal{D}(A^*)\subset X_0\to X_0$, where

$$\mathcal{D}(A^*)=\left\{ \begin{array}{l} (\varphi,\psi)\in\left(H^3(0,L)\right)^2; \psi(0)=\varphi(L)=\psi(L)=\varphi_x(0)=0, \\ \alpha_2\varphi(0)-(\alpha_3\varphi_x(L)+\psi_{xx}(0))=0 \text{ and } \psi_x(L)-\alpha_3(\varphi(0)+\varphi_x(L))=0 \end{array}\right\}.$$

Proposition 5.3. *The operators A and A^* are dissipative in X_0.*

Proof. Consider $(\eta,w)\in\mathcal{D}(A)$. By multiplying the first equation of the system (5.6.5) by η, the second one by w, integrating by parts in $(0,L)$ and using the boundary conditions (5.6.20), the following holds

$$\langle A(\eta,w),(\eta,w)\rangle_{X_0}=-\alpha_2|\eta(0)|^2-\alpha_3\left|\eta_x^2(L)\right|\le 0.$$

Hence A is dissipative in X_0. Analogously, we deduce that

$$\langle A^*(\varphi,\psi),(\varphi,\psi)\rangle_{X_0}=-\alpha_2|\varphi(0)|^2-\alpha_3|\varphi_x(L)|^2\le 0, \; \forall(\varphi,\psi)\in\mathcal{D}(A^*),$$

i.e., A^* is dissipative in X_0. □

Then, classical existence results give us the global well-posedness for (5.6.5) with boundary conditions (5.6.20) and initial conditions (5.6.7):

Theorem 5.27. *Let $(\eta^0,w^0)\in X_0$. Then, there exists a unique weak solution $(\eta,w)=S(\cdot)(\eta^0,w^0)$ of (5.6.5) with boundary conditions (5.6.20) and initial conditions (5.6.7) such that*

$$(\eta,w)\in C([0,T];X_0). \tag{5.6.21}$$

Moreover, if $(\eta^0,w^0)\in\mathcal{D}(A)$, then (5.6.5) with boundary conditions (5.6.20) and initial conditions (5.6.7) has a unique (classical) solution (η,w) such that

$$(\eta,w)\in C([0,T];\mathcal{D}(A))\cap C^1(0,T;X_0).$$

Additional regularity results for the weak solutions of (5.6.5) with boundary conditions (5.6.20)-(5.6.7) are given in the next theorem:

Theorem 5.28. *Let $(\eta^0,w^0)\in X_0$ and $(\eta,w)=S(\cdot)(\eta^0,w^0)$. Then, for any $T>0$*

$$\begin{aligned} \int_0^L\left(|\eta^0(x)|^2+|w^0(x)|^2\right)dx-\int_0^L\left(|\eta(T,x)|^2+|w(T,x)|^2\right)dx \\ =2\int_0^T\left(\alpha_2|\eta(t,0)|^2+\alpha_3|\eta_x(t,L)|^2\right)dt, \end{aligned} \tag{5.6.22}$$

and

$$\frac{T}{2}\int_0^L \left(\left|\eta^0(x)\right|^2+\left|w^0(x)\right|^2\right)dx=\frac{1}{2}\int_0^T\int_0^L\left(|\eta|^2+|w|^2\right)dxdt \\ +\int_0^T (T-t)\left(\alpha_2\left|\eta(t,0)\right|^2+\alpha_3\left|\eta_x(t,L)\right|^2\right)dt. \tag{5.6.23}$$

Moreover, there exist a positive constant $C=C(T,L)$ such that

$$\|(\eta,w)\|_{L^2\left(0,T;(H^1(0,L))^2\right)}\leq C\left\|\left(\eta^0,w^0\right)\right\|_{X_0}. \tag{5.6.24}$$

Remark 5.11. *Observe that Theorem 5.28, as Theorem 5.26, reveals a Kato smoothing effect. This is possible due to damping in the boundary condition of Neumann-Dirichlet type.*

Proof. To obtain (5.6.22) and (5.6.23) we proceed as in the proof of Theorem 5.26. Let us proceed to the proof of (5.6.24). Multiply the first equation of (5.6.5) by xw, the second by $x\eta$, integrate over $(0,T)\times(0,L)$. After some integrations by parts, we have that

$$\frac{3}{2}\int_0^T\int_0^L\left(w_x^2+\eta_x^2\right)dxdt=-\int_0^T\int_0^L x\left(\eta w\right)_t+\frac{1}{2}\int_0^T\int_0^L\left(w^2+\eta^2\right)dxdt \\ +\frac{L}{2}\int_0^T\left|\eta_x(t,L)\right|^2dt+\frac{L}{2}\int_0^T\left|w_x(t,L)\right|^2dt. \tag{5.6.25}$$

Using the boundary conditions (5.6.20), we obtain

$$\begin{aligned}\frac{L}{2}\int_0^T\left|w_x(t,L)\right|^2dt&=\frac{L\alpha_3^2}{2}\int_0^T\left|\eta(0)-\eta_x(L)\right|^2dt\\&=\frac{L\alpha_3^2}{2}\int_0^T\left(\left|\eta(0)\right|^2+\left|\eta_x(L)\right|^2-2\eta(0)\,\eta_x(L)\right)dt\\&\leq L\alpha_3^2\int_0^T\left(\left|\eta(0)\right|^2+\left|\eta_x(L)\right|^2\right)dt.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{L}{2}\int_0^T\left(\left|\eta_x(t,L)\right|^2+\left|w_x(t,L)\right|^2\right)dt&=L\alpha_3^2\int_0^T\left|\eta(0)\right|^2dt+\left(\frac{L}{2}+L\alpha_3^2\right)\int_0^T\left|\eta_x(L)\right|^2dt\\&=\frac{L\alpha_3^2}{\alpha_2}\int_0^T\alpha_2\left|\eta(0)\right|^2dt\\&\quad+\left(\frac{L}{2\alpha_3}+L\alpha_3\right)\int_0^T\alpha_3\left|\eta_x(L)\right|^2dt\\&\leq K\int_0^T\left(\alpha_2\left|\eta(0)\right|^2+\alpha_3\left|\eta_x(L)\right|^2\right)dt,\end{aligned} \tag{5.6.26}$$

where $K=\max\left\{\frac{L\alpha_3^2}{\alpha_2},\left(\frac{L}{2\alpha_3}+L\alpha_3\right)\right\}$.

Note that

$$\int_0^T\int_0^L \left(w^2+\eta^2\right)dxdt \leq 2\int_0^T E\left(0\right)dt = 2TE\left(0\right), \tag{5.6.27}$$

where

$$E\left(t\right)=\frac{1}{2}\int_0^L \left(w^2+\eta^2\right)dx$$

is the energy associated to the system (5.6.5). Furthermore,

$$\begin{aligned}
-\int_0^T\int_0^L x\left(\eta w\right)_t &= -\int_0^L x\eta\left(T,x\right)w\left(T,x\right)dx + \int_0^L x\eta^0\left(x\right)w^0\left(x\right)dx \\
&\leq L\int_0^L\left(\frac{\left|\eta\left(T,x\right)\right|^2+\left|w\left(T,x\right)\right|^2}{2}\right)dx + L\int_0^L\left(\frac{\left|\eta^0\right|^2+\left|w^0\right|^2}{2}\right)dx \\
&\leq LE\left(T\right)+L\int_0^L\left(\frac{\left|\eta^0\right|^2+\left|w^0\right|^2}{2}\right)dx \leq 2LE\left(0\right),
\end{aligned} \tag{5.6.28}$$

since $E\left(t\right)$ is decreasing. Thus, from (5.6.25)-(5.6.28), we obtain that

$$\frac{3}{2}\int_0^T\int_0^L \left(w_x^2+\eta_x^2\right)dxdt \leq 2TE\left(0\right)+2LE\left(0\right)+K\int_0^T\left(\alpha_2\left|\eta\left(0\right)\right|^2+\alpha_3\left|\eta_x\left(L\right)\right|^2\right)dt. \tag{5.6.29}$$

Then, (5.6.24) follows from (5.6.29) and (5.6.22). □

5.6.2 Adjoint System

This section is devoted to study the properties of the adjoint system of (5.6.5), namely

$$\begin{cases} \varphi_t+\psi_x+\psi_{xxx}=0 & \text{in } (0,T)\times(0,L), \\ \psi_t+\varphi_x+\varphi_{xxx}=0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.6.30}$$

satisfying the boundary conditions

$$\begin{cases} \varphi\left(t,0\right)=\varphi\left(t,L\right)=\varphi_x\left(t,0\right)=0 & \text{in } (0,T) \\ \psi\left(t,0\right)=\psi\left(t,L\right)=0,\ \psi_x\left(t,L\right)-\alpha_1\varphi_x\left(t,L\right)=0 & \text{in } (0,T), \end{cases} \tag{5.6.31}$$

and the initial conditions

$$\varphi\left(T,x\right)=\varphi^1\left(x\right),\ \ \psi\left(T,x\right)=\psi^1\left(x\right) \qquad \text{in } (0,L). \tag{5.6.32}$$

Remark that the change of variable $t\mapsto T-t$ reduces system (5.6.30)-(5.6.32) to

$$\begin{cases} \varphi_t-\psi_x-\psi_{xxx}=0 & \text{in } (0,T)\times(0,L), \\ \psi_t-\varphi_x-\varphi_{xxx}=0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.6.33}$$

satisfying the boundary conditions

$$\begin{cases} \varphi\left(t,0\right)=\varphi\left(t,L\right)=\varphi_x\left(t,0\right)=0 & \text{in } (0,T) \\ \psi\left(t,0\right)=\psi\left(t,L\right)=0,\ \psi_x\left(t,L\right)-\alpha_1\varphi_x\left(t,L\right)=0 & \text{in } (0,T), \end{cases} \tag{5.6.34}$$

and the initial conditions

$$\varphi(0,x)=\varphi^0(x), \ \ \psi(0,x)=\psi^0(x) \qquad \text{in } (0,L). \tag{5.6.35}$$

Thus, (5.6.33)-(5.6.35) is equivalent to

$$\begin{cases} (\varphi,\psi)_t = A^*(\varphi,\psi); \\ (\varphi,\psi)(0)=(\varphi^0,\psi^0), \end{cases}$$

where A^* is given by (5.6.10). Observe that the properties of the solutions of (5.6.33)-(5.6.35) are similar to the ones deduced in Theorem 5.25 and Theorem 5.26. More precisely, we have

Theorem 5.29. *Let* $(\varphi^0,\psi^0)\in X_0$. *Then there exist a unique weak solution* $(\varphi,\psi)=S(\cdot)(\varphi^0,\psi^0)$ *of (5.6.33)-(5.6.35) such that*

$$(\varphi,\psi)\in C([0,T];X_0)\cap L^2\left(0,T;\left(H^1(0,L)\right)^2\right) \tag{5.6.36}$$

and the following estimates holds

$$\|(\varphi,\psi)\|_{L^2\left(0,T;(H^1(0,L))^2\right)} \le c_1 \left\|\left(\varphi^0,\psi^0\right)\right\|_{X_0}, \tag{5.6.37}$$

$$\begin{aligned} \int_0^L \left(\left|\varphi^0(x)\right|^2+\left|\psi^0(x)\right|^2\right)dx - \int_0^L \left(|\varphi(T,x)|^2+|\psi(T,x)|^2\right)dx \\ = 2\alpha_1\int_0^T |\varphi_x(t,L)|^2\,dt \end{aligned} \tag{5.6.38}$$

and

$$\begin{aligned} \frac{T}{2}\int_0^L \left(\left|\varphi^0(x)\right|^2+\left|\psi^0(x)\right|^2\right)dx = \frac{1}{2}\int_0^T\int_0^L \left(|\varphi|^2+|\psi|^2\right)dxdt \\ +\alpha_1\int_0^T (T-t)\,|\varphi_x(t,L)|^2\,dt. \end{aligned} \tag{5.6.39}$$

where c_1 *and* α_1 *is a positive constants.*

Remark 5.12. *As in Theorems 5.27 and 5.28, we can extend the result of the above theorem to the boundary conditions given by (5.6.20).*

