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Twisted groupoid KR-theory

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Abstract

In his 1966's paper "*K-theory and Reality*", Atiyah introduced a variant of K-theory of complex vector bundles called KR-theory, which, in some sense, is a mixture of complex K-theory KU, real K-theory (also called orthogonal K-theory) KO, and Anderson's self-conjugate K-theory KSc. The main purpose of this thesis is to generalize that theory to the non-commutative framework of *twisted groupoid K-theory*. We then introduce *twisted groupoid KR-theory* by using the powerful machineries of Kasparov's "*real*" KK-theory. Specifically, we deal with the K-theory of $\mathbb{Z}_2$-graded C^*-algebras associated with groupoid dynamical systems endowed with involutions. Such dynamical systems are classified by the *Real graded Brauer group* to be defined and computed in terms of Čech cohomology classes. In this new K-theory, we give the analogues of the fundamental results in K-theory such as the *Mayer-Vietoris exact sequences*, the *Bott periodicity* and the *Thom isomorphism theorem*.

Résumé

Dans son article de 1966 intitulé "*K-theory and Reality*", Atiyah introduit une variante de la K-théorie des fibrés vectoriels complexes, notée KR, qui, d'une certaine manière, englobe à la fois la K-théorie complexe KU, la K-théorie réelle KO (dite aussi *orthogonale*), et la K-théorie auto-conjuguée KSc d'Anderson. Dans cette thèse, nous généralisons cette théorie au cadre non-commutatif de la K-théorie tordue des groupoïdes topologiques. Nous développons ainsi la *KR-théorie tordue des groupoïdes* en nous servant principalement des outils de la KK-théorie "réelle" de Kasparov. Il s'agit notamment de l'étude de la K-théorie des C^*-algèbres graduées associées à des systèmes dynamiques de groupoïdes munis de certaines involutions. Les classes d'équivalence de tels systèmes génèrent le *groupe de Brauer Réel gradué* que nous définissons et calculons en termes de classes de cohomologie de Čech. Nous donnons dans cette nouvelle théorie les analogues des résultats classiques en K-théorie tels que les suites exactes de Mayer-Vietoris, la périodicité de Bott et le théorème d'isomorphisme de Thom.

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Contents

1 Introduction

1.1 Overview of twisted K-theory

Twisted K-theory was introduced in the early 1970s by P. Donovan and M. Karoubi in [28] as *K-theory with local coefficients*, by analogy with *cohomology with local coefficients*. Given a compact space X, they defined twisted orthogonal K-theory $KO^*_\alpha(X)$ for

$$\alpha \in \widehat{\mathrm{BrO}}(X) := H^0(X, \mathbb{Z}_8) \times H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2),$$

and twisted complex K-theory $KU^*_\alpha(X)$ for

$$\alpha \in \widehat{\mathrm{Br}}(X)_{Tors} := H^0(X, \mathbb{Z}_2) \times H^1(X, \mathbb{Z}_2) \times \mathrm{Tors}\, H^3(X, \mathbb{Z}),$$

where $\widehat{\mathrm{Br}}(X)_{Tors}$ is the torsion subgroup of $\widehat{\mathrm{Br}}(X) := H^0(X, \mathbb{Z}_2) \times H^1(X, \mathbb{Z}_2) \times H^3(X, \mathbb{Z})$.

The main motivation of their work was that twisted K-theory has appeared to be the most natural way to display *Thom isomorphism* in K-theory. Indeed, if V is a real vector bundle over X, the KO-theory of its Thom space is isomorphic to $KO^*_\alpha(X)$, where

$$\alpha = -(\dim V \mod 8, w_1(V), w_2(V)) \in \widehat{\mathrm{BrO}}(X),$$

and where $w_i(V), i = 1, 2$ are the first Stiefel-Whitney classes. More precisely, $KO^*_\alpha(X)$ is defined to be the K-theory of the Banach algebra $\mathcal{C}(X; Cl(V))$ of continuous sections of the real Clifford bundle $Cl(V)$, which is a bundle of $\mathbb{Z}_2$-graded real Matrix algebras [42]. Thom isomorphism for complex bundles expresses analogously (one needs then to consider the complexified Clifford bundle $\mathbb{C}l(V) := Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$). Given that not all α arise from Clifford bundles, it had proved interesting to generalize that definition for the whole $\widehat{\mathrm{BrO}}(X)$ and $\widehat{\mathrm{Br}}(X)_{Tors}$. Fortunately, these groups classify bundles of simple central $\mathbb{Z}_2$-graded algebras over $\mathbb{R}$ and $\mathbb{C}$, respectively; such bundles are known as *Azumaya bundles* [33]. More concretely, a bundle $\mathcal{A} \longrightarrow X$ is Azumaya if its typical fibre is a graded Matrix algebra. The *graded orthogonal Brauer group* $\widehat{\mathrm{BrO}}(X)$ (resp. the *complex graded Brauer group* $\widehat{\mathrm{Br}}(X)_{Tors}$) is isomorphic to the set of Morita equivalence classes of real (resp. complex) Azumaya bundles on X [28, 92], equipped with the operation of graded tensor products. Then if $\mathcal{A}$ is any representative of the class corresponding to α, $KO_{\mathcal{A}}^{-n}(X) = KO_{\alpha}^{-n}(X) := KO_n(\mathcal{C}(X;\mathcal{A}))$, and $KU_{\mathcal{A}}^{-n}(X) = KU_{\alpha}^{-n}(X) := K_n(\mathcal{C}(X;\mathcal{A}))$ in the complex case [28, 43]. For non compact spaces, one takes the algebra $\mathcal{C}_0(X;\mathcal{A})$ of continuous sections vanishing at infinity. There is a pairing

$$K_{\alpha}^{-n}(K) \otimes K_{\beta}^{-m}(X) \longrightarrow K_{\alpha+\beta}^{-n-m}(X),$$

where K is either KO or KU, and if α is trivial, one recovers the ordinary K-theory. Moreover, twisted K-theory provides a generalization of Thom isomorphism.

Theorem 1.1.1. *([28, Theorem 6.15], [43, Theorem 4.2]). Let $\pi : V \longrightarrow X$ be a Euclidean vector bundle on X, and $\mathcal{A}$ be an Azumaya bundle on X. Then*

$$K^*_{\mathcal{A}\hat{\otimes} Cl(V)}(X) \cong K^*_{\pi^*\mathcal{A}}(V).$$

In investigating continuous-trace C^*-algebras [1] in the late 1980s, J. Rosenberg was led to introduce twisted K-theory when the twisting is an ungraded non-torsion element of $\widehat{\mathrm{Br}}(X)$ ([78]). The idea came from the pioneering work of J. Dixmier and A. Douady linking separable continuous-traces C^*-algebras and Čech cohomology ([27]). Indeed, from [27, Lemme 22,Théorèmes 11, 12, 13, & 14], we have the following correspondences:

Theorem 1.1.2. *Let X be a locally compact second-countable Hausdorff space. Then*

$$\left\{\begin{array}{l}\text{Stable separable}\\ \text{continuous}-\text{trace}\\ C^*-\text{algebras}\\ \text{with spectrum } X\end{array}\right\} \cong \left\{\begin{array}{l}\text{Elementary}\\ C^*-\text{bundles}\\ \text{over } X\end{array}\right\} \cong \left\{\begin{array}{l}\mathrm{PU}(\mathcal{H})-\text{principal}\\ \text{bundles}\\ \text{over } X\end{array}\right\} \cong \check{H}^2(X,\mathbb{S}^1).$$

Recall that an elementary C^*-bundle is a bundle $\mathcal{A} \longrightarrow X$ of elementary C^*-algebras (*i.e.* $\mathcal{A}_x \cong \mathcal{K}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space) satisfying Fell's condition (see [78, 75, 29]), and with structure group $\mathrm{Aut}(\mathcal{K}(\mathcal{H})) \cong \mathrm{PU}(\mathcal{H})$.

Given $\alpha \in \check{H}^2(X,\mathbb{S}^1) \cong \check{H}^3(X,\mathbb{Z})$, K-theory of X twisted by α is defined ([78, §2]) by

$$K_{\alpha}^{-i}(X) := K_i(A) = K_i(\mathcal{C}_0(X;\mathcal{A})),$$

[1] A C^*-algebra A is continuous-trace if its spectrum $\hat{A}$ is Hausdorff and if the *continuous-trace elements*

$$\{a \in A_+ \mid \mathrm{Tr}\pi(a) < \infty, \forall \pi \in \hat{A}, \text{ and } \pi \longmapsto \mathrm{Tr}\pi(a) \text{ is continuous on } \hat{A}\}$$

are dense in A_+ (see [78, 75] for instance).

where A (resp. $\mathcal{A}$) is any stable separable continuous-trace C^*-algebra with spectrum X (resp. any elementary C^*-bundle over X) realizing α in the sense of the above bijections. A topological interpretation of these groups have been given as follows. Denote by $\mathrm{Fred}^{(0)}$ the space of Fredholm operators on the infinite-dimensional separable Hilbert space $\mathcal{H}$, and let $\mathrm{Fred}^{(1)}$ denote the subspace of self-adjoint operators in Fred^0. Rosenberg has proved the following ([78, Proposition 2.1]))

Proposition 1.1.3. *If α corresponds to a $\mathrm{PU}(\mathcal{H})$-principal bundle $P \longrightarrow X$, then*

$$\begin{aligned} K^0_\alpha(X) &\cong \left[P, \mathrm{Fred}^{(0)}\right]^{\mathrm{PU}(\mathcal{H})}, \\ K^1_\alpha(X) &\cong \left[P, \mathrm{Fred}^{(1)}\right]^{\mathrm{PU}(\mathcal{H})}, \end{aligned}$$

where the right hand sides are the sets of homotopy classes of $\mathrm{PU}(\mathcal{H})$-equivariant continuous functions, with homotopy being in $\mathrm{Fred}^{(i)}$, $i = 0, 1$.

The classification of Theorem 1.1.2 was extended by E. Parker [70] to the graded case, offering then a C^*-algebraic picture of $\widehat{\mathrm{Br}}(X)$:

Theorem 1.1.4 (E. Parker 1988)**.** *The set $\widehat{Br}(X)$ of isomorphism classes of stable separable continuous-trace $\mathbb{Z}_2$-graded C^*-algebras with spectrum X, subject to the operation of graded tensor products, is isomorphic to $\check{H}^0(X, \mathbb{Z}_2) \times \check{H}^1(X, \mathbb{Z}_2) \times \check{H}^3(X, \mathbb{Z})$.*

Since the late 1990s, twisted K-theory has played a prominent part in mathematical physics, especially in string theory. Indeed, it has proved a good candidate for a mathematical attempt to classify, for instance D-brane charges and Ramond-Ramond fields [95, 37, 60, 13, 16, 15]. Further generalizations and various versions of twisted K-theory have then been elaborated during the last decade, such as K-theory twisted by graded infinite-dimensional twist (Atiyah-Segal [8]), K-theory of *bundle gerbes* (see Bouwknegt *et al.* [14]), twisted KO-theory and K-theory of real bundle gerbes (Mathai, Murray, and Stevenson [58]), equivariant K-theory twisted by discrete torsions and twisted orbifold K-theory (Adem-Ruan [1, 2]), and twisted K-theory of differentiable stacks using the theory of groupoids (Tu-Xu-Laurent [90], see also Freed-Hopkins-Teleman [30] and J.-L. Tu [87]). The latter is more general in the complex case, in that it contains all of the other versions of twisted complex K-theory. Indeed, the concept of *groupoid* is, in a sense, the natural generalization of *space*, *group*, and the data of a space acted upon by a group.

Abstractly, a *groupoid* ([63, 76]) is a *small category* [2] in which all morphisms are invertible. More concretely, a groupoid consists of a *unit space* $\mathcal{G}^{(0)}$ consisting of objects, a space of *arrows* $\mathcal{G}^{(1)}$, two maps $s, r : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$, called *source* and *range* (or *target*)

[2] A small category is a category for which the collection $\mathcal{C}^0$ of objects and the collection $\mathcal{C}^1$ of morphisms are sets.

maps, respectively, an *inversion* $i : \mathcal{G}^{(1)} \ni g \longmapsto g^{-1} \in \mathcal{G}^{(1)}$, and a *partial multiplication* $m : \mathcal{G}^{(2)} \ni (g,h) \longmapsto gh \in \mathcal{G}^{(1)}$, where $\mathcal{G}^{(2)}$ is the set of *composable arrows*

$$\mathcal{G}^{(2)} := \left\{(g,h) \in \mathcal{G}^{(1)} \times \mathcal{G}^{(1)} \mid s(g) = r(h)\right\}.$$

Such a data is often symbolized by $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$, where $X = \mathcal{G}^{(0)}$, or just by $\mathcal{G}$. Also, for the sake of simplicity, one writes $\mathcal{G}$ instead of $\mathcal{G}^{(1)}$. A *topological groupoid* is a groupoid $\mathcal{G}$ for which the unit space and space of morphisms are topological spaces, and all of the above maps (*the structural maps*) are continuous. From now on, by a groupoid we will mean a topological one.

Twistings of groupoid complex K-theory are defined in terms of elementary complex C^*-bundles $\mathcal{A}$ over the unit space endowed with an action of the space of morphisms $\mathcal{G}$ by *-automorphisms; *i.e.*, there is a family $\alpha = (\alpha)_{g\in\mathcal{G}}$ of *-isomorphisms $\alpha_g : \mathcal{A}_{s(g)} \xrightarrow{\cong} \mathcal{A}_{r(g)}$ with the property that $\alpha_g \circ \alpha_h = \alpha_{gh}$ whenever the product makes sense, and $\alpha_{g^{-1}} = (\alpha_g)^{-1}$. Such bundles are called a *Dixmier-Douady bundles* over $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ and generate an Abelian group $Br(\mathcal{G})$ called *the Brauer group*, which was introduced by Kumjian-Muhly-Renault-Williams in [49]. This group has been shown to classify $\mathbb{S}^1$-central extensions of groupoids *up to Morita equivalence* and to be isomorphic to the second groupoid Čech cohomology with coefficients in $\mathbb{S}^1$. We shall note that the notion of groupoid $\mathbb{S}^1$-central extension "*up to Morita equivalence*", generalizes Murray's *bundle gerbes* [67] and offers a geometric interpretation of the groupoid cohomology $\check{H}^2(\mathcal{G}_\bullet, \mathbb{S}^1)$.

The graded analogue $\widehat{\mathrm{Br}}(\mathcal{G})$ of $Br(\mathcal{G})$ has been given for instance in [30] and [87], where similar isomorphism as that of E. Parker (cf. Theorem 1.1.4) was established. Elements of $\widehat{\mathrm{Br}}(\mathcal{G})$ are then represented by $\mathbb{Z}_2$-graded Dixmier-Douady bundles over $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$; the automorphisms α_g are then required to by isomorphisms of $\mathbb{Z}_2$-graded C^*-algebras.

Given $\mathcal{A} \in \widehat{\mathrm{Br}}(\mathcal{G})$, one defines the *reduced* C^**-algebra* $\mathcal{A} \rtimes_r \mathcal{G}$ as the reduced C^*-algebra of the Fell bundle $\mathcal{A}'$ ([48]) defined as the pull-back $s^*\mathcal{A} \longrightarrow \mathcal{G}$. This is naturally a $\mathbb{Z}_2$-graded C^*-algebra where the grading is induced from the fibers of $\mathcal{A}$. The groups $K^*_{\mathcal{A}}(\mathcal{G}^\bullet)$ ([87]) are henceforth defined by means of Kasparov's KK-theory ([46, 45, 47, 9, 39]):

$$K^{-i}_{\mathcal{A}}(\mathcal{G}^\bullet) := KK_i(\mathbb{C}, \mathcal{A} \rtimes_r \mathcal{G}).$$

This 2-periodic theory admits a topological interpretation in the case where $\mathcal{G}$ is a proper groupoid (see [30, 90]); *i.e.*, for all compact subspace $K \subset \mathcal{G}^{(0)}$, the space $\mathcal{G}^K$ of arrows whose targets are in K is compact in $\mathcal{G}$. In the particular case of twists that are torsions, there is the notion *twisted vector bundle* over a proper groupoid, which may provide a geometric picture of groupoid twisted K-theory ([90]). For instance, if $\mathcal{G}$ reduces to a locally compact Hausdorff space X, then an extension up to Morita equivalence over X is nothing but a bundle gerbe over X, and a twisted vector bundle is nothing but a bundle gerbe module in the sense of [14]. In that case, groupoid twisted K-theory coincides with

K-theory of bundle gerbes. All of these points will be clarified in this thesis.

It is worth noting that in the torsion case, twisted K-theory of locally compact spaces behaves almost as K-theory of complex vector bundles. Roughly speaking, it has been proved by Mathai, Melrose, and Singer in [55] that for α a (ungraded) torsion in the sheaf cohomology group $\check{H}^2(X,\mathbb{S}^1)\cong\check{H}^3(X,\mathbb{Z})$, $K^0_\alpha(X)$ is isomorphic to the Grothendieck group of isomorphism classes of α-twisted vector bundles over X. By definition, if an n-torsion α is represented on an open cover $(U_i)_{i\in I}$ of X, an α-twisted vector bundle is the data of a family $(E_i)_{i\in I}$ of complex vector bundles $E_i \longrightarrow U_i$ of the same fixed rank, and transition functions $h_{ij}: U_{ij} \longrightarrow \mathrm{U}(n)$ satisfying $h_{ij}(x)\circ h_{jk}(x) = \alpha_{ijk}(x)h_{ik}(x), \forall x \in U_{ijk} = U_i\cap U_j\cap U_k$. In particular, when X is a (compact) manifold, it makes sense to talk about *index theory* in twisted K-theory (see Mathai-Melrose-Singer [55, 56, 57], or Nistor-Troitsky [68] for the equivariant case)

Twisted vector bundles are of particular interest when it comes to the K-theory of separable continuous-trace C^*-algebras all representations of which are of finite dimension.

Example 1.1.5. *Let G be a compact Lie group, and $\widehat{G}$ its dual space. A basic fact in the theory of group representations is that all irreducible representations of G are of finite rank, and the spectrum of the group C^*-algebra $C^*(G)$ is homeomorphic to $\widehat{G}$ (see for instance [26]). For $d\in\mathbb{N}$, let $\widehat{G}_d$ be the open subset of $\widehat{G}$ generated by all irreducible representations of dimension m. Then there are Azumaya bundles $\mathcal{A}_d \longrightarrow \widehat{G}_d$, $d\in\mathbb{N}$, such that $C^*(G)\cong\bigoplus_d \mathcal{C}_0(\widehat{G}_d;\mathcal{A}_d)$ (see [68, Corollary 5.5]). Therefore,*

$$K_i(C^*(G))\cong\bigoplus_d K^{-i}_{\mathcal{A}_d}(\widehat{G}_d),$$

and since the $\mathcal{A}_d$ are Azumaya, they are torsion elements in the Brauer group, and hence the right hand side of that isomorphism furnishes a geometric interpretation of the K-theory of $C^*(G)$ in terms of twisted vector bundles.

1.2 Why twisted groupoid KR-theory ?

The theory to be developed here is a first attempt to study groupoid KO-theory twisted by *Dixmier-Douady real C^*-bundles*. Our strategy is to construct a variant of twisted K-theory in which we will not lose the informations of twisted complex K-theory. For this end, a twisted version of Atiyah's KR-theory [6] appears to be the ideal candidate, for in the untwisted case, it is a theory combining all the known K-theories of vector bundles; *i.e.* K-theory of complex vector bundles, K-theory of real vector bundles, and K-theory of *self-conjugate* vector bundles, denoted by KSC (Anderson [5], Smith-Stong [86]).

KR-theory is defined in the category of *Real spaces*. A *Real* [3] space is a space X endowed with an involution $\tau : X \longrightarrow X$; *i.e.* τ is an isomorphism such that $\tau^2 = \mathrm{Id}$. Of course here the word "isomorphism" depends on the category we are working in: it is a homeomorphism in the category of topological spaces, a diffeomorphism in the category of real manifolds, a holomorphic or anti-holomorphic diffeomorphism in the category of complex manifolds (Landweber [51]). The fixed point set of X is called the Real part of X. Let us give some simple examples:

1. For $n = p + q \in \mathbb{N}$, give $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ the involution
$$(x_1, \dots, x_p, y_1, \dots, y_q) \longmapsto (x_1, \dots, x_p, -y_1, \dots, -y_q).$$
Then with respect to this involution $\mathbb{R}^n$ is a topological Real space, denoted by $\mathbb{R}^{p,q}$; the Real part of $\mathbb{R}^{p,q}$ is $\mathbb{R}^p$. In particular, $\mathbb{R}^{p,p}$ is the same as $\mathbb{C}^p$ endowed with the involution consisting of complex conjugation. Note that the Real part of $\mathbb{R}^{0,q}$ is $\{0\}$.

2. With respect to the involution given by coordinatewise complex conjugation
$$[z_1, \dots, z_n] \longmapsto [\bar{z}_1, \dots, \bar{z}_n],$$
the complex projective space $\mathbb{PC}^n$ is a Real space whose Real part identifies with the real projective space $\mathbb{RP}^n$.

3. Let M be a complex analytic manifold, and let M^{op} denote its conjugate complex analytic manifold. Then $M \cup M^{\mathrm{op}}$ is given the structure of Real complex analytic manifold with an empty Real part, by considering the involution consisting of switching M and M^{op} ([51]).

Let X be a locally compact Hausdorff Real space. A Real vector bundle over X is a complex vector bundle $E \longrightarrow X$ which is itself a Real space, with the property that the the projection intertwines the involutions, and that for all $x \in X$, the induced isomorphism $\tau_x : E_x \longrightarrow E_{\tau(x)}$ is conjugate-linear; *i.e.* $\tau_x(\lambda e) = \bar{\lambda}\tau_x(e)$, $\forall \lambda \in \mathbb{C}, e \in E_x$. An isomorphism of Real vector bundles is an isomorphism of complex vector bundles compatible with the involutions in the obvious sense. The group $KR(X)$ is defined to be the Grothendieck group of the isomorphism classes of Real vector bundles over X. The higher KR-groups are defined by $KR^{q-p}(X) := KR(X \times \mathbb{R}^{p,q})$, where the involution on $X \times \mathbb{R}^{p,q}$ is the product of the ones of X and $\mathbb{R}^{p,q}$. In particular, if $\tau = \mathrm{Id}$, then each fibre of a Real vector bundle E is in fact the complexification of a real vector, so that $E = E_{\mathbb{R}} \otimes \mathbb{C}$ is the complexification of a real vector bundle $E_{\mathbb{R}} \longrightarrow X$ defined as the fixed points of E. It follows that there is an equivalence between the category of Real vector bundles over X and the category of real vector bundles, and therefore $KR(X) \cong KO(X)$.

[3] Note the capitalization, used here to avoid any confusions with real manifolds or vector spaces over $\mathbb{R}$.

However, as we are dealing with non-commutative spaces, Kasparov's generalization of KR-theory is more relevant to the study of twisted groupoid KR-theory. Indeed, in his 1980's founding article [4] of KK-theory, Kasparov focused on Real graded C^*-algebras; *i.e.*, $\mathbb{Z}_2$-graded complex C^*-algebras A which are the complexifications of real graded C^*-algebras ([53, 81]) $A_{\mathbb{R}}$ (the *Real part* of A), or equivalently, $\mathbb{Z}_2$-graded complex C^*-algebras endowed with conjugate linear involutions respecting the gradings. Given two such C^*-algebras A and B, the group $KKR(A,B)$ is defined to be the subgroup of $KK(A,B)$ generated by all Kasparov A,B-modules (E,φ,F) that are complexifications of real Kasparov $A_{\mathbb{R}},B_{\mathbb{R}}$-modules in a sense to be determined later. For instance, for $p,q\in\mathbb{N}$, the complex Clifford algebra $\mathbb{C}l_{p,q}$ ([7, 42, 81]), endowed with the involution induced from the involution on $\mathbb{R}^{p,q}\otimes_{\mathbb{R}}\mathbb{C}$ (where $\mathbb{C}$ is given the complex conjugation), is a Real graded C^*-algebra whose Real part identifies with the real Clifford algebra $Cl_{p,q}$. The higher KKR-groups are given by

$$KKR_{p-q}(A,B) := KKR(A\hat{\otimes}\mathbb{C}l_{p,q},B)\cong KKR(A,B\hat{\otimes}\mathbb{C}l_{q,p}).$$

In view of the 8-periodic property of complex Clifford algebras ([7]), Bott periodicity in KKR-theory expresses the following way

$$KKR_{i+8}(A,B)\cong KKR(A,B),\, i\in\mathbb{Z}.$$

Example 1.2.1. *Let X be a Real space, and let the C^*-algebra $\mathcal{C}_0(X)$ of complex valued continuous functions on X be endowed with the involution $\mathcal{C}_0(X)\ni f\longmapsto \bar{f}\in\mathcal{C}_0(X)$, with $\bar{f}(x):=\overline{f(\tau(x))}, x\in X$. Then $\mathcal{C}_0(X)$ is a Real (trivially) graded C^*-algebra. Moreover, it is not hard to check that $KKR(\mathbb{C},\mathcal{C}_0(X))\cong KR(X)$. In particular, if τ is trivial, then $\mathcal{C}_0(X)_{\mathbb{R}}\cong\mathcal{C}_0(X;\mathbb{R})$, and in that case $KKR(\mathbb{C},\mathcal{C}_0(X))\cong KKO(\mathbb{R},\mathcal{C}_0(X;\mathbb{R}))\cong KO(X)$.*

It then would make sense to think of Kasparov's KKR-theory as the non-commutative analog of Atiyah's KR. For a Real graded C^*-algebra B, we define its KR-theory to be $KR_i(B):=KKR(\mathbb{C},B)$, where $\mathbb{C}$ is equipped with the involution consisting of complex conjugation.

Note that although $KR_*(B)$ is a subgroup of $K_*(B)$, one captures, to a certain extent, all the informations of the latter from the study of the former. Indeed, we have shown the following decomposition [5]

Proposition 1.2.2. *Let B be a Real graded C^*-algebras. Then*

$$K_i(B)\otimes\mathbb{Z}[1/2]\cong (KR_i(B)\oplus KR_{i-2}(B))\otimes\mathbb{Z}[1/2].$$

[4] [46] is the English translation of the original paper written in Russian: Kasparov, G., *The operator K-functor and extensions of C^*-algebras.* Izv. Akad. Nauk. SSSR Ser. Mat. 44 (1980).

[5] See Proposition 6.4.5 and Proposition 6.4.7.

Let us now set about introducing twisted KR-theory of locally compact second countable Hausdorff groupoids. By a *Real groupoid* we mean a topological groupoid together with a groupoid isomorphism $\tau : \mathcal{G} \longrightarrow \mathcal{G}$ such that $\tau^2 = \mathrm{Id}$. For instance, a Real space X may be thought of as a Real groupoid with unit space and space of arrows identified with X. Also, a group with involution is a Real groupoid with unit space reduced to one point; *e.g.* the unit sphere $\mathbb{S}^1$ endowed with the complex conjugation is a Real groupoid $\mathbb{S}^1 \rightrightarrows \cdot$ which plays an important role in the study of twists of KR-theory. Morphisms of Real groupoids are defined as functors intertwining the involutions. Besides such morphisms, there is the notion of *generalized morphisms* of Real groupoids which is defined almost as in the usual case ([90, 63]). Such generalized morphisms may also be connected by some notion of Real morphisms. Real groupoids are then the objects of a 2-category in which 1-morphisms are generalized morphisms and 2-morphisms are morphisms between generalized morphisms. Moreover, we will be working in a category $\mathfrak{RG}$ in which objects are Real groupoids and morphisms are 2-isomorphism classes of 1-morphisms.

Definition 1.2.3 (Definition 4.1.1)**.** *By a* Real graded D-D bundle *over a Real groupoid* $\mathcal{G}$ *we mean a Dixmier-Douady bundle* $\pi : \mathcal{A} \longrightarrow X$ *over* $\mathcal{G}$ *endowed with an involution* $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ *such that* $\pi(\sigma(a)) = \tau(\pi(a)), \forall a \in \mathcal{A}$*, the induced maps* $\sigma_x : \mathcal{A}_x \longrightarrow \mathcal{A}_{\tau(x)}$ *are anti-linear graded* **-isomorphisms, and* $\alpha_{\tau(g)}(\sigma(a)) = \sigma_{r(g)}(a), \forall g \in \mathcal{G}, a \in \mathcal{A}_{s(g)}$.

The *Real graded Brauer group* $\widehat{\mathrm{BrR}}(\mathcal{G})$ of $\mathcal{G}$ is defined as the set of *Morita equivalence* classes of Real graded D-D bundles, subject to the operation of Real graded tensor product over the unit space X. We denote by $\widehat{\mathrm{BrR}}_0(\mathcal{G})$ the subgroup of $\widehat{\mathrm{BrR}}(\mathcal{G})$ generated by Real graded D-D bundles $\mathcal{A}$ satisfying the property: for every $x \in X$, there exists a neighborhood U of x which is invariant under τ such that the Real space $\pi^{-1}(U)$ is homeomorphic to te Real space $U \times \widehat{\mathfrak{K}}_0$, where $\widehat{\mathfrak{K}}_0$ is the Real graded C^*-algebra $\mathcal{K}(\hat{\mathcal{H}})$ of compact operators over the graded Hilbert space $\hat{\mathcal{H}} := l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ endowed with the involution given by the coordinatewise complex conjugation with respect to its canonical orthonormal basis, and the grading $(h,k) \longmapsto (k,h)$. In fact, elements of $\widehat{\mathrm{BrR}}_0(\mathcal{G})$ expresses as generalized morphisms as follows. Let $\hat{U}(\hat{\mathcal{H}})$ be the group of homogeneous unitaries on $\hat{\mathcal{H}}$, endowed with the involution induced from $\hat{\mathcal{H}}$. Then $\hat{\mathcal{H}}$ is a Real groupoid with an obvious action by $\mathbb{S}^1$ which is compatible with the involutions. Let $\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) := \hat{\mathrm{U}}(\hat{\mathcal{H}})/\mathbb{S}^1$. Then

Theorem 1.2.4 (Theorem 4.7.4)**.** *Let* $\mathcal{G}$ *be a locally compact second countable Hausdorff Real groupoid with Haar system ([76]). Then*

$$\widehat{BrR}_0(\mathcal{G}) \cong \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st},$$

where $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}$ *is the subset of* stable *elements of* $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$.

In order to establish a cohomological formula of $\widehat{\mathrm{BrR}}(\mathcal{G})$, we need to define a cohomology theory $\check{H}R^*$ relevant to Real groupoids. This is a variant of J.-L. Tu's groupoid cohomology ([88]) which takes values on sheaves endowed with involutions. Although this theory behaves like a $\mathbb{Z}_2$-equivariant Čech cohomology, it is not.

We define a peculiar Real group $\mathsf{Inv}\hat{\mathfrak{K}}$ generated by triples of the form $(\hat{\mathbb{K}}, \hat{\mathbb{K}}^-, \mathfrak{t})$, where $\hat{\mathbb{K}}$ and $\hat{\mathbb{K}}^-$ are graded elementary C^*-algebras and $\mathfrak{t} : \hat{\mathbb{K}} \longrightarrow \hat{\mathbb{K}}^-$ is a graded conjugate linear *-isomorphism. One of the main result of this thesis is the following

Theorem 1.2.5. *Let $\mathcal{G}$ be a locally compact Hausdorff second countable Real groupoid with Haar system. There is a natural isomorphism*

$$\widehat{BrR}(\mathcal{G}) \cong \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}}) \times \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1),$$

where the group $\mathbb{Z}_2$ is given the trivial involution and $\mathbb{S}^1$ is given the complex conjugation.

In the particular case of a fixed point free involution τ on $\mathcal{G}$, the complex graded Brauer goup $\widehat{\mathrm{Br}}(\mathcal{G})$ and $\widehat{\mathrm{BrR}}(\mathcal{G})$ are related through the nice decomposition

$$\widehat{\mathrm{Br}}(\mathcal{G}) \otimes \mathbb{Z}[1/2] \cong \left(\widehat{\mathrm{BrR}}(\mathcal{G}) \oplus \widehat{\mathrm{Br}}(\mathcal{G}/_\tau)\right) \otimes \mathbb{Z}[1/2],$$

where $\mathcal{G}/_\tau$ is the groupoid obtained by identifying every $g \in \mathcal{G}$ with its image $\tau(g)$. The other extreme case is when τ is the trivial involution. In that case, we show that $\widehat{\mathrm{BrR}}(\mathcal{G})$ offers a generalization of Donovan-Karoubi's graded orthogonal Brauer group $\widehat{\mathrm{BrO}}$ (see Theorem 4.3.6).

Now for $\mathcal{A} \in \widehat{\mathrm{BrR}}(\mathcal{G})$, the reduced C^*-algebra $\mathcal{A} \rtimes \mathcal{G}$ is actually a Real graded C^*-algebra (see Chapter 5). Then, the *twisted Real K-theory* of $\mathcal{G}$ is defined as

$$KR^{-i}_{\mathcal{A}}(\mathcal{G}^\bullet) := KR_i(\mathcal{A} \rtimes_r \mathcal{G}).$$

From Proposition 1.2.2 we deduce

Theorem 1.2.6 (Theorem 6.4.1)**.** *Twisted complex K-theory and twisted Real K-theory are related by the following decomposition*

$$K^*_{\mathcal{A}}(\mathcal{G}^\bullet) \otimes \mathbb{Z}[1/2] \cong \left(KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \oplus KR^{*-2}_{\mathcal{A}}(\mathcal{G}^\bullet)\right) \otimes \mathbb{Z}[1/2].$$

Of course by taking τ to be the trivial involution, twisted KR-theory is nothing but a generalization of twisted KO-theory for topological groupoids.

We give various interpretations of this theory: topological in terms of Real Fredholm operators, as well as geometric in terms of Real graded twisted vector bundles. For the topological one, we show the following result

Theorem 1.2.7 (Theorem 7.4.2)**.** *Suppose the Real groupoid $\mathcal{G}$ is proper. Let $\mathcal{A} \in \widehat{BrR}_0(\mathcal{G})$. Then associated to $\mathcal{A}$, there is a Real proper groupoid $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ together 1-isomorphism $\mathcal{G} \longrightarrow \Gamma$, and for $p, q \in \mathbb{N}$, a generalized morphism $\mathbb{P}^{p-q} : \Gamma \longrightarrow \widehat{\mathrm{PU}}_{p-q}(\hat{\mathcal{H}})$, such that*

$$KR^{q-p}_{\mathcal{A}}(\mathcal{G}^\bullet) \cong \left[\mathbb{P}^{p-q}/\Gamma, \hat{\mathfrak{F}}^{p-q}\right]_R^{\widehat{\mathrm{PU}}_{p-q}(\hat{\mathcal{H}})},$$

where the right hand side is the set of homotopy classes of Real $\widehat{\mathrm{PU}}_{p-q}(\hat{\mathcal{H}})$-equivariant continuous functions. Here $\hat{\mathcal{H}}$ is endowed with some Real graded $\mathbb{C}l_{p,q}$-action; $\widehat{\mathrm{PU}}_{p-q}(\hat{\mathcal{H}})$ is the Real subgroup of $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ consisting of equivalence classes of unitaries commuting with the $\mathbb{C}l_{p,q}$-action, $\hat{\mathfrak{F}}^{p-q}$ is the Real space of degree 1 Fredholm operators on $\hat{\mathcal{H}}$.

This identification allows us to define the multiplicative structure

$$KR^{-i}_{\mathcal{A}}(\mathcal{G}^\bullet)\otimes KR^{-j}_{\mathcal{B}}(\mathcal{G}^\bullet)\longrightarrow KR^{-i-j}_{\mathcal{A}+\mathcal{B}}(\mathcal{G}^\bullet)$$

for Real proper groupoids (Proposition 7.5.1).

Finally, we introduce Real groupoid-equivariant KK-theory via *correspondences*, which provides an elegant proof of *Thom isomorphism* in twisted groupoid K-theory (cf. Theorem 9.8.1), and allows us to prove that twisted KR-theory is a covariant functor in the category of Real proper groupoids.

1.3 General plan

We have tried, to the extent possible, to write this thesis in a self-contained way. It is organized as follows:

- Chapter 2 is devoted to elementary notions and results about Real groupoids and Real graded extensions.
- In Chapter 3 we define Real Čech cohomology theory for Real groupoids, and then connect it to Real graded central extensions.
- In chapter 4 we introduce the *Real graded Brauer group* of a Real groupoid, and then give it a Real cohomological formula. We shall however point out that this chapter requires familiarity with Real graded elementary C^*-algebras and Real fields of graded C^*-algebras discussed in Appendices A, B and C. Mainly, the classification of Real graded elementary C^*-algebras established in Appendix A is a prerequisite.
- Chapter 5 is an expository about Real graded Fell bundles and their associated Real graded C^*-algebras. Especially, we prove the analog of the well-known Renault' equivalence theorem, for the reduced Real graded C^*-algebras of Morita equivalent Real graded Fell systems. This result is important in the study of twisted KR-theory, especially when it comes to geometric interpretation.
- Chapter 6 is an introduction to twisted KR-theory of locally compact second countable Hausdorff Real groupoids. Tha analogues of the fundamental results of K-theory are established: Bott periodicity, Mayer-Vietoris exact sequence, and extension maps.

- In Chapter 7 we give a topological formulation of twisted KR-theory in the proper case.
- In Chapter 8 we focus on the case where the twists are torsions elements of the Real graded Brauer goup. We then introduce K-theory of Real graded twisted vector bundles, which we compare to twisted KR-theory. The last section of this chapter is devoted to the case of transformation Real groupoids.
- Chapter 9 is aimed at investigating groupoid equivariant Real KK-theory using C^*-correspondences, which we use to prove Thom isomorphism in twisted KR-theory.

2

Real groupoids

2.1 Definitions and Examples

Recall that a *strict homomorphism* between two groupoids $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ and $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ is a functor $\varphi : \Gamma \longrightarrow \mathcal{G}$ given by a map $\Gamma^{(0)} \longrightarrow \mathcal{G}^{(0)}$ on objects and a map $\Gamma^{(1)} \longrightarrow \mathcal{G}^{(1)}$ on arrows, both denoted again by φ, which together preserve the groupoid structure maps, i.e. $\varphi(s(\gamma)) = s(\varphi(\gamma))$, $\varphi(r(\gamma)) = r(\varphi(\gamma))$, $\varphi(\mathbf{1}_y) = \mathbf{1}_{\varphi(y)}$ and $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1)\varphi(\gamma_2)$ (this means that also $\varphi(\gamma^{-1}) = \varphi(\gamma)^{-1}$), for any $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and any $y \in \Gamma^{(0)}$. Unless otherwise specified, all our groupoids are topological groupoids which are supposed to be Hausdorff and locally compact. We can now give the following definition.

Definition 2.1.1. *A* `Real` *groupoid is a groupoid* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *together with a strict* 2*-periodic homeomorphism* $\rho : \mathcal{G} \longrightarrow \mathcal{G}$. *The homeomorphism* ρ *is called a* `Real structure on` $\mathcal{G}$. *Such a groupoid will be denoted by a pair* $(\mathcal{G}, \rho)$.

Example 2.1.2. *Any topological Real space* (X, ρ) *in the sense of Atiyah ([6]) can be viwed as a Real groupoid whose the unit space and the space of morphisms are identified with* X*; i.e, the operations in this Real groupoid is defined by* $s(x) = r(x) = x$, $x.x = x$, $x^{-1} = x$.

Example 2.1.3. *Any group with involution can be viewed as a Real groupoid with unit space identified with the unit element. Such a group will be called Real.*

Real abelian groups will play an important role in the study of twisted K-theory of Real groupoids.

Lemma 2.1.4. *Let* G *be an abelian group equipped with an involution* $\tau : G \longrightarrow G$ *(i.e. a Real structure). Set*

$$\Re(\tau) := \{g \in G \mid \tau(g) = g\} = {}^{\mathbb{R}}G, \quad \Im(\tau) := \{g \in G \mid \tau(g) = -g\}.$$

Then,

$$G \otimes \mathbb{Z}[\frac{1}{2}] \cong (\Re(\tau) \oplus \Im(\tau)) \otimes \mathbb{Z}[\frac{1}{2}]. \tag{2.1}$$

If τ is understood, we will write ${}^{\mathbb{J}}G$ for $\Im(\tau)$. We call $\Re(\tau)$ and $\Im(\tau)$ the Real part and the imaginary part of G, respectively.

Proof. For all $g \in G$, one has $g + \tau(g) \in {}^{\mathbb{R}}G$, and $g - \tau(g) \in {}^{\mathbb{J}}G$. Therefore, after tensoring G with $\mathbb{Z}[1/2]$, every $g \in G$ admits a unique decomposition

$$g = \frac{g+\tau(g)}{2} + \frac{g-\tau(g)}{2} \in \mathbb{Z}[1/2] \otimes \left({}^{\mathbb{R}}G \oplus {}^{\mathbb{J}}G\right).$$

□

Example 2.1.5 (The Real spaces $\mathbb{R}^{p,q}$ and $\mathbf{S}^{p,q}$). *Let $n \in \mathbb{N}^*$. Suppose ρ is a Real structure on the additive group $\mathbb{R}^n$. Then, every $u \in \mathbb{R}^n$ decomposes into a unique sum $v + w$ such that $\rho(v) = v$ and $\rho(w) = -w$. Indeed, $u = \frac{u+\rho(u)}{2} + \frac{u-\rho(u)}{2}$, so that $\mathbb{R}^n = \ker(\frac{\mathbf{1}-\rho}{2}) \oplus \mathrm{Im}(\frac{\mathbf{1}-\rho}{2})$, and with respect to this decomposition, ρ is given by $\rho(v,w) = (v,-w)$. It then follows that there exists a unique decomposition $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ such that ρ is determined by the forumula*

$$\rho(x,y) = (\mathbf{1}_p \oplus (-\mathbf{1}_q))(x,y) := (x,-y),$$

for all $(x,y) = (x_1,\cdots,x_p,y_1,\cdots,y_q) \in \mathbb{R}^p \oplus \mathbb{R}^q$.
For each pair $(p,q) \in \mathbb{N}$, we will write $\mathbb{R}^{p,q}$ for the additive group $\mathbb{R}^{p+q}$ equipped with the Real structure $(\mathbf{1}_p \oplus (-\mathbf{1}_q))$.

Now, we define the Real space $\mathbf{S}^{p,q}$ as the invariant subset (i.e. invariant under the Real structure) of $\mathbb{R}^{p,q}$ consisting of those $u \in \mathbb{R}^{p+q}$ such that $\|u\| = 1$, where as usual, $\|u\|^2 = x_1^2 + \cdots + x_p^2 + y_1^2 + \cdots + y_q^2$. For $q = p$, $\mathbf{S}^{p,p}$ is clearly identified with the Real space $\mathbb{S}^p$ whose Real structure is given by the coordinatewise complex conjugation. Notice that ${}^r\mathbf{S}^{p,q} = \mathbf{S}^{p,0}$.

Example 2.1.6. *Let (X,ρ) be a topological Real space. Let us consider the fundamental groupoid $\pi_1(X)$ over X whose arrows from $x \in X$ to $y \in X$ are homotopy classes of paths (relative to end-points) from x to y and the partial multiplication given by the concatenation of paths. The involution ρ induces a Real structure on the groupoid as follows: if $[\gamma] \in \pi_1(X)$, we set $\rho([\gamma])$ the homotopy classes of the path $\rho(\gamma)$ defined by $\rho(\gamma)(t) := \rho(\gamma(t))$ for $t \in [0,1]$.*

Let us fix some notations and conventions.

Notations 2.1.7. *From now on, by a Real structure on a groupoid, we will mean a representative of a conjugation class of Real structures. Moreover, for the sake of simplicity, we will put $\bar{g} := \rho(g)$, and we will just write $\mathcal{G}$ instead of $(\mathcal{G},\rho)$ when ρ is understood.*

Definition 2.1.8. *Two Real structures ρ and ρ' on $\mathcal{G}$ are said to be* conjugate *if there exists a strict homeomorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}$ such that $\rho' = \phi \circ \rho \circ \phi^{-1}$. In this case we say that the Real groupoids $(\mathcal{G},\rho)$ and $(\mathcal{G},\rho')$ are equivalent.*

Lemma 2.1.9. *Let $\mathcal{G}$ and Γ be Real groupoids, and let $\phi:\Gamma\longrightarrow\mathcal{G}$ be a Real groupoid homomorphism, then $\phi({}^r\Gamma)$ is a full subgroupoid of ${}^r\mathcal{G}\rightrightarrows{}^rX$. If in addition ϕ is an isomorphism, then ${}^r\Gamma\cong{}^r\mathcal{G}\rightrightarrows{}^rX$.*
In particular, if ρ_1 and ρ_2 are two conjugate Real structures on $\mathcal{G}$, then ${}^{\rho_1}\mathcal{G}\cong{}^{\rho_2}\mathcal{G}$.

Proof. This is obvious since $\phi(\bar\gamma)=\overline{\phi(\gamma)}$ for all $\gamma\in\Gamma$. □

Remark 2.1.10. *Note that the converse of the second statement of the above lemma is false in general. For instance, consider the Real group $\mathbb{S}^1$ whose Real structure is given by the complex conjugation, and the Real group $\mathbb{Z}_2$ (with the trivial Real structure). We have ${}^{\mathbb{R}}\mathbb{S}^1=\{\pm1\}\cong\mathbb{Z}_2={}^{\mathbb{R}}\mathbb{Z}_2$.*

The following is an example of groupoids with equivalent Real structures.

Example 2.1.11 (**Symmetric Riemannian manifold**)**.** *Recall ([34, IV.3]) that a Riemannian manifold X is called `globally symmetric` if each point $x\in X$ is an isolated fixed point of an involutive isometry $s_x:X\longrightarrow X$; i.e. s_x is a diffeomorphism verifying $s_x^2=\mathrm{Id}_X$ and $s_x(x)=x$. Moreover, for every two points $x,y\in X$, s_x and s_y are related through the formula $s_x\circ s_y\circ s_x=s_{s_x(y)}$.*
Given such a space, each point $x\in X$ defines a Real structure on X which leaves x fixed. However, let x and y be two different points in X and let $z\in X$ be such that $y=s_z(x)$. Then, we get $s_z\circ s_x\circ s_z=s_y$ which means that the diffeomorphism $s_z:X\longrightarrow X$ implements an equivalence $s_x\sim s_y$. But since x and y are arbitrary, it turns out that all of the Real structures s_x are equivalent. Thus, all of the Real spaces (X,s_x) are equivalent to each others.

Now, recall ([34, IV. Theorem 3.3]) that if G denotes the identity component of $I(X)$, where the latter is the group of isometries on X, then the map $\sigma_{x_0}:g\longmapsto s_{x_0}gs_{x_0}$ is an involutive automorphism in G, for any arbitrary $x_0\in X$. It follows that all of the points of X give rise to equivalent Real groups (G,σ_x).

Definition 2.1.12. ***Real covers*** *Let (X,ρ) be a Real space. We say that an open cover $\mathcal{U}=\{U_i\}_{i\in I}$ of X is* Real *if $\mathcal{U}$ is invariant with respect to the Real structure ρ; i.e. $\rho(U_i)\in\mathcal{U},\forall i\in I$. Alternatively, $\mathcal{U}$ is Real if I is equipped with an involution $i\longmapsto\bar i$ such that $U_{\bar i}=\rho(U_i)$ for all $i\in I$.*

Remark 2.1.13. *Observe that Real open covers always exist for all locally compact Real space X. Indeed, let $\mathcal{V}=\{V_{i'}\}_{i'\in I'}$ be an open cover of the space X. Let $I:=I'\times\{\pm1\}$ be endowed with the involution $(i',\pm1)\longmapsto(i',\mp1)$. Next, put $U_{(i',\pm1)}:=\rho^{(\pm1)}(V_{i'})$, where $\rho^{(+1)}(g):=g$, and $\rho^{(-1)}(g):=\rho(g)$ for $g\in\mathcal{G}$.*

From now on, by a Real structure we will mean (a representative of) an equivalence class of Real structures.

Definition 2.1.14 (Real action). *Let (Z,τ) be a locally compact Hausdorff Real space. A (continuous) right Real action of $(\mathcal{G},\rho)$ on (Z,τ) is given by a continuous open map $\mathfrak{s}: Z \longrightarrow \mathcal{G}^{(0)}$ (called the `generalized source map`) and a continous map $Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G} \longrightarrow Z$, denoted by $(z,g) \longmapsto zg$, such that*

(a) $\tau(zg) = \tau(z)\rho(g)$ *for all* $(z,g) \in Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$*;*

(b) $\rho(\mathfrak{s}(z)) = \mathfrak{s}(\tau(z))$ *for all* $z \in Z$*;*

(c) $\mathfrak{s}(zg) = s(g)$*;*

(d) $z(gh) = (zg)h$ *for* $(z,g) \in Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$ *and* $(g,h) \in \mathcal{G}^{(2)}$*;*

(e) $z\mathfrak{s}(z) = z$ *for any* $z \in Z$ *where we identify* $\mathfrak{s}(z)$ *with its image in* $\mathcal{G}$ *by the inclusion* $\mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$*.*

If such a Real action is given, we say that (Z,τ) is a (right) Real $\mathcal{G}$-space.

Likewise a (continuous) left Real action of $(\mathcal{G},\rho)$ on (Z,τ) is determined by a continuous Real open surjection $\mathfrak{r}: Z \longrightarrow \mathcal{G}^{(0)}$ (the `generalized range map` of the action) and a continuous Real map $\mathcal{G} \times_{s,\mathcal{G}^{(0)},\mathfrak{r}} Z \longrightarrow Z$ satisfying the appropriate analogues of conditions (a), (b), (c), (d) and (e) above.

Given a right Real action of $(\mathcal{G},\rho)$ on (Z,τ) with respect to $\mathfrak{s}$, let $\Psi: Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G} \longrightarrow Z \times Z$ be defined by the formula $\Psi(z,g) = (z,zg)$. Then we say that the action is `free` if this map is one-to-one (or in other words if the equation $zg = z$ implies $g = \mathfrak{s}(z)$. The action is called `proper` if Ψ is proper.

Notations 2.1.15. *If we are given such a right (resp. left) Real action of $(\mathcal{G},\rho)$ on (Z,τ), and if there is no risk of confusion, we will write $Z * \mathcal{G}$ (resp. $\mathcal{G} * Z$) for $Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$ (resp. for $\mathcal{G} \times_{s,\mathcal{G}^{(0)},\mathfrak{r}} Z$).*

2.2 Real $\mathcal{G}$-bundles

Definition 2.2.1. *Let $(\mathcal{G},\rho)$ be a Real groupoid. A Real (right) $\mathcal{G}$-bundle over a Real space (Y,ϱ) is a Real (right) $\mathcal{G}$-space (Z,τ) with respect to a map $\mathfrak{s}: Z \longrightarrow \mathcal{G}^{(0)}$, together with a Real map $\pi: Z \longrightarrow Y$ satisfying the relation $\pi(zg) = \pi(z)$ for any $(z,g) \in Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$, and such that for any $y \in Y$, the induced map*

$$\tau_y : Z_y \longrightarrow Z_{\varrho(y)}$$

on the fibres is $\mathcal{G}$-antilinear in the sense that for $(z,g) \in Z_y \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$ we have

$$\tau_y(zg) = \tau_y(z)\rho(g)$$

as an element in $Z_{\varrho(y)}$.

Such a bundle (Z,τ) *is said to be* principal *if*

(i) $\pi : Z \longrightarrow Y$ *is* `locally split` *(means that it is surjective and admits local sections), and*

(ii) *the map* $Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G} \longrightarrow Z \times_Y Z$, $(z,g) \longmapsto (z,zg)$ *is a Real homeomorphism.*

Remarks 2.2.2.

(1). ***The unit bundle.*** *Given a Real groupoid* $(\mathcal{G},\rho)$, *its space of arrows* $\mathcal{G}^{(1)}$ *is a* $\mathcal{G}$*-principal Real bundle over* $\mathcal{G}^{(0)}$. *Indeed, the projection is the range map* $r : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$, *the generalized source map is given by* s *and the action is just the partial multiplication on* $\mathcal{G}$. *This bundle is denoted by* $U(\mathcal{G})$ *and is called the* `unit` *bundle of* $\mathcal{G}$ *(cf. [63]).*

(2). ***Pull-back.*** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\mathfrak{s}} & \mathcal{G}^{(0)} \\ {\scriptstyle \pi}\downarrow & & \\ Y & & \end{array}$$

be a $\mathcal{G}$*-principal Real bundle and* $f : Y' \longrightarrow Y$ *be a Real continuous map. Then the pull-back* $f^*Z := Y' \times_Y Z$ *equipped with the involution* (ϱ',τ) *has the structure of a* $\mathcal{G}$*-principal Real bundle over* Y'. *Indeed, the right Real* $\mathcal{G}$*-action is given by the* $\mathcal{G}$*-action on* Z *and the generalized source map is* $\mathfrak{s}'(y',z) := \mathfrak{s}(z)$.

(3). ***Trivial bundles.*** *From the previous two remarks, we see that if* (Z,τ) *is any Real space together with a Real map* $\varphi : Z \longrightarrow \mathcal{G}^{(0)}$, *then we get a* $\mathcal{G}$*-principal Real bundle* $\varphi^* U(\mathcal{G})$ *over* Z; *its total space being the space* $Z \times_{\varphi,\mathcal{G}^{(0)},r} \mathcal{G}$. *A Bundle of this form is called* `trivial` *while a* $\mathcal{G}$*-principal Real bundle which is locally of this form is called* `locally trivial`.

2.3 Generalized morphisms of Real groupoids

Definition 2.3.1. *A generalized morphism from a Real groupoid* (Γ,ϱ) *to a Real groupoid* $(\mathcal{G},\rho)$ *consists of a Real space* (Z,τ), *two maps*

$$\Gamma^{(0)} \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} \mathcal{G}^{(0)},$$

a left (Real) action of Γ *with respect to* $\mathfrak{r}$, *a right (Real) action of* $\mathcal{G}$ *with respect to* $\mathfrak{s}$, *such that*

(i) *the actions commute, i.e. if* $(z,g) \in Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$ *and* $(\gamma,z) \in \Gamma \times_{s,\Gamma^{(0)},\mathfrak{r}} Z$ *we must have* $\mathfrak{s}(\gamma z) = \mathfrak{s}(z)$, $\mathfrak{r}(zg) = \mathfrak{r}(z)$ *so that* $\gamma(zg) = (\gamma z)g$;

(ii) *the maps* $\mathfrak{s}$ *and* $\mathfrak{r}$ *are Real in the sense that* $\mathfrak{s}(\tau(z)) = \rho(\mathfrak{s}(z))$ *and* $\mathfrak{r}(\tau(z)) = \varrho(\mathfrak{r}(z))$ *for any* $z \in Z$*;*

(iii) $\mathfrak{r}: Z \longrightarrow \Gamma^{(0)}$ *is a locally trivial* $\mathcal{G}$*-principal Real bundle.*

Example 2.3.2. *Let* $f : \Gamma \longrightarrow \mathcal{G}$ *be a Real strict morphism. Let us consider the fibre product* $Z_f := \Gamma^{(0)} \times_{f,\mathcal{G}^{(0)},r} \mathcal{G}$ *and the maps* $\mathfrak{r} : Z_f \longrightarrow \Gamma^{(0)}$, $(y, g) \longmapsto y$ *and* $\mathfrak{s} : Z_f \longrightarrow \mathcal{G}^{(0)}$, $(y, g) \longmapsto s(g)$*. For* $(\gamma, (y, g)) \in \Gamma \times_{s,\Gamma^{(0)},\mathfrak{r}} Z_f)$*, we set* $\gamma.(y, g) := (r(\gamma), f(\gamma)g)$ *and for* $((y, g), g') \in Z_f \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$ *we set* $(y, g).g' := (y, gg')$*. Using the definition of a strict morphism, it is easy to check that these maps are well defined and make* Z_f *into a generalized morphism from* Γ *to* $\mathcal{G}$*. Furthermore, the map* τ *on* Z_f *defined by* $\tau(y, g) := (\varrho(y), \rho(g))$ *is a Real involution and then* Z_f *is a Real generalized morphism.*

Definition 2.3.3. *A morphism between two such morphisms* (Z, τ) *and* (Z', τ') *is a* Γ*-*$\mathcal{G}$*-equivariant Real map* $\varphi : Z \longrightarrow Z'$ *such that* $\mathfrak{s} = \mathfrak{s}' \circ \varphi$ *and* $\mathfrak{r} = \mathfrak{r}' \circ \varphi$*. We say that the Real generalized homomorphism* (Z, τ) *and* (Z', τ') *are* `isomorphic` *if there exists such a* φ *which is at the same time a homeomorphism.*

Compositions of Real generalized morphisms are defined by the following proposition.

Proposition 2.3.4. *Let* (Z', τ') *and* $(Z", \tau")$ *be Real generalized homomorphisms from* (Γ, ϱ) *to* $(\mathcal{G}', \rho')$ *and from* $(\mathcal{G}', \rho')$ *to* $(\mathcal{G}, \rho)$ *respectively. Then*

$$Z = Z' \times_{\mathcal{G}'} Z" := (Z' \times_{\mathfrak{s}',\mathcal{G}'^{(0)},\mathfrak{r}"} Z")/_{(z',z")\sim(z'g',g'^{-1}z")}$$

with the obvious Real involution, defines a Real generalized morphism from $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *to* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *.*

Proof. Let us describe at first the structure maps

$$\Gamma^{(0)} \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} \mathcal{G}^{(0)}$$

and the actions.

For $(z', z") \in Z$ we set $\mathfrak{r}(z', z") := \mathfrak{r}'(z')$ and $\mathfrak{s}(z', z") := \mathfrak{s}"(z")$. These are well defined and since $\mathfrak{s}(z'g', g'^{-1}z") = \mathfrak{s}"(g'^{-1}z") = \mathfrak{s}"(z")$ and $\mathfrak{r}(z'g', g'^{-1}z") = \mathfrak{r}'(z'g') = \mathfrak{s}'(z')$ from the point (i) of Definition 2.3.1. The actions are defined by $\gamma.(z', z") := (\gamma z', z")$ and $(z', z").g := (z', z"g)$ for $(\gamma, (z', z")) \in \Gamma \times_{s,\Gamma^{(0)},\mathfrak{r}} Z$ and $((z', z"), g) \in Z \times_{\mathfrak{s},\mathcal{G}^{(0)},r} \mathcal{G}$ while the Real involution is the obvious one: $\tau(z', z") := (\tau'(z'), \tau"(z"))$.

Now to show the local triviality of Z, notice that from (3) of Remarks 2.2.2, Z' and $Z"$ are locally of the form $U \times_{\varphi',\mathcal{G}'^{(0)},r'} \mathcal{G}'$ and $V \times_{\varphi",\mathcal{G}^{(0)},r} \mathcal{G}$ respectively, where $\varphi' : U \longrightarrow \mathcal{G}'^{(0)}$ and $\varphi" : V \longrightarrow \mathcal{G}^{(0)}$ are Real continuous maps, U and V subspaces of $\Gamma^{(0)}$ and $\mathcal{G}'^{(0)}$ respectively. It turns out that by construction, Z is locally of the form $W \times_{\varphi,\mathcal{G}'^{(0)},r} \mathcal{G}$ where $W = U \times_{\varphi',\mathcal{G}'^{(0)}} V$. □

Definition 2.3.5. *Given two Real generalized morphisms* $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G}',\rho')$ *and* $(Z',\tau'):(\mathcal{G}',\rho')\longrightarrow(\mathcal{G},\rho)$*, we define their composition* $(Z'\circ Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ *to be* $(Z\times_{\mathcal{G}'}Z',\tau\times\tau')$.

Remark 2.3.6. *It is easy to check that the composition of Real generalized morphisms is associative. For instance, if*

$$\Gamma\xrightarrow{(Z_1,\rho_1)}\mathcal{G}_1\xrightarrow{(Z_2,\rho_2)}\mathcal{G}_2\xrightarrow{(Z_3,\rho_3)}\mathcal{G}$$

are given Real generalized morphisms, we get two Real generalized morphisms $Z=Z_1\times_{\mathcal{G}_1}(Z_2\times_{\mathcal{G}_2}Z_3)$ *and* $Z'=(Z_1\times_{\mathcal{G}_1}Z_2)\times_{\mathcal{G}_2}Z_3$ *between* (Γ,ϱ) *and* $(\mathcal{G},\rho)$*; notice that here* Z *and* Z' *carry the obvious Real involutions. Moreover, the map* $Z\longrightarrow Z'$, $(z_1,(z_2,z_3))\longmapsto((z_1,z_2),z_3)$ *is a* Γ*-*$\mathcal{G}$*-equivariant Real homeomorphism. Hence, there exists a category* $\mathfrak{RG}$ *whose objects are Real locally compact groupoids and morphisms are isomorphism classes of Real generalized homomorphisms.*

Lemma 2.3.7. *Let* $f_1,f_2:\Gamma\to\mathcal{G}$ *be two Real strict homomorphisms. Then* f_1 *and* f_2 *define isomorphic Real generalized homomorphisms if and only if there exists a Real continuous map* $\varphi:\Gamma^{(0)}\longrightarrow\mathcal{G}$ *such that* $f_2(\gamma)=\varphi(r(\gamma))f_1(\gamma)\varphi(s(\gamma))^{-1}$.

Proof. Le $\Phi:Z_{f_1}\longrightarrow Z_{f_2}$ be a Real Γ-$\mathcal{G}$-equivariant homeomorphism, where $Z_{f_i}=\Gamma^{(0)}\times_{f_i,\mathcal{G}^{(0)},r}\mathcal{G}$. Then from the commutative diagrams

$$\begin{tikzcd}
\Gamma^{(0)} & Z_{f_1} \arrow[l, "pr_1"'] \arrow[r, "s\circ pr_2"] \arrow[d, "\Phi"] & \mathcal{G}^{(0)} \\
& Z_{f_2} \arrow[ul, "pr_1"] \arrow[ur, "s\circ pr_2"'] &
\end{tikzcd}$$

we have $\Phi(x,g)=(x,h)$ with $s(g)=s(h)$; and then there exists a unique element $\varphi(x)\in\mathcal{G}$ such that $h=\varphi(x)g$. To see that this defines a continuous map $\varphi:\Gamma^{(0)}\longrightarrow\mathcal{G}$, notice that for any $x\in\Gamma^{(0)}$, the pair $(x,f_1(x))$ is an element in Z_{f_1}, then $\varphi(x)$ is the unique element in $\mathcal{G}$ such that $\Phi(x,f_1(x))=(x,\varphi(x)f_1(x))$. Furthermore, since Φ is Real, $\Phi(\varrho(x),\rho(f_1(x)))=(\varrho(x),\rho(\varphi(x))\rho(f_1(x)))$ which shows that $\varphi(\varrho(x))=\rho(\varphi(x))$ for any $x\in\Gamma^{(0)}$; i.e. φ is Real.
Now for $\gamma\in\Gamma$, take $x=s(\gamma)$, then from the Γ-equivariance of Φ, we have

$$\Phi(\gamma.(s(\gamma),f_1(s(\gamma))))=\Phi(r(\gamma),f_1(\gamma))=\gamma.\Phi(s(\gamma),f_1(s(\gamma)));$$

so that

$$(r(\gamma),\varphi(r(\gamma))f_1(\gamma))=(r(\gamma),f_2(\gamma)\varphi(s(\gamma)))$$

and $f_2(\gamma).r(\varphi(s(\gamma)))=\varphi(r(\gamma))f_1(\gamma)\varphi(s(\gamma))$; but $r(\varphi(s(\gamma)))=s(f_2(\gamma))$ by definition of φ and this gives the desired relation.
The converse is easy to check by working backwards. □

2.4 Morita equivalences

Let (Γ,ϱ) and $(\mathcal{G},\rho)$ be two Real groupoids. Suppose that $f:(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ is an isomorphism in the category $\mathfrak{RG}_s$. In this case, we say that (Γ,ϱ) and $(\mathcal{G},\rho)$ are `strictly equivalent` and we write $(\Gamma,\varrho)\sim_{strict}(\mathcal{G},\rho)$. Now, consider the induced Real generalized morphisms $(Z_f,\tau_f):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ and $(Z_{f^{-1}},\tau_{f^{-1}}):(\mathcal{G},\rho)\longrightarrow(\Gamma,\varrho)$. Define `the inverse of` Z_f by $Z_f^{-1}:=\mathcal{G}\times_{r,\mathcal{G}^{(0)},f}\Gamma^{(0)}$ with the obvious Real structure also denoted by τ_f. The map $Z_{f^{-1}}\longrightarrow Z_f^{-1}$ defined by $(x,\gamma)\longmapsto(f(\gamma),f^{-1}(x))$ is clearly a $\mathcal{G}$-Γ-equivariant Real homeomorphism; hence, $(Z_{f^{-1}},\tau_{f^{-1}})$ and (Z_f^{-1},τ_f) are isomorphic Real generalized morphisms from $(\mathcal{G},\rho)$ to (Γ,ϱ). Notice that Z_f^{-1} is Z_f as space; thus, (Z_f,τ_f) is at the same time a Real generalized morphism from (Γ,ϱ) to $(\mathcal{G},\rho)$ and from $(\mathcal{G},\rho)$ to (Γ,ϱ). Furthermore, it is simple to check that $Z_f\circ Z_f^{-1}$ and $Z_{\mathrm{Id}_\mathcal{G}}$ define isomorphic Real generalized morphisms from $(\mathcal{G},\rho)$ into itself, and likewise, $Z_f^{-1}\circ Z_f$ and Z_{Id_Γ} are isomorphic Real generalized morphisms from (Γ,ϱ) into itself.

Definition 2.4.1. *Two Real groupoids (Γ,ϱ) and $(\mathcal{G},\rho)$ are said to be `Morita equivalent` if there exists a Real space (Z,τ) that is at the same time a Real generalized morphism from Γ to $\mathcal{G}$ and from $\mathcal{G}$ to Γ; that is to say that $\Gamma^{(0)}\xleftarrow{\mathfrak{r}}Z$ is a $\mathcal{G}$-principal `Real` bundle and $Z\xrightarrow{\mathfrak{s}}\mathcal{G}^{(0)}$ is a Γ-principal Real bundle.*

Remark 2.4.2. *Given a Morita equivalence $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$, its inverse, denoted by (Z^{-1},τ), is (Z,τ) as Real space, and if $\flat:(Z,\tau)\longrightarrow(Z^{-1},\tau)$ is the identity map, the left Real $\mathcal{G}$-action on (Z^{-1},τ) is given by $g.\flat(z):=\flat(z.g^{-1})$, and the right Real Γ-action is given by $\flat(z).\gamma:=\flat(\gamma^{-1}.z)$; (Z^{-1},τ) is the corresponding Real generalized morphism from $(\mathcal{G},\rho)$ to (Γ,ϱ).*

The discussion before Definition 2.4.1 shows that the Real generalized morphism induced by a Real strict morphism is actually a Morita equivalence. However, the converse is not true. Moreover, there is a functor

$$\mathfrak{RG}_s\longrightarrow\mathfrak{RG}, \tag{2.2}$$

where $\mathfrak{RG}_s$ is the category whose objects are Real locally compact groupoids and whose morphisms are Real strict morphisms, given by

$$f\longmapsto Z_f.$$

Definition 2.4.3. ***(Real cover groupoid).*** *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be a Real groupoid. Let $\mathcal{U}=\{U_j\}$ be a Real open cover of X. Consider the disjoint union $\coprod_{j\in J}U_j=\{(j,x)\in J\times X\ :\ x\in U_j\}$ with the Real structure $\rho^{(0)}$ given by $\rho^{(0)}(j,x):=(\bar{j},\rho(x))$ and define a Real local homeomorphism given by the projection $\pi:\coprod_j U_j\longrightarrow X,\ (j,x)\longmapsto x$. Then the set*

$$\mathcal{G}[\mathcal{U}]:=\{(j_0,g,j_1)\in J\times\mathcal{G}\times J\ :\ r(g)\in U_{j_0},s(g)\in U_{j_1}\},$$

endowed with the involution $\rho^{(1)}(j_0, g, j_1) := (\bar{j_0}, \rho(g), \bar{j_1})$ *has a structure of a* `Real` *locally compact groupoid whose unit space is* $\coprod_j U_j$. *The range and source maps are defined by* $\tilde{r}(j_0, g, j_1) := (j_0, r(g))$ *and* $\tilde{s}(j_0, g, j_1) := (j_1, s(g))$; *two triples are composable if they are of the form* (j_0, g, j_1) *and* (j_1, h, j_2), *where* $(g, h) \in \mathcal{G}^{(2)}$, *and their product is given by* $(j_0, g, j_1).(j_1, h, j_2) := (j_0, gh, j_2)$. *The inverse of* (j_0, g, j_1) *is* (j_1, g^{-1}, j_0).

It is a matter of simple verifications to check the following

Lemma 2.4.4. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a Real groupoid, and* $\mathcal{U}$ *a Real open cover of* X. *Then the Real generalized morphism* $Z_\iota : \mathcal{G}[\mathcal{U}] \longrightarrow \mathcal{G}$ *induced from the canonical Real morphism*

$$\iota : \mathcal{G}[\mathcal{U}] \longrightarrow \mathcal{G},\ (j_0, g, j_1) \longmapsto g,$$

is a Morita equivalence between $(\mathcal{G}[\mathcal{U}], \rho)$ *and* $(\mathcal{G}, \rho)$.

Definition 2.4.5. *Let*

$$\begin{tikzcd} Z \arrow[r, "\mathfrak{s}"] \arrow[d, "\pi"'] & \mathcal{G}^{(0)} \\ Y & \end{tikzcd}$$

be a locally trivial $\mathcal{G}$*-principal Real bundle. A section* $\mathsf{s} : Y \longrightarrow Z$ *is said to be Real if* $\mathsf{s} \circ \varrho = \tau \circ \mathsf{s}$. *Moreover, given a Real open cover* $\{U_j\}_{j \in J}$ *of* Y, *we say that a family of local sections* $\mathsf{s}_j : U_j \longrightarrow Z$ *is* globally Real *if for any* $j \in J$, *we have*

$$\mathsf{s}_{\bar{j}} \circ \varrho = \tau \circ \mathsf{s}_j. \tag{2.3}$$

Lemma 2.4.6. *Any locally trivial* $\mathcal{G}$*-principal Real bundle* $\pi : Z \longrightarrow Y$ *admits a globally Real family of local sections* $\{\mathsf{s}_j\}_{j \in J}$ *over some Real open cover* $\{U_j\}$.

Proof. Choose a `local trivialization` $(U_i, \varphi_i)_{i \in I}$ of Z; i.e. $\varphi_i : U_i \longrightarrow \mathcal{G}^{(0)}$ are continuous maps such that $\pi^{-1}(U_i) =: Z_{U_i} \cong U_i \times_{\varphi_i, \mathcal{G}^{(0)}, r} \mathcal{G}$ with $\tau_{Z_{U_i}} = (\varrho, \rho)$. It turns out that $Z_{U_{(i,\epsilon)}} \cong U_{(i,\epsilon)} \times_{\varphi_i^\epsilon, \mathcal{G}^{(0)}, r} \mathcal{G}$, where $\varphi_i^\epsilon := \rho^\epsilon \circ \varphi_i \circ \varrho^\epsilon : U_{(i,\epsilon)} \longrightarrow \mathcal{G}^{(0)}$ is a well defined continuous map and $U_{(i,\epsilon)} := \varrho^\epsilon(U_i)$ for $(i, \epsilon) \in I \times \mathbb{Z}_2$. However, for $(i, \epsilon) \in I \times \mathbb{Z}_2$, there is a homeomorphism $U_{(i,\epsilon)} \times_{\varphi_i^\epsilon, \mathcal{G}^{(0)}, r} \mathcal{G} \xrightarrow{(\varrho,\rho)} U_{\overline{(i,\epsilon)}} \times_{\varphi_i^{\epsilon+1}, \mathcal{G}^{(0)}, r} \mathcal{G}$. Now, putting $\mathsf{s}_{(i,\epsilon)} : U_{(i,\epsilon)} \longrightarrow Z,\ x \longmapsto (x, \varphi_i^\epsilon(x))$, we obtain the desired sections. □

For the remainder of this subsection we will need the following construction.
Let (Z, τ) be a Real space and (Γ, ϱ) a Real groupoid together with a continuous Real map $\varphi : Z \longrightarrow \Gamma^{(0)}$. Then we define an induced groupoid $\varphi^*\Gamma$ over Z in which the arrows from z_1 to z_2 are the arrows in Γ from $\varphi(z_1)$ to $\varphi(z_2)$; i.e.

$$\varphi^*\Gamma := Z \times_{\varphi, \Gamma^{(0)}, r} \Gamma \times_{s, \Gamma^{(0)}, \varphi} Z\ ,$$

and the product is given by $(z_1,\gamma_1,z_2).(z_2,\gamma_2,z_3)=(z_1,\gamma_1\gamma_2,z_3)$ whenever γ_1 and γ_2 are composable, while the inverse is given by $(z,\gamma,z')^{-1}=(z',\gamma^{-1},z)$. Moreover, the triple (ρ,ϱ,ρ) defines a Real structure $\varphi^*\varrho$ on $\varphi^*\Gamma$ making it into a Real groupoid $(\varphi^*\Gamma,\varphi^*\varrho)$ that we will call `the pull-back` of Γ over Z via φ.

Lemma 2.4.7. *Given a continuous locally split Real open map $\varphi: Z\longrightarrow\Gamma^{(0)}$, then the Real groupoids Γ and $\varphi^*\Gamma$ are Morita equivalent.*

Proof. Consider the Real strict homomorphism $\tilde\varphi:\varphi^*\Gamma\longrightarrow\Gamma$ defined by $(z_1,\gamma,z_2)\longmapsto\gamma$. Then by Example 2.3.2 we obtain a Real generalized homomorphism $Z\xleftarrow{\pi_1}Z_{\tilde\varphi}\xrightarrow{s\circ\pi_2}\Gamma^{(0)}$ with $Z_{\tilde\varphi}:=Z\times_{\tilde\varphi,\Gamma^{(0)},r}\Gamma$, π_1 and π_2 the obvious projections, and where $Z\hookrightarrow\varphi^*\Gamma$ by $z\longmapsto(z,\varphi(z),z)$. Now using the constructions of Example 2.3.2, it is very easy to check that $Z_{\tilde\varphi}$ is in fact a Morita equivalence.

□

Proposition 2.4.8. *Two Real groupoids (Γ,ϱ) and $(\mathcal{G},\rho)$ are Morita equivalent if and only if there exist a Real space (Z,τ) and two continuous Real maps $\varphi:Z\longrightarrow\Gamma^{(0)}$ and $\varphi':Z\longrightarrow\mathcal{G}^{(0)}$ such that $\varphi^*\Gamma\cong(\varphi')^*\mathcal{G}$ under a Real (strict) homeomorphism.*

Proof. Let $\Gamma^{(0)}\xleftarrow{\mathfrak{r}}Z\xrightarrow{\mathfrak{s}}\mathcal{G}^{(0)}$ be a Morita equivalence. Let us define

$$\Gamma\ltimes Z*Z\rtimes\mathcal{G}:=\{(\gamma,z_1,z_2,g)\in(\Gamma\times_{s,\Gamma^{(0)},\mathfrak{r}}Z)\times(Z\times_{\mathfrak{s},\mathcal{G}^{(0)},r}\mathcal{G})\mid z_1g=\gamma z_2\}\,.$$

This defines a Real groupoid over Z whose range and source maps are defined by the second and the third projection respectively, the product is given by

$$(\gamma,z_1,z_2,g).(\gamma',z_2,z_3,g')=(\gamma\gamma',z_1,z_3,gg'),$$

provided that $\gamma,\gamma'\in\Gamma^{(2)}$ and $g,g'\in\mathcal{G}^{(2)}$, and the inverse of (γ,z_1,z_2,g) is $(\gamma^{-1},z_2,z_1,g^{-1})$. Now, for a given triple $(z_1,\gamma,z_2)\in\mathfrak{r}^*\Gamma$, the relations $\mathfrak{r}(z_1)=r(\gamma)$ and $\mathfrak{r}(z_2)=s(\gamma)$ give $\mathfrak{r}(\gamma z_2)=\mathfrak{r}(z_1)$; then since $\mathfrak{r}:Z\longrightarrow\Gamma^{(0)}$ is a Real $\mathcal{G}$-principal bundle, there exists a unique $g\in\mathcal{G}$ such that $\gamma z_2=z_1g$. This gives an injective homomorphism $\Psi:\mathfrak{r}^*\Gamma\longrightarrow\Gamma\ltimes Z*Z\rtimes\mathcal{G}\,,(z_1,\gamma,z_2)\longmapsto(\gamma,z_1,z_2,g)$ which respects the Real structures. In the other hand, the map $\Phi:\Gamma\ltimes Z*Z\rtimes\mathcal{G}\longrightarrow\mathfrak{r}^*\Gamma\,,(\gamma,z_1,z_2,g)\longmapsto(z_1,\gamma,z_2)$ is a well defined Real homomorphism that is injective and Real. Moreover, these two maps are, by construction, inverse to each other so that we have a Real homeomorphism $\mathfrak{r}^*\Gamma\cong\Gamma\ltimes Z*Z\rtimes\mathcal{G}$. Furthermore, since $\mathfrak{s}:Z\longrightarrow\mathcal{G}^{(0)}$ is a Real Γ-principal bundle, we can use the same arguments to show that $\mathfrak{s}^*\mathcal{G}\cong\Gamma\ltimes Z*Z\rtimes\mathcal{G}$ under a Real homeomorphism.
Conversely, if $\varphi:Z\longrightarrow\Gamma^{(0)}$ and $\varphi':Z\longrightarrow\mathcal{G}^{(0)}$ are given continuous Real maps and $f:\varphi^*\Gamma\longrightarrow(\varphi')^*\mathcal{G}^{(0)}$ is a Real homeomorphism of groupoids, then the induced Real generalized homomorphism $\varphi^*\Gamma\xrightarrow{Z_f}(\varphi')^*\mathcal{G}$ is a Morita equivalence and Lemma 2.4.7 ends the proof. □

Proposition 2.4.9 (cf. Proposition 2.3 [90]). *Any Real generalized morphism*

$$\Gamma^{(0)} \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} \mathcal{G}^{(0)}$$

is obtained by composition of the canonical Morita equivalence between (Γ,ϱ) and $(\Gamma[\mathcal{U}],\varrho)$, where $\mathcal{U}$ is an open cover of $\Gamma^{(0)}$, with a Real strict morphism $f_{\mathcal{U}}:\Gamma[\mathcal{U}]\longrightarrow\mathcal{G}$ (i.e. its induced morphism in the category $\mathfrak{RG}$).

Proof. From Lemma 2.4.7, there is a Real Morita equivalence $Z_{\tilde{\mathfrak{r}}}:\mathfrak{r}^*\Gamma\longrightarrow\Gamma$ and the Real homeomorphism $\mathfrak{r}^*\Gamma\cong\Gamma\ltimes Z*Z\rtimes\mathcal{G}$ induces a Real strict homomorphism $f:\mathfrak{r}^*\Gamma\longrightarrow\mathcal{G}$ given by the fourth projection, and hence a Real generalized homomorphism $Z_f:\mathfrak{r}^*\Gamma\longrightarrow\mathcal{G}$. Furthermore, by using the construction of these generalized homomorphisms, it is easy to check that the composition $Z_{\tilde{\mathfrak{r}}}\times_\Gamma Z$ is $\mathfrak{r}^*\Gamma$-$\mathcal{G}$-equivariently homeomorphic to Z (under a Real homeomorphism); i.e, the diagram

$$\begin{tikzcd} \Gamma & \mathfrak{r}^*\Gamma \arrow[l, "Z_{\tilde{\mathfrak{r}}}"', "\cong"] \arrow[d, "Z_f"] \\ & \mathcal{G} \arrow[from=1-1, "Z"'] \end{tikzcd}$$

is commutative in the category $\mathfrak{RG}$.
Now, consider a Real open cover $\mathcal{U}=\{U_j\}$ of $\Gamma^{(0)}$ together with a globally Real family of local sections $\mathsf{s}_j:U_j\longrightarrow Z$ of $\mathfrak{r}:Z\longrightarrow\Gamma^{(0)}$. Then, setting $(j_0,\gamma,j_1)\longmapsto(\mathsf{s}_{j_0}(r(\gamma)),\gamma,\mathsf{s}_{j_1}(s(\gamma)))$ for $(j_0,\gamma,j_1)\in\Gamma[\mathcal{U}]$, we get a Real strict homomorphism $\tilde{\mathfrak{s}}:\Gamma[\mathcal{U}]\longrightarrow\mathfrak{r}^*\Gamma$ such that the composition $\Gamma[\mathcal{U}]\longrightarrow\mathfrak{r}^*\Gamma\longrightarrow\Gamma$ is the canonical map ι described in Example 2.4.3. Then, $f\circ\tilde{\mathfrak{s}}:\Gamma[\mathcal{U}]\longrightarrow\mathcal{G}$ is the desired Real strict homomorphism. □

This proposition leads us to think of a Real generalized morphism from a Real groupoid (Γ,ϱ) to a Real groupoid $(\mathcal{G},\rho)$ as a Real strict morphism $f_{\mathcal{U}}:(\Gamma[\mathcal{U}],\varrho)\longrightarrow(\mathcal{G},\rho)$, where $\mathcal{U}$ is a Real open cover of $\Gamma^{(0)}$.

To refine this point of view, given two Real groupoids (Γ,ϱ) and $(\mathcal{G},\rho)$, let Ω denote the collection of such pairs $(\mathcal{U},f_{\mathcal{U}})$. We say that two pairs $(\mathcal{U},f_{\mathcal{U}})$ and $(\mathcal{U}',f_{\mathcal{U}'})$ are `isomorphic` provided that $Z_{f_{\mathcal{U}}}\circ Z^{-1}_{\iota_{\mathcal{U}}}\cong Z_{f_{\mathcal{U}'}}\circ Z^{-1}_{\iota_{\mathcal{U}'}}$, where $\iota_{\mathcal{U}}:(\Gamma[\mathcal{U}],\varrho)\longrightarrow(\Gamma,\varrho)$ and $\iota_{\mathcal{U}'}:(\Gamma[\mathcal{U}'],\varrho)\longrightarrow(\Gamma,\varrho)$ are the canonical morphisms; this clearly defines an equivalence relation. We denote by $\Omega\big((\Gamma,\varrho),(\mathcal{G},\rho)\big)$ the set of isomorphism classes of elements of Ω.

Suppose that $(\mathcal{U},f_{\mathcal{U}}):(\Gamma,\varrho)\longrightarrow(\mathcal{G}'.\rho')$ is an equivalence class in $\Omega\big((\Gamma,\varrho),(\mathcal{G}',\rho')\big)$ and $(\mathcal{V},f_{\mathcal{V}}):(\mathcal{G}',\rho')\longrightarrow(\mathcal{G},\rho)$ is an element in $\Omega\big((\mathcal{G}',\rho'),(\mathcal{G},\rho)\big)$. Let $\iota_{\mathcal{G}'}:\mathcal{G}'[\mathcal{V}]\longrightarrow\mathcal{G}'$ be the canonical morphism, and let $Z^{-1}_{\iota_{\mathcal{G}'}}:(\mathcal{G}',\rho')\longrightarrow(\mathcal{G}'[\mathcal{V}],\rho')$ be the inverse of $Z_{\iota_{\mathcal{G}'}}$. Next, we apply Proposition 2.4.9 to the Real generalized morphism $Z^{-1}_{\iota_{\mathcal{G}'}}\circ Z_{f_{\mathcal{U}}}:\Gamma[\mathcal{U}]\longrightarrow\mathcal{G}'[\mathcal{V}]$ to get a Real open cover $\mathcal{U}'$ of $\Gamma^{(0)}$ containing $\mathcal{U}$ and a Real strict morphism $\varphi_{\mathcal{U}'}:(\Gamma[\mathcal{U}'],\varrho)\longrightarrow$

$(\mathcal{G}'[\mathcal{V}],\rho')$. Then, we pose

$$(\mathcal{V},f_{\mathcal{V}})\circ(\mathcal{U},f_{\mathcal{U}}):=(\mathcal{U}',f_{\mathcal{U}'}),\tag{2.4}$$

with $f_{\mathcal{U}'}=f_{\mathcal{V}}\circ\varphi_{\mathcal{U}'}$; thus we get an element of $\Omega\big((\Gamma,\varrho),(\mathcal{G},\rho)\big)$. It follows that there exists a category $\mathfrak{RG}_\Omega$ whose objects are Real groupoids, and in which a morphism from (Γ,ϱ) to $(\mathcal{G},\rho)$ is a class $(\mathcal{U},f_{\mathcal{U}})$ in $\Omega\big((\Gamma,\varrho),(\mathcal{G},\rho)\big)$.

Example 2.4.10. *Any Real strict morphism $f:(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ can be identified with the pair $(\Gamma^{(0)},f)$, by considering the trivial Real open cover $\Gamma^{(0)}$ consisting of one set, and by viewing the groupoid Γ as the cover groupoid $\Gamma[\Gamma^{(0)}]$. In particular, $\mathfrak{RG}_s$ is a subcategory of $\mathfrak{RG}_\Omega$.*

Example 2.4.11. *Suppose that $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ is a Real generalized morphism. Then, Proposition 2.4.9 provides a unique class $(\mathcal{U},f_{\mathcal{U}})\in\Omega((\Gamma,\varrho),(\mathcal{G},\rho))$.*

Remark 2.4.12. *Note that a class $(\mathcal{U},f_{\mathcal{U}})\in\Omega\big((\Gamma,\varrho),(\mathcal{G},\rho)\big)$ is an isomorphism in $\mathfrak{RG}_\Omega$ if there exists $(\mathcal{V},f_{\mathcal{V}})\in\Omega\big((\mathcal{G},\rho),(\Gamma,\varrho)\big)$ such that*

$$Z_{f_{\mathcal{U}}}\circ Z^{-1}_{\iota_{\mathcal{U}}}\circ Z_{f_{\mathcal{V}}}\cong Z_{\iota_{\mathcal{V}}}\text{ and }Z_{f_{\mathcal{V}}}\circ Z^{-1}_{\iota_{\mathcal{V}}}\circ Z_{f_{\mathcal{U}}}\cong Z_{\iota_{\mathcal{U}}},\tag{2.5}$$

where $\iota_{\mathcal{U}}:(\Gamma[\mathcal{U}],\varrho)\longrightarrow(\Gamma,\varrho)$ and $\iota_{\mathcal{V}}:(\mathcal{G}[\mathcal{U}],\rho)\longrightarrow(\mathcal{G},\rho)$ are the canonical morphisms.

Proposition 2.4.13. *Define $\mathsf{F}:\mathfrak{RG}\longrightarrow\mathfrak{RG}_\Omega$ by*

$$\mathsf{F}(Z,\tau):=(\mathcal{U},f_{\mathcal{U}}),\tag{2.6}$$

where, if $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ is a class of Real generalized morphisms, $(\mathcal{U},f_{\mathcal{U}})$ is the class of pairs corresponding to (Z,τ).