5.6.3 The nonhomogeneous system

Now we use the adjoint system to define our solution by transposition. Consider the nonhomogeneous system given by

$$\begin{cases} \eta_t+w_x+w_{xxx}=0 & \text{in } (0,T)\times(0,L), \\ w_t+\eta_x+\eta_{xxx}=0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.6.40}$$

satisfying the boundary conditions

$$\begin{cases} \eta(t,0)=0, \ \ \eta(t,L)=0, \ \ \eta_x(t,0)=0 & \text{in } (0,T) \\ w(t,0)=0, \ \ w(t,L)=0, \ \ w_x(t,L)+\alpha_1\eta_x(t,L)=g_2(t) & \text{in } (0,T), \end{cases} \tag{5.6.41}$$

and the initial conditions

$$\eta(0,x)=\eta^0(x),\ \ w(0,x)=w^0(x) \qquad \text{in } (0,L). \tag{5.6.42}$$

From (5.2.9), (5.6.34) and (5.6.41), we have that

$$\begin{aligned}0&=\int_0^L[\eta\varphi+w\psi]_0^T\,dx-\int_0^T w_x(t,L)\,\varphi_x(t,L)\,dt-\int_0^T\eta_x(t,L)\,\psi_x(t,L)\,dt\\&=\int_0^L[\eta\varphi+w\psi]_0^T\,dx-\int_0^T w_x(t,L)\,\varphi_x(t,L)\,dt-\int_0^T\eta_x(t,L)\,(\alpha_1\varphi_x(t,L))\,dt\\&=\int_0^L[\eta\varphi+w\psi]_0^T\,dx-\int_0^T\varphi_x(t,L)\,[w_x(t,L)+\alpha_1\eta_x(t,L)]\,dt.\end{aligned} \tag{5.6.43}$$

Therefore,

$$0=\int_0^L[\eta\varphi+w\psi]_0^T\,dx-\int_0^T\varphi_x(t,L)\,g_2(t)\,dt. \tag{5.6.44}$$

Definition 5.5. *Given $T>0$, $(\eta^0,w^0)\in X_0$ and $g_2\in L^2(0,T)$, we call a solution by transposition of (5.6.40)-(5.6.42), a function*

$$(\eta,w)\in\left[L^2((0,T)\times(0,L))\right]^2, \tag{5.6.45}$$

satisfying

$$\int_0^L(\eta(t)\,\varphi(t)+w(t)\,\psi(t))\,dx=-\int_0^t\varphi_x(s,L)\,g_2(s)\,ds+\int_0^L\left(\eta^0\varphi(0)+w^0\psi(0)\right)dx,$$

where (φ,ψ) is a solution of

$$\begin{cases}\varphi_t+\psi_x+\psi_{xxx}=0 & in\ (0,T)\times(0,L),\\ \psi_t+\varphi_x+\varphi_{xxx}=0 & in\ (0,T)\times(0,L),\end{cases} \tag{5.6.46}$$

satisfying the boundary conditions

$$\begin{cases}\varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0 & in\ (0,T)\\ \psi(t,0)=\psi(t,L)=0,\ \psi_x(t,L)-\alpha_1\varphi_x(t,L)=0 & in\ (0,T),\end{cases} \tag{5.6.47}$$

and the initial conditions

$$\varphi(T,x)=0,\ \ \psi(T,x)=0 \qquad in\ (0,L). \tag{5.6.48}$$

5.7 Exact Boundary Controllability Results: The Linear System With Boundary Damping

This section is devoted to the analysis of the exact controllability property for the linear system (5.6.5) with boundary control of Neumann type. More precisely, given $T>0$ and

(η^0, w^0), $\left(\eta^T, w^T\right) := (\eta^1, w^1) \in X_0$, we study the existence a control $g_2(t) \in L^2(0,T)$, such that the solution (η, w) of the system

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x) & \text{in } (0,L), \end{cases} \tag{5.7.1}$$

satisfying the boundary conditions

$$\begin{cases} \eta(t,0) = 0, \quad \eta(t,L) = 0, \quad \eta_x(t,0) = 0 & \text{in } (0,T) \\ w(t,0) = 0, \quad w(t,L) = 0, \quad w_x(t,L) + \alpha_1 \eta_x(t,L) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.7.2}$$

satisfies

$$\eta(T,\cdot) = \eta^1, \quad w(T,\cdot) = w^1 \text{ in } L^2(0,L). \tag{5.7.3}$$

From now on, we shall consider only the case $\eta^0 = w^0 = 0$.

5.7.1 Single Control of Neumann type

In this section we study the exact controllability, in time T, for the system (5.7.1)-(5.7.2). We first give an equivalent condition for the exact controllability property:

Lemma 5.21. *Let $(\eta^1, w^1) \in X_0$. Then, there exist a control $g_2(t) \in L^2(0,T)$, such that the solution (η, w) of (5.7.1)-(5.7.2) satisfies (5.7.3) if and only if*

$$\int_0^L \left(\eta^1 \varphi^1 + w^1 \psi^1\right) dx = -\int_0^T \varphi_x(t,L)\, g_2(t)\, dt + \int_0^L \left(\eta^0 \varphi(0) + w^0 \psi(0)\right) dx, \tag{5.7.4}$$

for any $(\varphi^1, \psi^1) \in X_0$, (φ, ψ) being the solution of the backward system (5.6.30)-(5.6.32).

Proof. Identity (5.7.4) is obtained multiplying the equations in (5.7.1) by the solution (φ, ψ) of (5.6.30)-(5.6.32) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result:

Theorem 5.30. *Let $\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2+kl+l^2}{3}} : k, l \in \mathbb{N}^* \right\}$. Then, $\forall\ L \in (0,+\infty) \setminus \mathcal{N}$ and $\forall$ $T > 0$, $\exists\ C(T,L) > 0$ such that*

$$\left\| \left(\varphi^1, \psi^1\right) \right\|_{X_0}^2 \le C \int_0^T \left|\varphi_x(t,L)\right|^2 dt, \tag{5.7.5}$$

holds for any $(\varphi^1, \psi^1) \in X_0$, where (φ, ψ) is the solution of (5.6.30)-(5.6.32) with initial data (φ^1, ψ^1).

Proof. The change of variable $t = T - t$ transforms (5.6.30)-(5.6.32) into (5.6.33)-(5.6.35). Hence, inequality (5.7.5) is equivalent to

$$\left\| \left(\varphi^0, \psi^0\right) \right\|_{X_0}^2 \le C \int_0^T \left|\varphi_x(t,L)\right|^2 dt, \tag{5.7.6}$$

for any $(\varphi^0,\psi^0)\in X_0$, where (φ,ψ) is the solution of (5.6.33)-(5.6.35) with initial data (φ^0,ψ^0). To prove the observability (5.7.6), we proceed as in previous cases. Let us suppose that (5.7.6) does not hold. In this case, it follows that there exists a sequence $\{(\varphi_n^0,\psi_n^0)\}_{n\in\mathbb{N}}$ in X_0 such that

$$\left\|\left(\varphi_n^0,\psi_n^0\right)\right\|_{X_0}=1 \tag{5.7.7}$$

and

$$\int_0^T\left|\varphi_{n,x}(t,L)\right|^2dt\to 0 \text{ in } L^2(0,T)\text{, as } n\to\infty, \tag{5.7.8}$$

where (φ_n,ψ_n) is the solution of (5.6.33)-(5.6.35) with initial data (φ_n^0,ψ_n^0). From Theorem 5.29 we obtain that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;(H^1(0,L))^2\right)$ and from (5.6.33) we have that $\{((\varphi_n)_t,(\psi_n)_t)\}_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;(H^{-2}(0,L))^2\right)$. Since

$$\left(H^1(0,L)\right)^2\hookrightarrow_{cc}\left[L^2(0,L)\right]^2\hookrightarrow\left(H^{-2}(0,L)\right)^2,$$

being the first embedding compact, it follows that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;X_0)$. Therefore, there exists a subsequence, still denoted by the same index, such that

$$(\varphi_n,\psi_n)\longrightarrow(\varphi,\psi)\ \text{ in } L^2(0,T;X_0).$$

Moreover, by Theorem 5.29 and (5.7.8), we see that $\{(\varphi_n^0,\psi_n^0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X_0, hence for some pair $(\varphi^0,\psi^0)\in X_0$, we have that

$$\left(\varphi_n^0,\psi_n^0\right)\longrightarrow\left(\varphi^0,\psi^0\right)\ \text{ in } X_0. \tag{5.7.9}$$

From (5.6.39) and (5.7.8) we infer that

$$\varphi_x(t,L)=0 \tag{5.7.10}$$

and

$$\left\|\left(\varphi^0,\psi^0\right)\right\|_{X_0}=1. \tag{5.7.11}$$

Hence, (φ,ψ) is a solution of

$$\begin{cases}\varphi_t-\psi_x-\psi_{xxx}=0 & \text{in } (0,T)\times(0,L),\\ \psi_t-\varphi_x-\varphi_{xxx}=0 & \text{in } (0,T)\times(0,L),\\ \varphi(0,x)=\varphi^0(x),\ \ \psi(0,x)=\psi^0(x) & \text{in } (0,L),\end{cases} \tag{5.7.12}$$

satisfying the boundary conditions

$$\begin{cases}\varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=0 & \text{in } (0,T)\\ \psi(t,0)=\psi(t,L)=0,\ \psi_x(t,L)-\alpha_1\varphi_x(t,L)=0 & \text{in } (0,T),\end{cases} \tag{5.7.13}$$

and, in addition,

$$\varphi_x(\cdot,L)=0. \tag{5.7.14}$$

Remark that (5.7.11) implies that the solutions of (5.7.12)-(5.7.14) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following results: □

Lemma 5.22. *For any $T>0$, let N_T denote the space of the initial states $(\varphi^0,\psi^0)\in X_0$ such that the solution $(\varphi,\psi)=S(\cdot)(\varphi^0,\psi^0)$ of (5.7.12)-(5.7.13) satisfies (5.7.14). Then, for $L\in(0,+\infty)\setminus\mathcal{N}$, $N_T=\{0\}$, $\forall T>0$.*

Lemma 5.23. *Let $L>0$. Consider the assertion*

$$(\mathcal{A}) \qquad \exists\lambda\in\mathbb{C},\ \exists\left(\varphi^0,\psi^0\right)\in\left(H^3(0,L)\right)^2\setminus\{(0,0)\} \text{ such that}$$

$$\begin{cases} \lambda\varphi^0=\left[(\psi^0)'+(\psi^0)'''\right], \\ \lambda\psi^0=\left[(\varphi^0)'+(\varphi^0)'''\right], \\ \varphi^0(0)=\varphi^0(L)=(\varphi^0(0))'=(\varphi^0(L))'=0, \\ \psi^0(0)=\psi^0(L)=(\psi^0(L))'=0. \end{cases}$$

Then, $(\mathcal{A})$ holds if and only if $L\in\mathcal{N}$.

The proofs of Lemmas 5.22 and 5.23 follow exactly the same techniques used in [62, Lemma 3.4] and in Lemma 5.7, so it is omitted. Thus, with the Lemmas 5.22 and 5.23 in hands, Theorem 5.30 follows.

The following theorem gives a positive answer for the control problem (5.7.1)-(5.7.2):

Theorem 5.31. *Let $T>0$ and $L\in(0,+\infty)\setminus\mathcal{N}$. Then the system (5.7.1)-(5.7.2) is exactly controllable in time T.*

Proof. Let us define the following functional

$$\Lambda\left(\varphi^1,\psi^1\right)=\frac{1}{2}\left\|\varphi_x(\cdot,L)\right\|^2_{L^2(0,T)}-\int_0^L\left(\eta^1(x)\varphi^1(x)+w^1(x)\psi^1(x)\right)dx \tag{5.7.15}$$

where $(\varphi^1,\psi^1)\in X_0$ and (φ,ψ) is the solution of the backward system (5.6.30)-(5.6.32) with initial data (φ^1,ψ^1).

Let $\left(\hat{\varphi}^1,\hat{\psi}^1\right)\in X_0$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.7.4) is satisfied with $g_2\in L^2(0,T)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.7.4) and (5.7.5), it follows that

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right)&=\frac{1}{2}\left\|\varphi_x(\cdot,L)\right\|^2_{L^2(0,T)}-\int_0^L\left(\eta^1(x)\varphi^1(x)+w^1(x)\psi^1(x)\right)dx\\&\geq C^{-1}\left\|\left(\varphi^1,\psi^1\right)\right\|_{X_0}.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.31 is complete. □

5.7.2 Double Control Mixed Type

This section is devoted to the analysis of the exact controllability property for the linear system (5.6.5) with mixed boundary controls. More precisely, given $T>0$ and (η^0,w^0),

$\left(\eta^T, w^T\right) := (\eta^1, w^1) \in X_0$, we study the existence of controls $h_0, g_2 \in L^2(0,T)$ such that the solution (η, w) of the system

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_t + \eta_x + \eta_{xxx} = 0 & \text{in } (0,T)\times(0,L), \\ \eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x) & \text{in } (0,L), \end{cases} \tag{5.7.16}$$

satisfying the boundary conditions

$$\begin{cases} \eta(t,L) = 0, \;\; \eta_x(t,0) = 0, \;\; \alpha_2\eta(t,0) + \alpha_3\eta_x(t,L) + w_{xx}(t,0) = h_0(t) & \text{in } (0,T) \\ w(t,0) = 0, \;\; w(t,L) = 0, \;\; w_x(t,L) - \alpha_3(\eta(t,0) - \eta_x(t,L)) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.7.17}$$

satisfies

$$\eta(T,\cdot) = \eta^1, \quad w(T,\cdot) = w^1 \text{ in } L^2(0,L). \tag{5.7.18}$$

From now on, we shall consider only the case $\eta^0 = w^0 = 0$.

Definition 5.6. *Given $T > 0$, $(\eta^0, w^0) \in X_0$ and $(h_0, g_2) \in \left(L^2(0,T)\right)^2$, we call a solution by transposition of (5.7.16) with boundary condition (5.7.17), a function*

$$(\eta, w) \in \left[L^2((0,T)\times(0,L))\right]^2, \tag{5.7.19}$$

satisfying

$$\int_0^L (\eta(t)\varphi(t) + w(t)\psi(t))\,dx = -\int_0^t (\varphi(s,0)h_0(s) + \varphi_x(s,L)g_2(s))\,ds + \int_0^L \left(\eta^0\varphi(0) + w^0\psi(0)\right)dx,$$

where (φ, ψ) is a solution of (5.6.46) with boundary condition

$$\begin{cases} \varphi(t,L) = \varphi_x(t,0) = 0, \; \alpha_3(\varphi(t,0) + \varphi_x(t,L)) - \psi_x(t,L) = 0 & \text{in } (0,T) \\ \psi(t,0) = \psi(t,L) = 0, \; \alpha_2\varphi(t,0) - \alpha_3\varphi_x(t,L) - \psi_{xx}(t,0) = 0 & \text{in } (0,T), \end{cases} \tag{5.7.20}$$

and initial conditions

$$\varphi(T,x) = 0, \quad \psi(T,x) = 0 \qquad \text{in } (0,L). \tag{5.7.21}$$

Definition 5.7. *Let $T > 0$. System (5.7.16) is exact controllable in time T if for any initial and final data (η^0, w^0), $(\eta^1, w^1) \in X_0$, there exist controls functions $h_0, g_2 \in L^2(0,T)$ such that the solution of (5.7.16)-(5.7.17) satisfies (5.7.18).*

We first give an equivalent condition for the exact controllability property:

Lemma 5.24. *Let $(\eta^1, w^1) \in X_0$. Then, there exist two controls $g_0(t), h_2(t) \in L^2(0,T)$, such that the solution (η, w) of (5.7.16)-(5.7.17) satisfies (5.7.18) if and only if*

$$\int_0^L \left(\eta^1\varphi^1 + w^1\psi^1\right)dx = -\int_0^T (\varphi(t,0)h_0(t) + \varphi_x(t,L)g_2(t))\,dt + \int_0^L \left(\eta^0\varphi(0) + w^0\psi(0)\right)dx, \tag{5.7.22}$$

for any $(\varphi^1, \psi^1) \in X_0$, (φ, ψ) being the solution of the backward system (5.6.30)-(5.6.32).

Proof. Identity (5.7.22) is obtained multiplying the equations in (5.7.16) by the solution (φ, ψ) of (5.6.30)-(5.6.32) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result:

Theorem 5.32. *For any $T > 0$ and $L \in \mathcal{N}$ there exists $C = C(T, L) > 0$ such that the inequality*

$$\left\|\left(\varphi^1, \psi^1\right)\right\|_{X_0}^2 \leq C \int_0^T \left(\left|\varphi_x(t, L)\right|^2 + \left|\varphi(t, 0)\right|^2\right) dt, \tag{5.7.23}$$

holds for any $(\varphi^1, \psi^1) \in X_0$, where (φ, ψ) is the solution of (5.6.30)-(5.6.32) with initial data (φ^1, ψ^1).

Proof. The change of variable $t = T - t$ transforms (5.6.30)-(5.6.32) into (5.6.33)-(5.6.35). Hence, inequality (5.7.23) is equivalent to

$$\left\|\left(\varphi^0, \psi^0\right)\right\|_{X_0}^2 \leq C \int_0^T \left(\left|\varphi_x(t, L)\right|^2 + \left|\varphi(t, 0)\right|^2\right) dt, \tag{5.7.24}$$

for any $(\varphi^0, \psi^0) \in X_0$, where (φ, ψ) is the solution of (5.6.33)-(5.6.35) with initial data (φ^0, ψ^0).

We assume that (5.7.24) does not holds. Then, there exists a sequence $\{(\varphi_n^0, \psi_n^0)\}_{n\in\mathbb{N}} \in X_0$ such that

$$\left\|\left(\varphi_n^0, \psi_n^0\right)\right\|_{X_0} = 1, \tag{5.7.25}$$

$$\left\|\varphi_{n,x}(\cdot, L)\right\|_{L^2(0,T)}^2 \longrightarrow 0, \text{ as } n \to \infty \tag{5.7.26}$$

and

$$\left\|\varphi_n(\cdot, 0)\right\|_{L^2(0,T)}^2 \longrightarrow 0, \text{ as } n \to \infty \tag{5.7.27}$$

where (φ_n, ψ_n) is the solution of (5.6.33)-(5.6.35) with initial data (φ_n^0, ψ_n^0). Hence, proceeding as in the proof of the Theorem 5.30 (see also Remark 5.12) we obtain (φ, ψ) solution of

$$\begin{cases} \varphi_t - \psi_x - \psi_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \psi_t - \varphi_x - \varphi_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \varphi(0,x) = \varphi^0(x), \ \ \psi(0,x) = \psi^0(x) & \text{in } (0,L), \end{cases} \tag{5.7.28}$$

satisfying

$$\left\|\left(\varphi^0, \psi^0\right)\right\|_{X_0} = 1, \tag{5.7.29}$$

$$\begin{cases} \varphi(t, L) = \varphi_x(t, 0) = 0, \ \alpha_3\left(\varphi(t, 0) + \varphi_x(t, L)\right) - \psi_x(t, L) = 0 & \text{in } (0,T) \\ \psi(t, 0) = \psi(t, L) = 0, \ \alpha_2\varphi(t, 0) - \alpha_3\varphi_x(t, L) - \psi_{xx}(t, 0) = 0 & \text{in } (0,T), \end{cases} \tag{5.7.30}$$

and, in addition,

$$\varphi_x(\cdot, L) = \varphi(\cdot, 0) = 0. \tag{5.7.31}$$

Therefore,

$$\begin{cases} \varphi(t,0)=\varphi(t,L)=\varphi_x(t,0)=\varphi_x(t,L)=0 & \text{in } (0,T) \\ \psi(t,0)=\psi(t,L)=\psi_x(t,L)=\psi_{xx}(t,0)=0 & \text{in } (0,T), \end{cases} \tag{5.7.32}$$

Remark that (5.7.29) implies that the solutions of (5.7.28)-(5.7.31) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following result:

Lemma 5.25. *For any $T>0$, let N_T denote the space of the initial states $(\varphi^0,\psi^0)\in X_0$ such that the solution $(\varphi,\psi)=S(\cdot)(\varphi^0,\psi^0)$ of (5.7.28)-(5.7.30) satisfies (5.7.31). Then, for $L\in(0,+\infty)\setminus\mathcal{N}$, where $\mathcal{N}$ is defined by (5.4.59), $N_T=\{0\}$, $\forall T>0$.*

Proof. The proof is very similar to the one of [62, Lemma 3.4] and so it is omitted. Then, to finish the proof of the theorem we need to prove the following lemma:

Lemma 5.26. *Let $L>0$. Consider the assertion*

$$(\mathcal{A}_1) \qquad \exists\lambda\in\mathbb{C},\ \exists\left(\varphi^0,\psi^0\right)\in\left(H^3(0,L)\right)^2\setminus\{(0,0)\}\ \text{such that}$$

$$\begin{cases} \lambda\varphi^0=\left[(\psi^0)'+(\psi^0)'''\right], \\ \lambda\psi^0=\left[(\varphi^0)'+(\varphi^0)'''\right], \\ \varphi^0(0)=\varphi^0(L)=(\varphi^0(0))'=(\varphi^0(L))'=0, \\ \psi^0(0)=\psi^0(L)=(\psi^0(L))'=(\psi^0(0))''=0. \end{cases}$$

Then, $(\mathcal{A}_1)$ holds if and only if $L\in\mathcal{N}$.