Then F is a functor; furthermore, F is an isomorphism of categories.

Proof. Suppose that $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G}',\rho')$, $(Z',\tau'):(\mathcal{G}',\rho')\longrightarrow(\mathcal{G},\rho)$ are morphisms in $\mathfrak{RG}$. Let $\mathsf{F}(Z'\circ Z,\tau\times\tau')=(\mathcal{U},f_{\mathcal{U}})\in\Omega\big((\Gamma,\varrho),(\mathcal{G},\rho)\big)$, $\mathsf{F}(Z,\tau)=(\mathcal{U}',f_{\mathcal{U}'})\in\Omega\big((\Gamma,\varrho),(\mathcal{G}',\rho')\big)$, and $\mathsf{F}(Z',\tau')=(\mathcal{V},f_{\mathcal{V}})\in\Omega\big((\mathcal{G}',\rho'),(\mathcal{G},\rho)\big)$. Consider a Real open cover $\tilde{\mathcal{U}}$ of $\Gamma^{(0)}$ containing $\mathcal{U}'$ and a Real morphism $\varphi_{\tilde{\mathcal{U}}}:(\Gamma[\tilde{\mathcal{U}}],\varrho)\longrightarrow(\mathcal{G}'[\mathcal{V}],\rho')$ such that $Z_{\varphi_{\tilde{\mathcal{U}}}}\circ Z_i^{-1}\cong Z^{-1}_{\iota_{\mathcal{V}}}\circ Z_{f_{\mathcal{U}'}}$ as Real generalized morphisms from $(\Gamma[\mathcal{U}'],\varrho)$ to $(\mathcal{G}'[\mathcal{V}],\rho')$, where $i:(\Gamma[\tilde{\mathcal{U}}],\varrho)\longrightarrow(\Gamma[\mathcal{U}'],\varrho)$ and $\iota_{\mathcal{V}}:(\mathcal{G}'[\mathcal{V}],\rho')\longrightarrow(\mathcal{G}',\rho')$ are the canonical morphisms. Note that if $\iota_{\tilde{\mathcal{U}}}:(\Gamma[\tilde{\mathcal{U}}],\varrho)\longrightarrow(\Gamma,\varrho)$ is the canonical morphism, then $\iota_{\tilde{\mathcal{U}}}=\iota_{\mathcal{U}'}\circ i$; hence, $Z^{-1}_{\iota_{\tilde{\mathcal{U}}}}\cong Z_i^{-1}\circ Z^{-1}_{\iota_{\mathcal{U}'}}$ by functoriality.

On the other hand, $\mathsf{F}(Z',\tau')\circ\mathsf{F}(Z,\tau)=(\mathcal{V},f_{\mathcal{V}})\circ(\mathcal{U},f_{\mathcal{U}})=(\tilde{\mathcal{U}},f_{\tilde{\mathcal{U}}})$, where $f_{\tilde{\mathcal{U}}}=f_{\mathcal{V}}\circ\varphi_{\tilde{\mathcal{U}}}$. Henceforth,

$$Z_{f_{\tilde{\mathcal{U}}}}\circ Z^{-1}_{\iota_{\tilde{\mathcal{U}}}}\cong Z_{f_{\mathcal{V}}}\circ Z_{\varphi_{\tilde{\mathcal{U}}}}\circ Z_i^{-1}\circ Z^{-1}_{\iota_{\mathcal{U}'}}\cong Z_{f_{\mathcal{V}}}\circ Z^{-1}_{\iota_{\mathcal{V}}}\circ Z_{f_{\mathcal{U}'}}\circ Z^{-1}_{\iota_{\mathcal{U}'}}\cong Z'\circ Z,$$

which shows that $\mathsf{F}(Z'\circ Z,\tau\times\tau')\cong\mathsf{F}(Z',\tau')\circ\mathsf{F}(Z,\tau)$, and thus F is a functor. Now, it is not hard to see that we get an inverse functor for F by defining

$$\mathsf{Z}:\mathfrak{RG}_\Omega\longrightarrow\mathfrak{RG},(\mathcal{U},f_{\mathcal{U}})\longmapsto(Z_{f_{\mathcal{U}}}\circ Z^{-1}_{\iota_{\mathcal{U}}},\tau),\tag{2.7}$$

where τ is defined in an obvious way.

□

2.5 *Real* Graded twists

This section is devoted to the study of *Real graded twists*. We should point out that these elements have been already treated in many references in the usual case and are usually called *central extensions* of groupoids ([49], [30], and [90]).

Definition 2.5.1. *Let* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *be a Real groupoid and let* $\mathbf{S}$ *be a Real Abelian group. A* Real graded $\mathbf{S}$-groupoid $(\widetilde{\Gamma}, \delta)$ *over* Γ *is the data of*

1. *a Real groupoid* $\widetilde{\Gamma}$ *whose unit space is* Y*, together with a Real strict homomorphism* $\pi : \widetilde{\Gamma} \longrightarrow \Gamma$ *which restricts to the identity in* Y*,*

2. *a (left) Real action of* $\mathbf{S}$ *on* $\widetilde{\Gamma}$ *which is compatible with the partial product in* $\widetilde{\Gamma}$ *making* $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma$ *a (left) Real* $\mathbf{S}$*-principal bundle, and*

3. *a strict homomorphism* $\delta : \Gamma \longrightarrow \mathbb{Z}_2$*, called* the grading*, such that* $\delta(\bar{\gamma}) = \delta(\gamma)$ *for any* $\gamma \in \Gamma$.

In this case we refer to the triple $(\widetilde{\Gamma}, \Gamma, \delta)$ *as a* Real graded $\mathbf{S}$-twist*, and it is sometimes symbolized by the "extension"*

$$\begin{tikzcd} \mathbf{S} \arrow[r] & \widetilde{\Gamma} \arrow[r, "\pi"] & \Gamma \arrow[d, "\delta"] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

Example 2.5.2 (The trivial twist)**.** *Given any Real groupoid* Γ*, we form the product groupoid* $\Gamma \times \mathbf{S}$ *and we endow it with the Real structure* $\overline{(\gamma, \lambda)} := (\bar{\gamma}, \bar{\lambda})$ *for. Let* $\mathbf{S}$ *act on* $\Gamma \times \mathbf{S}$ *by multiplication with the second factor. Then* $\mathcal{T}_0 := (\Gamma \times \mathbf{S}, 0)$ *is a Real graded twist of* Γ*, where* $0 : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ *is the zero map. This element is called* the trivial Real graded $\mathbf{S}$-groupoid over Γ.

Example 2.5.3. *Let* Y *be a locally compact Real space and* $\{U_i\}_{i \in I \times \{\pm 1\}}$ *be a good Real open. Let us consider the Real groupoid* $Y[\mathfrak{U}] \rightrightarrows \coprod_i U_i$ *, and the space* $Y \times \mathbf{S}$ *together with the Real structure* $(y, \lambda) \longmapsto (\bar{y}, \bar{\lambda})$ *and the Real* $\mathbf{S}$*-action given by the multiplication on the second factor. There is a cononical Real morphism* $\delta : Y[U_i] \longrightarrow \mathbb{Z}_2$ *given by* $\delta(x_{i_0 i_1}) := \varepsilon_0 + \varepsilon_1$ *for* $i_0 = (i'_0, \varepsilon_0)$, $i_1 = (i'_1, \varepsilon_1) \in I$*. Then, a Real graded* $\mathbf{S}$*-twist* $(\widetilde{\Gamma}, Y[U_i], \delta)$ *consists of a family of principal Real* $\mathbf{S}$*-bundles* $\widetilde{\Gamma}_{ij} \cong U_{ij} \times \mathbf{S}$ *subject to the multiplication*

$$(x_{i_0 i_1}, \lambda_1) \cdot (x_{i_1 i_2}, \lambda_2) = (x_{i_0 i_2}, \lambda_1 \lambda_2 c_{i_0 i_1 i_2}(x)),$$

where $c = \{c_{i_0 i_1 i_2}\}$ *is a family of continuous maps* $c_{i_0 i_1 i_2} : U_{i_0 i_1 i_2} \longrightarrow \mathbf{S}$ *satisfying the cocycle condition (see the next chapter), and such that* $c_{\bar{i}_0 \bar{i}_1 \bar{i}_2}(\bar{x}) = \overline{c_{i_0 i_1 i_2}(x)}$ *for all* $x \in U_{i_0 i_1 i_2} = U_{i_0} \cap U_{i_1} \cap U_{i_2}$*. The pair* (δ, c) *will be called* the Dixmier-Douady class of $(\widetilde{\Gamma}, Y[U_i], \delta)$.

Example 2.5.4. *Let* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *be a Real groupoid, and let* $J : \Lambda \longrightarrow Y$ *be a Real* $\mathbf{S}$*-principal bundle. Then the tensor product* $r^*\Lambda \otimes \overline{s^*\Lambda}$*, which is a Real* $\mathbf{S}$*-principal bundle over* Γ*, naturally admits the structure of Real groupoid over* Y*, so that* $(r^*\Lambda \otimes \overline{s^*\Lambda}, 0)$ *is a Real graded* $\mathbf{S}$*-groupoid over* Γ.

There is an obvious notion of strict morphism of Real graded $\mathbf{S}$-twists. For instance, two Real graded $\mathbf{S}$-twists $(\widetilde{\Gamma}_1, \Gamma, \delta_1)$ and $(\widetilde{\Gamma}_2, \Gamma, \delta_2)$ are isomorphic if there exists a Real $\mathbf{S}$-equivariant isomorphism of groupoids $f : \widetilde{\Gamma}_1 \longrightarrow \widetilde{\Gamma}_2$ such that the diagram

$$\begin{tikzcd} \widetilde{\Gamma}_1 \arrow[r, "\pi_1"] \arrow[d, "f"'] & \Gamma \\ \widetilde{\Gamma}_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

commutes in the category $\mathfrak{RG}_s$. In particular, we say that $(\widetilde{\Gamma}, \delta)$ is *strictly trivial* if it isomorphic to the trivial Real graded groupoid $(\Gamma \times \mathbf{S}, 0)$. By $\widehat{\mathrm{TwR}}(\Gamma, \mathbf{S})$ we denote the set of strict isomorphism classes of Real graded $\mathbf{S}$-groupoids over Γ. The class of $(\widetilde{\Gamma}, \delta)$ in $\widehat{\mathrm{TwR}}(\Gamma, \mathbf{S})$ is denoted by $[\widetilde{\Gamma}, \delta]$.

Definition 2.5.5. *Given two Real graded* $\mathbf{S}$*-groupoids* $\mathcal{T}_1 = (\widetilde{\Gamma}_1, \delta_1)$ *and* $\mathcal{T}_2 = (\widetilde{\Gamma}_2, \delta_2)$ *over* $\mathcal{G}$*, we define their tensor product* $\mathcal{T}_1 \hat{\otimes} \mathcal{T}_2 = (\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2, \delta_1 + \delta_2)$ *by the* Baer sum *of* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *defined as follows. Define the groupoid* $\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2$ *as the quotient*

$$\widetilde{\Gamma}_1 \times_\Gamma \widetilde{\Gamma}_2 / \mathbf{S} := \{(\tilde{\gamma_1}, \tilde{\gamma_2}) \in \widetilde{\Gamma}_1 \times_{\pi_1, \Gamma, \pi_2} \widetilde{\Gamma}_2\} /_{(\tilde{\gamma_1}, \tilde{\gamma_2}) \sim (\lambda \tilde{\gamma_1}, \lambda^{-1} \tilde{\gamma_2})}, \tag{2.8}$$

where $\lambda \in \mathbf{S}$*, together with the obvious Real structure. The projection* $\pi_1 \otimes \pi_2$ *is just* π_i *and* $\delta = \delta_1 + \delta_2$ *is given by* $\delta(\gamma) = \delta_1(\gamma) + \delta_2(\gamma)$.
The product in the Real groupoid $\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2$ *is*

$$(\tilde{\gamma_1}, \tilde{\gamma_2})(\tilde{\gamma'_1}, \tilde{\gamma'_2}) := (-1)^{\delta_2(\gamma_2)\delta_1(\gamma'_1)} (\tilde{\gamma_1}\tilde{\gamma'_1}, \tilde{\gamma_2}\tilde{\gamma'_2}), \tag{2.9}$$

whenever this does make sense and where $\gamma_i = \pi_2(\tilde{\gamma_i})$, $i = 1, 2$.

Lemma 2.5.6. *Given* $[\widetilde{\Gamma}_i, \delta_i] \in \widehat{TwR}(\Gamma, \mathbf{S})$, $i = 1, 2$, *set*

$$[\widetilde{\Gamma}_1, \delta_1] + [\widetilde{\Gamma}_2, \delta_2] := [\widetilde{\Gamma}_1 \hat{\otimes} \widetilde{\Gamma}_2, \delta_1 + \delta_2].$$

Then, under this sum, $\widehat{TwR}(\Gamma, \mathbf{S})$ *is an Abelian group whose zero element is given by the class of the trivial element* $\mathcal{T}_0 = (\mathcal{G} \times \mathbf{S}, 0)$.

Proof. The tensor product defined above is commutative in $\widehat{\mathrm{TwR}}(\Gamma, \mathbf{S})$. Indeed, the groupoid $\widetilde{\Gamma}_2 \hat{\otimes} \widetilde{\Gamma}_1 = \widetilde{\Gamma}_2 \times_\Gamma \widetilde{\Gamma}_1 / \mathbf{S}$ is endowed with the multiplication

$$(\tilde{\gamma_2}, \tilde{\gamma_1})(\tilde{\gamma'_2}, \tilde{\gamma'_1}) = (-1)^{\delta_1(\gamma_1)\delta_2(\gamma'_2)} (\tilde{\gamma_2}\tilde{\gamma'_2}, \tilde{\gamma_1}\tilde{\gamma'_1}).$$

Then the map

$$\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2 \longrightarrow \widetilde{\Gamma}_2\hat{\otimes}\widetilde{\Gamma}_1\,, (\tilde{\gamma}_1,\tilde{\gamma}_2)\longmapsto (-1)^{\delta_1(\gamma_1)\delta_2(\gamma_2)}(\tilde{\gamma}_2,\tilde{\gamma}_1)$$

is a Real $\mathbf{S}$-equivariant isomorphism of groupoids.
Now define the inverse of $(\widetilde{\Gamma},\delta)$ is $(\widetilde{\Gamma}^{op},\delta)$ where $\widetilde{\Gamma}^{op}$ is $\widetilde{\Gamma}$ as a set but, together with the same Real structure, but the $\mathbf{S}$-principal bundle structure is replaced by the conjugate one, i.e. $\lambda\tilde{\gamma}^{op}=(\bar{\lambda}\tilde{\gamma})^{op}$, and the product $*_{op}$ in $\widetilde{\Gamma}^{op}$ is

$$\tilde{\gamma}*_{op}\tilde{\gamma}' := (-1)^{\delta(\gamma)\delta(\gamma')}\tilde{\gamma}\tilde{\gamma}'.$$

Now it is easy to see that the map

$$\Gamma\times\mathbf{S}\longrightarrow \widetilde{\Gamma}\times_{\Gamma}\widetilde{\Gamma}^{op}/\mathbf{S}\,, (\gamma,\lambda)\longmapsto (\lambda\tilde{\gamma},\tilde{\gamma})\,,$$

where $\tilde{\gamma}\in\widetilde{\Gamma}$ is any lift of $\gamma\in\Gamma$, is an isomorphism. □

We have the following criteria of strict triviality; the proof is the same as in [90, Proposition 2.8].

Proposition 2.5.7. *Let $(\widetilde{\Gamma},\delta)$ be a Real graded $\mathbf{S}$-groupoid over the Real groupoid $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$. The following are equivalent:*

(i) $(\widetilde{\Gamma},\delta)$ is strictly trivial.

(ii) There exists a Real strict homomorphism $\sigma:\Gamma\longrightarrow\widetilde{\Gamma}$ such that $\pi\circ\sigma=\mathrm{Id}$.

(iii) There exists a Real $\mathbf{S}$-equivariant groupoid homomorphism $\varphi:\widetilde{\Gamma}\longrightarrow\mathbf{S}$.

Example 2.5.8. *Let $J:\Lambda\longrightarrow Y$ be a Real $\mathbf{S}$-principal bundle with a Real (left) Γ-action that is compatible with the $\mathbf{S}$-action; in other words $Y \xleftarrow[J]{} \Lambda \longrightarrow \star$ is a Real generalized homomorphism from Γ to $\mathbf{S}$. Then, the Real Γ-action induces an $\mathbf{S}$-equivariant isomorphism $\Lambda_{s(\gamma)}\ni v\longmapsto\gamma\cdot v\in\Lambda_{r(\gamma)}$ for every $\gamma\in\Gamma$. Hence, there is a Real $\mathbf{S}$-equivariant groupoid isomorphism $\varphi:r^*\Lambda\otimes\overline{s^*\Lambda}\longrightarrow\Gamma\times\mathbf{S}$ defined as follows. If $(v,\flat(w))\in\Lambda_{r(\gamma)}\otimes\overline{\Lambda}_{s(\gamma)}$, there exists a unique $\lambda\in\mathbf{S}$ such that $\gamma\cdot w=v\cdot\lambda$. We then set*

$$\varphi([v,\flat(w)]):=(\gamma,\lambda).$$

The inverse of φ is $\varphi'(\gamma,\lambda):=[v_\gamma,\overline{\gamma^{-1}\cdot v_\gamma}]$, where for $\gamma\in\Gamma$, v_γ is any lift of $r(\gamma)$ through the projection J.

Observe that the set of Real graded $\mathbf{S}$-groupoids of the from $(r^*\Lambda\otimes\overline{s^*\Lambda},0)$ over Γ (cf. Example 2.5.4) is a subgroup of $\widehat{\mathrm{TwR}}(\Gamma,\mathbf{S})$. By $\widehat{\mathrm{extR}}(\Gamma,\mathbf{S})$ we denote the quotient of $\widehat{\mathrm{TwR}}(\Gamma,\mathbf{S})$ by this subgroup.

Let us show that $\widehat{\mathrm{extR}}(\cdot,\mathbf{S})$ is functorial in the category $\mathfrak{RG}_s$. Let Γ, Γ' be two Real groupoids, and let $f:\Gamma'\longrightarrow\Gamma$ be a morphism in $\mathfrak{RG}_s$. Suppose that $\mathcal{T}=(\widetilde{\Gamma},\delta)$ is a Real graded $\mathbf{S}$-groupoid over Γ. Then, the pull-back $f^*\widetilde{\Gamma} := \widetilde{\Gamma} \times_{\pi,\Gamma,f} \Gamma'$ of the Real $\mathbf{S}$-principal bundle $\pi : \widetilde{\Gamma} \longrightarrow \Gamma$, on which the Real groupoid structure is the one induced from the product Real groupoid $\widetilde{\Gamma} \times \Gamma'$, defines a Real graded twist

$$f^*\mathcal{T} := \begin{tikzcd} \mathbf{S} \arrow[r] & f^*\widetilde{\Gamma} \arrow[r, "f^*\pi"] & \Gamma' \arrow[d, "f^*\delta"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{2.10}$$

where $f^*\pi(\tilde{\gamma},\gamma') := \gamma'$, $f^*\delta(\gamma') := \delta(f(\gamma')) \in \mathbb{Z}_2$, and the Real left $\mathbf{S}$-action on $f^*\widetilde{\Gamma}$ being given by $\lambda \cdot (\tilde{\gamma},\gamma') = (\lambda\tilde{\gamma},\gamma')$. Suppose now that $\mathcal{T}_i = (\widetilde{\Gamma}_i,\delta_i)$, $i = 1,2$ are representatives in $\widehat{\text{extR}}(\Gamma,\mathbf{S})$. Then, $f^*(\mathcal{T}_1\hat{\otimes}\mathcal{T}_2) = f^*\mathcal{T}_1\hat{\otimes} f^*\mathcal{T}_2$; indeed,

$$f^*(\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2) = \left(\widetilde{\Gamma}_1 \times_\Gamma \widetilde{\Gamma}_2/\mathbf{S}\right) \times_\Gamma \Gamma' \cong \left((\Gamma_1 \times_\Gamma \Gamma') \times_\Gamma (\widetilde{\Gamma}_2 \times_\Gamma \Gamma')\right)/\mathbf{S} = f^*\widetilde{\Gamma}_1\hat{\otimes} f^*\widetilde{\Gamma}_2.$$

Moreover, it is easily seen that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent in $\widehat{\text{extR}}(\Gamma,\mathbf{S})$, then so are $f^*\mathcal{T}_1$ and $f^*\mathcal{T}_2$. Thus, f induces a morphism of Abelian groups $f^* : \widehat{\text{extR}}(\Gamma,\mathbf{S}) \longrightarrow \widehat{\text{extR}}(\Gamma',\mathbf{S})$. We then have proved this

Lemma 2.5.9. *The correspondence*

$$\widehat{ext}\mathrm{R}(\cdot,\mathbf{S}) : \mathfrak{RG}_s \longrightarrow \mathfrak{Ab}, \Gamma \longmapsto \widehat{ext}\mathrm{R}(\Gamma,\mathbf{S}), f \longmapsto f^*, \tag{2.11}$$

where $\mathfrak{Ab}$ is the category of Abelian groups, is a contravariant functor. In particular, $\widehat{ext}\mathrm{R}(\mathcal{G},\mathbf{S})$ is invariant under Real strict isomorphisms.

2.6 *Real* Graded central extensions

In this section we introduce Real graded central extensions of Real groupoids, by adapting the exposition presented in [87] to our context.

Definition 2.6.1. *Let $(\widetilde{\Gamma}_i,\Gamma_i,\delta_i), i = 1,2$, be Real graded $\mathbf{S}$-twists. Then a Real generalized homomorphism $Z : \widetilde{\Gamma}_1 \longrightarrow \widetilde{\Gamma}_2$ is said to be $\mathbf{S}$-equivariant if there is a Real action of $\mathbf{S}$ on Z such that*

$$(\lambda\tilde{\gamma}_1)\cdot z\cdot\tilde{\gamma}_2 = \tilde{\gamma}_1\cdot(\lambda z)\cdot\tilde{\gamma}_2 = \tilde{\gamma}_1\cdot z\cdot(\lambda\tilde{\gamma}_2),$$

for any $(\lambda,\tilde{\gamma}_1,z,\tilde{\gamma}_2) \in \mathbf{S}\times\widetilde{\Gamma}_1\times Z\times\widetilde{\Gamma}_2$ such that these products make sense. We refer to $Z : (\widetilde{\Gamma}_1,\Gamma_1,\delta_1) \longrightarrow (\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$ as a generalized morphism of Real graded $\mathbf{S}$-twists. In particular, if Z is an isomorphism, the two Real graded $\mathbf{S}$-twists are said to be Morita equivalent*; in this case we write $(\widetilde{\Gamma}_1,\Gamma_1,\delta_1) \sim (\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$.*

Lemma 2.6.2. *Let* $Z : (\widetilde{\Gamma}_1, \Gamma_1, \delta_1) \longrightarrow (\widetilde{\Gamma}_2, \Gamma_2, \delta_2)$ *be a generalized morphism. Then the* $\mathbf{S}$*-action on* Z *is free and the Real space* $Z/\mathbf{S}$ *(with the obvious involution) is a Real generalized homomorphism from* Γ_1 *to* Γ_2*.*

Proof. Same as [90, Lemma 2.10]. □

Definition 2.6.3. *Let* $\mathcal{G}$ *be a Real groupoid and* $\mathbf{S}$ *an abelian Real group. A* Real graded $\mathbf{S}$-central extension *of* $\mathcal{G}$ *consists of a triple* $(\widetilde{\Gamma}, \Gamma, \delta, P)$*, where* $(\widetilde{\Gamma}, \Gamma, \delta)$ *is a Real graded* $\mathbf{S}$*-twist, and* P *is a (Real) Morita equivalence* $\Gamma \longrightarrow \mathcal{G}$*.*

Definition 2.6.4. *We say that* $(\widetilde{\Gamma}_1, \Gamma_1, \delta_1, P_1)$ *and* $(\widetilde{\Gamma}_2, \Gamma_2, \delta_2, P_2)$ *are Morita equivalent if there exists a Morita equivalence* $Z : (\widetilde{\Gamma}_1, \Gamma_1, \delta_1) \longrightarrow (\widetilde{\Gamma}_2, \Gamma_2, \delta_2)$ *such that the diagrams*

$$\begin{tikzcd} \Gamma_1 \arrow[r, "Z/\mathbf{S}"] \arrow[dr, "P_1"'] & \Gamma_2 \arrow[d, "P_2"] \\ & \mathcal{G} \end{tikzcd} \tag{2.12}$$

and

$$\begin{tikzcd} \Gamma_1 \arrow[r, "Z/\mathbf{S}"] \arrow[dr, "\delta_1"'] & \Gamma_2 \arrow[d, "\delta_2"] \\ & \mathbb{Z}_2 \end{tikzcd} \tag{2.13}$$

commute in the category $\mathfrak{RG}$*. Such a* Z *is also called* an equivalence bimodule *of Real graded* $\mathbf{S}$*-central extensions. The set of Morita equivalence classes of Real graded* $\mathbf{S}$*-central extensions of* $\mathcal{G}$ *is denoted by* $\widehat{Ext\mathrm{R}}(\mathcal{G}, \mathbf{S})$*.*

The set $\widehat{\mathrm{ExtR}}(\mathcal{G}, \mathbf{S})$ admits a natural structure of abelian group described in the following way. Assume that $\mathbb{E}_i = (\widetilde{\Gamma}_i, \Gamma_i, \delta_i, P_i)$, $i = 1, 2$, are two given Real graded $\mathbf{S}$-central extensions of $\mathcal{G}$, then $Y_1 \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} Y_2$ is a Morita equivalence between Γ_1 and Γ_2, where $Z = P_1 \times_{\mathcal{G}} P_2$. But from Proposition 2.4.8 there exists a Real homeomorphism $f : \mathfrak{s}^*\Gamma_2 \longrightarrow \mathfrak{r}^*\Gamma_1$. Now one can see that the maps $\pi : \mathfrak{r}^*\widetilde{\Gamma}_1 \longrightarrow \mathfrak{r}^*\Gamma_1$, $(z, \tilde{\gamma}_1, z') \longmapsto (z, \pi_1(\tilde{\gamma}_1), z')$ and $\pi' : \mathfrak{s}^*\widetilde{\Gamma}_2 \longrightarrow \mathfrak{r}^*\Gamma_1(z, \tilde{\gamma}_2, z') \longmapsto \pi \circ f(z, \tilde{\gamma}_2, z')$ define two Real $\mathbf{S}$-principal bundles and then $(\mathfrak{r}^*\widetilde{\Gamma}_1, \delta)$ and $(\mathfrak{s}^*\widetilde{\Gamma}_2, \delta)$, where $\delta := \delta_1 \circ pr_2$, define elements of $\widehat{\mathrm{extR}}(\mathfrak{r}^*\Gamma_1, \mathbf{S})$. Therefore, we can form the tensor product $(\mathfrak{r}^*\widetilde{\Gamma}_1 \hat{\otimes} \mathfrak{s}^*\widetilde{\Gamma}_2, \delta \otimes \delta)$ are Real graded $\mathbf{S}$-groupoid over $\mathfrak{r}^*\Gamma_1$. Moreover, $\mathfrak{r}^*\Gamma_1 \sim_{Morita} \Gamma_1$; then, if $P : \mathfrak{r}^*\Gamma_1 \longrightarrow \mathcal{G}$ is a Real Morita equivalence, we obtain a Real graded $\mathbf{S}$-central extension of $\mathcal{G}$ by setting

$$\mathbb{E}_1 \hat{\otimes} \mathbb{E}_2 := (\mathfrak{r}^*\widetilde{\Gamma}_1 \hat{\otimes} \mathfrak{s}^*\widetilde{\Gamma}_2, \mathfrak{r}^*\Gamma_1, \delta, P), \tag{2.14}$$

that we will call *the tensor product of* $\mathbb{E}_1$ *and* $\mathbb{E}_2$. Thus, we define the sum

$$[\mathbb{E}_1] + [\mathbb{E}_2] := [\mathbb{E}_1 \hat{\otimes} \mathbb{E}_2],$$

which is easily seen to be well defined in $\widehat{\mathrm{Ext}}\mathrm{R}(\mathcal{G},\mathbf{S})$. The inverse $\mathbb{E}^{op}$ of $\mathbb{E}$ is $(\widetilde{\Gamma}^{op},\Gamma,\delta,P)$. Notice that $\widehat{\mathrm{ext}}\mathrm{R}(\mathcal{G},\mathbf{S})$ is naturally a subgroup of $\widehat{\mathrm{Ext}}\mathrm{R}(\mathcal{G},\mathbf{S})$ by identifying a Real graded **S**-twist $(\widetilde{\Gamma},\mathcal{G},\delta)$ with the Real graded **S**-central extension $(\widetilde{\Gamma},\mathcal{G},\delta,\mathcal{G})$. We summarize this in the next lemma.

Lemma 2.6.5. *Under the sum defined above, $\widehat{Ext}\mathrm{R}(\mathcal{G},\mathbf{S})$ is an abelian group whose zero element is the class of the trivial Real graded* **S***-central extension* $(\mathcal{G}\times\mathbf{S},\mathcal{G},0,\mathcal{G})$.

When the Real structure is trivial, then we recover the usual definition of graded central extensions (see [30] for instance) of $\mathcal{G}$ by the group $\mathbb{Z}_2$. More precisely, we get a generalization of what Mathai, Murray, and Stevenson have called *real bundle gerbes* in [58].

Proposition 2.6.6. *Suppose that* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *is equipped with a trivial Real structure. Then*

$$\widehat{Ext}\mathrm{R}(\mathcal{G},\mathbf{S}^{1,1})\cong\widehat{Ext}(\mathcal{G},\mathbb{Z}_2).$$

2.7 Functoriality of $\widehat{\mathbf{Ext}}\mathrm{R}(\cdot,\mathbf{S})$

The aim of this section is to show that $\widehat{\mathrm{Ext}}\mathrm{R}(\cdot,\mathbf{S})$ is functorial in the category $\mathfrak{RG}$, and hence that the group $\widehat{\mathrm{Ext}}\mathrm{R}(\mathcal{G},\mathbf{S})$ invariant under Morita equivalence. To do this, we will need the following

Proposition 2.7.1. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a Real groupoid. Then, there is an isomorphism of abelian groups*

$$\widehat{Ext}\mathrm{R}(\mathcal{G},\mathbf{S})\cong\varinjlim_{\mathcal{U}}\widehat{ext}\mathrm{R}(\mathcal{G}[\mathcal{U}],\mathbf{S}). \tag{2.15}$$

Before giving the proof of this proposition, we have to describe the sum in the inductive limit $\varinjlim_{\mathcal{U}}\widehat{\mathrm{ext}}\mathrm{R}(\mathcal{G}[\mathcal{U}],\mathbf{S})$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two Real open covers of X, and let $\mathcal{T}_i=(\tilde{\mathcal{G}}_i,\mathcal{G}[\mathcal{U}_i],\delta_i)$ be Real graded **S**-groupoids over $\mathcal{G}[\mathcal{U}_i]$, $i=1,2$. Let $(\mathcal{V},f_{\mathcal{V}})\in\Omega(\mathcal{G}[\mathcal{U}_1],\mathcal{G}[\mathcal{U}_2])$ be the unique class corresponding to the Real Morita equivalence $Z^{-1}_{\iota_{\mathcal{U}_1}}\circ Z_{\iota_{\mathcal{U}_2}}$ from $\mathcal{G}[\mathcal{U}_1]$ to $\mathcal{G}[\mathcal{U}_2]$. $\mathcal{V}$ is a Real open cover of X containing $\mathcal{U}_1$, and $f_{\mathcal{V}}:\mathcal{G}[\mathcal{V}]\longrightarrow\mathcal{G}[\mathcal{U}_2]$ is a Real strict morphism. Denote by $\iota_{\mathcal{V},\mathcal{U}_1}$ the canonical Real morphism $\mathcal{G}[\mathcal{V}]\longrightarrow\mathcal{G}[\mathcal{U}_1]$. Then, the tensor product of $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_1\hat{\otimes}\mathcal{T}_2:=\iota^*_{\mathcal{V},\mathcal{U}_1}\mathcal{T}_1\hat{\otimes}f^*_{\mathcal{V}}\mathcal{T}_2, \tag{2.16}$$

which defines a Real graded **S**-groupoids over the Real groupoid $\mathcal{G}[\mathcal{V}]$.

Proof of Proposition 2.7.1. For a Real graded $\mathbf{S}$-central extension $\mathbb{E} = (\widetilde{\Gamma}, \Gamma, \delta, P)$ of $\mathcal{G}$, let $(\mathcal{V}, f_{\mathcal{V}}) \in \Omega(\mathcal{G}, \Gamma)$ be the isomorphism in $\mathfrak{RG}_{\Omega}$ corresponding to the Morita equivalence $P^{-1} : \mathcal{G} \longrightarrow \Gamma$. Setting

$$\begin{tikzcd} \mathcal{T}_{\mathbb{E}} := \mathbf{S} \arrow[r] & f_{\mathcal{V}}^{*}\widetilde{\Gamma} \arrow[r, "f_{\mathcal{V}}^{*}\pi"] & \mathcal{G}[\mathcal{V}] \arrow[d, "\delta \circ f_{\mathcal{V}}"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{2.17}$$

we get a Real graded $\mathbf{S}$-groupoid over $\mathcal{G}[\mathcal{V}]$. It is not hard to check that this provides us the desired isomorphism of abelian groups; the inverse is given by the formula

$$\mathbb{E}_{\mathcal{T}} := (\tilde{\mathcal{G}}, \mathcal{G}[\mathcal{U}], \delta, Z_{\iota_{\mathcal{U}}}), \tag{2.18}$$

for a Real graded $\mathbf{S}$-twist $\mathcal{T} = (\tilde{\mathcal{G}}, \mathcal{G}[\mathcal{U}], \delta)$. □

From this proposition, it is now possible to define the *pull-back* of a Real graded $\mathbf{S}$-central extension via a Real generalized morphism. More precisely, we have

Definition and Proposition 2.7.2. *Let $\mathcal{G}$ and $\mathcal{G}'$ be Real groupoids, and let $Z : \mathcal{G}' \longrightarrow \mathcal{G}$ be a Real generalized morphism. Suppose that $\mathbb{E} = (\widetilde{\Gamma}, \Gamma, \delta, P)$ is a representative in $\widehat{Ext}R(\mathcal{G}, \mathbf{S})$, and that $\mathcal{T}_{\mathbb{E}} = (f_{\mathcal{V}}^{*}\widetilde{\Gamma}, \mathcal{G}[\mathcal{V}], \mathcal{G}[\mathcal{V}], \delta \circ f_{\mathcal{V}})$ is its image in $\varinjlim_{\mathcal{U}} \widehat{ext}R(\mathcal{G}[\mathcal{U}], \mathbf{S})$ (see the proof of Proposition 2.7.1). Let $(\mathcal{W}, f_{\mathcal{W}}) \in \Omega\big(\mathcal{G}', \mathcal{G}[\mathcal{V}]\big)$ be the morphism in $\mathfrak{RG}_{\Omega}$ corresponding to the Real generalized morphism $Z_{\iota_{\mathcal{V}}}^{-1} \circ Z : \mathcal{G}' \longrightarrow \mathcal{G}[\mathcal{V}]$. Then*

$$Z^{*}\mathbb{E} := \mathbb{E}_{f_{\mathcal{W}}^{*}\mathcal{T}_{\mathbb{E}}}. \tag{2.19}$$

is a Real graded $\mathbf{S}$-central extension of the Real groupoid $\mathcal{G}'$; it is called the pull-back of $\mathbb{E}$ along Z

Now the following is straightforward.

Corollary 2.7.3. *There is a contravariant functor*

$$\widehat{Ext}R(\cdot, \mathbf{S}) : \mathfrak{RG} \longrightarrow \mathfrak{Ab}, \tag{2.20}$$

which sends a Real groupoid $\mathcal{G}$ to the abelian group $\widehat{Ext}R(\mathcal{G}, \mathbf{S})$. In particular, $\widehat{Ext}R(\mathcal{G}, \mathbf{S})$ is invariant under Morita equivalences.

2.8 Rg bundle gerbes

We now restrict ourselves to Real spaces. Mainly, we study the relationship between elements of $\widehat{\mathrm{ExtR}}(X, \mathbb{S}^1)$ with bundle gerbes ([67]).

Definition 2.8.1. *Let X be a locally compact Hausdorff Real space. A* Rg bundle gerbe *over X is the data of a locally compact Hausdorff Real space Y, a continuous locally split Real open map $\varphi : Y \longrightarrow X$ and a Rg $\mathbb{S}^1$-twist ($\mathbb{S}^1 \longrightarrow \widetilde{\Gamma} \xrightarrow{\pi} Y^{[2]}, \delta$) over the Real groupoid $Y^{[2]} \rightrightarrows Y$ defined as the fiber-product $Y^{[2]} := Y \times_{\varphi,X,\varphi} Y$ together with the Real structure $\overline{(y_1, y_2)} := (\bar{y}_1, \bar{y}_2)$, with the source and range maps $s(y_1, y_2) := y_2, r(y_1, y_2) := y_1$, and the inversion given by $(y_1, y_2)^{-1} := (y_2, y_1)$.*
We shall denote such a Rg bundle gerbe by $(\widetilde{\Gamma}, Y^{[2]}, \delta)$.

More concretely, given such a $(\widetilde{\Gamma}, Y^{[2]}, \delta)$, the bundle $\pi : \widetilde{\Gamma} \longrightarrow Y^{[2]}$ admits a *product*, that is, there is an ismorphism of $\mathbb{S}^1$-spaces

$$\widetilde{\Gamma}_{(y_1,y_2)} \otimes \widetilde{\Gamma}_{(y_2,y_3)} \longrightarrow \widetilde{\Gamma}_{(y_1,y_3)},$$

over all composable pairs $(y_1, y_2), (y_2, y_3) \in Y^{[2]}$. These isomorphisms are compatible with the Real structures in the sense that we have commutative diagram

$$\begin{tikzcd}
\widetilde{\Gamma}_{(y_1,y_2)} \otimes \widetilde{\Gamma}_{(y_2,y_3)} \arrow[r] \arrow[d, "bar \times bar"'] & \widetilde{\Gamma}_{(y_1,y_3)} \arrow[d, "bar"] \\
\widetilde{\Gamma}_{(\bar{y}_1,\bar{y}_2)} \otimes \widetilde{\Gamma}_{(\bar{y}_2,\bar{y}_3)} \arrow[r] & \widetilde{\Gamma}_{(\bar{y}_1,\bar{y}_3)}
\end{tikzcd}$$

Moreover, since the projection π is a strict morphism of Real groupoid, the product is associative whenever triple products are defined; i.e. there is a commutative diagram

$$\begin{tikzcd}
\widetilde{\Gamma}_{(y_1,y_2)} \otimes \widetilde{\Gamma}_{(y_2,y_3)} \otimes \widetilde{\Gamma}_{(y_3,y_4)} \arrow[r] \arrow[d] & \widetilde{\Gamma}_{(y_1,y_3)} \otimes \widetilde{\Gamma}_{(y_3,y_4)} \arrow[d] \\
\widetilde{\Gamma}_{(y_1,y_2)} \otimes \widetilde{\Gamma}_{(y_2,y_4)} \arrow[r] & \widetilde{\Gamma}_{(y_1,y_4)}
\end{tikzcd}$$

over all composable pairs $(y_1, y_2), (y_2, y_3), (y_3, y_4) \in Y^{[2]}$.

Definition 2.8.2. *A morphism of graded Rg bundle gerbes from $(\widetilde{\Gamma}, Y^{[2]}, \delta)$ to $(\widetilde{\Gamma}', Y'^{[2]}, \delta')$ consists of a pair (f, h) where $f : \widetilde{\Gamma} \longrightarrow (\widetilde{\Gamma}'$ is a Real morphism, and $h : Y \longrightarrow Y'$ is a continuous Real map, such that we have commutative diagrams*

$$\begin{tikzcd}
\widetilde{\Gamma} \arrow[r, "\pi"] \arrow[dd, "f"'] & Y^{[2]} \arrow[dr, "\delta"] \arrow[dd, "h^{[2]}"] & \\
 & & \mathbb{Z}_2 \\
\widetilde{\Gamma}' \arrow[r, "\pi'"'] & Y'^{[2]} \arrow[ur, "\delta'"'] &
\end{tikzcd}
\text{ and }
\begin{tikzcd}
Y \arrow[dr, "\varphi"] \arrow[dd, "h"'] & \\
 & X \\
Y' \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

We are now going to compare graded Real bundle gerbes over X with elements of $\widehat{\mathrm{ExtR}}(X, \mathbb{S}^1)$. For this end, we need the following result.

Proposition 2.8.3. *Let* $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ *be a locally compact Hausdorff Real groupoid and let* X *be a locally compact Hausdorff Real space. Then the following are equivalent:*

(a) Γ *is Morita equivalent to* X*.*

(b) There exists a locally compact Hausdorff Real space (Z,τ)*, and a continuous locally split Real open map* $\varphi : Z, \longrightarrow X$ *such that* Γ *is isomorphic to* $Z^{[2]} \rightrightarrows Z$ *in the category* $\mathfrak{RG}_s$*.*

Proof. $(a) \Longrightarrow (b)$. Suppose $Y \xleftarrow{\mathfrak{r}} Z \xrightarrow{\mathfrak{s}} X$ is a Real Morita equivalence. Then the orbit space Z/X, is identified with the Real space Z. But then the map $\mathfrak{r}$ induces a Real homeomorphism $Z \longrightarrow Y$. On the other hand, applying Proposition 2.4.8, one has

$$Z^{[2]} = (\mathfrak{s}^* X \cong \mathfrak{r}^* \Gamma = \Gamma,$$

where $Z^{[2]} := Z \times_{\mathfrak{s},X,\mathfrak{s}} Z$. It follows that with Z and the Real map $\mathfrak{s}$, the assertion (b) is held. $(b) \Longrightarrow (a)$. If $\varphi : Z \longrightarrow X$ is locally split, then Lemma 2.4.7 implies that the pullback $Z^{[2]} = \varphi^* X$ is Morita equivalent to the Real groupoid X. □

An immediate consequence of Proposition 2.8.3 is that any Rg bundle gerbe $(\tilde{\Gamma}, Y^{[2]}, \delta)$ over X defines a Rg $\mathbb{S}^1$-central extension of X; and conversely, any Rg $\mathbb{S}^1$-central extension of X comes from a Rg bundle gerbe over X. We can summarize this situation as follows:

Corollary 2.8.4. *Let* X *be a locally compact Hausdorff Real space. Then the group* $\widehat{Ext\mathrm{R}}(X, \mathbb{S}^1)$ *is isomorphic to the group* $\widehat{Rbg}(X)$ *of stable isomorphism classes of Rg bundle gerbes over* X *equipped with the operation of tensor product.*

Remark 2.8.5. *It is worth noting that in the case of a trivial involution on* X*, then our definition coincides with the notion of* real *bundle gerbes discussed in [58].*

3

Čech cohomology of Real groupoids

J. L. Tu gives in [88] a concise exposition of a Čech cohomology theory $\check{H}^*$ for groupoids as the Čech cohomology of simplicial topological spaces, and he shows, for instance, that the well-known isomorphism between $\mathbb{S}^1$-central extensions of a groupoid $\mathcal{G}$ and the second cohomology group (see [76]) of $\mathcal{G}$ with coefficient in the sheaf $\mathcal{S}$ of germs of $\mathbb{S}^1$-valued functions holds; *i.e.*, $\mathrm{Ext}(\mathcal{G},\mathbb{S}^1) \cong \check{H}^2(\mathcal{G}_\bullet,\mathbb{S}^1)$. Our purpose here is to study an analogous theory $\check{H}R^*$ for Real groupoids and then to find a link to the Abelian group $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$.

3.1 *Real* simplicial spaces

We start by recalling some preliminary notions. For each zero integer $n \in \mathbb{N}$, we set $[n] = \{0,\ldots,n\}$. Recall ([88]) that the simplicial (resp. pre-simplicial) category Δ (resp. Δ') is the category whose objects are the sets $[n]$, and whose morphisms are the nondecreasing (resp. increasing) maps $f : [m] \longrightarrow [n]$. For $n \in \mathbb{N}$, we denote by $\Delta^{(N)}$ the N-truncated full subcategory of Δ whose objects are those $[k]$ with $k \leq N$.

Definition 3.1.1. *A* Real simplicial (resp. pre-simplicial, N-simplicial) topological space *consists of a contravariant functor from Δ (resp. Δ', $\Delta^{(N)}$) to the category $\mathfrak{RTop}$ whose objects are topological Real spaces and morphisms are continuous Real maps. A morphism of Real simplicial (resp. pre-simplicial,...) spaces is a morphism of such functors.*

More concretely, a Real (pre-)simplicial space is given by a family $(X_\bullet,\rho_\bullet) = (X_n,\rho_n)_{n\in\mathbb{N}}$ of topological Real spaces, and for every map $f : [m] \longrightarrow [n]$ we are given a continuous Real map (called *face* or *degeneracy map* depending which of m and n is larger) $\tilde{f} : (X_n,\rho_n) \longrightarrow (X_m,\rho_m)$, satisfying the relation $\widetilde{f\circ g} = \tilde{g}\circ\tilde{f}$ whenever f and g are composable.

Definition 3.1.2 (N-skeleton)**.** *Let $(X_\bullet,\rho_\bullet)$ be a Real simplicial space. For any $N\in\mathbb{N}$, we define the N*-skeleton *of $(X_\bullet,\rho_\bullet)$ as the Real simplicial space $(X_\bullet,\rho_\bullet)^N$ "of dimension N"; that is, $(X_n,\rho_n)^N=(X_n,\rho_n)$ for $n\leq N$, and $(X_n,\rho_n)^N=(X_N,\rho_N)$ for all $n\geq N+1$.*

Let $\varepsilon_i^n:[n-1]\longrightarrow[n]$ be the unique increasing injective map that avoids i, and let $\eta_i^n:[n+1]\longrightarrow[n]$ be the unique nondecreasing surjective map such that i is reached twice; that is,

$$\varepsilon_i^n(k)=\begin{cases}k, & \text{if } k\leq i-1,\\ k+1, & \text{if } k\geq i,\end{cases}\quad \text{and } \eta_i^n(k)=\begin{cases}k, & \text{if } k\leq i;\\ k-1, & \text{if } k\geq i+1.\end{cases}\tag{3.1}$$

We will omit the superscript n if there is no ambiguity.

If $(X_\bullet,\rho_\bullet)$ is a Real simplicial space, the face and degeneracy maps

$$\tilde{\varepsilon}_i^n:(X_n,\rho_n)\longrightarrow(X_{n-1},\rho_{n-1}),\quad \text{and } \tilde{\eta}_i^n:(X_n,\rho_n)\longrightarrow(X_{n+1},\rho_{n+1}),\ i=0,\dots,n$$

clearly satisfy the following *simplicial identities*:

$$\begin{array}{l}\tilde{\varepsilon}_i^{n-1}\tilde{\varepsilon}_j^n=\tilde{\varepsilon}_{j-1}^{n-1}\tilde{\varepsilon}_i^n \text{ if } i\leq j-1,\ \tilde{\eta}_i^{n+1}\tilde{\eta}_j^n=\tilde{\eta}_{j+1}^{n+1}\tilde{\eta}_i^n \text{ if } i\leq j,\ \tilde{\varepsilon}_i^{n+1}\tilde{\eta}_j^n=\tilde{\eta}_{j-1}^{n-1}\tilde{\varepsilon}_i^n \text{ if } i\leq j-1,\\ \tilde{\varepsilon}_i^{n+1}\tilde{\eta}_j^n=\tilde{\eta}_j^{n-1}\tilde{\varepsilon}_{i-1}^n \text{ if } i\geq j+2, \text{ and } \tilde{\varepsilon}_j^{n+1}\tilde{\eta}_j^n=\tilde{\varepsilon}_{j+1}^{n+1}\tilde{\eta}_j^n=\mathrm{Id}_{X_n}.\end{array}\tag{3.2}$$

Conversely, let $(X_n,\rho_n)_{n\in\mathbb{N}}$ be a sequence of topological Real spaces together with maps satisfying (3.2). Then, thanks to Theorem 5.2 of [54], there is a unique Real simplicial structure on $(X_\bullet,\rho_\bullet)$ such that $\tilde{\varepsilon}_i$ and $\tilde{\eta}_i$ are the face and degeneracy maps respectively.

We now give an example of Real simplicial space, which is inspired by [91, 2.3], that will play a central rôle in the sequel.

Example 3.1.3 (**Real simplicial structure on a Real groupoid**)**.** *Consider the pair groupoid $[n]\times[n]\rightrightarrows[n]$; that is, the product is $(i,j)(j,k)):=(i,k)$ and the inverse of (i,j) is (j,i).*

If $(\mathcal{G},\rho)$ is a topological Real groupoid, we define

$$\mathcal{G}_n:=\mathrm{Hom}([n]\times[n],\mathcal{G})$$

as the space of strict morphisms from the groupoid $[n]\times[n]\rightrightarrows[n]$ to $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$. We obtain a Real structure on $\mathcal{G}_n$ by defining $\rho_n(\varphi):=\rho\circ\varphi$, for $\varphi\in\mathcal{G}_n$. Any $f\in\mathrm{Hom}_\Delta([m],[n])$ (or $f\in\mathrm{Hom}_{\Delta'}([m],[n])$) naturally gives rise to a strict morphism $f\times f:[m]\times[m]\longrightarrow[n]\times[n]$, which, in turn, induces a Real map $\tilde{f}:(\mathcal{G}_n,\rho_n)\longrightarrow(\mathcal{G}_m,\rho_m)$ given by $\tilde{f}(\varphi):=\varphi\circ(f\times f)$ for $\varphi\in\mathcal{G}_n$. Hence, we obtain a Real simplicial space $(\mathcal{G}_\bullet,\rho_\bullet)$.

Notice that the groupoid $[n]\times[n]\rightrightarrows[n]$ is generated by elements $(i-1,i), 1\leq i\leq n$; indeed, given an element $(i,j)\in[n]\times[n]$, we can suppose that $i\leq j$ (otherwise, we take its inverse (j,i)), and then $(i,j)=(i,i+1)\dots(j-1,j)$. It turns out that any strict morphism $\varphi:[n]\times[n]\longrightarrow\mathcal{G}$ is uniquely determined by its images $\varphi(i-1,i)\in\mathcal{G}$; hence, the well defined Real map

$$\mathcal{G}_n\longrightarrow\mathcal{G}^{(n)},\ \varphi\longmapsto(g_1,\dots,g_n),$$

where $g_i := \varphi(i-1,i)$, $1 \leq i \leq n$, *and* $\mathcal{G}^{(n)} := \{(h_1,...,h_n) \mid s(h_i) = r(h_{i-1}),\ i = 1,...,n\}$, *identifies* $(\mathcal{G}_n, \rho_n)$ *with* $(\mathcal{G}^{(n)}, \rho^{(n)})$, *where* $\rho^{(n)}$ *is the obvious Real structure on the fibred product* $\mathcal{G}^{(n)}$. *Therefore, using this identification, the face maps* $\tilde{\varepsilon}_i^n : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_{n-1}, \rho_{n-1})$ *of* $(\mathcal{G}_\bullet, \rho_\bullet)$ *are given by:*

$$\begin{aligned}
&\tilde{\varepsilon}_0^n(g_1,g_2,...,g_n) = (g_2,...,g_n),\\
&\tilde{\varepsilon}_i^n(g_1,g_2,...,g_n) = (g_1,...,g_ig_{i+1},...,g_n),\ 1 \leq i \leq n-1,\\
&\tilde{\varepsilon}_n^n(g_1,g_2,...,g_n) = (g_1,...,g_{n-1}),
\end{aligned} \tag{3.3}$$

and for $n = 1$, *by* $\tilde{\varepsilon}_0^1(g) = s(g)$, $\tilde{\varepsilon}_1^1(g) = r(g)$; *while the degeneracy maps* $\tilde{\eta}_i^n : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_{n+1}, \rho_{n+1})$ *are given by:*

$$\begin{aligned}
&\tilde{\eta}_0^n(g_1,g_2,...,g_n) = (r(g_1),g_1,...,g_n),\\
&\tilde{\eta}_i^n(g_1,g_2,...,g_n) = (g_1,...,s(g_i),g_{i+1},...,g_n),\ 1 \leq i \leq n,
\end{aligned} \tag{3.4}$$

and $\tilde{\eta}_0^0 : \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ *is the unit map of the Real groupoid.*

Now for $n \in \mathbb{N}$, we define the space $(E\mathcal{G})_n$ of $(n+1)$-tuples of elements of $\mathcal{G}$ that map to the same unit; *i.e.* $(E\mathcal{G})_n := \{(\gamma_0,...,\gamma_n) \in \mathcal{G}^{n+1} \mid r(\gamma_0) = r(\gamma_1) = ... = r(\gamma_n)\}$. Suppose we are given $(g_1,...,g_n) \in \mathcal{G}_n$. Then we can choose an $(n+1)$-tuple $(\gamma_0,...,\gamma_n) \in (E\mathcal{G})_n$ such that $g_i = \gamma_{i-1}^{-1}\gamma_i$ for each $i = 1,...,n$. If $(\gamma'_0,...\gamma'_n)$ is another $(n+1)$-tuples verifying these identities, then $s(\gamma'_i) = s((\gamma'_{i-1})^{-1}\gamma'_i) = s(\gamma_{i-1}^{-1}\gamma_i) = s(\gamma_i)$, for all $i = 1,...,n$, and that means that there exists a unique $g \in \mathcal{G}$, such that $s(g) = r(\gamma_i)$ and $\gamma'_i = g\cdot\gamma_i$. This hence gives us a well defined injective map

$$\mathcal{G}_n \longrightarrow (E\mathcal{G})_n/_\sim,\ (g_1,...,g_n) \longmapsto [\gamma_0,...,\gamma_n],$$

where $(\gamma_0,...,\gamma_n) \sim (g\cdot\gamma_0,...,g\cdot\gamma_n)$. Moreover, this map is surjective, for if $(\gamma_0,...,\gamma_n) \in (E\mathcal{G})_n$, one can consider morphisms g_i from $s(\gamma_i)$ to $s(\gamma_{i-1})$, $i = 1,...,n$, so that we have

$$\gamma_1 = \gamma_0 g_1,\ \gamma_2 = \gamma_1 g_2 = \gamma_0 g_1 g_2, ..., \gamma_n = \gamma_0 g_1 \cdots g_n,$$

and then

$$[\gamma_0,...,\gamma_n] = [r(g_1), g_1, g_1g_2, ..., g_1\cdots g_n]$$

which gives the inverse $(E\mathcal{G})_n/_\sim \ni [\gamma_0,...,\gamma_n] \longmapsto (g_1,...,g_n) \in \mathcal{G}_n$. It hence turns out that we can identify $\mathcal{G}_n$ with the quotient $(E\mathcal{G})_n$. Note that the quotient space $(E\mathcal{G})_n/_\sim$ naturally inherits the Real structure ρ_{n+1} and that the isomorphism defined above is compatible with the Real structures.

Henceforth, an element of $\mathcal{G}_n$ will be represented by a vector $\overrightarrow{g} = (g_1,...,g_n)$, where we view $\overrightarrow{g}$ as a morphism $[n]\times[n] \longrightarrow \mathcal{G}$, and $g_i = \overrightarrow{g}(i-1,i)$, $i = 1,...,n$, or $\overrightarrow{g} = [\gamma_0,...,\gamma_n]$ as a class in $(E\mathcal{G})_n/_\sim$. For the first picture, if $f \in \mathrm{Hom}_\Delta([m],[n])$, then the Real face/degeneracy map $\tilde{f} : (\mathcal{G}_n, \rho_n) \longrightarrow (\mathcal{G}_m, \rho_m)$ is given by:

$$\tilde{f}(\overrightarrow{g}) = \left(\overrightarrow{g}\left(f(0), f(1)\right), ..., \overrightarrow{g}\left(f(m-1), f(m)\right)\right). \tag{3.5}$$

For instance, if f in injective, then

$$\overrightarrow{g}\left(f(i-1), f(i)\right) = \overrightarrow{g}\left(f(i-1), f(i-1)+1\right) \cdots \overrightarrow{g}\left(f(i)-1, f(i)\right) \text{ for } f(i) \geq 1,$$

and thus

$$\tilde{f}(\overrightarrow{g}) = (g_{f(0)+1} \cdots g_{f(1)}, \ldots, g_{f(m-1)+1} \cdots g_{f(m)}). \tag{3.6}$$

However, the second picture offers a more general formula for the face and degeneracy maps; roughly speaking, for any $f \in \mathrm{Hom}_\Delta([m],[n])$, we have $\overrightarrow{g}(i,j) = \gamma_i^{-1}\gamma_j$ for every $(i,j) \in [n] \times [n]$. In particular, $\overrightarrow{g}(f(k-1), f(k)) = \gamma_{f(k-1)}^{-1}\gamma_{f(k)}$, for every $k \in [m]$; then (3.5) gives :

$$\tilde{f}(\overrightarrow{g}) = [\gamma_{f(0)}, \ldots, \gamma_{f(m)}]. \tag{3.7}$$

3.2 *Real* sheaves on *Real* simplicial spaces

In this section we will follow closely [88, §3] to study Real sheaves on Real (pre-)simplicial spaces. We start by introducing some preliminary notions.

Let $\mathfrak{C}$ be a topological category. We define the category $\mathfrak{C}_R$, that we will abusively call *the pseudo-Real category* associated to $\mathfrak{C}$, by setting:

- $Ob(\mathfrak{C}_R)$ consists of triples (A, σ_A, A'), where $A,\ A' \in Ob(\mathfrak{C})$ and $\sigma_A \in \mathrm{Hom}_{\mathfrak{C}}(A, A')$;
- $\mathrm{Hom}_{\mathfrak{C}_R}\left((A,\sigma_A,A'),(B,\sigma_B,B')\right)$ consists of pairs $(f,\tilde{f})$ of morphisms $f: A \longrightarrow B$, $\tilde{f}: A' \longrightarrow B'$ in $\mathfrak{C}$ such that the diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma_A"'] & B \arrow[d, "\sigma_B"] \\ A' \arrow[r, "\tilde{f}"] & B' \end{tikzcd}$$

commute.

Now, let $\phi: \mathfrak{C} \longrightarrow \mathfrak{C}$ be a functor. Then we define the subcategory $\mathfrak{C}_\phi$ of $\mathfrak{C}_R$ whose objects are pairs $(A, \phi(A))$, where $A \in Ob(\mathfrak{C})$, and in which a morphism from $(A,\phi(A))$ to $(B,\phi(B))$ is a pair $(f,\tilde{f})$ of morphisms $f: A \longrightarrow B$, $\tilde{f}: \phi(A) \longrightarrow \phi(B)$ such that $\tilde{f}\circ\phi = \phi\circ f$. A fundamental example of this is the category $\mathfrak{OB}(X)$ of open subsets of a given topological Real space (X,ρ). Recall that objects of this category are the collection of the open sets $U \subset X$, and morphisms are the canonical injections $V \hookrightarrow U$ when $V \subset U$. Given such a Real space (X,ρ), the map ρ induces a functor (which is an isomorphism) $\rho: \mathfrak{OB}(X) \longrightarrow \mathfrak{OB}(X)$ given by

$$\left(V \xhookrightarrow{\iota} U\right) \longmapsto \left(\rho(V) \xhookrightarrow{\rho\circ\iota\circ\rho} \rho(U)\right).$$

Definition 3.2.1 (Real presheaves)**.** *Let (X,ρ) be a topological Real space, and let $\mathfrak{C}$ be a topological category. A* Real presheaf $(\mathfrak{F},\sigma)$ *on (X,ρ) with values in $\mathfrak{C}$ is a contravariant functor from $\mathfrak{O}\mathfrak{B}(X)_\rho$ to $\mathfrak{C}_R$; a morphism of Real presheaves is a morphism of such functors.*

Specifically, from the fact that $\rho : X \longrightarrow X$ is a homeomorphism and from the canonical properties of the injections $V \hookrightarrow U$ of open sets $V \subset U \subset X$, a Real presheaf on (X,ρ) with values in $\mathfrak{C}$ assigns to each open subset $U \subset X$ a triple $(\mathfrak{F}(U), \sigma_U, \mathfrak{F}(\rho(U)))$, where $\mathfrak{F}(U)$, $\mathfrak{F}(\rho(U))$ are objects of $\mathfrak{C}$, and $\sigma_U \in \mathrm{Isom}_{\mathfrak{C}}(\mathfrak{F}(U), \mathfrak{F}(\rho(U)))$, and for $V \subset U$ we are given two morphisms $\varphi_{V,U} : \mathfrak{F}(U) \longrightarrow \mathfrak{F}(V)$ and $\varphi_{\rho(V),\rho(U)} : \mathfrak{F}(\rho(U)) \longrightarrow \mathfrak{F}(\rho(V))$, called the restriction morphisms, such that:

- $\varphi_{U,U} = \mathrm{Id}_{\mathfrak{F}(U)}$;
- $\sigma_V \circ \varphi_{V,U} = \varphi_{\rho(V),\rho(U)} \circ \sigma_U$,
- $\varphi_{W,U} = \varphi_{W,V} \circ \varphi_{V,U}$, and $\varphi_{\rho(W),\rho(U)} = \varphi_{\rho(W),\rho(V)} \circ \varphi_{\rho(V),\rho(U)}$.

A morphism of Real presheaves $\phi : (\mathfrak{F}, \sigma^{\mathfrak{F}}) \longrightarrow (\mathfrak{G}, \sigma^{\mathfrak{G}})$ is then a family of morphisms $\phi_U \in \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{F}(U), \mathfrak{G}(U))$ such that, for all pairs of open sets U, V with $V \subset U$, the diagrams below commute:

$$\begin{tikzcd}
\mathfrak{F}(\rho(U)) \arrow[d, "\phi_{\rho(U)}"] & \mathfrak{F}(U) \arrow[l, "\sigma_U^{\mathfrak{F}}"'] \arrow[r, "\varphi_{V,U}^{\mathfrak{F}}"] \arrow[d, "\phi_U"] & \mathfrak{F}(V) \arrow[d, "\phi_V"] \\
\mathfrak{G}(\rho(U)) & \mathfrak{G}(U) \arrow[l, "\sigma_U^{\mathfrak{G}}"'] \arrow[r, "\varphi_{V,U}^{\mathfrak{G}}"] & \mathfrak{G}(V)
\end{tikzcd} \tag{3.8}$$

As in the standard case, if $(\mathfrak{F},\sigma)$ is a Real presheaf over X, and if U is an open subset of X, an element $\mathsf{s} \in \mathfrak{F}(U)$ is called a *section of* $(\mathfrak{F},\sigma)$ *on* U, and for $x \in X$. If V is an open subset of U, and $\mathsf{s} \in \mathfrak{F}(U)$, one often writes $\mathsf{s}_{|V}$ for $\varphi_{V,U}(\mathsf{s})$.

Definition 3.2.2. *([44, Definition 2.2]). A* Real sheaf *over (X,ρ) with values in $\mathfrak{C}$ is a Real presheaf $(\mathfrak{F},\sigma)$ satisfying the following conditions:*

(i) For any open set $U \subset X$, any open cover $U = \bigcup_{i\in I} U_i$, any section $\mathsf{s} \in \mathfrak{F}(U)$, $\mathsf{s}_{|U_i} = 0$ for all i implies $\mathsf{s} = 0$.

(ii) For any open set $U \subset X$, any open cover $U = \bigcup_{i\in I} U_i$, any family of sections $\mathsf{s}_i \in \mathfrak{F}(U_i)$ satisfying $\mathsf{s}_{i|U_{ij}} = \mathsf{s}_{j|U_{ij}}$ for all nonempty intersection U_{ij}, there exists $\mathsf{s} \in \mathfrak{F}(U)$ such that $\mathsf{s}_{|U_i} = \mathsf{s}_i$ for all i.

A morphism of Real sheaves is a morphism of the underlying presheaves. We denote by $\mathfrak{C}_R(X)$ (or simply by $\mathsf{Sh}_\rho(X)$ if there is no risk of confusion) for the category of Real sheaves on (X,ρ) with values in $\mathfrak{C}$.

Notice that if $(\mathfrak{F},\sigma)$ is a Real sheaf (resp. presheaf) on (X,ρ), then $\mathfrak{F}$ is a sheaf (resp. presheaf) on X in the usual sense. Recall that the *stalk* of $\mathfrak{F}$ at a point $x \in X$, denoted by $\mathfrak{F}_x$, is the direct limit of the direct system $(\mathfrak{F}(U),\varphi_{V,U})$ where U runs along the family of open neighborhoods of x; *i.e.*

$$\mathfrak{F}_x := \varinjlim_{x\in U} \mathfrak{F}(U), \tag{3.9}$$

The image of a section $\mathsf{s} \in \mathfrak{F}(U)$ in $\mathfrak{F}_x$ by the canonical morphism $\mathfrak{F}(U) \longrightarrow \mathfrak{F}_x$ (where $x \in U$) is called the *germ* of s at x and denoted by s_x.

Note that if U is an open neighborhood of x, $\rho(U)$ is an open neighborhood of $\rho(x)$, and the isomorphism $\sigma_U : \mathfrak{F}(U) \ni \mathsf{s} \longmapsto \sigma_U(\mathsf{s}) \in \mathfrak{F}(\rho(U))$ extends to an isomorphism $\sigma_x : \mathfrak{F}_x \longrightarrow \mathfrak{F}_{\rho(x)}$, defined by $\sigma_x(\mathsf{s}_x) = (\sigma_U(\mathsf{s}))_{\rho(x)}$, whose inverse is $\sigma_{\rho(x)}$. We thus have a well defined 2-periodic isomorphism, also denoted by σ, on the topological [1] space $\mathcal{F} := \coprod_{x\in X} \mathfrak{F}_x$, given by

$$\sigma : \mathcal{F} \longrightarrow \mathcal{F},\ (x,\mathsf{s}_x) \longmapsto (\rho(x),\sigma_x(\mathsf{s}_x)) \tag{3.10}$$

which gives a Real space $(\mathcal{F},\sigma)$.

Example 3.2.3. *Let (X,ρ) be a Real space. Then the space $C(X)$ of continuous complex values functions on X defines a Real sheaf of abelian groups on (X,ρ) by $(U,\rho(U)) \longmapsto (C(U),\tilde{\rho}_U, C(\rho(U)))$, where $\tilde{\rho}_U(f)(\rho(x)) := \overline{f(x)}$.*

Definition 3.2.4 (Pushforward, pullback)**.** *Let (X,ρ), (Y,ϱ) be topological Real spaces, and let $f : (Y,\varrho) \longrightarrow (X,\rho)$ be a a continuous Real map. Suppose that $(\mathfrak{F},\sigma)$ and $(\mathfrak{G},\varsigma)$ are Real sheaves on (X,ρ) and (Y,ϱ) respectively, with values in the same category $\mathfrak{C}$.*

(i) The pushforward *of $(\mathfrak{G},\varsigma)$ by f, denoted by $(f_*\mathfrak{G}, f_*\varsigma)$, is the Real sheaf on (X,ϱ) defined by the contravariant functor:*

$$\mathfrak{O}\mathfrak{B}(X)_\rho \longrightarrow \mathfrak{C}_R,\ (U,\rho(U)) \longmapsto \big(f_*\mathfrak{G}(U), f_*\varsigma_U, f_*\mathfrak{G}(\rho(U))\big), \tag{3.11}$$

where $f_*\mathfrak{G}(U) := \mathfrak{G}(f^{-1}(U))$, $f_*\varsigma_U := \varsigma_{f^{-1}(U)}$, and

$$f_*\mathfrak{G}(\rho(U)) = \mathfrak{G}(f^{-1}(\rho(U))) \cong \mathfrak{G}(\varrho(f^{-1}(U))).$$

(ii) The pullback *of $(\mathfrak{F},\sigma)$ along f, denoted by $(f^*\mathfrak{F}, f^*\sigma)$, is the Real sheaf on (Y,ϱ) associated to the Real presheaf defined by:*

$$\mathfrak{O}\mathfrak{B}(Y)_\varrho \longrightarrow \mathfrak{C}_R,\ (V,\varrho(V)) \longmapsto (f^*\mathfrak{F}(V), f^*\sigma_V, f^*\mathfrak{F}(\varrho(V))), \tag{3.12}$$

[1] Recall that if $\mathfrak{F}$ is a presheaf over X, any section $\mathsf{s} \in \mathfrak{F}(U)$ induces a map $[\mathsf{s}] : U \longrightarrow \coprod_x \mathfrak{F}_x$, $y \longmapsto \mathsf{s}_y$. We give $\mathcal{F} := \coprod_{x\in X} \mathfrak{F}_x$ the largest topology such that all the maps $[\mathsf{s}]$ are continuous. On the other hand, associated to $\mathfrak{F}$, there is a sheaf $\widehat{\mathfrak{F}}$ given by $\widehat{\mathfrak{F}}(U) := \Gamma(U,\mathcal{F})$, and we have that $\mathfrak{F}(U) \cong \Gamma(U,\mathcal{F})$ if and only if $\mathfrak{F}$ is a sheaf. Then, given a Real presheaf $(\mathfrak{F},\sigma)$, one can define its associated Real sheaf in the same fashion.

where $f^*\mathfrak{F}(V) := \varinjlim_{\substack{f(V)\subset U\subset X\\ U\ open}} \mathfrak{F}(U)$, *and* $f^*\sigma_V : f^*\mathfrak{F}(V) \longrightarrow f^*\mathfrak{F}(\varrho(V))$ *is the morphism in* $\mathfrak{C}$ *extending functorially* $\sigma_U : \mathfrak{F}(U) \longrightarrow \mathfrak{F}(\rho(U))$ *along the family of open neighborhoods of* $f(V)$ *in* X.

It immediately follows from this definition that we have a covariant functor

$$\mathfrak{RTop} \longrightarrow \mathfrak{RSh}, \left((Y,\varrho) \xrightarrow{f} (X,\rho) \right) \longmapsto \left(\mathsf{Sh}_\varrho(Y) \xrightarrow{f_*} \mathsf{Sh}_\rho(X) \right), \tag{3.13}$$

and a contravariant functor

$$\mathfrak{RTop} \longrightarrow \mathfrak{RSh}, \left((Y,\varrho) \xrightarrow{f} (X,\rho) \right) \longmapsto \left(\mathsf{Sh}_\rho(X) \xrightarrow{f^*} \mathsf{Sh}_\varrho(Y) \right), \tag{3.14}$$

where $\mathfrak{RSh}$ is the category whose objects are the categories of Real sheaves on given Real spaces and morphisms are functors of such categories.

We will also need the following proposition.

Proposition 3.2.5. *Let* $f : (Y,\varrho) \longrightarrow (X,\rho)$ *be a a continuous Real map. Suppose that* $(\mathfrak{F},\sigma)$ *and* $(\mathfrak{G},\varsigma)$ *are Real sheaves on* (X,ρ) *and on* (Y,ϱ) *respectively, with values in the same category* $\mathfrak{C}$. *Then*

$$\mathrm{Hom}_{\mathsf{Sh}_\rho(X)}\big((\mathfrak{F},\sigma),(f_*\mathfrak{G},f_*\varsigma)\big) \cong \mathrm{Hom}_{\mathsf{Sh}_\varrho(Y)}((f^*\mathfrak{F},f^*\sigma),(\mathfrak{G},\varsigma)). \tag{3.15}$$

Proof. The proof is the same as in the general case where Real structures are not concerned (see for instance [44, Proposition 2.3.3]). □

Definition 3.2.6. *Given a continuous Real map* $f : (Y,\varrho) \longrightarrow (X,\rho)$ *and Real sheaves* $(\mathfrak{F},\sigma)$ *and* $(\mathfrak{G},\varsigma)$ *as above, we define the set* $\mathrm{Hom}_f(\mathfrak{F},\mathfrak{G})_{\sigma,\varsigma}$ *of* Real f-morphisms *from* $(\mathfrak{F},\sigma)$ *to* $(\mathfrak{G},\varsigma)$ *to be* $\mathrm{Hom}_{\mathsf{Sh}_\rho(X)}\big((\mathfrak{F},\sigma),(f_*\mathfrak{G},f_*\varsigma)\big) = \mathrm{Hom}_{\mathsf{Sh}_\varrho(Y)}\big((f^*\mathfrak{F},f^*\sigma),(\mathfrak{G},\varsigma)\big)$

Definition 3.2.7. *Let* $(X_\bullet,\rho_\bullet)$ *be a Real simplicial (resp. pre-simplicial) space. A Real sheaf on* $(X_\bullet,\rho_\bullet)$ *is a family* $(\mathfrak{F}^n,\sigma^n)_{n\in\mathbb{N}}$ *such that* $(\mathfrak{F}^n,\sigma^n)$ *is a Real sheaf on* (X_n,ρ_n) *for all* n, *and such that for each morphism* $f : [m] \longrightarrow [n]$ *in* Δ *(resp.* Δ'*) we are given Real* $\tilde{f}$*-morphisms* $\tilde{f}^* \in Hom_{\tilde{f}}(\mathfrak{F}^m,\mathfrak{F}^n)_{\sigma^m,\sigma^n}$ *such that*

$$\widetilde{f\circ g}^* = \tilde{f}^* \circ \tilde{g}^*, \tag{3.16}$$

whenever f *and* g *are composable.*

One can use the definition of the push-forward to give a concrete interpretation of this definition. Roughly speaking, a sequence $(\mathfrak{F}^n,\sigma^n)_{n\in\mathbb{N}}$ is a Real sheaf on a Real simplicial (resp. pre-simplicial, ...) space $(X_\bullet,\rho_\bullet)$, if for a given morphism $f:[m]\longrightarrow[n]$ in Δ (resp. Δ', ...), then for any pair of open sets $U\subset X_n$ and $V\subset X_m$ such that $\tilde f(U)\subset V$ there is a *restriction map* $\tilde f^*:\mathfrak{F}^m(V)\longrightarrow\mathfrak{F}^n(U)$ such that the diagram

$$\begin{array}{ccc}\mathfrak{F}^m(V) & \xrightarrow{\tilde f^*} & \mathfrak{F}^n(U)\\ {\scriptstyle\sigma^m_V}\downarrow & & \downarrow{\scriptstyle\sigma^n_U}\\ \mathfrak{F}^m(\rho(V)) & \xrightarrow{\tilde f^*} & \mathfrak{F}^n(\rho(U))\end{array} \tag{3.17}$$

commute, and $\tilde f^*\circ\tilde g^* = \widetilde{f\circ g}^*:\mathfrak{F}^k(W)\longrightarrow\mathfrak{F}^n(U)$ whenever $\tilde g(V)\subset W\subset X_k$. Morphisms of Real sheaves over $(X_\bullet,\rho_\bullet)$ are defined in the obvious way; we denote by $\mathsf{Sh}_{\rho_\bullet}(X_\bullet)$ for the category of Real sheaves over $(X_\bullet,\rho_\bullet)$.

3.3 *Real* $\mathcal{G}$-sheaves and reduced *Real* sheaves

Definition 3.3.1. *(i) A Real space (Y,ϱ) is said to be* étale *over (X,ρ) if there exists an* étale *Real map $f:(Y,\varrho)\longrightarrow(X,\rho)$; that is to say, every point $y\in Y$ has an open neighborhood V such that $f_{|V}:V\longrightarrow U$ is homeomorphism, where U in an open neighborhood of $f(y)$ in X.*

(ii) A Real groupoid $(\mathcal{G},\rho)$ is étale if the range (equivalently the source) map is étale.

(iii) A morphism $\pi_\bullet:(Y_\bullet,\varrho_\bullet)\longrightarrow(X_\bullet,\rho_\bullet)$ of Real (pre-)simplicial spaces is étale if for all n, $\pi_n:(Y_n,\varrho_n)\longrightarrow(X_n,\rho_n)$ is étale.

Example 3.3.2. *Any Real sheaf $(\mathfrak{F},\sigma)$ on (X,ρ) can be viewed as an étale Real space over (X,ρ). Indeed, considering the underlying topological Real space $(\mathcal{F},\sigma)$, it is easy to check that the canonical projection*

$$\mathcal{F}\longrightarrow X,\ (x,\mathsf{s}_x)\longmapsto x$$

is an étale Real map.

Definition 3.3.3. *Let $(\mathcal{G},\rho)$ be a topological Real groupoid. A Real $\mathcal{G}$-sheaf (or an étale Real $\mathcal{G}$-space) is an étale Real space $(\mathcal{E}_0,\nu_0)$ over $(\mathcal{G}^{(0)},\rho)$ equipped with a continuous Real $\mathcal{G}$-action.*

We say that $(\mathcal{E}_0,\nu_0)$ is an Abelian Real $\mathcal{G}$-sheaf if in addition it is an Abelian Real sheaf on $(\mathcal{G}^{(0)},\rho)$ such that the action $\alpha_g:(\mathcal{E}_0)_{s(g)}\longrightarrow(\mathcal{E}_0)_{r(g)}$ is a group homomorphism, for any $g\in\mathcal{G}$.

A morphism of Real $\mathcal{G}$-sheaves $(\mathcal{E}_0,\nu_0)$ and $(\mathcal{E}'_0,\nu'_0)$ is a $\mathcal{G}$-equivariant continuous Real map $\psi:(\mathcal{E}_0,\nu_0)\longrightarrow(\mathcal{E}'_0,\nu'_0)$ such that $p'\circ\psi=p$.

The category of Real $\mathcal{G}$*-sheaves is denoted by* $\mathfrak{B}_\rho\mathcal{G}$*, and is called the* classifying topos *of* $(\mathcal{G},\rho)$.

Examples 3.3.4. 1. *Considering a Real space* (X,ρ) *as a Real groupoid, a Real* X*-sheaf is the same thing as a Real sheaf over* (X,ρ)*; in other words we have that* $\mathfrak{B}_\rho X \cong \mathsf{Sh}_\rho(X)$.

2. *If* $(\mathcal{G},\rho)$ *is a Real group, then a Real* $\mathcal{G}$*-sheaf is just a Real space equipped with a continuous Real* $\mathcal{G}$*-action.*

Lemma 3.3.5. *Any generalized Real morphism* $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ *induces a morphism of toposes*

$$Z^*:\mathfrak{B}_\rho(\mathcal{G})\longrightarrow\mathfrak{B}_\varrho(\Gamma).$$

Consequently, there is a contravariant functor

$$\mathfrak{B}:\mathfrak{RG}\longrightarrow\mathfrak{RBG},$$

defined by

$$(\ (\Gamma,\varrho)\xrightarrow{(Z,\tau)}(\mathcal{G},\rho)\ )\longmapsto(\ \mathfrak{B}_\rho\mathcal{G}\xrightarrow{Z^*}\mathfrak{B}_\varrho\Gamma\ ),$$

where $\mathfrak{RBG}$ *is the category whose objects are classifying toposes of Real groupoids.*

Proof. As noted in [62, 2.2] for the usual case, any Real morphism $f:(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ gives rise to a functor $f^*:\mathfrak{B}_\rho\mathcal{G}\longrightarrow\mathfrak{B}_\varrho\Gamma$. Indeed, if $(\mathcal{E}_0,\nu_0)$ is a Real $\mathcal{G}$-sheaf through an étale Real $\mathcal{G}$-map $p:(\mathcal{E}_0,\nu_0)\longrightarrow(\mathcal{G}^{(0)},\rho)$, then we obtain a Real Γ-sheaf $(f^*\mathcal{E}_0,f^*\nu_0)$ by pulling back $(\mathcal{E}_0,\nu_0)$ along f; *i.e.* $f^*\mathcal{E}_0=\Gamma^{(0)}\times_{f,\mathcal{G}^{(0)},p}\mathcal{E}_0$, $f^*\nu_0=\varrho\times\nu_0$, $f^*p(y,e):=y$, and the right Real Γ-action is $\gamma\cdot(s(\gamma),e):=(r(\gamma),f(\gamma)\cdot e)$ when $p(e)=s(f(\gamma))$. If $\psi:(\mathcal{E}_0,\nu_0)\longrightarrow(\mathcal{E}'_0,\nu'_0)$ is a morphism of Real $\mathcal{G}$-sheaves, then the map $f^*\psi:(f^*\mathcal{E}_0,f^*\nu_0)\longrightarrow(f^*\mathcal{E}'_0,f^*\nu'_0)$ defined by $f^*\psi(y,e):=(y,\psi(e))$ is obviously a morphism a Real Γ-sheaves. It follows that any $(\mathcal{U},f_\mathcal{U})\in\mathrm{Hom}_{\mathfrak{RG}_\Omega}((\Gamma,\varrho),(\mathcal{G},\rho))$ gives rise to a covariant functor $f^*_\mathcal{U}:\mathfrak{B}_\rho\mathcal{G}\longrightarrow\mathfrak{B}_\varrho\Gamma[\mathcal{U}]$. Now if (Z,τ) corresponds to $(\mathcal{U},f_\mathcal{U})$, and if as in the previous chapter, $\iota:\Gamma[\mathcal{U}]\longrightarrow\Gamma$ is the canonical Real morphism, then we can push forward $(f^*_\mathcal{U}\mathcal{E}_0,f^*_\mathcal{U}\nu_0)$ through ι to get a Real Γ-sheaf $(Z^*\mathcal{E}_0,Z^*\nu_0)$; i.e

$$Z^*\mathcal{E}_0:=\iota_*f^*_\mathcal{U}\mathcal{E}_0, \tag{3.18}$$

and the Real structure $Z^*\nu_0$ is the obvious one. □

Lemma 3.3.6. *Let* $(\mathcal{G},\rho)$ *be a topological Real groupoid. Then, any Real* $\mathcal{G}$*-sheaf canonically defines a Real sheaf over the Real simplicial space* $(\mathcal{G}_n,\rho_n)_{n\in\mathbb{N}}$.

To prove this Lemma, we need some more preliminary notions.

Definition 3.3.7. *([88]). A morphism $\pi_\bullet : (\mathcal{E}_\bullet, \nu_\bullet) \longrightarrow (X_\bullet, \rho_\bullet)$ of Real simplicial spaces is called* reduced *if for all m, n and for all $f \in \mathrm{Hom}_\Delta([m],[n])$, the morphism $\tilde{f}$ induces an isomorphism*

$$(\mathcal{E}_n, \nu_n) \cong (X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{E}_m, \rho_n \times \nu_m).$$

In this case, we say that $(\mathcal{E}_\bullet, \nu_\bullet)$ is a reduced Real simplicial space over $(X_\bullet, \rho_\bullet)$.

Morphisms of reduced Real simplicial spaces over $(X_\bullet, \rho_\bullet)$ are defined in the obvious way.

Definition 3.3.8. *([88]). We say that a Real sheaf $(\mathfrak{F}^\bullet, \sigma^\bullet)$ over a Real simplicial space $(X_\bullet, \rho_\bullet)$ is* reduced *if for all m, n and all $f \in \mathrm{Hom}_\Delta([m],[n])$, $\tilde{f}^* \in \mathrm{Hom}\big((\tilde{f}^*\mathfrak{F}^m, \tilde{f}^*\sigma^m), (\mathfrak{F}^n, \sigma^n)\big)$ is an isomorphism.*

Lemma 3.3.9. *([88, Lemma 3.5]). Let $(X_\bullet, \rho_\bullet)$ be a Real simplicial space. Then, there is a one-to-one correspondence between reduced Real sheaves over $(X_\bullet, \rho_\bullet)$ and reduced étale Real simplicial spaces over $(X_\bullet, \rho_\bullet)$.*

Proof. Suppose that we are given a Real sheaf $(\mathfrak{F}^\bullet, \sigma^\bullet)$ over the Real simplicial space $(X_\bullet, \rho_\bullet)$, and let $(\mathcal{F}_n, \sigma_n)_{n\in\mathbb{N}}$ be its underlying sequence of topological Real spaces. We already know from Example 3.3.2 that each of the canonical projection maps $\pi_n : (\mathcal{F}_n, \sigma_n) \longrightarrow (X_n, \rho_n)$ is étale. Now suppose that $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is reduced; that is to say that for any morphism $f \in \mathrm{Hom}_\Delta([m],[n])$, and every open set $V \subset X_m$, $\tilde{f}^* : \mathfrak{F}^m(V) \longrightarrow \mathfrak{F}^n(\tilde{f}^{-1}(V))$ is an isomorphism, so that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{F}^m(V) \arrow[r, "\tilde{f}^*", "\cong"'] \arrow[d, "\sigma^m_V"'] & \mathfrak{F}^n(\tilde{f}^{-1}(V)) \arrow[d, "\sigma^n_{f^{-1}(V)}"] \\
\mathfrak{F}^m(\rho^m(V)) \arrow[r, "\tilde{f}^*", "\cong"'] & \mathfrak{F}^n(\rho^n(\tilde{f}^{-1}(V)))
\end{tikzcd} \tag{3.19}$$

Let $x \in X_n$, $y \in X_m$ such that $\tilde{f}(x) = y$, and let $U \subset X_n$ and $V \subset X_m$ be open neighborhoods of x and y respectively such that $\tilde{f}(U) \subset V$. Then, for a section $\mathsf{s}^m \in \mathfrak{F}^m(V)$, we have an element $(x, (y, \mathsf{s}^m_y)) \in X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{F}_m$ to which we assign an element $(x, \mathsf{s}^n_x) \in \mathcal{F}_n$ as follows: since $U \subset \tilde{f}^{-1}(V)$, the section $\mathsf{s}^m \in \mathfrak{F}^m(V) \cong \mathfrak{F}^n(\tilde{f}^{-1}(V))$ has a restriction $\mathsf{s}^n := \mathsf{s}^m_U \in \mathfrak{F}^n(U)$. In this way we get a well defined map $X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{F}_m \longrightarrow \mathcal{F}_n$. Moreover, it is easy to check that this map is an isomorphism; the inverse is the map

$$\mathcal{F}_n \ni (x, \mathsf{s}^n_x) \longmapsto (x, (\tilde{f}(x), (\tilde{f}^*\mathsf{s}^n)_{\tilde{f}(x)})) \in X_n \times_{\tilde{f}, X_m, \pi_m} \mathcal{F}_m,$$

where if $x \in U \subset X_n$ and $\tilde{f}(U) \subset V \subset X_m$, $\tilde{f}^*\mathsf{s}^n$ is any section in $\mathfrak{F}^m(V) \cong \mathfrak{F}^n(\tilde{f}^{-1}(V))$ that has the same class as s^n at the point x when restricted to $\mathfrak{F}^n(U)$ through the restriction map $\mathfrak{F}^n(\tilde{f}^{-1}(V)) \longrightarrow \mathfrak{F}^n(U)$. Furthermore, for every $f \in \mathrm{Hom}_\Delta([m],[n])$, there is a face/degeneracy map $\tilde{f}:(\mathcal{F}_n,\sigma_n)\longrightarrow(\mathcal{F}_m,\sigma_m)$ given by $\tilde{f}(x,\mathsf{s}_x):=(\tilde{f}(x),(\tilde{f}^*\mathsf{s})_{\tilde{f}(x)})$; hence $(\mathcal{F}_\bullet,\sigma_\bullet)$ is a reduced étale Real simplicial space over $(X_\bullet,\rho_\bullet)$.