Proof. We use an argument which is similar to the one used in [62, Lema 3.5]. Assume that (φ^0,ψ^0) satisfies $(\mathcal{F}_1)$ and let us denote by $(\varphi,\psi)\in\left(H^2(\mathbb{R})\right)^2$ their prolongation by 0 to $\mathbb{R}$. Then,

$$\begin{cases} -\lambda\varphi+\psi'+\psi'''=\psi'(0)(\delta_0)'-\psi''(L)\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda\psi+\varphi'+\varphi'''=\varphi''(0)\delta_0-\varphi''(L)\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \end{cases} \tag{5.7.33}$$

where δ_{x_0} and $(\delta_{x_0})'$ denote the Dirac measure at x_0. Note that the $(\mathcal{A}_1)$ is equivalent to the existence of complex numbers α, α', β, γ, γ', λ with $(\alpha,\alpha',\beta,\gamma,\gamma')\neq(0,0,0,0,0)$ and $(\varphi,\psi)\in\left(H^2(\mathbb{R})\right)^2$ with compact support in $[-L,L]$ such that

$$\begin{cases} -\lambda(\varphi+\psi)+(\psi+\varphi)'+(\psi+\varphi)'''=\alpha\delta_0+\beta(\delta_0)'+\gamma\delta_L & \text{in } \mathcal{D}'(\mathbb{R}), \\ -\lambda(\varphi-\psi)+(\varphi-\psi)'+(\varphi-\psi)'''=\alpha'\delta_0-\beta(\delta_0)'+\gamma'\delta_L & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases} \tag{5.7.34}$$

Let us introduce the notation $\hat{\varphi}(\xi)=\int_0^L\varphi(\xi)e^{-ix\xi}dx$ and $\hat{\psi}(\xi)=\int_0^L\psi(\xi)e^{-ix\xi}dx$. Then, taking the Fourier transform in (5.7.33) we obtain

$$-\lambda\hat{\varphi}(\xi)+(i\xi)\hat{\psi}(\xi)+(i\xi)^3\hat{\psi}(\xi)+\psi''(L)e^{-iL\xi}-(i\xi)\psi'(0)=0 \tag{5.7.35}$$

and

$$-\lambda\hat{\psi}(\xi)+(i\xi)\hat{\varphi}(\xi)+(i\xi)^3\hat{\varphi}(\xi)-\varphi''(0)+\varphi''(L)e^{-iL\xi}=0. \tag{5.7.36}$$

Then, adding (5.7.35) and (5.7.36) the following identity holds

$$\left(-\lambda+(i\xi)+(i\xi)^{3}\right)\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)=\varphi''(0)+(i\xi)\psi'(0)+\left(-\psi''(L)-\varphi''(L)\right)e^{-iL\xi}.$$

We denote

$$\hat{u}(\xi):=\left(\hat{\varphi}(\xi)+\hat{\psi}(\xi)\right)=\frac{\varphi''(0)+(i\xi)\psi'(0)+\left(-\psi''(L)-\varphi''(L)\right)e^{-iL\xi}}{-\lambda+(i\xi)+(i\xi)^{3}}. \quad (5.7.37)$$

We also take the difference between (5.7.35) and (5.7.36) to obtain

$$\left(-\lambda-(i\xi)-(i\xi)^{3}\right)\left(\hat{\varphi}(\xi)-\hat{\psi}(\xi)\right)=-\varphi''(0)+(i\xi)\psi'(0)+\left(-\psi''(L)+\varphi''(L)\right)e^{-iL\xi}.$$

Here, we denote

$$\hat{v}(\xi):=\left(\hat{\varphi}(\xi)-\hat{\psi}(\xi)\right)=\frac{-\varphi''(0)+(i\xi)\psi'(0)+\left(-\psi''(L)+\varphi''(L)\right)e^{-iL\xi}}{-\lambda-(i\xi)-(i\xi)^{3}}.$$

Introducing the change of variable $\xi\longmapsto-\xi$, we have that

$$\hat{v}(-\xi)=\frac{-\varphi''(0)-(i\xi)\psi'(0)+\left(-\psi''(L)+\varphi''(L)\right)e^{iL\xi}}{-\lambda+(i\xi)+(i\xi)^{3}}. \quad (5.7.38)$$

Setting $\lambda=ip$, we write (5.7.37) and (5.7.38) as

$$\hat{u}(\xi)=i\frac{\alpha+(i\xi)\beta+\gamma e^{-iL\xi}}{\xi^{3}-\xi+p} \quad (5.7.39)$$

and

$$\hat{v}(-\xi)=i\frac{\alpha'-(i\xi)\beta+\gamma' e^{iL\xi}}{\xi^{3}-\xi+p}. \quad (5.7.40)$$

Then, the proof is obtained proceeding as in Lemma 5.7. Thus, the Lemma 5.25 and Theorem 5.32 hold. □

The following theorem gives a positive answer for the control problem (5.7.16)-(5.7.17):

Theorem 5.33. *Let $T>0$ and $L\in(0,+\infty)\setminus\mathcal{N}$. Then, the system (5.7.16)-(5.7.17) is exactly controllable in time T.*

Proof. Let us define the following functional

$$\Lambda\left(\varphi^{1},\psi^{1}\right)=\frac{1}{2}\left\|\varphi_{x}(\cdot,L)\right\|_{L^{2}(0,T)}^{2}+\frac{1}{2}\left\|\varphi(\cdot,0)\right\|_{L^{2}(0,T)}^{2}-\int_{0}^{L}\left(\eta^{1}(x)\varphi^{1}(x)+w^{1}(x)\psi^{1}(x)\right)dx \quad (5.7.41)$$

where $(\varphi^{1},\psi^{1})\in X_{0}$ and (φ,ψ) is the solution of the backward system (5.6.30)-(5.6.32) with initial data (φ^{1},ψ^{1}).

Let $\left(\hat{\varphi}^{1},\hat{\psi}^{1}\right)\in X_{0}$ be a minimizer of Λ. By differentiating Λ, we obtain that (5.7.22) is satisfied with $h_{0},\ g_{2}\in L^{2}(0,T)$. Hence, in order to get the controllability result it is sufficient to prove that Λ has at least one minimum point. But from (5.7.22) and (5.7.23), it follows that

$$\begin{aligned}\Lambda\left(\varphi^1,\psi^1\right) &= \frac{1}{2}\left\|\varphi_x\left(\cdot,L\right)\right\|^2_{L^2(0,T)}+\frac{1}{2}\left\|\varphi\left(\cdot,0\right)\right\|^2_{L^2(0,T)}-\int_0^L\left(\eta^1\left(x\right)\varphi^1\left(x\right)+w^1\left(x\right)\psi^1\left(x\right)\right)dx\\ &\geq C^{-1}\left\|\left(\varphi^1,\psi^1\right)\right\|_H.\end{aligned}$$

Hence, by Lax-Milgram Theorem, Λ is invertible and the proof of Theorem 5.33 is complete. □

5.8 Boundary Controllability Result: The Nonlinear System

Now we can study the controllability of the nonlinear system (5.6.1), satisfying the boundary conditions

$$\begin{cases}\eta\left(t,0\right)=0,\ \ \eta\left(t,L\right)=0,\ \ \eta_x\left(t,0\right)=0 & \text{in }\ (0,T)\\ w\left(t,0\right)=0,\ \ w\left(t,L\right)=0,\ \ w_x\left(t,L\right)+\alpha_1\eta_x\left(t,L\right)=g_2\left(t\right) & \text{in }\ (0,T),\end{cases}\tag{5.8.1}$$

and the initial conditions

$$\eta\left(0,x\right)=\eta^0\left(x\right),\ \ w\left(0,x\right)=w^0\left(x\right)\qquad \text{in }\ (0,L).\tag{5.8.2}$$

Let $U=(\eta,w)$, $(S\left(t\right))_{t\geq 0}$ the semigroup generated by the linear part of the system (5.6.1), $U^0=(\eta^0,w^0)$ and $N\left(U\right)=-\left(\left(\eta w\right)_x,ww_x\right)$. Then, system (5.6.1) with boundary conditions (5.8.1)-(5.8.2) may be recast in the following integral form

$$\begin{cases}U\left(t\right)=AU+N\left(U\right),\\ U\left(0\right)=U^0,\end{cases}\tag{5.8.3}$$

with the boundary conditions (5.8.1). Then, the solution of (5.8.3) has the form

$$U\left(t\right)=S\left(t\right)U^0+\int_0^t S\left(t-s\right)N\left(U\left(s\right)\right).\tag{5.8.4}$$

Using the Kato smoothing effect established in Theorem 5.26, we prove that (5.8.4) is locally well posed in the space X_0.

Theorem 5.34. *For any $(\eta^0,w^0)\in X_0$, there exist a time $T>0$ and a unique solution*

$$(\eta,w)\in C\left([0,T];X_0\right)\cap L^2\left(0,T;\left(H^1\left(0,L\right)\right)^2\right)$$

of (5.8.4).

Proof. For any $(f,g)\in L^1\left(0,T;X_0\right)$, consider the problem

$$\begin{cases}\eta_t+w_x+w_{xxx}=f & \text{in }\ (0,T)\times(0,L),\\ w_t+\eta_x+\eta_{xxx}=g & \text{in }\ (0,T)\times(0,L),\end{cases}\tag{5.8.5}$$

satisfying the boundary conditions (5.8.1)-(5.8.2). Since problem (5.8.5) has a regular solutions, we can consider smooth initial data and conclude the next estimates by density arguments. A density argument yields that $(\eta, w) \in L^2\left(0, T; \left(H^1(0,L)\right)^2\right)$.

First, observe that

$$(\eta, w)(t) = S(t)\left(\eta^0, w^0\right) + \int_0^t S(t-s)\left(f(s,\cdot), g(s,\cdot)\right) ds,$$

where $(S(t))_{t\geq 0}$ is defined by Theorem 5.27. Therefore, from Claims 1 and 2 below, there exist a constant $C = C(T) > 0$ such that

$$\|(\eta, w)\|_{C([0,T];X_0)} + \|(\eta, w)\|_{L^2\left(0,T;(H^1(0,L))^2\right)} \leq C\left\{\left\|\left(\eta^0, w^0\right)\right\|_{X_0} + \int_0^T \|(f,g)\|_{X_0} ds\right\}. \tag{5.8.6}$$

Indeed, remark that solution of the problem (5.8.5) can be written as

$$(\eta, w) = (\eta_1, w_1) + (\eta_2, w_2),$$

where (η_1, w_1) and (η_2, w_2) are solutions, respectively, of

$$\begin{cases} \eta_{1,t} + w_{1,x} + w_{1,xxx} = 0 & \text{in } (0,T)\times(0,L), \\ w_{1,t} + \eta_{1,x} + \eta_{1,xxx} = 0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.8.7}$$

satisfying the boundary conditions

$$\begin{cases} \eta_1(t,0) = 0, \;\; \eta_1(t,L) = 0, \;\; \eta_{1,x}(t,0) = 0 & \text{in } (0,T) \\ w_1(t,0) = 0, \;\; w_1(t,L) = 0, \;\; w_{1,x}(t,L) + \alpha_1 \eta_{1,x}(t,L) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.8.8}$$

and the initial conditions

$$\eta_1(0,x) = \eta_1^0, \;\; w_1(0,x) = w_1^0 \quad \text{in } (0,L), \tag{5.8.9}$$

and

$$\begin{cases} \eta_{2,t} + w_{2,x} + w_{2,xxx} = f & \text{in } (0,T)\times(0,L), \\ w_{2,t} + \eta_{2,x} + \eta_{2,xxx} = g & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.8.10}$$

satisfying the boundary conditions

$$\begin{cases} \eta_2(t,0) = 0, \;\; \eta_2(t,L) = 0, \;\; \eta_{2,x}(t,0) = 0 & \text{in } (0,T) \\ w_2(t,0) = 0, \;\; w_2(t,L) = 0, \;\; w_{2,x}(t,L) + \alpha_1 \eta_{2,x}(t,L) = 0 & \text{in } (0,T), \end{cases} \tag{5.8.11}$$

and the initial conditions

$$\eta_2(0,x) = 0, \;\; w_2(0,x) = 0 \quad \text{in } (0,L). \tag{5.8.12}$$

Therefore, for the system (5.8.7)-(5.8.9), using the Theorem 5.26, we have that

$$\|(\eta_1, w_1)\|_{C([0,T];X_0)} + \|(\eta_1, w_1)\|_{L^2\left(0,T;(H^1(0,L))^2\right)} \leq C\left\|\left(\eta_1^0, w_1^0\right)\right\|_{X_0}.$$