Conversely, if $\pi_\bullet:(\mathcal{E}_\bullet,\nu_\bullet)\longrightarrow(X_\bullet,\rho_\bullet)$ is a reduced étale morphism of Real simplicial spaces, we let $\mathfrak{F}^n(U)$ be the space $C(U,\mathcal{E}_n)$ of continuous sections over U (where U is an open subset of X_n) of the projection $\pi_n:(\mathcal{E}_n,\nu_n)\longrightarrow(X_n,\rho_n)$. Next we define $\sigma^n_U:\mathfrak{F}^n(U)\longrightarrow\mathfrak{F}^n(\rho^n(U))$ by $\sigma^n_U(\mathsf{s})(\rho^n(x)):=\nu_n(\mathsf{s}(x))$. Notice that since the π_n's are étale, one can recover the Real spaces $(\mathcal{E}_n,\nu_n)$ by considering the underlying Real spaces of the Real sheaves $(\mathfrak{F}^n,\sigma^n)$. Now for any $f\in\mathrm{Hom}_\Delta([m],[n])$ and for any open set $V\subset X_m$, we have an isomorphism $\tilde{f}^*:\mathfrak{F}^m(V)\longrightarrow\mathfrak{F}^n(\tilde{f}^{-1}(V))$, $\mathsf{s}\longmapsto\tilde{f}^*\mathsf{s}$, where $(\tilde{f}^*\mathsf{s})(x)=(x,\mathsf{s}(\tilde{f}(x)))\in X_n\times_{\tilde{f},X_m,\pi_m}\mathcal{E}_m\cong\mathcal{E}_m$. □

Using the same construction as in the second part of this proof, we deduce the following

Lemma 3.3.10. *Any reduced Real simplicial space over $(X_\bullet,\rho_\bullet)$, étale or not, determines a Real sheaf over $(X_\bullet,\rho_\bullet)$.*

Proof of Lemma 3.3.6. Let (Z,τ) be a Real $\mathcal{G}$-sheaf, and let $\pi:(Z,\tau)\longrightarrow(\mathcal{G}^{(0)},\rho)$ be an étale Real map. Put for all $n\geq 0$, $\mathcal{E}_n:=(\mathcal{G}\ltimes Z)_n:=\mathcal{G}_n\times_{\tilde{\pi}_n,\mathcal{G}^{(0)},\pi}Z$, where $\tilde{\pi}_n(g_1,\dots,g_n)=\tilde{\pi}_n[\gamma_0,\dots,\gamma_n]=s(\gamma_n)=s(g_n)$, and define $\nu_n:=\rho_n\times\tau$. We thus obtain a Real simplicial space $(\mathcal{E}_n,\nu_n)$: the simplicial structure is given by

$$\mathcal{E}_n\ni\big([\gamma_0,\dots,\gamma_n],z\big)\longmapsto\Big((\gamma_{f(0)},\dots,\gamma_{f(m)}),\gamma^{-1}_{f(m)}\gamma_n\cdot z\Big)\in\mathcal{E}_m,\tag{3.20}$$

for $f\in\mathrm{Hom}_\Delta([m],[n])$. Furthermore, it is straightforward to see that the projections $\pi_n:\mathcal{E}_n\longrightarrow\mathcal{G}_n$ are compatible with the Real structures ν_n and ρ_n, and that they define a morphism of Real simplicial spaces. If $f\in\mathrm{Hom}_\Delta([m],[n])$, then the assignment

$$([\gamma_0,\dots,\gamma_n],z)\longmapsto\Big([\gamma_0,\dots,\gamma_n],([\gamma_{f(0)},\dots,\gamma_{f(m)}],\gamma^{-1}_{f(m)}\gamma_n\cdot z)\Big)$$

obviously defines a Real homeomorphism $\mathcal{E}_n\cong\mathcal{G}_n\times_{\tilde{f},\mathcal{G}_m,\pi_m}\mathcal{E}_m$ which shows that $(\mathcal{E}_\bullet,\nu_\bullet)$ is a reduced Real simplicial space over $(\mathcal{G}_n,\rho_n)$. It follows from Lemma 3.3.10 that $(\mathcal{E}_\bullet,\nu_\bullet)$ determines an object of $\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet)$. □

Remark 3.3.11. *Notice that in the proof above we did not use the fact that (Z,τ) is étale. In fact, the Real $\mathcal{G}$-action suffices for (Z,τ) to give rise to a Real sheaf over $(\mathcal{G}_\bullet,\rho_\bullet)$. However, the property of being étale will be necessary to show that the Real sheaf obtained is reduced (as it is mentioned in the following corollary).*

Corollary 3.3.12. *Let $(\mathcal{G},\rho)$ be a topological Real groupoid. Then there is a functor*

$$\mathcal{E}:\mathfrak{B}_\rho\mathcal{G}\longrightarrow\mathfrak{red}\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet),$$

where $\mathfrak{red}\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet)$ is the full subcategory of $\mathsf{Sh}_{\rho_\bullet}(\mathcal{G}_\bullet)$ consisting of all reduced Real sheaves over $(\mathcal{G}_\bullet,\rho_\bullet)$.

Proof. Let us keep the same notations as in the proof of Lemma 3.3.6. Since π is étale, so is π_n for all n. The reduced Real simplicial space $(\mathcal{E}_\bullet, \nu_\bullet)$ is then étale over $(\mathcal{G}_\bullet, \nu_\bullet)$. Now, it suffices to apply Lemma 3.3.9. □

3.4 *Real* $\mathcal{G}$-modules

Definition 3.4.1. *(Compare with [88, Definition 3.9]). Let $(\mathcal{G}, \rho)$ be a topological Real groupoid. A Real $\mathcal{G}$-module is a topological Real groupoid $(\mathcal{M}, \bar{\ })$, with unit space $(\mathcal{G}^{(0)}, \rho)$, and with source and range maps equal to a Real map $\pi : (\mathcal{M}, \bar{\ }) \longrightarrow (\mathcal{G}^{(0)}, \rho)$, such that*

- *$\mathcal{M}_x$ $(= \mathcal{M}^x = \mathcal{M}_x^x)$ is an abelian group for all $x \in \mathcal{G}^{(0)}$;*
- *for all $x \in \mathcal{G}^{(0)}$, the map $(\bar{\ }) : \mathcal{M}_x \longrightarrow \mathcal{M}_{\rho(x)}$ is a group morphism;*
- *as a Real space, $(\mathcal{M}, \bar{\ })$ is endowed with a Real $\mathcal{G}$-action $\alpha : \mathcal{G} \times_{s,\pi} \mathcal{M} \longrightarrow \mathcal{M}$;*
- *for each $g \in \mathcal{G}$, the map $\alpha_g : \mathcal{M}_{s(g)} \longrightarrow \mathcal{M}_{r(g)}$ given by the action is a group morphism.*

By Remark 3.3.11, any Real $\mathcal{G}$-module $(\mathcal{M}, \bar{\ })$ determines an abelian Real sheaf $(\mathfrak{F}^\bullet, \sigma^\bullet)$ on $(\mathcal{G}_\bullet, \rho_\bullet)$ constructed as follows: consider the reduced Real simplicial space $(\mathcal{E}_\bullet, \nu_\bullet) = ((\mathcal{G} \ltimes \mathcal{M})_n, \rho_n \times (\bar{\ }))$, where the Real simplicial structure is given by:

$$\tilde{f}\left([\gamma_0, \ldots, \gamma_n], t\right) = \left([\gamma_{f(0)}, \ldots, \gamma_{f(m)}], \gamma_{f(m)}^{-1}\gamma_n \cdot t\right),$$

for any $f \in \mathrm{Hom}_\Delta([m], [n])$. Next, $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is defined as the sheaf of germs of continuous sections of the projections $\pi_\bullet : (\mathcal{E}_\bullet, \nu_\bullet) \longrightarrow (\mathcal{G}_\bullet, \rho_\bullet)$.

Example 3.4.2. *Let $(\mathcal{G}, \rho)$ be a topological Real groupoid and let $\mathcal{M} = \mathcal{G}^{(0)} \times \mathbb{S}^1$ be endowed with the canonical Real structure $\overline{(x, \lambda)}) := (\rho(x), \bar{\lambda})$, and Real $\mathcal{G}$-action $g \cdot (s(g), \lambda) = (r(g), \lambda)$. Then $(\mathcal{M}, \bar{\ })$ is a Real $\mathcal{G}$-module. The corresponding Real sheaf is called the constant sheaf of germs of $\mathbb{S}^1$-valued functions and denoted (abusively) $\mathbb{S}^1$. More generally, if S is any Real group, $\mathcal{G}^{(0)} \times S$ is a Real $\mathcal{G}$-module, and the induced Real sheaf over $(\mathcal{G}_\bullet, \rho_\bullet)$ is denoted by S.*

3.5 Pre-simplicial *Real* covers

Definition 3.5.1 (Compare with Definition 4.1 [88]). . *Let $(X_\bullet, \rho_\bullet)$ be a Real pre-simplicial space. A* Real open cover *of $(X_\bullet, \rho_\bullet)$ is a sequence $\mathcal{U}_\bullet = (\mathcal{U}_n)_{n \in \mathbb{N}}$ such that $\mathcal{U}_n = (U_j^n)_{j \in J_n}$ is a Real open cover of (X_n, ρ_n).*

We say that $\mathcal{U}_\bullet$ *is* pre-simplicial *if* $(J_\bullet, {}^-) = (J_n, {}^-)_{n\in\mathbb{N}}$ *is a Real pre-simplicial set such that for all* $f \in \mathrm{Hom}_{\Delta'}([m],[n])$ *and for all* $j \in J_n$*, one has* $\tilde{f}(U^n_j) \subseteq U^m_{\tilde{f}(j)}$*. In the same way, one defines the notions of simplicial Real cover and* N*-simplicial Real cover.*

We will use the same construction as in [88, §4.1] to show the following lemma.

Lemma 3.5.2. *Any Real open cover* $\mathcal{U}_\bullet$ *of a Real (pre-)simplicial space* $(X_\bullet, \rho_\bullet)$ *gives rise to a pre-simplicial Real open cover* $_\natural\mathcal{U}_\bullet$*.*

Proof. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \bigcup_{k=0}^n \mathcal{P}^k_n$, where $\mathcal{P}^k_n = \mathrm{Hom}_{\Delta'}([k],[n])$. Let $\mathcal{P} = \bigcup_n \mathcal{P}_n$, and let Λ_n (or $\Lambda_n(J_\bullet)$ if there is a risk of confusion) be the set of maps

$$\lambda : \mathcal{P} \longrightarrow \bigcup_k J_k \text{ such that } \lambda(\mathcal{P}^k_n) \in J_k, \text{ for all } k. \tag{3.21}$$

It is immediate to see that Λ_n is non-empty; indeed, for each $k \in \mathbb{N}$, we fix a map $\vec{j}^{\,k} : [n] \longrightarrow J_k$ which can be written as $\vec{j}^{\,k} = (j^k_0, ..., j^k_n)$. Next, we define $\vec{j} = (\vec{j}^{\,k})_{k\in\mathbb{N}}$. Then the map $\lambda : \mathcal{P} \longrightarrow \bigcup_k J_k$ given by $\lambda(\varphi) := \vec{j} \circ \varphi$ lies in Λ_n. Moreover, Λ_n has a Real structure defines as follows: if $\varphi \in \mathcal{P}^k_n$, then we set

$$\bar{\lambda}(\varphi) := \overline{\lambda(\varphi)} \in J_k \tag{3.22}$$

Now, for all $\lambda \in \Lambda_n$, we let

$$U^n_\lambda := \bigcap_{k\leq n} \bigcap_{\varphi\in\mathcal{P}^k_n} \tilde{\varphi}^{-1}(U^k_{\lambda(\varphi)}). \tag{3.23}$$

Let $x \in X_n$. For each $k \leq n$ and $\varphi \in \mathcal{P}^k_n$, there is $j^k_\varphi \in J_k$ such that $\tilde{\varphi}(x) \in U^k_{j^k_\varphi} \subset X_k$. Define the map $\lambda_x : \mathcal{P} \longrightarrow \bigcup_k J_k$ by $\lambda_x(\varphi) := (j^k_\varphi)_k$. Then, one can see that $x \in \bigcap_{k\leq n}\bigcap_{\varphi\in\mathcal{P}^k_n} \tilde{\varphi}^{-1}(U^k_{\lambda_x(\varphi)}) = U^n_{\lambda_x}$. Furthermore, $\rho_n(U^n_\lambda) = U^n_{\bar{\lambda}}$; hence, $(U^n_\lambda)_{\lambda\in\Lambda_n}$ is a Real open cover of (X_n, ρ_n). If for any $f \in \mathrm{Hom}_{\Delta'}([m],[n])$, we define a map $\tilde{f} : \Lambda_n \longrightarrow \Lambda_m$ by

$$(\tilde{f}\lambda)(\varphi) := \lambda(f \circ \varphi), \text{ for all } \lambda \in \Lambda_n, \text{ and } \varphi \in \mathcal{P}^k_n,$$

one sees that $\tilde{f}(U^n_\lambda) \subseteq U^m_{\tilde{f}(\lambda)}$. Thus, $_\natural\mathcal{U}_\bullet = ((U^n_\lambda)_{\lambda\in\Lambda_n})_{n\in\mathbb{N}}$ is a pre-simplicial Real open cover of $(X_\bullet, \rho_\bullet)$. □

In the same way, for all $N \in \mathbb{N}$ and $n \leq N$, we denote by Λ^N_n the set of all maps $\lambda : \bigcup_{k\leq n} \mathrm{Hom}_\Delta([k],[n]) \longrightarrow \bigcup_{k\leq n} J_k$ that satify $\lambda(\mathrm{Hom}_\Delta([k],[n])) \subset J_k$, and we set

$$U^n_\lambda := \bigcap_{k\leq n} \bigcap_{\varphi\in\mathrm{Hom}_\Delta([k],[n])} \tilde{\varphi}^{-1}(U^n_{\lambda(\varphi)}).$$

Then, we provide $\Lambda^N_\bullet$ with the Real structure defined in the same fashion, and we give it the N-simplicial structure defined as follows: for any $f \in \mathrm{Hom}_{\Delta^N}([m],[n])$, the map $\tilde{f} : \Lambda^N_n \longrightarrow \Lambda^N_m$ is $(\tilde{f}\lambda)(\varphi) := \lambda(f \circ \varphi)$. We thus obtain a N-simplicial Real cover $_{\natural^N}\mathcal{U}_\bullet = (_{\natural^N}\mathcal{U}_n)_{n\in\mathbb{N}}$ of the N-skeleton of $(X_\bullet, \rho_\bullet)$, where $_{\natural^N}\mathcal{U}_n = (U^n_\lambda)_{\lambda\in\Lambda^N_n}$.

We endow the collection of Real open covers of $(X_\bullet, \rho_\bullet)$ with the partial pre-order given by the following definition.

Definition 3.5.3. *Let $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ be Real open covers of a Real simplicial space $(X_\bullet, \rho_\bullet)$, with $\mathcal{U}_n = (U_j^n)_{j\in J_n}$ and $\mathcal{V}_n = (V_i^n)_{i\in I_n}$. We say that $\mathcal{V}_\bullet$ is* finer *than $\mathcal{U}_\bullet$ if for each $n \in \mathbb{N}$, there exists a Real map $\theta_n : (I_n, \bar{\ }) \longrightarrow (J_n, \bar{\ })$ such that $V_i^n \subseteq U^n_{\theta_n(i)}$ for every $i \in I_n$. The Real map $\theta_\bullet = (\theta_n)_{n\in\mathbb{N}}$ is required to be pre-simplicial (resp. N-simplicial) if $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ are pre-simplicial (resp. N-simplicial).*

3.6 "Real" Čech cohomology

Definition 3.6.1 (Real local sections)**.** *Let $(\mathfrak{F}, \sigma)$ be an abelian Real (pre-)sheaf over (X, ρ) and let $\mathcal{U} = (U_j)_{j\in J}$ be a Real open cover of (X, ρ). We say that a family $\mathsf{s}_j \in \mathfrak{F}(U_j)$ is a* globally Real family *of local sections of $(\mathfrak{F}, \sigma)$ over $\mathcal{U}$ if for every $j \in J$, $\mathsf{s}_{\bar{j}}$ is the image of s_j in $\mathfrak{F}(U_{\bar{j}})$ by σ_{U_j}.*

We define $CR_{ss}(\mathcal{U}, \mathfrak{F})_{\rho,\sigma}$ to be the set of all globally Real families of local sections of $(\mathfrak{F}, \sigma)$ relative to $\mathcal{U}$; i.e.

$$CR_{ss}(\mathcal{U}, \mathfrak{F})_{\rho,\sigma} := \left\{ (\mathsf{s}_j)_{j\in J} \subset \prod_{j\in J} \mathfrak{F}(U_j) \mid \mathsf{s}_{\bar{j}} = \sigma_{U_j}(\mathsf{s}_j),\ \forall j \in J \right\}.$$

To avoid irksome notations, we will write $CR_{ss}(\mathcal{U}, \mathfrak{F})$ or $CR_{ss}(\mathcal{U}, \mathfrak{F})_\sigma$ instead of $CR_{ss}(\mathcal{U}, \mathfrak{F})_{\rho,\sigma}$. It is clear that $CR_{ss}(\mathcal{U}, \mathfrak{F})$ is an abelian group.

Now let $(X_\bullet, \rho_\bullet)$ be a Real simplicial space, and let $\mathcal{U}_\bullet$ be a pre-simplicial Real open cover of $(X_\bullet, \rho_\bullet)$. Suppose $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is a (pre-simplicial) abelian Real (pre-)sheaf over $(X_\bullet, \rho_\bullet)$.

Definition 3.6.2. *We define the complex $CR^*_{ss}(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)_{\rho_\bullet,\sigma^\bullet}$, also denoted by $CR^*_{ss}(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$ if there is no risk of confusion, by*

$$CR^n_{ss}(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) := CR_{ss}(\mathcal{U}_n, \mathfrak{F}^n)_{\rho_n,\sigma^n},\ \text{for } n \in \mathbb{N}. \tag{3.24}$$

A Real n-cochain *of $(X_\bullet, \rho_\bullet)$ relative to a pre-ssimplicial Real open cover $\mathcal{U}_\bullet$ with coefficients in $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is an element in $CR^n_{ss}(\mathcal{U}_\bullet, \mathfrak{F}^\bullet)$.*

Let us consider again the maps $\varepsilon_k : [n] \longrightarrow [n+1]$ defined by (3.1), for $k = 0, ..., n+1$. We have Real maps $\tilde{\varepsilon}_k : (J_{n+1}, \bar{\ }) \longrightarrow (J_n, \bar{\ })$, $\tilde{\varepsilon}_k : (X_{n+1}, \rho_{n+1}) \longrightarrow (X_n, \rho_n)$, and $\tilde{\varepsilon}_k : (\mathfrak{F}^{n+1}, \sigma^{n+1}) \longrightarrow (\mathfrak{F}^n, \sigma^n)$; and since $\tilde{\varepsilon}_k(U_j^{n+1}) \subseteq U^n_{\tilde{\varepsilon}_k(j)}$ for every $j \in J_{n+1}$, we have a restriction map

$$\tilde{\varepsilon}_k^* : \mathfrak{F}^n(U^n_{\tilde{\varepsilon}(j)}) \longrightarrow \mathfrak{F}^{n+1}(U_j^{n+1})$$

such that $\sigma^{n+1}_{U_j^{n+1}} \circ \tilde{\varepsilon}_k^* = \tilde{\varepsilon}_k^* \circ \sigma^n_{U^n_{\tilde{\varepsilon}_k(j)}}$.

Definition 3.6.3. *Let* $\mathfrak{U}_\bullet$ *be a pre-simplicial Real open cover of* $(X_\bullet, \rho_\bullet)$. *For* $n \geq 0$, *we define the* differential map

$$d^n : CR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^{n+1}_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \tag{3.25}$$

also denoted by d, *by setting for* $c = (c_j)_{j \in J_n} \in CR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ *and for* $j \in J_{n+1}$*:*

$$(dc)_j := \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^* (c_{\tilde{\varepsilon}_k(j)}). \tag{3.26}$$

Remark 3.6.4. *The differential* d *of* (3.26) *do maps* $CR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ *to* $CR^{n+1}_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$*; indeed, combining the fact that the* $\tilde{\varepsilon}_k$ *are Real maps and the discussion preceeding the last definition, one has*

$$(dc)_{\bar{j}} = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^* (c_{\tilde{\varepsilon}_k(\bar{j})}) = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^* (\sigma^n_{U^n_{\tilde{\varepsilon}_k(j)}} c_{\tilde{\varepsilon}_k(j)}) = \sigma_{U^{n+1}_j}((dc)_j).$$

Lemma 3.6.5. *The differential maps* d *are group homomorphisms that satisfy* $d^n \circ d^{n-1} = 0$ *for* $n \geq 1$.

Proof. That for any $n \in \mathbb{N}$, d^n is a group homomorphism is straightforward. Let $(c_{j'})_{j' \in J_{n-1}} \in CR^{n-1}_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$. Then, for $j \in J_{n+1}$ one has

$$\begin{aligned}
(d^n d^{n-1} c)_j &= \sum_{l=0}^{n+1} (-1)^l (\tilde{\varepsilon}^{n+1}_l)^* \left(\sum_{k=0}^{n} (-1)^k (\tilde{\varepsilon}^n_k)^* (c_{\tilde{\varepsilon}^n_k \circ \tilde{\varepsilon}^{n+1}_l (j)}) \right) \\
&= \sum_{l=0}^{n+1} \sum_{k=0}^{n} (-1)^{l+k} (\tilde{\varepsilon}^{n+1}_l)^* \circ (\tilde{\varepsilon}^n_k)^* (c_{\tilde{\varepsilon}^n_k \circ \tilde{\varepsilon}^{n+1}_l (j)}) \\
&= \sum_{p=0}^{n} \sum_{k=0, k \leq 2p}^{n} (\tilde{\varepsilon}^{n+1}_{2p-k})^* (\tilde{\varepsilon}^n_k)^* (c_{\tilde{\varepsilon}^n_k \circ \tilde{\varepsilon}^{n+1}_{2p-k} (j)}) \\
&- \sum_{p=0}^{n} \sum_{k=0, k \leq 2p+1}^{n} (\tilde{\varepsilon}^{n+1}_{2p+1-k})^* \circ (\tilde{\varepsilon}^n_k)^* (c_{\tilde{\varepsilon}^n_k \circ \tilde{\varepsilon}^{n+1}_{2p+1-k} (j)}) \\
&= 0, \text{since } \varepsilon^{n+1}_r \circ \varepsilon^n_q = \varepsilon^{n+1}_{r+1} \circ \varepsilon^n_q, \text{ for any } r, q \leq n.
\end{aligned}$$

∎

We thus can give the following

Definition 3.6.6. *A Real* n*-cochain* c *in the kernel of* d^n *is called a* Real n-cocycle *relative to the pre-simplicial Real open cover* $\mathfrak{U}_\bullet$ *with coefficients in* $(\mathfrak{F}^\bullet, \sigma^\bullet)$*; the Real* n*-cocyles form a subgroup* $ZR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ *of* $CR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$. *The Real* n*-cochains belonging to the image of* d^{n-1} *are called* Real n-coboundaries *relative to* $\mathfrak{U}_\bullet$ *and form a subgroup* $BR^n_{ss}(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ *(since*

$d^2=0$). The n^{th} Real cohomology group *of the pre-simplicial Real open cover $\mathfrak{U}_\bullet$ with coefficients in $(\mathfrak{F}^\bullet,\sigma^\bullet)$ is defined by the n^{th} cohomology group of the complex*

$$\ldots \xrightarrow{d^{n-2}} CR^{n-1}_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet) \xrightarrow{d^{n-1}} CR^{n}_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet) \xrightarrow{d^{n}} CR^{n+1}_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet) \xrightarrow{d^{n+1}} \ldots$$

That is,

$$HR^n_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet) := \frac{ZR^n_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)}{BR^n_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)} := \frac{\ker d^n}{\operatorname{Im} d^{n-1}}.$$

Example 3.6.7. *(Compare with [88, Example 4.3]). Let $(X_\bullet,\rho_\bullet)$ be the constant Real simplicial space associated with a topological Real space (X,ρ); that is $(X_n,\rho_n)=(X,\rho)$ for every $n\geq 0$. Suppose $\mathfrak{U}=:\mathfrak{U}_0=(U^0_j)_{j\in J_0}$ is a Real open cover of (X,ρ). Define $J_n:=J_0^{n+1}$ together with the obvious Real structure. Then $(J_n,\,^-)$ is admits a simplicial structure by*

$$\tilde{f}(j_0,\ldots,j_n) := (j_{f(0)},\ldots,j_{f(m)}),\ \textit{for all } f\in \mathrm{Hom}_\Delta([m],[n]).$$

Let $U^n_{(j_0,\ldots,j_n)} := U^0_{j_0}\cap\ldots\cap U^0_{j_n}$ and $\mathfrak{U}_n=(U^n_j)_{j\in J_n}$. Of course $\mathfrak{U}_n$ is a Real open cover of (X_n,ρ_n), and for any $f\in \mathrm{Hom}_\Delta([m],[n])$ one has $\tilde{f}(U^n_{(j_0,\ldots,j_n)}) = U^n_{(j_0,\ldots,j_n)} \subseteq U^0_{f(0)}\cap\ldots\cap U^0_{f(m)} = U^m_{\tilde{f}(j_0,\ldots,j_n)}$; hence $\mathfrak{U}_\bullet$ is a simplicial Real open cover of $(X_\bullet,\rho_\bullet)$.
*Let $(\mathcal{F},\sigma)$ be an Abelian Real sheaf on (X,ρ) and let $(\mathfrak{F}^n,\sigma^n):=(\mathfrak{F},\sigma)$ for all $n\geq 0$. Then, $HR^*_{ss}(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)$ can be viewed as the "Real" analogue of the usual (*i.e.*, when all the Real structures are trivial) cohomology group $H^*(\mathfrak{U}_0,\mathfrak{F})$ and is denoted by $HR^*(\mathfrak{U},\mathfrak{F})$. A Real 0-cochain is a globally Real family $(\mathsf{s}_j)_{j\in J}$ of local sections. Given such a family, the differential d^0 gives: $(d^0\mathsf{s})_{(j_0,j_1)} = \mathsf{s}_{j_1|U_{j_0j_1}} - \mathsf{s}_{j_0|U_{j_0j_1}}$; it hence defines a Real 0-cocycle if there exists a Real global section $f\in\Gamma(X,\mathfrak{F})$ such that $\mathsf{s}_j=f_{U_j}$ for all $j\in J$.*

A Real 1-coboundary is then a family $(c_{j_0j_1})_{j_0,j_1\in J}$ of sections $c_{j_0j_1}\in\mathfrak{F}(U_{j_0j_1})\cong\Gamma(U_{j_0j_1},\mathcal{F})$ verifying $c_{\bar{j}_0\bar{j}_1}(\rho(x))=\sigma(c_{j_0j_1}(x))$ for every $x\in U_{j_0j_1}$, and such that there exists a globally Real family $(\mathsf{s}_j)_{j\in J}$ of sections $\mathsf{s}_j\in\Gamma(U_j,\mathcal{F})$ such that $c_{j_0j_1}=\mathsf{s}_{j_1}-\mathsf{s}_{j_0}$ over all non-empty intersection $U_{j_0j_1}$.

Finally, a Real 1-cochain $c=(c_{j_0j_1})\in CR^1_{ss}(\mathfrak{U},\mathfrak{F})$ can be seen as a family of sections $c_{j_0j_1}\in\Gamma(U_{j_0j_1},\mathcal{F})$ satisfying $c_{\bar{j}_0\bar{j}_1}(\rho(x))=\sigma(c_{j_0j_1}(x))$. Such a cocyle is 1-cocyle if and only if one has $(dc)_{j_0j_1j_2}=0$ for all $j_0,j_1,j_2\in J$; in other words, $c_{j_0j_1}+c_{j_1j_2}=c_{j_0j_2}$ over all non-empty intersection $U_{j_0j_1j_2}$.

We can apply Lemma 3.5.2 to generalize the definition of the Real cohomology groups relative to pre-simplicial Real open covers to arbitrary Real open covers of $(X_\bullet,\rho_\bullet)$.

Definition 3.6.8. *Let $(X_\bullet,\rho_\bullet)$ be a Real (pre-)simplicial space and let $(\mathfrak{F}^\bullet,\sigma^\bullet)\in Ob(\mathsf{Sh}_{\rho_\bullet}(X_\bullet))$. For any Real open cover $\mathfrak{U}_\bullet$ of $(X_\bullet,\mathfrak{F}^\bullet)$, we let*

$$CR^*(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet) := CR^*_{ss}(\natural\mathfrak{U}_\bullet,\mathfrak{F}^\bullet), \tag{3.27}$$

and we define the Real cohomology *groups of* $\mathfrak{U}_\bullet$ *with coefficients in* $(\mathfrak{F}^\bullet,\sigma^\bullet)$ *by*

$$HR^*(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet) := HR^*_{ss}(\natural\mathfrak{U}_\bullet,\mathfrak{F}^\bullet). \tag{3.28}$$

We head now toward the definition of the *Real Čech cohomology*; roughly speaking, given an Abelian Real (pre-)sheaf $(\mathfrak{F}^\bullet,\sigma^\bullet)$ over a Real simplicial space $(X_\bullet,\rho_\bullet)$, we want to define the Real cohomology groups $HR^n(X_\bullet,\mathfrak{F}^\bullet)$ as the inductive limit of the groups $HR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)$ over some category of Real open covers of $(X_\bullet,\rho_\bullet)$. To do this, we need some preliminaries elements.

Lemma 3.6.9. *Let* $(X_\bullet,\rho_\bullet)$ *and* $(\mathfrak{F}^\bullet,\sigma^\bullet)$ *be as above. Assume* $\mathfrak{U}_\bullet$ *and* $\mathcal{V}_\bullet$ *are Real open covers of* $(X_\bullet,\rho_\bullet)$, *with* $\mathfrak{U}_n=(U^n_j)_{j\in J_n}$ *and* $\mathcal{V}_n=(V^n_i)_{i\in I_n}$. *Then all refinements* $\theta_\bullet:(I_\bullet,{}^-)\longrightarrow(J_\bullet,{}^-)$ *induces group homomorphisms*

$$\theta^*_n : HR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)\longrightarrow HR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet). \tag{3.29}$$

Proof. In virtue of Lemma 3.5.2, one can assume that $\mathfrak{U}_\bullet$ and $\mathcal{V}_\bullet$ are pre-simplicial, and so that $\theta_\bullet$ is a pre-simplicial Real map. Define $\theta^*_n : CR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)\longrightarrow CR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$ as follows: for any $c=(c_j)_{j\in J_n}\in CR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)$, we put

$$(\theta^*_n c)_i := c_{\theta_n(i)|V^n_i};$$

i.e. $(\theta^*_n c)_i$ is the image of $c_{\theta_n(i)}$ by the canonical restriction $\mathfrak{F}^n(U^n_{\theta_n(i)})\longrightarrow\mathfrak{F}^n(V^n_i)$. A straightforward calculation shows that this well defines an element in $CR^n(\mathcal{V}_\bullet,\mathfrak{F}^bullet)$. Moreover, it is clear that θ^*_n is a group homomorphism for any n. Moreover, since $\theta_\bullet$ is pre-simplicial, $\tilde{\varepsilon}_k\circ\theta_{n+1}=\theta_n\circ\tilde{\varepsilon}_k$. Then, for $i\in I_{n+1}$, one has

$$(d\theta^*_n(c))_i=\sum_{k=0}^{n+1}(-1)^k\tilde{\varepsilon}^*_k(c_{\theta_n\circ\tilde{\varepsilon}_k(i)|V^n_{\tilde{\varepsilon}_k(i)}})=\sum_{k=0}^{n+1}(-1)^k\tilde{\varepsilon}^*_k(c_{\tilde{\varepsilon}_k\circ\theta_{n+1}(i)})_{|V^{n+1}_i}=(\theta^*_{n+1}d(c)),$$

then $d^n\circ\theta^*_n=\theta^*_{n+1}\circ d^n$ for all $n\in\mathbb{N}$. It turns out that θ^*_n maps $ZR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)$ into $ZR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$ and maps $BR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)$ into $BR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$. Consequently, θ^*_n passes through the quotients: $\theta^*_n([c]):=[\theta^*_n(c)]$, for $c\in ZR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)$. □

As noted in [88], the map $HR^*(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)\longrightarrow HR^*(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$ may depends on the choice of the given refinement.

Definition 3.6.10. *Let* $(X_\bullet,\rho_\bullet)$ *and* $(\mathfrak{F}^\bullet,\sigma^\bullet)$ *be as previously. Let* $\mathfrak{U}_\bullet$ *and* $\mathcal{V}_\bullet$ *be Real open covers of* $(X_\bullet,\rho_\bullet)$. *Let* $\phi_n,\psi_n : CR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)\longrightarrow CR^n(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$ *be two families of group homomorphisms commuting with* d. *We say that* $(\phi_n)_{n\in\mathbb{N}}$ *and* $(\psi_n)_{n\in\mathbb{N}}$ *are* equivalent *(resp.* N-equivalent, *for a given* $N\in\mathbb{N}$ *such that the* N*-keleton of* $\mathcal{V}_\bullet$ *admits an* N*-simplicial Real structure) if for all* $n\in\mathbb{N}$ *(resp. for all* $n\le N$*), there exists a group homomorphism* $h^n : CR^n(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)\longrightarrow CR^{n-1}(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)$, *with the convention that* $CR^{-1}(\mathcal{V}_\bullet,\mathfrak{F}^\bullet)=\{0\}$ *(and* $h^{N+1}=h^N$ *in case of* N*-equivalence), such that*

$$\phi_n-\psi_n=d^{n-1}\circ h^n+h^{n+1}\circ d^n,\ \forall n\in\mathbb{N}\ (\textit{resp.}\ \forall n\le N). \tag{3.30}$$

Observe that such N-equivalent families $\phi_\bullet$ and $\psi_\bullet$ induces group homomorphisms

$$HR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet),$$

also denoted by ϕ_n and ψ_n respectively, and given by $\phi_n([c]) := [\phi_n(c)]$, and $\psi_n([c]) := [\psi_n(c)]$ for all $c \in ZR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$. Assume $h^n : CR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^{n-1}(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ is such that (3.30) holds for all $n \leq N$, then for all $c \in ZR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$, one has

$$(\phi_n - \psi_n)([c]) = [d^{n-1}(h^n c)] + [h^{n+1}(d^n c)] = 0;$$

in other words, ϕ_n and ψ_n define the same homomorphism from $HR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)$ to $HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ when $n \leq N$.
It is clear that $(N$-)equivalence of morphisms $\phi_n : CR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow CR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$ is an equivalence relation. We also denote by $\phi_\bullet$ for the $(N$-)class of $\phi_\bullet$.

Definition 3.6.11. *Denote by $\mathfrak{N}$ the collection of all Real open covers of $(X_\bullet, \rho_\bullet)$. Let $\mathfrak{U}_\bullet$, $\mathcal{V}_\bullet \in \mathfrak{N}$. We say that $\mathcal{V}_\bullet$ is h-*finer *than $\mathfrak{U}_\bullet$ if $\mathcal{V}_\bullet$ is finer than $\mathfrak{U}\bullet$ in the sense of Definition 3.5.3, and if there exists $N \in \mathbb{N}$ such that the N-skeleton of $\mathcal{V}_\bullet$ admits an N-simplicial Real strucutre. In this case, we will write $\mathfrak{U}_\bullet \preceq_N \mathcal{V}_\bullet$ or $\mathfrak{U}_\bullet \preceq_h \mathcal{V}_\bullet$.*

We refer to [88, Lemma 4.5]) for the proof of the following

Lemma 3.6.12. *Let $\mathfrak{U}_\bullet$ and $\mathcal{V}_\bullet$ be Real open covers of $(X_\bullet, \rho_\bullet)$ such that $\mathfrak{U}_\bullet \preceq_N \mathcal{V}_\bullet$. If $\theta_\bullet, \theta'_\bullet : (I_\bullet, ^-) \longrightarrow (J_\bullet, ^-)$ are two arbitrary refinements, then their induced group homomorphisms $\theta^*_\bullet$ and $(\theta'_\bullet)^*$ are N-equivalent. Consequently, there is a canonical morphism*

$$HR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$$

for each $n \leq N$.

Example 3.6.13. *By Lemma 3.5.2, from anyy Real open cover $\mathfrak{U}_\bullet$ of $(X_\bullet, \rho_\bullet)$ and anyy $N \in \mathbb{N}$, one can form an N-simplicial Real open cover $_{\natural^N}\mathfrak{U}_\bullet$ of the N-skeleton of $(X_\bullet, \rho_\bullet)$. Next, we define a new Real open cover $_\natural\mathfrak{U}^N_\bullet$ by setting*

$$_\natural\mathfrak{U}^N_n := \begin{cases} _{\natural^N}\mathfrak{U}_n, & \text{if } n \leq N \\ \mathfrak{U}_n, & \text{if } n \geq N+1 \end{cases} \tag{3.31}$$

It is clear that the N-skeleton of $_\natural\mathfrak{U}^N_\bullet$ admits an N-simplicial Real structure. Recall that $_\natural\mathfrak{U}^N_\bullet$ is indexed by $I_\bullet$, with $I_n = \Lambda^N_n$ if $n \leq N$ and $I_n = J_n$ if $n \geq N+1$. Now we get a refinement $_N\theta_\bullet : (I_\bullet, ^-) \longrightarrow (J_\bullet, ^-)$ by setting

$$_N\theta_n := \begin{cases} \Lambda^N_n \longrightarrow J_n,\ \lambda \longmapsto \lambda(\mathrm{Id}_{[n]}), & \text{if } n \leq N \\ \mathrm{Id} : J_n \longrightarrow J_n, & \text{if } n \geq N+1 \end{cases} \tag{3.32}$$

hence $\mathfrak{U}_\bullet \preceq_N {}_\natural\mathfrak{U}^N_\bullet$ for all $N \in \mathbb{N}$. In particular, $\mathfrak{U}_\bullet \preceq_0 \mathfrak{U}_\bullet$.

We deduce from the example above that "$\preceq_h$" is a pre-order in the collection $\mathfrak{N}$. Suppose that $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet \preceq_h \mathcal{W}$ and $K_\bullet \xrightarrow{\theta'_\bullet} I_\bullet \xrightarrow{\theta_\bullet} J_\bullet$ are refinements. Then it is easy to check that the maps $\theta^*_\bullet$ and $(\theta'_\bullet)^*$ defined by (3.29) verify the relation $(\theta_n \circ \theta'_n)^* = (\theta'_n)^* \circ \theta^*_n$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we denote by $\mathfrak{N}(n)$ the collection of all elements $\mathcal{U}_\bullet \in \mathfrak{N}$ such that $\mathcal{U}_\bullet \preceq_N \mathcal{U}_\bullet$ for some $N \geq n+1$; *i.e.*, $\mathcal{U}_\bullet \in \mathfrak{N}(n)$ if there is $N \geq n+1$ such that the N-skeleton of $\mathcal{U}_\bullet$ admits an N-simplicial Real structure. It is obvious that "$\preceq_h$" is also a preorder in $\mathfrak{N}(n)$. Furthermore, Lemma 3.6.12, states that if $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet$ in $\mathfrak{N}(n)$, there is a canonical map $HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n(\mathcal{V}_\bullet, \mathfrak{F}^\bullet)$. It follows that for all $n \in \mathbb{N}$, the collection

$$\left\{ HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \mid \mathcal{U}_\bullet \in \mathfrak{N}(n) \right\}$$

is a directed system of groups; this allows us to give the following definition.

Definition 3.6.14. *We define the n^{th}* Čech cohomology group *of $(X_\bullet, \rho_\bullet)$ with coefficients in $(\mathfrak{F}^\bullet, \sigma^\bullet)$ to be the direct limit*

$$\check{H}R^n(X_\bullet, \mathfrak{F}^\bullet) := \varinjlim_{\mathcal{U}_\bullet \in \mathfrak{N}(n)} HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet). \tag{3.33}$$

Lemma 3.6.15. *For every $\mathcal{U}_\bullet \in \mathfrak{N}$, pre-simplicial or not, there is a canonical group homomorphism*

$$\theta_{\mathcal{U}_\bullet} : HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow \check{H}R^n(X_\bullet, \mathfrak{F}^\bullet),$$

for all $n \in \mathbb{N}$.

Proof. For every $\mathcal{U}_\bullet \in \mathfrak{N}$ (simplicial or not), and for every $n \in \mathbb{N}$, we define the map $\theta_{\mathcal{U}_\bullet}$: $HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow \check{H}R^n(X_\bullet, \mathfrak{F}^\bullet)$ by composing the canonical homomorphism

$${}_N\theta^*_n : HR^n(\mathcal{U}_\bullet, \mathfrak{F}^\bullet) \longrightarrow HR^n({}_\natural\mathcal{U}^N_\bullet, \mathfrak{F}^\bullet)$$

with the canonical projection

$$p^N_{\mathcal{U}_\bullet} : HR^n({}_\natural\mathcal{U}^N_\bullet, \mathfrak{F}^\bullet) \longrightarrow \check{H}R^n(X_\bullet, \mathfrak{F}^\bullet),$$

for some $N \geq n+1$; *i.e.* $\theta_{\mathcal{U}_\bullet} = p^N_{\mathcal{U}_\bullet} \circ {}_N\theta^*_n$ (recall that ${}_N\theta_n$ is defined by (3.32)). □

3.7 Comparison with usual groupoid cohomologies

In this section we compare our cohomology with the usual cohomology theory in some special cases, especially with that developed in [88].

Proposition 3.7.1. *Suppose* $\mathbf{S}$ *is an Abelian Real group. Let* ${}^r\mathbf{S}$ *be the fixed point subgroup of* $\mathbf{S}$*. Let* $(\mathcal{G},\rho)$ *be a Real groupoid. Then if* ρ *is trivial , we have*

$$\check{H}R^*(\mathcal{G}_\bullet,\mathbf{S}) = \check{H}^*(\mathcal{G}_\bullet,{}^r\mathbf{S}).$$

In particular, if $\mathbf{S}$ *has no non-trivial fixed point, we have* $\check{H}R^*(\mathcal{G}_\bullet,\mathbf{S}) = 0$.

Proof. Let $(c_\lambda) \in ZR^n(\mathfrak{U}_\bullet,\mathbf{S})$. Since $\rho = \mathrm{Id}$, we may take the involution on $J_\bullet$ to be trivial. For every $\overrightarrow{g} \in U^n_\lambda$, we have

$$c_\lambda(\overrightarrow{g}) = c_\lambda(\overline{\overrightarrow{g}}) = \overline{c_\lambda(\overrightarrow{g})} \in {}^r\mathbf{S}.$$

Thus $c_\lambda \in ZR^n(\mathfrak{U}_\bullet,{}^r\mathbf{S})$.

Conversely, we obviously have $\check{H}^n(\mathcal{G}_\bullet,{}^r\mathbf{S}) \subset \check{H}R^n(\mathcal{G}_\bullet,\mathbf{S})$ since ρ is trivial. □

Corollary 3.7.2. *If* ρ *and the Real structure of* $\mathbf{S}$ *are trivial, then* $\check{H}^*(\mathcal{G}_\bullet,\mathbf{S}) = \check{H}^*(\mathcal{G}_\bullet,\mathbf{S})$.

Let us focus now on the case where $\mathcal{G}$ reduces to a Real space (X,τ) and $\mathbf{S} = \mathbb{Z}^{0,1}$. Then τ induces an action of $\mathbb{Z}_2$ on X by $(-1)\cdot x := \tau(x), (+1)\cdot x := x$.

Proposition 3.7.3. *(i)* $\check{H}R^*(X,\mathbb{Z}^{0,1}) \cong \check{H}^*_{(\mathbb{Z}_2,-)}(X,\mathbb{Z})$*, where the sign "−" stands for the* $\mathbb{Z}_2$*-equivariant cohomology with respect to the action of* $\mathbb{Z}_2$ *on* $\mathbb{Z}$ *given by* $(-1)\cdot n := -n, (+1)\cdot n := n$.

(ii) $\check{H}^*(X,\mathbb{Z}) \cong_{\mathbb{Q}} \check{H}^*_{(\mathbb{Z}_2,-)}(X,\mathbb{Z}) \oplus \check{H}^*_{(\mathbb{Z}_2,+)}(X,\mathbb{Z})$*, where the sign "+" means the trivial* $\mathbb{Z}_2$*-action on* $\mathbb{Z}$.

Proof. (i) Let $c \in \check{H}R^n(X,\mathbb{Z}^{0,1})$ be represented on the Real open cover (U_j) of X. Then $c_{\bar{j}_0\dots\bar{j}_n}(\tau(x)) = -c_{j_0\dots j_n}(x)$ implies $\tau^* c_{j_0\dots j_n}(x) = -c_{j_0\dots j_n}(x), \forall x \in X$; in other words, c is $\mathbb{Z}_2$-equivariant with respect to the $\mathbb{Z}_2$-action "−" on $\mathbb{Z}$. The converse is easy to check.

(ii) We define the involution $\tilde{\tau}$ on $\check{H}^n(X,\mathbb{Z})$ by $\tilde{\tau}(c) := -\tau^* c$. Then it is straightforward that the Real part ${}^r\check{H}^n(X,\mathbb{Z}) \cong \check{H}R^n(X,\mathbb{Z}^{0,1})$, while the imaginary part ${}^{\mathcal{I}}\check{H}^n(X,\mathbb{Z})$ is exactly $\check{H}^n_{(\mathbb{Z}_2,+)}(X,\mathbb{Z})$.

□

3.8 Long exact sequences

Let $(\mathfrak{F}^\bullet,\sigma^\bullet)$ and $(\mathfrak{G}^\bullet,\varsigma^\bullet)$ be Abelian Real sheaves on a Real simplicial space $(X_\bullet,\rho_\bullet)$. Suppose that $\phi_\bullet = (\phi_n)_{n\in\mathbb{N}} : (\mathfrak{F}^\bullet,\sigma^\bullet) \longrightarrow (\mathfrak{G}^\bullet,\varsigma^\bullet)$ is a morphism of Abelian Real (pre)sheaves, and that $\mathfrak{U}_\bullet$ is a Real open cover of $(X_\bullet,\rho_\bullet)$. Consider the pre-simplicial Real open cover

${}_{\natural}\mathfrak{U}_\bullet$ associated to $\mathfrak{U}_\bullet$. Then for any $n \in \mathbb{N}$, and any $\lambda \in \Lambda_n$, there is a morphism of Abelian groups

$$\tilde{\phi}_n : \mathfrak{F}^n(U^n_\lambda) \longrightarrow \mathfrak{G}^n(U^n_\lambda), \mathsf{s}_\lambda \longmapsto \phi_{n|U^n_\lambda}(\mathsf{s}_\lambda), \tag{3.34}$$

satisfying $\varsigma^n_{U^n_\lambda} \circ \tilde{\phi}_n = \tilde{\phi}_n \circ \sigma_{U^n_{\bar{\lambda}}}$. This gives a group homomorphism

$$\tilde{\phi}_n : CR^n_{ss}({}_{\natural}\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \longrightarrow CR^n_{ss}({}_{\natural}\mathfrak{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet}.$$

Moreover, for any $\lambda \in \Lambda_{n+1}$ and any $k \in [n+1]$, one has a commutative diagram

$$\begin{array}{ccc}
\mathfrak{F}^n(U^n_{\tilde{\varepsilon}_k(\lambda)}) & \xrightarrow{\phi_{n|U^n_{\tilde{\varepsilon}_k(\lambda)}}} & \mathfrak{G}^n(U^n_{\tilde{\varepsilon}_k(\lambda)}) \\
{\scriptstyle \tilde{\varepsilon}^*_k}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}^*_k} \\
\mathfrak{F}^{n+1}(U^{n+1}_\lambda) & \xrightarrow{\phi_{n+1|U^{n+1}_\lambda}} & \mathfrak{G}^{n+1}(U^{n+1}_\lambda)
\end{array}$$

Thus, $d^n \circ \tilde{\phi}_n = \tilde{\phi}_{n+1} \circ d^n$; i.e. one has a commutative diagram

$$\begin{array}{ccc}
CR^n_{ss}({}_{\natural}\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} & \xrightarrow{d^n} & CR^{n+1}_{ss}({}_{\natural}\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \\
\downarrow{\scriptstyle \tilde{\phi}_n} & & \downarrow{\scriptstyle \tilde{\phi}_{n+1}} \\
CR^n_{ss}({}_{\natural}\mathfrak{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet} & \xrightarrow{d^n} & CR^{n+1}_{ss}({}_{\natural}\mathfrak{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet}
\end{array} \tag{3.35}$$

that shows that ϕ gives rise to a homomorphism of Abelian groups

$$(\phi_{\mathfrak{U}_\bullet})_* : HR^n(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \longrightarrow HR^n(\mathfrak{U}_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet},\ [c] \longmapsto [\tilde{\phi}_n(c)]; \tag{3.36}$$

and therefore a group homomorphism $\phi_* : \check{H}R^n(X_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \longrightarrow \check{H}R^n(X_\bullet, \mathfrak{G}^\bullet)_{\varsigma^\bullet}$ defined in the obvious way. We thus have shown that $\check{H}R^*$ is functorial in the category $\mathsf{Sh}_{\rho_\bullet}(X_\bullet)$.

Proposition 3.8.1. *Suppose $(X_\bullet, \rho_\bullet)$ is a Real simplicial space such that each X_n is paracompact. If*

$$0 \longrightarrow (\mathfrak{F}'^\bullet, \sigma'^\bullet) \xrightarrow{\phi'_\bullet} (\mathfrak{F}^\bullet, \sigma^\bullet) \xrightarrow{\phi_\bullet} (\mathfrak{F}''^\bullet, \sigma''^\bullet) \longrightarrow 0$$

is an exact sequence of Real (pre-)sheaves over $(X_\bullet, \rho_\bullet)$, then there is a long exact sequence of Abelian groups

$$0 \longrightarrow \check{H}R^0(X_\bullet, \mathfrak{F}'^\bullet) \xrightarrow{\phi'_*} \check{H}R^0(X_\bullet, \mathfrak{F}^\bullet) \xrightarrow{\phi_*} \check{H}R^0(X_\bullet, \mathfrak{F}''^\bullet) \xrightarrow{\partial} \check{H}R^1(X_\bullet, \mathfrak{F}'^\bullet) \xrightarrow{\phi'_*} \cdots$$

The proof of this proposition is almost the same as in [88, §4].

3.9 The group $\check{H}R^0$

We shall recall the notations of [88, Section 4] that we will use throughout the rest of the chapter. Let $\mathfrak{U}_\bullet$ be a Real open cover of a Real simplicial space $(X_\bullet, \rho_\bullet)$ and let $_\natural\mathfrak{U}_\bullet$ be its associated pre-simplicial Real open cover. Recall that any $\varphi \in \mathcal{P}_n^k$ is represented by its image in $[n]$; *i.e.* $\varphi = \{\varphi(0), ..., \varphi(k)\}$. Then $\mathcal{P}_n$ is nothing but the collection of all non empty subsets of $[n]$. Henceforth, any subset $S = \{i_0, ..., i_k\} \subseteq [n]$, with $i_0 \le ... \le i_k$, designates the maps $\varphi \in \mathcal{P}_n^k$ such that $\varphi(0) = i_0, ..., \varphi(k) = i_k$.

Notations 3.9.1. *With the above observations, any element $\lambda \in \Lambda_n$ is represented by a $(2^{n+1} - 1) - tuple$ $(\lambda_S)_{\emptyset \neq S \subseteq [n]}$, where the subsets S are ordered first by cardinality, then by lexicographic order;* i.e.

$$S \in \{\{0\}, ..., \{n\}, \{0,1\}, ..., \{0,n\}, \{1,2\}, ..., \{1,n\}, ..., \{0,1,2\}, ...\{0,1,n\}, ..., \{0, ..., n\}\},$$

and $\lambda_S := \lambda(S)$. For instance, any element $\lambda \in \Lambda_1$ is represented by a triple $(\lambda_0, \lambda_1, \lambda_{01})$, with $\lambda_0 = \lambda(\{0\})$, $\lambda_1 = \lambda(\{1\})$ and $\lambda_{01} = \lambda(\{0,1\})$.

Recall that if $(\mathfrak{F}^\bullet, \sigma^\bullet)$ is an abelian Real sheaf over $(X_\bullet, \rho_\bullet)$, we are given two "restriction" maps on the space of global Real sections $\tilde{\varepsilon}_0^*, \tilde{\varepsilon}_1^* : \mathfrak{F}^0(X_0)_{\sigma^0} \longrightarrow \mathfrak{F}^1(X_1)_{\sigma^1}$. Let us set

$$\Gamma_{\rm inv}(\mathfrak{F}^\bullet)_{\sigma^\bullet} := \ker\left(\mathfrak{F}^0(X_0)_{\sigma^0} \overset{\tilde{\varepsilon}_0^*}{\underset{\tilde{\varepsilon}_1^*}{\rightrightarrows}} \mathfrak{F}^1(X_1)_{\sigma^1} \right) = \left\{ \mathsf{s} \in \mathfrak{F}^0(X_0)_{\sigma^0} \mid \tilde{\varepsilon}_0^*(\mathsf{s}) = \tilde{\varepsilon}_1^*(\mathsf{s}) \right\}.$$

Proposition 3.9.2. *([88, Proposition 5.1]) Let $(\mathfrak{F}^\bullet, \sigma^\bullet)$ be an abelian Real sheaf over $(X_\bullet, \rho_\bullet)$ and let $\mathfrak{U}_\bullet$ be a Real open cover of $(X_\bullet, \rho_\bullet)$. Then*

$$\check{H}R^0(X_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \cong HR^0(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet} \cong \Gamma_{inv}(\mathfrak{F}^\bullet)_{\sigma^\bullet}. \tag{3.37}$$

Proof. One identifies Λ_0 with J_0. Note that $\mathcal{P}_1 = \{\varepsilon_0^1, \varepsilon_1^1, \mathrm{Id}_{[1]}\}$, and that for any $\lambda = (\lambda_0, \lambda_1, \lambda_{01})$ in Λ_1 one has $\tilde{\varepsilon}_0(\lambda) = \lambda(\varepsilon_0) = \lambda_1$, $\tilde{\varepsilon}_1(\lambda) = \lambda(\varepsilon_1) = \lambda_0$. We thus have $U_\lambda^1 = U_{\lambda_{01}}^1 \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_1}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_0}^0)$. Now, let $(\mathsf{s}_{\lambda_0})_{\lambda_0 \in J_0} \in ZR^0(\mathfrak{U}_\bullet, \mathfrak{F}^\bullet)_{\sigma^\bullet}$. Then

$$0 = (d\mathsf{s})_{(\lambda_0, \lambda_1, \lambda_{01})} = \tilde{\varepsilon}_0^*(\mathsf{s}_{\lambda_1}) - \tilde{\varepsilon}_1^*(\mathsf{s}_{\lambda_0}), \text{ on } U_\lambda^1, \tag{3.38}$$

Therefore, $\tilde{\varepsilon}_0^*(\mathsf{s}_{\lambda_1}) = \tilde{\varepsilon}_1^*(\mathsf{s}_{\lambda_0})$ on $\tilde{\varepsilon}_0^{-1}(U_{\lambda_1}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_0}^0)$, and $\tilde{\varepsilon}_0^*(\mathsf{s}_{\bar{\lambda}_1}) = \tilde{\varepsilon}_1^*(\mathsf{s}_{\bar{\lambda}_0})$ on $\tilde{\varepsilon}_0^{-1}(U_{\bar{\lambda}_1}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\bar{\lambda}_0}^0)$, for all $\lambda_0, \lambda_1 \in J_0$. Applying $\tilde{\eta}_0^*$ to both sides of the above identity, we get that $\mathsf{s}_{\lambda_0} = \mathsf{s}_{\lambda_1}$ and $\mathsf{s}_{\bar{\lambda}_0} = \mathsf{s}_{\bar{\lambda}_1}$; in other words, $\mathsf{s}_{\lambda_0} = \mathsf{s}_{\lambda_1}$ on $U_{\lambda_0}^0 \cap U_{\lambda_1}^0$ for all $\lambda_0, \lambda_0 \in J_0$. Since $(\mathfrak{F}^0, \sigma^0)$ is a Real sheaf on (X_0, ρ_0), there exists a global Real sections $\mathsf{s} \in \mathfrak{F}^0(X_0)_{\sigma^0}$ such that $\mathsf{s}_{U_{\lambda_0}^0} = \mathsf{s}_{\lambda_0}$ for all $\lambda_0 \in J_0$. Now, equation (3.38) is equivalent to $\tilde{\varepsilon}_0^*(\mathsf{s}) = \tilde{\varepsilon}_1^*(\mathsf{s})$; *i.e.*, $\mathsf{s} \in \Gamma_{\rm inv}(\mathfrak{F}^\bullet)_{\sigma^\bullet}$ and this ends the proof. □

3.10 $\check{H}R^1$ and the Real Picard group

Let us consider the same data as in the previous section. Let $\mathfrak{U}_\bullet$ be a Real open cover of $(X_\bullet,\rho_\bullet)$. For $\lambda=(\lambda_0,\lambda_1,\lambda_2,\lambda_{01},\lambda_{02},\lambda_{12},\lambda_{012})\in\Lambda_2$, one has

$$U^2_\lambda=\tilde{\varphi}_{00}^{-1}(U^0_{\lambda_0})\cap\tilde{\varphi}_{01}^{-1}(U^0_{\lambda_1})\cap\tilde{\varphi}_{02}^{-1}(U^0_{\lambda_2})\cap\tilde{\varepsilon}_2^{-1}(U^1_{\lambda_{01}})\cap\tilde{\varepsilon}_1^{-1}(U^1_{\lambda_{02}})\cap\tilde{\varepsilon}_0^{-1}(U^1_{\lambda_{12}})\cap U^2_{\lambda_{012}}, \tag{3.39}$$

where $\varphi_{00}=\varepsilon^2_1\circ\varepsilon^1_1,\varphi_{01}=\varepsilon^2_0\circ\varepsilon^1_0$ and $\varphi_{02}=\varepsilon^2_1\circ\varepsilon^1_0$.
Let $c=(c_\lambda)_{\lambda\in\Lambda_1}\in ZR^1(\mathfrak{U}_\bullet,\mathfrak{F}^\bullet)_{\sigma^\bullet}$. Then

$$0=(dc)_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}\lambda_{012}}=\tilde{\varepsilon}_0^*c_{\lambda_1\lambda_2\lambda_{12}}-\tilde{\varepsilon}_1^*c_{\lambda_0\lambda_2\lambda_{02}}+\tilde{\varepsilon}_2^*c_{\lambda_0\lambda_1\lambda_{02}},\text{ on }U^2_\lambda, \tag{3.40}$$

and of course we get a similar identities for $(dc)_{\bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_{01}\bar{\lambda}_{02}\bar{\lambda}_{12}\bar{\lambda}_{012}}$ on $U^2_{\bar{\lambda}}$. Now applying $\tilde{\eta}_1^*$ to (3.10), we obtain

$$c_{\lambda_0\lambda_1\lambda_{01}}=c_{\lambda_0\lambda_1\lambda_{02}}-c_{\lambda_1\lambda_2\lambda_{12}}$$

on $\tilde{\varepsilon}_1^{-1}(U^0_{\lambda_0})\cap\tilde{\varepsilon}_0^{-1}(U^0_{\lambda_1})\cap\tilde{\varepsilon}_0^{-1}(U^0_{\lambda_2})\cap U^1_{\lambda_{01}}\cap U^1_{\lambda_{02}}\cap U^1_{\lambda_{12}}\cap\tilde{\eta}_1^{-1}(U^2_{\lambda_{012}})$, which means that for any $\lambda_0,\lambda_1,\lambda_{01}\in J_0$, $s_{\lambda_0\lambda_1\lambda_{01}}$ does not depends on the choice of λ_{01}. Therefore, there exists a Real family $(f_{\lambda_0\lambda_1})\in\prod_{\lambda_0,\lambda_1\in\Lambda_0}\mathfrak{F}^1\left(\tilde{\varepsilon}_1^{-1}(U^0_{\lambda_0})\cap\tilde{\varepsilon}_0^{-1}(U^0_{\lambda_1})\right)$ such that $f_{\lambda_0\lambda_1|U^1_{\lambda_0\lambda_1\lambda_{01}}}=c_{\lambda_0\lambda_1\lambda_{01}}$ for any $(\lambda_0,\lambda_1,\lambda_{01})\in\Lambda_1$. Now, the cocycle relation (3.10) becomes

$$\tilde{\varepsilon}_0^*f_{\lambda_1\lambda_2}-\tilde{\varepsilon}_1^*f_{\lambda_0\lambda_2}+\tilde{\varepsilon}_2^*f_{\lambda_0\lambda_1} \tag{3.41}$$

on $U^1_{\lambda_0\lambda_1\lambda_{01}}\cap U^1_{\lambda_{02}}\cap U^1_{\lambda_{12}}$.

Let $(\mathcal{G},\rho)$ be a locally compact Hausdorff Real groupoid. We are interested in the 1^{st} Real Čech cohomology group of $(\mathcal{G}_\bullet,\rho_\bullet)$ with coefficients in the Abelian Real sheaf $(\mathcal{S}^\bullet,\sigma^\bullet)=(\mathbf{S},\sigma)$ over $(\mathcal{G}_\bullet,\rho_\bullet)$ associated to the Real $\mathcal{G}$-module $(\mathcal{G}^{(0)}\times\mathbf{S},\rho\times{}^-)$, where $(\mathbf{S},{}^-)$ is an Abelian group endowed with the trivial $\mathcal{G}$-action. Note that in this case, for any pre-simplicial Real open cover $\mathfrak{U}_\bullet\in\mathfrak{N}(n)$ of $(\mathcal{G}_\bullet,\rho_\bullet)$, elements of the group $CR^n(\mathfrak{U}_\bullet,\mathcal{S}^\bullet)$ are of the form $(c_\lambda)_{\lambda\in\Lambda_n}$, where $c_\lambda\in\Gamma(U^n_\lambda,\mathbf{S})$ are such that $c_{\bar{\lambda}}(\rho_n(\vec{g}))=\overline{c_\lambda(\vec{g})}\in\mathbf{S}$ for any $\vec{g}\in U^n_\lambda\subset\mathcal{G}_n$.

Proposition 3.10.1. *With the above notations, the Real Čech cohomology group $\check{H}R^1(\mathcal{G}_\bullet,\mathbf{S})$ is isomorphic to the group* $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathbf{S})$ *of isomorphism classes of Real generalized homomorphisms $(\mathcal{G},\rho)\longrightarrow(\mathbf{S},{}^-)$.*

Proof. The operations in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},S)$ are defined as follows. If $(Z,\tau),(Z',\tau'):(\mathcal{G},\rho)\longrightarrow(S,{}^-)$ are Real generalized homomorphisms, their sum is

$$(Z,\tau)+(Z',\tau'):=Z\times_{\mathcal{G}^{(0)}}Z'/_\sim \tag{3.42}$$

where $(z,z')\sim(z\cdot t^{-1},z'\cdot t)$ for all $t\in\mathbf{S}$, together with the obvious Real structure $\tau\times\tau'$. The inverse of (Z,τ) is (Z^{-1},τ), where Z^{-1} is Z as a topological space, and if $\flat:Z\hookrightarrow Z^{-1}$ is the identity map, then the S-action on Z^{-1} is defined by $\flat(z)\cdot t := \flat(z\cdot t^{-1})$ and the $\mathcal{G}$-action is defined as follows: $(g,\flat(z)) \in \mathcal{G}\ltimes Z^{-1}$ if and only if $(g,z)\in\mathcal{G}\ltimes Z$, in which case we set $g\cdot\flat(z) := \flat(g\cdot z)$. Finally, the Real structure on Z^{-1} is $\tau(\flat(z)) := \flat(\tau(z))$. Then we define the sum in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},S)$ by $[Z,\tau]+[Z',\tau'] := [(Z,\tau)+(Z',\tau')]$, and we put $[Z,\tau]^{-1} := [(Z^{-1},\tau)]$. It is not hard to check that subject to these operations, $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},S)$ is an Abelian group.

Now, suppose we are given a Real open cover $\mathcal{U}_0 = (U^0_j)_{j\in J_0}$ of $(\mathcal{G}^{(0)},\rho)$ trivializing the Real generalized homomorphism $(Z,\tau):(\mathcal{G},\rho)\longrightarrow(S,\,^-)$. Let $(\mathsf{s}_j)_{j\in J_0}$ be a Real family of local sections of the S-principal Real bundle $\mathfrak{r}:(Z,\tau)\longrightarrow(\mathcal{G}^{(0)},\rho)$. Form a pre-simplicial Real open cover $\mathcal{U}_\bullet$ of the Real simplicial space $(\mathcal{G}_\bullet,\rho_\bullet)$ by setting $J_n := J_0^{n+1}$, $\mathcal{U}_n := (U^n_{(j_0,\dots,j_n)})_{(j_0,\dots,j_n)\in J_n}$, where

$$U^n_{(j_0,\dots,j_n)} := \left\{(g_1,\dots,g_n)\in\mathcal{G}_n \mid r(g_1)\in U^0_{j_0},\dots,r(g_n)\in U^0_{j_{n-1}}, s(g_n)\in U^0_{j_n}\right\}. \tag{3.43}$$

Then, for all $g\in U^1_{(j_0,j_1)}$, $\mathfrak{r}(g\cdot\mathsf{s}_{j_1}(s(g))) = r(g) = \mathfrak{r}(\mathsf{s}_{j_0}(r(g)))$; hence, there exists a unique element $c_{j_0j_1}(g)\in S$ such that $g\cdot\mathsf{s}_{j_1}(s(g)) = \mathsf{s}_{j_0}(r(g))\cdot c_{j_0j_1}(g)$. We then obtain a family of continuous functions $c_{j_0j_1}: U^1_{(j_0,j_1)}\longrightarrow S$ such that

$$g\cdot\mathsf{s}_{j_1}(s(g)) = \mathsf{s}_{j_0}(r(g))\cdot c_{j_0j_1}(g),\ \forall g\in U^1_{(j_0,j_1)}. \tag{3.44}$$

Furthermore, notice that $U^1_{(j_0,j_1)} = \tilde{\varepsilon}_0^{-1}(U^0_{j_1})\cap\tilde{\varepsilon}_1^{-1}(U^0_{j_0})$. Let $(g_1,g_2)\in U^2_{(j_0,j_1,j_2)}$. Then

$$\begin{aligned}(g_1g_2)\cdot\mathsf{s}_{j_2}(s(g_2)) &= g_1\cdot\mathsf{s}_{j_1}(r(g_2))\cdot c_{j_1j_2}(g_2) = g_1\cdot\mathsf{s}_{j_1}(s(g_1))\cdot c_{j_1j_2}(g_2)\\ &= \mathsf{s}_{j_0}(r(g_1))\cdot c_{j_0j_1}(g_1)\cdot c_{j_1j_2}(g_2);\end{aligned}$$

hence $c_{j_0j_2}(g_1g_2) = c_{j_0j_1}(g_1)\cdot c_{j_1j_2}(g_2)$. In other words,

$$\tilde{\varepsilon}_0^* c_{\tilde{\varepsilon}_0(j_0,j_1,j_2)}\cdot(\tilde{\varepsilon}_1^* c_{\tilde{\varepsilon}_1(j_0,j_1,j_2)})^{-1}\cdot\tilde{\varepsilon}_2^* c_{\tilde{\varepsilon}_2(j_0,j_1,j_2)} = 1$$

over all $U^2_{(j_0,j_1,j_2)}$. Moreover, we clearly have $c_{\bar{j}_0\bar{j}_1}(\rho(g)) = \overline{c_{j_0j_1}(g)}\in S$. This gives us a Real 1-cocycle $(c_{j_0j_1})_{(j_0,j_1)\in J_1}\in ZR^1(\mathcal{U}_\bullet,\mathcal{S}^\bullet)$.

Suppose $f:(Z,\tau)\longrightarrow(Z',\tau')$ is an isomorphism of Real generalized morphisms (see chapter 2). Up to a refinement, we can choose $\mathcal{U}_0$ in such a way that we have two Real families $(\mathsf{s}_j)_{j\in J_0}$, $(\mathsf{s}')_{j\in J_0}$ of local sections of the Real projections $\mathfrak{r}:(Z,\tau)\longrightarrow(X,\rho)$ and $\mathfrak{r}':(Z',\tau')\longrightarrow(X,\rho)$ respectively. Since for all $j\in J_0$ and $x\in U_j$, $\mathfrak{r}'(f_{U_j}(\mathsf{s}_j)(x)) = \mathfrak{r}(\mathsf{s}_j(x)) = x = \mathfrak{r}'(\mathsf{s}'_j(x))$, there exists a unique element $\varphi_j(x)\in S$ such that $\mathsf{s}'_j(x) = f_{U_j}(\mathsf{s}_j(x))\cdot\varphi_j(x)$, and this gives a Real family of continuous functions $\varphi_j: U_j\longrightarrow S$. It follows that if $c=(c_{j_0j_1})$ and $c'=(c'_{j_0j_1})$ are the Real 1-cocycle associated to (Z,τ) and (Z',τ') respectively. Then, over $U^1_{(j_0,j_1)}$, one has

$$g\cdot f_{U_{j_1}}(\mathsf{s}_{j_1}(s(g)))\cdot\varphi_{j_1} = f_{U_{j_0}}(\mathsf{s}_{j_0}(r(g)))\cdot\varphi_{j_0}(r(g))\cdot c'_{j_0j_1}(g);$$

But, since f is $\mathcal{G}$-S-equivariant, we get

$$f_{U_{j_0}(\mathsf{s}_{j_0}(r(g)))}\cdot c_{j_0j_1}(g)\cdot\varphi_{j_1}(s(g)) = f_{U_{j_0}}(\mathsf{s}_{j_0}(r(g)))\cdot\varphi_{j_0}(r(g))\cdot c'_{j_0j_1}(g);$$

thus $c'_{j_0j_1}(g)\cdot c^{-1}_{j_0j_1}(g) = \varphi_{j_1}(s(g))\cdot\varphi_{j_0}(r(g))^{-1}$, or $(c'\cdot c^{-1})_{(j_0,j_1)} = \tilde{\varepsilon}_0^*\varphi_{\tilde{\varepsilon}_0(j_0,j_1)}\cdot\tilde{\varepsilon}_1^*\varphi^{-1}_{\tilde{\varepsilon}_1(j_0,j_1)}$ for all $(j_0,j_1)\in J_1$. This shows that $c'.c^{-1}\in BR^1(\mathcal{U}_\bullet,\mathbf{S})$. We then deduce a well defined group homomorphism

$$c_1 : \mathrm{Hom}_{\mathfrak{R}\mathfrak{G}}(\mathcal{G},\mathbf{S}) \longrightarrow \check{H}R^1(\mathcal{G}_\bullet,\mathbf{S}),\ c_1([Z,\tau]) := [c_{j_0j_1}]\in HR^1(\mathcal{U}_\bullet,\mathbf{S}), \tag{3.45}$$

where $\mathcal{U}_\bullet$ is the Real open cover defined from any Real local trivialization of (Z,τ).

Conversely, given a Real Čech 1-cocycle $c=(c_{\lambda_0\lambda_1})$ over a pre-simplicial Real open cover $\mathcal{U}_\bullet\in\mathfrak{N}(1)$, we let $Z:=\coprod_{\lambda_0\in\Lambda_0}U_{\lambda_0}\times\mathbf{S}$, together with the Real structure ν defined by $\nu(x,t):=(\rho(x),\bar{t})$, and equipped with the Real $\mathcal{G}$-action $g\cdot(s(g),t):=(r(g),c_{\lambda_0\lambda_1}(g)\cdot t)$ for any $g\in U^1_{\lambda_0\lambda_1\lambda_{01}}$, $t\in S$, and the obvious Real S-action. It is easy to see that the canonical projections define a Real generalized morphism $(Z,\nu):(\mathcal{G},\rho)\longrightarrow(\mathbf{S},\,^-)$. One can check that if $[c]=[c']$ then $(Z,\tau)\cong(Z',\tau')$ by working backwards. □

Remark 3.10.2. *Suppose that $(\mathbf{S},\sigma)$ is a non-abelian Real group. Then we still can talk about Čech Real 1-cocycles on $(\mathcal{G}_\bullet,\rho_\bullet)$ with coefficients on the non-Abelian Real sheaf $(\mathcal{S}^\bullet,\sigma^\bullet)$, and then form in the same way $\check{H}R^1(\mathcal{G}_\bullet,\mathcal{S}^\bullet)$ as a set. However, there is no reason for $\check{H}R^1(\mathcal{G}_\bullet,\mathbf{S})$ to be an Abelian group, it is not even a group since the sum of a Real 1-cocycle is not necessarily a Real 1-cocycle. Nevertheless, the result above remains valid in the sense that there is a bijection between the set* $\mathrm{Hom}_{\mathfrak{R}\mathfrak{G}}(\mathcal{G},\mathbf{S})$ *of isomorphism classes of generalized Real morphism $(\mathcal{G},\rho)\longrightarrow(\mathbf{S},\sigma)$ and the set $\check{H}R^1(\mathcal{G}_\bullet,\mathbf{S})$.*
We will study in the next chapter (see Section 4.5) the case of a non abelian Real group for which this set admits an Abelian monoid structure.

A particular example of Proposition 3.10.1 is when $S=\mathbb{S}^1$ together with the complex conjugation as Real structure; in this case, the associated Real sheaf is denoted by $\mathbb{S}^1$ as mentioned earlier. It is well known that the *Picard* group Pic(X) of a locally compact topological space X is isomorphic to the 1^{st} sheaf cohomology group $H^1(X,\underline{\mathbb{S}^1}_X)$ (see for instance [19, chap.2]). In the Real case, we shall introduce the Real Picard group PicR$(\mathcal{G})$ of a Real groupoid, and we will apply Proposition 3.10.1 to get an analogous result.

Definition 3.10.3 (Real line $\mathcal{G}$-bundle)**.** 1. *By a* Real line $\mathcal{G}$-bundle *we mean a Real $\mathcal{G}$-space $(\mathcal{L},\nu)$, and a continuous surjective Real map $\pi:(\mathcal{L},\nu)\longrightarrow(\mathcal{G}^{(0)},\rho)$ such that $\pi:\mathcal{L}\longrightarrow\mathcal{G}^{(0)}$ is a complex vector bundle of rank 1, and such that for every $x\in\mathcal{G}^{(0)}$, the induced isomorphism $\nu_x:\mathcal{L}_x\longrightarrow\mathcal{L}_{\rho(x)}$ is $\mathbb{C}$-anti-linear in the sense that $\nu_x(v\cdot z)=\nu_x(v)\cdot\bar{z}$.*

2. *A homomorphism from a Real line $\mathcal{G}$-bundle $(\mathcal{L},\nu)$ to a Real line $\mathcal{G}$-bundle $(\mathcal{L}',\nu')$ is a homormophism of complex vector bundles $\phi:\mathcal{L}\longrightarrow\mathcal{L}'$ intertwining the Real structures and which is $\mathcal{G}$-equivariant;* i.e. *$\phi(g\cdot v)=g\cdot\phi(v)$ for any $(g,v)\in\mathcal{G}\ltimes\mathcal{L}$.*

3. *We say that a Real line* $\mathcal{G}$*-bundle* $(\mathcal{L}, \nu)$ *is* locally trivial *if there exists a Real open cover* $\mathcal{U}$ *of* $(\mathcal{G}^{(0)}, \rho)$*, and a family of isomorphisms of complex vector bundles* $\varphi_j : U_j \times \mathbb{C} \longrightarrow \mathcal{L}_{|U_j}$ *such that*

   - $\varphi_{\bar{j}}(\rho(x), \bar{z}) = \nu_{U_j}(\varphi_j(x,z))$ *for all* $x \in U_j$ *and* $(x,z) \in U_j \times \mathbb{C}$*,*
   - *if* $r(g) \in U_{j_0}$ *and* $s(g) \in U_{j_1}$*, then one has* $g.\varphi_{j_1}(s(g), z) = \varphi_{j_0}(r(g), z)$*.*

Example 3.10.4. *The trivial action* $\mathcal{G}$ *on* $\mathcal{G}^{(0)} \times \mathbb{C}$ *(*i.e. $g \cdot (s(g), z) := (r(g), z)$*) is Real; moreover, the canonical projection* $\mathcal{G}^{(0)} \times \mathbb{C} \longrightarrow \mathcal{G}^{(0)}$ *defines a Real line* $\mathcal{G}$*-bundle that we call* trivial.

Definition 3.10.5 (Real hermitian $\mathcal{G}$-metric)**.** *Let* $(\mathcal{L}, \nu)$ *be a locally trivial Real line* $\mathcal{G}$*-bundle. A* Real hermitian $\mathcal{G}$-metric *on* $(\mathcal{L}, \nu)$ *is a continuous function* $h : \mathcal{L} \longrightarrow \mathbb{R}_+$ *such that*

- $h(\nu(v)) = h(v)$*, and* $h(v \cdot z) = h(v) \cdot |z|^2$*, for all* $v \in \mathcal{L},\ z \in \mathbb{C}$*;*
- $h(g \cdot v) = h(v)$*, for all* $(g, v) \in \mathcal{G} \ltimes \mathcal{L}$*, and*
- $h(v) > 0$ *whenever* $v \in \mathcal{L}^+ := \mathcal{L} \setminus 0$*, where* $0 : \mathcal{G}^{(0)} \hookrightarrow \mathcal{L}$ *is the zero-section.*

If such h *exists,* $(\mathcal{L}, \nu, h)$ *is called a* hermitian Real line $\mathcal{G}$-bundle *(we will often omit the metric).*

Definition 3.10.6 (The Real Picard group)**.** *The* Real Picard group *of* $(\mathcal{G}, \rho)$ *is defined as the set of isomorphism classes of locally trivial hermitian Real line* $\mathcal{G}$*-bundles. This "group" is denoted by* $Pic\mathrm{R}(\mathcal{G})$*.*

Theorem 3.10.7. *(compare with [19, Theorem 2.1.8]). Let* $(\mathcal{G}, \rho)$ *be a locally compact Hausdorff Real groupoid. Then* $Pic\mathrm{R}(\mathcal{G})$ *is an Abelian group. Furthermore,*

$$Pic\mathrm{R}(\mathcal{G}) \cong \check{H}R^1(\mathcal{G}_\bullet, \mathcal{S}^1).$$

Proof. Associated to any hermitian Real line $\mathcal{G}$-bundle $\pi : (\mathcal{L}, \nu) \longrightarrow (\mathcal{G}^{(0)}, \rho)$, there is a Real generalized morphism $(\mathcal{L}^1, \nu) : (\mathcal{G}, \rho) \longrightarrow (\mathbb{S}^1, {}^-)$ obtained by setting

$$\mathcal{L}^1 := \left\{ v \in \mathcal{L} \mid \mathrm{h}(v) = 1 \right\}. \tag{3.46}$$

$\pi : (\mathcal{L}^1, \nu) \longrightarrow (\mathcal{G}^{(0)}, \rho)$ is indeed an $\mathbb{S}^1$-principal Real bundle, and $\mathcal{L}^1$ is invariant under the action of $\mathcal{G}$. Hence $(\mathcal{L}^1, \nu)$ is indeed a Real generalized morphism. Conversely, if $(\tilde{\mathcal{L}}, \tilde{\nu}) : (\mathcal{G}, \rho) \longrightarrow (\mathbb{S}^1, {}^-)$ is a Real generalized morphism, define $\mathcal{L} := \tilde{\mathcal{L}} \times_{\mathbb{S}^1} \mathbb{C}$, where $\mathbb{S}^1$ acts by multiplication on $\mathbb{C}$; $\nu(v, z) := (\tilde{\nu}(v), \bar{z})$, $g \cdot (v, z) := (g \cdot v, z)$ for $(g, v) \in \mathcal{G} \ltimes \tilde{\mathcal{L}}$, and $\mathrm{h}(v, z) := |z|^2$. Then $(\mathcal{L}, \nu, \mathrm{h})$ is a hermitian Real line $\mathcal{G}$-bundle. Moreover, it is not hard to check that if $(\mathcal{L}, \nu, \mathrm{h})$ and $(\mathcal{L}', \nu', \mathrm{h}')$ are isomorphic hermitian Real line $\mathcal{G}$-bundles, then their associated Real generalized homomorphisms $(\mathcal{L}^1, \nu)$ and $((\mathcal{L}')^1, \nu')$ are isomorphic. We then have a map

$$\mathrm{PicR}(\mathcal{G}) \longrightarrow H^1(\mathcal{G}, \mathbb{S}^1)_\rho,\ [(\mathcal{L}, \nu, \mathrm{h})] \longmapsto [\mathcal{L}^1, \nu] \tag{3.47}$$

which is clearly an isomorphism of Abelian groups. Now, applying Proposition 3.10.1, we get the desired result. □

3.11 Ungraded Real extensions

Let us consider the subgroup $\widehat{\mathrm{extR}}^+(\Gamma,\mathbf{S})$ of ungraded Real $\mathbf{S}$-twists of the Real groupoid Γ; *i.e.* $(\widetilde{\Gamma},\delta)\in\widehat{\mathrm{extR}}^+(\Gamma,\mathbf{S})$ if $\delta=0$. Similarly, we define the subgroup $\widehat{\mathrm{ExtR}}^+(\mathcal{G},S)$ of $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbf{S})$ of ungraded Real $\mathbf{S}$-central extensions over $\mathcal{G}$. Elements of $\widehat{\mathrm{ExtR}}^+(\mathcal{G},\mathbf{S})$ will then be denoted by pairs of the form $(\widetilde{\Gamma},\Gamma)$.

Let $\mathcal{T}=\ \mathbf{S}\longrightarrow\tilde{\mathcal{G}}\xrightarrow{\pi}\mathcal{G}[\mathcal{U}_0]\ \in\widehat{\mathrm{extR}}^+(\mathcal{G}[\mathcal{U}_0],\mathbf{S})$ be an ungraded Real S-twist, for a fixed Real open cover $\mathcal{U}_0=(U_j^0)_{j\in J_0}$. Consider again the pre-simplicial Real open cover $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet,\rho_\bullet)$ defined by (3.43). Recall that the groupoid $\mathcal{G}[\mathcal{U}_0]$ is defined by

$$\mathcal{G}[\mathcal{U}_0]=\left\{(j_0,g,j_1)\in J_0\times\mathcal{G}\times J_0\mid g\in U^1_{(j_0,j_1)}\right\}.$$

Suppose that the $\mathbf{S}$-principal Real bundle $\pi:(\tilde{\mathcal{G}},\tilde{\rho})\longrightarrow(\mathcal{G}[\mathcal{U}_0],\rho)$ admits a Real family of local continuous sections $\mathsf{s}_{j_0j_1}$ relative to the Real open cover $\mathcal{V}_1$ of $(\mathcal{G}[\mathcal{U}_0],\rho)$ given by $\mathcal{V}_1=(V^1_{(j_0,j_1)})_{(j_0,j_1)\in J_1}$, where

$$V^1_{(j_0,j_1)}:=\{j_0\}\times U^1_{(j_0,j_1)}\times\{j_1\}.$$

Then, for any $(g_1,g_2)\in U^2_{(j_0,j_1,j_2)}$, we have that

$$\begin{aligned}\pi(\mathsf{s}_{j_0j_1}(j_0,g_1,j_1)\cdot\mathsf{s}_{j_1j_2}(j_1,g_2,j_2))&=\pi(\mathsf{s}_{j_0j_1}(j_0,g_1,j_1))\cdot\pi(\mathsf{s}_{j_1j_2}(j_1,g_2,j_2))\\&=(j_0,g_1g_2,j_2)=\pi(\mathsf{s}_{j_0j_2}(j_0,g_1g_2,j_2));\end{aligned}$$

thus, there exists a unique element $\omega_{(j_0,j_1,j_2)}(g_1,g_2)\in S$ such that

$$\mathsf{s}_{j_0j_2}(j_0,g_1g_2,j_2)=\omega_{(j_0,j_1,j_2)}(g_1,g_2)\cdot\mathsf{s}_{j_0j_1}(j_0,g_1,j_1).\mathsf{s}_{j_1j_2}(j_1,g_2,j_2).\tag{3.48}$$

This provides a family of continuous functions $\omega_{(j_0,j_1,j_2)}:U^2_{(j_0,j_1,j_2)}\longrightarrow\mathbf{S}$ determined by (3.48) and that verifies clearly $\omega_{(\bar{j}_0,\bar{j}_1,\bar{j}_2)}(\rho(g_1),\rho(g_2))=\overline{\omega_{(j_0,j_1,j_2)}(g_1,g_2)},\forall(g_1,g_2)\in U^2_{(j_0,j_1,j_2)}\subset\mathcal{G}_2$. It is straightforward that the family $(\omega_{(j_0,j_1,j_2)})$ verifies the cocycle condition; hence we obtain a Real Čech 2-cocycle

$$\omega(\mathcal{T}):=(\omega_{(j_0,j_1,j_2)})_{(j_0,j_1,j_2)\in J_2}\in ZR^2(\mathcal{U}_\bullet,\mathbf{S})\tag{3.49}$$

associated to $\mathcal{T}$.

In fact, this construction generalizes for arbitrary Real open cover $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet,\rho_\bullet)$.

Lemma 3.11.1 (Compare Proposition 5.6 in [88]). *Let $(\mathcal{G},\rho)$ be a topological Real groupoid. Given a Real open cover $\mathcal{U}_\bullet$ of $(\mathcal{G}_\bullet,\rho_\bullet)$, let $\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0],\mathbf{S})$ denote the subgroup of all twists $\mathbf{S}\longrightarrow\tilde{\mathcal{G}}\xrightarrow{\pi}\mathcal{G}[\mathcal{U}_0]\ \in\widehat{\mathrm{extR}}^+(\mathcal{G}[\mathcal{U}_0],\mathbf{S})$ such that π admits a Real family of local continuous sections $\mathsf{s}_\lambda:\{\lambda_0\}\times U_\lambda\times\{\lambda_1\}\longrightarrow\tilde{\mathcal{G}}$ relative to the Real open cover*

$$\mathcal{V}_1:=(\{\lambda_0\}\times U^1_{(\lambda_0,\lambda_1,\lambda_{01})}\times\{\lambda_1\})_{(\lambda_0,\lambda_1,\lambda_{01})\in\Lambda_1}$$

of $(\mathcal{G}[\mathfrak{U}_0],\rho)$. *Then the canonical map*

$$\widehat{ext\mathrm{R}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbf{S}) \longrightarrow HR^2(\mathfrak{U}_\bullet,\mathbf{S}),\ [\mathcal{T}] \longmapsto [\omega(\mathcal{T})], \tag{3.50}$$

is a group isomorphism.

Proof. First of all, we shall prove that $\widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbf{S})$ is a subgroup of $\widehat{\mathrm{extR}}^+(\mathcal{G}[\mathfrak{U}_0],\mathbf{S})$. Let

$$\mathcal{T} = (\ \mathbf{S} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G}[\mathfrak{U}_0]\ ),\ \mathcal{T}' = (\ S \longrightarrow \tilde{\mathcal{G}}' \xrightarrow{\pi'} \mathcal{G}[\mathfrak{U}_0]\ )$$

be representatives in $\widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbf{S})$. Then their tensor product (cf. (2.8)) is

$$\mathcal{T}\hat{\otimes}\mathcal{T}' := (\ \mathbf{S} \longrightarrow \tilde{\mathcal{G}}\hat{\otimes}\tilde{\mathcal{G}}' \xrightarrow{\pi} \mathcal{G}[\mathfrak{U}_0]\ ,0),$$

where $\tilde{\mathcal{G}}\hat{\otimes}\tilde{\mathcal{G}}' = \tilde{\mathcal{G}} \times_{\mathcal{G}[\mathfrak{U}_0]} \tilde{\mathcal{G}}'/\mathbf{S}$. Let $f_\lambda : \{\lambda_0\}\times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}$ and $f'_\lambda : \{\lambda_0\}\times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}'$ be Real families of continuous local sections of π and π' respectively. Then we get a Real family of continuous local sections $\mathsf{s}_\lambda : \{\lambda_0\}\times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}\hat{\otimes}\tilde{\mathcal{G}}'$ for π by setting

$$\mathsf{s}_\lambda(\lambda_0,g,\lambda_1) := \left[(f_\lambda(\lambda_0,g,\lambda_1), f'_\lambda(\lambda_0,g,\lambda_1))\right],$$

which implies that $\mathcal{T}\hat{\otimes}\mathcal{T}' \in \widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbf{S}$.

Now let $\mathcal{T}$ be an (ungraded) Real twist of $(\mathcal{G}[\mathfrak{U}_0],\rho)$ such that π verifies the condition of the lemma. Assume that $\mathcal{T}'$ is any Real twist of $(\mathcal{G}[\mathfrak{U}_0],\rho)$ isomorphic to $\mathcal{T}$. Let $f : \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}'$ be a Real $\mathbf{S}$-equivariant isomorphism that makes the following diagram

$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[r, "\pi"] \arrow[d, "f"'] & \mathcal{G}[\mathfrak{U}_0] \\ \tilde{\mathcal{G}}' \arrow[ur, "\pi'"'] & \end{tikzcd} \tag{3.51}$$

commute. Thus, given a Real family $\mathsf{s}_\lambda : \{\lambda_0\}\times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}$, the maps $f\circ \mathsf{s}_\lambda : \{\lambda_0\}\times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{\mathcal{G}}'$ define a Real family of local continuous sections for π'; hence the class $[\mathcal{T}] \in \widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbb{S}^1)$.

Suppose we are given a representative

$$\mathcal{T} =\ \mathbf{S} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G}[\mathfrak{U}_0]$$

in $\widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbf{S})$. Recall that for $(\lambda_0,\lambda_1,\lambda_{01}) \in \Lambda_1$, $U^1_{\lambda_0\lambda_1\lambda_{01}} = U^1_{\lambda_{01}} \cap r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_1})$, and for any $\lambda = (\lambda_0,\lambda_1,\lambda_2\lambda_{01},\lambda_{02},\lambda_{12},\lambda_{012}) \in \Lambda_2$, we have from (3.39) that

$$U^2_\lambda = \tilde{\varepsilon}_1^{-1}\circ r^{-1}(U^0_{\lambda_0})\cap\tilde{\varepsilon}_0^{-1}\circ s^{-1}(U^0_{\lambda_1})\cap\tilde{\varepsilon}_1^{-1}\circ s^{-1}(U^0_{\lambda_2})\cap\tilde{\varepsilon}_2^{-1}(U^1_{\lambda_{01}})\cap\tilde{\varepsilon}^{-1}(U^1_{\lambda_{02}})\cap\tilde{\varepsilon}_0^{-1}(U^1_{\lambda_{12}})\cap U^2_{\lambda_{012}}.$$

Then, for all $(g_1,g_2) \in U^2_\lambda$, one has

- $g_1g_2 = \tilde{\varepsilon}_1(g_1,g_2) \in r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_2}) \cap U^1_{\lambda_{02}} = U^1_{\lambda_0\lambda_2\lambda_{02}}$,
- $g_1 = \tilde{\varepsilon}_2(g_1,g_2) \in U^1_{\lambda_{01}}$, $g_2 = \tilde{\varepsilon}_0(g_1,g_2) \in s^{-1}(U^0_{\lambda_1}) \cap U^1_{\lambda_{12}}$; and hence
$$g_1 \in r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_1}) \cap U^1_{\lambda_{01}} = U^1_{\lambda_0\lambda_1\lambda_{01}}, \text{ and}$$
$$g_2 \in r^{-1}(U^0_{\lambda_1}) \cap s^{-1}(U^0_{\lambda_2}) \cap U^1_{\lambda_{12}} = U^1_{\lambda_1\lambda_2\lambda_{12}}.$$

Then as in the discussion before the lemma (cf. (3.49)), there exists a Real family of functions $\omega_\lambda : U^2_\lambda \longrightarrow \mathbb{S}^1$ such that

$$\mathsf{s}_{\lambda_0\lambda_2\lambda_{02}}(\lambda_0, g_1g_2, \lambda_2) = \omega_\lambda(g_1,g_2)\cdot \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(\lambda_0,g_1,\lambda_1)\cdot \mathsf{s}_{\lambda_1\lambda_2\lambda_{12}}(\lambda_1,g_2,\lambda_2) \tag{3.52}$$

and $\omega_{\bar\lambda}(\rho(g_1),\rho(g_2)) = \overline{\omega_\lambda(g_1,g_2)}$, for all $(g_1,g_2) \in U^2_{\lambda_0\lambda_1\lambda_2\lambda_{01}\lambda_{02}\lambda_{12}\lambda_{012}}$. Moreover, it is easy to verify by a routine calculation that $(\omega_\lambda)_{\lambda\in\Lambda_2}$ verify the cocycle condition on

$$U^3_{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_{01}\lambda_{02}\lambda_{03}\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{0123}} \subset \mathcal{G}_2;$$

therefore, we have constructed a Real Čech 2-cocyle $(\omega_\lambda)_{\lambda\in\Lambda_2} \in ZR^2(\mathcal{U}_\bullet, \mathbf{S})$ associated to $\mathcal{T}$.

Assume that $(\tilde{\mathsf{s}}_\lambda)_{\lambda\in\Lambda_2}$ is another Real family of continuous local sections of π, and that $(\tilde\omega_\lambda)_{\lambda\in\Lambda_2} \in ZR^2(\mathcal{U}_\bullet,\mathbf{S})$ is its associated Real Čech 2-cocycle. Then for any $(\lambda_0,\lambda_1,\lambda_{01}) \in \Lambda_1$ and $g \in U^1_{\lambda_0\lambda_1\lambda_{01}}$, there exists a unique $c_{\lambda_0\lambda_1\lambda_{01}}(g) \in \mathbf{S}$ such that

$$\tilde{\mathsf{s}}_{\lambda_0\lambda_1\lambda_{01}}(g) = c_{\lambda_0\lambda_1\lambda_{01}}(g)\cdot \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g), \tag{3.53}$$

where we abusively write, for instance, $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g)$ for $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(\lambda_0,g,\lambda_1)$. Since $(\tilde{\mathsf{s}}_{\lambda_0\lambda_1\lambda_{01}})$ and $\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}$ are Real families, we have that

$$c_{\bar\lambda_0\bar\lambda_1\bar\lambda_{01}}(\rho(g)) = \overline{c_{\lambda_0\lambda_1\lambda_{01}}(g)} \text{ for all } g \in U^1_{\lambda_0\lambda_1\lambda_{01}}.$$

It turns out that the $c_{\lambda_0\lambda_1\lambda_{01}}$'s define an element in $CR^1(\mathcal{U}_\bullet,\mathbf{S})$. Moreover, for $\lambda \in \Lambda_2$ as previously, and for $(g_1,g_2) \in U^2_\lambda$, we obtain from (3.52) and (3.53)

$$\mathsf{s}_{\lambda_0\lambda_2\lambda_{02}}(g_1g_2) = c_{\lambda_0\lambda_2\lambda_{02}}(g_1g_2)^{-1}\cdot c_{\lambda_0\lambda_1\lambda_{01}}(g_1)\cdot c_{\lambda_1\lambda_2\lambda_{12}}(g_2)\cdot\tilde\omega_\lambda(g_1,g_2)\cdot \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g_1)\cdot \mathsf{s}_{\lambda_1\lambda_2\lambda_{12}}(g_2);$$

and

$$(\omega_\lambda\cdot\tilde\omega_\lambda^{-1})(g_1,g_2) = c_{\lambda_0\lambda_2\lambda_{02}}(g_1g_2)^{-1}.c_{\lambda_0\lambda_1\lambda_{01}}(g_1)\cdot c_{\lambda_1\lambda_2\lambda_{12}}(g_2) = (dc)_\lambda(g_1,g_2);$$

hence $((\omega\cdot\tilde\omega^{-1})_\lambda)_{\lambda\in\Lambda_2} \in BR^2(\mathcal{U}_\bullet,\mathbb{S}^1)$. In other words, the class in $HR^2(\mathcal{U}_\bullet,\mathbf{S})$ of the Real 2-cocycle (ω_λ) does not depend on the choice of the Real family of local sections of π.

We want now to check that the map (3.50) is well defined. To do so, suppose that $\mathcal{T} \cong \mathcal{T}'$ in $\widehat{\mathrm{extR}}_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0],\mathbf{S})$, and that $(\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}})$ and $\mathsf{s}'_{\lambda_0\lambda_1\lambda_{01}}$ are Real family of local continuous sections of π and π'. Let us keep the diagram (3.51). Let $(\omega_\lambda)_{\lambda\in\Lambda_2}$ and $(\omega'_\lambda)_{\lambda\in\Lambda_2}$ be the associated Real 2-cocycles in $ZR^2(\mathcal{U}_\bullet,\mathbf{S})$ of $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then we define an element

$(b_{\lambda_0\lambda_1\lambda_{01}}) \in CR^1(\mathcal{U}_\bullet, \mathbf{S})$ as follows: for any $g \in U^1_{\lambda_0\lambda_1\lambda_{01}}$, $b_{\lambda_0\lambda_1\lambda_{01}}(g)$ is the unique element of $\mathbf{S}$ such that

$$\mathsf{s}'_{\lambda_0\lambda_1\lambda_{01}}(g) = b_{\lambda_0\lambda_1\lambda_{01}}(g) \cdot f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g). \tag{3.54}$$

This is well defined since $\pi'(\mathsf{s}'_{\lambda_0\lambda_1\lambda_{01}}(g)) = \pi(\mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g)) = \pi'(f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g))$. Furthermore, $(f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}})_{(\lambda_0,\lambda_1,\lambda_{01})\in\Lambda_1}$ is a Real family of local continuous sections of π. Then, for all $\lambda \in \Lambda_2$ and all $(g_1, g_2) \in U^2_\lambda$, we can write

$$f \circ \mathsf{s}_{\lambda_0\lambda_2\lambda_{02}}(g_1 g_2) = \omega_\lambda(g_1, g_2) \cdot f \circ \mathsf{s}_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot f \circ \mathsf{s}_{\lambda_1\lambda_2\lambda_{12}}(g_2),$$

up to a multiplication of ω_λ by a Real 2-coboundary. It then follows that

$$\omega_\lambda(g_1, g_2) \cdot \omega'_\lambda(g_1, g_2)^{-1} = b_{\lambda_0\lambda_2\lambda_{02}}(g_1 g_2)^{-1} \cdot b_{\lambda_0\lambda_1\lambda_{01}}(g_1) \cdot b_{\lambda_1\lambda_2\lambda_{12}}(g_2) = (db)_\lambda(g_1, g_2).$$

Consequently, $(\omega_\lambda)_{\lambda\in\Lambda_2}$ depends only on the class of $\mathcal{T}$ in $\widehat{\mathrm{extR}}_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathbf{S})$. The fact that $(\delta_{\lambda_0\lambda_1\lambda_{01}})$ also depends only on the class of $\mathcal{T}$ is straightforward. We then have proved that any element $[\mathcal{T}]$ in $\widehat{\mathrm{extR}}_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathbf{S})$ determines a unique cohomology class

$$[\omega(\mathcal{T})] \in HR^2(\mathcal{U}_\bullet, \mathbf{S}). \tag{3.55}$$

Conversely, given a pair $(\omega_\lambda)_{\lambda\in\Lambda_2} \in ZR^2(\mathcal{U}_\bullet, \mathbf{S})$, we want to construct an ungraded Real extension of $(\mathcal{G}[\mathcal{U}_0], \rho)$ which is in $\widehat{\mathrm{extR}}^+_{\mathcal{U}}(\mathcal{G}[\mathcal{U}_0], \mathbf{S})$. For this we proceed as in the proof of Proposition 5.6 in [88]. For $\lambda \in \Lambda_2$, put

$$\begin{aligned}
\mu_{01} &:= (\lambda_0, \lambda_{01}, \lambda_1),\\
\mu_{02} &:= (\lambda_0, \lambda_{02}, \lambda_2),\\
\mu_{12} &:= (\lambda_1, \lambda_{12}, \lambda_2).
\end{aligned}$$

Let $\mathfrak{c}_{\mu_{01}\mu_{02}\mu_{12}} := \omega_\lambda$. We have $\mathcal{V}_1 = (V^1_{\mu_{01}})_{i\in I_1}$, where I_1 consists of triples $\mu_{01} = (\lambda_0, \lambda_{01}, \lambda_1)$ and $V^1_{\mu_{01}} := \{\lambda_0\} \times U^1_{\lambda_0\lambda_1\lambda_{01}} \times \{\lambda_1\}$. I_1 is equipped with the obvious involution, so that $\mathcal{V}_1$ is a Real open cover of $\mathcal{G}[\mathcal{U}_0]$. We set

$$\widetilde{\Gamma}^\omega := \coprod_{\mu_{01}\in I_1} \{(t, g, \mu_{01}) \mid t \in \mathbf{S}, g \in V^1_{\mu_{01}}\}/\sim,$$

subject to the product law

$$[t_1, g_1, \mu_{01}] \cdot [t_1, g_2, \mu_{12}] = [t_1 \cdot t_2 \cdot \mathfrak{c}_{\mu_{01}\mu_{02}\mu_{12}}(g_1, g_2), g_1 g_2, \mu_{02}],$$

where

$$(t, g, \mu_{12}) \sim (\mathfrak{c}_{\mu_{01}\mu_{01}\mu_{01}}(r(g), r(g))^{-1} \cdot t \cdot \mathfrak{c}_{\mu_{01}\mu_{02}\mu_{12}}(r(g), g), g, \mu_{02}). \tag{3.56}$$

The projection $\pi : \widetilde{\Gamma}^\omega \longrightarrow \mathcal{G}[\mathcal{U}_0]$ is defined by $\pi([t, g, \mu_{01}]) := g$, and the Real structure is

$$\overline{[t, g, \mu_{01}]} := [\bar{t}, \rho(g), \overline{\mu_{01}}].$$

It is straightforward to see that these operations give $\widetilde{\Gamma}^\omega$ the structure of ungraded Real $\mathbf{S}$-twist of $\mathcal{G}[\mathfrak{U}_0]$; what is more, the maps $\mathsf{s}_{\mu_{01}}: V^1_{\mu_{01}} \longrightarrow \widetilde{\Gamma}^\omega$ defined by $\mathsf{s}_{\mu_{01}}(g) := [0,g,\mu_{01}]$ are a Real family of continuous sections of π, so that the Real extension

$$\mathcal{T} = \ \mathbf{S} \longrightarrow \widetilde{\Gamma}^\omega \xrightarrow{\pi} \mathcal{G}[\mathfrak{U}_0]$$

is in $\widehat{\mathrm{extR}}^+_{\mathfrak{U}}(\mathcal{G}[\mathfrak{U}_0],\mathbf{S})$. It is also clear that $[\omega(\mathcal{T})]=[\omega]$. □

Corollary 3.11.2. *We have* $\widehat{Ext\mathrm{R}}^+(\mathcal{G},\mathbf{S})\cong \check{H}R^2(\mathcal{G}_\bullet,\mathbf{S})$.

3.12 The cup-product $\check{H}R^1(\cdot,\mathbb{Z}_2)\times\check{H}R^1(\cdot,\mathbb{Z}_2)\to\check{H}R^2(\cdot,\mathbb{S}^1)$

Let $\delta,\delta'\in \check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)$, and let L and L' be representatives of their corresponding classes in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathbb{Z}_2)$ (cf. Proposition 3.10.1). Then by viewing $\mathbb{Z}_2=\{\mp1\}$ as a Real subgroup of $\mathbb{S}^1$ (identifying -1 with $(-1,0)$ and $+1$ with $(1,0)$), we define the tensor product $r^*L\otimes\overline{s^*L'}\longrightarrow\mathcal{G}$, and and using the same reasoning as in Example 2.5.8, we see that this is clearly a Real $\mathbb{Z}_2$-principal bundle; thus we have an ungraded Real $\mathbb{Z}_2$-central extension

$$\mathbb{Z}_2\longrightarrow r^*L\otimes\overline{s^*L'}\longrightarrow\mathcal{G}.$$

Therefore, we get an ungraded Real $\mathbb{S}^1$-central extension $(L\smile L',\mathcal{G})$ given by

$$L\smile L' := (r^*L\otimes\overline{s^*L'})\times_{\mathbb{Z}_2}\mathbb{S}^1, \tag{3.57}$$

together with the evident Real structure and Real $\mathbb{S}^1$-action.

Definition 3.12.1. *We define the cup product*

$$\smile\ :\ \check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\times\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\longrightarrow\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)$$

by

$$\delta\smile\delta' := \omega(L\smile L'),$$

where $L\smile L'$ *is determined by equation* (3.57).

Lemma 3.12.2. *The cup product* $\smile$ *defined above is a well defined bilinear map;* i.e.

$$(\delta_1+\delta_2)\smile(\delta_1'+\delta_2')=\delta_1\smile\delta_1'+\delta_1\smile\delta_2'+\delta_2\smile\delta_1'+\delta_2\smile\delta_2'.$$

Proof. If δ_i is realized by the generalized Real homomorphism $L_i:\mathcal{G}\longrightarrow\mathbb{Z}_2$, then $\delta_1+\delta_2$ is realized by L_1+L_2. The result follows from the easy to check bilinearity of the tensor product $r^*L\otimes\overline{s^*L'}$ with respect to the sum in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathbb{Z}_2)$. □

3.13 Cohomological picture of the group $\widehat{\mathbf{ExtR}}(\mathcal{G},\mathbb{S}^1)$

Our purpose in this section is to provide a Real Čech cohomological picture of $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$, for any given locally compact Hausdorff Real groupoid $(\mathcal{G},\rho)$.

Let $\mathcal{T}=(\tilde{\mathcal{G}},\delta)\in\widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}_0],\mathbb{S}^1)$, where as usual $\mathcal{U}_0$ is a Real open cover of $\mathcal{G}^{(0)}$. Let $\mathcal{U}_\bullet$ be the pre-simplicial Real open cover of $(\mathcal{G}_\bullet,\rho_\bullet)$ defined as in (3.43).

Define a continuous map $\delta_{j_0j_1}:U^1_{(j_0,j_1)}\longrightarrow\mathbb{Z}_2$ over all $U^1_{(j_0,j_1)}\in\mathcal{U}_1$ by $\delta_{j_0j_1}(g):=\delta(j_0,g,j_1)$. Then, over all $U^2_{(j_0,j_1,j_2)}$, we have that $\delta_{j_0j_2}(g_1g_2)=\delta((j_0,g_1,j_1)\cdot(j_1,g_2,j_2))=\delta_{j_0j_1}(g_1)\cdot\delta_{j_1j_2}(g_2)$. Moreover, since δ is a Real morphism, we have that $\delta_{\bar{j}_0\bar{j}_1}(\rho(g))=\delta_{j_0j_1}(g)$; hence $\mathcal{T}$ determines a Real Čech 1-cocycle

$$\delta(\mathcal{T}):=(\delta_{j_0j_1})_{(j_0,j_1)\in J_1}\in ZR^1(\mathcal{U}_\bullet,\mathbb{Z}_2),\tag{3.58}$$

Then, (3.58) gives a Real Čech 1-cocycle $(\delta_{\lambda_0\lambda_1\lambda_{01}})\in ZR^1(\mathcal{U}_\bullet,\mathbb{Z}_2)$ defined by $\delta_{\lambda_0\lambda_1\lambda_{01}}(g):=\delta(\lambda_0,g,\lambda_1)$ for any $g\in U^1_{\lambda_0\lambda_1\lambda_{01}}$; this does make sense, for we know from Section 3.10 that Real Čech 1-cocycles do not depend on λ_{01}.

If $\mathcal{T}'$ is another Rg $\mathbb{S}^1$-central extension over $\mathcal{G}$, we may suppose it is represented by a Rg $\mathbb{S}^1$-twisted $(\tilde{\mathcal{G}}',\delta')$ of $\mathcal{G}[\mathcal{U}_0]$. Then by definition of the grading of $\mathcal{T}\hat{\otimes}\mathcal{T}'$, we have $\delta(\mathcal{T}\hat{\otimes}\mathcal{T}')=\delta(\mathcal{T})+\delta(\mathcal{T}')$.

Theorem 3.13.1 (Compare Proposition 2.13 [30]). *Let $(\mathcal{G},\rho)$ be a locally compact Hausdorff Real groupoid. There is a set-theoretic split-exact sequence*

$$0\longrightarrow\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)\hookrightarrow\widehat{Ext\mathrm{R}}(\mathcal{G},\mathbb{S}^1)\xrightarrow{\delta}\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\longrightarrow 0\tag{3.59}$$

so that we have a canonical group isomorphism

$$dd:\widehat{Ext\mathrm{R}}(\mathcal{G},\mathbb{S}^1)\cong\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\ltimes\check{H}R^2(\mathcal{G}_\bullet,\mathcal{S}^1),\tag{3.60}$$

where the semi-direct product $\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\ltimes\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)$ is defined by the operation

$$(\delta,\omega)+(\delta',\omega'):=(\delta+\delta',(\delta\smile\delta')\cdot\omega\cdot\omega').$$

Proof. The first arrow is the canonical inclusion $\widehat{\mathrm{ExtR}}^+(\mathcal{G},\mathbb{S}^1)\subset\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$, and hence is injective. The exactness of the sequence (3.59) is obvious, by definition of δ and $\widehat{\mathrm{ExtR}}^+(\mathcal{G},\mathbb{S}^1)$.

The map δ is well defined; indeed, if $\mathcal{T}\sim\mathcal{T}'$ in $\widehat{\mathrm{extR}}(\mathcal{G}[\mathcal{U}_0],\mathbb{S}^1)$, they differ from a twist coming from an element of $\mathrm{PicR}(\mathcal{G}[\mathcal{U}_0])$, and hence by construction of δ, one has $\delta(\mathcal{T})=\delta(\mathcal{T}')$. Moreover, δ is surjective, for if $L\in\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathbb{Z}_2)$ represents the Real 1-cocycle $(\varepsilon_{j_0j_1})\in ZR^1(\mathcal{U}_\bullet,\mathbb{Z}_2)$, then $L\smile L$ is graded as follows:

$$L\smile L:=(\mathbb{S}^1\longrightarrow(r^*L\otimes\overline{s^*L})\times_{\mathbb{Z}_2}\mathbb{S}^1\longrightarrow\mathcal{G}[\mathcal{U}_0],\delta'),$$

where

$$\delta'((j_0,\gamma,j_1)) := \varepsilon_{j_0 j_1}(\gamma).$$

We see that $\delta(L \smile L) = \varepsilon$. Finally, note that the operation law comes from the definition of the sum in $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$. $\square$

3.14 The proper case

In this section, we are interested in some particular Abelian Real sheaves on $(\mathcal{G}_\bullet,\rho_\bullet)$, where $(\mathcal{G},\rho)$ is a proper groupoid. More precisely, we aim to generalize a result by Crainic (see [23, Proposition 1]) stating that for a proper Lie groupoid $\mathcal{G}$, and "*representation*" E of $\mathcal{G}$ ([23, 1.2]), the *differentiable* cohomology $H^n_d(\mathcal{G},E) = 0$ for all $n \geq 1$. Let us first introduce some few notions and properties.

Definition 3.14.1 (Real Haar measure)**.** *Let $(\mathcal{G},\rho)$ be a locally compact Real groupoid, and let $\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$ be a (left)* Haar system *for $\mathcal{G}$ (cf. [76, §.2]). Define a new family $\{\mu^x_\rho\}_{x\in\mathcal{G}^{(0)}}$ of measures μ^x_ρ, with support $\mathcal{G}^x$ for all $x \in \mathcal{G}^{(0)}$, defined by*

$$\mu^x_\rho(C) := \mu^{\rho(x)}(\rho(C)),\ \textit{for all measurable subset } C \subset \mathcal{G}^x. \tag{3.61}$$

We say that $\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$ is **Real** *if*

$$\mu^x = \mu^x_\rho,\ \forall x \in \mathcal{G}^{(0)}. \tag{3.62}$$

Lemma 3.14.2. *Any Haar system for $\mathcal{G}$ gives rise to a Real one.*

Proof. Assume $\{\mu^x\}$ is a Haar system for $\mathcal{G}$. For every $x \in \mathcal{G}^{(0)}$, we set

$$\tilde{\mu}^x := \frac{1}{2}(\mu^x + \mu^x_\rho). \tag{3.63}$$

It is clear that $\{\tilde{\mu}^x\}_{x\in\mathcal{G}^{(0)}}$ is a Haar system for $\mathcal{G}$; measurable subsets for $\tilde{\mu}^x$ being exactly those for μ^x. Moreover, one has

$$\tilde{\mu}^x_\rho = \frac{1}{2}\left(\mu^{\rho(x)}\circ\rho + \mu^{\rho(x)}_\rho\circ\rho\right) = \frac{1}{2}\left(\mu^x_\rho + \mu^x\right) = \tilde{\mu}^x,\ \forall x \in \mathcal{G}^{(0)}.$$

$\square$

Remark 3.14.3. *From the lemma above, we will always assume Haar systems for $\mathcal{G}$ to be Real.*

In what follows, the Real group $\mathbb{K}$ is either the additive group $\mathbb{R}$ equipped with the Real structure $t \longmapsto \bar{t} := -t$, or the additive group $\mathbb{C}$ equipped with the complex conjugation $z \longmapsto \bar{z}$ as Real structure.

Definition 3.14.4. *Let* $(\mathcal{G},\rho)$ *be a locally compact Real groupoid. A* Real representation *of* $(\mathcal{G},\rho)$ *is a locally trivial Real* $\mathbb{K}$*-vector bundle* $\pi:(E,\nu)\longrightarrow(\mathcal{G}^{(0)},\rho)$ *endowed with a (left) continuous Real* $\mathcal{G}$*-action; that is a Real open cover* (U_j) *of* $(\mathcal{G}^{(0)},\rho)$ *and isomorphisms* $\phi_j: U_j\times\mathbb{K}^r\longrightarrow E_{|U_j}$ *such that* $\nu(\phi_j(x,(a_1,...,a_r)))=\phi_{\bar{j}}(\rho(x),(\bar{a}_1,...,\bar{a}_r))$, $\forall x\in U_j,(a_1,...,a_r)\in\mathbb{K}^r$*, and*

- $\forall x\in\mathcal{G}^{(0)}$*, the induced isomorphism* $\nu_x:E_x\longrightarrow E_{\rho(x)}$ *is* $\mathbb{K}$*-antilinear:*

$$\nu_x(\xi\cdot a)=\nu_x(\xi)\cdot\bar{a},\ \forall\xi\in E_x,a\in\mathbb{K};$$

- $\forall g\in\mathcal{G}$*, the isomorphism* $E_{s(g)}\longrightarrow E_{r(g)}$*, induced by the* $\mathcal{G}$*-action, is linear.*

Note that such a Real representation (E,ν) can be viewed as a Real $\mathcal{G}$-module in the following way: E is the groupoid $E\rightrightarrows\mathcal{G}^{(0)}$ with $r_E(\xi)=s_E(\xi):=\pi(\xi)$ for every $\xi\in E$, for any $x\in\mathcal{G}^{(0)}$, $E_x=E^x=E^x_x$ is isomorphic to the group $\mathbb{K}$, then the product in E is defined by the sum on the fibres. The Real sheaf on $(\mathcal{G}_\bullet,\rho_\bullet)$ associated to the Real $\mathcal{G}$-module (E,ν) will be denoted $(E^\bullet,\nu^\bullet)$.

Definition 3.14.5. *([90, Definition 2.20]) A locally compact Real groupoid* $(\mathcal{G},\rho)$ *is said to be* proper *if any of the following equivalent conditions is satisfied:*

(i) the Real map $(s,r):\mathcal{G}\longrightarrow\mathcal{G}^{(0)}\times\mathcal{G}^{(0)}$ *is proper;*

(ii) for every $K\subset\mathcal{G}^{(0)}$ *compact,* $\mathcal{G}^K_K$ *is compact.*

Proper Real groupoids can be characterized by the following (we refer to Propositions 6.10 and 6.11 in [89] for a proof)

Proposition 3.14.6. *Let* $(\mathcal{G},\rho)$ *be a locally compact Real groupoid with a Haar system* $\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$*. Then* $(\mathcal{G},\rho)$ *is proper if and only it admits a* `cutoff` *Real function; that is, a function* $x:\mathcal{G}^{(0)}\longrightarrow\mathbb{R}_+$ *such that*

(i) $\forall x\in\mathcal{G}^{(0)}$, $c(\rho(x))=c(x)$*;*

(ii) $\forall x\in\mathcal{G}^{(0)}$, $\int_{\mathcal{G}^x}c(s(g))d\mu^x(g)=1$*;*

(iii) the map $r:supp(c\circ s)\longrightarrow\mathcal{G}^{(0)}$ *is proper; i.e. for every* $K\subset\mathcal{G}^{(0)}$ *compact,* $supp(c)\cap s(\mathcal{G}^K)$ *is compact.*

Theorem 3.14.7. *Suppose* $(\mathcal{G},\rho)$ *is a locally compact proper Real groupoid with a Haar system. Then, for any Real representation* (E,ν) *of* $(\mathcal{G},\rho)$*, we have*

$$\check{H}R^n(\mathcal{G}_\bullet,E^\bullet)=0,\ \forall n\geq1.$$

To prove this result, we shall recall fundamentals of vector-valued integration exposed, for instance, in [94, Appendix B.1], and then adapt them to the case when we deal with Real structures. Let X be a locally compact Hausdorff space, and let B be a separable Banach space. Let μ be a Radon measure on X. Then measurable functions $f : X \longrightarrow B$ are defined as usual, and such function is *integrable* if

$$\|f\|_1 := \int_X \|f(x)\| d\mu(x) < \infty.$$

The collection of all B-valued integrable functions on X is denoted by $\mathcal{L}^1(X,B)$, and the set of equivalence classes of functions in $\mathcal{L}^1(X,B)$ is a Banach space denoted by $L^1(X,B)$ ([94, Proposition B.31]). Furthermore, $\mathcal{C}_c(X,B)$ is dense in $L^1(X,B)$ The B-valued integration of elements of $L^1(X,B)$ is defined as a linear map $I : \mathcal{C}_c(X,B) \longrightarrow B$ given by

$$I(f) := \int_X f(x) d\mu(x), \text{ and } \|I(f)\| \leq \|f\|_1. \tag{3.64}$$

Moreover, this integral is characterized by the following

Proposition 3.14.8. *(cf. Proposition B.34 [94]) Let μ be a Radon measure on X, and let B be a Banach space. Then, the integral is characterized by*

(a) for all $f \in \mathcal{C}_c(X,B)$ and $\varphi \in B^*$,

$$\varphi\left(\int_X f(x) d\mu(x)\right) = \int_X \varphi(f(x)) d\mu(x);$$

(b) if $L : B \longrightarrow B'$ is any bounded linear map between two Banach spaces, than

$$L\left(\int_X f(x) d\mu(x)\right) = \int_X L(f(x)) d\mu(x).$$

Now suppose (X,ρ) is a locally compact Hausdorff Real space, μ is a Real Radon measure; *i.e.* $\mu(\rho(C)) = \rho(C)$ for every measurable set $C \subset X$. Let (B,ς) be a separable Real Banach space. Then from the above, we deduce the

Lemma 3.14.9. *Let $\mathcal{C}_c(X,B)$ be equipped with the Real structure denoted by $\tilde{\rho} : \mathcal{C}_c(X,B) \longrightarrow \mathcal{C}_c(X,B)$, and given by $\rho(f)(x) := \varsigma(f(\rho(x)))$. Then, under the above assumption, the integral $\int : \mathcal{C}_c(X,B) \longrightarrow B$ is Real, in that it commutes with the Real structures ς and $\tilde{\rho}$; i.e*

$$\int_X \varsigma(f(\rho(x))) d\mu(x) = \varsigma\left(\int_X f(x) d\mu(x)\right), \forall f \in \mathcal{C}_c(X,B). \tag{3.65}$$

Proof. For any $\varphi \in B^*$, define $\bar{\varphi} \in B^*$ by $\bar{\varphi}(b) := \overline{\varphi(\varsigma(b))}$. Then, from Proposition 3.14.8 (a) and the definition of $\bar{\varphi}$, one has

$$\overline{\varphi\left(\varsigma\left(\int_X f(x) d\mu(x)\right)\right)} = \int_X \overline{\varphi(\varsigma(f(x)))} d\mu(x) = \overline{\int_X \varphi(\varsigma(f(x))) d\mu(x)}.$$

Thus,

$$\varphi\left(\varsigma\left(\int_X f(x)d\mu(x)\right)\right) = \int_X \varphi(\varsigma(f(x)))d\mu(x).$$

Again from (b) of Proposition 3.14.8 and from the fact that μ is Real, we then get

$$\varphi\left(\varsigma\left(\int_X f(x)d\mu(x)\right)\right) = \varphi\left(\int_X \varsigma(f(\rho(x)))d\mu(x)\right), \forall \varphi \in B^*,$$

and the result holds. □

Now let us investigate the case of a Real groupoid $(\mathcal{G},\rho)$ together with a Real representation (E,ν). Let $\mu = \{\mu^x\}_{x\in\mathcal{G}^{(0)}}$ be a Real Haar system for $(\mathcal{G},\rho)$. For any $x \in \mathcal{G}^{(0)}$, we can apply (3.64) to E_x and get the integral $\int_{\mathcal{G}^x} : \mathcal{C}_c(\mathcal{G}^x, E_x) \longrightarrow E_x$. Further, it is very easy to check that

$$\nu_x\left(\int_{\mathcal{G}^x} f(\gamma)d\mu^x(\gamma)\right) = \int_{\mathcal{G}^{\rho(x)}} \nu_x(f(\rho(\gamma)))d\mu^{\rho(x)}(\gamma),\ \forall f \in \mathcal{C}_c(\mathcal{G}^x, E_x). \tag{3.66}$$

Proof of Theorem 3.14.7. Fix a Real Haar system $\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$ for $(\mathcal{G},\rho)$ and a cutoff Real function $c : \mathcal{G}^{(0)} \longrightarrow \mathbb{R}_+$. Let $\mathcal{U}_\bullet$ be a Real open cover of $(\mathcal{G}_\bullet, \rho_\bullet)$. Let $\lambda := (\lambda_0, \lambda_1, \dots, \lambda_{01\dots n}) \in \Lambda_n$ and $U^n_\lambda \in {}_\natural\mathcal{U}_n$. Denote by $\Lambda_{n+1|\lambda}$ the subset of Λ_{n+1} consisting of those $\tilde{\lambda} \in \Lambda_{n+1}$ such that $\tilde{\lambda}(S) = \lambda_S$ for all $\emptyset \neq S \subseteq [n]$. Then, if for any $x \in U^0_{\lambda_n}$, we denote

$$(U^n_\lambda \star \mathcal{G}^x) \cap \operatorname{supp}(c\circ s) := \{(g_1,\dots,g_n,\gamma) \in U^n_\lambda \times (\mathcal{G}^x \cap \operatorname{supp}(c\circ s)) \mid s(g_n) = r(\gamma) = x\},$$

we have that

$$(U^n_\lambda \star \mathcal{G}^x) \cap \operatorname{supp}(c\circ s) \subset \bigcup_{\tilde{\lambda}\in\Lambda_{n+1|\lambda}} U^{n+1}_{\tilde{\lambda}}. \tag{3.67}$$

Notice that for $\tilde{\lambda}$ running over $\Lambda_{n+1|\lambda}$, only its images $\tilde{\lambda}_S \in \Lambda_{\#S-1}$, for $S \subseteq [n+1]$ containing $n+1$, are led to vary. On the other hand, since $\mathcal{G}^x \cap \operatorname{supp}(c\circ s)$ is compact in $\mathcal{G}$ (by (iii) of Proposition 3.14.6), the union (3.67) is finite. In particular, for every $S \in S(n+1) := \{S \subseteq [n+1] \mid n+1 \in S \neq \emptyset\}$, where elements of $S(n+1)$ are ranged in cardinality and in lexicographic order, there is $\tilde{\lambda}^{l_S}_S \in \Lambda_{\#S-1}$, $l_S = 0,\dots,m_S$, such that

$$(U^n_\lambda \star \mathcal{G}^x) \cap \operatorname{supp}(c\circ s) \subset \bigcup_{l=(l_S)_{S\in S(n+1)}} U^{n+1}_{\lambda^l}, \tag{3.68}$$

where for any $l = (l_S)_{S\in S(n+1)} \in \mathbb{N}^{2^{n+1}}$ written as

$$l = \left(l_{\{n+1\}}, l_{\{0,n+1\}}, l_{\{1,n+1\}}, \dots, l_{\{n,n+1\}}, \dots, l_{\{1,\dots,n+1\}}, l_{\{0,1,\dots,n+1\}}\right),$$

the element $\lambda^l \in \Lambda_{n+1|\lambda}$ is given by the following

$$\begin{cases} \lambda^l(S) := \lambda_S, & \text{for any } S \subseteq [n]; \\ \lambda^l(S) := \lambda^{l_S}_S, & \text{for any } S \in S(n+1). \end{cases} \tag{3.69}$$

Now for each $S \in S(n+1)$, $\varepsilon_S^{n+1} =: \varepsilon_S : [\#S-1] \longrightarrow [n+1]$ denotes the unique morphism in $\mathrm{Hom}_{\Delta'}([\#S-1],[n+1])$ whose range is exactly S. It is then clear that

$$\tilde{\varepsilon}_S((U_\lambda^n \star \mathcal{G}^x) \cap \mathrm{supp}(\mathrm{c} \circ s)) \subset \bigcup_{l_S} U^{\#S-1}_{\lambda_S^{l_S}}, \ \forall S \in S(n+1). \tag{3.70}$$

Next, choose for every $S \in S(n+1)$, a partition of unity

$$\varphi_{\lambda_S^{l_S}} : \tilde{\varepsilon}_S((U_\lambda^n \star \mathcal{G}^x) \cap \mathrm{supp}(\mathrm{c} \circ s)) \longrightarrow \mathbb{R}_+$$

subordinate to the open covering $\left(U^{\#S-1}_{\lambda_S^{l_S}}\right)_{l_S=0}^{m_S}$.

For all $n \geq 1$, we define the map $h^n : CR^{n+1}_{ss}(\mathfrak{U}_\bullet, E^\bullet) \longrightarrow CR^n_{ss}(\mathfrak{U}_\bullet, E^\bullet)$ by

$$\begin{aligned}(h^n f)_\lambda(g_1,\ldots,g_n) := (-1)^{n+1} \int_{\mathcal{G}^{s(g_n)}} \sum_{l=(l_S)_{S\in S(n+1)}} f_{\lambda^l}(g_1,\ldots,g_n,\gamma) \cdot \prod_{S\in S(n+1)} \\ \prod_{l_S} \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S(g_1,\ldots,g_n,\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma).\end{aligned} \tag{3.71}$$

Notice that

$$(U_{\bar{\lambda}}^n \star \mathcal{G}^{\rho(x)}) \cap \mathrm{supp}(\mathrm{c} \circ s \circ \rho) \subset \bigcup_{l=(l_S)_{S\in S(n+1)}} U^{n+1}_{\bar{\lambda}^l},$$

where the $\bar{\lambda}^l$ ' s are defined in the obvious way. Hence, we get a partition of unity of $\tilde{\varepsilon}_S((U_{\bar{\lambda}}^n \star \mathcal{G}^{\rho(x)}) \cap \mathrm{supp}(\mathrm{c} \circ s \circ \rho))$ subordinate to the open covering $\left(U^{\#S-1}_{\bar{\lambda}_S^{l_S}}\right)_{l_S=0}^{m_S}$ by setting

$$\varphi_{\bar{\lambda}_S^{l_S}}(\tilde{\varepsilon}_S(\rho(g_1),\ldots,\rho(g_n))) := \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S(g_1,\ldots,g_n)).$$

Next, using (3.66), it is straightforward that

$$(h^n f)_{\bar{\lambda}}(\rho(g_1),\ldots,\rho(g_n)) = \nu_{|U_\lambda^n} \circ (h^n f)_\lambda(g_1,\ldots,g_n),$$

which means that $((h^n f)_\lambda)_{\lambda\in\Lambda_n} \in CR^n_{ss}(\mathfrak{U}_\bullet, E^\bullet)$.

Assume now that $(f_\lambda)_{\lambda\in\Lambda_n} \in CR^n_{ss}(\mathfrak{U}_\bullet, E^\bullet)$. Then, for every $U_\lambda^n \in {}_\natural\mathfrak{U}_n$ and $(g_1,\ldots,g_n) \in U_\lambda^n$, one has

$$\begin{aligned}(h^n d^n f)_\lambda(g_1,\ldots,g_n) = (-1)^{n+1} \int_{\mathcal{G}^{s(g_n)}} \sum_{(l_S)_{S\in S(n+1)}} (d^n f)_{\lambda^l}(g_1,\ldots,g_n,\gamma) \cdot \prod_{S\in S(n+1)} \\ \prod_{l_S} \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S^{n+1}(g_1,\ldots,g_n,\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma) \\ = f_\lambda(g_1,\ldots,g_n) - A_\lambda(g_1,\ldots,g_n),\end{aligned} \tag{3.72}$$

where

$$A_\lambda(g_1,\ldots,g_n) := (-1)^n \sum_{k=0}^n (-1)^k \int_{\mathcal{G}^{s(g_n)}} \sum_{(l_S)_{S\in S(n+1)}} f_{\tilde{\varepsilon}_k^{n+1}(\lambda^l)}(\tilde{\varepsilon}_k^{n+1}(g_1,\ldots,g_n,\gamma)) \cdot \prod_{S\in S(n+1)}$$

$$\prod_{l_S} \varphi_{\lambda_S^{l_S}}(\tilde{\varepsilon}_S^{n+1}(g_1,\ldots,g_n,\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma).$$

We want to show that

$$A_\lambda(g_1,\ldots,g_n) = (d^{n-1}h^{n-1}f)_\lambda(g_1,\ldots,g_n). \tag{3.73}$$

One has

$$\begin{aligned}(d^{n-1}h^{n-1}f)_\lambda(g_1,\ldots,g_n) = (-1)^n \sum_{k=0}^{n-1} \int_{\mathcal{G}^{s(g_n)}} \sum_{r_k:=(r_{k,T})_{T\in S(n)}} & f_{\tilde{\varepsilon}_k^n(\lambda)^{r_k}}(\tilde{\varepsilon}_k^n(g_1,\ldots,g_n),\gamma) \cdot \prod_{T\in S(n)} \\ & \prod_{r_{k,T}} \varphi_{\tilde{\varepsilon}_k^n(\lambda)_T^{r_{k,T}}}(\tilde{\varepsilon}_T^n(\tilde{\varepsilon}_k^n(g_1,\ldots,g_n),\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma) \\ & + \int_{\mathcal{G}^{s(g_{n-1})}} \sum_{r_n:=(r_{n,T})_{T\in S(n)}} f_{\tilde{\varepsilon}_n^n(\lambda)^{r_n}}(g_1,\ldots,g_{n-1},\gamma) \cdot \prod_{T\in S(n)} \\ & \prod_{r_{n,T}} \varphi_{\tilde{\varepsilon}_n^n(\lambda)_T^{r_{n,T}}}(\tilde{\varepsilon}_T^n(g_1,\ldots,g_{n-1},\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma) \\ & = B_\lambda(g_1,\ldots,g_n) + C_\lambda(g_1,\ldots,g_n).\end{aligned} \tag{3.74}$$

Notice that by the left-invariance of $\{\mu^x\}_{x\in\mathcal{G}^{(0)}}$, the second integral C_λ in the right hand side of (3.74) can be written as

$$\begin{aligned}C_\lambda(g_1,\ldots,g_n) &= \int_{\mathcal{G}^{s(g_n)}} \int_{(r_{n,T})_{T\in S(n)}} f_{\tilde{\varepsilon}_n^n(\lambda)^{r_n}}(g_1,\ldots,g_{n-1},g_n\gamma) \cdot \prod_{T\in S(n)} \\ & \prod_{r_{n,T}} \varphi_{\tilde{\varepsilon}_n^n(\lambda)_T^{r_{n,T}}}(\tilde{\varepsilon}_T^n(g_1,\ldots,g_{n-1},g_n\gamma)) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma) \\ &= \int_{\mathcal{G}^{s(g_n)}} \sum_{(r_{n,T})_{T\in S(n)}} f_{\tilde{\varepsilon}_n^n(\lambda)^{r_n}}(\tilde{\varepsilon}_n^{n+1}(g_1,\ldots,g_n,\gamma)) \cdot \prod_{T\in S(n)} \\ \prod_{r_{n,T}} \varphi_{\tilde{\varepsilon}_n^n(\lambda)_T^{r_{n,T}}}&(\tilde{\varepsilon}_T^n(\tilde{\varepsilon}_n^{n+1}(g_1,\ldots,g_{n-1},g_n,\gamma))) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma).\end{aligned} \tag{3.75}$$

On the other hand, for any $k = 0,\ldots,n-1$, one has $(\tilde{\varepsilon}_k^n(g_1,\ldots,g_n),\gamma) = \tilde{\varepsilon}_k^{n+1}(g_1,\ldots,g_n,\gamma)$; hence

$$\begin{aligned}B_\lambda(g_1,\ldots,g_n) = (-1)^n \sum_{k=0}^{n-1} (-1)^k \int_{\mathcal{G}^{s(g_n)}} \sum_{(r_{k,T})_{T\in S(n)}} & f_{\tilde{\varepsilon}_k^n(\lambda)^{r_k}}(\tilde{\varepsilon}_k^{n+1}(g_1,\ldots,g_n,\gamma)) \cdot \prod_{T\in S(n)} \\ \prod_{r_{k,T}} \varphi_{\tilde{\varepsilon}_k^n(\lambda)_T^{r_{k,T}}} & (\tilde{\varepsilon}_T^n(\tilde{\varepsilon}_k^{n+1}(g_1,\ldots,g_n,\gamma))) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma).\end{aligned}$$

Thus, (3.74) becomes

$$\begin{aligned}(d^{n-1}h^{n-1}f)_\lambda(g_1,\ldots,g_n) = (-1)^n \sum_{k=0}^{n} (-1)^k \int_{\mathcal{G}^{s(g_n)}} \sum_{(r_{k,T})_{T\in S(n)}} & f_{\tilde{\varepsilon}_k^n(\lambda)^{r_k}}(\tilde{\varepsilon}_k^{n+1}(g_1,\ldots,g_n,\gamma)). \\ \prod_{T\in S(n)} \prod_{r_{k,T}} \varphi_{\tilde{\varepsilon}_k^n(\lambda)_T^{r_{k,T}}} & (\tilde{\varepsilon}_T^n(\tilde{\varepsilon}_k^{n+1}(g_1,\ldots,g_n,\gamma))) \cdot c(s(\gamma)) d\mu^{s(g_{n-1})}(\gamma).\end{aligned} \tag{3.76}$$

Now, for any $k=0,\dots,n$, $r_k=(r_{k,T})_{T\in S(n)}$, let $\gamma\in\mathcal{G}^{s(g_n)}$ such that $\tilde{\varepsilon}^{n+1}_k(g_1,\dots,g_n,\gamma)\in U^n_{\tilde{\varepsilon}^n_k(\lambda)^{r_k}}$. Then, there exists $l=(l_S)_{S\in S(n+1)}$ such that $(g_1,\dots,g_n,\gamma)\in U^{n+1}_{\lambda^l}$, so that

$$\tilde{\varepsilon}^{n+1}_k(g_1,\dots,g_n,\gamma)\in U^n_{\tilde{\varepsilon}^n_k(\lambda)^{r_k}}\bigcup U^n_{\tilde{\varepsilon}^{n+1}_k(\lambda^l)}.$$

One can then suppose that for any $k\in[n]$ and any family $r_k=(r_{k,T})_{T\in S(n)}$, there exists a family $l=(l_S)_{S\in S(n+1)}$ such that $\tilde{\varepsilon}^n_k(\lambda)^{r_k}=\tilde{\varepsilon}^{n+1}_k(\lambda^l)$. Moreover, in virtue to the identities (3.2), it is straightforward that for each $k\in[n]$ and any $T\in S(n)$, there exists a unique $S\in S(n+1)$ such that $\varepsilon^{n+1}_S=\varepsilon^{n+1}_k\circ\varepsilon^n_T$, so that $\tilde{\varepsilon}^{n+1}_S=\tilde{\varepsilon}^n_T\circ\tilde{\varepsilon}^{n+1}_k$. Therefore, we obtain from (3.76) that

$$\begin{aligned}(d^{n-1}h^{n-1}f)_\lambda(g_1,\dots,g_n)=(-1)^n\sum_{k=0}^{n}(-1)^k\int_{\mathcal{G}^{s(g_n)}}\sum_{(l_S)_{S\in S(n+1)}}f_{\tilde{\varepsilon}^{n+1}(\lambda^l)}(\tilde{\varepsilon}^{n+1}_k(g_1,\dots,g_n,\gamma)).\\ \prod_{S\in S(n+1)}\prod_{l_S}\varphi_{\lambda^{l_S}_S}(\tilde{\varepsilon}^{n+1}_S(g_1,\dots,g_n,\gamma)).c(s(\gamma))d\mu^{s(g_n)}(\gamma)\\ =A_\lambda(g_1,\dots,g_n).\end{aligned}\tag{3.77}$$

Combining with (3.72), we thus have shown that

$$h^n\circ d^n+d^{n-1}\circ h^{n-1}=\mathrm{Id}_{CR^n_{ss}(\mathfrak{U}_\bullet,E^\bullet)},\ \forall n\geq 1;\tag{3.78}$$

i.e. $h^\star$ defines a contraction of $CR^\star_{ss}(\mathfrak{U}_\bullet,E^\bullet)$ for any Real open cover $\mathfrak{U}_\bullet$ of $(\mathcal{G}_\bullet,\rho_\bullet)$ and this ends our proof. □

4

The *Real* graded Brauer group

In this chapter, we investigate Real graded groupoid C^*-dynamical systems. From these we define the Real graded Brauer group $\widehat{\mathrm{BrR}}(\mathcal{G})$ of a Real groupoid. The main purpose is to establish a cohomological formula of $\widehat{\mathrm{BrR}}(\mathcal{G})$.

4.1 Rg Dixmier-Douady bundles

We refer the reader to Appendix C for the basics of Real graded Banach bundles.

Definition 4.1.1. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a second countable locally compact Hausdorff Real groupoid. Let $p : \mathcal{A} \longrightarrow X$ be a locally trivial Rg u.s.c. Banach bundle. A $\mathcal{G}$-action by isomorphisms α on $\mathcal{A}$ is a collection $(\alpha_g)_{g\in\mathcal{G}}$ of graded isomorphisms (resp. $*$-isomorphisms) $\alpha_g : \mathcal{A}_{s(g)} \longrightarrow \mathcal{A}_{r(g)}$ such that*

(a) $g \cdot a := \alpha_g(a)$ makes $(\mathcal{A}, \sigma)$ into a (left) Real $\mathcal{G}$-space with respect to p;

(b) the induced anti-linear graded isomorphisms $\tau_x : \mathcal{A}_x \longrightarrow \mathcal{A}_{\bar{x}}$ verify $\tau_{r(g)} \circ \alpha_g = \alpha_{\bar{g}} \circ \tau_{s(g)} : \mathcal{A}_{s(g)} \xrightarrow{\cong} \mathcal{A}_{r(\bar{g})}$, for every $g \in \mathcal{G}$;

(c) $\alpha_{gg'} = \alpha_g \circ \alpha_{g'}$ for any $(g, g') \in \mathcal{G}^{(2)}$.

We say that $(\mathcal{A}, \alpha)$ is a Rg u.s.c. Banach $\mathcal{G}$-bundle. If the field $\tilde{\mathcal{A}} = \coprod_X \mathcal{A}_x \longrightarrow X$ is continuous, then $(\mathcal{A}, \alpha)$ is called a Rg Banach $\mathcal{G}$-bundle.

One also defines Rg u.s.c. C^*-$\mathcal{G}$-bundles, and Rg u.s.c. Hilbert bundles. In the case of C^*-$\mathcal{G}$-bundles, the isomorphisms α_g are required to be *-isomorphisms, while in the case of Hilbert $\mathcal{G}$-bundles, they are required to be isometries.

Definition 4.1.2. *A morphism of Rg Banach $\mathcal{G}$-bundles (resp. C^*-$\mathcal{G}$-bundles) from $(\mathcal{A},\alpha)$ from $(\mathcal{B},\beta)$ is a morphism $\Phi:\mathcal{A}\longrightarrow\mathcal{B}$ of Rg Banach bundles (resp. C^*-bundles) which is $\mathcal{G}$-equivariant; i.e., $\Phi_{r(g)}\circ\alpha_g=\beta_g\circ\Phi_{s(g)}$ for all $g\in\mathcal{G}$.*

Remark 4.1.3. *Notice that if $(\mathcal{A},\alpha)$ is Rg Banach $\mathcal{G}$-bundle , then $\alpha_x=\mathrm{Id}_{\mathcal{A}_x}$ for all $x\in X$. Indeed, we have for every $x\in X$, $\alpha_x:\mathcal{A}_x\longrightarrow\mathcal{A}_x$ is a graded automorphism, and $\alpha_x=\alpha_{x.x}=\alpha_x\circ\alpha_x$. In particular, if we put $x=gg^{-1}\in X$ for $g\in\mathcal{G}$, we obtain $\alpha_{gg^{-1}}=\alpha_g\circ\alpha_{g^{-1}}=\mathrm{Id}$ and then $\alpha_{g^{-1}}=\alpha_g^{-1}$ for every $g\in\mathcal{G}$.*

Definition 4.1.4. *A* Rg Dixmier-Douady (D-D for short) bundle *over $(\mathcal{G},\tau)$ is a Rg C^*-$\mathcal{G}$-bundle $(\mathcal{A},\alpha)$ such that $\mathcal{A}\longrightarrow X$ is a Rg elementary C^*-bundle that satisfies Fell's condition. Denote by $\widehat{\mathfrak{BrR}}(\mathcal{G})$ the collection all Rg D-D bundles over $\mathcal{G}$.*

Suppose α is a $\mathcal{G}$-action by isomorphisms on the Rg C^*-$\mathcal{G}$-bundle $\mathcal{A}$. Consider the Rg X-algebra $A=\mathcal{C}_0(X;\mathcal{A})$. Then α induces a Rg $\mathcal{C}_0(\mathcal{G})$-linear isomorphism $\alpha:s^*A\longrightarrow r^*A$ defined by $\alpha(f)(g):=\alpha_g(f(g))$ for $f\in s^*A$ and $g\in\mathcal{G}$.

Example 4.1.5. *Let $X\times\mathbb{C}$ be endowed with the Real structure $\overline{(x,t)}:=(\tau(x),\bar{t})$, and the $\mathcal{G}$-action by automorphisms $g\cdot(s(g),t):=(r(g),t)$. Then with respect to the projection $X\times\mathbb{C}\longrightarrow X$, $X\times\mathbb{C}$ is a Rg D-D bundle over $\mathcal{G}\underset{s}{\overset{r}{\rightrightarrows}}X$.*

Example 4.1.6. *Let $\mu=\{\mu^x\}_{x\in X}$ be a Real Haar system on $\mathcal{G}\underset{s}{\overset{r}{\rightrightarrows}}X$. Let the graded Hilbert space $\hat{\mathcal{H}}=l^2(\mathbb{N})\oplus l^2(\mathbb{N})$ be equipped with a fixed Real structure of type $J_{\mathbb{R},0}$ (see Appendix A ??). For $x\in X$, we put $\widehat{\mathcal{H}}_{p,x}:=L^2(\mathcal{G}^x)\hat{\otimes}\hat{\mathcal{H}}$, together with the scalar product $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle(x)$ given by*

$$\langle\!\langle\xi\,,\eta\rangle\!\rangle(x):=\int_{\mathcal{G}^x}\langle\xi(g),\eta(g)\rangle_{\mathbb{C}}\,d\mu_{\mathcal{G}}^x(g),\ \text{for}\ \xi,\eta\in L^2(\mathcal{G}^x;\hat{\mathcal{H}})\cong L^2(\mathcal{G}^x)\hat{\otimes}\hat{\mathcal{H}}.\qquad(4.1)$$

Let $\hat{\mathcal{H}}_{\mathcal{G}}:=\coprod_{x\in X}\hat{\mathcal{H}}_x$ be equipped with the action $g\cdot(s(g),\varphi\hat{\otimes}\xi):=(r(g),(\varphi\circ g^{-1})\hat{\otimes}\xi)\in\hat{\mathcal{H}}_{r(g)}$. Define the Real structure on $\widehat{\mathcal{H}}_{\mathcal{G}}$ by $(x,\varphi\hat{\otimes}\xi)\longmapsto(\tau(x),\tau(\varphi)\hat{\otimes}J_{\mathbb{R},0}(\xi))$. Then one shows that there exists a unique topology on $\hat{\mathcal{H}}_{\mathcal{G}}$ such that the canonical projection $\widehat{\mathcal{H}}_{\mathcal{G}}\longrightarrow\mathcal{G}^{(0)}$ defines a locally trivial Rg Hilbert $\mathcal{G}$-bundle.

Now, let $\widehat{\mathcal{K}}_x:=\mathcal{K}(\hat{\mathcal{H}}_x)$ be equipped with the operator norm topology, and put

$$\widehat{\mathcal{K}}_{\mathcal{G}}:=\coprod_{x\in X}\widehat{\mathcal{K}}_x$$

together with the Real structure given by $\overline{(x,T)}:=(\bar{x},\bar{T})$, where $\bar{T}\in\widehat{\mathcal{K}}_{\bar{x}}$ is defined by $\bar{T}(\varphi\hat{\otimes}\xi):=T(\tau(\varphi)\hat{\otimes}Ad_{J_{\mathbb{R},0}}(\xi))$ for any $\varphi\hat{\otimes}\xi\in\hat{\mathcal{H}}_{\bar{x}}$. Next, define the Real $\mathcal{G}$-action θ on $\widehat{\mathcal{K}}_{\mathcal{G}}$ by

$$\theta_g(s(g),T):=(r(g),gTg^{-1}).$$

We then have a Rg D-D bundle $(\widehat{\mathcal{K}}_{\mathcal{G}},\theta)$ over $\mathcal{G}$ given by the canonical projection

$$\widehat{\mathcal{K}}_{\mathcal{G}}\longrightarrow X,\ (x,T)\longmapsto x.$$

Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ and $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ be Real groupoids and let

$$\begin{tikzcd} Z \arrow[r, "\mathfrak{s}"] \arrow[d, "\mathfrak{r}"] & X \\ Y & \end{tikzcd}$$

be a generalized Real. Let $p : \mathcal{A} \longrightarrow X$ is a Rg (u.s.c.) Banach bundle. Then $\mathfrak{s}^*\mathcal{A} = Z \times_{\mathfrak{s},X,p} \mathcal{A}$ is a principal (right) $\mathcal{G}$-space by:

$$(z, a).g := (zg, \alpha_g^{-1}(a))$$

for $r(g) = z$. It is obviously a Real space with respect to the involution $\overline{(z,a)} := (\bar{z}, \bar{a})$. Next, define $\mathcal{A}^Z$ to be the quotient space $\mathfrak{s}^*\mathcal{A}/\mathcal{G}$ together with the induced Real structure defined by

$$\overline{[z,a]} := [\overline{(z,a)}],$$

where we use the notation $[z, a]$ to denote the orbit of (z, a) in $\mathcal{A}^Z$. Consider the continuous surjective map $\mathfrak{r} \circ pr_1 : \mathfrak{s}^*\mathcal{A} \longrightarrow Y$, $(z, a) \longmapsto \mathfrak{r}(z)$, where, as usual, pr_1 denotes the first projection. Since $\mathfrak{r}(zg) = \mathfrak{r}(z)$ for $(z, g) \in Z \times_{\mathfrak{s},X,r} \mathcal{G}$ (condition (i) of Definition 2.3.1), we get a well defined continuous surjection $p^Z : \mathcal{A}^Z \longrightarrow Y$ given by $p^Z([z, a]) := \mathfrak{r}(z)$. Furthermore, since $\mathfrak{r}$ is Real, one has

$$p^Z(\overline{[z,a]}) = p^Z([\bar{z}, \bar{a}]) = \mathfrak{r}(\bar{z}) = \overline{\mathfrak{r}(z)} = \overline{p^Z([z,a])}.$$

Thus, $p^Z : \mathcal{A}^Z \longrightarrow Y$ is a continuous Real surjection. Moreover, it is not hard to check that $p^Z : \mathcal{A}^Z \longrightarrow \Gamma^{(0)}$ is open and the map $a \longmapsto [z, a]$ defines a graded isomorphism from $\mathcal{A}_{\mathfrak{s}(z)}$ onto $\mathcal{A}^Z_{\mathfrak{r}(z)}$ (see [49, p.14]).

Proposition 4.1.7. *. Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ and $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ be locally compact Real groupoids. Suppose that $Z : \Gamma \longrightarrow \mathcal{G}$ is a generalized Real homomorphism and that $(\mathcal{A}, \alpha)$ is a Rg Banach $\mathcal{G}$-bundle. Then, with the constructions above, $p^Z : \mathcal{A}^Z \longrightarrow Y$ is a Rg Banach bundle. Furthermore,*

$$\alpha^Z_\gamma[z, a] := [\gamma \cdot z, a], \ \textit{for}\ \mathfrak{r}(z) = s(\gamma), \tag{4.2}$$

defines a Real left Γ-action on $\mathcal{A}^Z$ making $(\mathcal{A}^Z, \alpha^Z)$ into a Rg Banach Γ-bundle called the pull-back *of $(\mathcal{A}, \alpha)$ along Z.*

In particular, if $(\mathcal{A}, \alpha) \in \widehat{\mathfrak{Br}\mathfrak{R}}(\mathcal{G})$, then $(\mathcal{A}^Z, \alpha^Z) \in \widehat{\mathfrak{Br}\mathfrak{R}}(\Gamma)$.

The proof of this proposition is almost the same as that of [49, Proposition 2.15], hence we omit it.

Corollary 4.1.8. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *and* $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ *be locally compact Real groupoids. Suppose that* $Z : \mathcal{G} \longrightarrow \Gamma$ *is a Real Morita equivalence. Then the map* $\Phi^Z : \widehat{\mathfrak{BrR}}(\mathcal{G}) \longrightarrow \widehat{\mathfrak{BrR}}(\Gamma)$ *given by*

$$\Phi^Z(\mathcal{A},\alpha) := (\mathcal{A}^Z,\alpha^Z) \tag{4.3}$$

is a bijection.

We close this section by recalling a construction we will use in the sequel.

Definition 4.1.9. *(cf. [49, p.20]). Let* $(\mathcal{A},\alpha)$ *be a Rg (u.s.c.) Banach* $\mathcal{G}$*-bundle. Define the* conjugate bundle $(\overline{\mathcal{A}},\bar{\alpha})$ *of* $(\mathcal{A},\alpha)$ *as follows. Let* $\overline{\mathcal{A}}$ *be the topological Real space* $\mathcal{A}$ *and let* $\flat : \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ *be the identity map. Then* $\overline{p} : \overline{\mathcal{A}} \longrightarrow X$ *defined by* $\bar{p}(\flat(a)) = \flat(p(a))$ *is a Rg (u.s.c.) Banach bundle with fibre* $\overline{\mathcal{A}}_x$ *identified with the conjugate graded Banach algebra of* $\mathcal{A}_x$ *(the grading is* $\overline{\mathcal{A}}_x^i = \overline{\mathcal{A}_x^i}, i = 0,1$*). Furthermore, endowed with the Real* $\mathcal{G}$*-action by automorphisms* $\bar{\alpha}_g(\flat(a)) := \flat(\alpha_g(a))$ *for* $g \in \mathcal{G}, a \in \mathcal{A}_{s(g)}$, $\overline{\mathcal{A}} \longrightarrow X$ *becomes a Rg (u.s.c.) Banach* $\mathcal{G}$*-bundle. If* $(\mathcal{A},\alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$*, then* $(\overline{\mathcal{A}},\bar{\alpha}) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$.

4.2 The group $\widehat{\mathbf{Br}\mathrm{R}}(\mathcal{G})$

In this section we define the Brauer group of Real graded D-D bundles over a locally compact Real groupoid $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ with a paracompact base space.

Definition 4.2.1. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be second countable locally compact Hausdorff Real groupoid. Two element* $(\mathcal{A},\alpha)$ *and* $(\mathcal{B},\beta)$ *of* $\widehat{\mathfrak{BrR}}(\mathcal{G})$ *are* Morita equivalent *if there is a Rg* $\mathcal{A}$*-*$\mathcal{B}$*-imprimitivity bimodule* $\mathcal{X} \longrightarrow X$ *which admits a Real* $\mathcal{G}$*-action* V *by isomorphisms such that*

$$\begin{aligned} {}_{\mathcal{A}_{s(g)}}\langle V_g(\xi), V_g(\eta)\rangle &= \alpha_g({}_{\mathcal{A}_{s(g)}}\langle \xi,\eta\rangle); \textit{and} \\ \langle V_g(\xi), V_g(\eta)\rangle_{\mathcal{B}_{s(g)}} &= \beta_g(\langle \xi,\eta\rangle_{\mathcal{B}_{s(g)}}) \end{aligned} \tag{4.4}$$

for all $g \in \mathcal{G}$*, and* $\xi,\eta \in \mathcal{X}_{s(g)}$.
In this case we write $(\mathcal{A},\alpha) \sim_{(\mathcal{X},V)} (\mathcal{B},\beta)$.

Example 4.2.2. *Suppose that* $\Phi : (\mathcal{A},\alpha) \longrightarrow (\mathcal{B},\beta)$ *is an isomorphism of Rg D-D bundles over* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *. Then* $(\mathcal{A},\alpha) \sim_{(\mathcal{B},\beta)} (\mathcal{B},\beta)$.

Lemma 4.2.3. *Morita equivalence of Rg DD-bundles over* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *is an equivalence relation in* $\widehat{\mathfrak{BrR}}(\mathcal{G})$.

Proof. Again the proof is a the same as in [49] (Lemma 3.2). Let us just recall how to prove that the relation is symmetric. Suppose $(\mathcal{A},\alpha)\sim_{(\mathcal{X},V)}(\mathcal{B},\beta)$. Then define the structure of Rg $\mathcal{B}$-$\mathcal{A}$-bimodule on the conjugate bundle $(\overline{\mathcal{X}},\overline{V})$ (cf.Definition 4.1.9) by setting:

$$\begin{aligned} b\cdot\iota(\xi) &:= \iota(\xi\cdot b^*), \quad {}_{\mathcal{B}_x}\langle\iota(\xi),\iota(\eta)\rangle := \langle\xi,\eta\rangle_{\mathcal{B}_x}, \\ \iota(\xi)\cdot a &:= \iota(a^*\cdot\xi), \quad \langle\iota(\xi),\iota(\eta)\rangle_{\mathcal{A}_x} := {}_{\mathcal{A}_x}\langle\xi,\eta\rangle, \end{aligned}$$

for all $x\in\mathcal{G}^{(0)}$, $a\in\mathcal{A}_x$, $b\in\mathcal{B}_x$ and $\xi,\eta\in\mathcal{X}_x$. With these operations, each fibre of $\overline{\mathcal{X}}$ becomes a graded $\mathcal{B}$-$\mathcal{A}$-imprimitivity bimodule which satisfies conditions (a) and (b) of Definition **??**. It is moreover easy to verify that relations (4.4) hold if we replace $(\mathcal{A},\alpha)$ by $(\mathcal{B},\beta)$ and $(\mathcal{X},V)$ by $(\overline{\mathcal{X}},\overline{V})$; so that $(\mathcal{B},\beta)\sim_{(\overline{\mathcal{X}},\overline{V})}(\mathcal{A},\alpha)$. □

Definition 4.2.4. *Let* $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ *be a second countable locally compact Hausdorff Real groupoid.* The Real graded Brauer group of $\mathcal{G}$ $\widehat{\mathrm{BrR}}(\mathcal{G})$ *is defined as the set of Morita equivalence classes of Rg D-D bundles over* $\mathcal{G}$*. The class of* $(\mathcal{A},\alpha)$ *in* $\widehat{\mathrm{BrR}}(\mathcal{G})$ *is denoted by* $[\mathcal{A},\alpha]$.

Example 4.2.5. *Let* $\mathcal{G}$ *consist of the point* $\{*\}$ *together with the trivial involution. Then, every Rg DD-bundle over* $\{*\}$ *is trivial; i.e. it is given by a Rg elementary* C^**-algebra* $\widehat{\mathcal{K}}_p$*. We thus recover the Real graded Brauer group of the point* $\widehat{\mathrm{BrR}}(*)\cong\mathbb{Z}_8$ *described in A.5.*

Let $(\mathcal{A},\alpha)$ and $(\mathcal{B},\beta)$ be Rg DD-bundles. We have already defined the tensor product $\mathcal{A}\hat{\otimes}_X\mathcal{B}$ which is a Rg C^*-bundle over X. We want to equip this tensor product with a Real $\mathcal{G}$-action $\alpha\hat{\otimes}\beta$ such that $(\mathcal{A}\hat{\otimes}_X\mathcal{B},\alpha\hat{\otimes}\beta)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$. We define $\alpha\hat{\otimes}\beta$ as follows. For all $g\in\mathcal{G}$, we put $\alpha_g\hat{\otimes}\beta_g:\mathcal{A}_{s(g)}\hat{\otimes}\mathcal{B}_{s(g)}\longrightarrow\mathcal{A}_{r(g)}\hat{\otimes}\mathcal{B}_{r(g)},\ a\hat{\otimes}b\longmapsto\alpha_g(a)\hat{\otimes}\beta_g(b)$. Note that from the definition of a Real $\mathcal{G}$-action on a Rg C^*-bundle, $\alpha_g\hat{\otimes}\beta_g$ is a graded $*$-isomorphism that clearly verifies conditions (b) and (c) of Definition 4.1.1. Therefore, the same arguments used in [49, p.18] can be used here to show that $\alpha\hat{\otimes}\beta$ is continuous; thus, its restriction $\alpha\hat{\otimes}\beta$ on the closed subset $\mathcal{A}\hat{\otimes}_X\mathcal{B}$ of $\mathcal{A}\hat{\otimes}\mathcal{B}$ defines a Real $\mathcal{G}$-action. Furthermore, it can be shown that this operation is Morita equivalence preserving ([49, p.19]).

Proposition 4.2.6. *Let* $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ *be a locally compact Real groupoid such that* X *is paracompact. Then* $\widehat{\mathrm{BrR}}(\mathcal{G})$ *is an abelian group with respect to the operations*

$$[\mathcal{A},\alpha]+[\mathcal{B},\beta]:=[\mathcal{A}\hat{\otimes}_X\mathcal{B},\alpha\hat{\otimes}\beta]. \tag{4.5}$$

The identity of $\widehat{\mathrm{BrR}}(\mathcal{G})$ *is given by the class* $0:=[X\times\mathbb{C},\tau\times bar]$ *of the Rg D-D bundle defined in Example 4.1.5. The inverse of* $[\mathcal{A},\alpha]$ *is* $[\overline{\mathcal{A}},\bar{\alpha}]$.

Proof. See Proposition 3.6 and Theorem 3.7 in [49]. □

For the sake of simplicity we will often use the following notations.

Notations 4.2.7. *We will write $\mathcal{A}$ for the class $[\mathcal{A},\alpha]$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$; we will also leave out the actions when we are working in the group $\widehat{\mathrm{BrR}}(\mathcal{G})$: for instance we will write $\mathcal{A}+\mathcal{B}$ instead of $[\mathcal{A},\alpha]+[\mathcal{B},\beta]$.*

Lemma 4.2.8. *Let $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$ and let $(\widehat{\mathcal{K}}_{\mathcal{G}},\theta)$ be the Rg D-D bundle defined in Example 4.1.6. Then $\mathcal{A}+\widehat{\mathcal{K}}_{\mathcal{G}}=\mathcal{A}$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$.*

Proof. Recall that the Real $\mathcal{G}$-action θ is given by Ad_{Θ}, where Θ is the Real $\mathcal{G}$-action on the Rg Hilbert $\mathcal{G}$-bundle $\hat{\mathcal{H}}_{\mathcal{G}}\longrightarrow X$ (see Example 4.1.6); i.e. $\theta_g(T)=\Theta_g T\Theta_g^{-1}$. The Rg Banach $\mathcal{G}$-bundle $(\mathcal{A}\hat{\otimes}_X\hat{\mathcal{A}},\alpha\hat{\otimes}\Theta)$ is easily checked to be a Morita equivalence

$$(\mathcal{A}\hat{\otimes}_X\widehat{\mathcal{K}}_{\mathcal{G}},\alpha\hat{\otimes}\theta)\sim(\mathcal{A},\alpha)$$

in $\widehat{\mathfrak{BrR}}(\mathcal{G})$ with respect to the pointwise actions and inner-products operations:

$$\begin{aligned}
(a\hat{\odot}T)\cdot(b\hat{\otimes}\xi) &:= (-1)^{\partial T\cdot\partial b}ab\hat{\odot}T\xi, \text{ and}\\
{}_{\mathcal{A}_x\hat{\otimes}\widehat{\mathcal{K}}_x}\langle b\hat{\odot}\xi,d\hat{\odot}\eta\rangle &:= (-1)^{\partial\xi\partial d}bd^*\hat{\odot}T_{\xi,\eta};\\
(b\hat{\odot}\xi)\cdot c &:= bc\hat{\odot}\xi, \text{ and}\\
\langle b\hat{\odot}\xi,d\hat{\odot}\eta\rangle_{\mathcal{A}_x} &:= \langle\!\langle\xi,\bar{\eta}\rangle\!\rangle(x)\cdot b^*d,
\end{aligned}$$

for $x\in X$, $a\hat{\odot}T\in\mathcal{A}_x\hat{\otimes}\widehat{\mathcal{K}}_x, b\hat{\odot}\xi,d\hat{\odot}\eta\in\mathcal{A}_x\hat{\otimes}\widehat{\mathcal{H}}_x$, and $c\in\mathcal{A}_x$. □

Lemma 4.2.9. *Let $\mathcal{G}\underset{s}{\overset{r}{\rightrightarrows}}X$ be a locally compact Hausdorff Real groupoid with paracompact base space, and let $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$. Then $\mathcal{A}=0$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$ if and only if there exists a Rg Hilbert $\mathcal{G}$-bundle $(\mathscr{H},U)$ such that $(\mathcal{A},\alpha)\cong(\mathcal{K}(\mathscr{H}),Ad_U)$ in $\widehat{\mathfrak{BrR}}(\mathcal{G})$.*

Proof. If $(\mathcal{X},V)$ is a Morita equivalence between $(\mathcal{A},\alpha)$ and the trivial bundle $X\times\mathbb{C}$, then each fibre $\mathcal{X}_x$ is a graded Hilbert space; and since $\mathcal{X}_x$ is a full graded Hilbert $\mathcal{A}_x$-module and since $\mathcal{X}$ is a Real Morita equivalence, there is an isomorphism of graded C^*-algebras $\varphi_x:\mathcal{A}_x\longrightarrow\mathcal{K}(\mathcal{X}_x)$ such that $\varphi_x({}_{\mathcal{A}_x}\langle\xi,\eta\rangle)=T_{\xi,\eta}$, for all $\xi,\eta\in\mathcal{X}_x$, and $\varphi_{\bar{x}}(a)=\overline{\varphi_x(\bar{a})}$ for all $a\in\mathcal{A}_x$. Moreover, in view of relations (4.4), we have

$$\varphi_{r(g)}(\alpha_g({}_{\mathcal{A}_{s(g)}}\langle\xi,\eta\rangle))=\varphi_{r(g)}({}_{\mathcal{A}_{r(g)}}\langle V_g(\xi),V_g(\eta)\rangle)=T_{V_g\xi,V_g\eta}=Ad_{V_g}(T_{\xi,\eta}),$$

for every $\gamma\in\mathcal{G}$ and $\xi,\eta\in\mathcal{A}_{s(g)}$. It follows that the family $(\varphi_x)_{x\in X}$ is an isomorphism of Real graded D-D bundles $\varphi:(\mathcal{A},\alpha)\longrightarrow(\mathcal{K}(\mathcal{X}),Ad_V)$.

Conversely, using the same operations as in the proof of Lemma 4.2.8, the Rg Hilbert $\mathcal{G}$-bundle $(\mathscr{H},U)$ defines a Morita equivalence of Rg D-D bundles between $(\mathcal{K}(\mathscr{H}),Ad_U)$ and the trivial one $X\times\mathbb{C}\longrightarrow X$. □

From this lemma we deduce the following characterization of Morita equivalent Rg D-D bundles.

Corollary 4.2.10. *Let* $(\mathcal{A},\alpha)$ *and* $(\mathcal{B},\beta)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$. *Then* $\mathcal{A}=\mathcal{B}$ *in* $\widehat{\mathrm{BrR}}(\mathcal{G})$ *if and only if there exists a Rg Hilbert* $\mathcal{G}$*-bundle* $(\mathscr{H},U)$ *such that* $(\mathcal{A}\hat{\otimes}_X\overline{\mathcal{B}},\alpha\hat{\otimes}\bar{\beta})\cong(\mathcal{K}(\mathscr{H}),Ad_U)$ *in* $\widehat{\mathfrak{BrR}}(\mathcal{G})$.

4.3 Complex and orthogonal Brauer groups

The purpose of this section is to compare the group $\widehat{\mathrm{BrR}}(\mathcal{G})$ of a Real groupoid $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ we defined in the previous section with the well-known graded complex and Brauer group $\widehat{\mathrm{Br}}(\mathcal{G})$ of the groupoid $\mathcal{G}$ (see [70], [28], [87], and [30]), as well as with a generalization of Donovan-Karoubi's *graded orthogonal Brauer group* $\widehat{\mathrm{BrO}}(X)$ ([28]).

Recall [87] that the *graded complex Brauer group* $\widehat{\mathrm{Br}}(\mathcal{G})$ is defined as the set of Morita equivalence classes of *graded complex D-D bundles* [1] over the groupoid $\mathcal{G}$. Moreover, there is an interpretation of $\widehat{\mathrm{Br}}(\mathcal{G})$ in terms of Čech cohomology classes; more precisely, there is an isomorphism

$$\widehat{\mathrm{Br}}(\mathcal{G})\cong\check{H}^0(\mathcal{G}_\bullet,\mathbb{Z}_2)\oplus\left(\check{H}^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\ltimes\check{H}^2(\mathcal{G}_\bullet,\mathcal{S}^1)\right). \tag{4.6}$$

For topological spaces, this group was denoted by $\mathsf{GBr}^\infty(X)$ in Parker's paper [70].

In order to defined *twisted K-theory*, Donovan and Karoubi have defined in their fundamental paper [28] two groups $\mathsf{GBrU}(X)$ and $\mathsf{GBrO}(X)$ respectively called *the graded unitary Brauer group* and *the graded orthogonal Brauer group* of the sapce X. The former is just the finite-dimensional version of $\mathsf{GBr}^\infty(X)$, while the latter is the set of equivalence classes of *graded real simple algebra bundles*. They show that

$$\mathsf{GBrO}X\cong\check{H}^0(X,\mathbb{Z}_8)\oplus\left(\check{H}^1(X,\mathbb{Z}_2)\ltimes\check{H}^2(X,\mathbb{Z}_2)\right). \tag{4.7}$$

We will define an infinite-dimensional analog of $\mathsf{GBrO}(X)$ for groupoids, and show later an isomorphism analogous to (4.7).

Proposition 4.3.1. *Suppose that* $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ *is a Real groupoid which can be written as the disjoint union of two locally compact groupoids* $\mathcal{G}_1\rightrightarrows X_1$ *and* $\mathcal{G}_2\rightrightarrows X_2$ *such that* $\tau(g_1)\in\mathcal{G}_2,\tau(g_2)\in\mathcal{G}_1,\forall g_1\in\mathcal{G}_1,g_2\in\mathcal{G}_2$. *Then*

$$\widehat{Br\mathrm{R}}(\mathcal{G})\cong\widehat{Br}(\mathcal{G}_1)\cong\widehat{Br}(\mathcal{G}_2).$$

Proof. Observe first that τ induces an isomorphism $\mathcal{G}_1\cong\mathcal{G}_2$, so that $\widehat{\mathrm{Br}}(\mathcal{G}_1)\cong\widehat{\mathrm{Br}}(\mathcal{G}_2)$.

[1] Elements of $\widehat{\mathrm{Br}}(\mathcal{G})$ are defined in the same way as that of $\widehat{\mathrm{BrR}}(\mathcal{G})$ except that no Real structures are involved

Let $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$. Then $\mathcal{A}=\mathcal{A}_1\oplus\mathcal{A}_2$, where $\mathcal{A}_1\longrightarrow X_1$ and $\mathcal{A}_2\longrightarrow X_2$ are graded complex elementary C^*-bundles. It is straightforward that the graded action α of $\mathcal{A}$ induces a $\mathcal{G}_i$-action α_i on $\mathcal{A}_i, i=1,2$, making $(\mathcal{A}_i,\alpha_i)$ into a graded complex D-D bundle over $\mathcal{G}_i$. However, since the projection $p:\mathcal{A}\to X_1\sqcup X_2$ intertwines the Real structure of $\mathcal{A}$ and that of X, we have $\bar{a}_1\in\mathcal{A}_2$ and $\bar{a}_2\in\mathcal{A}_1$ for all $a_1\in\mathcal{A}_1, a_2\in\mathcal{A}_2$. Indeed, over all $x\in X_1$, the involution induces the conjugate linear isomorphism

$$\tau_x:\mathcal{A}_x=(\mathcal{A}_1)_x\longrightarrow\mathcal{A}_{\bar{x}}=(\mathcal{A}_2)_{\bar{x}}.$$

It turns out that the Real structure of $\mathcal{A}$ induces an isomorphism of graded complex D-D bundles

$$\tau:(\mathcal{A}_2,\alpha_2)\xrightarrow{\cong}(\overline{\mathcal{A}_1},\overline{\alpha_1})$$

over the groupoid $\mathcal{G}_2$. In fact $(\overline{\mathcal{A}},\bar{\alpha})$ is isomorphic to the Rg D-D bundle $(\mathcal{A}',\alpha')=(\tau^*\mathcal{A},\tau^*\alpha)$; i.e. $(\mathcal{A}',\alpha')$ is such that

$$\begin{cases}\mathcal{A}'_x=(\mathcal{A}_2)_{\bar{x}}, & \text{if } x\in X_1;\\ \mathcal{A}'_x=(\mathcal{A}_1)_{\bar{x}}, & \text{if } x\in X_2;\\ \alpha'_{g_1}=\alpha_{\bar{g}_1}:(\mathcal{A}_2)_{s(\bar{g}_1)}\longrightarrow(\mathcal{A}_2)_{r(\bar{g}_1)}, & g_1\in\mathcal{G}_1;\\ \alpha'_{g_2}=\alpha_{\bar{g}_2}:(\mathcal{A}_1)_{s(\bar{g}_2)}\longrightarrow(\mathcal{A}_1)_{r(\bar{g}_2)}, & g_2\in\mathcal{G}_2.\end{cases}\tag{4.8}$$

Note that the same is true for every Rg Banach bundle over $\mathcal{G}$. Define the map

$$\begin{array}{rccc}\Phi_{12}: & \widehat{\mathrm{BrR}}(\mathcal{G}) & \longrightarrow & \widehat{\mathrm{Br}}(\mathcal{G}_1)\\ & [\mathcal{A},\alpha] & \longmapsto & [\mathcal{A}_1,\alpha_1],\end{array}$$

Φ_{12} is well-defined since if $(\mathcal{A},\alpha)\sim_{(\mathcal{X},V)}(\mathcal{B},\beta)$ in $\widehat{\mathfrak{BrR}}(\mathcal{G})$, then the restriction $(\mathcal{X}_1,V_1)$ of $(\mathcal{X},V)$ over $\mathcal{G}_1$ induces a Morita equivalence of graded complex D-D bundles $(\mathcal{A}_1,\alpha_1)\sim(\mathcal{B}_1,\beta_1)$ over $\mathcal{G}_1$. Moreover from the identifications (4.8) we see that $\Phi_{12}(-[\mathcal{A},\alpha])=-\Phi([\mathcal{A},\alpha])$. Furthermore, we have clearly $(\mathcal{A}\hat{\otimes}_X\mathcal{B})_i=\mathcal{A}_i\hat{\otimes}_{X_i}\mathcal{B}_i$ for $i=1,2$, and that the involution induces an isomorphism of graded complex D-D bundles $\mathcal{A}_2\hat{\otimes}_{X_2}\mathcal{B}_2\xrightarrow{\cong}\overline{\mathcal{A}_1\hat{\otimes}_{X_1}\mathcal{B}_1}=\overline{\mathcal{A}_1}\hat{\otimes}_{X_2}\overline{\mathcal{B}_1}$ over $\mathcal{G}_2\rightrightarrows X_2$, which shows Φ_{12} is a group homomorphism.
Conversely, if $(\mathcal{A}_1,\alpha_1)$ is a graded complex D-D bundle over $\mathcal{G}_1$, we define the Real graded D-D bundle over $\mathcal{G}$ by setting $\mathcal{A}:=\mathcal{A}_1\oplus\overline{\tau^*_{|X_2}\mathcal{A}_1}$, and $\alpha:=\alpha_1\oplus\overline{\tau^*_{|\mathcal{G}_2}\alpha_1}$; then we define

$$\begin{array}{rccc}\Phi'_{12}: & \widehat{\mathrm{Br}}(\mathcal{G}_1) & \longrightarrow & \widehat{\mathrm{BrR}}(\mathcal{G})\\ & [\mathcal{A}_1,\alpha_1] & \longmapsto & [\mathcal{A}_1\oplus\overline{\tau^*_{|X_2}\mathcal{A}_1},\alpha_1\oplus\overline{\tau^*_{|\mathcal{G}_2}\alpha_1}].\end{array}$$

It is clear that Φ_{12} and Φ'_{12} are inverse of each other. □

Corollary 4.3.2. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be a locally compact Hausdorff groupoid with paracompact unit space. Let the product groupoid $\mathcal{G}\times\mathbf{S}^{0,1}\rightrightarrows X\times\mathbf{S}^{0,1}$ be equipped with the Real structure $(g,\pm1)\longmapsto(g,\mp1)$. Then*

$$\widehat{Br}\mathrm{R}(\mathcal{G}\times\mathbf{S}^{0,1})\cong\widehat{Br}(\mathcal{G}).$$

Proof. Apply Proposition 4.3.1 to $\mathcal{G} = (\mathcal{G}\times\{+1\})\sqcup(\mathcal{G}\times\{-1\})$. $\square$

Example 4.3.3 (Computation of $\widehat{\mathrm{BrR}}(\mathbf{S}^{0,1})$). *The groupoid* $\mathbf{S}^{0,1} \rightrightarrows \mathbf{S}^{0,1}$ *identifies with* $\{pt\}\times\{\pm 1\}$. *Thus from Corollary 4.3.2 we get*

$$\widehat{Br}\mathrm{R}(\mathbf{S}^{0,1}) \cong \widehat{Br}(\{pt\}) \cong \mathbb{Z}_2.$$

Definition 4.3.4. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *be a locally compact groupoid. A* graded real [2] D-D bundle $(\mathcal{A},\alpha)$ *over* $\mathcal{G}$ *consists of a locally trivial* C^**-bundle* $p:\mathcal{A}\longrightarrow X$*, a family of isomorphisms of graded* $\mathbb{R}$*-*C^**-algebras* $\alpha_g:\mathcal{A}_{s(g)}\longrightarrow\mathcal{A}_{r(g)}$*, such that*

(a) the operation $g\cdot a := \alpha_g(a)$ *makes* $\mathcal{A}$ *into a* $\mathcal{G}$*-space with respect to the projection* p*;*

(b) $\alpha_{gh} = \alpha_g\circ\alpha_h, \forall (g,h)\in\mathcal{G}^{(2)}$*;*

(c) the complexification $(\mathcal{A}_{\mathbb{C}},\alpha_{\mathbb{C}})$ *of* $(\mathcal{A},\alpha)$ *defines an element of the collection* $\widehat{\mathfrak{Br}}(\mathcal{G})$ *of graded complex D-D bundles over* $\mathcal{G}$*, where* $\mathcal{A}_{\mathbb{C}} := \mathcal{A}\otimes_{\mathbb{R}}\mathbb{C}\longrightarrow X$ *is the bundle with fibre* $(\mathcal{A}_{\mathbb{C}})_x := \mathcal{A}_x\otimes_{\mathbb{R}}\mathbb{C}$*, and for* $g\in\mathcal{G}$*,* $(\alpha_{\mathbb{C}})_g := \alpha\otimes\mathrm{Id}_{\mathbb{C}}$.