For the system (5.8.10)-(5.8.12), we prove that

Claim 1. There exist $C>0$ such that

$$\|(\eta_2,w_2)\|_{C([0,T];X_0)}\leq C\,\|(f,g)\|_{L^1(0,T;X_0)}\,. \tag{5.8.13}$$

Indeed, note that

$$(\eta_2,w_2)(t)=\int_0^t S(t-s)(f,g)(s,\cdot)\,ds,$$

where $(S(t))_{t\geq 0}$ is C_0-semigroup defined in Theorem 5.27. Remark that

$$\left\|\mathcal{X}_{[0,t]}S(t-s)(f,g)(s,\cdot)\right\|_{X_0}\leq C\,\|(f,g)(s,\cdot)\|_{X_0}\,,$$

where $\mathcal{X}$ denotes the characteristic function. Thus, by Lesbesgue's Theorem, we have that

$$(\eta_2,w_2)\in C([0,T];X_0)\,. \tag{5.8.14}$$

Furthermore,

$$\|(\eta_2,w_2)(t,\cdot)\|_{X_0}\leq\int_0^t\|(f,g)(s,\cdot)\|_{X_0}\,ds\leq\|(f,g)\|_{L^1(0,T;X_0)}\,. \tag{5.8.15}$$

Therefore, from (5.8.14) and (5.8.15), Claim 1 holds.
Claim 2. There exist $C>0$ such that

$$\|(\eta_2,w_2)\|_{L^2\left(0,T;(H^1(0,L))^2\right)}\leq C\,\|(f,g)\|_{L^1(0,T;X_0)}\,. \tag{5.8.16}$$

Indeed, we proceed as in the proof of Theorem 5.26. Multiplying the first equation of (5.8.10) by xw_2, the second one by $x\eta_2$ and integrating by parts in $(0,T)\times(0,L)$, we obtain that

$$\begin{aligned}\int_0^T\int_0^L\left(|\eta_{2,x}|^2+|w_{2,x}|^2\right)dxdt\leq\frac{L}{3}\,\|(f,g)\|^2_{L^1(0,T;X_0)}+\frac{T}{3}\,\|(f,g)\|^2_{L^1(0,T;X_0)}\\ \frac{2L}{3}\int_0^T\|(f,g)\|_{X_0}\,\|(\eta_2,w_2)\|_{X_0}\,dt.\end{aligned}$$

Then, from (5.8.15), we get

$$\int_0^T\int_0^L\left(|\eta_{2,x}|^2+|w_{2,x}|^2\right)dxdt\leq\left(L+\frac{T}{3}\right)\|(f,g)\|^2_{L^1(0,T;X_0)}\,,$$

i. e., (5.8.16) holds.

Now, we prove that (5.8.4) has a unique solution. For this, we consider the map Γ defined by

$$(\Gamma U)(t)=S(t)\,U^0+\int_0^t S(t-s)\,N(U(s))\,ds,$$

where $(S(t))_{t\geq 0}$ is C_0-semigroup defined by Theorem 5.27 and $N(U)=\left(-(\eta w)_x\,,-ww_x\right)$.

We prove that Γ has a unique fixed point in $\overline{B_R(0)}\subset E:=L^2\left(0,T,\left(H^1(0,L)\right)^2\right)$, where $\overline{B_R(0)}$ is the closed ball of radius R in E endowed of your usually norm. We started proving the following result:

Claim 3: There exists a constant $K > 0$, such that

$$\|N(U_1) - N(U_2)\|_{X_0} \le K \left(\|U_1\|_{(H^1(0,L))^2} + \|U_2\|_{(H^1(0,L))^2} \right) \|U_1 - U_2\|_{(H^1(0,L))^2}, \tag{5.8.17}$$

for all $U_1, U_2 \in (H^1(0,L))^2$.
Indeed, observe that

$$\|w\eta_x\|_{L^2(0,L)} \le \|w\|_{L^\infty(0,L)} \|\eta_x\|_{L^2(0,L)} \le C \|w\|_{H^1(0,L)} \|\eta\|_{H^1(0,L)}, \tag{5.8.18}$$

for all $(\eta, w) \in (H^1(0,L))^2$, with $C > 0$. Therefore, if $U_1 = (\eta_1, w_1)$ and $U_2 = (\eta_2, w_2)$, we have that

$$\begin{aligned}
\|N(U_1) - N(U_2)\|_{X_0}^2 &= \|((\eta_2 w_2)_x - (\eta_1 w_1)_x, w_2 w_{2,x} - w_1 w_{1,x})\|_{X_0}^2 \\
&\le \|w_2(\eta_{2,x} - \eta_{1,x})\|_{L^2(0,L)}^2 + \|\eta_{1,x}(w_2 - w_1)\|_{L^2(0,L)}^2 \\
&+ \|\eta_2(w_{2,x} - w_{1,x})\|_{L^2(0,L)}^2 + \|w_{1,x}(\eta_1 - \eta_2)\|_{L^2(0,L)}^2 \\
&+ \|w_{2,x}(w_2 - w_1)\|_{L^2(0,L)}^2 + \|w_1(w_{2,x} - w_{1,x})\|_{L^2(0,L)}^2 \\
&\le C^2 \left(\|\eta_1\|_{H^1(0,L)}^2 + \|w_1\|_{H^1(0,L)}^2 \right) \|w_2 - w_1\|_{H^1(0,L)}^2 \\
&+ C^2 \left(\|\eta_2\|_{H^1(0,L)}^2 + \|w_2\|_{H^1(0,L)}^2 \right) \|w_2 - w_1\|_{H^1(0,L)}^2 \\
&+ C^2 \left(\|\eta_1\|_{H^1(0,L)}^2 + \|w_1\|_{H^1(0,L)}^2 \right) \|\eta_2 - \eta_1\|_{H^1(0,L)}^2 \\
&+ C^2 \left(\|\eta_2\|_{H^1(0,L)}^2 + \|w_2\|_{H^1(0,L)}^2 \right) \|\eta_2 - \eta_1\|_{H^1(0,L)}^2 \\
&= C^2 \left(\|U_1\|_{(H^1(0,L))^2}^2 + \|U_2\|_{(H^1(0,L))^2}^2 \right) \|U_1 - U_2\|_{(H^1(0,L))^2}^2.
\end{aligned}$$

Therefore,

$$\|N(U_1) - N(U_2)\|_{X_0} \le C \left(\|U_1\|_{(H^1(0,L))^2} + \|U_2\|_{(H^1(0,L))^2} \right) \|U_1 - U_2\|_{(H^1(0,L))^2},$$

that is, Claim 3 holds.

Now, let $T > 0$, $R > 0$ to be real numbers (specified later) and consider the open ball of radius R, $B_R(0) \subset E$. From Claim 3, we obtain that

$$\int_0^T \|N(U)\|_{X_0} dt \le K \int_0^T \|U\|_{(H^1(0,L))^2}^2 dt = C \|U\|_E^2 \le CR^2 < \infty,$$

thus,

$$N(U) \in L^1(0,T;X_0).$$

Therefore, by (5.8.6), the map $\Gamma U \in E$. Observe that Claim 2 ensures that

$$\left\| \int_0^t S(t-s) N(U(s)) ds \right\|_E \le C \int_0^T \|N(U(s))\|_{X_0} ds.$$

Thus,

$$\begin{aligned}\|\Gamma U\|_E &\leq \left\|S(\cdot)U^0\right\|_E + C\int_0^T \|N(U(s))\|_{X_0}\,ds\\&\leq \left\|S(\cdot)U^0\right\|_E + KC\int_0^T \|U\|^2_{(H^1(0,L))^2}\,ds\\&= \left\|S(\cdot)U^0\right\|_E + KC\|U\|^2_E\,.\end{aligned}$$

Taking $R = 2\|S(\cdot)U^0\|_E$ we obtain, from Theorem 5.26, that $R \leq \overline{C}(T)\|U^0\|_{X_0}$. Furthermore,

$$\|\Gamma U\|_E \leq \frac{R}{2} + KCR^2 \leq \frac{R}{2} + KC\overline{C}(T)\left\|U^0\right\|_{X_0} R = \left(\frac{1}{2} + KC\overline{C}(T)\left\|U^0\right\|_{X_0}\right)R,$$

where $K, C > 0$. Thus, as $\overline{C}(T) \leq K\sqrt{T}$, for $K > 0$, it follows that, for $T > 0$ small enough, Γ is a map in $B_R(0) \hookrightarrow B_R(0)$.

Finally, let $U_1, U_2 \in B_R(0) \subset E$. From Claim 2, we have that

$$\begin{aligned}\|\Gamma U_1 - \Gamma U_2\|_E &\leq C\|N(U_1(s)) - N(U_2(s))\|_E\\&= C\int_0^T \|N(U_1(s)) - N(U_2(s))\|_{X_0}\,ds\\&\leq CK\int_0^T \left(\|U_1\|_{(H^1(0,L))^2} + \|U_2\|_{(H^1(0,L))^2}\right)\|U_1 - U_1\|_{(H^1(0,L))^2}\,ds\\&\leq 2RKC\int_0^T \|U_1 - U_2\|^2_{(H^1(0,L))^2}\,ds\\&\leq 2RKC\sqrt{T}\|U_1 - U_2\|_E\,.\end{aligned}$$

Therefore, for $R > 0$ small enough, Γ is an contraction in $B_R(0)$ in itself. Then, by Banach fixed-point Theorem, there exist a unique solution $U \in E$ of (5.8.4). Moreover,

$$\begin{aligned}\|U(t)\|_{X_0} &\leq \left\|S(t)U^0\right\|_{X_0} + \int_0^t \|S(t-s)N(U(s))\|_{X_0}\,ds\\&\leq \left\|U^0\right\|_{X_0} + \int_0^t \|N(U(s))\|_{X_0}\,ds\\&\leq \left\|U^0\right\|_{X_0} + K\int_0^t \|U(s)\|^2_{(H^1(0,L))^2}\,ds.\end{aligned}$$

Thus,

$$\begin{aligned}\|U\|_{C([0,T];X_0)} &= \max_{t\in[0,T]}\|U(t)\|_{X_0} \leq \left\|U^0\right\|_{X_0} + K\max_{t\in[0,T]}\int_0^t \|U(s)\|^2_{(H^1(0,L))^2}\,ds\\&= \left\|U^0\right\|_{X_0} + K\|U\|^2_E < \infty,\end{aligned}$$

that is, $U \in C([0,T];X_0)$ and the proof ends. $\square$

We may now prove the main result of this chapter.