Definition 4.3.5. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *be a locally compact groupoid. The* graded orthogonal Brauer group $\widehat{\mathrm{BrO}}(\mathcal{G})$ *of* $\mathcal{G}$ *is defined to be the set or Morita equivalence classes of graded real D-D bundles over* $\mathcal{G}$*, where two such bundles* $(\mathcal{A},\alpha)$ *and* $(\mathcal{B},\beta)$ *are said to be Morita equivalent if and only if their complexifications* $(\mathcal{A}_{\mathbb{C}},\alpha_{\mathbb{C}})$ *and* $(\mathcal{B}_{\mathbb{C}},\beta_{\mathbb{C}})$ *are Morita equivalent in* $\widehat{\mathfrak{Br}}(\mathcal{G})$.

We will use the same notations in $\widehat{\mathrm{Br}}(\mathcal{G})$ and $\widehat{\mathrm{BrO}}(\mathcal{G})$ as in Notations 4.2.7.

Theorem 4.3.6. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *be a locally compact Hausdorff Real groupoid with* X *paracompact.*

1. *If the Real structure* τ *is fixed point free, then we have an isomorphism*

$$\widehat{Br}(\mathcal{G})\otimes\mathbb{Z}[1/2] \cong (\widehat{Br}\mathrm{R}(\mathcal{G})\oplus\widehat{Br}(\mathcal{G}/_\tau))\otimes\mathbb{Z}[1/2], \tag{4.9}$$

   where $\mathcal{G}/_\tau$ *is the groupoid* $\mathcal{G}/_\tau \rightrightarrows X/_\tau$ *obtained from* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *by identifying every point* $g\in\mathcal{G}$ *with its image by* τ.

2. *It* τ *is trivial, then every element* $\mathcal{A}\in\widehat{Br}\mathrm{R}(\mathcal{G})$ *is a* 2*-torsion; i.e.*

$$2\mathcal{A} = 0.$$

Furthermore, $\widehat{Br}\mathrm{R}(\mathcal{G})\cong\widehat{Br}\mathrm{O}(\mathcal{G})$*. In particular,* $\widehat{Br}\mathrm{O}(\mathcal{G})$ *is an abelian group under the obvious operations, zero element being given by the trivial bundle* $X\times\mathbb{R}\longrightarrow X$ *with the* $\mathcal{G}$*-action* $g\cdot(s(g),t) := (r(g),t)$.

[2] Here "real" with a lowercase "r" is to emphasize that the fibers of $\mathcal{A}$ are $\mathbb{R}$-C^*-algebras.

We shall mention that roperty 2 was already proved by D. Saltman in the special case of Azumaya algebras with involution (see [79, Theorem 4.4 (a)]). Our result is then a generalization of this to infinite-dimensional Real bundles of algebras.

To prove Theorem 4.3.6 we need the

Lemma 4.3.7. *Let $(\mathcal{G},\tau)$ be a locally compact Hausdorff Real groupoid with paracompact base space. Then the assignment $(\mathcal{A},\alpha)\longmapsto(\overline{\tau^*\mathcal{A}},\overline{\tau^*\alpha})$ defines an involution on the group*

$$\begin{array}{rccc}\hat{\tau}: & \widehat{Br}(\mathcal{G}) & \longrightarrow & \widehat{Br}(\mathcal{G})\\ & \mathcal{A} & \longmapsto & -\tau^*\mathcal{A}\end{array}$$

such that the Real part $\widehat{Br}(\mathcal{G})_\tau$ is isomorphic to $\widehat{BrR}(\mathcal{G})$ after tensoring with $\mathbb{Q}$; more precisely,

$$\widehat{Br}(\mathcal{G})_\tau\otimes\mathbb{Z}[\frac{1}{2}]\cong\widehat{BrR}(\mathcal{G})\otimes\mathbb{Z}[\frac{1}{2}].$$

Proof. That $\hat{\tau}$ is a group homomorphism follows from the functorial property of the abelian $\widehat{\mathrm{Br}}(\mathcal{G})$ ([49]). Now let

$$\Phi_{\mathbb{C}}:\widehat{\mathrm{BrR}}(\mathcal{G})\longrightarrow\widehat{\mathrm{Br}}(\mathcal{G}),\mathcal{A}\longmapsto\mathcal{A}$$

be the map consisting of "forgetting the Real structures", and let

$$\begin{array}{rccc}\Phi_R: & \widehat{\mathrm{Br}}(\mathcal{G}) & \longrightarrow & \widehat{\mathrm{BrR}}(\mathcal{G})\\ & \mathcal{A} & \longmapsto & \mathcal{A}+\hat{\tau}(\mathcal{A})\end{array}$$

That $\Phi_{\mathbb{C}}$ is a well-defined group homomorphism is clear.

To prove that Φ_R is well defined, we shall first verify that $(\mathcal{A}\hat{\otimes}_X\overline{\tau^*\mathcal{A}},\alpha\hat{\otimes}\overline{\tau^*\alpha})\in\widehat{\mathfrak{BrR}}(\mathcal{G})$ for all $(\mathcal{A},\alpha)\in\widehat{\mathfrak{Br}}(\mathcal{G})$. Let $\sigma=(\sigma_x)_x$ be the family of conjugate-linear isomorphisms of graded complex C^*-algebras $\sigma_x:\mathcal{A}_x\hat{\otimes}\overline{\mathcal{A}_{\bar{x}}}\longrightarrow\mathcal{A}_{\bar{x}}\hat{\otimes}\overline{\mathcal{A}_x}$ given on homogeneous tensors by

$$\sigma_x(a\hat{\odot}\flat(b)):=(-1)^{\partial a\cdot\partial b}(b\hat{\odot}\flat(a)).\qquad(4.10)$$

Then σ is a Real structure on the bundle $\mathcal{A}\hat{\otimes}_X\overline{\tau^*\mathcal{A}}\longrightarrow X$, and it is a matter of simple verifications to see that conditions (a)-(c) in Definition 4.1.1 are satisfied when $(\mathcal{A}\hat{\otimes}_X\overline{\tau^*\mathcal{A}},\alpha\hat{\otimes}\overline{\tau^*\alpha})$ is equipped with the involution σ.

Suppose now that $(\mathcal{A},\alpha)\sim_{(\mathcal{A},V)}(\mathcal{B},\beta)$ in $\widehat{\mathfrak{Br}}(\mathcal{G})$. By using the same reasoning as the one we used above for graded complex D-D bundle, one verifies that the graded complex Banach $\mathcal{G}$-bundle $(\mathcal{X}\hat{\otimes}_X\overline{\tau^*\mathcal{X}},V\hat{\otimes}\overline{\tau^*V})$ admits a Real structure $\sigma^{\mathcal{X}}$ making it into a Rg Banach $\mathcal{G}$-bundle. Note that this bundle implements a Morita equivalence $(\mathcal{A}\hat{\otimes}_X\overline{\tau^*\mathcal{A}},\alpha\hat{\otimes}\overline{\tau^*\alpha})\sim(\mathcal{B}\hat{\otimes}_X\overline{\tau^*\mathcal{B}},\beta\hat{\otimes}\overline{\tau^*\beta})$ in $\widehat{\mathfrak{Br}}(\mathcal{G})$. Moreover, since by definition

$${}_{\mathcal{A}_x\hat{\otimes}\overline{\mathcal{A}_{\bar{x}}}}\langle\xi\hat{\odot}\flat(\eta),\xi'\hat{\odot}\flat(\eta')\rangle={}_{\mathcal{A}_x}\langle\xi,\xi'\rangle\hat{\odot}_{\overline{\mathcal{A}_{\bar{x}}}}\langle\flat(\eta),\flat(\eta')\rangle,\ \textit{and}\ \langle\xi\hat{\odot}\flat(\eta),\xi'\hat{\odot}\flat(\eta')\rangle_{\mathcal{B}_x\hat{\otimes}\overline{\mathcal{B}_{\bar{x}}}}$$

for every $x\in X,\xi,\xi',\eta,\eta'\in\mathcal{X}_x$, then we see that the inner products ${}_{\mathcal{A}\hat{\otimes}_X\overline{\tau^*\mathcal{A}}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{B}\hat{\otimes}_X\overline{\tau^*\mathcal{B}}}$ of $\mathcal{X}\hat{\otimes}_X\overline{\tau^*\mathcal{X}}$ intertwine the Real structures; hence we have a Morita equivalence

$$(\mathcal{A}\hat{\otimes}_X\overline{\tau^*\mathcal{A}},\alpha\hat{\otimes}\overline{\tau^*\alpha})\sim_{(\mathcal{X}\hat{\otimes}_X\overline{\tau^*\mathcal{X}},V\hat{\otimes}\overline{\tau^*V})}(\mathcal{B}\hat{\otimes}_X\overline{\tau^*\mathcal{B}},\beta\hat{\otimes}\overline{\tau^*\beta})$$

in $\widehat{\mathfrak{BrR}}(\mathcal{G})$, so that Φ_R is well defined.

Φ_R is a group homomorphism since $\widehat{\mathrm{Br}}(\mathcal{G})$ is an abelian group and since $\hat{\tau}$ is linear; i.e. for every $\mathcal{A},\mathcal{B}\in\widehat{\mathrm{Br}}(\mathcal{G})$,

$$(\mathcal{A}+\hat{\tau}(\mathcal{A}))+(\mathcal{B}+\hat{\tau}(\mathcal{B}))=(\mathcal{A}+\mathcal{B})+\hat{\tau}(\mathcal{A}+\mathcal{B}).$$

Let us verify that up to inverting 2, Φ'_R and $\Phi_{\mathbb{C}}$ are inverse of each other, where Φ'_R is the restriction of Φ_R on the fixed points $\widehat{\mathrm{Br}}(\mathcal{G})_{\mathbb{R}}$ of $\hat{\tau}$. First observe that if $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$, then the Real structure of $\mathcal{A}$ induces an isomorphism $(\mathcal{A},\alpha)\cong(\overline{\tau^*\mathcal{A}},\overline{\tau^*\alpha})$ in $\widehat{\mathfrak{Br}}(\mathcal{G})$. Thus, for $\mathcal{A}\in\widehat{\mathrm{BrR}}(\mathcal{G})$, we get $(\Phi_R\circ\Phi_{\mathbb{C}})(\mathcal{A})=2\mathcal{A}$. Suppose now that $\mathcal{A}\in\widehat{\mathrm{Br}}(\mathcal{G})_{\mathbb{R}}$. Then $(\Phi_{\mathbb{C}}\circ\Phi'_R)(\mathcal{A})=\Phi_{\mathbb{C}}(2\mathcal{A})=2\mathcal{A}$, which completes the proof. □

Remark 4.3.8. *It is straightforward, by using Lemma 4.3.7, that one has a similar characterization for graded complex D-D bundles as that of Corollary 4.2.10.*

Proof of Theorem 4.3.6. 1. It suffices to show that the imaginary part ${}^{\mathfrak{I}}\widehat{\mathrm{Br}}(\mathcal{G})$ with respect to the involution $\hat{\tau}:\widehat{\mathrm{Br}}(\mathcal{G})\longrightarrow\widehat{\mathrm{Br}}(\mathcal{G})$ of Lemma 4.3.7 is isomorphic to $\widehat{\mathrm{Br}}(\mathcal{G}/_{\tau})$ (afer inverting 2), and then we will apply Lemma 2.1.4.

Assume $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$ is such that $\hat{\tau}(\mathcal{A})=-\mathcal{A}$. Then thanks to Corollary 4.2.10, there exists a Rg Hilbert $\mathcal{G}$-bundle $(\hat{\mathscr{H}},U)$ and an isomorphism of Rg D-D bundles

$$(\tau^*\mathcal{A}\hat{\otimes}_X\mathcal{K}(\hat{\mathscr{H}}),\overline{\tau^*\alpha}\hat{\otimes}Ad_U)\xrightarrow{\cong}(\mathcal{A}\hat{\otimes}_X\mathcal{K}(\hat{\mathscr{H}}),\bar{\alpha}\hat{\otimes}Ad_U).\tag{4.11}$$

We then obtain a Rg D-D bundle $(\mathcal{A}/_{\tau},\alpha^{\tau})$ over $\mathcal{G}/_{\tau}\rightrightarrows X/_{\tau}$ by setting

$$\mathcal{A}/_{\tau}:=\mathcal{A}\hat{\otimes}_X\mathcal{K}(\hat{\mathscr{H}})\hat{\otimes}_X\mathcal{K}(\tau^*\hat{\mathscr{H}}),\ \text{and}\ \alpha^{\tau}:=\alpha\hat{\otimes}Ad_U\hat{\otimes}Ad_{\tau^*U}),\tag{4.12}$$

with projection $p_{\tau}:\mathcal{A}/_{\tau}\longrightarrow X/_{\tau}$ given by

$$p_{\tau}(a\hat{\odot}T\hat{\odot}T')=p(a),\ \text{for}\ a\hat{\odot}T\hat{\odot}T'\in\mathcal{A}_x\hat{\otimes}\mathcal{K}(\hat{\mathscr{H}}_x)\hat{\otimes}\mathcal{K}(\hat{\mathscr{H}}_{\bar{x}}).$$

Next define the map

$$\begin{array}{cccc}\Psi_{\tau}: & {}^{\mathfrak{I}}\widehat{\mathrm{BrR}}(\mathcal{G}) & \longrightarrow & \widehat{\mathrm{Br}}(\mathcal{G}/_{\tau})\\ & \mathcal{A} & \longmapsto & \mathcal{A}/_{\tau}.\end{array}$$

This definition does not depend on the choice of $(\hat{\mathscr{H}},U)$, for if $(\hat{\mathscr{H}}',U')$ is another Rg Hilbert $\mathcal{G}$-bundle such that $(\tau^*\mathcal{A}\hat{\otimes}_X\mathcal{K}(\hat{\mathscr{H}}'),\tau^*\alpha\hat{\otimes}Ad_{U'})\cong(\mathcal{A}\hat{\otimes}_X\mathcal{K}(\hat{\mathscr{H}}'),\alpha\hat{\otimes}Ad_{U'})$, then putting

$$\mathcal{A}'/_{\tau}:=\mathcal{A}\hat{\otimes}_X\mathcal{K}(\hat{\mathscr{H}}')\hat{\otimes}_X\mathcal{K}(\tau^*\hat{\mathscr{H}}'),$$

we get

$$\begin{aligned}\mathcal{A}/_{\tau}\hat{\otimes}_{X/_{\tau}}\overline{\mathcal{A}'/_{\tau}}&\cong\mathcal{A}\hat{\otimes}_X\overline{\mathcal{A}}\hat{\otimes}_X\hat{\otimes}\mathcal{K}(\hat{\mathscr{H}}\hat{\otimes}_X\tau^*\hat{\mathscr{H}}\hat{\otimes}_X\overline{\hat{\mathscr{H}}'}\hat{\otimes}_X\overline{\tau^*\hat{\mathscr{H}}'})\\ &\cong\mathcal{K}(\hat{\mathscr{H}}\hat{\otimes}_X\hat{\mathscr{H}}\hat{\otimes}_X\tau^*\hat{\mathscr{H}}\hat{\otimes}_X\overline{\hat{\mathscr{H}}'}\hat{\otimes}_X\overline{\tau^*\hat{\mathscr{H}}'}).\end{aligned}$$

Moreover, $\mathcal{K}(\mathscr{H}\hat{\otimes}_X\mathscr{H}\hat{\otimes}_X\tau^*\mathscr{H}\hat{\otimes}_X\overline{\mathscr{H}'}\hat{\otimes}_X\overline{\tau^*\mathscr{H}'}$ defines a graded Hilbert $\mathcal{G}/_\tau$-bundle. Hence, by Corollary 4.2.10 and Remark 4.3.8 we see that $\mathcal{A}/_\tau = \mathcal{A}'/_\tau$ in $\widehat{\mathrm{Br}}(\mathcal{G}/_\tau)$. Ψ_τ is a group homomorphism by commutativity of the graded tensor product.

Conversely, denote by $\pi_\tau : \mathcal{G} \longrightarrow \mathcal{G}/_\tau$ the canonical projection. Then the pull-back of a graded complex D-D bundle $(\mathcal{A},\alpha) \in \widehat{\mathfrak{Br}}(\mathcal{G}/_\tau)$ is a graded complex D-D bundle $(\mathcal{A}',\alpha') := (\pi_\tau^*\mathcal{A},\pi_\tau^*\alpha) \in \widehat{\mathfrak{Br}}(\mathcal{G})$ which clearly verifies $(\tau^*\mathcal{A}',\tau^*\alpha') \cong (\mathcal{A}',\alpha')$ in $\widehat{\mathfrak{Br}}(\mathcal{G})$ (this is because for all $x \in X$ we have $\mathcal{A}'_x = \mathcal{A}'_{\bar{x}}$); so $\hat{\tau}(\mathcal{A}') = -\mathcal{A}'$ and $\mathcal{A}' \in {}^{\mathfrak{I}}\widehat{\mathrm{Br}}(\mathcal{G})$. Thus the pull-back map π_τ^* induces a group homomorphism

$$\begin{array}{cccc} \pi_\tau^* : & \widehat{\mathrm{Br}}(\mathcal{G}/_\tau) & \longrightarrow & {}^{\mathfrak{I}}\widehat{\mathrm{Br}}(\mathcal{G}) \\ & \mathcal{A} & \longmapsto & \mathcal{A}' := \pi_\tau^*\mathcal{A}. \end{array}$$

Now, for all $\mathcal{A} \in \widehat{\mathrm{Br}}(\mathcal{G}/_\tau)$ we have $(\pi_\tau^*\mathcal{A})/_\tau = \mathcal{A}$ since $\tau^*\pi_\tau^*\mathcal{A} = \pi_\tau^*\mathcal{A}$ and so that a graded Hilbert $\mathcal{G}$-bundle $\mathscr{H}$ such that relation (4.11) holds for the graded complex D-D bundle $(\pi_\tau^*\mathcal{A},\pi_\tau^*\alpha)$ is the trivial one $X\times\mathbb{C} \longrightarrow X$. This shows that $\Psi_\tau\circ\pi_\tau^* = \mathrm{Id}$. Also, one clearly has $\pi_\tau^*\circ\Psi_\tau = \mathrm{Id}$, which gives the isomorphism ${}^{\mathfrak{I}}\widehat{\mathrm{Br}}(\mathcal{G}) \cong \widehat{\mathrm{Br}}(\mathcal{G}/_\tau)$. Combining Lemma 4.3.7 and Lemma 2.1.4, we obtain the desired isomorphism (4.9).

2. We always have $\mathcal{A}+\overline{\mathcal{A}} = 0$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$ for all $(\mathcal{A},\alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$. Moreover, we have already seen in the end of the proof of Lemma 4.3.7 that the Real structure of $\mathcal{A}$ induces an isomorphism of Rg D-D bundles $(\mathcal{A},\alpha) \cong (\overline{\tau^*\mathcal{A}},\overline{\tau^*\alpha})$. In particular, if $\tau : \mathcal{G} \longrightarrow \mathcal{G}$ is trivial, we have $(\mathcal{A},\alpha) \cong (\overline{\mathcal{A}},\bar{\alpha})$; hence $\mathcal{A} = -\mathcal{A}$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$.
Furthermore, τ being trivial, each fibre of $\mathcal{A}$ is in fact a Rg elementary C^*-algebra, and then the complexification of a graded real elementary C^*-algebra. $(\mathcal{A},\alpha)$ is then the complexification of a graded real D-D bundle over $\mathcal{G}$. Conversely, every complexification $(\mathcal{A}_\mathbb{C},\alpha_\mathbb{C})$ of a graded real D-D bundle $(\mathcal{A},\alpha)$ over $\mathcal{G}$ is a Rg D-D bundle whose Real structure is carried out by $\mathbb{C}$; i.e. $\overline{a\otimes_\mathbb{R}\lambda} := a\otimes_\mathbb{R}\bar{\lambda}$ for $a\otimes_\mathbb{R}\lambda \in \mathcal{A}_x\otimes_\mathbb{R}\mathbb{C}$. This process is easily seen to provide an isomorphism $\widehat{\mathrm{BrR}}(\mathcal{G}) \cong \widehat{\mathrm{BrO}}(\mathcal{G})$. □

Observe that Rg D-D bundle $(\mathcal{A},\alpha)$ can also be considered as a graded real D-D bundle $(\mathcal{A}_{real},\alpha_{real})$ by forgetting the complex structure of the fibers. Moreover, the conjugate bundle of real C^*-algebras $(\mathcal{A}_{real},\alpha_{real})$ is itself. Hence, if the involution τ of $\mathcal{G}$ is fixed point free, we have $\overline{\tau^*\mathcal{A}_{real}} = \tau^*\mathcal{A}_{real} \cong \mathcal{A}_{real}$, which means that $(\mathcal{A}_{real},\alpha_{real})$ is a bundle of graded real elementary C^*-algebras over the quotient groupoid $\mathcal{G}/_\tau \rightrightarrows X/_\tau$. We therefore have the

Proposition 4.3.9. *Suppose $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ is endowed with a fixed point free involution τ. Then there is a group homomorphism*

$$\Psi_{real} : \widehat{Br\mathrm{R}}(\mathcal{G}) \longrightarrow \widehat{Br\mathrm{O}}(\mathcal{G}/_\tau)$$

obtained by "forgetting the complex structures" of Rg graded D-D bundles over $\mathcal{G}$.

Remark 4.3.10. *Beware that Ψ_{real} is not injective; indeed $\Psi_{real}(\overline{\mathcal{A}}) = \Psi_{real}(\mathcal{A})$ for all $\mathcal{A} \in \widehat{Br}\mathrm{R}(\mathcal{G})$, while in general $\overline{\mathcal{A}} \neq \mathcal{A}$ in $\widehat{Br}\mathrm{R}(\mathcal{G})$.*

4.4 Elementary involutive triples and types of Rg D-D bundles

In this section we define the *type* of a Rg D-D bundle over a Real groupoid. We start by introducing some notions.

Definition 4.4.1. *An* elementary involutive triple $(\hat{\mathbb{K}}, \hat{\mathbb{K}}^-, \mathbf{t})$ *consists of a graded elementary C^*-algebra $\hat{\mathbb{K}}$, a graded C^*-algebra $\hat{\mathbb{K}}^-$ isomorphic to the conjugate C^*-algebra of $\hat{\mathbb{K}}$, and a conjugate linear isomorphism $\mathbf{t}: \hat{\mathbb{K}} \longrightarrow \hat{\mathbb{K}}^-$ of graded C^*-algebra. Such triple will be represented by the map $\mathbf{t}$. Denote by $\hat{\mathfrak{K}}$ the collection of all elementary involutive triples.*
A morphism from $\mathbf{t}$ to $\mathbf{t}'$ is the data of homomorphisms of graded C^*-algebras $\varphi: \hat{\mathbb{K}} \longrightarrow \hat{\mathbb{K}}'$, and $\varphi^-: \hat{\mathbb{K}}^- \longrightarrow \hat{\mathbb{K}}'^-$ such that the following diagram commutes

$$\begin{tikzcd} \hat{\mathbb{K}} \arrow[r, "\varphi"] \arrow[d, "\mathbf{t}"] & \hat{\mathbb{K}}' \arrow[d, "\mathbf{t}'"] \\ \hat{\mathbb{K}}^- \arrow[r, "\varphi^-"] & \hat{\mathbb{K}}'^- \end{tikzcd} \tag{4.13}$$

Finally, we define the sum in $\hat{\mathfrak{K}}$ by:

$$\mathbf{t} + \mathbf{t}' := (\hat{\mathbb{K}} \hat{\otimes} \hat{\mathbb{K}}', \hat{\mathbb{K}}^- \hat{\otimes} \hat{\mathbb{K}}'^-, \mathbf{t} \hat{\otimes} \mathbf{t}').$$

Example 4.4.2. *The Real structure "bar" of $\widehat{\mathcal{K}}_0$ induces an isomorphism of $\widehat{\mathcal{K}}_0$ into its conjugate algebra. We then have an elementary involutive triple $\mathbf{t}_0 = (\widehat{\mathcal{K}}_0, \widehat{\mathcal{K}}_0, bar)$.*

Definition and Lemma 4.4.3. *Two elements $\mathbf{t}, \mathbf{t}' \in \hat{\mathfrak{K}}$ are said to be* stably isomorphic *if and only if $\mathbf{t} + \mathbf{t}_0$ is isomorphic to $\mathbf{t}' + \mathbf{t}_0$; in this case, we write $\mathbf{t} \cong_s \mathbf{t}'$. The set of stable isomorphism classes of elements of $\hat{\mathfrak{K}}$ forms an abelian group $\mathsf{Inv}\hat{\mathfrak{K}}$ under the sum defined above. The inverse of $\mathbf{t}$ in $\mathsf{Inv}\hat{\mathfrak{K}}$ is the stable isomorphisms class of*

$$-\mathbf{t} := (\hat{\mathbb{K}}^-, \hat{\mathbb{K}}, \mathbf{t}^{-1}).$$

The class of $\mathbf{t}$ in $\mathsf{Inv}\hat{\mathfrak{K}}$ will also be denoted by $\mathbf{t}$.

Proof. It is straightforward that $\mathbf{t} + \mathbf{t}' = \mathbf{t}' + \mathbf{t}$ in $\mathsf{Inv}\hat{\mathfrak{K}}$. Moreover, we have

$$\mathbf{t} - \mathbf{t} = (\hat{\mathbb{K}} \hat{\otimes} \hat{\mathbb{K}}^-, \hat{\mathbb{K}}^- \hat{\otimes} \hat{\mathbb{K}}, \mathbf{t} \hat{\otimes} \mathbf{t}^{-1}) \cong (\hat{\mathbb{K}} \hat{\otimes} \hat{\mathbb{K}}^-, \hat{\mathbb{K}} \hat{\otimes} \hat{\mathbb{K}}^-, \mathbf{t}'),$$

via the isomorphism $(\mathrm{Id}_{\hat{\mathbb{K}} \hat{\otimes} \hat{\mathbb{K}}^-}, \varphi')$, where $\varphi': \hat{\mathbb{K}}^- \hat{\otimes} \hat{\mathbb{K}} \longrightarrow \hat{\mathbb{K}} \hat{\otimes} \hat{\mathbb{K}}^-$ is the canonical isomorphism $\varphi'(T \hat{\otimes} T') := (-1)^{\partial T \partial T'} T' \hat{\otimes} T$, and $\mathbf{t}' := \varphi' \circ (\mathbf{t} \hat{\otimes} \mathbf{t}^{-1})$. Thus, $\mathbf{t} - \mathbf{t} \cong_s \mathbf{t}_0$. □

We can recover the group $\widehat{\mathrm{BrR}}(*)$ from $\mathsf{Inv}\hat{\mathfrak{K}}$. More precisely, suppose $\mathbf{t} = -\mathbf{t}$, and

$$(\varphi,\varphi') : (\hat{\mathbb{K}}\hat{\otimes}\hat{\mathcal{K}}_0, \hat{\mathbb{K}}^-\hat{\otimes}\hat{\mathcal{K}}_0, \mathbf{t}\hat{\otimes}bar) \longrightarrow (\hat{\mathbb{K}}^-\hat{\otimes}\hat{\mathcal{K}}_0, \hat{\mathbb{K}}\hat{\otimes}\hat{\mathcal{K}}_0, \mathbf{t}^{-1}\hat{\otimes}bar)$$

is an isomorphism. Then, $\varphi'\circ(\mathbf{t}\hat{\otimes}bar) = (\mathbf{t}^{-1}\hat{\otimes}bar)\circ\varphi$ is a Real structure on the graded elementary C^*-algebra $\hat{\mathbb{K}}\hat{\otimes}\mathcal{K}(\hat{\mathcal{H}})$. Moreover, if (φ_0,φ_0') is another isomorphism, it is easy to check that $\varphi'\circ(\mathbf{t}\hat{\otimes}bar)$ and $\varphi_0'\circ(\mathbf{t}\hat{\otimes}bar)$ are conjugate, hence define the same element of $\widehat{\mathrm{BrR}}(*)$. Conversely, any Real graded elementary C^*-algebra is obviously a 2-torsion of $\mathsf{Inv}\hat{\mathfrak{K}}$. We then have proved the following

Lemma 4.4.4. *The group $\widehat{Br\mathrm{R}}(*)$ is isomorphic to the subgroup of $\mathsf{Inv}\hat{\mathfrak{K}}$ of elements of order* 2.

Now let us return to the study if Rg D-D bundles over Real groupoids.

Proposition 4.4.5. *Let $(\mathcal{A},\alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$. Then each fiber $\mathcal{A}_x$ gives rise to an element $\mathbf{t}_x^{\mathcal{A}} \in \mathsf{Inv}\hat{\mathfrak{K}}$, and the family $\mathbf{t}^{\mathcal{A}} := (\mathbf{t}_x^{\mathcal{A}})_{x\in X}$ defines a cohomology class in $\check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}})$. This process defines a group homomorphism*

$$\mathbf{t} : \widehat{Br\mathrm{R}}(\mathcal{G}) \longrightarrow \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}}),$$

which is surjective.

Proof. Denote by τ the Real structure of $\mathcal{A}$. Over all $x\in X$, there is a conjugate linear isomorphism of graded C^*-algebras $\tau_x : \mathcal{A}_x \longrightarrow \mathcal{A}_{\bar{x}}$. Then the graded elementary (complex) C^*-algebras $\mathcal{A}_x$ and $\mathcal{A}_{\bar{x}}$ are of the same parity. Let $(\mathcal{U},\varphi)$ be a local trivialization of the graded elementary complex C^*-bundle $\mathcal{A}$ such that $\mathcal{U} = (U_i)$ is a Real open cover of X. Then the isomorphisms $\varphi_i : U_i \times \hat{\mathbb{K}}_i \longrightarrow \mathcal{A}_{|U_i}$ induces a family of graded isomorphisms $\varphi_x : \hat{\mathbb{K}}_x \longrightarrow \mathcal{A}_x$. Then $\mathbf{t}_x := (\hat{\mathbb{K}}_x, \hat{\mathbb{K}}_{\bar{x}}, t_x)$, where $t_x := \varphi_{\bar{x}}\circ\tau_x\circ\varphi_x$, is an element of $\mathsf{Inv}\hat{\mathfrak{K}}$, and the assignment $X \ni x \longmapsto \mathbf{t}_x \in \mathsf{Inv}\hat{\mathfrak{K}}$ is a locally constant $\mathcal{G}$-invariant Real function. Indeed, the $\mathcal{G}$-invariance (i.e. $\mathbf{t}_{r(g)} = \mathbf{t}_{s(g)}$ in $\mathsf{Inv}\hat{\mathfrak{K}}$ for all $g\in\mathcal{G}$) comes from the commutative diagram

$$\begin{tikzcd}
\hat{\mathbb{K}}_{s(g)} \arrow[r, "\varphi_{s(g)}"] \arrow[d, "\mathbf{t}_{s(g)}"] & \mathcal{A}_{s(g)} \arrow[r, "\alpha_g"] \arrow[d, "\tau_{s(g)}"] & \mathcal{A}_{r(g)} \arrow[r, "\varphi^{-1}_{r(g)}"] \arrow[d, "\tau_{r(g)}"] & \hat{\mathbb{K}}_{r(g)} \arrow[d, "\mathbf{t}_{r(g)}"'] \\
\hat{\mathbb{K}}_{s(\bar{g})} \arrow[r, "\varphi_{s(\bar{g})}"] & \mathcal{A}_{s(\bar{g})} \arrow[r, "\alpha_{\bar{g}}"] & \mathcal{A}_{\bar{g}} \arrow[r, "\varphi_{r(\bar{g})}"] & \hat{\mathbb{K}}_{r(\bar{g})}
\end{tikzcd}$$

Moreover, since τ is a continuous function, $\mathbf{t}^{\mathcal{A}} : X \ni x \longmapsto \mathbf{t}_x \in \mathsf{Inv}\hat{\mathfrak{K}}$ is locally constant. Hence $\mathbf{t}^{\mathcal{A}} \in \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}})$.
That $\mathbf{t}^{\bar{\mathcal{A}}} = -\mathbf{t}^{\mathcal{A}}$ and $\mathbf{t}^{\mathcal{A}+\mathcal{B}} = \mathbf{t}^{\mathcal{A}} + \mathbf{t}^{\mathcal{B}}$ is clear from the definition of the sum and the inverse in $\mathsf{Inv}\hat{\mathfrak{K}}$, and from the definition of the conjugate bundle and the tensor product of Rg D-D bundles. Observe that from the construction of $\hat{\mathcal{K}}_{\mathcal{G}}$, $\mathbf{t}^{\hat{\mathcal{K}}_{\mathcal{G}}} = \mathbf{t}_0 = 0$ since $\hat{\mathcal{K}}_{\mathcal{G}} \cong$

$\coprod_{x\in X}\mathcal{K}(L^2(\mathcal{G}^x))\otimes\widehat{\mathcal{K}}_0$ with involution given by $\widehat{\mathcal{K}}_x\ni\varphi\otimes T\longmapsto\tau(\varphi)\otimes Ad_{J_{0,\mathbb{R}}}(T)\in\widehat{\mathcal{K}}_{\bar{x}}$. Thus, if $\mathcal{A}=\mathcal{B}$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$, we have (thanks to Lemma 4.2.8 and Lemma 4.2.10) $\mathcal{A}+\bar{\mathcal{B}}+\widehat{\mathcal{K}}_{\mathcal{G}}=\mathcal{K}(\hat{\mathcal{H}}_{\mathcal{G}}\hat{\otimes}_X\mathscr{H})=0$; hence $\mathbf{t}^{\mathcal{A}-\mathcal{B}}=\mathbf{t}^{\mathcal{A}}-\mathbf{t}^{\mathcal{B}}=0$, which shows that $\mathbf{t}:\widehat{\mathrm{BrR}}(\mathcal{G})\longrightarrow\check{H}R^0(\mathcal{G}_\bullet,\mathsf{Inv}\hat{\mathfrak{K}})$ is group homomorphism. It is surjective since for all $\mathbf{t}\in\check{H}R^0(\mathcal{G}_\bullet,\mathsf{Inv}\hat{\mathfrak{K}})$,

$$\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}}:=\coprod_{x\in X}\widehat{\mathcal{K}}_x\hat{\otimes}\mathbb{K}_{\mathbf{t}_x},\tag{4.14}$$

equipped with the obvious involution and $\mathcal{G}$-action, defines a Rg D-D bundle over $\mathcal{G}$. □

Definition 4.4.6. *For $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$, the element $\mathbf{t}^{\mathcal{A}}$ of Proposition 4.4.5 is called* the type of $(\mathcal{A},\alpha)$. *The homomorphism $\mathbf{t}:\widehat{Br\mathrm{R}}(\mathcal{G})\longrightarrow\check{H}R^0(\mathcal{G}_\bullet,\mathsf{Inv}\hat{\mathfrak{K}})$ is called* the type map.

Definition 4.4.7. *A Rg D-D bundle $(\mathcal{A},\alpha)$ is said to be* of type $i \mod 8$ *if $\mathbf{t}^{\mathcal{A}}$ is the constant function $\mathbf{t}^{\mathcal{A}}=i\in\mathbb{Z}_8\subset\mathsf{Inv}\hat{\mathfrak{K}}$. By $\widehat{Br\mathrm{R}}_i(\mathcal{G})$ we denote the set of Morita equivalence classes of Rg D-D bundles of type $i \mod 8$ over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$. Next, we define*

$$\widehat{Br\mathrm{R}}_*(\mathcal{G}):=\bigoplus_{i=0}^{7}\widehat{Br\mathrm{R}}_i(\mathcal{G}).$$

Example 4.4.8. *Let $\hat{\mathcal{H}}$ be, as usual, equipped with the Real structure $J_{\mathbb{R}}$. Then $\widehat{\mathcal{K}}_0\longrightarrow\cdot$ is a Rg D-D bundle of type 0 over $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})\rightrightarrows\cdot$, where the Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-action is given by Ad;* i.e. *$[u]\cdot T:=Ad_u(T)$, for $[u]\in\widehat{\mathrm{PU}}(\hat{\mathcal{H}})\cong Aut^{(0)}(\widehat{\mathcal{K}}_0)$, $T\in\widehat{\mathcal{K}}_0$.*

We have the following easy result which shows that the study of $\widehat{\mathrm{BrR}}(\mathcal{G})$ reduces to that of Rg D-D bundles of type 0.

Proposition 4.4.9. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be as usual. Then $\widehat{Br\mathrm{R}}_0(\mathcal{G})$ is a subgroup of $\widehat{Br\mathrm{R}}(\mathcal{G})$. Furthermore, the sequences of groups*

$$0\longrightarrow\widehat{Br\mathrm{R}}_0(\mathcal{G})\xrightarrow{\iota_0}\widehat{Br\mathrm{R}}(\mathcal{G})\xrightarrow{\mathbf{t}}\check{H}R^0(\mathcal{G}_\bullet,\mathsf{Inv}\hat{\mathfrak{K}})\longrightarrow 0\tag{4.15}$$

$$0\longrightarrow\widehat{Br\mathrm{R}}_0(\mathcal{G})\xrightarrow{\iota_0}\widehat{Br\mathrm{R}}_*(\mathcal{G})\xrightarrow{\mathbf{t}}\check{H}R^0(\mathcal{G}_\bullet,\mathbb{Z}_8)\longrightarrow 0,\tag{4.16}$$

where ι_0 is the inclusion homomorphism, are split-exact. Therefore, we have two isomorphisms of abelian groups

$$\widehat{Br\mathrm{R}}(\mathcal{G})\cong\check{H}R^0(\mathcal{G}_\bullet,\mathsf{Inv}\hat{\mathfrak{K}})\oplus\widehat{Br\mathrm{R}}_0(\mathcal{G}),\ \text{and}\quad\widehat{Br\mathrm{R}}_*(\mathcal{G})\cong\check{H}R^0(\mathcal{G}_\bullet,\mathbb{Z}_8)\oplus\widehat{Br\mathrm{R}}_0(\mathcal{G}).$$

Proof. We only prove for the first sequence, from which we deduce the second one. It is clear that $\mathbf{t}\circ\iota_0=0$ and ι_0 is an injective homomorphism. We also proved in Proposition 4.4.5 that $\mathbf{t}$ was surjective. To show the sequence splits, we only have to verify that the correspondence $\mathbf{t}\longmapsto\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}}$, where $\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}}\longrightarrow X$ is the Rg D-D bundle given by (4.14) defines a group homomorphism $\check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}}) \longrightarrow \widehat{\mathrm{BrR}}(\mathcal{G})$. This is immediate from construction: we have $\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}+\mathbf{t}'} \cong \widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}} \hat{\otimes}_X \widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}'}$, and a routine verification shows that any isomorphism $\mathbf{t}+\mathbf{t}_0 \cong \mathbf{t}'+\mathbf{t}_0$ induces an isomorphism of Rg D-D bundles $\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}+\mathbf{t}_0} \cong \widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}'+\mathbf{t}_0}$ so that $\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}} = \widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}'}$ in $\widehat{\mathrm{BrR}}(\mathcal{G})$ if $\mathbf{t} \sim_s \mathbf{t}'$. Also from the definition of $-\mathbf{t}$, we have $\widehat{\mathcal{K}}_{\mathcal{G},-\mathbf{t}} = \overline{\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}}} = -\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}}$. Finally it is obvious that $\widehat{\mathcal{K}}_{\mathcal{G},\mathbf{t}}$ is of type $\mathbf{t}$. □

4.5 Generalized classifying morphisms

In this section we are dealing with the Rg D-D bundle $(\widehat{\mathcal{K}}_0, Ad)$ of type 0 over the Real groupoid $\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \rightrightarrows \cdot$, where $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ is equipped with the compact-open topology and the usual involution induced by the degree 0 Real structure $J_{0,\mathbb{R}}$ on $\hat{\mathcal{H}}$.

Definition 4.5.1. *Let $(\mathcal{A},\alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$ of type 0. A* generalized classifying morphism *for $(\mathcal{A},\alpha)$ is a generalized Real homomorphism $P : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ such that $(\mathcal{A},\alpha) \cong (\widehat{\mathcal{K}}_0^P, Ad^P)$ as Rg D-D bundles.*

Remark 4.5.2. *Note that $\widehat{\mathcal{K}}_0^P = P \times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})} \widehat{\mathcal{K}}_0 := P \times \widehat{\mathcal{K}}_0/_\sim$, where the equivalence relation is $(\varphi, T) \sim (\varphi\cdot[u], [u^{-1}]\cdot T)$ for $[u] \in \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. The Real $\mathcal{G}$-action by automorphisms is given by the Real (left) $\mathcal{G}$-action on P.*

Before going on the study of generalized classifying morphisms, we shall say something about generalized Real homomorphisms $\mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. First of all, recall from Remark 3.10.2 that although the Real group $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ is not abelian, it still is possible to define Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-valued Čech 1-cocycles over any Real groupoid $\mathcal{G}$, and hence form the set $\check{H}R^1(\mathcal{G}_\bullet, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$. Furthermore, as we had already pointed out, using the same arguments as in Proposition 3.10.1, $\check{H}R^1(\mathcal{G}_\bullet, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ and $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ are set-theoretically bijective.

However, when identified with $\mathrm{Hom}_{\mathfrak{RG}_\Omega}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$, the set $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ admits the structure of abelian monoid defined as follows. Fix an isomorphism

$$\hat{\mathcal{H}} \hat{\otimes} \hat{\mathcal{H}} \xrightarrow{\cong} \hat{\mathcal{H}}$$

of Rg Hilbert spaces. Then the map

$$\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \ni ([u_1],[u_2]) \longmapsto [u_1 \hat{\otimes} u_1] \in \widehat{\mathrm{PU}}(\hat{\mathcal{H}} \hat{\otimes} \hat{\mathcal{H}}) \cong \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$$

is a Real a homeomorphism, where the unitary $u_1 \hat{\otimes} u_2$ is given on $\hat{\mathcal{H}} \hat{\otimes} \hat{\mathcal{H}}$ by

$$(u_1 \hat{\otimes} u_2)(\xi_1 \hat{\otimes} \xi_2) := (-1)^{\partial u_2 \cdot \partial u_1} u_1(\xi_1) \hat{\otimes} u_2(\xi_2).$$

Given $p_1, p_2 \in \mathrm{Hom}_{\mathfrak{RG}_\Omega}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$, we may, without loss of generality, assume they are represented on the same open Real cover $\mathcal{U}$ of X; *i.e.* $p_1, p_2 : \mathcal{G}[\mathcal{U}] \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ are strict Real morhisms. Henceforth, the map

$$\begin{aligned} p_1 \hat{\otimes} p_2 : \ \mathcal{G}[\mathcal{U}] &\longrightarrow \ \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \\ \gamma &\longmapsto \ p_1(\gamma) \hat{\otimes} p_2(\gamma) \end{aligned} \tag{4.17}$$

becomes a well defined strict Real homomorphism. Therefore we have:

Definition and Lemma 4.5.3. *For* $[P_1], [P_2] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$. *We define the sum*

$$[P_1] + [P_2] := [P_1 \otimes P_2],$$

where $P_1 \otimes P_2$ *is the generalized homomorphism from* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *to* $\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \rightrightarrows \cdot$ *obtained by composing the corresponding morphism* $p_1 \hat{\otimes} p_2 \in \mathrm{Hom}_{\mathfrak{RG}_\Omega}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *with the generalized Real morphism induced by a canonical Morita equivalence* $\mathcal{G} \sim \mathcal{G}[\mathcal{U}]$. *Then* $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *is an abelian monoid with respect to this operation.*

Remark 4.5.4. *The same reasoning applies to the Real group* $\mathrm{U}^0(\hat{\mathcal{H}})$*: the operation of tensor product of cocycles makes the set* $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathrm{U}^0(\hat{\mathcal{H}})) \cong \check{H}R^1(\mathcal{G}_\bullet, \mathrm{U}^0(\hat{\mathcal{H}}))$ *into an abelian monoid. Similarly the corresponding operation in* $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathrm{U}^0(\hat{\mathcal{H}}))$ *is denoted additively.*

We list some simple properties for generalized classifying morphisms.

Proposition 4.5.5. *If* P_1 *and* P_2 *are generalized classifying morphisms for* $(\mathcal{A}_1, \alpha_1)$ *and* $(\mathcal{A}_2, \alpha_2)$*, respectively, then* $P_1 \otimes P_2$ *is a generalized classifying morphism for the Real graded tensor product* $(\mathcal{A}_1 \hat{\otimes}_X \mathcal{A}_2, \alpha_1 \hat{\otimes} \alpha_2)$.

Proof. Up to considering the pull-back of $\mathcal{A}_i, i = 1, 2$ along the canonical Real inclusion $\mathcal{G}[\mathcal{U}] \hookrightarrow \mathcal{G}$, we can suppose that the $(\mathcal{A}_i, \alpha_i)$ are Rg D-D bundles over the Real cover groupoid $\mathcal{G}[\mathcal{U}]$, where $\mathcal{U}$ is an open Real cover such that the morphisms $p_i \in \mathrm{Hom}_{\mathfrak{RG}_\Omega}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ corresponding to P_i are represented. The isomorphisms $(\mathcal{A}_i, \alpha_i) \cong (\widehat{\mathcal{K}}_0^{P_i}, Ad^{P_i})$ mean that the pull-back $(p_i^* \widehat{\mathcal{K}}_0, p_i^* Ad)$ is isomorphic to $(\mathcal{A}_i, \alpha_i), i = 1, 2$. We thus have reduced the proposition to show that

$$(\mathcal{A}_1 \hat{\otimes}_Y \mathcal{A}_2, \alpha_1 \hat{\otimes} \alpha_2) \cong ((p_1 \hat{\otimes} p_2)^* \widehat{\mathcal{K}}_0, (p_1 \hat{\otimes} p_2)^* Ad),$$

where $Y = \coprod_{i \in I} U_i$. But this is clear by using functorial property of $\widehat{\mathfrak{BrR}}(\cdot)$ in the category $\mathfrak{RG}_s$ and the isomorphism of Rg D-D bundles $(\widehat{\mathcal{K}}_0 \hat{\otimes} . \widehat{\mathcal{K}}_0, Ad \hat{\otimes} Ad) \cong (\widehat{\mathcal{K}}_0, Ad)$ over $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. ∎

Proposition 4.5.6. *Suppose* $Z : \Gamma \longrightarrow \mathcal{G}$ *is a generalized Real homomorphism. Let* $(\mathcal{A}, \alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$ *of type* 0*. If* P *is a generalized classifying morphism for* $(\mathcal{A}, \alpha)$*, then* $P \circ Z$ *is a generalized classifying morphism for* $(\mathcal{A}^Z, \alpha^Z) \in \widehat{\mathfrak{BrR}}(\Gamma)$.

Proof. This is a consequence of the cofunctorial property of $\widehat{\mathfrak{BrR}}(\cdot)$ in the category $\mathfrak{RG}$: *i.e.* $\mathcal{A}\cong\widehat{\mathcal{K}}_0^P$ implies $\mathcal{A}^Z\cong(\widehat{\mathcal{K}}_0^P)^Z\cong\widehat{\mathcal{K}}_0^{P\circ Z}$. □

Proposition 4.5.7. *Let* $(\mathcal{A}_1,\alpha_1),(\mathcal{A}_2,\alpha_2)$ *be isomorphic Rg D-D bundles of type* 0 *over* $\mathcal{G}\underset{s}{\overset{r}{\rightrightarrows}}X$. *If* P_1 *and* P_2 *are generalized classifying morphisms for* $(\mathcal{A}_1,\alpha_1)$ *and* $(\mathcal{A}_2,\alpha_2)$, *respectively, there exists a* $\mathcal{G}$-$\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$*-equivariant Real isomorphism* $P_1\cong P_2$.

Proof. Let $f_i:\mathcal{A}_i\longrightarrow P_i\times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})}\widehat{\mathcal{K}}_0, i=1,2$, and $\phi:\mathcal{A}_1\longrightarrow\mathcal{A}_2$ be isomorphisms of Rg D-D bundles. Then $h:=f_2\circ\phi\circ f_1^{-1}:P_1\times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})}\widehat{\mathcal{K}}_0\longrightarrow P_2\times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})}\widehat{\mathcal{K}}_0$ is an isomorphism of Rg D-D bundles over $\mathcal{G}$. $P_1\longrightarrow X$ and $P_2\longrightarrow X$ being $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-principal bundles, it follows from the theory of principal bundles (see for instance Husemöller [38, §4.6]) that there exists an isomorphism of $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-principal bundles $f:P_1\longrightarrow P_2$ over X such that $h([\varphi,T])=[f(\varphi),T]$ for all $[\varphi,T]\in P_1\times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})}\widehat{\mathcal{K}}_0$. Moreover, f must be Real since h is. Also, since h is $\mathcal{G}$-equivariant, we have $h([g\cdot\varphi,T])=g\cdot h([\varphi,T])=g\cdot[f(\varphi),T]=[g\cdot f(\varphi),T]$, so that $[f(g\cdot\varphi),T]=[g\cdot f(\varphi),T],\forall[\varphi,T]\in P_1\times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})}\widehat{\mathcal{K}}_0$. f is thus an isomorphism of generalized Real homomorphisms $P_1\cong P_2:\mathcal{G}\longrightarrow\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. □

From Proposition 4.5.7 we deduce, among other things, the following

Corollary 4.5.8. *If there exists a generalized classifying morphism for* $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$, *then it is unique up to isomorphism of generalized Real homomorphisms.*

The existence of generalized classifying morphisms is the content of the nex section.

4.6 Construction of the classifying morphism P

It is known (see [87]) that graded complex D-D bundles are, in some sense, classified by the groupoid $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})\rightrightarrows\cdot$; i.e. giving a graded complex D-D bundle $\mathcal{A}$ over $\mathcal{G}$ is equivalent to giving a generalized morphism $P\in\mathrm{Hom}_{\mathfrak{G}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$, where $\mathfrak{G}$ is the category of topological groupoids and isomorphism classes of generalized morphisms. In view of the isomorphism established in Lemma 4.3.7, it is natural to expect a similar correspondence in the category of Real spaces. We show that the $\mathcal{G}$-equivariant $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-principal bundle associated to a Rg D-D bundle admits a natural involution turning it into an element in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$, where the Real groupoid $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})\rightrightarrows\cdot$ is given the compact-open topology (which is equivalent to the *-strong operator topology) and the usual involution $Ad_{J_{0,\mathbb{R}}}$.

A Rg D-D bundle $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$ being of type 0 means that the fibres $\mathcal{A}_x$ are isomorphic to the graded complex elementary C^*-algebra $\widehat{\mathcal{K}}_{ev}=\mathcal{K}(\hat{\mathcal{H}})$, and there is a Real local trivialization $(U_j,\varphi_j)_{j\in J}$ with commutative diagrams

$$\begin{tikzcd} \mathcal{A}_{|U_j} \arrow[r, "h_j"] \arrow[d, "\tau_{U_j}"'] & U_j\times\widehat{\mathcal{K}}_0 \arrow[d, "\tau\times bar"] \\ \mathcal{A}_{|U_{\bar{j}}} \arrow[r, "h_{\bar{j}}"] & U_{\bar{j}}\times\widehat{\mathcal{K}}_0 \end{tikzcd} \tag{4.18}$$

and a Real family of continuous function $a_{ij}:U_{ij}\longrightarrow \mathrm{Aut}^{(0)}(\widehat{\mathcal{K}}_0)=\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$, over every non-empty intersection $U_{ij}=U_i\cap U_j$, such that the homeomorphism

$$h_i\circ h_j^{-1}:U_{ij}\times\widehat{\mathcal{K}}_0\longrightarrow U_{ij}\times\widehat{\mathcal{K}}_0$$

sends (x,T) to $(x,a_{ij}(x)T)$. Notice that $(a_{ij})\in ZR^1(\mathcal{U},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$. We then obtain the "Real" analog of the well known *Dixmier-Douady class* (see [27], [75]). What is more, we get a Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-valued Čech 1-cocyle μ over $\mathcal{G}$ as follows. From the Real open cover $\mathcal{U}=(U_j)_{j\in J}$, from the Real open cover $\mathcal{U}_1=(U^1_{(j_0,j_1)})$ of $\mathcal{G}$ by setting $U^1_{(j_0,j_1)}=\{g\in\mathcal{G}\mid r(g)\in U_{j_0}, s(g)\in U_{j_1}\}$ (cf. (3.43)). Using the isomorphism of Rg C^*-bundles $s^*\mathcal{A}\longrightarrow r^*\mathcal{A}$ over $\mathcal{G}$ induced by the Real $\mathcal{G}$-action α and the commutative diagram (4.18), there is a Real family of continuous family $\mu_{(j_0,j_1)}:U^1_{(j_0,j_1)}\longrightarrow \mathrm{Aut}^{(0)}(\widehat{\mathcal{K}}_0)=\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ such that

$$\mu_{(j_0,j_1)}(g)=h_{j_0|_{r(g)}}\circ\alpha_g\circ h^{-1}_{j_1|_{s(g)}},\forall g\in U^1_{(j_0,j_1)}, \tag{4.19}$$

where $h_{j_0|_{r(g)}}:\mathcal{A}_{r(g)}\longrightarrow\{r(g)\}\times\widehat{\mathcal{K}}_0$, and $h_{j_1|_{s(g)}}:\{s(g)\}\times\widehat{\mathcal{K}}_0\longrightarrow\mathcal{A}_{s(g)}$ are the restrictions of the isomorphisms $h_{j_0}:r^*\mathcal{A}_{|U^1_{(j_0,j_1)}}\longrightarrow U_{j_0}\times\widehat{\mathcal{K}}_0$ and $h^{-1}_{j_1}:U_{j_1}\times\widehat{\mathcal{K}}_0\longrightarrow s^*\mathcal{A}_{|U^1_{(j_0,j_1)}}$. It is easy to verify that $\mu^{\mathcal{A}}=(\mu_{(j_0,j_1)})$ is a Real 1-coboundary. We are going to show that the generalized Real homomorphism corresponding to the class to the class of $\mu^{\mathcal{A}}$ in $\check{H}R^1(\mathcal{G}_\bullet,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ is actually a classifying generalized morphism for $(\mathcal{A},\alpha)$.

We first give further constructions. For $x\in X$, let $P_x:=\mathrm{Isom}^{(0)}(\widehat{\mathcal{K}}_0,\mathcal{A}_x)$. Put

$$P:=\coprod_{x\in X}P_x. \tag{4.20}$$

For $g\in\mathcal{G}$ and $p=(s(g),\varphi)\in P_{s(g)}$, the $\mathcal{G}$-action α of $\mathcal{A}$ provides the element $g\cdot p\in P_{r(g)}$ given by

$$g\cdot p:=(r(g),\alpha_g\circ\varphi). \tag{4.21}$$

We wish to define a topology on P such that not only the canonical projection $P\ni(x,\varphi)\longmapsto x\in X$ is continuous but also the formula (4.21) defines a continuous action of $\mathcal{G}$ on P with respect to the projection just given. To do so, we first consider the pull-backs $s^*P\longrightarrow\mathcal{G}$ and $r^*P\longrightarrow\mathcal{G}$ of $P\longrightarrow X$ along the range and source maps. Then we look at the fibred-product $s^*P\times_{\mathcal{G}}r^*P\longrightarrow\mathcal{G}$.

Lemma 4.6.1. *The $\mathcal{G}$-action α of $\mathcal{A}$ induces a (set-theoretical) embedding*

$$s^*P \times_{\mathcal{G}} r^*P \hookrightarrow \mathcal{G} \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}}).$$

Proof. If $((g,\varphi),(g,\psi)) \in s^*P \times_{\mathcal{G}} r^*P$, then $\psi^{-1} \circ \alpha_g \circ \varphi \in \mathrm{Aut}^{(0)}(\widehat{\mathcal{K}}_0) = \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. It is straightforward to see that the correspondence

$$\begin{array}{ccc} s^*P \times_{\mathcal{G}} r^*P & \longrightarrow & \mathcal{G} \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \\ ((g,\varphi),(g,\psi)) & \longmapsto & (g, \psi^{-1} \circ \alpha_g \circ \varphi) \end{array}$$

is a well-defined injection map. □

Definition 4.6.2. *Let $(\mathcal{A},\alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$ of type 0, and let P be given by (4.20). Let the space $s^*P \times_{\mathcal{G}} r^*P$ be given the topology induced from the product topology of $\mathcal{G} \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ via the embedding of Lemma 4.6.1. Then we endow P with the topology induced from the embedding*

$$\begin{array}{ccc} P & \hookrightarrow & s^*P \times_{\mathcal{G}} r^*P \\ (x,\varphi) & \longmapsto & ((x,\varphi),(x,\varphi)) \end{array}$$

In this way, P is looked at as a subspace of $\mathcal{G} \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$.

From this definition, it is obvious that the projection $P \longrightarrow X$ is an open continuous map with respect to which the formula (4.21) defines a continuous $\mathcal{G}$-action on P. Moreover, P is a Real $\mathcal{G}$-space with respect to the involution $P \ni (x,\varphi) \longmapsto (\bar{x},\bar{\varphi})$, where for $\varphi \in \mathrm{Isom}^{(0)}(\widehat{\mathcal{K}}_0, \mathcal{A}_x)$, the isomorphism $\bar{\varphi}$ is defined by $\bar{\varphi}(T) := \overline{\varphi(\bar{T})}$ for all $T \in \widehat{\mathcal{K}}_0$.

Proposition 4.6.3. *Let $u \in \hat{\mathrm{U}}(\hat{\mathcal{H}})$ and $[u]$ its class in the group $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. For $\varphi \in P_x$ we put $\varphi \cdot [u] := \varphi \circ Ad_u \in P_x$. Then the map $P \ni (x,\varphi) \longmapsto (x, \varphi \cdot [u]) \in P$ defines a principal Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-action on P compatible with the $\mathcal{G}$-action with respect to the projection $P \longrightarrow X$. In other words, we have a generalized Real homomorphism*

$$\begin{array}{ccc} P & \longrightarrow & \cdot \\ \downarrow & & \\ X & & \end{array}$$

from $\mathcal{G}$ to $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$.

Proof. The continuity of the map $\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \times P \ni ([u],(x,\varphi)) \longmapsto (x, \varphi \cdot [u]) \in P$ is a direct consequence of the construction of the topology of P. It respects the Real structures since for all $T \in \widehat{\mathcal{K}}_0$,

$$\overline{\varphi \cdot [u]}(T) = \overline{\varphi(u\bar{T}u^{-1})} = \overline{\varphi \circ Ad_u}(\bar{R}) = \bar{\varphi} \circ Ad_{\bar{u}}(T), = (\varphi \cdot [\bar{u}])(T).$$

It only reminds to check that the $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-action is principal; *i.e.* that condition

(i) in Definition 2.2.1) is satisfied for the map $P \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \longrightarrow P \times_X P$ given by

$$((x,\varphi),[u]) \longmapsto ((x,\varphi),(x,\varphi\cdot[u])).$$

But this is clear; indeed that map has inverse $P \times_X P \longrightarrow P \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ defined by

$$((x,\varphi),(x,\psi)) \longmapsto ((x,\varphi),\varphi^{-1}\circ\psi).$$

□

Proposition 4.6.4. *The class of* $[P] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *in* $\check{H}R^1(\mathcal{G}_\bullet,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *is* $\mu^{\mathcal{A}}$*, the latter being given by* (4.19).

Proof. The Real local trivialization (U_j,h_j) of (4.18) gives rise to a Real family of local sections $\mathsf{s}_j : U_j \longrightarrow P$ such that $\mathsf{s}_j(x) = h^{-1}_{j|_x} \in \mathrm{Isom}^{(0)}(\hat{\mathcal{K}}_0,\mathcal{A}_x)$. For $g \in U^1_{(j_0,j_1)}$, we have $g\cdot h^{-1}_{j_1|_{s(g)}} = \alpha_g \circ h^{-1}_{j_1|_{s(g)}}$, hence $g\cdot \mathsf{s}_{j_1}(s(g)) = \mathsf{s}_{j_0}(r(g))\cdot\mu_{(j_0,j_1)}$, which proves the result (cf. the proof of Proposition 3.10.1 for the construction of the class of such cohomology class). □

Proposition 4.6.5. *Every Rg D-D bundle* $(\mathcal{A},\alpha)$ *of type* 0 *over* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *admits a generalized classifying morphism* $\mathbb{P}(\mathcal{A})$*. Furthermore, the assignment*

$$[P] \longmapsto \mathbb{A}([P]) := P \times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})} \hat{\mathcal{K}}_0$$

induces a well defined surjective homomorphism of abelian monoids

$$\mathbb{A} : \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}_\bullet,\widehat{\mathrm{PU}}(\hat{\mathcal{H}})) \longrightarrow \widehat{BrR}_0(\mathcal{G}). \tag{4.22}$$

Proof. Let $P : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ be the generalized Real homomorphism defined above (cf (4.20)). Then the family of fibrewise maps

$$P \times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})} \hat{\mathcal{K}}_0 \ni ((x,\varphi),T) \longmapsto \varphi(T) \in \mathcal{A}_x$$

is clearly an isomorphism of Rg D-D bundles over $\mathcal{G}$. Therefore, $P : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ is a generalized classifying morphism for $(\mathcal{A},\alpha)$. The uniqueness of P is guaranteed by Corollary 4.5.8.

The map $\mathbb{A}$ is well defined since an isomorphism of generalized Real homomorphisms $P \cong P'$ obviously induces an isomorphism between the associated Rg D-D bundles. It is a homomorphism of abelian monoids, for if $[P],[P'] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ then, thanks to Proposition 4.5.5 and the uniqueness of the generalized classifying morphism, $P\otimes P'$ is a generalized classifying morphism for $\mathbb{A}([P])\hat{\otimes}_X\mathbb{A}([P'])$ and for $\mathbb{A}([P]+[P'])$ at the same time; so that $\mathbb{A}([P\otimes P']) \cong \hat{\mathcal{K}}_0^{P\otimes P'} \cong \mathbb{A}([P])\hat{\otimes}_X\mathbb{A}([P'])$, which implies $\mathbb{A}([P]+[P']) = \mathbb{A}([P]) + \mathbb{A}([P'])$ in $\widehat{\mathrm{BrR}}_0(\mathcal{G})$. The surjectivity of $\mathbb{A}$ is a consequence of the existence of the generalized classifying morphism we just proved. □

Remark 4.6.6. *Let X be a locally compact Hausdorff space. Recall that Atiyah and Segal defined the monoid* $\mathrm{Proj}^{\pm}(X)$ *to be the set of infinite dimensional projective graded complex Hilbert space bundles on X (see in [8, pp.11-12]) subjected to the operation of graded tensor products, and showed that as a set,* $\mathrm{Proj}^{\pm}(X) \cong \check{H}^1(X,\mathbb{Z}_2) \times \check{H}^2(X,\mathbb{Z})$. *Note that if X is endowed with a Real structure τ, then* $\mathrm{Hom}_R\, G(X,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *is nothing but the Real analog of* $\mathrm{Proj}^{\pm}(X)$. *We thus may expect to have a similar result as in the complex case; this will be discussed in the next sections.*

4.7 Intermediate isomorphism theorem

Consider once again the abelian monoids $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{U}}^0(\hat{\mathcal{H}}))$ and $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$. There is a canonical monomorphism

$$pr : \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\mathrm{U}^0(\hat{\mathcal{H}})) \longrightarrow \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$$

induced by the canonical Real projection $\widehat{\mathrm{U}}^0(\hat{\mathcal{H}}) \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$; *i.e.*, if $\mathbf{U} : \mathcal{G} \longrightarrow \mathrm{U}^0(\hat{\mathcal{H}})$ is a generalized Real homomorphism, then we obtain a generalized Real homomorphism

$$pr \circ \mathbf{U} := \mathbf{U} \times_{\mathrm{U}^0(\hat{\mathcal{H}})} \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}}).$$

Definition 4.7.1. *An element* $[P] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *is called* trivial *if* $[P] = [pr \circ \mathbf{U}]$ *for some* $\mathbf{U} : \mathcal{G} \longrightarrow \mathrm{U}^0(\hat{\mathcal{H}})$.

Define an equivalence relation in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ *by saying that* $P_1, P_2 : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ *are* stably isomorphic *if there exists a trivial generalized Real homomorphism Q such that*

$$[P_1] + [Q] = [P_2] + [Q].$$

In that case we write $[P_1] \sim_{st} [P_2]$. *We define*

$$\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st} := \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))/_{\sim_{st}}.$$

The class of $[P]$ *with respect to "$\sim_{st}$" is denoted by* $[P]_{st}$

Lemma 4.7.2. $[P]$ *is trivial if and only if P is the generalized classifying morphism of a Rg D-D bundle of the form* $(\mathcal{K}(\hat{\mathscr{H}}), Ad_U)$ *where* $(\hat{\mathscr{H}}, u)$ *is a Rg Hilbert $\mathcal{G}$-bundle.*

Proof. Assume $P \cong pr \circ \mathbf{U}$ trivial. Let $[\omega] \in \check{H}R^1(X,\widehat{\mathrm{U}}^0(\hat{\mathcal{H}}))$ be the class of the Real $\mathrm{U}^0(\hat{\mathcal{H}})$-principal bundle $\mathbf{U} \longrightarrow X$, and let $[c] \in \check{H}R^1(\mathcal{G}_\bullet,\mathrm{U}^0(\hat{\mathcal{H}}))$ be the class of $\mathbf{U}$ as Real $\mathrm{U}^0(\hat{\mathcal{H}})$-principal $\mathcal{G}$-bundle. Suppose, without loss of generality, that $\mathcal{U} = (U_j)_{j\in J}$ is a Real open cover of X on which ω is represented, and such that c is represented on the Real open cover $\mathcal{U}_1 = (U^1_{(j_0,j_1)})_{j_0,j_1\in J}$. Then we get a Rg Hilbert $\mathcal{G}$-bundle $(\hat{\mathscr{H}}, u)$ by setting:

$$\hat{\mathscr{H}} := \coprod_{j\in J} U_j \times \hat{\mathcal{H}}_{/\sim} \tag{4.23}$$

where $U_i \times \hat{\mathcal{H}} \ni (x,\xi) \sim (x, \omega_{ij}(x)\xi) \in U_i \times \hat{\mathcal{H}}$; if $[x,\xi]_j$ denotes the class in $\mathscr{H}$ of $(x,\xi) \in U_j \times \hat{\mathcal{H}}$, we define the Real structure $[x,\xi]_j \longmapsto [\bar{x},\bar{\xi}]_{\bar{j}}$ (where as usual the "bar" in $\hat{\mathcal{H}}$ is the Real structure $J_{0,\mathbb{R}}$), and the projection $\pi : \mathscr{H} \longrightarrow X$ by $\pi([x,\xi]_j) = x$; the Real $\mathcal{G}$-action U is

$$U_g([s(g),\xi]_{j_1}) := [r(g), c_{(j_0,j_1)}(g)\xi]_{j_0}. \tag{4.24}$$

By construction, we see that $pr \circ \mathbf{U} = \mathbf{U} \times_{\mathrm{U}^0(\hat{\mathcal{H}})} \hat{\mathcal{K}}_0$ is a generalized classifying morphism for $(\mathcal{K}(\mathscr{H}), Ad_U)$ and that the class $\mu^{\mathcal{K}(\mathscr{H})}$ (recall (4.19)) in $\check{H}R^1(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ is Ad_c, where $(Ad_c)_{(j_0,j_1)} := Ad_{c_{(j_0,j_1)}}$.

Conversely, a Rg Hilbert $\mathcal{G}$-bundle $(\mathscr{H}, U)$ gives rise to a class $[\omega] \in \check{H}R^1(\mathcal{G}_\bullet, \mathrm{U}^0(\hat{\mathcal{H}}))$, hence to a generalized Real homomorphism $\mathbf{U} : \mathcal{G} \longrightarrow \mathrm{U}^0(\hat{\mathcal{H}})$. It follows from Proposition 4.6.4 that $pr \circ \mathbf{U} : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ is a generalized classifying morphism for $(\mathcal{K}(\mathscr{H}), Ad_U)$; therefore $P \cong pr \circ \mathbf{U}$ by Corollary 4.5.8. $\square$

Lemma 4.7.3. $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}$ *is an abelian group with respect to the sum; the inverse of* $[P]_{st}$ *is* $[P^*]_{st}$ *where* P^* *is the generalized classifying morphism for the conjugate bundle of* $\mathbb{A}([P])$.

Proof. We only need to verify the existence of the inverse. $P \otimes P^*$ is a generalized classifying morphism for the Rg D-D bundle $\mathbb{A}([P]) \hat{\otimes}_X \overline{\mathbb{A}([P])}$. From Corollary 4.2.10, $P \otimes P^*$ is then a generalized classifying morphism for $(\mathcal{K}(\mathscr{H}), Ad_U)$ where $(\mathscr{H}, U)$ is a Rg Hilbert $\mathcal{G}$-bundle. Therefore, $[P \otimes P^*]$ is trivial, by Lemma 4.7.2. $\square$

The main result of this section is the following

Theorem 4.7.4. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *be a locally compact second-countable Real groupoid with Real Haar system. Then* $\widehat{Br}\mathrm{R}_0(\mathcal{G}) \cong \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}$.

The proof is based on the following lemma.

Lemma 4.7.5. *(Compare [87]). The sequence of abelian monoids*

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathrm{U}^0(\hat{\mathcal{H}})) \xrightarrow{pr} \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) \xrightarrow{\mathbb{A}} \widehat{Br}\mathrm{R}_0(\mathcal{G}) \longrightarrow 0 \tag{4.25}$$

is exact.

Proof. We have already seen that pr was a monomorphism of abelian monoids, and $\mathbb{A}$ was an epimorphism of abelian monoids. It then remains to show that $\ker \mathbb{A} = \mathrm{Im}(pr)$.

$\mathrm{Im}(pr) \subset \ker \mathbb{A}$: indeed, from Lemma 4.7.2 and Corollary 4.5.8, for all $[\mathbf{U}] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \mathrm{U}^0(\hat{\mathcal{H}}))$, the Rg D-D bundle $\mathbb{A}([pr \circ \mathbf{U}])$ is of the form $(\mathcal{K}(\mathscr{H}), Ad_U)$, hence $\mathbb{A}([pr \circ \mathbf{U}]) = 0$ in $\widehat{\mathrm{BrR}}_0(\mathcal{G})$ by Corollary 4.2.10.

$\ker \mathbb{A} \subset \mathrm{Im}(pr)$: if $\mathbb{A}([P]) = 0$ then P is the generalized classifying morphism for some $(\mathcal{K}(\mathscr{H}), Ad_U)$. So, by Lemma 4.7.2, $[P]$ is trivial; in other words, $[P] = [pr \circ \mathbf{U}] \in \mathrm{Im}(pr)$. $\square$

Proof of Theorem 4.7.4. First of all, observe that there is a canonical isomorphism of abelian monoids
$$\frac{\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))}{\mathrm{Im}(pr)} \cong \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st},$$
the quotient monoid is then an abelian group. Moreover, from the exact sequence (4.25) we deduce an isomorphism of abelian monoids
$$\mathscr{P}:\widehat{\mathrm{BrR}}_0(\mathcal{G}) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st},$$
such that $\mathscr{P}(\mathcal{A})$ is the class in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}$ of the generalized classifying morphism P of $(\mathcal{A},\alpha)$. Furthermore, by definition of the inverse in $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}$, we see that this isomorphism respects the inversion; it therefore is an isomorphism of abelian groups, this completes the proof. □

4.8 Example: computation of $\widehat{\mathrm{BrR}}_G(*)$

Here we apply the observations of the previous sections to compute the Rg Brauer group of a locally compact Real group $G \rightrightarrows \cdot$.

If $\mathcal{G} = G \rightrightarrows \cdot$ is a Real group, and if S is a Real group, then $\mathrm{Hom}_{\mathfrak{RG}}(G,S)$ identifies with the set $\mathrm{Hom}(G,S)_R$ of continuous Real group homomorphisms from G to S. In particular
$$\mathrm{Hom}_{\mathfrak{RG}}(G,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st} \cong \frac{\mathrm{Hom}(G,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_R}{\mathrm{Hom}(G,\mathrm{U}^0(\hat{\mathcal{H}}))_R}.$$
For instance, if G is given the trivial Real structure, then
$$\mathrm{Hom}_{\mathfrak{RG}}(G,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st} \cong \frac{\mathrm{Hom}(G,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}_{\mathbb{R}}))}{\mathrm{Hom}(G,\mathrm{U}^0(\hat{\mathcal{H}}_{\mathbb{R}}))}.$$

Moreover, a Rg D-D bundle over $G \rightrightarrows \cdot$ is obviously of constant type since it is given by a Real bundle over the point together with a Real action of G; so $\widehat{\mathrm{BrR}}(G) = \widehat{\mathrm{BrR}}_*(G)$. It is convenient to write $\widehat{\mathrm{BrR}}_G(*)$ instead of $\widehat{\mathrm{BrR}}(G)$ since it is exactly the Rg Brauer group of the point with the trivial Real G-action. Similarly, we write $\widehat{\mathrm{BrO}}_G(*)$ and $\widehat{\mathrm{Br}}_G(*)$.

Now, applying Proposition 4.4.9 and Theorem 4.7.4 to G, we get

Proposition 4.8.1. *Let G be a locally compact Real group. Then*
$$\begin{aligned}\widehat{BrR}_G(*) &\cong \mathbb{Z}_8 \oplus \frac{\mathrm{Hom}(G,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_R}{\mathrm{Hom}(G,\mathrm{U}^0(\hat{\mathcal{H}}))_R},\\ \widehat{BrO}_G(*) &\cong \mathbb{Z}_8 \oplus \frac{\mathrm{Hom}(G,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}_{\mathbb{R}}))}{\mathrm{Hom}(G,\mathrm{U}^0(\hat{\mathcal{H}}_{\mathbb{R}}))}.\end{aligned}$$

4.9 The main isomorphisms

The purpose of this section is to establish the main result of this chapter. Namely, we prove the following theorem.

Theorem 4.9.1. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a locally compact Hausdorff second-countable Real groupoid with Real Haar system. Then*

$$\begin{aligned} \widehat{Br\mathrm{R}}(\mathcal{G}) &\cong \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}}) \oplus \left(\check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)\right); \\ \widehat{Br\mathrm{R}}_*(\mathcal{G}) &\cong \check{H}R^0(\mathcal{G}_\bullet, \mathbb{Z}_8) \oplus \left(\check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)\right). \end{aligned}$$

We first deduce from Theorem 4.9.1 and Theorem 3.14.7 the following corollary.

Corollary 4.9.2. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a locally compact Hausdorff second-countable Real proper groupoid with Real Haar system. Then*

$$\begin{aligned} \widehat{Br\mathrm{R}}(\mathcal{G}) &\cong \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}}) \oplus \left(\check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}R^3(\mathcal{G}_\bullet, \mathbb{Z}^{0,1})\right); \\ \widehat{Br\mathrm{R}}_*(\mathcal{G}) &\cong \check{H}R^0(\mathcal{G}_\bullet, \mathbb{Z}_8) \oplus \left(\check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}R^3(\mathcal{G}_\bullet, \mathbb{Z}^{0,1})\right). \end{aligned}$$

We also deduce a generalization of Donovan-Karoubi's isomorphism (4.7):

Corollary 4.9.3. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a groupoid with paracompact unit space and Haar system. Then*

$$\widehat{Br\mathrm{O}}(\mathcal{G}) \cong \check{H}^0(\mathcal{G}_\bullet, \mathbb{Z}_8) \oplus \left(\check{H}^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}^2(\mathcal{G}_\bullet, \mathbb{Z}_2)\right).$$

Proof. This follows from Theorem 4.3.6 2), Theorem 4.9.1, and the fact that when $\mathcal{G}$ is given the trivial involution, $\check{H}R^n(\mathcal{G}_\bullet, \mathbb{S}^1) \cong \check{H}^n(\mathcal{G}_\bullet, \mathbb{Z}_2)$ (see **??**). □

The proof of Theorem 4.9.1 is divided into several steps that mainly consist of constructing an isomorphism $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st} \cong \widehat{\mathrm{Ext}\mathrm{R}}(\mathcal{G}, \mathbb{S}^1)$.

Let us consider the following "generic" Rg $\mathbb{S}^1$-central extension $\mathbb{E}_{\hat{\mathcal{K}}_0}$ of $\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \rightrightarrows \cdot$

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \hat{\mathrm{U}}(\hat{\mathcal{H}}) \arrow[r, "pr"] & \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \arrow[d, "\partial"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{4.26}$$

where $\partial([u])$ is the degree of the homogeneous unitary u.

Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ be a Real groupoid and let $P : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ be a generalized Real homomorphism. Then we get a Rg $\mathbb{S}^1$-central extension $P^*\mathbb{E}_{\hat{\mathcal{K}}_0}$ of $\mathcal{G}$ by pulling back $\mathbb{E}_{\hat{\mathcal{K}}_0}$ via P (see Definition and Proposition 2.7.2).

Lemma 4.9.4. *The assignment $P \longmapsto P^*\mathbb{E}_{\widehat{\mathcal{K}}_0}$ induces a well defined homomorphism of abelian monoids*

$$\begin{array}{rll}\mathscr{T}: & \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) & \longrightarrow \widehat{Ext\mathrm{R}}(\mathcal{G}, \mathbb{S}^1) \\ & [P] & \longmapsto [P^*\mathbb{E}_{\widehat{\mathcal{K}}_0}]\end{array}$$

Proof. Assume $P \cong P'$ are isomorphic generalized Real homomorphisms from $\mathcal{G}$ to $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. As usual, we may assume that P and P' are represented in $\mathrm{Hom}_{\mathfrak{RG}_\Omega}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ on the same Real open cover $\mathcal{U}$ of X by two Real strict homomorphisms $f : \mathcal{G}[\mathcal{U}] \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ and $f' : \mathcal{G}[\mathcal{U}] \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$, respectively. The pull-backs $P^*\mathbb{E}_{\widehat{\mathcal{K}}_0}$ and $(P')^*\mathbb{E}_{\widehat{\mathcal{K}}_0}$ are then Morita equivalent to

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \widehat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}), f} \mathcal{G}[\mathcal{U}] \arrow[r] & \mathcal{G}[\mathcal{U}] \arrow[d, "\partial\circ f"] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \widehat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}), f'} \mathcal{G}[\mathcal{U}] \arrow[r] & \mathcal{G}[\mathcal{U}] \arrow[d, "\partial\circ f'"] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

respectively, where in both cases, the projection is the canonical one onto the second factor. Since $P \cong P'$, there is an isomorphism of Real groupoids $\phi : \mathcal{G}[\mathcal{U}] \longrightarrow \mathcal{G}[\mathcal{U}]$ such that $f' = f \circ \phi$. Therefore, the map

$$\widehat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}), f} \mathcal{G}[\mathcal{U}] \longrightarrow \widehat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}), f'} \mathcal{G}[\mathcal{U}]$$

defined by $(u, \gamma) \longmapsto (u, \phi(\gamma))$ induces an isomorphism of Rg $\mathbb{S}^1$-central extension $P^*\mathbb{E}_{\widehat{\mathcal{K}}_0} \cong (P')^*\mathbb{E}_{\widehat{\mathcal{K}}_0}$. Hence $\mathscr{T}$ is well defined.

Let us check that $\mathscr{T}$ is a homomorphism. Let $[P_1], [P_2] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ and let $p_1, p_2 : \mathcal{G}[\mathcal{U}] \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ be Real strict morphisms representing $[P_1]$ and $[P_2]$, respectively in the category $\mathfrak{RG}_\Omega$. Then the map

$$\left(\widehat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}), p_1} \mathcal{G}[\mathcal{U}]\right) \times_{\mathcal{G}[\mathcal{U}]} \left(\widehat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}), p_2} \mathcal{G}[\mathcal{U}])\right) \longrightarrow \widehat{\mathrm{U}}(\hat{\mathcal{H}} \hat{\otimes} \hat{\mathcal{H}}) \times_{pr, \widehat{\mathrm{PU}}(\hat{\mathcal{H}} \hat{\otimes} \hat{\mathcal{H}}), p_1 \hat{\otimes} p_2} \mathcal{G}[\mathcal{U}])$$

defined by

$$\big((u_1, \gamma), (u_2, \gamma)\big) \longmapsto (u_1 \hat{\otimes} u_2, \gamma),$$

is easily checked to define an isomorphism of Rg $\mathbb{S}^1$-central extensions

$$(P_1^*\mathbb{E}_{\widehat{\mathcal{K}}_0}) \hat{\otimes} (P_2^*\mathbb{E}_{\widehat{\mathcal{K}}_0}) \cong (P_1 \otimes P_2)^*\mathbb{E}_{\widehat{\mathcal{K}}_0}.$$

Thus, $\mathscr{T}([P_1] + [P_2]) = \mathscr{T}([P_1]) + \mathscr{T}([P_2])$, and we are done. $\square$

Lemma 4.9.5. *If $[P]$ is trivial, then $\mathscr{T}([P]) = 0$ in $\widehat{Ext}\mathrm{R}(\mathcal{G},\mathbb{S}^1)$. Therefore, $\mathscr{T}$ induces a homomorphism of abelian groups*

$$\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st} \longrightarrow \widehat{Ext}\mathrm{R}(\mathcal{G},\mathbb{S}^1),$$

also denoted by $\mathscr{T}$.

Proof. Since $\mathscr{T}([P])$ depends only on the isomorphism class of P, it suffices to suppose $P = pr \circ \mathbf{U}$ for some generalized Real homomorphism $\mathbf{U} : \mathcal{G} \longrightarrow \mathrm{U}^0(\hat{\mathcal{H}})$. Let $u : \mathcal{G}[\mathcal{U}] \to \mathrm{U}^0(\hat{\mathcal{H}})$ be a Real strict homomorphism representing $\mathbf{U}$ in the category $\mathfrak{RG}_\Omega$. Then the Real groupoids morphism $p : \mathcal{G}[\mathcal{U}] \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ given by $p(\gamma) = [u_\gamma]$ represents P. It follows that the map

$$\begin{array}{ccc} \mathcal{G}[\mathcal{U}] & \longrightarrow & \hat{\mathrm{U}}(\hat{\mathcal{H}}) \times_{pr,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}),p} \mathcal{G}[\mathcal{U}] \times_{p,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}),pr} \hat{\mathrm{U}}(\hat{\mathcal{H}}) \\ \gamma & \longmapsto & (u_\gamma, \gamma, u_\gamma) \end{array}$$

is a well defined section of the projection of $(p^*\widehat{\mathrm{PU}}(\hat{\mathcal{H}}), \mathcal{G}[\mathcal{U}], \partial \circ p)$; the latter is then a strictly trivial Rg $\mathbb{S}^1$-twist (cf. Proposition 2.5.7). Therefore, $P^*\mathbb{E}_{\hat{\mathcal{K}}_0}$ is a trivial Rg $\mathbb{S}^1$-central extension of $\mathcal{G}$. □

At this point, we are following closely [90, §2.6] to construct a homomorphism $\mathscr{P}'$ in the other direction; and then we will show that $\mathscr{T}$ and $\mathscr{P}'$ are inverses of each other.

Let $\mathbb{E} = (\tilde{\Gamma}, \Gamma, \delta, Z)$ be a Rg $\mathbb{S}^1$-central extension of $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$.

Definition 4.9.6. *([90, Definition 2.37]). A function $\xi \in \mathcal{C}_c(\tilde{\Gamma})$ is said $\mathbb{S}^1$-equivariant if $\xi(\lambda\tilde{\gamma}) = \lambda^{-1}\xi(\tilde{\gamma})$ for any $\lambda \in \mathbb{S}^1$ and any $\tilde{\gamma} \in \tilde{\Gamma}$.*

Let $\mu = \{\mu^y\}_{y\in Y}$ be a Real Haar system of the Real groupoid $\tilde{\Gamma} \rightrightarrows Y$. For $y \in Y$, consider the graded Hilbert space $\mathcal{L}^2_y := L^2(\tilde{\Gamma}^y)^{\mathbb{S}^1}$ consisting of $\mathbb{S}^1$-equivariant functions on $\tilde{\Gamma}^y$ which are L^2 with respect to μ^y. Note that the $\mathbb{Z}_2$-grading of $\mathcal{L}^2_y$ is the one induced by δ; *i.e.*, for $\xi \in \mathcal{C}_c(\tilde{\Gamma}^y)^{\mathbb{S}^1}$, define $\delta\xi$ by

$$(\delta\xi)(\tilde{\gamma}) := (-1)^{\delta(\gamma)}\xi(\tilde{\gamma}), \tag{4.27}$$

where $\gamma \in \Gamma$ is such that $\pi(\tilde{\gamma}) = \gamma$. Let

$$\mathscr{H}_{\tilde{\Gamma},y} = \mathcal{L}^2_y \otimes \mathcal{H}, \quad \text{and } \mathscr{H}_{\tilde{\Gamma},y} := \coprod_y \mathscr{H}_{\tilde{\Gamma},y}, \tag{4.28}$$

where, as usual $\mathcal{H} = l^2(\mathbb{N})$ is the generic separable infinite dimensional Hilbert space, endowed with the Real structure $J_{\mathbb{R}}$ given by the complex conjugation with respect to the canonical basis. Then the countably generated continuous field of infinite dimensional graded Hilbert spaces $\tilde{\mathscr{H}}_{\tilde{\Gamma}} \longrightarrow Y$, is a locally trivial graded Hilbert bundle, and hence trivial thanks to [27, Théorème 5]. By identifying $\mathscr{H}_{\tilde{\Gamma},y}$ with the space

$$L^2(\tilde{\Gamma}^y;\mathcal{H})^{\mathbb{S}^1} := \{\xi \in \mathcal{C}_c(\tilde{\Gamma}^y;\mathcal{H}) \mid \xi(\lambda\tilde{\gamma}) = \lambda^{-1}\xi(\tilde{\gamma}), \forall \lambda \in \mathbb{S}^1, \tilde{\gamma} \in \tilde{\Gamma}^y\},$$

we define the Real structure on $\tilde{\mathscr{H}}_{\tilde{\Gamma}}$ by

$$\mathscr{H}_{\tilde{\Gamma},y} \ni \xi \longmapsto \bar{\xi} \in \mathscr{H}_{\tilde{\Gamma},\bar{y}}, \tag{4.29}$$

where $\bar{\xi}(\tilde{\gamma}) := \overline{\xi(\bar{\tilde{\gamma}})}$ for all $\tilde{\gamma} \in \tilde{\Gamma}^{\bar{y}}$. Together with this involution, $\tilde{\mathscr{H}}_{\tilde{\Gamma}}$ is clearly a Rg Hilbert bundle over Y.

For $y \in Y$, let $\hat{\mathscr{U}}_{\tilde{\Gamma},y} = \mathrm{U}^0(\hat{\mathcal{H}}, \mathscr{H}_{\tilde{\Gamma},y}) \bigcup \mathrm{U}^1(\hat{\mathcal{H}}, \mathscr{H}_{\tilde{\Gamma},y})$ be the space of homogeneous unitary operators from $\hat{\mathcal{H}}$ to $\mathscr{H}_{\tilde{\Gamma},y}$. Put

$$\hat{\mathscr{U}}_{\tilde{\Gamma}} = \coprod_{y\in Y} \mathscr{U}_{\tilde{\Gamma},y}.$$

The field $\hat{\mathscr{U}}_{\tilde{\Gamma}}$ is endowed with the topology induced from $\tilde{\mathscr{H}}$: a section $Y \ni y \longmapsto u_y \in \hat{\mathscr{U}}_{\tilde{\Gamma},y}$ is continuous if and only if for every $\xi \in \hat{\mathcal{H}}$, the map $y \longmapsto u_y\xi$ is a continuous section of $\tilde{\mathscr{H}}_{\tilde{\Gamma}} \longrightarrow Y$. The bundle $\hat{\mathscr{U}}_{\tilde{\Gamma}} \longrightarrow Y$ is, in an obvious way, a Real $\widehat{\mathrm{U}}(\hat{\mathcal{H}})$-principal bundle, with the Real structure $\hat{\mathscr{U}}_{\tilde{\Gamma},y} \ni u \longmapsto \bar{u} \in \hat{\mathscr{U}}_{\tilde{\Gamma},\bar{y}}$, where for $\xi \in \hat{\mathcal{H}}$, $\bar{u}(\xi) := \overline{u(\bar{\xi})} \in \mathscr{H}_{\tilde{\Gamma},\bar{y}}$. Notice that scalar multiplication with elements of the fibers $\hat{\mathscr{U}}_{\tilde{\Gamma},y}$ induces a Real $\mathbb{S}^1$-action on $\hat{\mathscr{U}}_{\tilde{\Gamma}}$. It follows that its quotient

$$\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}} := \hat{\mathscr{U}}_{\tilde{\Gamma}}/\mathbb{S}^1 \tag{4.30}$$

is a Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-principal bundle over Y. We write $[(x,u)]$ the class of $(x,u) \in \hat{\mathscr{U}}_{\tilde{\Gamma},y}$ in the quotient $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$.

One defines a Real left Γ-action on $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}} \longrightarrow Y$ in the following way: let $\gamma \in \Gamma$ and $[(s(\gamma),u)] \in P\hat{\mathscr{U}}_{\tilde{\Gamma}}$, then $\gamma \cdot [(s(g),u)]$ is the class $[(r(\gamma),\gamma\cdot u)]$ of the element $(r(\gamma),\gamma\cdot u) \in \hat{\mathscr{U}}_{\tilde{\Gamma},r(\gamma)}$, where for each $\xi \in \hat{\mathcal{H}}$, the function $(\gamma\cdot u)\xi \in L^2(\tilde{\Gamma}^{r(\gamma)};\hat{\mathcal{H}})^{\mathbb{S}^1}$ is given by

$$(\gamma\cdot u)\xi : \tilde{\Gamma}^{r(\gamma)} \ni h \longmapsto (u\xi)(\tilde{\gamma}^{-1}h),$$

where $\tilde{\gamma} \in \tilde{\Gamma}_\gamma$ is any lift of γ with respect to the projection $\tilde{\Gamma} \longrightarrow \Gamma$. It is easy to verify that with respect to this well defined Real action, $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$ is a Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-principal bundle over the Real groupoid $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$, in other words, it is a generalized Real homomorphism from Γ to $\widehat{\mathrm{PU}}(\hat{\mathcal{H}}) \rightrightarrows \cdot$.

Now the composite $\mathcal{G} \xrightarrow{Z^{-1}} \Gamma \xrightarrow{P\hat{\mathscr{U}}_{\tilde{\Gamma}}} \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ gives us the generalized Real homomorphism

$$P\mathbb{E} := \mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}} \circ Z^{-1} : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}}). \tag{4.31}$$

Remark 4.9.7. *Notice that the Real Γ-action on $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$ is induced by the Real $\tilde{\Gamma}$-action on $\hat{\mathscr{U}}_{\tilde{\Gamma}}$ defined by $\tilde{\gamma}\cdot(s(\tilde{\gamma}),u) := (r(\tilde{\gamma}),\tilde{\gamma}\cdot u)$, where for $\xi \in \hat{\mathcal{H}}$, $((\tilde{\gamma}\cdot u)\xi)(h) := (u\xi)(\tilde{\gamma}^{-1}h)$ for all $h \in \tilde{\Gamma}^{r(\tilde{\gamma})}$. In fact, $\hat{\mathscr{U}}_{\tilde{\Gamma}}$ is a Real $\widehat{\mathrm{U}}(\hat{\mathcal{H}})$-principal bundle over the Real groupoid $\tilde{\Gamma} \rightrightarrows Y$.*

Lemma 4.9.8. *Let $\mathbb{E}_i = (\widetilde{\Gamma}_i, \Gamma_i, \delta_i, Z_i), i = 1,2$ be Morita equivalent Rg $\mathbb{S}^1$-central extensions of $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, with equivalence implemented by an $\mathbb{S}^1$-equivarient Morita equivalence $Z : \widetilde{\Gamma}_1 \longrightarrow \widetilde{\Gamma}_2$ (see Definition 2.6.1). Denote by $Z' = Z/\mathbb{S}^1 : \Gamma_1 \longrightarrow \Gamma_2$ the induced Morita equivalence. Then $\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_2} \circ Z'$ and $\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1}$ are isomorphic.*

Proof. This is a consequence of [90, Lemma 2.39]. □

Lemma 4.9.9. *Assume $\mathbb{E}_1$ and $\mathbb{E}_2$ are Morita equivalent Rg $\mathbb{S}^1$-central extensions of $\mathcal{G}$. Then $[P\mathbb{E}_1] = [P\mathbb{E}_2]$.*

Proof. A Morita equivalence $\Gamma \sim_Z \mathcal{G}$ induces an isomorphism of abelian monoids

$$Z_* : \mathrm{hom}_{\mathfrak{RG}}(\Gamma, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) \longrightarrow \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$$

given by $Z_*[P] := [P \circ Z^{-1}]$ ([90, Proposition 2.35]).
Now if Z is a Morita equivalence from $\mathbb{E}_1$ and $\mathbb{E}_2$, then under the notations of Lemma 4.9.8, the commutative diagram of generalized Real homomorphisms

$$\begin{tikzcd} \Gamma_1 \arrow[rr, "Z'"] \arrow[dr, "Z_1"'] & & \Gamma_2 \arrow[dl, "Z_2"] \\ & \mathcal{G} & \end{tikzcd}$$

induces a commutative diagram of abelian monoids

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{RG}}(\Gamma_1, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) \arrow[rr, "Z'_*"] \arrow[dr, "Z_{1*}"'] & & \mathrm{Hom}_{\mathfrak{RG}}(\Gamma_2, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) \arrow[dl, "Z_{2*}"] \\ & \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) & \end{tikzcd}$$

Consequently, $\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_2} \circ (Z')^{-1} \circ Z_2^{-1} \cong \mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1} \circ Z_1^{-1} = P\mathbb{E}_1$. But from Lemma 4.9.8, $\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1} \circ (Z')^{-1} \cong \mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_2}$. Therefore, $P\mathbb{E}_2 \cong P\mathbb{E}_1$. □

Lemma 4.9.10. *The assignment $\mathbb{E} \longmapsto P\mathbb{E}$ induces group homomorphism*

$$\mathscr{P}' : \widehat{Ext\mathrm{R}}(\mathcal{G}, \mathbb{S}^1) \longmapsto \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}, [\mathbb{E}] \longmapsto [P\mathbb{E}]_{st}.$$

Moreover $\mathscr{T} \circ \mathscr{P}' = \mathrm{Id}$.

Proof. The second statement follows from [90, Proposition 2.38].

For the first statement, we shall check first that $[P\mathbb{E}]$ is trivial for $\mathbb{E}$ trivial. But, thanks to the previous lemma, it suffices to show that $P\mathbb{E}_0$ comes from a generalized Real homomorphism $\mathbf{U} : \mathcal{G} \longrightarrow \mathrm{U}^0(\hat{\mathcal{H}})$, where $\mathbb{E}_0$ is the trivial extension $(\mathcal{G} \times \mathbb{S}^1, \mathcal{G}, 0)$. This is obvious since $L^2(\mathcal{G}^x \times \mathbb{S}^1)^{\mathbb{S}^1} \cong L^2(\mathcal{G}^x)$, which implies in particular that there is a canonical Real graded

$\mathcal{G}$-action on the Real (trivially) graded Hilbert bundle $\mathscr{H}_{\mathcal{G}\times\mathbb{S}^1}$.

Let $[\mathbb{E}_1],[\mathbb{E}_2]\in\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$. For the sake of simplicity, we shall assume $\mathbb{E}_1$ and $\mathbb{E}_2$ are represented by Rg $\mathbb{S}^1$-twists $(\widetilde{\Gamma}_1,\Gamma,\delta_1)$ and $(\widetilde{\Gamma}_2,\Gamma,\delta_2)$ over the same Real groupoid $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ Morita equivalent to $\mathcal{G}$. Let μ_i be a Real Haar system of $\widetilde{\Gamma}_i$, $i=1,2$. Then $\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2$ is equipped with the Real Haar system $\mu_1\times_{\mathbb{S}^1}\mu_2$, and Γ is equipped with the image of the later (here $\mu_1\times_{\mathbb{S}^1}\mu_2$ is meant to say the product measure is invariant under the diagonal action by $\mathbb{S}^1$). For $y\in Y$, we denote by $\mathcal{C}_c(\widetilde{\Gamma}_1^y;\mathcal{H})^{\mathbb{S}^1}\hat{\otimes}_{\mathcal{C}_c(\Gamma^y)}\mathcal{C}_c(\widetilde{\Gamma}_2;\mathcal{H})^{\mathbb{S}^1}$ the completion, with respect to the inductive limit topology in $\mathcal{C}_c(\widetilde{\Gamma}_1^y\hat{\otimes}\widetilde{\Gamma}_2^y)^{\mathbb{S}^1}$, of the $\mathcal{C}_c(\Gamma^y)$-linear span of

$$\left\{\xi_1\hat{\odot}\xi_2 \mid \xi_1\in\mathcal{C}_c(\widetilde{\Gamma}_1^y)^{\mathbb{S}^1},\xi_2\in\mathcal{C}_c(\widetilde{\Gamma}_2^y)^{\mathbb{S}^1}\right\},$$

where $\xi_1\hat{\odot}\xi_2\in\mathcal{C}_c(\widetilde{\Gamma}_1^y\hat{\otimes}\widetilde{\Gamma}_2)^{\mathbb{S}^1}$ is defined by $[(\tilde{\gamma}_1,\tilde{\gamma}_2)]\longmapsto\xi_1(\tilde{\gamma}_1)\xi_2(\tilde{\gamma}_2)$, and where $\mathcal{C}_c(\Gamma^y)$ acts on $\mathcal{C}_c(\widetilde{\Gamma}_i^y)^{\mathbb{S}^1}$ by the formulas: $(\xi_1\cdot\phi)(\tilde{\gamma}_1)=\xi_1(\tilde{\gamma}_1)\phi(\pi_1(\tilde{\gamma}_1))$, and $(\phi\cdot\xi_2)(\tilde{\gamma}_2)=\phi(\pi_2(\tilde{\gamma}_2))\xi_2(\tilde{\gamma}_2)$ for $\xi_1\in\mathcal{C}_c(\widetilde{\Gamma}_1^y)^{\mathbb{S}^1},\phi\in\mathcal{C}_c(\Gamma^y),\xi_2\in\mathcal{C}_c(\widetilde{\Gamma}_2^y)$ and $\tilde{\gamma}_i\in\widetilde{\Gamma}_i^y$, $i=1,2$. Then, passing to the L^2-norms, the graded Hilbert spaces $L^2(\widetilde{\Gamma}_1^y\hat{\otimes}\widetilde{\Gamma}_2^y)^{\mathbb{S}^1}$ and $L^2(\widetilde{\Gamma}_1^y)^{\mathbb{S}^1}\hat{\otimes}_{L^2(\Gamma^y)}L^2(\widetilde{\Gamma}_2^y)^{\mathbb{S}^1}$ are isomorphic.

Define a generalized Real homomorphism $\mathscr{U}(\Gamma):\Gamma\longrightarrow\mathrm{U}^0(\hat{\mathcal{H}})$ by the Real filed

$$\mathscr{U}(\Gamma):=\coprod_{y\in Y}\mathrm{U}^0(\hat{\mathcal{H}},L^2(\Gamma^y)\otimes\mathcal{H}),$$

where the Real Γ-action is induced by the Real Γ-action on the Rg Hilbert Γ-bundle $\mathscr{H}(\Gamma)=\coprod_{y\in Y}L^2(\Gamma^y)\otimes\mathcal{H}\longrightarrow Y$ defined by $\gamma\cdot(s(\gamma),\xi):=(r(\gamma),\gamma\cdot\xi)$, with $(\gamma\cdot\xi)(h):=\xi(\gamma^{-1}h)$ for every $h\in\Gamma^{r(\gamma)}$ (note that the grading of $\mathscr{H}(\Gamma)$ is carried by $L^2(\Gamma^y)$ and is given by $(\delta_1\otimes\delta_2)\xi(\gamma):=(-1)^{\delta_1(\gamma)+\delta_2(\gamma)}\xi(\gamma)$). Then, remarking that $pr\circ\mathscr{U}(\Gamma)=\mathscr{U}(\Gamma)/\mathbb{S}^1$, we define an isomorphism of Real $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$-principal bundles over $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$

$$\begin{array}{ccc}\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1}\otimes\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_2}\otimes(pr\circ\mathscr{U}(\Gamma)) & \xrightarrow{\cong} & \mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2}\\ [(y,u_1)]\otimes[(y,u_2)]\otimes[(y,v)] & \longmapsto & [(y,v\cdot(u_1\hat{\odot}u_2))]\end{array}\tag{4.32}$$

as follows: given $\xi\in\hat{\mathcal{H}}$, we write $u_i(\xi)=\sum_j\phi_i^j\otimes\eta_i^j$, $i=1,2$, where $\phi_i^j\in L^2(\widetilde{\Gamma}_i^y)^{\mathbb{S}^1},\eta_i^j\in\mathcal{H}$, and similarly, $v(\xi)=\sum_j\psi^j\otimes\zeta^j\in L^2(\Gamma^y)\otimes\mathcal{H}$; then the unitary $v\cdot(u_1\hat{\odot}u_2)$ is defined by

$$\begin{array}{rl}(v\cdot(u_1\hat{\odot}u_2))(\xi) & :=\sum_j\left(\psi^j\cdot\phi_1^j\hat{\odot}\phi_2^j\right)\otimes\eta_1^j\otimes\eta_2^j\otimes\zeta^j\\ & \in L^2(\widetilde{\Gamma}_1^y\otimes\widetilde{\Gamma}_2^y)^{\mathbb{S}^1}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\cong L^2(\widetilde{\Gamma}_1^y\otimes\widetilde{\Gamma}_2^y)^{\mathbb{S}^1}\otimes\mathcal{H}.\end{array}\tag{4.33}$$

Thus $[\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2}]\sim_{st}[\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_1}\otimes\mathbb{P}\hat{\mathscr{U}}_{\widetilde{\Gamma}_2}]$ in $\mathrm{Hom}_{\mathfrak{RG}}(\Gamma,\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$. Therefore, by functoriality, we have $[P(\mathbb{E}_1\hat{\otimes}\mathbb{E}_1)]\sim_{st}[(P\mathbb{E}_2)\otimes(P\mathbb{E}_2)]$, which means that $\mathscr{P}'([\mathbb{E}_1]+[\mathbb{E}_2])=\mathscr{P}'([\mathbb{E}_1])+\mathscr{P}'([\mathbb{E}_2])$. ∎

Lemma 4.9.11. *We have $\mathscr{P}'\circ\mathscr{T}=\mathrm{Id}$; consequently, we have a group isomorphism*

$$\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G},\widehat{\mathrm{PU}}(\hat{\mathcal{H}}))_{st}\cong\widehat{Ext\mathrm{R}}(\mathcal{G},\mathbb{S}^1).$$

Proof. In view of Lemma 4.9.10, we only have to verify that $\mathscr{P}'\circ\mathscr{T} = \mathrm{Id}$. Let $[P] \in \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ be represented by a pair $[(\mathfrak{U}, p)] \in \mathrm{Hom}_{\mathfrak{RG}_\Omega}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$. Recall (see Proposition 2.4.13) that $P \cong P' \circ Z^{-1}_{\iota_\mathfrak{U}}$, where $\iota_\mathfrak{U} : \mathcal{G}[\mathfrak{U}] \hookrightarrow \mathcal{G}$ is the canonical Real inclusion and P' is the generalized Real homomorphism induced from the strict Real homomorphism $p : \mathcal{G}[\mathfrak{U}] \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$. Notice that $P' = \coprod_j U_j \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}})$ together with the Real $\mathcal{G}[\mathfrak{U}]$-action $g_{j_0j_1}\cdot (s(g)_{j_1}, [u]) := (r(g)_{j_0}, [u])$. On the other hand, there is canonical Real $\mathcal{G}[\mathfrak{U}]$-action on the Rg Hilbert bundle $\tilde{\mathscr{H}}_{p^*\hat{\mathrm{U}}(\hat{\mathcal{H}})}$, hence $P\hat{\mathscr{U}}_{p^*\hat{\mathrm{U}}(\hat{\mathcal{H}})} = \widehat{\mathrm{PU}}(\hat{\mathcal{H}}, \tilde{\mathscr{H}}_{p^*\hat{\mathrm{U}}(\hat{\mathcal{H}})}) \cong \coprod_j U_j \times \widehat{\mathrm{PU}}(\hat{\mathcal{H}}) = P'$. It follows that $P\hat{\mathscr{U}}_{p^*\hat{\mathrm{U}}(\hat{\mathcal{H}})} \circ Z^{-1}_{\iota_\mathfrak{U}} \cong P' \circ Z^{-1}_{\iota_\mathfrak{U}} = P$, and hence $\mathscr{P}'(\mathscr{T}([P])) = [P]$. □

Proof of Theorem 4.9.1. By Theorem **??**, Theorem 4.7.4, and Lemma 4.9.11: we have an isomorphism

$$dd \circ \mathscr{T} \circ \mathscr{P} : \widehat{\mathrm{BrR}}_0(\mathcal{G}) \longrightarrow \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}^2(\mathcal{G}_\bullet, \mathbb{S}^1).$$

The result then follows from 4.4.9. □

4.10 Oriented Rg D-D bundles

Definition 4.10.1. *For $\mathcal{A} \in \widehat{Br\mathrm{R}}(\mathcal{G})$, its image in $\check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{R}}) \oplus \big(\check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)\big)$ is called* the D-D class of $\mathcal{A}$ *and is denoted by $DD(\mathcal{A})$.*

Definition 4.10.2. *Given a Rg D-D bundle $(\mathcal{A}, \alpha)$ of type 0, the Rg $\mathbb{S}^1$-central extension obtained by the composite*

$$\widehat{Br\mathrm{R}}_0(\mathcal{G}) \xrightarrow{\mathscr{P}} \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}(\hat{\mathcal{H}})) \xrightarrow{\mathscr{T}} \widehat{Ext\mathrm{R}}(\mathcal{G}, \mathbb{S}^1)$$

is called the associated Rg extension *and is denoted by* $\mathbb{E}_\mathcal{A}$.

Definition 4.10.3. *A Rg D-D bundle $(\mathcal{A}, \alpha)$ is called* oriented *if its D-D class is of the form $(0, 0, \mathfrak{c})$; $(\mathcal{A}, \alpha)$ is then of type 0. By $\widehat{Br\mathrm{R}}^+(\mathcal{G})$ we denote the subset of $\widehat{Br\mathrm{R}}(\mathcal{G})$ consisting of oriented Rg D-D bundles.*

It should be noted that the associated Rg extension $\mathbb{E}_\mathcal{A}$ of an oriented Rg D-D bundle is *even* in the sense that the grading δ of $\mathbb{E}_\mathcal{A}$ is the zero function. Of course Morita equivalence and tensor product of Rg D-D bundles preserve orientation. Thus $\widehat{\mathrm{BrR}}^+(\mathcal{G})$ is a subgroup of $\widehat{\mathrm{BrR}}_0(\mathcal{G})$. Indeed, using similar arguments as in [42, §3.4], we obtain the "Real analog" of Kumjian-Muhly-Renault-Williams [49].

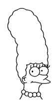
Theorem 4.10.4. *Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ be a locally compact Hausdorff second-countable Real groupoid with Real Haar system. Then*

$$\widehat{Br\mathrm{R}}^+(\mathcal{G}) \cong \mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, \widehat{\mathrm{PU}}^0(\hat{\mathcal{H}}))_{st} \cong \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1).$$

Remark 4.10.5. *We shall note that the above result generalizes Rosenberg classification of* real continuous-trace algebras *given in [78]. Indeed, let* (X,τ) *be a compact Real space. Then* $\widehat{Br}\mathrm{R}^+(X) \cong \check{H}R^2(X,\mathbb{S}^1) \cong \check{H}R^3(X,\mathbb{Z}^{0,1})$*. Thus, if* $\mathcal{A} \in \widehat{Br}\mathrm{R}^+(X)$ *with Dixmier-Douady class* $DD(\mathcal{A}) = \alpha \in \check{H}R^3(X,\mathbb{Z}^{0,1})$*, we have* $\tau^*\alpha = -\alpha$*, which coincides with [78, Proposition 3.1].*

5

Equivalence Theorem for *Real* Graded Fell systems



This chapter is aimed at investigating Fell bundles and their C^*-algebras in the graded and Real case (see [48] or [90, A.2] for an introduction to Fell bundles). We will study in §. 5.2 the graded Real reduced C^*-algebras of given graded Real Fell bundles and their Morita equivalences. More specifically, we will give in §. 5.6 the analogue of the Renault's equivalence theorem (cf. [64, Theorem 2.8]) for the graded Real reduced groupoid C^*-algebras. We refer to [77, 64] and more recently [65] for more details on the Renault's equivalence theorem for groupoid C^*-algebras and for groupoid crossed products. In Section 5.3 we will introduce the notion of *garded Real Fell pair*, and *equivalence of graded Real Fell bundles*, and then in Section 5.6, we will profit from the technical tools of Sims and Williams in [83] using the so-called *linking groupoid* (cf. [69]).
We should mention that the Renaul's equivalence theorem has been already proven by P. Muhly and D. Williams for trivially graded Fell bundles (see [66]). We however propose a different approach from theirs taking into account those not necessarily trivial gradings (and Real structures).

Throughout this chapter, $(\mathcal{G},\rho)$, (Γ,ϱ), etc. are locally compact, second countable, Hausdorff Real groupoids with open source and range maps, and μ (or $\mu_{\mathcal{G}}, \mu_\Gamma$, etc.) is a Real Haar system.

5.1 Rg Fell bundles and their full C^*-algebras

Let $p : \mathscr{E} \longrightarrow \mathcal{G}$ be a Rg Banach bundle with Real structure $\sigma : \mathscr{E} \longrightarrow \mathscr{E}$. Set

$$\mathscr{E}^{[2]} := \left\{(e_1, e_2) \in \mathscr{E} \times \mathscr{E} \mid (p(e_1), p(e_2)) \in \mathcal{G}^{(2)}\right\}.$$

Endow $\mathscr{E}^{[2]}$ with the obvious Real structure $\sigma^{[2]}$. If $\mathsf{m} : \mathcal{G}^{(2)} \longrightarrow \mathcal{G}$ denotes the partial multiplication of the groupoid $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, then the pull-back $(\mathsf{m}^*\mathscr{E}, \mathsf{m}^*\sigma)$ is a Rg Banach bundle over $(\mathcal{G}^{(2)}, \sigma^{(2)})$.

Definition 5.1.1 (Compare [48])**.** *A* multiplication *on* $(\mathscr{E}, \sigma)$ *is a continuous Real map*

$$\mathscr{E}^{[2]} \ni (e_1, e_2) \longmapsto \big((p(e_1), p(e_2)), e_1 e_2\big) \in \mathsf{m}^*\mathscr{E} \tag{5.1}$$

that satisfies the following properties:

(i) the induced map $\mathscr{E}_g \times \mathscr{E}_h \longrightarrow \mathscr{E}_{gh}$ *is bilinear and sends* $\mathscr{E}_g^i \times \mathscr{E}_h^j$ *to* $\mathscr{E}_{gh}^{i+j}$, $i, j = 0, 1$, *for* $(g, h) \in \mathcal{G}^{(2)}$*;*

(ii) (associativity) $(e_1 e_2) e_3 = e_1 (e_2 e_3)$ *whenever the multiplication is defined;*

(iii) $\|e_1 e_2\| \le \|e_1\| \|e_2\|$, *for every* $(e_1, e_2) \in \mathscr{E}^{[2]}$.

A *-involution *on* $(\mathscr{E}, \sigma)$ *is a continuous* 2*-periodic Real map* $(^*) : \mathscr{E} \ni e \longmapsto e^* \in \mathscr{E}$ *such that*

(iv) $p(e^*) = p(e)^{-1}$*;*

(v) the induced map $(^*) : \mathscr{E}_g \longrightarrow \mathscr{E}_{g^{-1}}$ *is conjugate linear and graded, for all* $g \in \mathcal{G}$.

Finally, we say that $p : (\mathscr{E}, \sigma) \longrightarrow (\mathcal{G}, \rho)$ *is a* Rg Fell bundle *if in addition the following conditions hold:*

(vi) $(e_1 e_2)^* = e_2^* e_1^*$, $\forall (e_1, e_2) \in \mathscr{E}^{[2]}$*;*

(vii) $\|e^* e\| = \|e\|^2$, $\forall e \in \mathscr{E}$*;*

(viii) $e^* e \ge 0$, $\forall e \in \mathscr{E}$*;*

*(ix) (*fullness*) the image of* $\mathscr{E}_g \times \mathscr{E}_h \longrightarrow \mathscr{E}_{gh}$ *spans a dense subspace of* $\mathscr{E}_{gh}$.

The pair $(\mathcal{G}, \mathscr{E})$ *is called a* Rg Fell system*; we will also write* $((\mathcal{G}, \rho), (\mathscr{E}, \sigma))$ *when the needs arise.*

Remark 5.1.2. *Conditions (v),(vi) and (vii) of the definition imply that for* $x \in X$, $\mathscr{E}_x$ *is a graded* C^**-algebra. In particular, the restriction* $\mathscr{E}^{(0)} := \mathcal{E}_X$*, together with the Real structure* σ*, defines a Rg* C^**-bundle over* X.

Given a Rg Fell bundle $\mathscr{E}$ over a second countable locally compact Hausdorff Real groupoid $\mathcal{G}$ with a Real Haar system μ, we want to turn the space $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ of compactly supported continuous sections into a Rg C^*-algebra $(C^*(\mathcal{G};\mathscr{E}),\sigma)$ that we will call the *full* Rg C^*-algebra associated to $(\mathscr{E},\sigma)$. For this end, observe first that $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ is graded by

$$\mathcal{C}_c(\mathcal{G};\mathscr{E}) \ni \xi \longmapsto \epsilon\circ\xi, \tag{5.2}$$

where ϵ is the (fiberwise) grading of $\mathscr{E}$, and has the natural Real structure given by

$$\mathcal{C}_c(\mathcal{G};\mathscr{E}) \ni \xi \longmapsto \sigma(\xi) := \sigma\circ\xi\circ\rho. \tag{5.3}$$

The following is a particular case of [32, Proposition 3.11].

Lemma 5.1.3. *Let $(\mathscr{E},\sigma)$ be a Rg Fell bundle over $\mathcal{G}$, and let B be a Real subspace of $(\mathcal{C}_0(\mathcal{G};\mathscr{E}),\sigma)$ that is invariant under the grading. Assume that*

(a) $\xi\in B$ and $\varphi\in\mathcal{C}_0(\mathcal{G})$ implies $\varphi\cdot\xi\in B$, where $(\varphi\cdot\xi)(g):=\varphi(g)\xi(g)$, and

(b) for each $g\in\mathcal{G}$, the set $\{\xi(g) : \xi\in B\}$ is dense, as a graded subspace, in $\mathscr{E}_g$.

Then, as a Rg subspace, (B,σ) is dense in $(\mathcal{C}_0(\mathcal{G};\mathscr{E}),\sigma)$.

Definition 5.1.4. *Let $(\mathscr{E},\sigma)$ be a Rg Fell bundle over $\mathcal{G}$. Given $f\in\mathcal{C}_c(\mathcal{G}^{(2)})$ and $\xi\in\mathcal{C}_0(\mathcal{G};\mathscr{E})$, we define the function $f\otimes\xi:\mathcal{G}^{(2)}\longrightarrow m^*\mathscr{E}$ by*

$$(f\otimes\xi)(g,h)=f(g,h)\xi(gh)\in\mathscr{E}_{gh},$$

for all $(g,h)\in\mathcal{G}^{(2)}$. Such a function will be called an elementary tensor. *Next, we set*

$$\mathcal{C}_c(\mathcal{G}^{(2)})\hat{\odot}\mathcal{C}_0(\mathcal{G};\mathscr{E}):=\operatorname{span}\left\{f\otimes\xi : f\in\mathcal{C}_c(\mathcal{G}^{(2)}),\xi\in\mathcal{C}_0(\mathcal{G};\mathscr{E})\right\}.$$

Proposition 5.1.5. *(Compare with [32, Proposition 3.40]). Let $(\mathscr{E},\sigma)$ be as above. Then, each elementary tensor $f\otimes\xi$ is a compactly supported continuous sections of $(m^*\mathcal{E},m^*\sigma)$ over $\mathcal{G}^{(2)}$ vanishing at infinity;* i.e. *$f\otimes\xi\in\mathcal{C}_c(\mathcal{G}^{(2)};m^*\mathscr{E})$. Furthermore, $\mathcal{C}_c(\mathcal{G}^{(2)})\hat{\odot}\mathcal{C}_0(\mathcal{G};\mathscr{E})$, equipped with the $\mathbb{Z}_2$-grading $1\otimes\epsilon$ and Real involution $\rho^{(2)}\otimes\sigma$, is dense in $(\mathcal{C}_0(\mathcal{G}^{(2)};m^*),m^*\sigma)$.*

Proof. That an elementary tensor $f\otimes\xi$ defines a continuous section in $\mathcal{C}(\mathcal{G}^{(2)};m^*\mathscr{E})$ is clear. Moreover, from the definition, one has $\|(f\otimes\xi)(g,h)\|=\|f(g,h)\|\|\xi(gh)\|$; hence $\operatorname{supp}(f\otimes\xi)\subset\operatorname{supp}(f)$, and this shows that $f\otimes\xi\in\mathcal{C}_c(\mathcal{G}^{(2)};m^*\mathscr{E})$. Note that the grading $1\otimes\epsilon$ and the Real structure $\rho^{(2)}\otimes\sigma$ of $\mathcal{C}_c(\mathcal{G}^{(2)})\hat{\odot}\mathcal{C}_0(\mathcal{G};\mathcal{E})$ are respectively given over elementary tensors by

$$(1\otimes\epsilon)(f\otimes\xi)(g,h)=f(g,h)\epsilon(\xi(gh)),\text{ and}$$
$$(\rho^{(2)}\otimes\sigma)(f\otimes\xi)(g,h)=\overline{f(\rho(g),\rho(h))}.\nu(\xi(\rho(gh)));$$

while that of $\mathcal{C}_0(\mathcal{G}^{(2)};m^*\mathscr{E})$ are

$$(m^*\epsilon(\zeta))(g,h) := \epsilon_{gh}(\zeta(g,h)), \text{ and}$$

$$m^*\sigma(\zeta)(g,h) := m^*\sigma(\zeta(\rho(g),\rho(h))).$$

It is easy to see that $m^*\epsilon(f\otimes\xi) = (1\otimes\epsilon)(f\otimes\xi)$ and $m^*\sigma(f\otimes\xi) = (\rho^{(2)}\otimes\sigma)(f\otimes\xi)$; therefore, $\mathcal{C}_c(\mathcal{G}^{(2)})\hat{\odot}\mathcal{C}_0(\mathcal{G};\mathscr{E})$ is a Rg subspace of $\mathcal{C}_0(\mathcal{G}^{(2)};m^*\mathscr{E})$.

To prove that $(\mathcal{C}_c(\mathcal{G}^{(2)})\hat{\odot}\mathcal{C}_0(\mathcal{G};\mathscr{E}),\rho^{(2)}\otimes\sigma)$ is dense in $\mathcal{C}_0(\mathcal{G}^{(2)};m^*\mathscr{E})$, we just have to check that the conditions (a) and (b) in Lemma 5.1.3 held for the Rg Fell bundle $(m^*\mathscr{E},m^*\nu)$ of $(\mathcal{G}^{(2)},\rho^{(2)})$ and the Rg subspace $\mathcal{C}_c(\mathcal{G}^{(2)})\hat{\odot}\mathcal{C}_0(\mathcal{G};\mathscr{E})$. For $\varphi\in\mathcal{C}_0(\mathcal{G}^2)$ and an elementary tensor $f\otimes\xi$, one has $(\varphi\cdot(f\otimes\xi))(g,h) = \varphi(g,h)f(g,h)\xi(gh)) = (\varphi f\otimes\xi)(g,h)$; then condition (a) holds. For condition (b), we are using the same arguments as in [32, Proposition 3.40]. Let $(g,h)\in\mathcal{G}^{(2)}$, and $e\in m^*\mathscr{E}_{(g,h)} = \mathscr{E}_{gh}$, and let $f\in C_c(\mathcal{G}^{(2)})$ such that $f(g,h)=1$. Choose a section $\xi\in\mathcal{C}_0(\mathcal{G};\mathscr{E})$ such that $\xi(gh)=e$. Then $(f\otimes\xi)(g,h)=e$, which completes the proof. □

Proposition 5.1.6. *Let $(\mathcal{G},\mathscr{E})$ be a Rg Fell system and let μ be a Real Haar system for $\mathcal{G}$. Then $(\mathcal{C}_c(\mathcal{G};\mathscr{E}),\sigma)$ is a Rg*-algebra with respect to the following operations:*

$$(\xi*\eta)(g) := \int_{\mathcal{G}^{r(g)}}\xi(\gamma)\eta(\gamma^{-1}g)\,d\mu^{r(g)}(\gamma),\ \textit{for}\ \xi,\eta\in\mathcal{C}_c(\mathcal{G};\mathscr{E}),\ \textit{and} \tag{5.4}$$

$$\xi^*(g) := \xi(g^{-1})^*. \tag{5.5}$$

The following lemma will be needed in the proof of the above proposition.

Lemma 5.1.7. *Let the above settings be given. Then, given a section $\chi\in\mathcal{C}_c(\mathcal{G}^{(2)};m^*\mathscr{E})$, the formula*

$$\psi(g) := \int_{\mathcal{G}^{s(g)}}\chi(\gamma,\gamma^{-1}g^{-1})^*\,d\mu^{s(g)}(\gamma), \tag{5.6}$$

provides a continuous section $\psi\in\mathcal{C}_c(\mathcal{G};\mathcal{E})$.

Proof. Observe that, from the definition of $m^*\mathscr{E}$, for every $\gamma\in\mathcal{G}^{s(g)}$,

$$\chi(\gamma,\gamma^{-1}g^{-1})\in\mathscr{E}_{\gamma\gamma^{-1}g^{-1}} = \mathscr{E}_{g^{-1}};$$

so that $\chi(\gamma,\gamma^{-1}g^{-1})^*\in\mathcal{E}_g$. Hence, for every $g\in\mathcal{G}$, the integral (5.6) takes values in $\mathscr{E}_g$, and ψ is then a section of $\mathscr{E}$. It then remains to check that ψ is continuous and compactly supported. However, from Proposition 5.1.5, χ is the uniform limit of a sum of the form $\sum_i f_i\otimes\xi_i$. On the other hand, it is easy to see that if $\chi_j\longrightarrow\chi$ with respect to the inductive limit topology in $\mathcal{C}_c(\mathcal{G}^{(2)};m^*\mathscr{E})$, then $\psi_j\longrightarrow\psi$ with respect to the inductive limit topology.

It turns out that we can restrict ourselves to the case when $\chi = f\otimes\xi$ is an elemantary tensor. In this case, we have

$$\psi(g) = \int_{\mathcal{G}^{s(g)}} \overline{f(\gamma,\gamma^{-1}g^{-1})}d\mu^{s(g)}(\gamma)\xi(g^{-1})^*.$$

Next, it is not hard, using the Stone-Weierstrass' Theorem, to check that span$\{f_1.f_2 : \mathcal{G}\times\mathcal{G}\ni(g_1,g_2)\longmapsto f_1(g_1)f_2(g_2) \mid f_1,f_2\in\mathcal{C}_c(\mathcal{G})\}$ is dense in $\mathcal{C}_c(\mathcal{G}\times\mathcal{G})$. Now, since $\mathcal{G}^{(2)}$ is a closed subset of the normal space $\mathcal{G}\times\mathcal{G}$, we can extend the function $f\in\mathcal{C}_c(\mathcal{G}^{(2)})$ to all of $\mathcal{G}\times\mathcal{G}$. Thus, we can suppose that ψ is of the form

$$\psi(g) = \int_{\mathcal{G}^{s(g)}} \overline{f_1(\gamma)f_2(\gamma^{-1}g^{-1})}d\mu^{s(g)}(\gamma)\xi(g^{-1})^*. \tag{5.7}$$

It follows that $\operatorname{supp}\psi\subset(\operatorname{supp}f_1)(\operatorname{supp}f_2)$, where the latter is the set

$$\{g_1g_2 \mid g_1\in\operatorname{supp}f_1, g_2\in\operatorname{supp}f_2\in\}.$$

This shows that ψ is compactly supported. Now we can use the same arguments as above and that of Renault in [76, Proposition II.1.1] to show the continuity of ψ. Extend the function $\mathcal{G}^{(2)}\ni(g,\gamma)\longmapsto f_2(\gamma^{-1}g^{-1})$ to a bounded continuous function $\tilde{f}_2 : \mathcal{G}\times\mathcal{G}\longrightarrow\mathbb{C}$. Since the function $\mathcal{G}\ni g\longmapsto l_g\in C_c(\mathcal{G})$, where $l_g(\gamma)\longmapsto\overline{f_1(\gamma)\tilde{f}_2(g,\gamma)}$ is continuous, so is the function

$$\mathcal{G}\times X\ni(g,x)\longmapsto\int_{\mathcal{G}^x}\overline{f_1(\gamma)\tilde{f}_2(g,\gamma)}d\mu^{s(g)}(\gamma)\xi(g^{-1})^*\in\mathcal{E}_g,$$

and in particular, its restriction to the subspace $\{(g,s(g))\in\mathcal{G}\times X \mid g\in\mathcal{G}\}$ is continuous; hence the function defined by (5.7) is continuous. Therefore, $\psi\in\Gamma_c(\mathcal{G};\mathcal{E})$. □

Remark 5.1.8. *Notice that in the above proof we have freely used the assumption that the involution in $\mathcal{C}_c(\mathcal{G};\mathcal{E})$ given by (5.1.6) is well defined. In fact, this is not hard to check as it will be fleshed out below.*

Proof of Proposition 5.1.6. We shall first show that the operations are well defined. Given $\xi\in\mathcal{C}_c(\mathcal{G};\mathcal{E})$, $\xi(g^{-1})\in\mathcal{E}_{g^{-1}}$; thus $\xi^*(g) = \xi(g^{-1})^*\in\mathcal{E}_g$ and ξ^* is well in $\mathcal{C}(\mathcal{G};\mathcal{E})$, the continuity of ξ^* coming from that of ξ and $\operatorname{supp}\xi^*$ being exactly

$$(\operatorname{supp}\xi)^{-1} := \{g^{-1}\in\mathcal{G} \mid g\in\operatorname{supp}\xi\}.$$

Note that the formula $\xi*\eta$ defined by (5.4) is just an application of Lemma 5.1.7 by setting:

$$\chi(g_1,g_2) := \eta(g_2)^*\xi(g_1)^*, \text{ for } (g_1,g_2)\in\mathcal{G}^{(2)};$$

so that

$$\xi*\eta(g) = \int_{\mathcal{G}^{s(g^{-1})}}\chi(\gamma,\gamma^{-1}g)^*d\mu^{s(g^{-1})}(\gamma).$$

Furthermore, if $g \in \operatorname{supp}(\xi * \eta)$, then there exists $\gamma \in \mathcal{G}^{r(g)}$ such that both of $\xi(\gamma)$ and $\eta(\gamma^{-1}g)$ are nonzero. Then $\gamma^{-1}g \in \operatorname{supp}\eta$, and then $\operatorname{supp}(\xi * \eta) \subset (\operatorname{supp}\xi)(\operatorname{supp}\eta)$.

It is routine to check that the convolution and the involution operations are compatible with both of the $\mathbb{Z}_2$-grading and the Real structure of $\mathcal{C}_c(\mathcal{G};\mathscr{E})$. Furthermore, the convolution is associative: if $\xi,\eta,\zeta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, and $g \in \mathcal{G}$, then

$$\begin{aligned}
(\xi * \eta) * \zeta(g) &= \int_{\mathcal{G}^{r(g)}} (\xi * \eta)(\gamma)\zeta(\gamma^{-1}g)\,d\mu^{r(g)}(\gamma)\\
&= \int_{\mathcal{G}^{r(g)}} \int_{\mathcal{G}^{r(\gamma)}} \xi(h)\eta(h^{-1}\gamma)\zeta(\gamma^{-1}g)\,d\mu^{r(\gamma)}(h)\,d\mu^{r(g)}(\gamma)\\
&= \int_{\mathcal{G}^{r(g)}} \xi(h) \int_{\mathcal{G}^{r(h)}} \eta(h^{-1}\gamma)\zeta(\gamma^{-1}g)\,d\mu^{r(h)}(\gamma)\,d\mu^{r(g)}(h)\\
&= \int_{\mathcal{G}^{r(g)}} \xi(h) \int_{\mathcal{G}^{s(h)}} \eta(\gamma)\zeta(\gamma^{-1}h^{-1}g)\,d\mu^{s(h)}\,d\mu^{r(g)}(h),
\end{aligned}$$

where we used the associativity of the multiplication on $(\mathscr{E},\sigma)$ to get the third equality and the left invariance of the Haar measure μ to obtain the last one. Then we get

$$\begin{aligned}
(\xi * \eta) * \zeta(g) &= \int_{\mathcal{G}^{r(g)}} \xi(h)(\eta * \zeta)(h^{-1}g)\,d\mu^{r(g)}(h)\\
&= \xi * (\eta * \zeta)(g).
\end{aligned}$$

We shall also verify that the operation (5.1.6) anti-commutes with the convolution. To do this, observe that since the *-involution on $(\mathscr{E},\mu)$ is antilinear, the same arguments we used in the proof of Lemma 3.14.9 would show that if $f \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, and $g \in \mathcal{G}$, then

$$\left(\int_{\mathcal{G}^{r(g)}} f(\gamma)\,d\mu^{r(g)}(\gamma)\right)^* = \int_{\mathcal{G}^{r(g)}} f(\gamma)^*\,d\mu^{r(g)}(\gamma).$$

It follows that if $\xi,\eta \in \Gamma_c(\mathcal{G};\mathcal{E})$ and $g \in \mathcal{G}$; then

$$\begin{aligned}
(\xi * \eta)^*(g) &= \left(\int_{\mathcal{G}^{s(g)}} \xi(\gamma)\eta(\gamma^{-1}g^{-1})\,d\mu^{s(g)}(\gamma)\right)^*\\
&= \int_{\mathcal{G}^{s(g)}} \eta(\gamma^{-1}g^{-1})^*\xi(\gamma)^*\,d\mu^{s(g)}(\gamma)\\
&= \int_{\mathcal{G}^{s(g)}} \eta^*(g\gamma)\xi^*(\gamma^{-1})\,d\mu^{s(g)}(\gamma)\\
&= \int_{r(g)} \eta^*(\gamma)\xi^*(\gamma^{-1}g)\,d\mu^{r(g)}(\gamma), \text{ by the left invariance of } \mu,\\
&= \eta^* * \xi^*(g).
\end{aligned}$$

At this point, it remains to show that these operations are continuous with respect to the inductive limit topology. Suppose that $\xi_i \longrightarrow \xi$ and $\eta_i \longrightarrow \eta$ in $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ with respect to the inductive limit topology, and that K and L are compact subsets of $\mathcal{G}$ such that, eventually,

$\operatorname{supp}\xi_i \subset K$ and $\operatorname{supp}\eta_i \subset L$. Then, eventually, $\operatorname{supp}(\xi_i * \eta_i) \subset KL$. Now, from (3.64), one has

$$\begin{aligned}\|\xi*\eta-\xi_i*\eta_i\| &\le \int_{\mathcal{G}^{r(g)}} \|\xi(\gamma)\eta(\gamma^{-1}g)-\xi_i(\gamma)\eta_i(\gamma^{-1}g)\|\,d\mu^{r(g)}(\gamma)\\ &\le \int_{\mathcal{G}^{r(g)}} \|\xi(\gamma)-\xi_i(\gamma)\|\|\eta(\gamma^{-1}g)\|\,d\mu^{r(g)}\\ &+\int_{\mathcal{G}^{r(g)}} \|\xi_i(\gamma)\|\|\eta(\gamma^{-1}g)-\eta_i(\gamma^{-1}g)\|\,d\mu^{r(g)}(\gamma).\end{aligned}$$

Therefore, $\xi_i*\eta_i \longrightarrow \xi*\eta$ in $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ with respect to the inductive limit topology. Moreover, eventually, $\operatorname{supp}\xi_i^* \subset K^{-1}$. Then, from the definition of the *-involution,

$$\|\xi^*(g)-\xi_i^*(g)\| = \|(\xi(g^{-1})-\xi_i(g^{-1}))^*\|;$$

hence $\xi_i^* \longrightarrow \xi^*$ with respect to the inductive limit topology. □

Remark 5.1.9. *Although we had not mentioned it before, we used in the last part of the proof the fact that given $\xi\in\mathcal{C}_c(\mathcal{G};\mathscr{E})$, the function $\mathcal{G}\ni g\longmapsto \|\xi(g)\|\in\mathbb{R}_+$ is μ-integrable since it is continuous and compactly supported.*

Definition 5.1.10. *Assume $(\mathcal{G},\mathscr{E})$ is a Rg Fell system. For $\xi\in\mathcal{C}_c(\mathcal{G};\mathscr{E})$, we set*

$$\|\xi\|_1 := \sup_{x\in X}\int_{\mathcal{G}^x}\|\xi(g)\|\,d\mu^x(\gamma).$$

Then, we define the I-norm *on $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ by*

$$\|\xi\|_I := \max\left\{\|\xi\|_1,\|\xi^*\|_1\right\}.$$

The proposition below can been seen as a generalisation of [76, Proposition II.1.4] or [32, Proposition 3.57]. For this reason, we will omit the proof.

Proposition 5.1.11. *Suppose $(\mathcal{G},\mathscr{E})$ is a Rg Fell system. Then the I-norm is a norm on $\mathcal{C}_c(\mathcal{G};\mathscr{E})$, compatible with the grading, invariant with respect to the involution σ, and defining a topology coarser than the inductive limit topology.*

Proposition 5.1.12. *Let $(\mathcal{G},\mathscr{E})$ be a Rg Fell system. Then the Rg *-algebra $(\Gamma_c(\mathcal{G};\mathscr{E}),\sigma)$ admits a self-adjoint two-sided Real approximate identity with respect to the inductive limit topology.*

The proof of Proposition 5.1.12 will be delayed until Section 5.5

Definition 5.1.13. *Let $(L^1(\mathcal{G};\mathscr{E}),\sigma)$ be the Rg Banach *-algebra consisting of the completion $L^1(\mathcal{G};\mathscr{E})$ of $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ with respect to the I-norm, and the grading and Real structure extended to $L^1(\mathcal{G};\mathscr{E})$. Then the* full Rg C^*-algebra of $(\mathscr{E},\sigma)$, *denoted by $(C^*(\mathcal{G};\mathscr{E}),\sigma)$, is defined as the enveloping C^*-algebra $C^*(\mathcal{G};\mathscr{E})$ of $L^1(\mathcal{G};\mathscr{E})$, provided with the grading and the Real structure of $L^1(\mathcal{G};\mathscr{E})$ extended*[1].

[1] Recall (see for instance [26, 2.7.2]) that given a *-algebra A with an approximate identity, and equipped

5.2 Reduced crossed products

Our goal in this section is to construct a graded Real C^*-algebra associated to a Rg Dixmier-Douady bundle $(\mathcal{A},\alpha)$ over a Real groupoid $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, by using some tools we will be developing here for Rg Fell bundles. Namely, we are constructing the *reduced C^*-algebra* of a Rg Fell system $(\mathcal{G},\mathscr{E})$. Note that this construction is known in the case where no grading nor Real structure are involved (see for instance [90, p.907], and in [48] for the special case of proper groupoids).

Proposition 5.2.1. *Let $(\mathcal{G},\mathscr{E})$ be a Rg Fell system. Consider the Rg C^*-algebra (A,σ), where $A := \mathcal{C}_0(X;\mathscr{E})$. Then,*

(i) the operation

$$(f.\xi)(g) := f(r(g))\xi(g), \text{ for } f \in A, \xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E}), \text{ and } g \in \mathcal{G}, \tag{5.8}$$

defines a Rg (left) A-action on $(\mathcal{C}_c(\mathcal{G};\mathscr{E}),\sigma)$.

(ii) the operation

$$_A\langle\xi,\eta\rangle(x) := \int_{\mathcal{G}^x} \xi^*(\gamma)\eta(\gamma^{-1})d\mu^x(g), \text{ for } \xi,\eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E}), \text{ and } x \in X, \tag{5.9}$$

defines a Rg A-valued inner product on $(\mathcal{C}_c(\mathcal{G};\mathscr{E}),\sigma)$.

Therefore, with these operations, $(\Gamma_c(\mathcal{G};\mathscr{E}),\sigma)$ is a graded Real pre-Hilbert A-module.

Proof. (i) In view of Definition 5.1.1, for every $g \in \mathcal{G}$, there is an action of $\mathscr{E}_{r(g)}$ on $\mathscr{E}_g$ such that

$$\begin{tikzcd} \mathscr{E}_{r(g)} \times \mathscr{E}_g \arrow[r] \arrow[d, "\sigma_{r(g)}\times\sigma_g"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{E}_g \arrow[d, "\sigma_g"] \\ \mathscr{E}_{\rho(r(g))} \times \mathscr{E}_{\rho(g)} \arrow[r] & \mathscr{E}_{\rho(g)} \end{tikzcd}$$

and such that $\mathscr{E}^i_{r(g)} \times \mathscr{E}^j_g$ is sent to $\mathscr{E}^{i+j}_g$, $i,j \in \{0,1\}$. It then follows that, given $f \in A$ and $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, we have $f.\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, and the operation is compatible with the gradings and the Real structures.

with a seminorm $\|\|$ such that $\|ab\| \le \|a\|\|b\|$, $\|a^*\| = \|a\|$, and $\|a^*a\| = \|a\|^2$, its envelopping C^*-algebra can be constructed as follows: set $I := \{a \in A \mid \|a\| = 0\}$; then the map $a \longmapsto \|a\|$ defines a norm on the quotient A/I, and the completion of A/I with respect to this norm a C^*-algebra. Now, if $\sigma : A \longrightarrow A$ is a Real structure on A, then the map $A/I \ni [a] \longmapsto \tilde{\sigma}([a]) := [\sigma(a)] \in A/I$ is well defined and provides a Real structure on the quotient A/I, and then a Real structure on the envelopping C^*-algebra. The same holds for a grading ϵ on A.

(ii)The formula well defines an element of A; indeed, given $\xi,\eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, put $F_{\xi.\eta} : \mathcal{G} \longrightarrow \mathcal{E}, \gamma \longmapsto \xi^*(\gamma)\eta(\gamma^{-1})$. Then one obtains ${}_A\langle\xi,\eta\rangle = \mu(F_{\xi,\eta}) \in A$ (cf. [76, Definition I.2.2]).
Let $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$. We verify that ${}_A\langle\xi,\xi\rangle \in A_+$, i.e. ${}_A\langle\xi,\xi\rangle(x)$ is a positive element of the C^*-algebra $\mathcal{E}_x$. For every $x \in X$:

$${}_A\langle\xi,\xi\rangle(x) = \int_{\mathcal{G}^x} \xi(\gamma^{-1})^*\xi(\gamma^{-1})d\mu^x(\gamma) \geq 0, \text{ thanks to (viii) Definition 5.1.1.}$$

Also, it is clear that if ${}_A\langle\xi,\xi\rangle(x) = 0, \forall x \in X$, then $\xi = 0$ on $\mathcal{G}$. The remaining properties of the inner product are very easy to check; for instance, for $\xi,\eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, one has

$${}_A\langle\xi,\eta\rangle^*(x) = ({}_A\langle\xi,\eta\rangle(x))^* = \int_{\mathcal{G}^x} \eta^*(\gamma)\xi(\gamma^{-1})d\mu^x(\gamma) = {}_A\langle\eta,\xi\rangle(x),\ \forall x \in X.$$

Let us verify that ${}_A\langle\cdot,\cdot\rangle$ is compatible with the $\mathbb{Z}_2$-gradings and the Real structures. The mere use of the properties defining a graded Real Fell bundle allows us to see that for $i,j \in \{0,1\}$, ${}_A\langle\mathcal{C}_c(\mathcal{G};\mathscr{E})^i,\mathcal{C}_c(\mathcal{G};\mathscr{E})^j\rangle \subseteq A^{i+j}$. Now, to show that $\sigma({}_A\langle\xi,\eta\rangle) = {}_A\langle\sigma(\xi),\sigma(\eta)\rangle$, we may apply the equality (3.66) to each fiber $\mathscr{E}_x$ of $(\mathscr{E}_{|X},\sigma)$ and the function $\mathcal{G}^x \ni \gamma \longmapsto \xi^*(\gamma)\eta(\gamma^{-1}) \in \mathscr{E}_x$. Then,

$$\begin{aligned}
\nu_x({}_A\langle\xi,\eta\rangle(x)) &= \int_{\mathcal{G}^{\rho(x)}} \sigma_x(\xi^*(\rho(\gamma))\eta(\rho(\gamma)^{-1}))d\mu^{\rho(x)}(\gamma)\\
&= \int_{\mathcal{G}^{\rho(x)}} \sigma(\xi)^*(\gamma)\sigma(\eta)(\gamma^{-1})d\mu^{\rho(x)}(\gamma)\\
&= {}_A\langle\sigma(\xi),\sigma(\eta)\rangle(\rho(x)), \forall x \in \mathcal{G}^{(0)};
\end{aligned}$$

thus $\sigma({}_A\langle(\xi,\eta)) = {}_A\langle\sigma(\xi),\sigma(\eta)\rangle, \forall \xi,\eta \in \mathcal{C}_c(\mathcal{G};\mathcal{E})$. □

Definition 5.2.2. *For $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, put*

$$\|\xi\|_2 := \|{}_A\langle\xi,\xi\rangle\|^{1/2};$$

where the norm $\|.\|$ in A is, of course, given by $\|f\| = \sup_{x\in X}\|f(x)\|_{\mathscr{E}_x}$, for $f \in A$. Then, we define the Rg Hilbert A-module $(L^2(\mathcal{G};\mathscr{E}),\sigma)$ as the completion of $(\mathcal{C}_c(\mathcal{G};\mathscr{E}),\sigma)$ with respect to $\|\cdot\|_2$.

Definition 5.2.3. *For every $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, define a Rg*-morphism*

$$\pi_l(\xi) : \mathcal{C}_c(\mathcal{G};\mathscr{E}) \longrightarrow \mathcal{C}_c(\mathcal{G};\mathscr{E})$$

by left multiplication; namely:

$$\pi_l(\xi)(\eta) := \xi * \eta, \text{ for } \eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E}). \tag{5.10}$$

Proposition 5.2.4. *The operation (5.10) provides a Rg*-monomorphism*

$$\pi_l : \mathcal{C}_c(\mathcal{G};\mathscr{E}) \longrightarrow \mathcal{L}_A(L^2(\mathcal{G};\mathscr{E})), \xi \longmapsto \pi_l(\xi). \tag{5.11}$$

Furthermore, π_l extends to a Rg*-representation of $(L^1(\mathcal{G};\mathscr{E}),\sigma)$, also denoted by π_l and called the left regular representation. *We will write $\pi_l^{\mathcal{G},\mathscr{E}}$ (or $\pi_l^{\mathcal{G}}$) when there is a risk of confusion.*

Proof. Let $\xi,\eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$. It is clear that $\pi_l(\xi)$ is A-linear. For $x \in X$, one has

$$\begin{aligned}
{}_A\langle \xi*\eta,\xi*\eta\rangle(x) &= \int_{\mathcal{G}^x}((\xi*\eta)(g^{-1}))^*(\xi*\eta)(g^{-1})d\mu^x(g)\\
&= \int_{\mathcal{G}^x}\int_{\mathcal{G}^{s(g)}}\eta(\gamma^{-1}g^{-1})^*\xi(\gamma)^*d\mu^{s(g)}(\gamma)\int_{s(g)}\xi(h)\eta(h^{-1}g^{-1})d\mu^{s(g)}(h)d\mu^x(g)\\
&= \int_{\mathcal{G}^x}\int_{\mathcal{G}^{s(g)}}\eta^*(g\gamma)\xi^*(\gamma^{-1})d\mu^{s(g)}(\gamma)\int_{\mathcal{G}^{s(g)}}\xi(h)\eta(h^{-1}g^{-1})d\mu^{s(g)}(h)d\mu^x(g)\\
&= \int_{\mathcal{G}^x}\int_{\mathcal{G}^{r(g)}}\eta^*(\gamma)\xi^*(\gamma^{-1}g)d\mu^{r(g)}(\gamma)\int_{\mathcal{G}^{r(g)}}\xi(g^{-1}h)\eta(h^{-1})d\mu^{r(g)}(h)d\mu^x(g),
\end{aligned}$$

by the left invariance of the measure μ.

It follows that for all $x \in X$

$$\begin{aligned}
\|{}_A\langle \xi*\eta,\xi*\eta\rangle(x)\| &\le \int_{\mathcal{G}^x}\left\|\left(\int_{\mathcal{G}^{r(g)}}\eta^*(\gamma)\xi^*(\gamma^{-1}g)d\mu^{r(g)}(\gamma)\right)\right.\\
&\qquad\left.\left(\int_{\mathcal{G}^{r(g)}}\xi(g^{-1}h)\eta(h^{-1})d\mu^{r(g)}(h)d\right)\right\|d\mu^x(g)\\
&\le \int_{\mathcal{G}^x}\int_{\mathcal{G}^{r(g)}}\|\xi^*(\gamma^{-1}g)\xi(g^{-1}\gamma)\|\|\eta^*(\gamma)\eta(\gamma^{-1})\|d\mu^{r(g)}(\gamma)d\mu^x(g)\\
&\le \|{}_A\langle\xi,\xi\rangle(x)\|\|{}_A\langle\eta,\eta\rangle(x)\|.
\end{aligned}$$

Therefore, $\|\pi_l(\xi)(\eta)\|_2 \le \|\xi\|_2\|\eta\|_2, \forall \eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, and then $\pi_l(\xi)$ is bounded with respect to the norm $\|.\|_2$. Thus $\pi_l(\xi)$ extends to a bounded A-linear operator on the completion $L^2(\mathcal{G};\mathscr{E})$.

Suppose that $\xi,\eta,\zeta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$. Then for all $x \in X$

$$\begin{aligned}
{}_A\langle\pi_l(\xi)\eta,\zeta\rangle(x) &= \int_{\mathcal{G}^x}(\eta^**\xi^*)(g)\zeta(g^{-1})d\mu^x(g)\\
&= \int_{\mathcal{G}^x}\int_{\mathcal{G}^{r(g)}}\eta^*(\gamma)\xi^*(\gamma^{-1}g)d\mu^{r(g)}(\gamma)\zeta(g^{-1})d\mu^x(g)\\
&= \int_{\mathcal{G}^x}\eta^*(\gamma)\int_{\mathcal{G}^{r(\gamma)}}\xi^*(\gamma^{-1}g)\zeta(g^{-1})d\mu^{r(\gamma)}(g)d\mu^x(\gamma),
\end{aligned}$$

where the last equality is obtained by switching the integrals. Now by applying the left invariance of the Real Haar measure μ, we get

$$\begin{aligned}
{}_A\langle\pi_l(\xi)\eta,\zeta\rangle(x) &= \int_{\mathcal{G}^x}\eta^*(\gamma)\left(\int_{\mathcal{G}^{s(\gamma)}}\xi^*(g)\zeta(g^{-1}\gamma^{-1})d\mu^{s(\gamma)}(g)\right)d\mu^x(\gamma)\\
&= \int_{\mathcal{G}^x}\eta^*(\gamma)(\xi^**\zeta)(\gamma^{-1})d\mu^x(\gamma)\\
&= {}_A\langle\eta,\pi_l(\xi)^*\zeta\rangle(x),
\end{aligned}$$

where $\pi_l(\xi)^* := \xi^* * (.)$. Hence, $\pi_l(\xi)$ is adjointable for all $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, and this shows that the map (5.10) is well define. Moreover, it is obvious that π_l is a *-monomorphism and is compatible with the gradings and Real structures. Therefore, the last part of the proposition follows. □

Definition 5.2.5. *We define the* Rg reduced C^*-algebra $(C_r^*(\mathcal{G};\mathscr{E}),\sigma)$ *of* $(\mathcal{G},\mathscr{E})$ *to be the closure (with respect to the operator norm) of the image of* $(L^1(\mathcal{G};\mathscr{E}),\sigma)$ *by the left regular representation, together with the grading and Real structure inherited from that of* $\mathcal{L}_A(L^2(\mathcal{G};\mathscr{E}))$*;* i.e.

$$C_r^*(\mathcal{G};\mathscr{E}) := \overline{\pi_l(L^1(\mathcal{G};\mathscr{E}))} \subset \mathcal{L}_A(L^2(\mathcal{G};\mathscr{E})).$$

Remark 5.2.6 (The reduced norm)**.** *Alternatively, we can think of* $(C_r^*(\mathcal{G};\mathscr{E}),\sigma)$ *as the completion of* $(\mathcal{C}_c(\mathcal{G};\mathscr{E}),\sigma)$ *with respect to the* reduced norm $\|\cdot\|_r$ *given by*

$$\|\xi\|_r := \sup_{\|\eta\|_2=1}\left\{\|\pi_l(\xi)\eta\|_2 \mid \eta\in\mathcal{C}_c(\mathcal{G};\mathscr{E})\right\},\ \textit{for}\ \xi\in\mathcal{C}_c(\mathcal{G};\mathscr{E}). \tag{5.12}$$

The following example will play a central role in the sequel.

Example 5.2.7. *Let* $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$ *with Real structure* σ*. Then the pull-back* $(s^*\mathcal{A},s^*\sigma)$ *of* $(\mathcal{A},\sigma)$ *by the source map* s *is a Rg Fell bundle over* $\mathcal{G}$ *as follows: for all* $(g,h)\in\mathcal{G}^{(2)}$*, the map* $s^*\mathcal{A}_g\times s^*\mathcal{A}_h\longrightarrow s^*\mathcal{A}_{gh}$ *is given by* $(a,b)\longmapsto\alpha_{h^{-1}}(a)b\in s^*\mathcal{A}_{gh}=\mathcal{A}_{s(gh)}=\mathcal{A}_{s(h)}$*, and the involution is* $\mathcal{A}_{s(g)}\ni a\longmapsto\alpha_g(a^*)\in s^*\mathcal{A}_{g^{-1}}=\mathcal{A}_{r(g)}$*. All the properties of Definition 5.1.1 are obvious.*

Definition 5.2.8. *Let* $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ *be a locally compact Hausdorff Real groupoid, and let* $(\mathcal{A},\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$*. Then, the* reduced crossed product $(\mathcal{G}\ltimes_r\mathcal{A},s^*\sigma)$ *of* $(\mathcal{A},\alpha)$ *with* $(\mathcal{G},\rho)$ *is the Rg* C^**-algebra given by*

$$\mathcal{A}\rtimes_r\mathcal{G} := C_r^*(\mathcal{G};s^*\mathcal{A}),$$

equipped with the Real structure $s^*\sigma$ *and the grading* $s^*\epsilon$*, where* ϵ *is the fiberwise grading of* $\mathcal{A}$*.*

Proposition 5.2.9. *Suppose that* $\mathcal{A}=\mathcal{B}$ *in* $\widehat{Br}\mathrm{R}(\mathcal{G})$*. Then, as Real graded* C^**-algebras,*

$$\mathcal{A}\rtimes_r\mathcal{G}\sim_{Morita}\mathcal{B}\rtimes_r\mathcal{G}.$$

Proof. First, observe that elements of $\mathcal{A}\rtimes_r\mathcal{G}$ are limits (with respect to the operator norm in $\mathcal{L}(L^2(\mathcal{G};s^*\mathcal{A}))$) of elements of the form $\pi_l(a)$, with $a\in\mathcal{C}_c(\mathcal{G};s^*\mathcal{A})$.
Assume that $(\mathcal{A},\alpha)\sim_{(\mathcal{X},V)}(\mathcal{B},\beta)$, where $\mathcal{X}$ has the involution κ. Set:

$$(\pi_l(a).\xi)(g) := a(g)\xi(g)\in s^*\mathcal{X}_g,\ \text{and} \tag{5.13}$$

$$(\xi.\pi_l(b))(g) := \xi(g)b(g)\in s^*\mathcal{X}_g=\mathcal{X}_{s(g)}, \tag{5.14}$$

for $\pi_l(a)\in\mathcal{A}\rtimes_r\mathcal{G},\pi_l(b)\in\mathcal{B}\rtimes_r\mathcal{G},\xi\in\Gamma_c(\mathcal{G};s^*\mathcal{X})$, and , $g\in\mathcal{G}$. And on the other hand, we set:

$${}_{\mathcal{A}\rtimes_r\mathcal{G}}\langle\xi,\eta\rangle := \pi_l({}_{\mathcal{A}}\langle\xi,\eta\rangle)\in\mathcal{A}\rtimes_r\mathcal{G},\ \text{and} \tag{5.15}$$

$$\langle\xi,\eta\rangle_{\mathcal{B}\rtimes_r\mathcal{G}} := \pi_l(\langle\xi,\eta\rangle_{\mathcal{B}})\in\mathcal{B}\rtimes_r\mathcal{G}, \tag{5.16}$$

for every $\xi,\eta \in \Gamma_c(\mathcal{G}; s^*\mathcal{X})$, where

$$_{\mathcal{A}}\langle \xi,\eta\rangle(g) = {}_{\mathcal{A}_{s(g)}}\langle \xi(g),\eta(g)\rangle \in \mathcal{A}_{s(g)}, \text{ and } \langle \xi,\eta\rangle_{\mathcal{B}}(g) = \langle \xi(g),\eta(g)\rangle_{\mathcal{B}_{s(g)}} \in \mathcal{B}_{s(g)}, \forall g \in \mathcal{G}.$$

It is not hard to verify, in a one hand, that (5.13) and (5.14) respectively define a continuous Rg left $\mathcal{A} \rtimes_r \mathcal{G}$-action and a continuous Rg right $\mathcal{B} \rtimes_r \mathcal{G}$-action on the Rg algebra $(\mathcal{C}_c(\mathcal{G}; s^*\mathcal{X}), s^*\kappa)$ making it a Rg $\mathcal{A} \rtimes_r \mathcal{G}$-$\mathcal{B} \rtimes_r \mathcal{G}$-bimodule; and on the other hand, that the formulas (5.15) and (5.16) are respectively well defined continuous Rg $\mathcal{A} \rtimes_r \mathcal{G}$-valued and $\mathcal{B} \rtimes_r \mathcal{G}$-valued full inner products on $(\mathcal{C}_c(\mathcal{G}; s^*\mathcal{X}), s^*\kappa)$. Therefore, $(\mathcal{C}_c(\mathcal{G};\mathcal{X}), s^*\kappa)$ implements a Rg $\mathcal{A} \rtimes_r \mathcal{G}$-$\mathcal{B} \rtimes_r \mathcal{G}$-imprimitivity bimodule. □

Alternatively, we will sometimes use another definition of the reduced norm, which is a generalization of that of [83]. Suppose we are given a right Rg Fell system $(\mathcal{G}, \mathscr{E})$. Then, for $x \in X$, consider the inclusion $i_x : \mathcal{G}_x \longrightarrow \mathcal{G}$. Then, as in [?, A.3], we define the (right) graded Hilbert A_x-module $L^2(\mathcal{G}_x; \mathscr{E})$ as the completion of $\mathcal{C}_c(\mathcal{G}_x; i_x^*\mathscr{E})$ with respect to the graded inner product $\langle \xi,\eta\rangle_{A_x} := \int_{\mathcal{G}_x} \xi(g)^*\eta(g)d\mu_x(g)$ (the right action being $(\xi \cdot a) : \mathcal{G}_x \ni g \longmapsto \xi(g)\cdot a \in \mathscr{E}_g$). Furthermore, the involution σ on $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ induces a conjugate linear isometry $\sigma_x : \mathcal{L}_{A_x}(L^2(\mathcal{G}_x;\mathscr{E})) \longrightarrow \mathcal{L}_{A_{\bar{x}}}(L^2(\mathcal{G}_{\bar{x}};\mathscr{E}))$ defined in the evident way. The following lemma is very easy to prove.

Lemma 5.2.10. *Let $(\mathcal{G}, \mathscr{E})$ be a Rg Fell system. Then for all $x \in X$, left multiplication by elements of $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ gives a graded *-representation $\pi_x^{\mathcal{G}} : \mathcal{C}_c(\mathcal{G};\mathscr{E}) \longrightarrow \mathcal{L}_{A_x}(L^2(\mathcal{G}_x;\mathscr{E}))$, and the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{C}_c(\mathcal{G};\mathscr{E}) & \xrightarrow{\pi_x^{\mathcal{G}}} & \mathcal{L}_{A_x}(L^2(\mathcal{G}_x;\mathscr{E})) \\
\sigma \downarrow & & \downarrow \sigma_x \\
\mathcal{C}_c(\mathcal{G};\mathscr{E}) & \xrightarrow{\pi_{\bar{x}}^{\mathcal{G}}} & \mathcal{L}_{A_{\bar{x}}}(L^2(\mathcal{G}_{\bar{x}};\mathscr{E}))
\end{array}$$

Moreover, we have

$$\|\xi\|_{C^*(\mathcal{G};\mathscr{E})} := \|\xi\|_r = \sup_{x\in X}\{\|\pi_x^{\mathcal{G}}(\xi)\|, \ \forall \xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$$

5.3 Rg Fell pairs and equivalences

Definition 5.3.1 (Fell action)**.** *Let $(\mathcal{G}, \rho)$ be as usual, and let (Z, τ) be a Real (right) principal $\mathcal{G}$-space. Suppose that $p : (\mathscr{E}, \sigma) \longrightarrow (\mathcal{G}, \rho)$ is a Rg Fell bundle, and that $\pi : (\mathscr{X}, \kappa) \longrightarrow (Z, \tau)$ is a Rg Banach bundle. Denote by $\alpha : Z * \mathcal{G} \longrightarrow Z$ the Real $\mathcal{G}$-action on (Z, τ), and endow the topological space*

$$\mathscr{X} * \mathscr{E} := \{(u,e) \in \mathscr{X} \times \mathscr{E} \mid (\pi(u), p(e)) \in Z * \mathcal{G}\}$$

with the ovious Real structure. A Fell action *of* $(\mathscr{E},\sigma)$ *on* $(\mathscr{X},\kappa)$ *consists of a continuous Rg (right)* $\mathcal{G}$*-action on* $(\mathscr{X},\kappa)$ *such that* π *is* $\mathcal{G}$*-equivariant, a continuous Real map* $\mathscr{X} * \mathscr{E} \ni (u,e) \longmapsto ue \in \alpha^*\mathscr{X}$ *such that*

(i) (bilinearity) *for all* $(z,g) \in Z * \mathcal{G}$*, the induced map* $\mathscr{X}_z \times \mathscr{E}_g \longrightarrow \mathscr{X}_{zg}$ *is bilinear,graded, and is compatible with the scalar multiplication; i.e.* $(tu)e = u(te) = t(ue)$*,* $\forall\ t \in \mathbb{C}, (u,e) \in \mathscr{X}_z \times \mathscr{E}_g$*;*

(ii) (associativity) *for* $(z,g) \in Z * \mathcal{G}$ *and* $(g,h) \in \mathcal{G}^{(2)}$*, one has*

$$u(e_1e_2) = (ue_1)e_2, \forall (u,e_1,e_2) \in \mathscr{X}_z \times \mathscr{E}_g \times \mathscr{E}_h;$$

(iii) $\|ue\| = \|u\|\|e\|, \forall (u,e) \in \mathscr{X}_z \times \mathscr{E}_g$*;*

(iv) for all $(u,g) \in \mathscr{X} * \mathcal{G}$*,* $\|u.g\| = \|u\|$*;*

(v) (faithfullness) *for all* $(z,g) \in Z * \mathcal{G}$*, the image of the induced map* $\mathscr{X}_z \times \mathscr{E}_g \longrightarrow \mathscr{X}_{zg}$ *spans a graded dense subspace of* $\mathscr{X}_{zg}$*.*

In this case we write $(\mathscr{X},\mathcal{E})$ *(or* $((\mathscr{X},\kappa),(\mathscr{E},\sigma))$ *if there is a risk of confusion), and we say that* $(\mathscr{X},\mathscr{E})$ *is a* (right) Rg Fell $\mathcal{G}$-pair *over* Z*.*

Remark 5.3.2.

1. *In the same token, given a left Real* $\mathcal{G}$*-space* Z*, one can define a left Fell action of* $(\mathscr{E},\sigma)$ *over* $(\mathscr{X},\kappa)$*. In that case, we write* $(\mathscr{E},\mathscr{X})$ *to emphasize the left action.*
2. *Observe that since the projection* $\pi : \mathscr{X} \longrightarrow Z$ *is* $\mathcal{G}$*-equivariant, there is an isomorphism* $\mathscr{X}_{zg} \longrightarrow \mathscr{X}_z$ *over all* $(z,g) \in Z * \mathcal{G}$ *given by* $u \longmapsto ug^{-1}$*. It then follows from condition (i) in Definition 5.3.1 that for all* $(z,g) \in Z * \mathcal{G}$*,* $\mathscr{X}_z$ *admits a structure of graded left* $\mathscr{E}_g$*-module.*

Now, suppose $Z : \Gamma \longrightarrow \mathcal{G}$ is an isomorphism in $\mathfrak{RG}$. Recall (cf. Remark 2.4.2) that $Z^{-1} : \mathcal{G} \longrightarrow \Gamma$ is then an isomorphism in $\mathfrak{RG}$. We will need some notions from [69, § 6.1.6].

Definition 5.3.3. *Define the (continuous) Real* Γ*-valued inner product*

$${}_\Gamma\langle\cdot,\cdot\rangle : Z \times_X Z^{-1} \longrightarrow \Gamma, (z,\flat(z')) \longmapsto {}_\Gamma\langle z,z'\rangle,$$

and the (continuous) Real $\mathcal{G}$*-valued inner product*

$$\langle\cdot,\cdot\rangle_{\mathcal{G}} : Z^{-1} \times_Y Z \longrightarrow \mathcal{G}, (\flat(z),z') \longmapsto \langle z,z'\rangle_{\mathcal{G}}$$

as follows:

- *for* $(z,\flat(z'))\in Z\times_X Z^{-1}$, ${}_{\Gamma}\langle z,z'\rangle$ *is the unique element of* Γ *such that* $z={}_{\Gamma}\langle z,z'\rangle\cdot z'$;
- *for* $(\flat(z),z')\in Z^{-1}\times_Y Z$, $\langle z,z'\rangle_{\mathcal{G}}$ *is the unique element of* $\mathcal{G}$ *such that* $z'=z\cdot\langle z,z'\rangle_{\mathcal{G}}$.

Remark 5.3.4. *It is routine to check that the* Γ*-valued inner product and the* $\mathcal{G}$*-valued inner product verify*

$$
\begin{aligned}
{}_{\Gamma}\langle z,z'\rangle^{-1}&={}_{\Gamma}\langle z',z\rangle, \forall\ (z,\flat(z'))\in Z\times_X Z^{-1}, \text{ and}\\
\langle z,z'\rangle_{\mathcal{G}}^{-1}&=\langle z',z\rangle_{\mathcal{G}}, \forall\ (\flat(z),z')\in Z^{-1}\times_Y Z.
\end{aligned}
$$

Further, the following lemma shows that the two inner products are compatible with the actions on Z.

Lemma 5.3.5. *Let* $Z:\Gamma\longrightarrow\mathcal{G}$ *be an isomorphism in* $\mathfrak{RG}$. *Then, the* Γ*-valued inner product and the* $\mathcal{G}$*-valued inner product are compatible with the Real* Γ*-action and the Real* $\mathcal{G}$*-action on* Z*; that is, that for all* $(z,z',z'')\in Z\times_X Z^{-1}\times_Y Z$, *one has*

$$
z\cdot\langle z',z''\rangle_{\mathcal{G}}={}_{\Gamma}\langle z,z'\rangle\cdot z''. \tag{5.17}
$$

Proof. This comes from a very simple calculation (see for instance [69, Proposition 6.1.35]). □

Proposition 5.3.6. *Let* $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ *be an isomorphism in the category* $\mathfrak{RG}$. *Then, any right Fell* $\mathcal{G}$*-pair* $(\mathscr{X},\mathscr{E})$ *over* (Z,τ) *gives rise to a left Fell* $\mathcal{G}$*-pair* $((\mathscr{E},\sigma),(\overline{\mathscr{X}},\overline{\kappa}))$ *over the inverse* $(Z^{-1},\tau^{\flat})$ *of* (Z,τ), *where* $\overline{\mathscr{X}}$ *is defined as the conjugate bundle of* $\mathscr{X}$. *Similarly, any left Fell* Γ*-pair over* (Z,τ) *gives rise to a right Fell* Γ*-pair over* $(Z^{-1},\tau^{\flat})$.

Proof. Of course it suffices to prove the proposition for a Fell $\mathcal{G}$-pair.The bundle $\overline{\mathscr{X}}$ is defined as follows: $\overline{\mathscr{X}}$ is identified with $\mathscr{X}$ as a topological space and if $\flat:\mathscr{X}\longrightarrow\overline{\mathscr{X}}$ is the identity map, the Real structure $\bar{\kappa}:\overline{\mathscr{X}}\longrightarrow\overline{\mathscr{X}}$ is $\bar{\kappa}(\flat(u)):=\flat(\kappa(u))$, the projection $\bar{\pi}:\overline{\mathscr{X}}\longrightarrow Z^{-1}$ is given by $\bar{\pi}(\flat(u)):=\flat(\pi(u))$, and finally, the fibre $\overline{\mathscr{X}}_{\flat(z)}$ is the conjugate algebra of $\mathscr{X}_z$ (*i.e.* $\overline{\mathfrak{X}}_{\flat(z)}=\overline{\mathscr{X}_z}$). Next, $(\mathcal{G},\rho)$ acts on the left on $(\overline{\mathscr{X}},\bar{\kappa})$ by $g\cdot\flat(u)=\flat(ug^{-1})$ for $(u,g^{-1})\in\mathscr{X}*\mathcal{G}$. It is clear that the projection $\bar{\pi}$ is Real and $\mathcal{G}$-equivariant. Denote by $\alpha^{\flat}:\mathcal{G}*Z^{-1}\longrightarrow Z^{-1}$ the left Real $\mathcal{G}$-action on $(Z^{-1},\tau^{\flat})$. It is routine to show that the map $\mathscr{E}*\overline{\mathscr{X}}\longrightarrow(\alpha^{\flat})^*\overline{\mathscr{X}}$, defined on the fibres by

$$
\mathscr{E}_g\times\overline{\mathscr{X}}_{\flat(z)}\ni(e,\flat(u))\longmapsto\flat(ue^*)\in\overline{\mathscr{X}}_{g.\flat(z)}, \tag{5.18}
$$

provides a Fell left action of $(\mathscr{E},\sigma)$ on $(\overline{\mathscr{X}},\bar{\kappa})$. □

Observe that, given a Morita equivalence $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$, the Real space $(Z\times_{\mathcal{G}^{(0)}}Z^{-1},\tau\times\tau^{\flat})$ is a locally compact Real groupoid over (Z,τ) as follows: the product is

$$
(z,\flat(z'))\cdot(z',\flat(z''))=(z,\flat(z'')),
$$

the source and range maps are $s(z,\flat(z'))=z'$, $r(z,\flat(z'))=z$ respectively. Similarly, $(Z^{-1}\times_{\Gamma^{(0)}}Z,\tau^{\flat}\times\tau)$ is a Real groupoid over $(Z^{-1},\tau^{\flat})$.

Definition 5.3.7. *Let (Z,τ) be as above. Then*

- *if $(\mathcal{G},\mathscr{E})$ and $(\Gamma,\mathscr{F})$ are Rg Fell systems, we write $(\mathscr{E}_{>\mathcal{G}},\sigma_{>\mathcal{G}})$ for the Rg Fell bundle over $(Z^{-1}\times_Y Z,\tau^\flat\times\tau)$ that consists of the pull-back of $\mathscr{E}\longrightarrow\mathcal{G}$ along the $\mathcal{G}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$.*
  By analogy, we write $(\mathscr{F}_{\Gamma<},\varsigma_{\Gamma<})$ for the graded Rg bundle over the Real groupoid $(Z\times_X Z^{-1},\tau\times\tau^\flat)$ given by the pull-back of $(\mathscr{F},\varsigma)$ along ${}_\Gamma\langle\cdot,\cdot\rangle$.

- *if $(\mathscr{X},\mathscr{E})$ is a Fell pair, we define the topological spaces*

$$\mathscr{X}*\overline{\mathscr{X}}=\{(u,\flat(u'))\in\mathscr{X}\times\overline{\mathscr{X}}\mid(\pi(u),\bar\pi(\flat(u')))\in Z\times_X Z^{-1}\},$$

$$\overline{\mathscr{X}}*\mathscr{X}=\{(\flat(u),u')\in\overline{\mathscr{X}}\times\mathscr{X}\mid(\bar\pi(\flat(u)),\pi(u'))\in Z^{-1}\times_Y Z\},$$

  and we endow them with the obvious Real structures.