Theorem 5.35. *Let* $T > 0$ *and* $L \in (0, +\infty) \backslash \mathcal{N}$*, where*

$$\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N}^* \right\}.$$

Then, there exists a constant $\delta > 0$ *such that for any initial data and final data* $\eta^0, w^0, \eta^1, w^1 \in L^2(0, L)$ *verifying*

$$\left\| \left(\eta^0, w^0\right) \right\|_{X_0} \leq \delta \text{ and } \left\| \left(\eta^1, w^1\right) \right\|_{X_0} \leq \delta,$$

there exist one control $g_2(t) \in L^2(0, T)$ *such that the solution*

$$(\eta, w) \in C([0, T], X_0) \cap L^2\left(0, T; \left(H^1(0, L)\right)^2\right) \cap H^1\left(0, L; \left(H^{-2}(0, L)\right)^2\right),$$

of

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & in\ (0, T) \times (0, L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & in\ (0, T) \times (0, L), \\ \eta(t, 0) = 0, \quad \eta(t, L) = 0, \quad \eta_x(t, 0) = 0 & in\ (0, T), \\ w(t, 0) = 0, \quad w(t, L) = 0, \quad w_x(t, L) + \alpha_1 \eta_x(t, L) = g_2(t) & in\ (0, T), \\ \eta(0, x) = \eta^0(x), \quad w(0, x) = w^0(x) & in\ (0, L), \end{cases} \tag{5.8.19}$$

verifies $\eta(T, \cdot) = \eta^1$, $w(T, \cdot) = w^1$.

Proof. To prove this result we apply a fixed point argument for a suitable map. First, we define

$$\begin{array}{rccc} \Psi: & \left(L^2(0, L)\right)^2 & \longrightarrow & L^2(0, T) \\ & \Psi\left(\eta^1, w^1\right) & = & g_2(t), \end{array}$$

where $g_2(t)$ is a control given by Theorem 5.31 which lead the solution of (5.8.19) from the initial data $(0, 0)$ to the final state (η^1, w^1).

More precisely, if $(\varphi^1, \psi^1) \in X_0$ is the minimizer of the functional Λ defined in Theorem 5.31 and (φ, ψ) is the solution of the backward system (5.6.30)-(5.6.32) with initial data (φ^1, ψ^1), then $g_2(t) \in L^2(0, T)$ is given by

$$g_2(t) = \varphi_x(t, L).$$

Since $\Lambda(\varphi^1, \psi^1) \leq \Lambda(0, 0) = 0$, from observability inequality (5.7.5) we deduce that Ψ is continuous.

First, we consider the following functional spaces

$$\begin{aligned} &\mathcal{Z} = L^2\left(0, T; \left(H^1(0, L)\right)^2\right); \\ &\mathcal{Y} = L^1\left(0, T; \left(L^2(0, L)\right)^2\right); \\ &\mathcal{G} = C\left([0, T]; \left(L^2(0, L)\right)^2\right) \cap L^2\left(0, T; \left(H^1(0, L)\right)^2\right). \end{aligned}$$

Now, we define the operator $\mathcal{F} : \mathcal{Z} \longrightarrow \mathcal{G}$ by

$$\mathcal{F}(\eta, w) = \Theta_1 \circ \Psi\left((\eta^1, w^1) - S(T)(\eta^0, w^0) + \Theta_2(-f, -g)(T, \cdot)\right) \\ +S(\cdot)(\eta^0, w^0) + \Theta_2(f, g).$$

Here $(f, g) = -\left((\eta w)_x, w w_x\right)$, the maps $\Theta_1 : L^2(0, T) \longrightarrow \mathcal{G}$ and $\Theta_2 : \mathcal{Y} \longrightarrow \mathcal{G}$ are defined, respectively, by

$$\Theta_1(g_2) = (\eta_1, w_1),$$

where (η_1, w_1) is the unique solution of (5.8.7)-(5.8.9), and

$$\Theta_2(f, g) = (\eta_2, w_2),$$

where (η_2, w_2) is the unique solution of (5.8.10)-(5.8.12).

Remark that, if (η, w) is a fixed point of $\mathcal{F}$, then (η, w) is a solution of (5.8.19) and satisfies

$$\eta(T, x) = \eta^1, w(T, x) = w^1,$$

that is, system (5.8.19) is controllable by (η^1, w^1).

We prove that there exists $\delta > 0$, small enough, such that if

$$\left\|\left(\eta^0, w^0\right)\right\|_{X_0} \leq \delta \text{ and } \left\|\left(\eta^1, w^1\right)\right\|_{X_0} \leq \delta,$$

the map $\mathcal{F}$ has a fixed point. To do this, it is sufficient to show that there exists $R > 0$, with the following properties:

1. $\mathcal{F}\left(\overline{B_R(0)}\right) \subset \overline{B_R(0)} \subset \mathcal{Z}$;
2. There exist a constant $c \in (0, 1)$ such that

$$\|\mathcal{F}(\eta, w) - \mathcal{F}(\hat{\eta}, \hat{w})\|_{\mathcal{G}} \leq c \|(\eta, w) - (\hat{\eta}, \hat{w})\|_{\mathcal{G}}, \forall (\eta, w), (\hat{\eta}, \hat{w}) \in \overline{B_R(0)},$$

where $\overline{B_R(0)}$ is the closed ball of radius R in $\mathcal{Z}$. Since Θ_1, Θ_2 and Ψ are continuous (see proofs of Theorems 5.34 and 5.30), there exists positive constants k_1, k_2 and k_3 such that

$$\begin{array}{l} \|\Theta_1(g_2)\|_{\mathcal{G}} \leq k_1 \|g_2\|_{L^2(0,T)}, \\ \|\Theta_2(f, g)\|_{\mathcal{G}} \leq k_2 \|(f, g)\|_{\mathcal{Y}}, \\ \|\Psi(\eta^1, w^1)\|_{(L^2(0,T))^2} \leq k_3 \|(\eta^1, w^1)\|_{(L^2(0,T))^2}. \end{array} \tag{5.8.20}$$

Let $R > 0$ (R will be chosen latter on) and let $(\eta, w) \in \overline{B_R(0)} \subset \mathcal{Z}$. We have that

$$\begin{aligned} \|\mathcal{F}(\eta, w)\|_{\mathcal{G}} \leq & \left\|\left(\eta^0, w^0\right)\right\|_{X_0} + k_1 k_3 \left\|\left(\eta^1, w^1\right) - S(T)\left(\eta^0, w^0\right) + \Theta_2(-f, -g)(T, \cdot)\right\|_{X_0} \\ & + k_2 \|(f, g)\|_{\mathcal{Y}} \\ \leq & \ \delta + 2k_1 k_3 \delta + k_1 k_2 k_3 C' \|(\eta, w)\|_{\mathcal{G}}^2 + C' k_2 \|(\eta, w)\|_{\mathcal{G}}^2 \\ \leq & \ \delta + 2k_1 k_3 \delta + (k_1 k_3 + 1) C' k_2 R^2. \end{aligned} \tag{5.8.21}$$

Therefore, $\mathcal{F}\left(\overline{B_R(0)}\right) \subset \overline{B_R(0)}$ for any $R > 0$ such that

$$(1 + 2k_1 k_3)\delta + (k_1 k_3 + 1) C' k_2 R^2 \leq R. \tag{5.8.22}$$

On the other hand, as

$$\mathcal{F}(\eta, w) - \mathcal{F}(\hat{\eta}, \hat{w}) = \Theta_2\left((f,g) - \left(\hat{f}, \hat{g}\right)\right) + \Theta_1 \circ \Psi\left(\Theta_2\left(\left(\hat{f}, \hat{g}\right) - (f,g)\right)\right),$$

we obtain

$$\begin{aligned}\|\mathcal{F}(\eta, w) - \mathcal{F}(\hat{\eta}, \hat{w})\|_{\mathcal{G}} &\leq k_2 C' \|(\eta, w) - (\hat{\eta}, \hat{w})\|_{\mathcal{G}}^2 + k_1 k_2 k_3 C' \|(\eta, w) - (\hat{\eta}, \hat{w})\|_{\mathcal{G}}^2 \\ &\leq 2k_2 C' R (1 + k_3 k_1) \|(\eta, w) - (\hat{\eta}, \hat{w})\|_{\mathcal{G}}^2 . \end{aligned} \tag{5.8.23}$$

Hence, $\mathcal{F}$ is a contraction if R verifies

$$2k_2 C' R (1 + k_3 k_1) < 1. \tag{5.8.24}$$

Now, if R satisfies (5.8.24), by choosing

$$\delta = \frac{R}{2(1 + 2k_1 k_3)},$$

we have that (5.8.22) also holds. Thus, for every (η^0, w^0), (η^1, w^1) such that

$$\left\|\left(\eta^0, w^0\right)\right\|_{X_0} \leq \delta \text{ and } \left\|\left(\eta^1, w^1\right)\right\|_{X_0} \leq \delta,$$

the map $\mathcal{F}$ has a fixed point and the proof ends. □

Now, we study the controllability of the nonlinear system (5.6.1), satisfying the boundary conditions

$$\begin{cases} \eta(t,L) = 0, \ \ \eta_x(t,0) = 0, \ \ \alpha_2 \eta(t,0) + \alpha_3 \eta_x(t,L) + w_{xx}(t,0) = h_0(t) & \text{in } (0,T) \\ w(t,0) = 0, \ \ w(t,L) = 0, \ \ w_x(t,L) - \alpha_3(\eta(t,0) - \eta_x(t,L)) = g_2(t) & \text{in } (0,T), \end{cases} \tag{5.8.25}$$

and the initial conditions

$$\eta(0,x) = \eta^0(x), \ \ w(0,x) = w^0(x) \qquad \text{in } (0,L). \tag{5.8.26}$$

Indeed, let $U = (\eta, w)$, $(S(t))_{t \geq 0}$ the semigroup generated by the linear part of the system (5.6.1) given by Theorem 5.27, $U^0 = (\eta^0, w^0)$ and $N(U) = -\left((\eta w)_x, w w_x\right)$. Then (5.6.1) with the boundary conditions (5.8.25)-(5.8.26) may be recast in the following integral form

$$\begin{cases} U(t) = AU + N(U), \\ U(0) = U^0, \end{cases} \tag{5.8.27}$$

with the boundary conditions (5.8.25). Then, the solution of (5.8.27) has the form

$$U(t) = S(t) U^0 + \int_0^t S(t-s) N(U(s)). \tag{5.8.28}$$

Using the Kato smoothing effect established in Theorem 5.28 and the same ideas of the proof of Theorem 5.34 the following theorem holds:

Theorem 5.36. *For any* $(\eta^0, w^0) \in X_0$*, there exists a time* $T > 0$ *and a unique solution*

$$(\eta, w) \in C([0,T]; X_0) \cap L^2\left(0,T; \left(H^1(0,L)\right)^2\right)$$

of (5.8.28).

With the result of solution of the nonlinear system in hands, we have an affirmative answer to the problem of control with the boundary conditions (5.8.25). This answer is given by the following theorem:

Theorem 5.37. *Let* $T > 0$ *and* $L \in (0, +\infty) \backslash \mathcal{N}$*, where*

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N}^* \right\}.$$

Then, there exists a constant $\delta > 0$ *such that for any initial data and final data* $\eta^0, w^0, \eta^1, w^1 \in L^2(0,L)$ *verifying*

$$\left\|\left(\eta^0, w^0\right)\right\|_{X_0} \leq \delta \text{ and } \left\|\left(\eta^1, w^1\right)\right\|_{X_0} \leq \delta,$$

there exist two controls $(h_0(t), g_2(t)) \in \left(L^2(0,T)\right)^2$ *such that the solution*

$$(\eta, w) \in C([0,T], X_0) \cap L^2\left(0,T; \left(H^1(0,L)\right)^2\right) \cap H^1\left(0,L; \left(H^{-2}(0,L)\right)^2\right),$$

of

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ \eta(t,L) = 0, \quad \eta_x(t,0) = 0 & \text{in } (0,T), \\ \alpha_2 \eta(t,0) + \alpha_3 \eta_x(t,L) + w_{xx}(t,0) = h_0(t) & \text{in } (0,T), \\ w(t,0) = 0, \quad w(t,L) = 0 & \text{in } (0,T), \\ w_x(t,L) - \alpha_3(\eta(t,0) - \eta_x(t,L)) = g_2(t) & \text{in } (0,T), \\ \eta(0,x) = \eta^0(x), \quad w(0,x) = w^0(x) & \text{in } (0,L), \end{cases} \tag{5.8.29}$$

verifies $\eta(T,\cdot) = \eta^1$*,* $w(T,\cdot) = w^1$*.*

Observe that the proof of the Theorem 5.37 is similar to the proof of Theorem 5.35, so it is omitted.

5.9 Stabilization of Boussinesq System

In order to study the exact controllability problem it was very important to obtain a gain of regularity for the solution, that is, to get the Kato smoothing effect. This result was obtained due to the choice of the boundary condition, that is, we consider the following system:

$$\begin{cases} \eta_t + w_x + (\eta w)_x + w_{xxx} = 0 & \text{in } (0,T) \times (0,L), \\ w_t + \eta_x + w w_x + \eta_{xxx} = 0 & \text{in } (0,T) \times (0,L), \end{cases} \tag{5.9.1}$$

satisfying two boundary conditions:

$$\begin{cases} \eta(t,0)=0, \;\; \eta(t,L)=0, \;\; \eta_x(t,0)=0 & \text{in } (0,T) \\ w(t,0)=0, \;\; w(t,L)=0, \;\; w_x(t,L)+\alpha_1\eta_x(t,L)=g_2(t) & \text{in } (0,T), \end{cases} \tag{5.9.2}$$

and

$$\begin{cases} \eta(t,L)=0, \;\; \eta_x(t,0)=0, \;\; \alpha_2\eta(t,0)+\alpha_3\eta_x(t,L)+w_{xx}(t,0)=h_0(t) & \text{in } (0,T) \\ w(t,0)=0, \;\; w(t,L)=0, \;\; w_x(t,L)-\alpha_3(\eta(t,0)-\eta_x(t,L))=g_2(t) & \text{in } (0,T), \end{cases} \tag{5.9.3}$$

where α_i are positive constants for $i=1,2,3$, and the initial conditions

$$\eta(0,x)=\eta^0(x), \;\; w(0,x)=w^0(x) \qquad \text{in } (0,L). \tag{5.9.4}$$

If we look at the above system (5.9.1) with boundary conditions (5.9.2)-(5.9.4) or (5.9.3)-(5.9.4) with $h_0(t)=g_2(t)=0$, another problem that arises in this context is related to the asymptotic behavior of the solutions for t sufficiently large. This question is entirely relevant because the energy associated of the linear system corresponding to (5.9.1) is negative in both cases. Indeed, under the above boundary conditions, if we multiply the first equation of (5.9.1) by η, the second one by w and integrating by parts over $(0,L)$, we obtain, respectively, that

$$\frac{d}{dt}E=-\alpha_1|\eta_x(t,L)|^2-\int_0^L(\eta w)_x\,\eta dx$$

and

$$\frac{d}{dt}E=-\alpha_2|\eta(t,0)|^2-\alpha_3|\eta_x(t,L)|^2-\int_0^L(\eta w)_x\,\eta dx,$$

where $E(t)=\frac{1}{2}\int_0^L(\eta^2+w^2)\,dx$ is the total energy associated to (5.9.1) and $\alpha_i>0$, for $i=1,2,3$. This indicates that the boundary conditions play the role of a feedback damping mechanism, at least for the linearized system. Therefore, the following questions arise:

1. Does $E(t)\to 0$ as $t\to+\infty$?
2. If it is the case, can we give the decay rate?

We will now give positive answers to these questions.

5.9.1 The Linear Problem

In this section we make use of the estimates derived in the previous sections to obtain the exponential decay of the linear problem associated to (5.9.1)-(5.9.3) when $h_0=g_2=0$. More precisely we consider the system

$$\begin{cases} \eta_t+w_x+w_{xxx}=0 & \text{in } (0,T)\times(0,L), \\ w_t+\eta_x+\eta_{xxx}=0 & \text{in } (0,T)\times(0,L), \end{cases} \tag{5.9.5}$$

satisfying, initially, the following boundary conditions

$$\begin{cases} \eta(t,0)=0, \;\; \eta(t,L)=0, \;\; \eta_x(t,0)=0 & \text{in } (0,T) \\ w(t,0)=0, \;\; w(t,L)=0, \;\; w_x(t,L)+\alpha_1\eta_x(t,L)=0 & \text{in } (0,T), \end{cases} \tag{5.9.6}$$

and the initial conditions

$$\eta(0,x)=\eta^0(x),\quad w(0,x)=w^0(x)\qquad \text{in } (0,L). \tag{5.9.7}$$

We are in a position to prove the exponential stability of the linearized system with boundary conditions (5.9.6).

Theorem 5.38. *Assume that* $\alpha_1>0$ *and* $L\in(0,+\infty)\setminus\mathcal{N}$, *where*

$$\mathcal{N}=\left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}}:k,l\in\mathbb{N}^*\right\}.$$

Then there exist two constants C_0, $\mu_0>0$, *such that for any* $(\eta^0,w^0)\in X_0$, *the solution of (5.9.5)-(5.9.7) satisfies*

$$\|(\eta(t),w(t))\|_{X_0}\le C_0e^{-\mu_0 t}\left\|\left(\eta^0,w^0\right)\right\|_{X_0},\ \forall t>0. \tag{5.9.8}$$

Proof. Observe that, using (5.9.6) and the classical argument, we only have to prove the following *observability inequality:*

$$\left\|\left(\eta^0,w^0\right)\right\|_{X_0}^2\le C\alpha_1\int_0^T|\eta_x(t,L)|^2\,dt. \tag{5.9.9}$$

Indeed, if (5.9.9) is proved, we have that

$$E(T)-E(0)\le-\frac{E(0)}{C},$$

that is,

$$E(T)\le E(0)-\frac{E(0)}{C}\le E(0)-\frac{E(T)}{C}.$$

Then,

$$E(T)\le\left(\frac{C}{C+1}\right)E(0).$$

Therefore, the semigroup associated of system (5.9.5)-(5.9.7) decays exponentially. Now, we prove (5.9.9) through several steps.

Step 1: (Compactness-Uniqueness Argument)

The proof is very similar of the result of controllability, more precisely, Theorem 5.30. Therefore, we will omit the details. Again, we argue by contradiction. If (5.9.9) is not true, there exist a sequence $\{(\eta_n^0,w_n^0)\}_{n\in\mathbb{N}}$ in X_0 such that

$$1=\left\|\left(\eta_n^0,w_n^0\right)\right\|_{X_0}^2=\int_0^L\left(\left|\eta_n^0\right|^2+\left|w_n^0\right|^2\right)dx>n\alpha_1\int_0^T|\eta_{n,x}(t,L)|^2\,dt. \tag{5.9.10}$$

From Theorem 5.26 and (5.9.10) we obtain that

$$\{(\eta_n,w_n)\}_{n\in\mathbb{N}}\text{ is bounded in }L^2\left(0,T;\left(H^1(0,L)\right)^2\right)$$

and (5.9.5) ensures that $\{((\eta_n)_t, (w_n)_t)\}_{n\in\mathbb{N}}$ is bounded in $L^2\left(0,T;(H^{-2}(0,L))^2\right)$. Hence, applying Aubin's lemma, we see that there exists a subsequence, still denoted by the same index, such that

$$(\eta_n, w_n) \longrightarrow (\eta, w) \text{ in } L^2(0,T;X_0).$$

Moreover, by (5.6.14) and (5.9.10), we see that $\{(\eta_n^0, w_n^0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X_0, hence for some pair $(\varphi^0,\psi^0)\in X_0$, we have that

$$\left(\eta_n^0, w_n^0\right) \longrightarrow \left(\eta^0, w^0\right) \text{ in } X_0. \tag{5.9.11}$$

Clearly, $(\eta, w) = S(\cdot)(\eta^0, w^0)$ and we infer from (5.9.10) that

$$\eta_x(t,L) = 0 \tag{5.9.12}$$

and

$$\left\|\left(\eta^0, w^0\right)\right\|_{X_0} = 1. \tag{5.9.13}$$

Step 2: (Reduction to a Spectral Problem)

We will use a similar argument to that used in Lemmas 5.22 and 5.23.

Lemma 5.27. *For any $T>0$, let N_T denote the space of the initial states $(\eta^0, w^0)\in X_0$ such that the solution $(\eta, w) = S(\cdot)(\eta^0, w^0)$ of (5.9.5)-(5.9.7) satisfies (5.9.12). If $N_T \neq \emptyset$ for some $T>0$ and $L\in(0,+\infty)\setminus\mathcal{N}$, then there exist $\lambda\in\mathbb{C}$ and $(\eta^0,w^0)\in(H^3(0,L))^2$ with $(\eta^0,w^0)\neq(0,0)$, such that*

$$\begin{cases} \lambda\eta^0 + (w^0)' + (w^0)''' = 0 \\ \lambda w^0 + (\eta^0)' + (\eta^0)''' = 0 \\ \eta^0(0) = \eta^0(L) = (\eta^0(0))' = 0 \\ w^0(0) = w^0(L) = 0 \\ (w^0(L))' = \alpha_1(\eta^0(L))' = 0. \end{cases} \tag{5.9.14}$$

Proof. The proof of Lemma follows exactly the same techniques used in [62, Lemma 3.4] and in the Lemma 5.22, so it is omitted.

To obtain the contradiction, it remains to prove that a triplet (λ, η^0, w^0) as above does not exist.

Step 3: (Nontrivial Solution for the Spectral Problem)

Lemma 5.28. *Let $\lambda\in\mathbb{C}$ and $(\eta^0,w^0)\in(H^3(0,L))^2$ fulfilling (5.9.14). Then $\eta^0 = w^0 = 0$.*

Proof. The proof of Lemma is given by Lemma 5.23. □

The exponential stability for the case of the linearized system with boundary conditions (5.9.3), when $h_0 = g_2 = 0$, is similar, therefore, it is omitted. Indeed, note that we need to prove the following observability inequality:

$$\left\|\left(\eta^0, w^0\right)\right\|_{X_0}^2 \leq C\int_0^T \left(\alpha_2|\eta(t,0)|^2 + \alpha_3|\eta_x(t,L)|^2\right)dt, \tag{5.9.15}$$

for $\alpha_2, \alpha_3 > 0$. The proof of (5.9.15) follows of Lemmas 5.25 and 5.26. Thus, the following result holds:

Theorem 5.39. *Assume that $\alpha_2, \alpha_3 > 0$ and L be as in Theorem 5.38. Then, there exist two constants C_0, $\mu_0 > 0$, such that for any $(\eta^0, w^0) \in X_0$, the solution of (5.9.5) with boundary condition (5.9.3), when $h_0 = g_2 = 0$, and initial conditions (5.9.7) satisfies*

$$\|(\eta(t), w(t))\|_{X_0} \leq C_0 e^{-\mu_0 t} \left\|\left(\eta^0, w^0\right)\right\|_{X_0}, \ \forall t > 0. \tag{5.9.16}$$

Definition 5.8. *For $0 \leq s \leq 3$, let X_s denote the collection of all functions $(\eta, w) \in (H^s(0, L))^2$ satisfying the s-compatibility conditions:*

$$\begin{array}{l} \eta(0) = w(0) = \eta(L) = w(L) = 0, \text{ when } 1/2 < s \leq 3/2 \\ \eta(0) = w(0) = \eta(L) = w(L) = \eta'(0) = w'(L) + \alpha_1 \eta'(L) = 0, \\ \text{when } 3/2 < s \leq 3. \end{array}$$

X_s is endowed with the Hilbertian norm

$$\|(\eta, w)\|_{X_s}^2 = \|\eta\|_{H^s(0,L)}^2 + \|w\|_{H^s(0,L)}^2 .$$

Now, we use Theorem 5.38 and some interpolation argument to prove an exponential stability result in each space X_s, for $0 \leq s \leq 3$.