Definition 5.3.8. *Let $(\mathcal{G},\mathscr{E})$ be a Rg Fell system. Suppose $(\mathscr{X},\mathscr{E})$ is a Fell pair over Z, where $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ is a Real Morita equivalence. A* Rg $\mathscr{E}$-valued inner product *on $(\mathcal{X},\kappa)$ is a continuous Real map $\langle\cdot,\cdot\rangle_{\mathscr{E}}:\overline{\mathscr{X}}*\mathscr{X}\longrightarrow\mathscr{E}_{>\mathcal{G}},(\flat(u),u')\longmapsto\langle u,u'\rangle_{\mathscr{E}}$ such that*

(a) for all $(\flat(z),z')\in Z^{-1}\times_Y Z$, the induced map $\langle\cdot,\cdot\rangle_{\mathscr{E}}:\overline{\mathscr{X}}_{\flat(z)}\times\mathscr{X}_{z'}\longrightarrow\mathscr{E}_{\langle z,z'\rangle_{\mathcal{G}}}$ is conjugate-linear in the first variable, bilinear in the second variable, and is graded (i.e. it maps $\overline{\mathscr{X}}^i_{\flat(z)}\times\mathscr{X}^j_{z'}$ to $\mathscr{E}^{i+j}_{<z,z'>_{\mathcal{G}}}$ for $i,j\in\{0,1\}$);

(b) ($\mathscr{E}$-linearity) over all $(\flat(z),z',g)\in Z^{-1}*Z*\mathcal{G}$, then the following diagram commutes:

$$\begin{tikzcd}
\overline{\mathscr{X}}_{\flat(z)}\times\mathscr{X}_z\times\mathscr{E}_g \arrow[r] \arrow[d] & \mathscr{E}_{<z,z'>_{\mathcal{G}}}\times\mathscr{E}_g \arrow[d] \\
\overline{\mathscr{X}}_{\flat(z)}\times\mathscr{X}_{z'.g} \arrow[r] & \mathscr{E}_{<z,z'>_{\mathcal{G}}.g}
\end{tikzcd} \tag{5.19}$$

in other words, $\langle u,u'\rangle_{\mathscr{E}}\cdot e=\langle u,u'\cdot e\rangle_{\mathscr{E}}$ in $\mathscr{E}_{<z,z'>_{\mathcal{G}}.g}$;

(c) (invariance) for all $(\flat(u),u')\in\overline{\mathscr{X}}_{\flat(z)}\times\mathscr{X}_{z'}$, $\langle ug^{-1},u'g\rangle_{\mathscr{E}}=\langle u,u'\rangle_{\mathscr{E}}$ whenever $(g,\flat(z))\in\mathcal{G}*Z^{-1}$;

(d) for all $(\flat(u),u')\in\overline{\mathscr{X}}_{\flat(z)}\times\mathscr{X}_{z'}$, $\langle u,u'\rangle^*_{\mathscr{E}}=\langle u',u\rangle_{\mathscr{E}}$ in $\mathscr{E}_{\langle z,z'\rangle^{-1}_{\mathcal{G}}}=\mathscr{E}_{\langle z',z\rangle_{\mathcal{G}}}$;

(e) (positivity) for all $z\in Z$ and $u\in\mathscr{X}_z$, $\langle u,u\rangle_{\mathscr{E}}\geq 0$ in $\mathscr{E}_{\langle z,z\rangle_{\mathcal{G}}}=\mathscr{E}_{\mathfrak{s}(z)}$; if $\langle u,u\rangle_{\mathscr{E}}=0$ then $u=0$.

In an obvious way, defines a Rg $\mathscr{F}$-valued inner product over $\mathscr{X}$ if $(\mathscr{F},\mathcal{X})$ is a left Fell pair over (Z,τ).

Remark 5.3.9. *Note that condition (d) of the definition implies that in particular,*

$$\langle u\cdot e, u'\rangle_{\mathcal{E}} = \langle u, u'\rangle_{\mathscr{E}} \cdot e^*,$$

whenever the multiplication and the inner product are defined.

Definition 5.3.10. *An* equivalence *between* $(\Gamma,\mathscr{F})$ *and* $(\mathcal{G},\mathscr{E})$ *consists of an isomorphism* $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$*, a right Fell pair* $((\mathscr{X},\kappa),(\mathscr{E},\sigma))$ *and a left Fell pair* $((\mathscr{F},\varsigma),(\mathscr{X},\kappa))$ *over* (Z,τ)*, a Rg*$\mathscr{E}$*-valued inner product*

$$\langle\cdot,\cdot\rangle_{\mathscr{E}}:\overline{\mathscr{X}}*\mathscr{X}\longrightarrow\mathscr{E}_{>_{\mathcal{G}}}, (\flat(u),u')\longmapsto\langle u,u'\rangle_{\mathscr{E}},$$

and a Rg$\mathscr{F}$*-valued inner product*

$${}_{\mathscr{F}}\langle\cdot,\cdot\rangle:\mathscr{X}*\overline{\mathscr{X}}\longrightarrow\mathscr{F}_{\Gamma<}, (u,\flat(u'))\longmapsto {}_{\mathscr{F}}\langle u,u'\rangle$$

on $(\mathscr{X},\kappa)$ *such that*

(i) (equivariance) for all $(\gamma,z,g)\in\Gamma*Z*\mathcal{G}$*, the multiplication* $\mathscr{F}_\gamma\times\mathscr{X}_z\times\mathscr{E}_g\longrightarrow\mathscr{X}_{\gamma z g}$ *is associative;* i.e. *for all* $(f,u,e)\in\mathscr{F}_\gamma\times\mathscr{X}_z\times\mathscr{E}_g$*, one has* $f(ue)=(fu)e$*;*

(ii) (compatibility) if $(z,\flat(z'),z'')\in Z\times_X Z^{-1}\times_Y Z$*, then the diagram*

$$\begin{tikzcd}
\mathscr{X}_z\times\overline{\mathscr{X}}_{\flat(z')}\times\mathscr{X}_{z''} \arrow[r]\arrow[d] & \mathscr{F}_{\Gamma<z,z'>}\times\mathscr{X}_{z''} \arrow[d]\\
\mathscr{X}_z\times\mathscr{E}_{<z',z''>_{\mathcal{G}}} \arrow[r] & \mathscr{X}_{z.<z',z''>_{\mathcal{G}}}
\end{tikzcd}\tag{5.20}$$

commutes; i.e. ${}_{\mathscr{F}}\langle u,u'\rangle\cdot u''=u\cdot\langle u',u''\rangle_{\mathscr{E}}$ *in* $\mathscr{X}_{z\cdot<z',z''>_{\mathcal{G}}}$, $\forall(u,\flat(u'),u'')\in\mathscr{X}_z\times\overline{\mathscr{X}}_{\flat(z')}\times\mathscr{X}_{z''}$*;*

(iii) ($\mathscr{E}$-fullness) for all $(\flat(z),z')\in Z^{-1}\times_Y Z$*, the induced map* $\overline{\mathscr{X}}_{\flat(z)}\times\mathscr{X}_{z'}\longrightarrow\mathscr{E}_{<z,z'>_{\mathcal{G}}}$ *is full; that is, that its image spans a dense graded subspace of* $\mathscr{E}_{<z,z'>_{\mathcal{G}}}$*;*

(iv) ($\mathscr{F}$-fullness) for all $(z,\flat(z'))\in Z\times_X Z^{-1}$*, the induced map* $\mathscr{X}_z\times\overline{\mathscr{X}}_{\flat(z')}\longrightarrow\mathscr{F}_{\Gamma<z,z'>}$ *is full.*

If such an equivalence exists, we write $((\Gamma,\varrho),(\mathscr{F},\varsigma))\sim_{((Z,\tau),(\mathscr{X},\kappa))}((\mathcal{G},\rho),(\mathscr{E},\sigma))$ *(we will write* $(\Gamma,\mathscr{F})\sim_{(Z,\mathscr{X})}(\mathcal{G},\mathscr{E})$ *when all of the structures are understood).*

Remark 5.3.11. *In the diagram* (5.20) *we have used Lemma* 5.3.5 *to observe that* $\mathfrak{X}_{\Gamma<z,z'>\cdot z''}=\mathfrak{X}_{z\cdot<z',z''>_{\mathcal{G}}}$.

Example 5.3.12. *The most relevant example to our work is the case of graded Real Dixmier-Douady bundles. Let then* $(\mathcal{B},\varsigma,\beta)\in\widehat{\mathfrak{BrR}}(\Gamma)$*, and* $(\mathcal{A},\sigma,\alpha)\in\widehat{\mathfrak{BrR}}(\mathcal{G})$*. If* $(s_\Gamma^*\mathcal{B},\Gamma)\sim_{(\mathscr{X},Z)}(s_\mathcal{G}^*\mathcal{A},\mathcal{G})$*, then conditions (iii) and (iv) in Definition 5.3.10 imply that, in particular, for all* $z\in Z$*,* $\mathscr{X}_z$ *is graded* $\mathcal{B}_{\mathfrak{s}(z)}$*-*$\mathcal{A}_{\mathfrak{r}(z)}$*-imprimitivity bimodule. Combined with the other conditions,* $\mathscr{X}$ *implements an equivalence of dynamical systems* $(\mathcal{B},\Gamma,\beta)\sim(\mathcal{B},\mathcal{G},\alpha)$ *in the sense of Muhly and Williams (cf. [65, Definition 5.1]).*

Remark 5.3.13. *The above example shows that by forgetting the gradings and the Real structures, one can think of our definition of equivalences of Fell systems as a generalization of that of [65].*

Another important example is the following.

Example 5.3.14. *Let* $(\mathcal{G},\mathscr{E})$ *be a Rg Fell system. Then there is an equivalence* $(\mathcal{G},\mathscr{E})\sim_{(\mathcal{G},\mathscr{E})}(\mathcal{G},\mathscr{E})$ *defined over the canonical Real Morita equivalence*

$$X \xleftarrow{r} Z_\mathcal{G} \xrightarrow{r} X\ ,$$

where $Z_\mathcal{G}:=\mathcal{G}$*,* $Z_\mathcal{G}^{-1}=\{g^{-1}\mid g\in\mathcal{G}\}$*, with the map* $\flat: Z_\mathcal{G}\longrightarrow Z_\mathcal{G}^{-1}$ *being defined by the inversion map, and* $(Z_\mathcal{G},\rho)$ *being equipped with the obvious Real left and right actions of* $(\mathcal{G},\rho)$*. We should mention that under these actions, we have the maps*

$$Z_\mathcal{G}\times_X Z_\mathcal{G}^{-1}\longrightarrow\mathcal{G},(g,h^{-1})\longmapsto gh^{-1},\ \text{and}$$

$$Z_\mathcal{G}^{-1}\times_X Z_\mathcal{G}\longrightarrow\mathcal{G},\ (g^{-1},h)\longmapsto g^{-1}h.$$

Note also that in this case, the Rg Banach bundle $(\overline{\mathscr{E}},\bar\sigma)$ *over* $(Z_\mathcal{G}^{-1},\rho^\flat)$ *is given fibrewise by* $\overline{\mathscr{E}}_{g^{-1}}=\{e^*\mid e\in\mathscr{E}_g\}$*. The left and right actions of* $(\mathscr{E},\sigma)$ *on itself over* $(Z_\mathcal{G},\rho)$ *come from the definition of* $(\mathscr{E},\sigma)$ *as a Rg Fell bundle. Moreover, the Rg inner products are given by*

$$\mathscr{E}_g\times\overline{\mathscr{E}}_{h^{-1}}\ni(e_1,e_2^*)\longmapsto e_1e_2^*\in\mathscr{E}_{gh^{-1}},\ \text{and}$$

$$\overline{\mathscr{E}}_g^{-1}\times\mathscr{E}_h\ni(e_1^*,e_2)\longmapsto e_1^*e_2.$$

It is strightforward to see that all of the conditions of Definition 5.3.10 are satisfied.

Proposition 5.3.15. *Equivalence of Rg Fell systems is an equivalence relation.*

Proof. 1. *Reflexivity.* The reflexivity of the relation is guaranteed by Example 5.3.14.

2. *Symmetry.* Suppose that $((\Gamma,\varrho),(\mathscr{F},\varsigma))\sim_{((Z,\tau),(\mathscr{X},\kappa))}((\mathscr{E},\sigma),(\mathcal{G},\rho))$. Recall that we have a Real Morita equivalence $(Z^{-1},\tau):(\mathcal{G},\rho)\longrightarrow(\Gamma,\varrho)$. Moreover, from Proposition 5.3.6,

we obtain a left Fell pair $((\mathscr{E},\sigma),(\overline{\mathscr{X}},\bar{\kappa}))$ and a right Fell pair $((\overline{\mathscr{X}},\bar{\kappa}),(\mathscr{F},\varsigma))$ over (Z^{-1},τ). Now, by replacing $\mathscr{X}$ by $\overline{\mathscr{X}}$ and vice versa in Definition 5.3.10, it is clear that

$$((\mathcal{G},\rho),(\mathscr{E},\sigma)) \sim_{((Z^{-1},\tau^\flat),(\overline{\mathscr{X}},\bar{\kappa}))} ((\Gamma,\varrho),(\mathscr{F},\varsigma)).$$

3. *Transitivity.* Assume that

$$((\Gamma,\varrho),(\mathscr{F},\varsigma)) \sim_{((Z_1,\tau_1),(\mathscr{X}_1,\kappa_1))} ((\mathcal{G}',\rho'),(\mathscr{E}',\sigma')) \sim_{((Z_2,\tau_2),(\mathscr{X}_2,\kappa_2))} ((\mathcal{G},\rho),(\mathscr{E},\sigma)).$$

Then the composition $(Z_1\times_{\mathcal{G}'}Z_2,\tau_1\times\tau_2)$ of (Z_1,τ_2) and (Z_2,τ_2) is an isomorphism $(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ in $\mathfrak{RG}$. Notice that from the $\mathcal{G}'$-invariance of the norm (cf. Definition 5.3.1 (iv)) of $\mathscr{X}_1$ and $\mathscr{X}_2$ and from the compatibility of the $\mathcal{G}'$-actions with the gradings and the Real structures, the quotient $\widetilde{\mathscr{X}_1}:=\mathscr{X}_1/\mathcal{G}'$ equipped with the Real involution $\kappa_1([u_1]):=[\kappa_1(u_1)]$ and the grading induced from that of $\mathscr{X}_1$ (resp. $\widetilde{\mathscr{X}_2}:\mathscr{X}_1/\mathcal{G}'$ equipped with the Real involution $\kappa_2([u_2]):=[\kappa_2(u_2)]$ and the grading induced from that of $\mathscr{X}_2$) is a Rg Banach bundle over $(Z_1/\mathcal{G}',\tau_1)$ (resp. over $(Z_2/\mathcal{G}',\tau_2)$). The projections $pr_1:Z_1\times_{\mathcal{G}'}Z_2\longrightarrow Z_1/\mathcal{G}',[(z_1,z_2)]\longmapsto[z_1]$ and $pr_2:Z_1\times_{\mathcal{G}'}Z_2\longrightarrow Z_2/\mathcal{G}',[(z_1,z_2)]\longmapsto[z_2]$ are well defined and are clearly compatible with the Real structures. Then we defined the graded Real tensor product of $(\mathscr{X}_1,\kappa_1)$ and $(\mathscr{X}_2,\kappa_2)$ over $(\mathcal{G}',\rho')$ by

$$\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2:=pr_1^*\widetilde{\mathscr{X}_1}\hat{\otimes}_{Z_1\times_{\mathcal{G}'}Z_2}pr_2^*\widetilde{\mathscr{X}_2}, \tag{5.21}$$

together with the Real structure denoted by $\kappa_1\hat{\otimes}_{\mathcal{G}'}\kappa_2$ and defined obviously. Now, we define the projection $\pi:(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2,\kappa_1\hat{\otimes}\kappa_2)\longrightarrow(Z_1\times_{\mathcal{G}'}Z_2,\tau_1\times\tau_2)$ by:

$$\pi\left([(z_1,z_2)],u_1\hat{\otimes}u_2\right)=[(\pi_1(u_1),\pi_2(u_2))]. \tag{5.22}$$

The fibrer of $\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2$ at a point $[(z_1,z_2)]\in Z_1\times_{\mathcal{G}'}Z_2$ is then identified with

$$\frac{(\mathscr{X}_1)_{z_1}\hat{\otimes}(\mathscr{X}_2)_{z_2}}{\mathcal{G}'},$$

where $u_1\hat{\otimes}u_2\sim(u_1g')\hat{\otimes}(g'^{-1}u_2)$ in $(\mathscr{X}_1)_{z_1}\hat{\otimes}(\mathscr{X}_2)_{z_2}$. Thus, we well have built a graded Real Banach bundle over $(Z\times_{\mathcal{G}'}Z_2,\tau_1\times\tau_2)$.
At this point, we have to verify the existence the left Real Γ-action and the right Real $\mathcal{G}$-action on $(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2,\kappa_1\hat{\otimes}\kappa_2)$. First, observe that setting

$$\widetilde{\mathscr{X}_1}*_{\mathcal{G}'}\widetilde{\mathscr{X}_2}:=\left\{([u_1],[u_2])\in\widetilde{\mathscr{X}_1}\times\widetilde{\mathscr{X}_1}\mid[(\pi_1(u_1),\pi_2(u_2))]\in Z_1\times_{\mathcal{G}'}Z_2\right\},$$

there is a well-defined continuous Real map

$$\widetilde{\mathscr{X}_1}*_{\mathcal{G}'}\widetilde{\mathscr{X}_2}\longrightarrow\mathcal{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2,\ ([u_1],[u_2])\longmapsto([(\pi_1(u_1),\pi_2(u_2))],[u_1]\hat{\otimes}[u_2]). \tag{5.23}$$

- We set

$$\gamma\cdot([(z_1,z_2)],[u_1]\hat{\otimes}[u_2]) := ([(\gamma\cdot z_1,z_2)],[\gamma\cdot u_1]\hat{\otimes}[u_2]), \tag{5.24}$$

  for $\gamma\in\Gamma$ and and elemantary tensor $([(z_1,z_2)],[u_1]\hat{\otimes}[u_2])\in\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2$ such that $(\gamma,z_1)\in\Gamma*Z_1$ and $(\gamma,u_1)\in\Gamma*\mathscr{X}_1$.

- Similarly, we set

$$([(z_1,z_2)],[u_1]\hat{\otimes}[u_2])\cdot g := ([(z_1,z_2\cdot g)],[u_1]\hat{\otimes}[u_2\cdot g]), \tag{5.25}$$

  for $g\in\mathcal{G}$ and and an elementary tensor $([(z_1,z_2)],[u_1]\hat{\otimes}[u_2])\in\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2$ such that $(z_2,g)\in Z_2*\mathcal{G}$ and $(u_2,g)\in\mathscr{X}*\mathcal{G}$.

We need to show that (5.24) and (5.25) define the desired continuous actions. We only do that for (5.24) since the same methods can be used to prove (5.25). Suppose that

$$(\gamma_i,([(z_{1,i},z_{2,i})],u_i)) \longrightarrow (\gamma,([(z_1,z_2)],u)),$$

in $\Gamma*(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)$. We want to verify that $\gamma_i\cdot(([z_{1,i},z_{2,i}],u_i))\longrightarrow\gamma\cdot([(z_1,z_2)],u)$ in $\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2$. Notice that the issue here is about the action on the tensor product; thus we only have to show that $\gamma_i\cdot u_i\longrightarrow\gamma\cdot u$. Fix $\epsilon>0$ and choose $v=\sum_{j=1}^n[v_1^j]\hat{\otimes}[v_2^j]\in(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)_{[(z_1,z_2)]}$ such that $\|u-v\|<\frac{\epsilon}{5}$. For each j, we can find elements $([v_{1,i}^j],[v_{2,i}^j])\in\widetilde{\mathscr{X}_1}*_{\mathcal{G}'}\widetilde{\mathscr{X}_2}$ such that $(([v_{1,i}^j],[v_{2,i}^j]))\longrightarrow([v_1^j],[v_2^j])$ in $\widetilde{\mathscr{X}_1}*_{\mathcal{G}'}\widetilde{\mathscr{X}_2}$. Using the continuity of the map (5.23) that $w_i:=\sum_{j=1}^n[v_{1,i}^j]\hat{\otimes}[v_{2,i}^j]\longrightarrow v$; thus eventually $\|w_i-v\|<\frac{\epsilon}{5}$. Therefore,

$$\gamma_i\cdot w_i=\sum_j[\gamma_i\cdot v_{1,i}^j]\hat{\otimes}[v_{2,i}^j]\longrightarrow\sum_j[\gamma\cdot v_1^j]\hat{\otimes}[v_2^j]=\gamma\cdot v,$$

and then one has $\|\gamma_i.w_i-\gamma\cdot v\|<\frac{\epsilon}{5}$. Therefore we eventually have

$$\begin{aligned}
\|\gamma_i\cdot u_i-\gamma\cdot u\| &\le \|\gamma_i\cdot u_i-\gamma_i\cdot w_i\|+\|\gamma_i\cdot w_i-\gamma\cdot v\|+\|\gamma\cdot v-\gamma\cdot u\|\\
&\le \|u_i-w_i\|+\|\gamma_i\cdot w_i-\gamma\cdot v\|+\|v-u\|\\
&\le \|u_i-u\|+2\|u-v\|+\|v-w_i\|+\|\gamma_i\cdot w_i-\gamma\cdot v\|\\
&<\epsilon.
\end{aligned}$$

Moreover, it is not hard to check that for $(\gamma,[(z_1,z_2)])\in\Gamma*(Z_1\times_{\mathcal{G}'}Z_2)$ and $([(z_1,z_2)],g)\in(Z_1\times_{\mathcal{G}'}Z_2)*\mathcal{G}$, from the obvious maps

$$\mathscr{F}_\gamma\times(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)_{[(z_1,z_2)]}\longrightarrow(\mathscr{X}_1\hat{\otimes}\mathscr{X}_2)_{[(\gamma\cdot z_1,z_2)]}, \tag{5.26}$$

$$(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)_{[(z_1,z_2)]}\times\mathscr{E}_g\longrightarrow(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)_{[(z_1,z_2.g)]}, \tag{5.27}$$

given respectively for elementary tensors by

$$(f,[u_1]\hat{\otimes}[u_2])\longmapsto[f\cdot u_1]\hat{\otimes}[u_2] \text{ and}$$

$$([u_1]\hat{\otimes}[u_2],e)\longmapsto [u_1]\hat{\otimes}[u_2.e],$$

we obtain two Fell pairs $((\mathscr{F},\varsigma),(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2,\kappa_1\hat{\otimes}\kappa_2))$ and $((\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2,\kappa_1\hat{\otimes}\kappa_2),(\mathscr{E},\sigma))$ over $(Z_1\times_{\mathcal{G}'}Z_2,\tau_1\times\tau_2)$ which are equivariant in the sense of Definition 5.3.10 (one shows, for instance, these maps are continuous by using Definition 5.3.1 (iii)). Now the rest of the proof is routine; we may however note that the inner products are obtained from the maps

$$(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)_{[(z_1,z_2)]}\times(\overline{\mathscr{X}}_1\hat{\otimes}_{\mathcal{G}'}\overline{\mathscr{X}}_2)_{[(\flat(z_1'),\flat z_2')]}\longrightarrow \mathscr{F}_{{}_\Gamma<z_1,z_1'>},\text{ and}$$
$$(\overline{\mathscr{X}}_1\hat{\otimes}_{\mathcal{G}'}\overline{\mathscr{X}}_2)_{[(\flat(z_1),\flat z_2)]}\times(\mathscr{X}_1\hat{\otimes}_{\mathcal{G}'}\mathscr{X}_2)_{[(z_1',z_2')]}\longrightarrow \mathscr{E}_{<z_2,z_2'>_{\mathcal{G}}}.$$

□

5.4 The Linking Fell bundle

We begin this section by making slight adjustments of some settings in [69, chapter 6] and [64, §.2] which are useful for our work.

Definition 5.4.1 (The linking Real groupoid)**.** *Suppose $(Z,\tau):(\Gamma,\varrho)\longrightarrow(\mathcal{G},\rho)$ is an isomorphism in $\mathfrak{RG}$. Let $\mathcal{M}$ be the locally compact Hausdorff space*

$$\mathcal{M}:=\Gamma\sqcup Z\sqcup Z^{-1}\sqcup\mathcal{G},$$

and let $\mathcal{M}^{(0)}:=\Gamma^{(0)}\sqcup\mathcal{G}^{(0)}$. Define the source and range maps of $\mathcal{M}$ as

$$s_{\mathcal{M}}:\mathcal{M}\longrightarrow\mathcal{M}^{(0)},\left\{\begin{array}{llll}\Gamma & \ni\gamma & \longmapsto s_\Gamma(\gamma) & \in\Gamma^{(0)}\\ Z & \ni z & \longmapsto \mathfrak{s}(z) & \in\mathcal{G}^{(0)}\\ Z^{-1} & \ni\flat(z) & \longmapsto \mathfrak{s}^\flat(\flat(z))=\mathfrak{r}(z) & \in\Gamma^{(0)}\\ \mathcal{G} & \ni g & \longmapsto s_{\mathcal{G}}(g) & \in\mathcal{G}^{(0)}\end{array}\right\},$$

and

$$r_{\mathcal{M}}:\mathcal{M}\longrightarrow\mathcal{M}^{(0)},\left\{\begin{array}{llll}\Gamma & \ni\gamma & \longmapsto r_\Gamma(\gamma) & \in\Gamma^{(0)}\\ Z & \ni z & \longmapsto \mathfrak{r}(z) & \in\Gamma^{(0)}\\ Z^{-1} & \ni\flat(z) & \longmapsto \mathfrak{r}^\flat(\flat(z))=\mathfrak{s}(z) & \in\mathcal{G}^{(0)}\\ \mathcal{G} & \ni g & \longmapsto r_{\mathcal{G}}(g) & \in\mathcal{G}^{(0)}\end{array}\right\}.$$

Then define the product in $\mathcal{M}$ as

$$\mathcal{M}^{(2)}\longrightarrow\mathcal{M},\left\{\begin{array}{llll}(\gamma_1,\gamma_2) & \in\Gamma^{(2)}: & \gamma_1\gamma_2 & \in\Gamma\\ (\gamma,z) & \in\Gamma*Z: & \gamma.z & \in Z\\ (z,\flat(z')) & \in Z\times_{\mathcal{G}^{(0)}}Z^{-1}: & z.\flat(z'):={}_\Gamma\langle z,z'\rangle & \in\Gamma\\ (z,g) & \in Z*\mathcal{G}: & z.g & \in Z\\ (\flat(z),z') & \in Z^{-1}\times_{\Gamma^{(0)}}Z: & \flat(z).z':=\langle z,z'\rangle_{\mathcal{G}} & \in\mathcal{G}\\ (\flat(z),\gamma) & \in Z^{-1}*\Gamma: & \flat(z).\gamma:=\flat(\gamma^{-1}.z) & \in Z^{-1}\\ (g,\flat(z)) & \in\mathcal{G}*Z^{-1}: & g.\flat(z):=\flat(zg^{-1}) & \in Z^{-1}\\ (g_1,g_2) & \in\mathcal{G}^{(2)}: & g_1g_2 & \in\mathcal{G}\end{array}\right\}.$$

Notice that with these definitions,

$$\mathcal{M}^{(2)} = \Gamma^{(2)} \sqcup \Gamma * Z \sqcup Z \times_{\mathcal{G}^{(0)}} Z^{-1} \sqcup Z * \mathcal{G} \sqcup Z^{-1} * \Gamma \sqcup Z^{-1} \times_{\Gamma^{(0)}} Z \sqcup \mathcal{G} * Z^{-1} \sqcup \mathcal{G}^{(2)}.$$

Proposition 5.4.2. *(Compare [69, Proposition 6.2.2]). $\mathcal{M}$ is a locally compact Hausdorff groupoid with open source and range maps. The inversion in $\mathcal{M}$ is the map*

$$\mathcal{M} \longrightarrow \mathcal{M}, \left\{ \begin{array}{llll} \Gamma & \ni \gamma & \longmapsto \gamma^{-1} & \in \Gamma \\ Z & \ni z & \longmapsto \flat(z) & \in Z^{-1} \\ Z^{-1} & \ni \flat(z) & \longmapsto z & \in Z \\ \mathcal{G} & \ni g & \longmapsto g^{-1} & \in \mathcal{G} \end{array} \right\}.$$

Moreover, $\mathcal{M}$ admits a Real structure $\rho_{\mathcal{M}}$ given by

$$\rho_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}, \left\{ \begin{array}{llll} \Gamma & \ni \gamma & \longmapsto \varrho(\gamma) & \in \Gamma \\ Z & \ni z & \longmapsto \tau(z) & \in Z \\ Z^{-1} & \ni \flat(z) & \longmapsto \flat(\tau(z)) & \in Z^{-1} \\ \mathcal{G} & \ni g & \longmapsto \rho(g) & \in \mathcal{G} \end{array} \right\}.$$

Now, suppose that $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ is an isomorphism. Then, from a Real Haar system $\mu_{\mathcal{G}}$ on $(\mathcal{G},\rho)$ and a Real Haar system μ_Γ on (Γ,ϱ), we want to construct a Real Haar system $\mu_{\mathcal{M}}$ on the linking Real groupoid $(\mathcal{M},\rho_{\mathcal{M}})$.

Definition 5.4.3. *(cf. [77, p.69] or [69, B.2.1]). Suppose that (Z,τ) and (X,ρ) are locally compact Hausdorff Real spaces, and that $\pi : (Z,\tau) \longrightarrow (X,\rho)$ is a continuous open Real map. Recall that a π-system on Z is a family $\mu_Z = \{\mu_Z^x\}_{x\in X}$ of measures such that*

(a) $\forall x \in X$, μ_Z^x is a measure on $\pi^{-1}(x)$,

*(c) (**continuity**) for all $\phi \in \mathcal{C}_c(Z)$,*

$$\mu_Z(\phi) : X \longrightarrow \mathbb{C},\ x \longmapsto \int_{\pi^{-1}(x)} \phi(z)\, d\mu_Z^x(z),$$

is an element of $\mathcal{C}_c(X)$.

Is is called `full` if $\operatorname{supp}\mu_Z^x = \pi^{-1}(x)$ for all $x \in X$. Finally, we say that μ_Z is `Real` if the family $\{\mu_Z^x\}_{x\in X}$ is Real; that is, $\mu_Z^{\rho(x)} \circ \tau = \mu_Z^x, \forall x \in X$.

Lemma 5.4.4. *Let $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ be an isomorphism in $\mathfrak{RG}$. There exists a full Real $\mathfrak{r}$-system $\mu_Z = \{\mu_Z^y\}_{y\in\Gamma^{(0)}}$ of Radon measures on (Z,τ) determined by*

$$\mu_Z^y(\phi) := \int_{\mathcal{G}^{\mathfrak{s}(z)}} \phi(z\cdot g)\, d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g), \tag{5.28}$$

for all $y \in \Gamma^{(0)}$ and $\phi \in \mathcal{C}_c(Z)$, where z is some arbitrary element of the fibre $Z_y = \mathfrak{r}^{-1}(y)$. Furthermore, μ_Z is a left Real Haar system on (Z,τ) for the left Real action of (Γ,ϱ); that is, for all $\gamma \in \Gamma$ and $\phi \in \mathcal{C}_c(Z)$, we have

$$\int_{Z_{r(\gamma)}} \phi(z)d\mu_Z^{r(\gamma)}(z) = \int_{Z_{s(\gamma)}} \phi(\gamma.z)d\mu_Z^{s(\gamma)}(z). \tag{5.29}$$

Proof. We refer to [69, Definition and Proposition 6.4.4]. □

Remark 5.4.5. *Similarly, considering the inverse $(Z^{-1},\tau^{\flat}) : (\mathcal{G},\rho) \longrightarrow (\Gamma,\varrho)$, the Real Haar system μ_Γ induces a left Real Haar system $\mu_{Z^{-1}} = \{\mu^x_{Z^{-1}}\}_{x\in\mathcal{G}^{(0)}}$ on $(Z^{-1},\tau^{\flat})$ for left Real action of $(\mathcal{G},\rho)$. Notice that that we have $\text{supp}\mu^x_{Z^{-1}} = (\mathfrak{r}^{\flat})^{-1}(x) = Z_x^{-1}$, and that for $\phi \in \mathcal{C}_c(Z^{-1})$ and $\flat(z) \in Z_x^{-1}$, we have*

$$\mu^x_{Z^{-1}}(\phi) := \int_{\Gamma^{s^{\flat}(\flat(z))}=\Gamma^{\mathfrak{r}(z)}} \phi(\flat(\gamma^{-1}z))d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma). \tag{5.30}$$

Proposition 5.4.6. *(cf. [69, Proposition 6.4.5], or [83, Lemma 4]). Let $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ be as previously. There is a left Real Haar system $\mu_{\mathcal{M}} = \{\mu^\omega_{\mathcal{M}}\}_{\omega\in\mathcal{M}^{(0)}}$ on the linking Real groupoid $(\mathcal{M},\rho_{\mathcal{M}})$ determined by*

$$\mu^\omega_{\mathcal{M}}(F) := \begin{cases} \mu^\omega_\Gamma(F_{|\Gamma}) + \mu^\omega_Z(F_{|Z}), & \text{if } \omega \in \Gamma^{(0)}, \text{ and} \\ \mu^\omega_{Z^{-1}}(F_{|Z^{-1}}) + \mu^\omega_{\mathcal{G}}(F_{|\mathcal{G}}), & \text{if } \omega \in \mathcal{G}^{(0)}, \end{cases} \tag{5.31}$$

for all $\omega \in \mathcal{M}^{(0)}$ and $F \in \mathcal{C}_c(\mathcal{M})$.

Definition and Proposition 5.4.7 (The linking Rg Fell bundle)**.** *Let $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ be an isomorphism in $\mathfrak{RG}$, and let $((\Gamma,\varrho),(\mathscr{F},\varsigma))$ and $((\mathcal{G},\rho),(\mathscr{E},\sigma))$ be two equivalent Rg Fell systems with equivalence $((Z,\tau),(\mathscr{X},\kappa))$. Then we define a Rg Banach bundle $(\mathscr{L},\sigma^{\mathscr{L}})$ over the linking Real groupoid $(\mathcal{M},\rho_{\mathcal{M}})$, where $\mathscr{L}$ is the topological space*

$$\mathscr{L} := \mathscr{F} \sqcup \mathscr{X} \sqcup \overline{\mathscr{X}} \sqcup \mathscr{E},$$

together with the obvious Real structure $\sigma^{\mathscr{L}} : \mathscr{L} \longrightarrow \mathscr{L}$ and the obvious grading; the projection is given by

$$p^{\mathscr{L}} : \mathscr{L} \longrightarrow \mathcal{M}, \left\{ \begin{array}{llll} \mathscr{F} & \ni f & \longmapsto p^{\mathscr{F}}(f) & \in \Gamma \\ \mathscr{X} & \ni u & \longmapsto \pi(u) & \in Z \\ \overline{\mathscr{X}} & \ni \flat(v) & \longmapsto \flat(\pi(v)) & \in Z^{-1} \\ \mathscr{E} & \ni e & \longmapsto p^{\mathscr{E}}(e) & \in \mathcal{G} \end{array} \right\}. \tag{5.32}$$

Moreover, $p^{\mathscr{L}}:(\mathscr{L},\sigma^{\mathscr{L}})\longrightarrow(\mathcal{M},\rho_{\mathcal{M}})$ *is a Rg Fell bundle with respect to the multiplication* $\mathscr{L}^{[2]}\longrightarrow \mathsf{m}^*\mathscr{L}$ *and involution* $(^*):\mathscr{L}\longrightarrow\mathscr{L}$ *respectively given by*

$$\left\{\begin{array}{llll}\mathscr{F}_{\gamma_1}\times\mathscr{F}_{\gamma_2} & \ni(f_1,f_2) & \longmapsto f_1f_2 & \in\mathscr{F}_{\gamma_1\gamma_2},\ \textit{for}\ (\gamma_1,g_2)\in\Gamma^{(2)}\\ \mathscr{F}_{\gamma}\times\mathscr{X}_z & \ni(f,u) & \longmapsto f\cdot u & \in\mathscr{X}_{\gamma\cdot z},\ \textit{for}\ (\gamma,z)\in\Gamma*Z\\ \mathscr{X}_{z_1}\times\overline{\mathscr{X}}_{\flat(z_2)} & \ni(u,\flat(v)) & \longmapsto {}_{\mathscr{F}}\langle u,v\rangle & \in\mathscr{F}_{\Gamma<z_1,z_2>},\ \textit{for}\ (z_1,\flat(z_2))\in Z\times_X Z^{-1}\\ \mathscr{X}_z\times\mathscr{E}_g & \ni(u,e) & \longmapsto u\cdot e & \in\mathscr{X}_{zg},\ \textit{for}\ (z,g)\in Z*\mathcal{G}\\ \overline{\mathscr{X}}_{\flat(z)}\times\mathscr{F}_{\gamma} & \ni(\flat(u),f) & \longmapsto\flat(f^*.u) & \in\overline{\mathscr{X}}_{\flat(\gamma^{-1}z)},\ \textit{for}\ (\flat(z),\gamma)\times Z^{-1}*\Gamma\\ \overline{\mathscr{X}}_{\flat(z_1)}\times\mathscr{X}_{z_2} & \ni(\flat(u),v) & \longmapsto\langle u,v\rangle_{\mathscr{E}} & \in\mathscr{E}_{<z_1,z_2>_{\mathcal{G}}},\ \textit{for}\ (\flat(z_1),z_2)\in Z^{-1}\times_Y Z\\ \mathscr{E}_g\times\overline{\mathscr{X}}_{\flat(z)} & \ni(e,\flat(u)) & \longmapsto\flat(u\cdot e^*) & \in\overline{\mathscr{X}}_{\flat(zg^{-1})},\ \textit{for}\ (g,\flat(z))\in\mathcal{G}*Z^{-1}\\ \mathcal{E}_g\times\mathcal{E}_h & \ni(e_1,e_2) & \longmapsto e_1e_2 & \in\mathcal{E}_{gh},\ \textit{for}\ (g,h)\in\mathcal{G}^{(2)}\end{array}\right\} \tag{5.33}$$

and

$$(^*):\mathscr{L}\to\mathscr{L},\ \left\{\begin{array}{llll}\mathscr{F}_{\gamma} & \ni f & \longmapsto f^* & \in\mathscr{F}_{\gamma^{-1}},\ \textit{for}\ \gamma\in\Gamma\\ \mathscr{X}_z & \ni u & \longmapsto\flat(u) & \in\overline{\mathscr{X}}_{\flat(z)},\ \textit{for}\ z\in Z\\ \overline{\mathscr{X}}_{\flat(z)} & \ni\flat(v) & \longmapsto v & \in\mathscr{X}_z,\ \textit{for}\ \flat(z)\in Z^{-1}\\ \mathscr{E}_g & \ni e & \longmapsto e^* & \in\mathscr{E}_{g^{-1}},\ \textit{for}\ g\in\mathcal{G}\end{array}\right\}. \tag{5.34}$$

Proof. It is clear that $p^{\mathscr{L}}:(\mathscr{L},\sigma^{\mathscr{L}})\longrightarrow(\mathcal{M},\rho_{\mathcal{M}})$ is a graded Real Banach bundle. Next, observe that all of the conditions of Definition 5.1.1 are verified by the operations (5.33) and (5.34) by merely applying Definition 5.3.10 to the equivalences $(Z,\mathscr{X})$ and $(Z^{-1},\overline{\mathscr{X}})$. □

At this point, we do integration on $(\mathcal{M},\rho_{\mathcal{M}})$ with values on the linking Fell bundle $\mathscr{L}$ with respect to the Real Haar system $\mu_{\mathcal{M}}$. We then can form the convolution Rg *-algebra $(\mathcal{C}_c(\mathcal{M};\mathscr{L}),\sigma^{\mathscr{L}})$. However, this *-algebra decomposes into direct sums of convolution *-algebras as we see with the following simple lemma.

Lemma 5.4.8. *As graded Real* **-algebras, we have*

$$\mathcal{C}_c(\mathcal{M};\mathscr{L})\cong\mathcal{C}_c(\Gamma;\mathscr{F})\oplus\mathcal{C}_c(Z;\mathscr{X})\oplus\mathcal{C}_c(Z^{-1};\overline{\mathscr{X}})\oplus\mathcal{C}_c(\mathcal{G};\mathscr{E}).$$

Therefore, an element $\xi\in\mathcal{C}_c(\mathcal{M};\mathscr{L})$ *can be viewed as a matrix*

$$\xi=\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix},$$

where $\xi_{11}:=\xi_{|\Gamma}\in\mathcal{C}_c(\Gamma;\mathscr{F}),\xi_{12}:=\xi_{|Z}\in\mathcal{C}_c(Z;\mathscr{X}),\xi_{21}:=\xi_{|Z^{-1}}\in\mathcal{C}_c(Z^{-1};\overline{\mathscr{X}})$*, and* $\xi_{22}:=\xi_{|\mathcal{G}}\in\mathcal{C}_c(\mathcal{G};\mathscr{E})$.

Remark 5.4.9. *Note that if $\xi \in \mathcal{C}_c(\mathcal{M};\mathscr{L})$, then for all $z \in Z$, one has*

$$\xi^*(z) = \xi(\flat(z))^* = \flat(\xi(\flat(z))) = \flat(\xi_{21}(\flat(z))).$$

Thus, $\xi^*_{21} := (\xi_{12})^* = \flat\circ\xi_{21}\circ\flat$. In the same way, we show that $\xi^*_{12} := (\xi_{21})^* = \flat\circ\xi_{12}\circ\flat$. It turns out that the involution in $(\mathcal{C}_c(\mathcal{M};\mathcal{L}),\sigma^{\mathcal{L}})$ is given by

$$\xi^* = \begin{pmatrix} \xi^*_{11} & \xi^*_{21} \\ \xi^*_{12} & \xi^*_{22} \end{pmatrix} = \begin{pmatrix} \xi^*_{11} & \flat\circ\xi_{21}\circ\flat \\ \flat\circ\xi_{12}\circ\flat & \xi^*_{22} \end{pmatrix},$$

where ξ^*_{11} and ξ^*_{22} are the images of ξ_{11} and ξ_{22} under the standard involutions in $\mathcal{C}_c(\Gamma;\mathscr{F})$ and $\mathcal{C}_c(\mathcal{G};\mathscr{E})$.

Before closing this section, we are giving a construction inspired by that of [64, p.11].

Proposition 5.4.10. *Let $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ be as above. Then $(\mathcal{C}_c(Z;\mathscr{X}),\kappa)$ admits a structure of a Rg $\mathcal{C}_c(\Gamma;\mathscr{F})$-$\mathcal{C}_c(\mathcal{G};\mathscr{E})$-bimodule under the operations defined as follows: for $\xi \in \mathcal{C}_c(\Gamma;\mathscr{F}), \eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, $\phi,\psi \in \mathcal{C}_c(Z;\mathscr{X})$, $\gamma \in \Gamma$, $g \in \mathcal{G}$, and $z \in Z$,*

$$(\xi\cdot\phi)(z) := \int_{\Gamma^{\mathfrak{r}(z)}} \xi(\gamma)\phi(\gamma^{-1}.z)\,d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma), \text{ and} \tag{5.35}$$

$$(\phi\cdot\eta)(z) := \int_{\mathcal{G}^{\mathfrak{s}(z)}} \phi(z\cdot g)\eta(g^{-1})\,d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g)\,. \tag{5.36}$$

Moreover, the operations

$${}_{\mathcal{C}_c(\Gamma;\mathscr{F})}\langle\phi,\psi\rangle(\gamma) := \int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathscr{F}}\big\langle \phi(z\cdot g),\psi(\gamma^{-1}\cdot z\cdot g)\big\rangle\,d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g), \text{ where } \mathfrak{r}(z) = r(\gamma), \tag{5.37}$$

and

$$\langle\phi,\psi\rangle_{\mathcal{C}_c(\mathcal{G};\mathscr{E})}(g) := \int_{\Gamma^{\mathfrak{r}(z)}} \big\langle \phi(\gamma^{-1}\cdot z),\psi(\gamma^{-1}\cdot z\cdot g)\big\rangle_{\mathscr{E}}\,d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma), \text{ where } \mathfrak{s}(z) = r(g), \tag{5.38}$$

define respectively a Rg $\mathcal{C}_c(\Gamma;\mathscr{F})$-valued pre-inner product and a Rg $\mathcal{C}_c(\mathcal{G};\mathscr{E})$-valued pre-inner product over $(\mathcal{C}_c(Z;\mathscr{X}),\kappa)$ with dense ranges. Furthermore, these pre-inner products are compatible with the the module structures (5.35) and (5.36) in the sense that

$${}_{\mathcal{C}_c(\Gamma;\mathscr{F})}\langle\phi,\psi\rangle\cdot\psi_1 = \phi\cdot\langle\psi,\psi_1\rangle_{\mathcal{C}_c(\mathcal{G};\mathscr{E})}, \forall \phi,\psi,\psi_1 \in \mathcal{C}_c(Z;\mathscr{X}). \tag{5.39}$$

Remark 5.4.11. *We shall note that the sense of "pre-inner product" used here is that of [75, p.15]; that is, the pairings ${}_{\mathcal{C}_c(\Gamma;\mathscr{F})}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{C}_c(\mathcal{G};\mathscr{E})}$ are required to verify ${}_{\mathcal{C}_c(\Gamma;\mathscr{F})}\langle\phi,\phi\rangle$ and $\langle\phi,\phi\rangle_{\mathcal{C}_c(\mathcal{G};\mathscr{E})}$ are positive in $C^*(\Gamma;\mathscr{F})$ and in $C^*(\mathcal{G};\mathscr{E})$, respectively.*

Notations 5.4.12.

1. *For the sake of simplicity, we write ${}_\star\langle\cdot\,,\cdot\rangle$ for ${}_{\mathcal{C}_c(\Gamma;\mathscr{F})}\langle\cdot,\cdot\rangle$ and $\langle\cdot\,,\cdot\rangle_\star$ for $\langle\cdot,\cdot\rangle_{\mathcal{C}_c(\mathcal{G};\mathscr{E})}$.*

2. *As in [83], if* $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E}), \eta \in \mathcal{C}_c(\Gamma;\mathscr{F})$, *and* $\phi,\psi \in \mathcal{C}_c(Z^{-1};\overline{\mathfrak{X}})$, *we write* $\xi : \phi$ *and* $\phi : \eta$ *for the left and right actions of* $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ *and* $\mathcal{C}_c(\Gamma;\mathscr{F})$ *on* $\mathcal{C}_c(Z^{-1};\overline{\mathscr{X}})$*, respectively, and we write* ${}_\star\langle\!\langle \phi\,,\psi\rangle\!\rangle$ *for* ${}_{\mathcal{C}_c(\mathcal{G};\mathscr{E})}\langle\phi,\psi\rangle$ *and* $\langle\!\langle \phi\,,\psi\rangle\!\rangle_\star$ *for* $\langle\phi,\psi\rangle_{\mathcal{C}_c(\Gamma;\mathscr{F})}$.

In the proof of Proposition 5.4.10, we will use the two following straightforward lemmas.

Lemma 5.4.13. *Suppose that* $\phi,\psi \in \mathcal{C}_c(Z;\mathscr{X})$. *Then*

$$F : Z\times_{\mathcal{G}^{(0)}} Z^{-1} \ni (z,z') \longmapsto \int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathscr{F}}\big\langle \phi(zg),\psi(z'g)\big\rangle\, d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g) \in \mathcal{C}_c\big(Z\times_{\mathcal{G}^{(0)}} Z^{-1};\mathscr{F}_{\Gamma^<}\big).$$

Lemma 5.4.14. *Suppose we are given the above settings. Let* $\xi \in \mathcal{C}_c(\Gamma;\mathscr{F})$, $\phi,\psi \in \mathcal{C}_c(Z;\mathscr{X})$, *and* $\eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$. *Then* $\flat\circ\phi\circ\flat$, $\flat\circ\psi\circ\flat \in \mathcal{C}_c(Z^{-1};\overline{\mathscr{X}})$. *Moreover, under the Notations 5.4.12, we have*

$$\eta : (\flat\circ\phi\circ\flat) = (\phi\cdot\eta^*)^* \qquad {}_\star\langle\!\langle \flat\circ\phi\circ\flat\,,\flat\circ\psi\circ\flat\rangle\!\rangle = \langle\phi\,,\psi\rangle_\star; \tag{5.40}$$

$$(\flat\circ\phi\circ\flat) : \xi = (\xi^*\cdot\phi)^* \qquad \langle\!\langle \flat\circ\phi\circ\flat\,,\flat\circ\psi\circ\flat\rangle\!\rangle_\star = {}_\star\langle\phi\,,\psi\rangle. \tag{5.41}$$

Proof of Proposition 5.4.10. First, notice that $(\xi\cdot\phi)(z) \in \mathscr{X}_z$ since for each $\gamma \in \Gamma^{\mathfrak{r}(z)}$, $\xi(\gamma)\phi(\gamma^{-1}\cdot z) \in \mathfrak{X}_{\gamma\gamma^{-1}\cdot z} = \mathfrak{X}_z$. Thus the formula (5.35) defines a section of $\mathscr{X}$. Also, $(\phi\cdot\eta)(z) \in \mathscr{X}_z$ for all $z \in Z$ and then (5.36) defines a section of $\mathfrak{X}$ as well. Since $(\xi\cdot\phi)(z)$ is nonzero only if there exists $\gamma \in \Gamma^{\mathfrak{r}(z)}$ such that $\xi(\gamma)$ and $\phi(\gamma^{-1}\cdot z)$ are nonzero, we see that

$$\operatorname{supp}(\xi\cdot\phi) \subset (\operatorname{supp}\xi)\cdot(\operatorname{supp}\phi) := \big\{\gamma z \mid (\gamma,z) \in (\operatorname{supp}\xi) * (\operatorname{supp}\phi)\big\};$$

hence $\xi\cdot\phi$ has compact support in Z. Similarly, we show that $\phi\cdot\eta$ has compact support. The continuity of $\xi\cdot\phi$ and $\phi\cdot\eta$ comes from the fact that μ_Γ and $\mu_{\mathcal{G}}$ are Haar measures. We then have shown that $\xi\cdot\phi$, $\phi\cdot\eta \in \mathcal{C}_c(Z;\mathscr{X})$. To see that the operation (5.35) is continuous, assume that $\xi_i \longrightarrow \xi$ with respect to the inductive limit topology in $\mathcal{C}_c(\Gamma;\mathscr{F})$ and $\phi_i \longrightarrow \phi$ with respect to the inductive limit topology in $\mathcal{C}_c(Z;\mathscr{X})$. So, for all $z \in Z$ and $\gamma \in \Gamma^{\mathfrak{r}(z)}$, $\xi_i(\gamma)\phi_i(\gamma^{-1}z) \longrightarrow \xi(\gamma)\phi(\gamma^{-1}z)$ since the multiplication $\mathscr{F}_\gamma \times \mathscr{X}_{\gamma^{-1}z} \longrightarrow \mathscr{X}_z$ is continuous. Hence, since μ_Γ is a Haar system, $\xi_i\cdot\phi_i \longrightarrow \xi\cdot\phi$. With a similar reasonning one verifies that (5.36) is continuous. The compatibility of these operations with the gradings and the Real structures is however straightforward.

Let us now check that (5.37) and (5.38) defines elements in $\mathcal{C}_c(\Gamma;\mathscr{F})$ and $\mathcal{C}_c(\mathcal{G};\mathscr{E})$, respectively. Since right Rg pair over (Z,τ) induces a left Rg pair over the inverse $(Z^{-1},\tau^\flat)$, it will be enough to verify the assertion for, say, (5.37). The formula does not depend on the choice of z; indeed, if $\mathfrak{r}(z') = \mathfrak{r}(z) = r(\gamma)$, then $z' = z\cdot\langle z,z'\rangle_{\mathcal{G}}$. Hence,

$${}_\star\langle\phi\,,\psi\rangle(z') = \int_{\mathcal{G}^{\mathfrak{s}(z')}} {}_{\mathscr{F}}\big\langle \phi(z'g),\psi(\gamma^{-1}z'\cdot g)\big\rangle\, d\mu_{\mathcal{G}}^{\mathfrak{s}(z')}(g)$$

$$= \int_{\mathcal{G}^{s(<z,z'>_{\mathcal{G}})}} {}_{\mathscr{F}}\big\langle \phi(z\cdot <z,z'>_{\mathcal{G}}\cdot g), \psi(\gamma^{-1}\cdot z\cdot <z,z'>_{\mathcal{G}}\cdot g)\big\rangle \, d\mu_{\mathcal{G}}^{s(<z,z'>_{\mathcal{G}})}(g)$$
$$= \int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathscr{F}}\big\langle \phi(zg), \psi(\gamma^{-1}zg)\big\rangle \, d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g),$$

where the last equality is obtained from the left invariance of $\mu_{\mathcal{G}}$ and by observing that $r(<z,z'>_{\mathcal{G}}) = \mathfrak{s}(z)$.
Suppose that $\gamma \in \operatorname{supp}({}_\star\langle\phi\,,\psi\rangle)$. Then for any fixed $z\in Z$ with $\mathfrak{r}(z) = r(\gamma)$, there exists $g\in\mathcal{G}^{\mathfrak{s}(z)}$ such that ${}_{\mathscr{F}}\langle\phi(\gamma^{-1}zg),\psi(zg)\rangle \neq 0$. It follows that $\gamma^{-1}zg\in\operatorname{supp}\phi$ and $zg\in\operatorname{supp}\psi$; thus $\gamma = {}_\Gamma\langle\gamma^{-1}zg,\ zg\rangle$ belongs to the image of $(\operatorname{supp}\phi)\times_X(\flat(\operatorname{supp}\psi))$ via the continuous map ${}_\Gamma\langle\cdot,\cdot\rangle\colon Z\times_X Z^{-1}\longrightarrow\Gamma$. So

$$\operatorname{supp}({}_\star\langle\phi\,,\psi\rangle)\subset {}_\Gamma\langle\cdot,\cdot\rangle\big((\operatorname{supp}\phi)\times_X(\flat(\operatorname{supp}\psi))\big)$$

is compact. Now we may use the same methods as in [64, p.11] to check the continuity of ${}_\star\langle\phi\,,\psi\rangle$: if $\gamma_\lambda\longrightarrow\gamma$ in Γ and $\mathfrak{r}(z) = r(\gamma)$, then since $\mathfrak{r}$ and r are open maps, we can find $(z_\lambda)_\lambda$ in Z with $z_\lambda\longrightarrow z$ (passing to a subnet if necessary) so that $\mathfrak{r}(z_\lambda) = r(\gamma_\lambda)$. Thus by observing that ${}_\star\langle\phi\,,\psi\rangle(\gamma_\lambda) = F(z_\lambda,\gamma_\lambda^{-1}z_\lambda)$, we get

$$\int_{\mathcal{G}^{\mathfrak{s}(z_\lambda)}} {}_{\mathscr{F}}\big\langle\phi(z_\lambda g),\psi(\gamma_\lambda^{-1}z_\lambda g)\big\rangle\, d\mu_{\mathcal{G}}^{\mathfrak{s}(z_\lambda)}(g)\longrightarrow\int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathscr{F}}\big\langle\phi(\gamma^{-1}zg),\psi(zg)\big\rangle\, d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g).$$

It now remains to verify the algebraic properties of the bimodule structure and the pre-inner products on $(\mathcal{C}_c(Z;\mathscr{X}),\kappa)$. Let

$$\xi\in\mathcal{C}_c(\Gamma;\mathscr{F}),\xi_1\in\mathcal{C}_c(\Gamma;\mathscr{F}),\ \eta\in\mathcal{C}_c(\mathcal{G};\mathscr{E}),\ \eta_1\in\mathcal{C}_c(\mathcal{G};\mathscr{E}),\ \text{and } \phi,\psi,\psi_1\in\mathcal{C}_c(Z;\mathscr{X}).$$

Then,

- $\forall\gamma\in\Gamma,\ (\xi * {}_\star\langle\phi\,,\psi\rangle)(\gamma) = \int_{\Gamma^{r(\gamma)}}\xi(h)\,{}_\star\langle\phi\,,\psi\rangle(h^{-1}\gamma)d\mu_\Gamma^{r(\gamma)}(h)$
  $= \int_{\Gamma^{r(\gamma)}}\xi(h)\int_{\mathcal{G}^{\mathfrak{s}(z_h)}} {}_{\mathscr{F}}\langle\phi(z_h\cdot g),\psi(\gamma^{-1}h\cdot z_h\cdot g)\rangle d\mu_{\mathcal{G}}^{\mathfrak{s}(z_h)}(g)d\mu_\Gamma^{r(\gamma)}(h),$
  where for each $h\in\Gamma^{r(\gamma)}, z_h\in Z$ with $\mathfrak{r}(z_h) = s(h)$,
  $= \int_{\Gamma^{r(\gamma)}}\xi(h)\int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathscr{F}}\langle\phi(h^{-1}zg),\psi(\gamma^{-1}zg)\rangle d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g)d\mu_\Gamma^{r(\gamma)}(h),$
  where $\mathfrak{r}(z) = \mathfrak{r}(hz_h) = r(\gamma), \forall h\in\Gamma^{r(\gamma)}$, and then $\mathfrak{s}(z) = \mathfrak{s}(z_h)$,
  $= \int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathscr{F}}\langle(\xi\cdot\phi)(zg)\psi(\gamma^{-1}zg)\rangle d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}$
  $= {}_\star\langle\xi\cdot\phi\,,\psi\rangle(\gamma);$ (5.42)

- $\forall z\in Z,\ ((\xi*\xi_1)\cdot\phi)(z) = \int_{\Gamma^{\mathfrak{r}(z)}}\int_{\Gamma^{r(\gamma)}}\xi(h)\xi_1(h^{-1}\gamma)d\mu_\Gamma^{r(\gamma)}(h)\phi(\gamma^{-1}z)d\mu_\Gamma^{\mathfrak{r}(z)}$
  $= \int_{\Gamma^{\mathfrak{r}(z)}}\xi(\gamma)\int_{\Gamma^\gamma}\xi_1(\gamma^{-1}h)\phi(h^{-1}z)d\mu_\Gamma^{r(\gamma)}(h)d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma)$

$$
\begin{aligned}
&= \int_{\Gamma^{\mathfrak{r}(z)}} \xi(\gamma) \int_{\Gamma^{s(\gamma)}} \xi_1(h)\phi(h^{-1}\gamma^{-1}z)\, d\mu_\Gamma^{s(\gamma)}(h)\, d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma) \\
&= \int_{\Gamma^{\mathfrak{r}(z)}} \xi(\gamma)(\xi_1\cdot\phi)(\gamma^{-1}z)\, d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma) \\
&= (\xi\cdot(\xi_1\cdot\phi))(z). \qquad (5.43)
\end{aligned}
$$

Similarly, it is easy to check that $(\langle\phi\,,\psi\rangle_\star * \eta)(g) = \langle\phi\,,\psi\cdot\eta\rangle_\star(g)$ for all $g\in\mathcal{G}$, and that $(\phi\cdot(\eta*\eta_1))(z) = ((\phi\cdot\eta)\cdot\eta_1)(z)$ for all $z\in Z$. Also, by routine computations, one verifies easily that ${}_\star\langle\phi\,,\psi\rangle^* = {}_\star\langle\psi\,,\phi\rangle$, and $\langle\phi\,,\psi\rangle_\star^* = \langle\psi\,,\phi\rangle_\star$. Furthermore, for all $z\in Z$, we have

- $$
\begin{aligned}
(\phi\cdot\langle\psi\,,\psi_1\rangle_\star)(z) &= \int_{\mathcal{G}^{\mathfrak{s}(z)}} \phi(zg)\langle\psi\,,\psi_1\rangle_\star(g^{-1})\, d\mu_\mathcal{G}^{\mathfrak{s}(z)}(g) \\
&= \int_{\mathcal{G}^{\mathfrak{s}(z)}} \phi(zg) \int_{\Gamma^{\mathfrak{r}(z')}} \langle\psi(\gamma^{-1}z'),\psi_1(\gamma^{-1}z'g^{-1})\rangle_{\mathscr{E}}\, d\mu_\Gamma^{\mathfrak{r}(z')}(\gamma)\, d\mu_\mathcal{G}^{\mathfrak{s}(z)}(g), \\
&\quad \text{where } z' \text{ is arbitrary such that } \mathfrak{s}(z') = s(g), \\
&= \int_{\mathcal{G}^{\mathfrak{s}(z)}} \phi(zg) \int_{\Gamma^{\mathfrak{r}(z)}} \langle\psi(\gamma^{-1}zg),\psi_1(\gamma^{-1}z)\rangle_{\mathscr{E}}\, d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma)\, d\mu_\mathcal{G}^{\mathfrak{s}(z)}(g) \\
&\quad \text{where we have taken in particular } z' = zg, \\
&= \int_{\mathcal{G}^{\mathfrak{s}(z)}} \int_{\Gamma^{\mathfrak{r}(z)}} {}_{\mathscr{F}}\langle\phi(zg),\psi(\gamma^{-1}zg)\rangle\cdot\psi_1(\gamma^{-1}z)\, d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma)\, d\mu_\mathcal{G}^{\mathfrak{s}(z)}(g) \\
&= ({}_\star\langle\phi\,,\psi\rangle\cdot\psi_1)(z). \qquad (5.44)
\end{aligned}
$$

The rest of the proof depends on the existence of approximates identities for the two actions; so we will complete it in the last part of the next section. □

It can be useful to express the convolution in $\mathcal{C}_c(\mathcal{M};\mathscr{L})$ in terms of matrices. Specifically, if $\xi,\eta\in\mathcal{C}_c(\mathcal{M};\mathscr{L})$, then we want to describe the entries of the matrix

$$
\xi*\eta = \begin{pmatrix} (\xi*\eta)_{11} & (\xi*\eta)_{12} \\ (\xi*\eta)_{21} & (\xi*\eta)_{22} \end{pmatrix} \in \begin{pmatrix} \mathcal{C}_c(\Gamma;\mathscr{F}) & \mathcal{C}_c(Z;\mathscr{X}) \\ \mathcal{C}_c(Z^{-1};\overline{\mathscr{X}}) & \mathcal{C}_c(\mathcal{G};\mathscr{E}) \end{pmatrix}.
$$

First, notice that for $y\in Y$ and $x\in X$, we have

$$
\mathcal{M}^y = \Gamma^y \sqcup Z_y,\ \mathcal{M}_y = \Gamma_y\sqcup Z_y^{-1},\ \mathcal{M}^x = Z_x^{-1}\sqcup\mathcal{G}^x, \text{ and } \mathcal{M}_x = Z_x\sqcup\mathcal{G}_x. \qquad (5.45)
$$

Thus,

- for all $\gamma\in\Gamma$,

$$
\begin{aligned}
(\xi*\eta)(\gamma) &= \int_{\Gamma^{r(\gamma)}} \xi_{11}(h)\eta_{11}(h^{-1}\gamma)\, d\mu_\Gamma^{r(\gamma)}(h) + \int_{Z_{r(\gamma)}} {}_{\mathcal{F}}\langle\eta_{12}(z),\flat(\eta_{21}(\flat(z).\gamma))\rangle\, d\mu_Z^{r(\gamma)}(z) \\
&= (\xi_{11}*\eta_{11})(\gamma) + \int_{\mathcal{G}^{\mathfrak{s}(z)}} {}_{\mathcal{F}}\langle\xi_{12}(zg),\flat(\eta_{21}(\flat(\gamma^{-1}zg)))\rangle\, d\mu_\mathcal{G}^{\mathfrak{s}(z)}(g) \\
&= (\xi_{11}*\eta_{11})(\gamma) + {}_\star\langle\xi_{12}\,,\eta_{21}^*\rangle(\gamma);
\end{aligned}
$$

- for all $z \in Z$,

$$
\begin{aligned}
(\xi * \eta)(z) &= \int_{\Gamma^{\mathfrak{r}(z)}} \xi_{11}(h)\eta_{12}(h^{-1}z)\,d\mu_\Gamma^{\mathfrak{r}(z)}(h) + \int_{Z_{\mathfrak{r}(z)}} \xi_{12}(z')\eta_{22}(< z, z' >_{\mathcal{G}}^{-1})\,d\mu_Z^{\mathfrak{r}(z)}(z') \\
&= (\xi_{11}\cdot\eta_{12})(z) + \int_{\mathcal{G}^{\mathfrak{s}(z)}} \xi_{12}(zg)\eta_{22}(g^{-1})\,d\mu_{\mathcal{G}}^{\mathfrak{s}(z)}(g) \\
&= (\xi_{11}\cdot\eta_{12})(z) + (\xi_{12}\cdot\eta_{22})(z);
\end{aligned}
$$

- for all $\flat(z) \in Z^{-1}$,

$$
\begin{aligned}
(\xi * \eta)(\flat(z)) &= \int_{Z^{-1}_{\mathfrak{r}^\flat(\flat(z))}} \xi_{21}(\flat(z'))\eta_{11}(< \flat(z), \flat(z') >_\Gamma^{-1})\,d\mu_{Z^{-1}}^{\mathfrak{r}^\flat(\flat(z))}(\flat(z')) \\
&\quad + \int_{\mathcal{G}^{\mathfrak{r}^\flat(\flat(z))}} \xi_{22}(g)\eta_{21}(g^{-1}.\flat(z))\,d\mu_{\mathcal{G}}^{\mathfrak{r}^\flat(\flat(z))}(g) \\
&= \int_{\Gamma^{\mathfrak{s}^\flat(\flat(z))}} \xi_{21}(\flat(z)\cdot\gamma)\eta_{11}(\gamma^{-1})\,d\mu_\Gamma^{\mathfrak{s}^\flat(\flat(z))}(\gamma) + (\xi_{22} : \eta_{21})(\flat(z)) \\
&= (\xi_{21} : \eta_{11})(\flat(z)) + (\xi_{22} : \eta_{21})(\flat(z)) \\
&= (\eta_{11}^*\cdot\xi_{21}^*)^*(\flat(z)) + (\eta_{21}^*\cdot\xi_{22}^*)^*(\flat(z));
\end{aligned}
$$

- for all $g \in \mathcal{G}$,

$$
\begin{aligned}
(\xi * \eta)(g) &= \int_{Z^{-1}_{r(g)}} \langle \flat(\xi_{21}(\flat(z))), \eta_{12}(\flat(z)^{-1}\cdot g)\rangle_{\mathcal{E}}\,d\mu_{Z^{-1}}^{r(g)}(\flat(z)) \\
&\quad + \int_{\mathcal{G}^{r(g)}} \xi_{22}(h)\eta_{22}(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h) \\
&= \int_{\Gamma^{\mathfrak{r}(z)}} \langle \xi_{21}^*(\gamma^{-1}z), \eta_{12}(\gamma^{-1}zg)\rangle_{\mathcal{E}}\,d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma) + (\xi_{22} * \eta_{22})(g) \\
&= \langle \xi_{21}^*, \eta_{12}\rangle_\star(g) + (\xi_{22} * \eta_{22})(g).
\end{aligned}
$$

We then have the following

Corollary 5.4.15. *The convolution in* $\mathcal{C}_c(\mathcal{M};\mathscr{L})$ *is given by*

$$
\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} * \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} = \begin{pmatrix} \xi_{11} * \eta_{11} + {}_\star\langle \xi_{12}, \eta_{21}^*\rangle & \xi_{11}\cdot\eta_{12} + \xi_{12}\cdot\eta_{22} \\ (\eta_{11}^*\cdot\xi_{21}^*)^* + (\eta_{21}^*\cdot\xi_{22}^*)^* & \langle \xi_{21}^*, \eta_{12}\rangle_\star + \xi_{22} * \eta_{22} \end{pmatrix}. \quad (5.46)
$$

5.5 Existence of approximate identities

We start this section with some more preliminaries on Rg C^*-algebras.

Definition 5.5.1. *Let (A,σ) be a Rg C^*-algebra. An approximate identity $\{a_i\}_{i\in I}$ for A is said to be* Real *if there is a Real structure $(\bar{\ }) : I \longrightarrow I$ on the directed set I such that $a_{\bar{i}} = \sigma(a_i)$ for all $i \in I$.*

Lemma 5.5.2. *Any Rg C^*-algebra (A,σ) admits a Real approximate identity.*

Proof. Take an approximate identity $\{a_{i'}\}_{i'\in I'}$, and define $I := I' \times \{0,1\}$, then set $\overline{(i',0)} := (i',1)$, $\overline{(i',1)} := (i',0)$, $a_{(i',0)} := a_{i'}$, and $a_{(i',1)} := \sigma(a_{i'})$. □

The following is an obvious modification of [83, Lemma 6.3]; we then omit the proof.

Lemma 5.5.3. *Let (A,σ) be a Rg C^*-algebra. Suppose that (X,κ) is a Rg Hilbert A-module. Then, sums of the form*

$$\sum_{i=1}^{n} {}_A\langle x_i, x_i\rangle$$

are dense in (A^0_+,σ); where A^0_+ is the sub-C^*-algebra of A consisting of the positive elements which are homogeneous of degree 0.

Corollary 5.5.4. *(Cf. [83, Corollary 6.5]) Set $(B,\varsigma) := (\mathcal{C}_0(Y;\mathscr{F}),\varsigma)$, $(A,\sigma) := (\mathcal{C}_0(X;\mathcal{E}),\sigma)$. Suppose that $b \in B^0_+$, that $\epsilon > 0$, and that $K \subset\subset Z$. Then, there are $\phi_1,\dots,\phi_n \in \mathcal{C}_c(Z;\mathscr{X})$ such that*

$$\|b(\mathfrak{r}(z)) - \sum_{i=1}^{n} {}_{\mathscr{F}}\langle \phi_i(z),\phi_i(z)\rangle\| < \epsilon, \ \forall z \in K. \tag{5.47}$$

Similar statement holds for A^0_+.

Proof. Notice that since $\mathscr{X}_z \times \overline{\mathcal{X}}_{\mathfrak{d}(z)} \longrightarrow \mathscr{F}_{\mathfrak{r}(z)}$ is full, we can view $\mathscr{X}_z$ as a full graded Hilbert $\mathscr{F}_{\mathfrak{r}(z)}$-module. It then follows from Lemma 5.5.3 that for all $z \in Z$, there are $\psi^z_1,\dots,\psi^z_{n_z} \in \mathcal{C}_c(Z;\mathcal{X})$ such that for all $z \in Z$,

$$\|b(\mathfrak{r}(z)) - \sum_{i=1}^{n_z} {}_{\mathcal{F}}\langle \psi^z_i(z),\psi^z_i(z)\rangle\| < \epsilon. \tag{5.48}$$

But since the map $\|\cdot\| : \mathscr{F} \longrightarrow \mathbb{R}_+$ is continuous, we can find an open neighborhood V_z of z such that (5.48) holds for all $z \in V_z$. Thus, there exists a finite open cover $\{V_j\}_{j=1}^m$ of the compact subset K, and sections $\psi^j_1,\dots,\psi^j_{n_j}$ for each $j = 1,\dots,m$, such that $\forall j$,

$$\|b(\mathfrak{r}(z)) - \sum_{i=1}^{n_j} {}_{\mathscr{F}}\langle \psi^j_i(z),\psi^j_i(z)\rangle\| < \epsilon, \ \forall z \in V_j. \tag{5.49}$$

Now take a partition of unity $\{\varphi_j\}_{j=1}^m$ subordinate to the cover $\{V_j\}_{j=1}^m$ with

$$\varphi_j \in \mathcal{C}_c(Z)_+,\ \operatorname{supp}\varphi_j \subset V_j\ \forall j,\ \sum_{j=1}^m \varphi_j(z) = 1,\ \forall z \in K, \text{ and } 0 \le \sum_{j=1}^m \varphi_j(z) \le 1, \forall z \in (Z-K).$$

Then (5.49) gives

$$\|b(\mathfrak{r}(z))-\sum_{j=1}^{m}\varphi_j(z)_{\mathscr{F}}\langle\psi_i^j(z),\psi_i^j(z)\rangle\|=\|b(\mathfrak{r}(z))-\sum_{i=1}^{n}{}_{\mathscr{F}}\langle\psi_i(z),\psi_i(z)\rangle\|<\epsilon,\ \forall z\in K,$$

where $\psi_i(z):=\sum_{j=1}^m\sqrt{\varphi_j(z)}\psi_i^j(z)$ for all $z\in K$, and $n:=\sum_{j=1}^m n_j$. □

The next lemma is an obvious modification of [83, Lemma 6.1]; then we omit the proof.

Lemma 5.5.5. *The graded Real C^*-algebra (B,ς) acts on $(\mathcal{C}_c(Z;\mathscr{X}),\kappa)$ in the natural way:*

$$(b\cdot\phi)(z):=b(\mathfrak{r}(z)).\phi(z)\in\mathscr{X}_z,\ \forall b\in B,\phi\in\mathcal{C}_c(Z;\mathscr{X}),z\in Z.$$

If $\{b_i\}_{i\in I}$ is a Real approximate identity for (B,ς), then for all $\phi\in\mathcal{C}_c(Z;\mathscr{X})$, $b_i.\phi\longrightarrow\phi$ in the inductive limit topology.

Definition 5.5.6. *A subset $U\subset\mathcal{G}$ is called* conditionally compact *(or* r-relatively compact *as in [77, p.56]) if $U\cap\mathcal{G}^K$ is relatively compact in $\mathcal{G}$ for every compact $K\subset\subset X$.*

Thanks to the proof of [77, Theorem 2.1.9], since X is paracompact, it has a fundamental system of conditionally compact neighborhoods in $\mathcal{G}$. Observe that since, as a subset of $\mathcal{G}$, X is invariant under ρ, then if U is a conditionally compact neighborhood of X, $\rho(U)$ is a conditionally compact neighborhood of X as well.

The next result is a generalisation of [66, Proposition 6.10] and [65, Proposition 6.8] (of course, once we forget the Real structures).

Proposition 5.5.7. *Assume one is given the aforementioned settings. Then, there is a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{C}_c(\Gamma;\mathscr{F})^0$ of elements of the form*

$$f_\lambda=\sum_{i=1}^{n_\lambda}{}_\star\langle\phi_i^\lambda,\phi_i^\lambda\rangle,$$

with each $\phi_i^\lambda\in\mathcal{C}_c(Z;\mathscr{X})$, which is a Real approximate identity with respect to the inductive limit topology for both the left Real action of $(\mathcal{C}_c(\Gamma;\mathscr{F}),\varsigma)$ on itself and on $(\mathcal{C}_c(Z;\mathscr{X}),\kappa)$. Similar statement holds for $(\mathcal{G},\mathscr{E})$.

Proof. In view of Example 5.3.14, it suffices to treat just the case of the Rg action of $(\mathcal{C}_c(\Gamma;\mathscr{F}))$ on $\mathcal{C}_c(Z;\mathscr{X})$. To do so, we we may use the same method as [66, Proposition 6.10]. However, some minor adaptations have to be done for our needs. A first step of the proof consists of providing for any given pair (i,F), where $i\in I$ with $\{b_i\}_{i\in I}$ a Real approximate identity for the Rg C^*-algebra (B,ς), and $F\subset\mathcal{C}_c(Z;\mathscr{X})$ is a finite subset, a net $\{f_{(i,F,U,\epsilon)}\}\subset\mathcal{C}_c(\Gamma;\mathscr{F})$, such that as U increases along the directed family of conditionally compact neighborhood of Y in Γ contained in a fixed conditionally compact neighborhood U_0 of Y in Γ, and as $\epsilon>0$ increases, $f_{(i,F,U,\epsilon)}\cdot\phi\longrightarrow b_i\cdot\phi$ in the inductive limit topology for each $\phi \in F$. To come to this end, one should use Corollary 5.5.4 and Lemma 5.5.5. Next, consider the family Λ consisting of 4-tuples (i, F, U, ϵ) (as above) directed by increasing i and F and decreasing F and ϵ. This family is endowed by the Real structure given by $\bar{\lambda} = \overline{(i, F, U, \epsilon)} := (\bar{i}, \kappa(F), \varrho(U), \epsilon)$, where

$$\kappa(F) := \left\{\kappa(\phi) \colon Z \ni z \longmapsto \kappa_z(\phi(\tau(z))) \in \mathscr{X}_{\tau(z)} \mid \phi \in F\right\}.$$

This Real structure on Λ is well defined, thanks to the discussion preceding the proposition. Then we put $f_{\bar{\lambda}} := \varsigma \circ f_\lambda$ for all $\lambda \in \Lambda$. The second step is to check that for $\phi \in \mathcal{C}_c(Z; \mathscr{X})$, $f_\lambda \cdot \psi \longrightarrow \psi$ in $\mathcal{C}_c(Z; \mathscr{X})$ for the inductive limit topology (see the last part of the proof of [66, Proposition 6.10]). □

An immediate consequence of Proposition 5.5.7, is Proposition 5.1.12:

Proof of Proposition 5.1.12. Given a graded Real Fell system $(\mathcal{G}, \mathscr{E})$, we have already seen that $(\mathcal{G}, \mathscr{E}) sim_{(\mathcal{G}, \mathscr{E})} (\mathcal{G}, \mathscr{E})$, where the Morita equivalence $(\mathcal{G}, \rho) : (\mathcal{G}, \rho) \longrightarrow (\mathcal{G}, \rho)$ is canonically defined by the source and range map $X \xleftarrow{r} \mathcal{G} \xrightarrow{s} X$. Now from Proposition 5.5.7, there is a Real approximate identity $\{f_\lambda\}$ for the left Real action of $\mathcal{C}_c(\mathcal{G}; \mathscr{E})$ on itself, with f_λ of the form

$$f_\lambda = \sum_{i=1}^{n_\lambda} {}_{\mathscr{E}}\langle \phi_i^\lambda, \phi_i^\lambda \rangle, \text{ where } \phi_i^\lambda \in \mathcal{C}_c(\mathcal{G}; \mathscr{E}).$$

But, using the right Real action of $\mathcal{C}_c(\mathcal{G}; \mathscr{E})$ on itself and the fact that $f_\lambda^* = f_\lambda$ for all $\lambda \in \Lambda$, it follows from Lemma 5.4.14 that $\{f_\lambda\}_{\lambda \in \Lambda}$ is also a Real approximate identity for the right Real action of $\Gamma_c(\mathcal{G}; \mathscr{E})$ on itself, and this completes the proof. □

End of the proof of proposition 5.4.10. In the proof of the proposition, we have only left to verify

(1) the positivity of the inner products, and

(2) the density of the range of the inner products.

By symmetry, it suffices to show for instance that ${}_\star\langle \psi , \psi \rangle \geq 0, \forall \psi \in \mathcal{C}_c(Z; \mathscr{X})$ to prove (1). Let $e_\lambda = \sum_{i=1}^{n_\lambda} \langle \phi_i^\lambda \rangle \in \mathcal{C}_c(\mathcal{G}; \mathscr{E})$ be a Real approcimate identity for the right action of $\mathcal{C}_c(\mathcal{G}; \mathscr{E})$ on itself and on $\mathcal{C}_c(Z; \mathscr{X})$ as in Proposition 5.5.7. Then

$${}_\star\langle \psi \cdot f_\lambda , \psi \rangle \longrightarrow {}_\star\langle \psi , \psi \rangle$$

with respect to the inductive limit topology. But

$$\begin{aligned} {}_\star\langle \psi \cdot e_\lambda , \psi \rangle &= \sum_i {}_\star\langle \psi \cdot \langle \phi_i^\lambda , \phi_i^\lambda \rangle , \psi \rangle_\star \\ &= \sum_i {}_\star\langle {}_\star\langle \psi , \phi_i^\lambda \rangle \cdot \phi_i^\lambda , \psi , \rangle \end{aligned}$$

$$= \sum_i {}_\star\langle \phi_i^\lambda , \psi\rangle^* * {}_\star\langle \phi_i^\lambda , \psi\rangle \geq 0 \text{ in } C^*(\Gamma;\mathcal{F});$$

hence ${}_\star\langle \psi , \psi\rangle \geq 0$ as the limit of a positive net in $C^*(\Gamma;\mathscr{F})$.

Let us check (2). Again by symmetry, it it suffices to check, for instance, that $\langle\cdot , \cdot\rangle_\star$ spans a dense range in $\mathcal{C}_c(\mathcal{G};\mathscr{E})$. If $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, then from Proposition 5.5.7, $e_\lambda * \xi \longrightarrow \xi$ in $\mathcal{C}_c(\mathcal{G};\mathscr{E})$ with respect to the inductive limit topology. But

$$e_\lambda * \xi = \sum_{i=1}^{n_\lambda} \langle \phi_i^\lambda , \phi_i^\lambda\rangle_\star * \xi = \sum_{i=1}^{n_\lambda} \langle \phi_i^\lambda , \phi_i^\lambda \cdot \xi\rangle_\star \in \langle \mathcal{C}_c(Z;\mathscr{X}) , \mathcal{C}_c(Z;\mathscr{X})\rangle_\star,$$

which completes the proof. □

5.6 The Equivalence Theorem

We start this section by the following observations. Let $((\mathcal{G},\rho),(\mathscr{E},\sigma))$ be a Rg Fell system and let $A := \mathcal{C}_0(X;\mathscr{E})$ be as usual. Suppose we are given a bounded continuous section $f \in \mathcal{C}_b(X;\mathscr{E})$. Then, for $\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, we define the element $L_f\xi =: f\xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$ by setting:

$$L_f\xi(g) := f(r(g))\xi(g) \in \mathcal{E}_g, \text{ for all } g \in \mathcal{G}. \tag{5.50}$$

Also, we define the element $\xi f \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$ by

$$\mathcal{G} \ni g \longmapsto \xi f(g) := \xi(g) f(s(g)) \in \mathscr{E}_g. \tag{5.51}$$

Notice that $\mathcal{C}_b(X;\mathcal{E})$ is a C^*-algebra under pointwise operations and the sup-norm (cf. [3, Lemma 3.2]). Furthermore, it admits the $\mathbb{Z}_2$-grading and the Real structure defined by $\varepsilon(f)(x) := \varepsilon(f(x))$, and $\sigma(f)(x) := \sigma(f(\rho(x)))$.

Lemma 5.6.1. *For all $f \in \mathcal{C}_b(X;\mathscr{E})$, we have $L_f \in \mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$, where L_f is the element defined by (5.50). Moreover, the map $L : \mathcal{C}_b(X;\mathscr{E}) \ni f \longmapsto L_f \in \mathcal{L}_A(L^2(\mathcal{G};\mathscr{E}))$ is a *-homomorphism which is Real and graded.*

Proof. L_f is clearly continuous; also it is bounded since f is a bounded section (it is straightforward that $\|L_f\|_{op} \leq \|f\|$, where $\|\cdot\|_{op}$ is the operator norm in $\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$). If $\xi,\eta \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$ and $x \in X$, then

$$\begin{aligned} {}_A\langle L_f\xi,\eta\rangle(x) &= \int_{\mathcal{G}^x} L_f\xi(g^{-1})^*\eta(g^{-1})d\mu_{\mathcal{G}}^x(g) \\ &= \int_{\mathcal{G}^x} \xi^*(g) f(r(g^{-1}))^*\eta(g^{-1})d\mu_{\mathcal{G}}^x(g) \\ &= {}_A\langle \xi, L_{f^*}\eta\rangle(x); \end{aligned}$$

hence, L_f is adjointable with adjoint $L_f^* := L_{f^*}$. Moreover, $L_{f_1f_2}(\xi) = L_{f_1}(L_{f_2}(\xi)), \forall \xi \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$; thus $L_{f_1f_2} = L_{f_1}L_{f_2}, \forall f_1, f_2 \in \mathcal{C}_b(X;\mathscr{E})$. Finally, it is obvious that L is compatible with the gradings and the Real structures of $\mathcal{C}_c(X;\mathscr{E})$ and $\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$. □

Proposition 5.6.2. *Every element of $\mathcal{C}_b(X;\mathscr{E})$ may be identified with an operator in $\mathcal{L}(L^2(\mathcal{G};\mathcal{E}))$ centralizing $C^*_r(\mathcal{G};\mathscr{E})$; therefore, $\mathcal{C}_b(X;\mathscr{E})$ is a Rg sub-C^*-algebra of $M(C^*_r(\mathcal{G};\mathscr{E}))$.*

Proof. If $\pi_l(\xi)\in C^*_r(\mathcal{G};\mathscr{E})$ and $f\in\mathcal{C}_b(X;\mathscr{E})$, we put

$$L_f(\pi_l\xi):=\pi_l(L_f\xi)=\pi_l(f\xi),\ \text{and}\ R_f(\pi_l\xi):=\pi_l(\xi f). \tag{5.52}$$

We verify that with these formulas, we obtain a double centralizer $(L_f,R_f)\in M(C^*_r(\mathcal{G};\mathscr{E}))$. To see that the assertion is true, observe that for $\xi,\eta\in\mathcal{C}_c(\mathcal{G};\mathscr{E})$ and $g\in\mathcal{G}$, one has

$$\begin{aligned}(f(\xi*\eta))(g)&=f(r(g))\int_{\mathcal{G}^{r(g)}}\xi(h)\eta(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h)\\&=\int_{\mathcal{G}^{r(g)}}f(r(h))\xi(h)\eta(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h)\\&=\int_{\mathcal{G}^{r(g)}}(f\xi)(h)\eta(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h)\\&=f\xi*\eta;\end{aligned}$$

and similarly one shows that $(\xi*\eta)f=\xi*\eta f$. Moreover, we have

$$\begin{aligned}(\xi f*\eta)(g)&=\int_{\mathcal{G}^{r(g)}}\xi(h)f(s(h))\eta(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h)\\&=\int_{\mathcal{G}^{r(g)}}\xi(h)f(r(h^{-1}g))\eta(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h)\\&=\int_{\mathcal{G}^{r(g)}}\xi(h)(f\eta)(h^{-1}g)\,d\mu_{\mathcal{G}}^{r(g)}(h)\\&=(\xi*f\eta)(g);\end{aligned}$$

so that $R_f(\pi_l(\xi))\pi_l(\eta)=\pi_l(\xi)L_f(\pi_l(\eta))$, and by continuity, for every $f\in\mathcal{C}_b(X;\mathscr{E})$, the pair (L_f,R_f) verifies $R_f(a)b=aL_f(b)$ for all $a,b\in C^*_r(\mathcal{G};\mathscr{E})$; i.e. $(L_f,R_f)\in M(C^*_r(\mathcal{G};\mathscr{E}))$. Now, since the *-homomorphism $\mathcal{C}_b(X;\mathscr{E})\ni f\longmapsto L_f\in\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$ is Real and graded, the map $\mathcal{C}_b(X;\mathscr{E})\ni f\longmapsto(L_f,R_f)\in M(C^*_r(\mathcal{G};\mathscr{E}))$ is a Rg *-homomorphism. □

Remark 5.6.3. *In what follows, we identify the double centralizer (L_f,R_f), and hence the element $f\in\mathcal{C}_b(X;\mathscr{E})$, with $L_f\in\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$, by considering L_f as a* multiplier *of $C^*_r(\mathcal{G};\mathscr{E})$ under the formulas: $L_f\pi_l(\xi):=\pi_l(L_f\xi)=\pi_l(f\xi)$, and $\pi_l(\xi)L_f:=R_f(\pi_l(\xi))=\pi_l(\xi f)$.*

Definition 5.6.4. *Consider the Real field of graded C^*-algebras*

$$M(\mathscr{E}):=\coprod_{x\in X}M(\mathscr{E}_x),$$

equipped with the grading inherited from that of $\mathscr{E}$ and the Real structure induced by that of X and $\mathscr{E}$, over (X,ρ). Then, we denote by $\mathcal{C}_b^{str}(X;M(\mathscr{E}))$ the unital C^*-algebra consisting of all the bounded strictly continuous sections *of $M(\mathscr{E})$ over X (cf. [3, p.7]). Together with the Real structure σ and the obvious grading, $(\mathcal{C}_b^{str}(X;M(\mathscr{E})),\sigma)$ is a graded Real unital C^*-algebra (under pointwise operations and the sup-norm).*

Notice that the unit $\mathbf{1} \in \mathcal{C}_b^{str}(X; M(\mathscr{E}))$ is the section given by

$$\mathbf{1} : X \ni x \longmapsto (\mathrm{Id}_{\mathscr{E}_x}, \mathrm{Id}_{\mathscr{E}_x}) \in M(\mathscr{E}_x),$$

where $\mathrm{Id}_{\mathscr{E}_x} : \mathscr{E}_x \longrightarrow \mathscr{E}_x$ is the identity map. From Proposition 5.6.2 we obtain the following corollary.