Corollary 5.3. *Let α_1 and L be as in Theorem 5.38. Then, for $s \in [0, 3]$, there exists a constant $C_s > 0$ such that for any $(\eta^0, w^0) \in X_s$, the solution $(\eta(t), w(t))$ of (5.9.5)-(5.9.7) belongs to $C(\mathbb{R}^+; X_s)$ and fulfills*

$$\|(\eta(t), w(t))\|_{X_0} \leq C_0 e^{-\mu_0 t} \left\|\left(\eta^0, w^0\right)\right\|_{X_0}, \ \forall t > 0. \tag{5.9.17}$$

Proof. Observe that, for $s = 0$, from Theorem 5.38, the result is true. We prove the case $s = 3$. Consider $(\eta^0, w^0) = U^0 \in X_3 := D(A)$ and define

$$U(t) := (\eta(t), w(t)) = S(\cdot) U^0,$$

where $(S(t))_{t \geq 0}$ is C_0-semigroup generated by Theorem 5.25. Then, if $V(t) := U_t(t)$, we have that

$$\begin{cases} V_t = AU_t = AV, \\ V(0) = AU(0) = AU_0 := V_0 \in X_0. \end{cases}$$

Therefore, by Theorem 5.38, there exists constants $C_0, \mu_0 > 0$, such that

$$\|V(t)\|_{X_0} \leq C_0 e^{-\mu_0 t} \|U_0\|_{X_0}, \text{ for } t \geq 0.$$

As $V = AU$ and $V_0 = AU_0$, we obtain that

$$\|AU(t)\|_{X_0} \leq C_0 e^{-\mu_0 t} \|AU_0\|_{X_0}, \text{ for } t \geq 0.$$

Since, $\|U\|_{X_3}$ and $\|U\|_{X_0} + \|AU\|_{X_0}$ are equivalent on X_3, we conclude that for some constant $C_3 > 0$ we have that

$$\|U\|_{X_3} \leq C_3 e^{-\mu_0 t} \left(\|U_0\|_{X_0} + \|AU_0\|_{X_0}\right) = C_3 e^{-\mu_0 t} \|U_0\|_{X_3},$$

for all $t > 0$. This proves (5.9.17) for $s = 3$. The fact (5.9.17) is still valid for $0 < s < 3$ follows by a standard interpolation argument, since

$$X_s = [X_0, X_3]_{s/3} .$$

□

Remark 5.13. *Observe that in [58], the authors consider system (5.9.5) with three dissipations in the boundary conditions. In this case it is possible to obtain the decay with less amount of damping. On the other hand, this fact is closely linked with the appearance of the critical set for the length L.*

Remark 5.14. *Another problem cited would be the stabilization of the system (5.9.5) with boundary condition (5.9.3), when $h_0 = g_2 = 0$, and initial conditions (5.9.7). The Theorem 5.39 gives us a positive answers to the exponential decay of the system, but observe that adding two dissipation in the boundary conditions, there exist a set of critical values for L. Proceeding as in the proof of Corollary 5.3 it is possible to prove the stabilization in X_s of the system (5.9.5) with boundary condition (5.9.3), when $h_0 = g_2 = 0$, and initial conditions (5.9.7).*

5.9.2 Well-posedness and Exponential Stability

Now we return our attention for the nonlinear system (5.9.1), satisfying the boundary conditions (5.9.2)-(5.9.4) when $h_0 = g_2 = 0$. Let $U = (\eta, w)$, $(S(t))_{t\geq 0}$ the semigroup generated by the linear part of the system (5.9.1), $U^0 = (\eta^0, w^0)$ and $N(U) = -\left((\eta w)_x, ww_x\right)$. Then, system (5.9.1) with boundary conditions (5.9.2)-(5.9.4) (or (5.9.3)-(5.9.4)) may be recast in the following integral form

$$\begin{cases} U(t) = AU + N(U), \\ U(0) = U^0, \end{cases} \tag{5.9.18}$$

with the boundary conditions (5.9.2)(or (5.9.3)). Then, the solution of (5.9.1) has the form

$$U(t) = S(t)U^0 + \int_0^t S(t-s)N(U(s)). \tag{5.9.19}$$

Using the Kato smoothing effect established in Theorem 5.26, Theorem 5.34 ensure the local well-posedness in the space X_0.

As cited in [58], due to a lack of a priori X_0-estimate, the issue of the global existence of solutions is difficult to address. However, the global existence together with the exponential stability may be established for small initial data. To that end, the Kato smoothing estimate and the exponential decay rate in X_1 are combined with a pointwise (in time) estimate. Therefore, using the same argument in [58], we can guarantee the existence and global exponential decay of the system (5.9.1) with boundary conditions (5.9.2)-(5.9.4) (or (5.9.3)-(5.9.4)). More precisely, it is possible to prove the following Lemma:

Lemma 5.29. *For any $\mu \in (0, \mu_0)$, there exist a constant $C = C(\mu) > 0$ such that for any $U^0 \in X_0$*

$$\left\|S(t)U^0\right\|_{X_1} \leq C\frac{e^{-\mu t}}{\sqrt{t}}\left\|U_0\right\|_{X_0}, \forall t > 0. \tag{5.9.20}$$

Proof. See [58, Lemma 3.2]. □

Finally, with the previous Lemma, we are in a position to prove the well-posedness and the exponential stability for solutions issued from small initial data in X_1. Fix a number $\mu \in (0, \mu_0)$, and let us introduce the space

$$F = \left\{ U = (\eta, w) \in C\left(\mathbb{R}^+; X_1\right); \left\| e^{\mu t} U\left(t\right) \right\|_{L^\infty(\mathbb{R}^+; X_1)} < \infty \right\}$$

endowed with its natural norm. Then, the next Theorem, proven in [58, Theorem 3.3], follows:

Theorem 5.40. *There exists a number $r_0 > 0$ such that for any $(\eta^0, w^0) \in X_1$ with $\|(\eta^0, w^0)\|_{X_1} \leq r_1$, the integral equation (5.9.19) admits a unique solution $(\eta, w) \in F$.*

Chapter 6

Conclusion and perspectives

6.1 Conclusion

The chapter 4 was devoted to the study of a coupled system introduced by Gear and Grimshaw in 1984 to model the interactions of two-dimensional, long, internal gravity waves propagation in a stratified fluid. It was proved by Chaves and Davila (see [25]) that, by introducing periodic boundary conditions the solutions are considered in $[H_p^s(0,1)]^2$, for any s positive integer. Moreover, the authors also give a simpler derivation of the conservation laws discovered by Gear and Grimshaw, and Bona et al [8]. With this result in hand, we studied, introducing a symmetric dissipative mechanism in both equations, the exponentially decaying of the total energy associated to the system is derived (Theorem 3.2). The proof is based on the construction of a Lyapunov function and several identities deduced from the infinite family of conservation laws which characterize this system.

The chapter 5 was devoted to the study of the controllability properties of Korteweg de Vries equation (KdV). The equation was first derived by Boussinesq [10] and Korteweg-de Vries [39] as a model for the propagation of water waves along a channel. The equation is also a very useful approximation model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems. While a lot of research has been devoted to the initial value problem, the boundary value problem and the controllability properties have received much less attention. Some of the most important results in this direction have been obtained in the paper [62] where the boundary controllability properties of the KdV equation have been studied. We studied the null and exact controllability problem with a distributed control on a bounded domain. The null controllability result, given in Theorem 4.1, is first established for a linearized system by following the classical duality approach (see [28, 42]), which reduces the null controllability to an observability inequality for the solutions of the adjoint system. To prove the observability inequality, we derive a new Carleman estimate with an internal observation in $(0,T) \times (l_1, l_2)$ and use some interpolation arguments inspired by those in [32], where the authors derived a similar result when the control acts on a neighborhood on the left endpoint (that is, $l_1 = 0$). The null controllability is extended to the nonlinear system by applying Kakutani fixed-point theorem. Nextly, an exact controllability property is studied in Theorems 4.2 and 4.3. In both cases, as for the boundary control, the localization of the distributed control plays crucial a role in the final results. While in the first theorem the exact controllability property holds in a weighted space $L^2_{\frac{1}{(L-x)}dx}$, in the second one only a "regional controllability" result can be established.

In chapter 6, we studied the controllability and stabilizability properties of a Boussinesq systems. The classical Boussinesq systems were first derived by Boussinesq, in [11], to describe the two-way propagation of small amplitude, long wavelength gravity waves on the surface of water in a canal. These systems and their higher-order generalizations also arise when modeling the propagation of long-crested waves on large lakes or on the ocean and in other contexts. In [6], the authors derived a four-parameter family of Boussinesq systems to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. In the first part of this chapter we studied the boundary controllability properties of the system. The controls belong to $L^2(0,T)$ and may be in number of two or one, the total number of boundary conditions being six (Theorems 5.2). The technique of proof is inspired by the study of the controllability properties for the KdV equation developed by Rosier in [62]. More precisely, the controllability of the linear systems is shown by reducing the problem to a unique continuation principle for the eigenfunctions of the corresponding differential operator. As in the case of the KdV equation, in order to ensure this unique continuation property, the length of the interval can be taken any positive real number excepting a countable set of values explicitly described in each case. We note that, to prove the controllability property for the nonlinear system, special boundary conditions have to be chosen in order to ensure the so called Kato smoothing effect and a higher regularity of the solutions (Theorems 5.3 and 5.4). In the last part of this chapter, a stabilizability property was derived. The chosen boundary conditions ensure the non increasing property of the associated energy and, finally, its exponential decay (Theorem 5.5).

6.2 Perspectives

Based on the work developed in this thesis, some questions arise naturally. We will therefore describe in this section some problems that we can interest thereafter.

6.2.1 New damping mechanism

It would be possible to reduce the dissipation in the system studied in the Chapter 4? More precisely, one could add a single dissipative mechanism in one equation? It was noted that with the approach used here (Lyapunov Method) would not be possible to obtain similar results in this case. Then, a natural idea is to consider first a local damping mechanism as in [72] where the same problem was addressed for the KdV equation.

We expect that such mechanism acting in both equations of the Gear-Grimshaw system give as the exponential decay of the energy, but this is an issue entirely open.

6.2.2 Internal Controllability for the Korteweg-de Vries Equation

In chapter 5 of this thesis was proven one result of "regional controllability" for the Korteweg-de Vries equation. More specifically, it was possible to show a controllability in the right side of the spacial domain. The issue whether u can also be controlled exactly on (l_1, l_2) is open.

6.2.3 Controllability for the non linear Boussinesq System in critical cases

In chapter 6, we prove the boundary controllability of the linear system of the Boussinesq system of KdV-KdV type. More precisely, we show that the controllability follows when $L \notin \mathcal{N}$ or $L \notin \mathcal{R}$. But, in [17], the authors shows that the non linear KdV equation is exactly controllable in critical cases. A natural question would be, if the techniques used by the authors can be used to obtain a similar result for the Boussinesq system of KdV-KdV type. The application of this method to this system is a completely open problem.

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