Corollary 5.6.5. *Let $(\mathcal{G}, \mathscr{E})$ be a Rg Fell system. Then $(\mathcal{C}_b^{str}(X; M(\mathscr{E})), \sigma)$ is a Rg sub-C^*-algebra of $(M(C_r^*(\mathcal{G}; \mathscr{E})), \sigma)$.*

Proof. The map $\mathcal{C}_b(X; \mathscr{E}) \ni f \longmapsto L_f \in M(C_r^*(\mathcal{G}; \mathscr{E}))$ is non-degenerate; indeed, observe that by considering the Rg left Fell pair $((\mathscr{E}, \sigma), (\mathscr{E}, \sigma))$ over $(\mathcal{G}, \rho)$ determined by the full maps $\mathscr{E}_g \times \mathscr{E}_h \longrightarrow \mathscr{E}_{gh}$, we see that for $f \in \mathcal{C}_b(X; \mathscr{E}) \subset A$ and $\xi \in \mathcal{C}_c(\mathcal{G}; \mathscr{E})$, the element $L_f \xi \in \mathcal{C}_c(\mathcal{G}; \mathscr{E})$ is nothing but the action of A on $\mathcal{C}_c(\mathcal{G}; \mathscr{E})$ defined in Lemma 5.5.5. It follows that if $\{a_i\}_{i \in I}$ is an approximate identity of $\mathcal{C}_b(X; \mathscr{E})$, then for all $\xi \in \mathcal{C}_c(\mathcal{G}; \mathscr{E})$ we have $L_{a_i} \pi_l(\xi) = \pi_l(a_i \cdot \xi) \longrightarrow \pi_l(\xi)$ thanks to Lemma 5.5.5. Whence, $L(\mathcal{C}_b(X; \mathscr{E})) C_r^*(\mathcal{G}; \mathscr{E})$ is dense in $C_r^*(\mathcal{G}; \mathscr{E})$. Now, from [71, §.3.12.10 and §.3.12.12], the map L extends to a unital strictly continuous *-homomorphism $M(\mathcal{C}_b(X; \mathscr{E})) \longrightarrow M(C_r^*(\mathcal{G}; \mathscr{E}))$; this map, also denoted by L, is clearly graded and Real. Furthermore, from [3, Lemma 3.1], we have $M(\mathcal{C}_0(X; \mathscr{E})) = \mathcal{C}_b^{str}(X; M(\mathscr{E}))$, which settles the result. □

Proposition 5.6.6. *Suppose that $(\Gamma, \mathscr{F}) \sim_{(Z, \mathscr{X})} (\mathcal{G}, \mathscr{E})$, and let $(\mathcal{M}, \mathscr{L})$ be the linking Rg Fell system as in the previous sections. Let χ_Y and χ_X be the characteristic functions of Y and X, respectively. Then we get two elements $\chi_Y \mathbf{1}$ and $\chi_X \mathbf{1}$ of $\mathcal{C}_b^{str}(\mathcal{M}^{(0)}; M(\mathscr{L}))$, where $\mathbf{1} \in \mathcal{C}_b^{str}(\mathcal{M}^{(0)}; M(\mathscr{L}))$ defined by scalar multiplication.*
Now define

$$p_\Gamma := L_{\chi_Y \mathbf{1}}, \text{ and } p_{\mathcal{G}} := L_{\chi_X \mathbf{1}} \in M(C_r^*(\mathcal{M}; \mathscr{L})).$$

Then p_Γ and $p_{\mathcal{G}}$ are complementary full projections [2] *which are homogeneous of degree* 0 *in $M(C_r^*(\mathcal{M}; \mathscr{L}))$; moreover, they are invariant under the Real structure.*

Proof. That $\chi_Y \mathbf{1}$, $\chi_X \mathbf{1} \in \mathcal{C}_b^{str}(\mathcal{M}^{(0)}; M(\mathscr{E}))$ is trivial. Also, it is easy to check that $p_\Gamma^2 = p_\Gamma^* = p_\Gamma$, and that $p_{\mathcal{G}}^2 = p_{\mathcal{G}}^* = p_{\mathcal{G}}$, so that p_Γ and $p_{\mathcal{G}}$ are projections in $M(C_r^*(\mathcal{M}; \mathscr{L}))$. That p_Γ and $p_{\mathcal{G}}$ are homogeneous of degree 0 and invariant under the Real structure is also straightforward. Let

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \in \mathcal{C}_c(\mathcal{M}; \mathscr{L}).$$

[2] Recall from [17] that a projection $p \in M(A)$ is said to be *full* if pAp is not contained in any proper closed two-sided ideal of A; that is, span$\{ApA\}$ is dense in A (see for instance [18] or [75, p.50]). In this case, we say that pAp is a *full corner* of A. Two projections $p, q \in M(A)$ are *complementary* if $p + q = 1$, in which case pAq is a pAp-qAq-imprimitivity bimodule; i.e. pAp and qAq are Morita equivalent. Conversely, two C^*-algebras A and B are Morita equivalent if and only if there is a C^*-algebra C with complementary full corners isomorphic to A and B, respectively (cf. [18, Theorem 1.1], [75, Theorem 3.19]).

Then

$$(\chi_Y \mathbf{1})\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ 0 & 0 \end{pmatrix}, \text{ and } (\chi_X \mathbf{1})\xi = \begin{pmatrix} 0 & 0 \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

Thus, $(p_\Gamma + p_\mathcal{G})(\pi_l^\mathcal{M}(\xi)) = \pi_l^\mathcal{M}(\xi)$, so that $p_\Gamma + p_\mathcal{G} = 1$ in $M(C_r^*(\mathcal{M};\mathscr{L}))$. Now, let $\xi,\eta \in \mathcal{C}_c(\mathcal{M};\mathscr{L})$. Then

$$\begin{aligned}\pi_l^\mathcal{M}(\xi)p_\Gamma\pi_l^\mathcal{M}(\eta) &= \pi_l^\mathcal{M}(\xi * p_\Gamma\eta) = \pi_l^\mathcal{M}(\xi p_\Gamma * \eta) \\ &= \pi_l^\mathcal{M}\begin{pmatrix} \xi_{11} * \eta_{11} & \xi_{11}\cdot\eta_{12} \\ & \\ \xi_{21} : \eta_{11} & \langle \xi_{21}^*, \eta_{12}\rangle_\star \end{pmatrix}.\end{aligned}$$

So to check that p_Γ is full, we just have to show that

$$\begin{aligned}\operatorname{span}\Bigg\{\pi_l^\mathcal{M}\begin{pmatrix} \xi_{11} * \eta_{11} & \xi_{11}\cdot\eta_{12} \\ \xi_{21} : \eta_{11} & \langle \xi_{21}^*, \eta_{12}\rangle_\star \end{pmatrix} \mid \xi_{11} \in \mathcal{C}_c(\Gamma;\mathscr{F}), \xi_{21} \in \mathcal{C}_c(Z^{-1};\overline{\mathfrak{X}}), \\ \eta_{12} \in \mathcal{C}_c(Z;\mathscr{X}), \eta_{11} \in \mathcal{C}_c(\Gamma;\mathscr{E})\Bigg\} \qquad (5.53)\end{aligned}$$

is dense in $C_r^*(\mathcal{M};\mathscr{L})$. But this is not hard to see by using the previous results. Indeed, the existence of Real approximate identities in $\mathcal{C}_c(\Gamma;\mathscr{F})$ (cf. Proposition 5.5.7) for both the left Real actions of $\mathcal{C}_c(\Gamma;\mathscr{F})$ on itself and on $\mathcal{C}_c(Z;\mathscr{X})$ shows that elements of the form $\xi_{11} * \eta_{11}$, for $\xi_{11},\eta_{11} \in \mathcal{C}_c(\Gamma;\mathscr{F})$ span a dense Rg subspace of $\mathcal{C}_c(\Gamma;\mathscr{F})$ and that elements of the form $\xi_{11}\cdot\eta_{12}$, for $\eta_{12} \in \mathcal{C}_c(Z;\mathscr{X})$, span a dense Rg subspace of $\mathcal{C}_c(Z;\mathscr{X})$. Also, that elements of the form $\xi_{21} : \eta_{11}$, where $\xi_{21} \in \mathcal{C}_c(Z^{-1};\overline{\mathscr{X}}), \eta_{11} \in \mathcal{C}_c(\Gamma;\mathscr{F})$, span a dense Rg subspace of $\mathcal{C}_c(Z^{-1};\overline{\mathscr{X}})$ follows from the existence of a Real approximate identity in $\mathcal{C}_c(\Gamma;\mathcal{F})$ for the right Rg action of $\mathcal{C}_c(\Gamma;\mathscr{F})$ on $\mathcal{C}_c(Z^{-1};\overline{\mathscr{X}})$. Finally, thanks to Proposition 5.4.10, the image of $\langle\cdot\,,\cdot\rangle_\star$ is a dense Rg subspace of $\mathcal{C}_c(\mathcal{G};\mathscr{E})$. Whence, we have shown that $C_r^*(\mathcal{M};\mathscr{L})p_\Gamma C_r^*(\mathcal{M};\mathscr{L})$ is dense in $C_r^*(\mathcal{M};\mathscr{L})$. In a similar fashion, we get that $C_r^*(\mathcal{M};\mathscr{L})p_\mathcal{G} C_r^*(\mathcal{M};\mathscr{L})$ is dense in $C_r^*(\mathcal{M};\mathscr{L})$, which completes the proof. □

The main result of the chapter is the following theorem.

Theorem 5.6.7 (The Renault's Equivalence)**.** *Let* (Γ,ϱ) *and* $(\mathcal{G},\rho)$ *be second countable locally compact Hausdorff Real groupoids. Suppose that*

$$((\Gamma,\varrho),(\mathscr{F},\varsigma)) \sim_{((Z,\tau),(\mathscr{X},\kappa))} ((\mathcal{G},\rho),(\mathscr{E},\sigma))$$

are equivalent Rg Fell systems. Then the isomorphisms of Rg convolution algebras

$$\mathcal{C}_c(\Gamma;\mathscr{F}) \ni \xi_{11} \longmapsto \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix} \in p_\Gamma\mathcal{C}_c(\mathcal{M};\mathscr{L})p_\Gamma, \tag{5.54}$$

and

$$\mathcal{C}_c(\mathcal{G};\mathscr{E}) \ni \eta_{22} \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & \eta_{22} \end{pmatrix} \in p_\mathcal{G}\mathcal{C}_c(\mathcal{M};\mathscr{L})p_\mathcal{G} \tag{5.55}$$

extend to two isomorphisms of Rg C^*-algebras

$$C_r^*(\Gamma;\mathscr{F}) \longrightarrow p_\Gamma C_r^*(\mathcal{M};\mathcal{L})p_\Gamma, \text{ and } C_r^*(\mathcal{G};\mathscr{E}) \longrightarrow p_\mathcal{G} C_r^*(\mathcal{M};\mathscr{L})p_\mathcal{G}. \tag{5.56}$$

In particular, $(C_r^*(\Gamma;\mathscr{F}),\varsigma)$ *and* $(C_r^*(\mathcal{G};\mathscr{E}),\sigma)$ *are Morita equivalent with Rg imprimitivity bimodule* $(p_\Gamma C_r^*(\mathcal{M};\mathscr{L})p_\mathcal{G},\sigma^{\mathscr{L}})$ *which is isometrically isomorphic to the completion* (X_r,κ) *of* $(\mathcal{C}_c(Z;\mathscr{X}),\kappa)$ *in the norm*

$$\|\phi\|_{\mathscr{E}} := \|\langle\phi\,,\phi\rangle_\star\|^{1/2}_{C_r^*(\mathcal{G};\mathscr{E})}, \text{ for } \phi\in\mathcal{C}_c(Z;\mathscr{X}).$$

In order to proof Theorem 5.6.7, we need some more constructions.

Suppose $(\Gamma,\mathscr{F})\sim_{(Z,\mathscr{X})}(\mathcal{G},\mathscr{E})$. For $x\in X$, we also denote by $\mathscr{X}\longrightarrow Z_x$ the pull-back of $\mathscr{X}\longrightarrow Z$ along the inclusion $Z_x\hookrightarrow Z$. Then we define $L^2(Z_x;\mathscr{X})$ as the completion of $\mathcal{C}_c(Z_x;\mathscr{X})$ with respect to the graded A_x-valued inner product $\langle\phi\,,\psi\rangle_\star(x) = \int_{\Gamma^{\mathfrak{r}(z)}}\langle\phi(\gamma^{-1}\cdot z),\psi(\gamma^{-1}\cdot z)\rangle_{\mathcal{E}}\,d\mu_\Gamma^{\mathfrak{r}(z)}(\gamma)$, where $\mathfrak{s}(z)=x$, and the right A_x-action $(\phi\cdot a)(z):=\phi(z)a$, for $\phi\in\mathcal{C}_c(Z_x;\mathscr{X}), a\in A_x$. Thus, $L^2(Z_x;\mathscr{X})$ is a graded Hilbert A_x-module. Similarly, for all $y\in Y$, one can form the graded Hilbert B_y-module $L^2(Z_y^{-1};\overline{\mathscr{X}})$. Note the involutions κ of $\mathscr{X}$ and τ of Z induce an obvious conjugate-linear graded isometry $\tau_x:\mathcal{C}_c(Z_x;\mathscr{X})\longrightarrow\mathcal{C}_c(Z_{\bar{x}};\mathscr{X}_{\bar{x}})$ defined by $\tau_x(\phi)(z):=\kappa_x(\phi(\tau(z)))$. What's more, we have $\tau_x(\phi\cdot a)=\tau_x(\phi)\cdot\sigma(a)$. Hence, there is an induced conjugate-linear graded *-isomorphism

$$\tau_x\colon\mathcal{L}_{A_x}(L^2(Z_x;\mathscr{X}))\longrightarrow\mathcal{L}_{A_{\bar{x}}}(L^2(Z_{\bar{x}};\mathscr{X})).$$

Proposition 5.6.8. *Suppose* $(\Gamma,\mathscr{F})\sim_{(Z,\mathscr{X})}(\mathcal{G},\mathscr{E})$. *For* $x\in X$, *the left action of* $\mathcal{C}_c(\Gamma;\mathscr{F})$ *on* $\mathcal{C}_c(Z_x;\mathscr{X})$ *induces a graded* **-representation* $R_x^\Gamma:\mathcal{C}_c(\Gamma;\mathscr{F})\longrightarrow\mathcal{L}_{A_x}(L^2(Z_x;\mathscr{X}))$ *that factors through the Rg* C^**-algebra* $C_r^*(\Gamma;\mathscr{F})$ *such that the following diagram commutes*

$$\begin{tikzcd} C_r^*(\mathcal{G};\mathscr{E}) \arrow[r, "R_x^\mathcal{G}"] \arrow[d, "\sigma"'] & \mathcal{L}_{A_x}(L^2(Z_x;\mathscr{X})) \arrow[d, "\tau_x"] \\ C_r^*(\mathcal{G};\mathscr{E}) \arrow[r, "R_{\bar{x}}^\mathcal{G}"] & \mathcal{L}_{A_{\bar{x}}}(L^2(Z_{\bar{x}};\mathscr{X})) \end{tikzcd} \tag{5.57}$$

Similarly, for all $y\in Y$, *we get a representation* $R_y^\mathcal{G}:C_r^*(\mathcal{G};\mathscr{E})\longrightarrow\mathcal{L}_{B_y}(L^2(Z_y^{-1};\overline{\mathscr{X}}))$, *together with a similar commutative diagram.*

Proof. Let $\xi\in\mathcal{C}_c(\Gamma;\mathscr{F})$; then for $\phi,\psi\in\mathcal{C}_c(Z_x;\mathscr{X})$, simple calculations give $\langle\xi\cdot\phi\,,\psi\rangle_\star(x)=\langle\phi\,,\xi^*\cdot\psi\rangle_\star(x)$. It follows that the A_x-linear operator $\mathcal{C}_c(Z_x;\mathscr{X})\ni\phi\longmapsto\xi\cdot\phi\in\mathcal{C}_c(Z_x;\mathcal{X})$ is adjointable, and then bounded with respect to the norm $\|\cdot\|_{L^2(Z_x;\mathscr{X})}$, which gives the *-representation $R_x^\Gamma:\mathcal{C}_c(\Gamma;\mathscr{F})\longrightarrow\mathcal{L}_{A_x}(L^2(Z_x;\mathscr{X})),\xi\longrightarrow(R_x^\Gamma(\xi):\phi\longmapsto\xi\cdot\phi)$.

Now, let $z_0\in Z_x$, and let $y:=\mathfrak{r}(z_0)$. Then, to complete the proof it suffices to check that for all $\xi\in\mathcal{C}_c(\Gamma;\mathscr{F})$, $\|R_x^\Gamma(\xi)\|\le\|\pi_y^\Gamma(\xi)\|$, where $\pi_y^\Gamma:\mathcal{C}_c(\Gamma;\mathscr{F})\longrightarrow\mathcal{L}_{B_y}(L^2(\Gamma_y;\mathscr{F}))$ is the representation defined in Lemma 5.2.10.

Consider the (left) Hilbert B_y-module $\mathcal{X}_{z_0}$, and form the interior tensor product $L^2(\Gamma_y;\mathscr{F})\otimes_{B_y}\mathscr{X}_{z_0}$ which is a right graded Hilbert A_x-module under the operations defined on simple tensors by: $(\xi\otimes u)\cdot a := \xi\otimes(ua)$, and $\langle \xi\otimes u, \eta\otimes v\rangle := \langle u, \langle\xi,\eta\rangle_{B_y}\cdot v\rangle_{A_x}$. Then, the map

$$u_{z_0} : L^2(\Gamma_y;\mathscr{F})\otimes_{B_y}\mathcal{X}_{z_0} \longrightarrow L^2(Z_x;\mathscr{X}),\ \sum_i \xi_i\otimes u_i \longmapsto \sum_i \xi_i\cdot u_i, \tag{5.58}$$

where for $\xi\in\mathcal{C}_c(\Gamma_y;\mathscr{F})$ and $u\in\mathcal{X}_{z_0}$, $(\xi\cdot u)(z) := \xi({}_\Gamma<z,z_0>)\cdot u\in\mathscr{X}_{{}_\Gamma<z,z_0>\cdot z_0}$, is an isomorphism of graded Hilbert A_x-modules. The map (5.58) is clearly A_x-linear and injective. To see that it is surjective, first notice that the well defined map $Z_x\ni z\longmapsto {}_\Gamma<z,z_0>\in\Gamma_y$, is a homeomorphism of Γ-spaces (its inverse being $\Gamma_y\ni\gamma\longmapsto\gamma\cdot z_0\in Z_x$). Next, for all $z\in Z_x$, the linear span of the image of $\mathcal{F}_{{}_\Gamma<z,z_0>}\times\mathscr{X}_{z_0}\ni(f,u)\longmapsto f\cdot u\in\mathcal{X}_{{}_\Gamma<z,z_0>\cdot z_0}$ is dense in $\mathscr{X}_{{}_\Gamma<z,z_0>\cdot z_0}$ by definition of a Fell pair; so that, using the Stone-Weierstrass theorem,

$$\operatorname{span}\{\eta\cdot u : Z_x\ni z\longmapsto \eta({}_\Gamma<z,z_0>)\cdot u\in\mathscr{X}_{{}_\Gamma<z,z_0>\cdot z_0}\mid \eta\in\mathcal{C}_c(\Gamma_y;\mathscr{F}), u\in\mathscr{X}_{z_0}\}$$

is dense in $\mathcal{C}_c(Z_x;\mathscr{X})$ in the inductive limit topology. It follows that any $\phi\in\mathcal{C}_c(Z_x;\mathcal{X})$ is the inductive limit of some $\sum_i\eta_i\cdot u_i = u_{z_0}(\sum_i\eta_i\otimes u_i)$. We then have an isomorphism of C^*-algebras

$$\tilde u_{z_0} : \mathcal{L}_{A_x}(L^2(\Gamma_y;\mathscr{F})\otimes_{B_y}\mathscr{X}_{z_0}) \longrightarrow \mathcal{L}_{A_x}(L^2(Z_x;\mathscr{X}))$$

such that $\tilde u_{z_0}(T)\left(\sum_i\xi_i\cdot u_i\right) := u_{z_0}\left(T\left(\sum_i\xi_i\otimes u_i\right)\right)$, for all $T\in\mathcal{L}_{A_x}(L^2(\Gamma_y;\mathscr{F})\otimes_{B_y}\mathscr{X}_{z_0})$. Furthermore, the following diagram is commutative

$$\begin{tikzcd}
\mathcal{C}_c(\Gamma;\mathscr{F}) \arrow[r, "R_x^\Gamma"] \arrow[d, "\pi_y^\Gamma"'] & \mathcal{L}_{A_x}(L^2(Z_x;\mathscr{X})) \\
\mathcal{L}_{B_y}(L^2(\Gamma_y;\mathscr{F})) \arrow[r] & \mathcal{L}_{A_x}(L^2(\Gamma_y;\mathscr{F})\otimes_{B_y}\mathcal{X}_{z_0}) \arrow[u, "\tilde u_{z_0}"']
\end{tikzcd}$$

where the lower horizontal arrow is the map $T\longmapsto T\otimes\mathrm{Id}$ (cf. for instance [50, p.50]). Indeed, let $\xi\in\mathcal{C}_c(\Gamma;\mathscr{F})$, and $\phi\in\mathcal{C}_c(Z_x;\mathscr{X})$. Without loss of generality, we can suppose that $\phi=\eta\cdot u$; then,

$$\tilde u_{z_0}(\pi_y^\Gamma(\xi)\otimes\mathrm{Id})\phi = (\pi_y^\Gamma(\xi)\otimes\mathrm{Id})(\eta\otimes u) = (\pi_y^\Gamma(\xi)\eta)\otimes u = (\xi*\eta)\cdot u = \xi\cdot(\eta\cdot u) = R_x^\Gamma(\xi)(\eta\otimes u) = R_x^\Gamma(\xi)\phi,$$

which completes the proof since u_{z_0} is an isomorphism and $\|\pi_y^\Gamma(\xi)\otimes\mathrm{Id}\|\le\|\pi_y^\Gamma(\xi)\|$ (see [50, p.50]). □

Proof of Theorem 5.6.7. That the maps defined by (5.54) and (5.55) are isomorphisms of convolutions *-algebras is obvious.
As previously, let us put $B := \mathcal{C}_0(Y;\mathscr{F})$ and $A := \mathcal{C}_0(X;\mathscr{E})$. Then

$$\mathcal{C}_0(\mathcal{M}^{(0)};\mathscr{L})\cong B\oplus A,$$

as Rg C^*-algebras. Now, with respect to this decomposition, simple calculations show that

$$ {}_{B\oplus A}\langle \xi,\eta\rangle = \big({}_B\langle \xi_{11},\eta_{11}\rangle + {}_{\star}\langle \xi_{21}^*, \eta_{21}^*\rangle_{|Y}\big) \oplus \big(\langle \xi_{12}, \eta_{12}\rangle_{\star|X} + {}_A\langle \xi_{22},\eta_{22}\rangle\big), \tag{5.59} $$

for all $\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$ in $\mathcal{C}_c(\mathcal{M};\mathscr{L})$. In particular, suppose that $\xi = \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix} \in p_\Gamma \mathcal{C}_c(\mathcal{M};\mathscr{L})p_\Gamma$, then

$$ {}_{B\oplus A}\left\langle \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix} \right\rangle = {}_B\langle \xi_{11},\xi_{11}\rangle \oplus 0, $$

so that

$$ \left\| \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix} \right\|_{L^2(\mathcal{M};\mathscr{L})} = \|\xi_{11}\|_{L^2(\Gamma;\mathscr{F})}; \tag{5.60} $$

thus, (5.54) extends to an isometric B-linear map u_Γ of B-modules

$$ u_\Gamma : L^2(\Gamma;\mathscr{F}) \longrightarrow p_\Gamma L^2(\mathcal{M};\mathscr{L})p_\Gamma, $$

where $p_\Gamma L^2(\mathcal{M};\mathcal{L})p_\Gamma$ is the completion of $p_\Gamma \mathcal{C}_c(\mathcal{M};\mathscr{L})p_\Gamma$ with respect to the norm of $L^2(\mathcal{M};\mathscr{L})$. Similarly, for $\xi_{22} \in \mathcal{C}_c(\mathcal{G};\mathscr{E})$, we get

$$ {}_{B\oplus A}\left\langle \begin{pmatrix} 0 & 0 \\ 0 & \xi_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \xi_{22} \end{pmatrix} \right\rangle = 0 \oplus {}_A\langle \xi_{22},\xi_{22}\rangle, $$

and hence

$$ \left\| \begin{pmatrix} 0 & 0 \\ 0 & \xi_{22} \end{pmatrix} \right\|_{L^2(\mathcal{M};\mathscr{L})} = \|\xi_{22}\|_{L^2(\mathcal{G};\mathcal{E})}; \tag{5.61} $$

so that (5.55) extends to an isometric A-linear map $u_\mathcal{G}$ of A-modules

$$ u_\mathcal{G} : L^2(\mathcal{G};\mathscr{E}) \longrightarrow p_\mathcal{G} L^2(\mathcal{M};\mathscr{L})p_\mathcal{G}. $$

Furthermore, since u_Γ and $u_\mathcal{G}$ are surjective, then from [**?**, Theorem 3.5], they are unitaries in $\mathcal{L}_B(L^2(\Gamma;\mathscr{F}), p_\Gamma L^2(\mathcal{M};\mathscr{L})p_\Gamma)$ and $\mathcal{L}_A(L^2(\mathcal{G};\mathscr{E}), p_\mathcal{G} L^2(\mathcal{M};\mathscr{L})p_\mathcal{G})$, respectively; in other words,

$$ L^2(\Gamma;\mathscr{F}) \approx p_\Gamma L^2(\mathcal{M};\mathscr{L})p_\Gamma $$

as Hilbert B-modules, and

$$ L^2(\mathcal{G};\mathscr{E}) \approx p_\mathcal{G} L^2(\mathcal{M};\mathscr{L})p_\mathcal{G} $$

as Hilbert A-modules, here the sign "≈" stands for *unitarily equivalent*. Moreover, it is very easy to see that the following diagrams commute:

$$\begin{tikzcd} \mathcal{C}_c(\Gamma;\mathscr{F}) \arrow[r, "\cong"] \arrow[d, "\pi_l^\Gamma"] & p_\Gamma \mathcal{C}_c(\mathcal{M};\mathscr{L}) p_\Gamma \arrow[d, "\pi_l^{\mathcal{M}}"] \\ \mathcal{L}\left(L^2(\Gamma;\mathscr{F})\right) \arrow[r, "\cong"] & \mathcal{L}\left(p_\Gamma L^2(\mathcal{M};\mathscr{L}) p_\Gamma\right) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_c(\mathcal{G};\mathscr{E}) \arrow[r, "\cong"] \arrow[d, "\pi_l^{\mathcal{G}}"'] & p_{\mathcal{G}} \mathcal{C}_C(\mathcal{M};\mathscr{L}) p_{\mathcal{G}} \arrow[d, "\pi_l^{\mathcal{M}}"] \\ \mathcal{L}\left(L^2(\mathcal{G};\mathscr{E})\right) \arrow[r, "\cong"] & \mathcal{L}\left(p_{\mathcal{G}} L^2(\mathcal{M};\mathscr{L}) p_{\mathcal{G}}\right) \end{tikzcd} \tag{5.62}$$

It then only remains to check that for $\xi = \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 & 0 \\ 0 & \eta_{22} \end{pmatrix}$, we have $\|\xi\|_{C_r^*(\mathcal{M};\mathscr{L})} = \|\xi_{11}\|_{C_r^*(\Gamma;\mathscr{F})}$ and $\|\eta\|_{C_r^*(\mathcal{M};\mathscr{L})} = \|\eta_{22}\|_{C_r^*(\mathcal{G};\mathscr{E})}$ which will lead to the desired isomorphisms of C^*-algebras (5.56) since p_Γ and $p_{\mathcal{G}}$ are complementary (cf. Proposition 5.6.6). However, by symmetry it suffices to check one of the latter equalities. To this end, we will use the constructions of Lemma 5.2.10.

Note that we have

$$\mathcal{C}_c(\mathcal{M}_\omega;\mathscr{L}) = \begin{cases} \mathcal{C}_c(\Gamma_y;\mathscr{F}) \oplus \mathcal{C}_c(Z_y^{-1};\overline{\mathscr{X}}), & \text{if } \omega = \mathrm{y} \in \mathrm{Y}; \\ \mathcal{C}_c(Z_x;\mathscr{X}) \oplus \mathcal{C}_c(\mathcal{G}_x;\mathscr{E}), & \text{if } \omega = \mathrm{x} \in \mathrm{X} \end{cases}$$

In other words, elements of $\mathcal{C}_c(\mathcal{M}_y;\mathscr{L})$, for $y \in Y$, are of the form $\begin{pmatrix} \eta_{11} & 0 \\ \eta_{21} & 0 \end{pmatrix}$ with $\eta_{11} \in \mathcal{C}_c(\Gamma_y;\mathscr{F})$ and $\eta_{21} \in \mathcal{C}_c(Z_y^{-1};\overline{\mathscr{X}})$, while elements of $\mathcal{C}_c(\mathcal{M}_x;\mathscr{L})$, for $x \in X$, are of the form $\begin{pmatrix} 0 & \eta_{12} \\ 0 & \eta_{22} \end{pmatrix}$ with $\eta_{12} \in \mathcal{C}_c(Z_x;\mathscr{X})$ and $\eta_{22} \in \mathcal{C}_c(\mathcal{G}_x;\mathscr{E})$. Then, for all $y \in Y$, and $\eta, \zeta \in \mathcal{C}_c(\mathcal{M}_y;\mathscr{L})$, one has

$$\langle \eta,\zeta\rangle_{B_y} = \int_{\Gamma_y} \eta_{11}(\gamma)^*\zeta_{11}(\gamma)^* d(\mu_\Gamma)_y(\gamma) + \int_{Z_y^{-1}} {}_{\mathscr{F}}\langle \eta_{21}(\flat(z)), \zeta_{21}(\flat(z))\rangle d(\mu_{Z^{-1}})_y(\flat(z)),$$

where $(\mu_{Z^{-1}})_y$ is the Radon measure on Z^{-1} with support Z_y^{-1}, which is the image of μ^y on Z under the "inversion" $Z^{-1} \longrightarrow Z, \flat(z) \longmapsto z$; it is then given by

$$(\mu_{Z^{-1}})_y(\phi) = \int_{\mathcal{G}^{\mathfrak{r}^\flat(\flat(z))}} \phi(g^{-1}\cdot\flat(z)) d\mu_{\mathcal{G}}^{\mathfrak{r}^\flat(\flat(z))}(g), \text{ for } \phi \in \mathrm{C_c}(\mathrm{Z}^{-1}).$$

So, by using Notations 5.4.12, we get $\langle \xi,\eta\rangle_{B_y} = \langle \eta_{11},\zeta_{11}\rangle_{B_y} + \langle\!\langle \eta_{21}, \zeta_{21}\rangle\!\rangle_\star(y)$; hence $L^2(\mathcal{M}_y;\mathscr{L}) = L^2(\Gamma;\mathscr{F}) \oplus L^2(Z_y^{-1};\overline{\mathscr{X}})$. In the same way, we verify that $L^2(\mathcal{M}_x;\mathcal{L}) = L^2(Z_x;\mathscr{X}) \oplus L^2(\mathcal{G}_x;\mathscr{E})$. Thus, for all $\xi \in \mathcal{C}_c(\mathcal{M};\mathscr{L})$, we have

$$\|\xi\|_{C_r^*(\mathcal{M};\mathscr{L})} = \max\left\{ \sup_{y\in Y} \|\pi_y^{\mathcal{M}}(\xi)\|, \sup_{x\in X} \|\pi_x^{\mathcal{M}}(\xi)\| \right\}.$$

In particular, if $\xi = \begin{pmatrix} \xi_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_c(\mathcal{M};\mathscr{L})$, and $y \in Y$, then $\pi_y^{\mathcal{M}}(\xi) = \pi_y^\Gamma(\xi_{11}) \oplus 0$, so that

$$\|\xi\|_{C_r^*(\mathcal{M};\mathscr{L})} = \max\left\{ \|\xi_{11}\|_{C_r^*(\Gamma;\mathscr{F})}, \sup_{x\in X} \|\pi_x^{\mathcal{M}}(\xi)\| \right\}. \tag{5.63}$$

Now, let $x \in X$, and suppose $\eta \in \mathcal{C}_c(\mathcal{M}_x; \mathscr{L})$ is such that $\|\eta\|_{L^2(\mathcal{M}_x;\mathcal{L})} \le 1$; *i.e.*

$$\max\left\{\|\eta_{12}\|_{L^2(Z_x;\mathcal{X})}, \|\eta_{22}\|_{L^2(\mathcal{G};\mathscr{E})}\right\} \le 1.$$

Then, from a simple calculation we obtain

$$\langle \pi_x^{\mathcal{M}}(\xi)\eta, \pi_x^{\mathcal{M}}(\xi)\eta\rangle_{A_x} = \langle \xi_{11}\cdot\eta_{12}\,,\xi_{11}\cdot\eta_{12}\rangle_\star(x) = \langle R_x^\Gamma(\xi_{11})\eta_{12}\,, R_x^\Gamma(\xi_{11})\eta_{12}\rangle_\star(x);$$

hence, by applying Proposition 5.6.8, we get $\|\pi_x^{\mathcal{M}}(\xi)\eta\|_{L^2(\mathcal{M}_x;\mathcal{L})} = \|R_x^\Gamma(\xi_{11})\eta_{12}\|_{L^2(Z_x;\mathscr{X})} \le \|\xi_{11}\|_{C_r^*(\Gamma;\mathscr{F})}$. Therefore, from (5.63), we get

$$\|\xi\|_{C_r^*(\mathcal{M};\mathscr{L})} = \|\xi_{11}\|_{C_r^*(\Gamma;\mathscr{F})}.$$

□

Corollary 5.6.9. *Assume that* $(\mathcal{A},\sigma,\alpha) \in \widehat{\mathfrak{BrR}}(\mathcal{G})$. *If* $(Z,\tau) : (\Gamma,\varrho) \longrightarrow (\mathcal{G},\rho)$ *is an isomorphism in* $\mathfrak{RG}$ *then*

$$(\mathcal{A} \rtimes_r \mathcal{G}, s_{\mathcal{G}}^*\sigma) \sim_{Morita} (\mathcal{A}^Z \rtimes_r \Gamma, s_\Gamma^*\sigma^Z).$$

Proof. Recall that the graded Real Dixmier-Douady bundle $(\mathcal{A}^Z,\sigma^Z,\alpha^Z)$ over (Γ,ϱ) is defined as $\mathcal{A}^Z := \mathfrak{s}^*\mathcal{A}/\mathcal{G}$. Observe that for $\gamma \in \Gamma$, the fibre $(s_\Gamma^*\mathcal{A}^Z)_\gamma = \mathcal{A}^Z_{s_\Gamma(\gamma)}$ is identified with $Z_{s_\Gamma(\gamma)} \times_{\mathcal{G}^{(0)}} \mathcal{A}/\mathcal{G}$. Consider the graded Real C^*-bundle $(\mathfrak{s}^*\mathcal{A},\mathfrak{s}^*\sigma)$ over (Z,τ). Then, the Fell system $((s_\Gamma^*\mathcal{A}^Z, s_\Gamma^*\sigma^Z),(\Gamma,\varrho))$ acts on $((\mathfrak{s}^*\mathcal{A},\mathfrak{s}^*\sigma),(Z,\tau))$ on the left via

$$\frac{Z_{s_\Gamma(\gamma)} \times_{\mathcal{G}^{(0)}} \mathcal{A}}{\mathcal{G}} \times \mathcal{A}_{\mathfrak{s}(z)} \ni ([z,a],b) \longmapsto ab \in \mathcal{A}_{\mathfrak{s}(\gamma z)} = \mathcal{A}_{\mathfrak{s}(z)}, \tag{5.64}$$

where $(\gamma,z) \in \Gamma * Z$. Also, we have a right Fell pair $((s_{\mathcal{G}}^*\mathcal{A},\mathcal{G}),(\mathfrak{s}^*\mathcal{A},Z))$ determined by the right action

$$\mathcal{A}_{\mathfrak{s}(z)} \times \mathcal{A}_{s_{\mathcal{G}}(g)} \ni (a,b) \longmapsto \alpha_g^{-1}(a)b \in \mathcal{A}_{\mathfrak{s}(zg)} = \mathcal{A}_{s_{\mathcal{G}}(g)}. \tag{5.65}$$

Next, define the inner products in the obvious way: if $(z,\flat(z')) \in Z\times_{\mathcal{G}^{(0)}} Z^{-1}$, we set

$$\mathcal{A}_{\mathfrak{s}(z)} \times \overline{\mathcal{A}_{\mathfrak{s}(z)}} \ni (a,\flat(b)) \longmapsto [z,a^*b] \in (s_\Gamma^*\mathcal{A}^Z)_{{}_\Gamma<z,z'>} = \frac{Z_{s_\Gamma({}_\Gamma<z,z'>)}\times_{\mathcal{G}^{(0)}}\mathcal{A}}{\mathcal{G}}, \tag{5.66}$$

and if $(\flat(z),z') \in Z^{-1}\times_{\Gamma^{(0)}} Z$, we put

$$\overline{\mathcal{A}_{\mathfrak{s}(z)}} \times \mathcal{A}_{\mathfrak{s}(z')} \ni (\flat(a),b) \longmapsto \alpha^{-1}_{<z,z'>_{\mathcal{G}}}(a)^*b \in (s_{\mathcal{G}}^*\mathcal{A})_{<z,z'>_{\mathcal{G}}} = \mathcal{A}_{s_{\mathcal{G}}(<z,z'>_{\mathcal{G}})} = \mathcal{A}_{\mathfrak{s}(z')}. \tag{5.67}$$

It is now straightforward that the settings (5.64), (5.65), (5.66), and (5.67) give an equivalence of graded Real Fell systems $(s_\Gamma^*\mathcal{A}^Z,\Gamma) \sim_{(\mathfrak{s}^*\mathcal{A},Z)} (s_{\mathcal{G}}^*\mathcal{A},\mathcal{G})$. We thus complete the proof by applying Theorem 5.6.7. □

5.7 The reduced C^*-algebra of an element of $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$

In this section we are following [90, §3.1] to construct the *reduced* C^*-algebra associated to a graded Real $\mathbb{S}^1$-central extension. We begin with the following simple lemma which provides another example of graded Real Fell bundles having direct bearing on our work.

Lemma 5.7.1. *Let* $\mathbb{E} = (\,\mathbb{S}^1 \longrightarrow \widetilde{\Gamma} \xrightarrow{\pi} \Gamma,\delta,P\,)$ *be a graded Real* $\mathbb{S}^1$*-central extension of* $(\mathcal{G},\rho)$*. Consider the associated line bundle* $L := \widetilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C}$ *of the* $\mathbb{S}^1$*-principal bundle* $\pi:\widetilde{\Gamma}\longrightarrow\Gamma$*, where*

$$\widetilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C} := \widetilde{\Gamma}\times\mathbb{C}/_{(\tilde{\gamma},z)\sim(t\tilde{\gamma},t^{-1}z)}.$$

Define the grading ϵ *and the Real structure* σ *on* L *by* $\epsilon([\tilde{\gamma},z]) = [\tilde{\gamma},z.\delta(\pi(\tilde{\gamma}))]$*, and* $\sigma([\tilde{\gamma},z]) := [\tilde{\varrho}(\tilde{\gamma}),\bar{z}]$*, respectively. Then*

(i) the projection $(L,\sigma)\longrightarrow(\Gamma,\varrho),[\tilde{g},t]\longmapsto\pi(\tilde{\gamma})$ *defines a graded Real line bundle over the Real groupoid* (Γ,ϱ)*;*

(ii) equipped with the product and the **-involution given respectively by*

- $L_{\gamma_1}\times L_{\gamma_2}\ni\big([\tilde{\gamma}_1,z_1],[\tilde{\gamma}_2,z_2]\big)\longmapsto[\tilde{\gamma}_1\tilde{\gamma}_2,z_1z_2]\in L_{\gamma_1\gamma_2}$*, for* $(\gamma_1,g_2)\in\Gamma^{(2)}$*, and*
- $L_\gamma\ni[\tilde{\gamma},z]\longmapsto[\tilde{\gamma}^{-1},\bar{z}]\in L_{\gamma^{-1}}$*,*

(L,ν) *is a graded Real Fell bundle over* $(\mathcal{G},\varrho)$*.*

Definition 5.7.2. *(cf. [90, Definition 3.1]). Let* $\mathbb{E}$ *be given as above. Then, the graded Real reduced* C^**-algebra* $(C^*_r(\Gamma;L),\sigma)$ *of* (L,σ) *is called* the reduced C^*-algebra *of* $\mathbb{E}$*, and is denoted by* $(C^*_r(\mathbb{E}),\sigma)$*.*

Theorem 5.7.3. *Suppose that* $[\mathbb{E}_1]=[\mathbb{E}_2]$ *in* $\widehat{Ext\mathrm{R}}(\mathcal{G},\mathbb{S}^1)$*. Then as Rg* C^**-algebras,*

$$(C^*_r(\mathbb{E}_1),\sigma_1)\sim_{Morita}(C^*_r(\mathbb{E}_2),\sigma_2).$$

Proof. Suppose $\mathbb{E}_i = (\,\mathbb{S}^1 \longrightarrow \widetilde{\Gamma}_i \xrightarrow{\pi_i} \Gamma_i\,,\delta_i,P_i)$, and $L_i := \widetilde{\Gamma}_1\times_{\mathbb{S}^1}\mathbb{C}$. Let $p_i : L_i\longrightarrow\Gamma_i$, $i=1,2$ be the Real projections. Set $\ker p_i := {L_i}_{|\Gamma_i^{(o)}} = p_i^{-1}(\Gamma_i^{(0)})$, where as usual, $\Gamma_i^{(0)}$ is identified with its image in Γ_i by the identity map $\Gamma_i^{(0)}\hookrightarrow\Gamma_i$. Observe that $\ker p_i\cong\ker\pi_i\times_{\mathbb{S}^1}\mathbb{C}$, where $\ker\pi_i := {\widetilde{\Gamma}_i}_{|\Gamma_i^{(0)}} = \Gamma_i^{(0)}\times\mathbb{S}^1$ is the restriction of the Real $\mathbb{S}^1$-principal bundle $\pi_i:\widetilde{\Gamma}_i\longrightarrow\Gamma_i$ to $\Gamma_i^{(0)}\subset\Gamma_i$. Moreover, together with the trivial grading on the fibers and the Real structure defined as the restriction of σ_i, $\ker p_i\longrightarrow\Gamma_i^{(0)}$, $i=1,2$, is a graded Real line bundle. Now let $(Z,\tau):(\widetilde{\Gamma}_1,\tilde{\varrho}_1)\longrightarrow(\widetilde{\Gamma}_2,\tilde{\varrho}_2)$ be a Real $\mathbb{S}^1$-equivariant Morita equivalent implementing the equivalence $\mathbb{E}_1\sim\mathbb{E}_2$ in $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$. Recall from Definition 2.6.4 that we have a Real Morita equivalence $(Z/\mathbb{S}^1,\tau):(\Gamma_1\varrho_1)\longrightarrow(\Gamma_2,\varrho_2)$. Define

$$\mathcal{X} := (Z/\mathbb{S}^1\times_{\Gamma_1^{(0)}}\ker p_1)\otimes(Z/\mathbb{S}^1\times_{\Gamma_2^{(0)}}\ker p_2), \tag{5.68}$$

together with the Real structure κ given by the unique extension of the involution

$$\begin{array}{ccc}(Z/\mathbb{S}^1\times_{\Gamma_1^{(0)}}\ker p_1)\times(Z/\mathbb{S}^1\times_{\Gamma_2^{(0)}}\ker p_2) & \longrightarrow & (Z/\mathbb{S}^1\times_{\Gamma_1^{(0)}}\ker p_1)\otimes(Z/\mathbb{S}^1\times_{\Gamma_2^{(0)}}\ker p_2)\\ (([z_1],u_1),([z_2],u_2)) & \longmapsto & ([\tau(z_1)],\sigma_1(u_1))\otimes([\tau(z_2)],\sigma_2(u_2))\end{array} \tag{5.69}$$

Then, $(\mathcal{X},\kappa)$ is a Real line bundle over $(Z/\mathbb{S}^1,\tau)$, and then is a graded Real Fell bundle (with the trivial grading). Furthermore, it is not hard to verify that $((\mathcal{X},\kappa),(Z/\mathbb{S}^1,\tau))$ implements an equivalence of graded Real Fell systems $\big((L_1,\sigma),(\Gamma_1,\varrho_1)\big)\sim\big((L_2,\sigma_2),(\Gamma_2,\varrho_2)\big)$. Therefore, our assertion follows from Theorem 5.6.7. $\square$

There is another picture of the reduced Rg C^*-algebra of a Rg $\mathbb{S}^1$-central extension. Indeed, let $\mathbb{E}=(\mathbb{S}^1\longrightarrow\widetilde{\Gamma}\xrightarrow{\pi}\Gamma,\delta,P)$ be as previously. Consider the trivial Rg Fell bundle $(\mathbf{1},\tilde{\varrho})\longrightarrow(\widetilde{\Gamma},\varrho)$, where $\mathbf{1}:=\widetilde{\Gamma}\times\mathbb{C}$, together with the grading $\tilde{\delta}$ and Real structure $\tilde{\varrho}$ respectively given by:

$$\tilde{\delta}(\tilde{\gamma},z)=(\tilde{\gamma},z.\delta(\pi(\tilde{\gamma}))),\ \text{and}\ \tilde{\varrho}(\tilde{\gamma},z)=(\tilde{\varrho}(\tilde{\gamma}),\bar{z}),\ \forall(\tilde{\gamma},z)\in\mathbf{1}.$$

Note that the *-involution on the fibers of $(\mathbf{1},\tilde{\varrho})$ is the complex conjugation. We will write $C_c(\widetilde{\Gamma})$ (resp. $C^*(\widetilde{\Gamma})$, $C^*_r(\widetilde{\Gamma})$) for $\Gamma_c(\widetilde{\Gamma};\mathbf{1})$ (resp. for $C^*(\widetilde{\Gamma};\mathbf{1})$, $C^*_r(\widetilde{\Gamma};\mathbf{1})$). The reason why we use these notations is that a compacty supported continuous section of $(\mathbf{1},\tilde{\varrho})$ can be seen as a compacty supported complex valued function on $\widetilde{\Gamma}$. Notice that our definitions of the C^*-algebras $C^*(\widetilde{\Gamma})$ and $C^*_r(\widetilde{\Gamma})$ does not differ from that given by Renault in [76]. However, the convolution algebra $C_c(\widetilde{\Gamma})$ is equipped with the grading $\xi\longmapsto\tilde{\delta}(\xi)$, with $\tilde{\delta}(\xi)(\tilde{\gamma}):=\xi(\tilde{\gamma}).\delta(\pi(\tilde{\gamma}))$ for $\tilde{\gamma}\in\widetilde{\Gamma}$, and the Real structure $\tilde{\varrho}$ given by $\tilde{\varrho}(\xi)(\tilde{\gamma})=\overline{\xi(\tilde{\varrho}(\tilde{\gamma}))}\in\mathbf{1}_{\tilde{\varrho}(\tilde{\gamma})}=\mathbb{C}$.

Definition 5.7.4. *Let* $\mathbb{E}=(\mathbb{S}^1\longrightarrow\widetilde{\Gamma}\xrightarrow{\pi}\Gamma,\delta,P)$ *be a Rg* $\mathbb{S}^1$*-central extension of* $(\mathcal{G},\rho)$. *Then we put*

$$C_c(\widetilde{\Gamma})^{\mathbb{S}^1}:=\left\{\xi\in C_c(\widetilde{\Gamma})\mid\xi(t\tilde{\gamma})=t^{-1}\xi(\tilde{\gamma}),\ \forall t\in\mathbb{S}^1,\tilde{\gamma}\in\widetilde{\Gamma}\right\}.$$

Lemma 5.7.5. *Let* $\mathbb{E}=(\mathbb{S}^1\longrightarrow\widetilde{\Gamma}\xrightarrow{\pi}\Gamma,\delta,P)$ *be as above. Then, endowed with the grading* $\tilde{\delta}$ *and the Real structure* $\tilde{\varrho}$ *induced from that of* $C_c(\widetilde{\Gamma})$, $(C_c(\widetilde{\Gamma})^{\mathbb{S}^1},\tilde{\varrho})$ *is a Rg convolution subalgebra of* $(C_c(\widetilde{\Gamma}),\tilde{\varrho})$.

Proof. If $\xi\in C_c(\widetilde{\Gamma})^{\mathbb{S}^1}$, then for all $\tilde{\gamma}\in\widetilde{\Gamma}$ and $t\in\mathbb{S}^1$, one has

$$(\tilde{\delta}\xi)(t\tilde{\gamma})=\xi(t\tilde{\gamma})\delta(\pi(t\tilde{\gamma}))=t^{-1}\xi(\tilde{\gamma})\delta(\pi(\tilde{\gamma})),\ \text{and}$$

$$(\tilde{\varrho}\xi)(t\tilde{\gamma})=\overline{\xi(\tilde{\varrho}(t\tilde{\gamma}))}=\overline{\xi(\bar{t}\tilde{\varrho}(\tilde{\gamma}))}=t^{-1}\overline{\xi(\tilde{\varrho}(\tilde{\gamma}))};$$

hence $\tilde{\delta}\xi,\tilde{\varrho}\xi\in C_c(\widetilde{\Gamma})^{\mathbb{S}^1},\forall\xi\in C_c(\widetilde{\Gamma})^{\mathbb{S}^1}$.
Now, suppose that $\xi,\eta\in C_c(\widetilde{\Gamma})^{\mathbb{S}^1}$. Then, for all $\tilde{\gamma}\in\widetilde{\Gamma}$ and $t\in\mathbb{S}^1$,

$$\xi*\eta(t\tilde{\gamma})=\int_{\widetilde{\Gamma}^{\tilde{\gamma}}}\xi(\tilde{h})\eta(\tilde{h}^{-1}t\tilde{\gamma})d\mu_{\widetilde{\Gamma}}^{r(\tilde{\gamma})}(\tilde{h})$$

$$= t^{-1}\int_{\widetilde{\Gamma}^{r(\tilde{\gamma})}} \xi(\tilde{h})\eta(\tilde{h}^{-1}\tilde{\gamma})d\mu_{\widetilde{\Gamma}}^{r(\tilde{\gamma})}(\tilde{h})$$
$$= t^{-1}(\xi*\eta)(\tilde{\gamma});$$

and $\xi^*(t\tilde{\gamma}) = \overline{(\xi(t^{-1}\tilde{\gamma}^{-1})} = \overline{t\xi(\tilde{\gamma}^{-1})} = t^{-1}\overline{\xi(\tilde{\gamma}^{-1})} = t^{-1}\xi^*(\tilde{\gamma})$. Thus, $C_c(\widetilde{\Gamma})^{\mathbb{S}^1}$ is closed under both the convolution and the involution in $C_c(\widetilde{\Gamma})$. □

Definition 5.7.6 (The reduced $\mathbb{S}^1$-equivariant C^*-algebra)**.** *The (Rg) reduced $\mathbb{S}^1$-equivariant C^*-algebra $(C^*_r(\widetilde{\Gamma})^{\mathbb{S}^1},\tilde{\varrho})$ of $\mathbb{E}$ is defined by setting*

$$C^*_r(\widetilde{\Gamma})^{\mathbb{S}^1} := \overline{\pi_l(C_c(\widetilde{\Gamma})^{\mathbb{S}^1})} \subset C^*_r(\widetilde{\Gamma};\mathbf{1}).$$

Example 5.7.7. *Let the Real groupoid $(\mathbb{S}^1,^-)$ be equipped with the Real normalized Haar system $dt := \frac{d\theta}{2\pi}$. Then all $f\in \mathcal{C}_c(\mathbb{S}^1)^{\mathbb{S}^1}$ is completely determined by its value at $1\in\mathbb{S}^1$; for if $t,t'\in\mathbb{S}^1$, then $f(tt') = t^{-1}f(t')$, and in particular,*

$$\forall t\in\mathbb{S}^1,\ f(t) = t^{-1}f(1).$$

This yields to an isomorphism of Real algebras

$$\mathcal{C}_c(\mathbb{S}^1)^{\mathbb{S}^1} \ni f \longmapsto f(1)\in\mathbb{C}.$$

Moreover, a simple calculation gives

$$\|f\|_{C^*_r(\mathbb{S}^1)^{\mathbb{S}^1}} = |f(1)|,$$

so that $C^*_r(\mathbb{S}^1)^{\mathbb{S}^1}\cong\mathbb{C}$.

Our purpose now is to connect $(C^*_r(\mathbb{E}),\sigma)$ to $(C^*_r(\widetilde{\Gamma})^{\mathbb{S}^1},\tilde{\varrho})$, since this latter seems to be likely an easy picture for us to work with.

Remark 5.7.8. *Note that $C^*_r(\widetilde{\Gamma})^{\mathbb{S}^1}$ can also be seen as the closure of $C_c(\widetilde{\Gamma})^{\mathbb{S}^1}$ in $C^*_r(\widetilde{\Gamma};\mathbf{1})$ with respect to the operator norm (see [90, p.865]).*

Lemma 5.7.9. *Given $\xi\in\mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1}$, set for all $\gamma\in\Gamma$:*

$$\tilde{\tilde{\xi}}(\gamma) := [(\tilde{\gamma},\xi(\tilde{\gamma}))]\in\widetilde{\Gamma}_\gamma\times_{\mathbb{S}^1}\mathbb{C}. \tag{5.70}$$

Then this formula provides a section $\tilde{\tilde{\xi}}\in\mathcal{C}_c(\Gamma;L)$. Moreover, it defines an isometric isomorphism of Rg convolution algebras $(\mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1},\tilde{\varrho}) \longrightarrow (\mathcal{C}_c(\Gamma;L),\sigma)$.

Proof. If $\tilde{\gamma},\tilde{\gamma}'\in\widetilde{\Gamma}_\gamma$, then there exists a unique $t\in\mathbb{S}^1$ such that $\tilde{\gamma}'=t\tilde{\gamma}$; so $[(\tilde{\gamma}',\xi(\tilde{\gamma}'))] = [(t\tilde{\gamma},t^{-1}\xi(\tilde{\gamma}))] = [(\tilde{\gamma},\xi(\tilde{\gamma}))] = \tilde{\tilde{\xi}}(\gamma)$. This shows that the section $\tilde{\tilde{\xi}}:\Gamma\longrightarrow L$ is well-defined, for all $\xi\in\mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1}$. Since ξ and π are continuous, so is $\tilde{\tilde{\xi}}$. Moreover, $\mathrm{supp}\tilde{\tilde{\xi}}$ is compact, for if $\gamma\in\mathrm{supp}\tilde{\tilde{\xi}}$, then $\widetilde{\Gamma}_\gamma = \pi^{-1}(\gamma)\subset\mathrm{supp}\xi$ and this means that $\mathrm{supp}\tilde{\tilde{\xi}}\subset\pi(\mathrm{supp}\xi)$. It is immediate that if $\xi_i \longrightarrow \xi$ in $\mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1}$ with respect to the inductive limit topology, then $\tilde{\xi}_i \longrightarrow \tilde{\xi}$ in $\mathcal{C}_c(\Gamma;L)$ with respect to the inductive limit topology. We then have a continuous map $\mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1} \longrightarrow \mathcal{C}_c(\Gamma;L), \xi \longmapsto \tilde{\xi}$. To see that this map respects the gradings and the Real structures, observe that

$$\epsilon(\tilde{\xi})(\gamma) = \epsilon([(\tilde{\gamma},\xi(\tilde{\gamma}))]) = [(\tilde{\gamma},\xi(\tilde{\gamma}).\delta(\gamma))] = [(\tilde{\gamma},(\tilde{\delta}\xi)(\tilde{\gamma}))] = (\widetilde{\tilde{\delta}\xi})(\gamma),$$

and

$$\sigma(\tilde{\xi})(\gamma) = \sigma(\tilde{\xi}(\gamma)) = \sigma([(\tilde{\varrho}(\tilde{\gamma}),\xi(\tilde{\varrho}(\tilde{\gamma})))]) = [(\tilde{\gamma},\overline{\xi(\tilde{\varrho}(\tilde{\gamma}))})] = [(\tilde{\gamma},\tilde{\varrho}(\xi)(\tilde{\gamma}))] = (\widetilde{\tilde{\varrho}\xi})(\gamma).$$

Now let $\xi,\eta \in \mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1}$. Then

$$\begin{aligned}(\tilde{\xi} * \tilde{\eta})(\gamma) &= \int_{\Gamma^{r(\gamma)}} [(\tilde{h},\xi(\tilde{h}))].[\tilde{h}^{-1}\tilde{\gamma},\eta(\tilde{h}^{-1}\tilde{\gamma})]\,d\mu_\Gamma^{r(\gamma)}(\dot{h}) \\ &= \int_{\Gamma^{r(\gamma)}} [(\tilde{\gamma},\xi(\tilde{h})\eta(\tilde{h}^{-1}\tilde{\gamma}))]d\mu_\Gamma^{r(\gamma)}(\dot{h}) \\ &= \left[\left(\tilde{\gamma},\int_{\Gamma^{r(\tilde{\gamma})}} \xi(\tilde{h})\eta(\tilde{h}^{-1}\tilde{\gamma})d\mu_\Gamma^{r(\tilde{\gamma})}(\dot{h})\right)\right] \\ &= [(\tilde{\gamma},(\xi*\eta)(\tilde{\gamma}))] = (\widetilde{\xi*\eta})(\gamma),\ \forall \gamma \in \Gamma.\end{aligned}$$

Also, $(\tilde{\xi})^*(\gamma) = \tilde{\xi}(\gamma^{-1})^* = [(\tilde{\gamma}^{-1},\xi(\tilde{\gamma}^{-1}))]^* = [(\tilde{\gamma},\overline{\xi(\tilde{\gamma}^{-1})})] = [(\tilde{\gamma},\xi^*(\tilde{\gamma}))] = (\widetilde{\xi^*})(\gamma)$. We then have shown that the map defined by (5.70) is a homomorphism of Rg convolution algebras. The remaining of the proof is obvious. □

Proposition 5.7.10. *Let* $\mathbb{E} = (\mathbb{S}^1 \longrightarrow \widetilde{\Gamma} \xrightarrow{\pi} \Gamma, \delta, P)$ *be a Rg* $\mathbb{S}^1$*-central extension of* $(\mathcal{G},\rho)$. *Then, as Rg* C^**-algebras,*

$$(C_r^*(\widetilde{\Gamma})^{\mathbb{S}^1},\tilde{\varrho}) \cong (C_r^*(\mathbb{E}),\sigma).$$

Proof. In view of Lemma 5.7.9, it suffices to show that for $\xi \in \mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1}$, we have

$$\|\xi\|_{C_r^*(\widetilde{\Gamma})} = \|\tilde{\xi}\|_{C_r^*(\Gamma;L)} =: \|\tilde{\xi}\|_{C_r^*(\mathbb{E})},$$

where $\tilde{\xi} \in \mathcal{C}_c(\Gamma;L)$ is the element defined by (5.70). We refer to [90, pp.865-866] for the proof of this equality. □

Corollary 5.7.11. *Let* $(\mathcal{G},\rho)$ *be as usual, and let the Real groupoid* $(\mathbb{S}^1,{}^-)$ *be equipped with the Real Haar system* dt *as in Example 5.7.7. We endow* $(\mathcal{G}\times\mathbb{S}^1 \rightrightarrows X, \rho\times{}^-)$ *with the Real Haar system define as the product* $\mu\times dt$*; that is,*

$$(\mu\times dt)^x(f) := \int_{\mathcal{G}^x}\int_{\mathbb{S}^1} f(g,t)\,dt\,d\mu^x(g),\ \forall f\in\mathcal{C}_c(\mathcal{G}\times\mathbb{S}^1),\ and\ x\in X. \tag{5.71}$$

Then, we have a Morita equivalence of Rg C^**-algebras (with trivial gradings):*

$$C_r^*(\mathcal{G}) \sim_{Morita} C_r^*(\mathcal{G}\times\mathbb{S}^1)^{\mathbb{S}^1}.$$

Proof. Consider the trivial Rg $\mathbb{S}^1$-central extension

$$\mathbb{E}_0 = (\ \mathbb{S}^1 \longrightarrow \mathcal{G}\times\mathbb{S}^1 \longrightarrow \mathcal{G}\ , 0, \mathcal{G})$$

of $(\mathcal{G},\rho)$. Then, the map

$$(\mathcal{G}\times\mathbb{S}^1)\times_{\mathbb{S}^1}\mathbb{C}\ni[(g,t),z]\longmapsto(g,tz)\in\mathcal{G}\times\mathbb{C}$$

defines an isomorphism of Real line bundles $L\xrightarrow{\cong}\mathcal{G}\times\mathbb{C}\ =: \mathbf{1}$ over $(\mathcal{G},\rho)$. Hence, by applying Proposition 5.7.10 and Theorem 5.6.7, we get

$$C_r^*(\mathcal{G}\times\mathbb{S}^1)^{\mathbb{S}^1}\cong C_r^*(\mathbb{E}_0):=C_r^*(\mathcal{G};L)\sim_{Morita}C_r^*(\mathcal{G};\mathbf{1})=:C_r^*(\mathcal{G}).$$

□

Remark 5.7.12. *Note that the last corollary could be proved directly by observing that the well-defined map*

$$\mathcal{C}_c(\mathcal{G})\ni f\longmapsto\left(\tilde{f}:(g,t)\longmapsto t^{-1}f(g)\right)\in\mathcal{C}_c(\mathcal{G}\times\mathbb{S}^1)^{\mathbb{S}^1},$$

is an isomorphism of Real convolution algebras which is isometric with respect to the reduced norms.

At this point, since we have an isomorphism of Abelian groups $\widehat{\mathrm{BrR}}_0(\mathcal{G})\cong\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$, we may study the relation between the two definitions of the Rg reduced C^*-algebra of an element of $\widehat{\mathrm{BrR}}_0(\mathcal{G})$ (cf. Definition 5.2.8 and that of an element of $\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$. (cf. Definition 5.7.2).

Theorem 5.7.13. *Let $\mathbb{E}\in\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$, and let $(\mathcal{A}_{\mathbb{E}},\alpha_{\mathbb{E}},\sigma_{\mathbb{E}})\in\widehat{\mathrm{BrR}}_0(\mathcal{G})$ be its corresponding Rg DD-bundle of type* 0. *Then, as Rg C^*-algebras, we have*

$$\mathcal{A}_{\mathbb{E}}\rtimes_r\mathcal{G}\sim_{Morita}C_r^*(\mathbb{E}^{\mathrm{op}}).$$

Proof. Write $\mathbb{E}=(\ \mathbb{S}^1\longrightarrow\tilde{\Gamma}\xrightarrow{\pi}\Gamma\ ,\delta,Z)$, where $(Z,\tau):(\mathcal{G},\rho)\longrightarrow(\Gamma,\varrho)$ is a Morita equivalence. Recall (see the previous chapter) that $\mathcal{A}_{\mathbb{E}}:=\mathcal{A}^Z$, where $(\mathcal{A},\alpha,\sigma)\in\widehat{\mathrm{BrR}}_0(\Gamma)$ is given by $\mathcal{A}:=\widehat{\mathcal{K}}(\mathsf{H}^{\tilde{\Gamma}})$, where $\mathsf{H}^{\tilde{\Gamma}}:=\coprod_{x\in\Gamma^{(0)}}L^2(\tilde{\Gamma}^x,\mathcal{H}_0)^{\mathbb{S}^1}$. Then, from Corollary 5.6.9, we have

$$\mathcal{A}_{\mathbb{E}}\rtimes_r\mathcal{G}\sim_{Morita}\mathcal{A}\rtimes_r\Gamma:=C_r^*(\Gamma;s^*\widehat{\mathcal{K}}(\mathsf{H}^{\tilde{\Gamma}})).$$

Thus, we only have to show that

$$C_r^*(\Gamma;s^*\mathcal{A})\sim_{Morita}C_r^*(\Gamma;L)=:C_r^*(\mathbb{E}),\ \text{where } L:=\tilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C},\tag{5.72}$$

and then we will apply Proposition 5.7.10. However, again in view of the Renault's equivalence Theorem 5.6.7, it suffices to built an equivalence between the Rg Fell systems $\left((s^*\mathcal{A},s^*\sigma),(\Gamma,\varrho)\right)$

and $\big((L,\tilde{\varrho}\times{}^{-}),(\Gamma,\varrho)\big)$.
Consider the Rg Banach bundle $\mathcal{X}:=s^*\mathsf{H}^{\widetilde{\Gamma}}$ over (Γ,ϱ) defined as the pull-back of the Rg Hilbert Γ-bundle $\mathsf{H}^{\widetilde{\Gamma}}\longrightarrow\Gamma^{(0)}$ through the source map of Γ. We claim that $\mathcal{X}$ implements the desired equivalence over (Γ,ϱ); that is, that

$$(s^*\mathcal{A},\Gamma)\sim_{(\mathcal{X},\Gamma)}(L,\Gamma). \tag{5.73}$$

First, recall that the Real Γ-action on $\mathsf{H}^{\widetilde{\Gamma}}$ is given by the graded unitaries

$$L^2(\widetilde{\Gamma}^{s(\gamma)},\mathcal{H}_0)^{\mathbb{S}^1}\ni\xi\longmapsto\big(\gamma\xi:\widetilde{\Gamma}^{r(\gamma)}\ni\tilde{h}\longmapsto\xi(\tilde{\gamma}^{-1}\tilde{h})\in\mathcal{H}_0\big)\in L^2(\widetilde{\Gamma}^{r(\gamma)},\mathcal{H}_0)^{\mathbb{S}^1},$$

where $\tilde{\gamma}$ is any lift of γ along the projection $\pi:\widetilde{\Gamma}\longrightarrow\Gamma$. We then have a Rg left (resp. right) action of $s^*\mathcal{A},s^*\sigma$ (resp. of L)on $\mathcal{X}$ from the well-defined maps

$$\begin{array}{ccc}\widehat{\mathcal{K}}(L^2(\widetilde{\Gamma}^{s(\gamma_1)},\mathcal{H}_0)^{\mathbb{S}^1})\times L^2(\widetilde{\Gamma}^{s(\gamma_2)},\mathcal{H}_0)^{\mathbb{S}^1} & \longrightarrow & L^2(\widetilde{\Gamma}^{s(\gamma_1\gamma_2)},\mathcal{H}_0)^{\mathbb{S}^1}\\ (T\quad,\quad\xi) & \longmapsto & T\cdot\xi:=\gamma_2^{-1}T(\gamma_2\xi),\end{array} \tag{5.74}$$

and

$$\begin{array}{ccc}L^2(\widetilde{\Gamma}^{s(\gamma_2)},\mathcal{H}_0)^{\mathbb{S}^1}\times\big(\widetilde{\Gamma}_{\gamma_3}\times_{\mathbb{S}^1}\mathbb{C}\big) & \longrightarrow & L^2(\widetilde{\Gamma}^{s(\gamma_2\gamma_3)},\mathcal{H}_0)^{\mathbb{S}^1}\\ (\xi\quad,\quad[\tilde{\gamma},\lambda]) & \longmapsto & \xi\cdot[\tilde{\gamma},\lambda]:=\gamma_3^{-1}\lambda\xi\end{array} \tag{5.75}$$

The maps (5.74) and (5.75) are continuous since the Γ-actions are continuous. Also, they are full and graded since the actions are, in fact, graded isomorphisms.
We now construct the $s^*\mathcal{A}$-valued and L-valued inner products $\mathcal{X}*\overline{\mathcal{X}}\longrightarrow s^*\mathcal{A}_{\Gamma<}$ and $\overline{\mathcal{X}}\times\mathcal{X}\longrightarrow L_{>_\Gamma}$, respectively. Observe that, as in Example 5.3.14, $\Gamma^{-1}=\{\gamma^{-1}\mid\gamma\in\Gamma\}$, if $(\gamma,\flat(\gamma'))\in\Gamma\times_{\Gamma^{(0)}}\Gamma^{-1}$ (in other words, $s(\gamma)=s(\gamma')$), then ${}_\Gamma<\gamma,\gamma'>=\gamma\gamma'^{-1}$, and if $(\flat(\gamma'),\gamma")\in\Gamma^{-1}\times_{\Gamma^{(0)}}\Gamma$ (i.e. $r(\gamma')=r(\gamma")$), then $<\gamma',\gamma">_\Gamma=\gamma'^{-1}\gamma"$. We then define these inner products as

$$\begin{array}{ccc}\mathcal{X}_\gamma\times\overline{\mathcal{X}}_{\gamma'^{-1}} & \longrightarrow & \mathcal{A}_{s(\gamma\gamma'^{-1})}=\widehat{\mathcal{K}}(L^2(\widetilde{\Gamma}^{r(\gamma')},\mathcal{H}_0))^{\mathbb{S}^1}\\ (\xi,\flat(\eta)) & \longmapsto & {}_{s^*\mathcal{A}}\langle\xi,\eta\rangle:=\theta_{\gamma'\xi,\gamma'\eta}\end{array} \tag{5.76}$$

where for $\zeta,\zeta'\in L^2(\widetilde{\Gamma}^x,\mathcal{H}_0)^{\mathbb{S}^1}$, $\theta_{\zeta,\zeta'}\in\widehat{\mathcal{K}}(L^2(\widetilde{\Gamma}^x,\mathcal{H}_0))^{\mathbb{S}^1}$ is the rank one operator

$$L^2(\widetilde{\Gamma}^x,\mathcal{H}_0)^{\mathbb{S}^1}\ni\zeta"\longmapsto(\langle\zeta',\zeta"\rangle_x)\zeta\in L^2(\widetilde{\Gamma}^x,\mathcal{H}_0)^{\mathbb{S}^1},\ x\in\Gamma^{(0)};$$

and

$$\begin{array}{ccc}\overline{\mathcal{X}}_{\gamma'^{-1}}\times\mathcal{X}_{\gamma"} & \longrightarrow & L_{\gamma'^{-1}\gamma"}=\widetilde{\Gamma}_{\gamma'^{-1}\gamma"}\times_{\mathbb{S}^1}\mathbb{C}\\ (\flat(\xi),\eta) & \longmapsto & \langle\xi,\eta\rangle_L:=\Big[\tilde{\gamma}'^{-1}\tilde{\gamma}",\langle\gamma'\xi,\gamma"\eta\rangle_{r(\gamma')}\Big]\end{array} \tag{5.77}$$

where $\tilde{\gamma}'$ and $\tilde{\gamma}"$ are any lifts of γ' and $\gamma"$, respectively. Recall (see the last chapter) that for $x\in Y$, the scalar product $\langle\cdot,\cdot\rangle_x$ on $\mathsf{H}^{\widetilde{\Gamma}}_x=\mathcal{X}_x$ is defined as

$$\langle\xi,,\eta\rangle_x=\int_{\widetilde{\Gamma}^x}\langle\xi(\tilde{h}),(\eta)(\tilde{h})\rangle_{\mathbb{C}}d\mu^x_{\widetilde{\Gamma}}(\tilde{h})\in\mathbb{C}.$$

The algebraic properties of these maps are easy to check. The map (5.76) is full, for

$$\operatorname{span}\left\{\theta_{\zeta,\zeta'} \mid \zeta,\zeta' \in L^2(\widetilde{\Gamma}^{r(\gamma')},\mathcal{H}_0)^{\mathbb{S}^1}\right\}$$

is the ideal of finite-rank operators on $L^2(\widetilde{\Gamma}^{r(\gamma')},\mathcal{H}_0)^{\mathbb{S}^1}$ and the graded map

$$L^2(\widetilde{\Gamma}^{s(\gamma')},\mathcal{H}_0)^{\mathbb{S}^1} \longrightarrow L^2(\widetilde{\Gamma}^{r(\gamma')},\mathcal{H}_0)^{\mathbb{S}^1}$$

given by the Γ-action is an isomorphism of Hilbert spaces. The map (5.77) is clearly surjective (then full, of course). It remains only to verify that the compatibility condition (cf. Definition 5.3.10 (ii)) holds; that is, for any triple $(\gamma,\gamma'^{-1},\gamma") \in \Gamma \times_{\Gamma^{(0)}} \Gamma^{-1} \times_{\Gamma^{(0)}} \Gamma$,

$$\xi \cdot \langle \xi',\xi"\rangle_L = {}_{s^*\mathcal{A}}\langle \xi,\xi'\rangle \cdot \xi",\ \forall (\xi,\flat(\xi'),\xi") \in \mathcal{X}_\gamma \times \overline{\mathcal{X}}_{\gamma'^{-1}} \times \mathcal{X}_{\gamma"}. \tag{5.78}$$

One has

$$\begin{aligned}
\xi \cdot \langle \xi',\xi"\rangle_L &= \xi \cdot \left[\tilde{\gamma}'^{-1}\tilde{\gamma}", \langle \gamma'\xi',\gamma"\xi"\rangle_{r(\gamma')}\right] \\
&= \gamma"^{-1}\gamma' \cdot (\langle \gamma'\xi',\gamma"\xi"\rangle_{r(\gamma')})\xi \\
&= \gamma"^{-1} \cdot (\langle \gamma'\xi',\gamma"\xi"\rangle_{r(\gamma')})(\gamma'\xi) \\
&= \gamma"^{-1} \cdot \theta_{\gamma'\xi,\gamma'\xi'}(\gamma"\xi") \\
&= {}_{s^*\mathcal{A}}\langle \xi,\xi'\rangle \cdot \xi",
\end{aligned}$$

and the proof is completed. □

6
First notions of Twisted KR-Theory

6.1 Definitions and basic properties

In this section, we introduce twisted Real K-theory of locally compact Hausdorff Real groupoids in an operator theoretic point of view, using Kasparov's Real KK-theory developped in [46].

Recall that for any fixed difference $p-q \mod 8$, there is a covariant functor KR_{p-q} from the category of Real graded C^*-algebras from the category $\mathfrak{Ab}$ of abelian groups defined by

$$KR_{p-q}(A) := KKR(\mathbb{C}l_{p,q}, A) \cong KKR(\mathbb{C}, A\hat{\otimes}\mathbb{C}l_{q,p}).$$

where $\mathbb{C}$ is equipped with the complex conjugation as Real structure, and as in Appendix A, $\mathbb{C}l_{k,l}$ is the complex Clifford algebra endowed with the the Real structure $cl_{k,l}$ such that $(\mathbb{C}l_{k,l})_{\mathbb{R}} = Cl_{k,l}$.

Definition 6.1.1. *Let $\mathcal{G}$ be a locally compact Hausdorff, second countable Real groupoid with Real Haar system. Let $\alpha = (\mathbf{t},\delta,\mathbf{c}) \in \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{R}}) \times \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbf{S}^{1,1})$, and let $[\mathcal{A}] \in \widehat{Br\mathrm{R}}(\mathcal{G})$ be such that $DD(\mathcal{A}) = \alpha$. We define the* twisted Real K-theory *of $\mathcal{G}$ by*

$$KR_\alpha^{q-p}(\mathcal{G}^\bullet) = KR_{\mathcal{A}}^{q-p}(\mathcal{G}^\bullet) := KR_{p-q}(\mathcal{A} \rtimes_r \mathcal{G})\ [1].$$

We will often write $KR_\alpha(\mathcal{G}^\bullet)$ instead of $KR_\alpha^0(\mathcal{G}^\bullet)$.

Remark 6.1.2. *It is clear from Theorem 5.6.7 that, up to isomorphisms, $KR_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet)$ depends only on the class of $\mathcal{A}$ in $\widehat{Br\mathrm{R}}(\mathcal{G})$, and hence on the class of α. In particular, if $\alpha = (0,0,0)$, then $\mathcal{A} \rtimes_r \mathcal{G}$ is Morita equivalent to $C_r^*(\mathcal{G})$; so that $KR_{(0,0,0)}^{-j}(\mathcal{G}^\bullet) \cong KR_j(C_r^*(\mathcal{G}))$.*

[1] As in [90], we have used the notation $\mathcal{G}^\bullet$ to specify that we are working with the groupoid $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, not the space $\mathcal{G}$.

Definition 6.1.3. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *be a Real groupoid. Define the* Real K-theory groups $KR^{-j}(\mathcal{G}^\bullet)$ *to be the twisted* KR*-theory of* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *when the twisting* α *is the trivial class* $(0,0,0) \in \check{H}R^0(\mathcal{G}_\bullet, \mathsf{Inv}\hat{\mathfrak{K}}) \times \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbf{S}^{1,1})$*; i.e.,*

$$KR^{-j}(\mathcal{G}^\bullet) := KR^{-j}_{(0,0,0)}(\mathcal{G}^\bullet) = KR_j(C^*_r(\mathcal{G})).$$

When $\mathcal{G}$ is just a Real space X, our definition of the groups $KR^{-j}(X)$ obviously coincides with the one given by Atiyah in [6].

The following result is immediately deduced from Definition 6.1.1 and from Theorem 5.7.13.

Proposition 6.1.4. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *be a Real groupoid. Suppose that* $\alpha = (0, \delta, \mathbf{c}) \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \ltimes \check{H}R^2(\mathcal{G}_\bullet, \mathbf{S}^{1,1})$ *and* $\mathcal{A}$ *is a Real graded D-D bundle of type* 0 *over* $\mathcal{G}$ *realizing* α *in* $\widehat{Br\mathrm{R}}_0(\mathcal{G})$*. Then,*

$$KR^{-j}_\alpha(\mathcal{G}^\bullet) \cong KR_j(C^*_r(\mathbb{E}^{\mathrm{op}}_{\mathcal{A}})), \forall j \in \mathbb{Z},$$

where as usual, $\mathbb{E}_{\mathcal{A}}$ *is the Real graded* $\mathbf{S}^{1,1}$*-central extension of* $\mathcal{G}$ *realizing* $[\mathcal{A}]$ *in* $\widehat{Ext\mathrm{R}}(\mathcal{G}, \mathbf{S}^{1,1})$.

We will need the following definition in the sequel.

Definition 6.1.5. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *and* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *be Real groupoids with Real Haar systems* $\mu_{\mathcal{G}}$ *and* μ_Γ*, respectively. A strict Real homomorphism* $f : \Gamma \longrightarrow \mathcal{G}$ *is said to be* compatible *with the Haar systems if*

$$\int_{f(\Gamma)^{f(y)}} \phi(g)\, d\mu^{f(y)}_{\mathcal{G}}(g) = \int_{\Gamma^y} \phi(f(\gamma))\mu^y_\Gamma(\gamma), \forall \phi \in \mathcal{C}_c(\mathcal{G}), y \in Y.$$

Now from the theory of Real graded D-D bundles and of Kasparov's KKR-theory, we deduce the following properties.

Theorem 6.1.6. *Let* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *,* α*, and* $\mathcal{A}$ *be as above. Then,*

1. *(*Formal Bott periodicity*). For all* $j \in \mathbb{Z}$*, we have*

$$KR^{-j-8}_{\mathcal{A}}(\mathcal{G}^\bullet) \cong KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet).$$

2. *(*Bott periodicity*). For all* $p, q \in \mathbb{N}$*,* $j \in \mathbb{Z}$*, we have*

$$KR^{-j+q-p}_{\mathcal{A}}(\mathcal{G}^\bullet) \cong KR^{-j}_{\mathcal{A}_{p,q}}((\mathcal{G} \times \mathbb{R}^{p,q})^\bullet),$$

   where $\mathcal{A}_{p,q}$ *is the Real graded D-D bundle over the Real groupoid*

$$\mathcal{G} \times \mathbb{R}^{p,q} \rightrightarrows X \times \mathbb{R}^{p,q}$$

   obtained by pulling back $\mathcal{A} \longrightarrow X$ *through the Real groupoid morphism* $\mathcal{G} \times \mathbb{R}^{p,q} \longrightarrow \mathcal{G}$ *given by the canonical projection into the first factor; the involution of* $\mathcal{G} \times \mathbb{R}^{p,q}$ *is the product one, and* $\mathbb{R}^{p,q}$ *is viewed as a Real space (i.e., we forget its group structure).*

3. *(*Functoriality in $\widehat{\mathfrak{BrR}}(\mathcal{G})$*). Let $\mathcal{G}$ be a fixed Real groupoid with Real Haar system μ. Let $\mathcal{A},\mathcal{B}$ be Real graded D-D bundles over $\mathcal{G}$. Then if*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\varphi"] \arrow[dr, "\pi_{\mathcal{A}}"'] & & \mathcal{B} \arrow[dl, "\pi_{\mathcal{B}}"] \\ & X & \end{tikzcd}$$

is a morphism of Real graded D-D bundles (see Chapter???), there is a canonical homomorphism of abelian groups

$$\varphi_* : KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \longrightarrow KR^*_{\mathcal{B}}(\mathcal{G}^\bullet). \tag{6.1}$$

Proof. 1) This is a consequence of [46, Theorem 5.5].
2) Thanks to the Bott periodicity theorem in KKR-theory ([46, Theorem 7.5]), one has

$$KR^{-j+q-p}_{\mathcal{A}}(\mathcal{G}^\bullet) \cong KKR_j(\mathbb{C}, \mathcal{C}_0(\mathbb{R}^{p,q}, \mathcal{A} \rtimes_r \mathcal{G})).$$

Therefore, the only thing to do is to verify that there is an isomorphism of Real graded C^*-algebras

$$\mathcal{C}_0(\mathbb{R}^{p,q}, \mathcal{A} \rtimes_r \mathcal{G}) \cong \mathcal{A}_{p,q} \rtimes_r (\mathcal{G} \times \mathbb{R}^{p,q}), \tag{6.2}$$

which is simple. Indeed, every $\xi \in \mathcal{C}_c(\mathcal{G} \times \mathbb{R}^{p,q}; s^*\mathcal{A} \times \mathbb{R}^{p,q})$ is of the form $\xi(g,t) = (f(g,t),t)$ for all $(g,t) \in \mathcal{G} \times \mathbb{R}^{p,q}$, where $f_t := f(\cdot,t) \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$. The map $\mathcal{C}_c(\mathcal{G} \times \mathbb{R}^{p,q}; s^*\mathcal{A} \times \mathbb{R}^{p,q}) \ni \xi \longmapsto (\mathbb{R}^{p,q} \ni t \longmapsto f(\cdot,t) \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})) \in \mathcal{C}_c(\mathbb{R}^{p,q}; \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A}))$ is an isomorphism of Real graded convolution algebras whose inverse is given by $\mathcal{G} \times \mathbb{R}^{p,q} \ni (g,t) \longmapsto (f(t)(g),t) \in \mathcal{A}_{s(g)} \times \mathbb{R}^{p,q}$ for $f \in \mathcal{C}_0(\mathbb{R}^{p,q}; \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A}))$. Moreover, $\mathcal{C}_c(\mathcal{G} \times \mathbb{R}^{p,q}; s^*\mathcal{A} \times \mathbb{R}^{p,q}) \cong \mathcal{C}_c(\mathbb{R}^{p,q}) \hat{\odot} \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$. Hence, since $\mathcal{A}_{p,q} \rtimes_r (\mathcal{G} \times \mathbb{R}^{p,q}) = C^*_r(\mathcal{G} \times \mathbb{R}^{p,q}; s^*\mathcal{A} \times \mathbb{R}^{p,q})$ and $\mathcal{C}_0(\mathbb{R}^{p,q}; \mathcal{A} \rtimes_r \mathcal{G}) \cong \mathcal{C}_0(\mathbb{R}^{p,q}) \hat{\otimes} C^*_r(\mathcal{G}; s^*\mathcal{A})$, we get (6.2) by passing to the completion of the Real graded subalgebra

$$\pi_l^{\mathcal{G} \times \mathbb{R}^{p,q}}(\mathcal{C}_c(\mathbb{R}^{p,q}) \hat{\odot} \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})) \cong \pi_l(\mathcal{C}_0(\mathbb{R}^{p,q})) \hat{\odot} \pi_l(\mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})) \subset \mathcal{L}(L^2(\mathcal{G} \times \mathbb{R}^{p,q}, s^*\mathcal{A} \times \mathbb{R}^{p,q})).$$

3) The map φ induces a homomorphism of Real graded convolution algebras

$$\varphi_* : \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A}) \longrightarrow \mathcal{C}_c(\mathcal{G}; s^*\mathcal{B})$$

by setting $(\varphi_*\xi)(g) := \varphi_{s(g)}(\xi(g))$ for $\xi \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A}), g \in \mathcal{G}$. Moreover, for all $\xi, \eta \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$ and $x \in X$, we have

$$\begin{aligned} \langle \varphi_*\xi, \varphi_*\eta \rangle_{\mathcal{B}_x} &= \int_{\mathcal{G}_x} \varphi_x(\xi(g)^*)\varphi_x(\eta(g))\, d\mu_x(g) \\ &= \varphi_x\left(\int_{\mathcal{G}_x} \xi(g)^*\eta(g)\, d\mu_x(g) \right) \\ &= \varphi_x(\langle \xi, \eta \rangle_{\mathcal{A}_x}); \end{aligned}$$

so that $\|\varphi_*\xi\|_{L^2(\mathcal{G}_x;s^*\mathcal{B})} \leq \|\varphi_x\| \cdot \|\xi\|_{L^2(\mathcal{G}_x;s^*\mathcal{A})}$. Hence we have

$$\begin{aligned}\|\varphi_*\xi\|_{\mathcal{B}\rtimes_r\mathcal{G}} =& \sup_{x\in X}\sup_{\|\eta\|\leq 1,\eta\in L^2(\mathcal{G};s^*\mathcal{B})}\|(\varphi_*\xi)*\eta\|_{L^2(\mathcal{G};s^*\mathcal{B})}\\ \leq& \sup_{x\in X}\sup_{\|\eta\|\leq 1,\zeta\in L^2(\mathcal{G};s^*\mathcal{A})}\|\varphi_*(\xi*\zeta)\|_{L^2(\mathcal{G};s^*\mathcal{B})}\\ \leq& \|\varphi\|\cdot\sup_{x\in X}\sup_{\|\zeta\|\leq 1,\zeta\in L^2(\mathcal{G};s^*\mathcal{A})}\|\xi*\zeta\|_{L^2(\mathcal{G};s^*\mathcal{A})} = \|\varphi\|\cdot\|\xi\|_{\mathcal{A}\rtimes_r\mathcal{G}}.\end{aligned}$$

Therefore, φ_* extends to a homomorphism of Real graded C^*-algebras $\varphi_* : \mathcal{A}\rtimes_r\mathcal{G} \longrightarrow \mathcal{G}\rtimes_r\mathcal{B}$, which yields the homomorphism $\varphi_* : KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \longrightarrow KR^*_{\mathcal{B}}(\mathcal{G})$. Moreover it is easy to check that if $\mathcal{A}\xrightarrow{\varphi}\mathcal{B}\xrightarrow{\psi}\mathcal{C}$ are two homomorphisms in $\widehat{\mathfrak{BrR}}(\mathcal{G})$, then $(\psi\circ\varphi)_* = \psi_*\circ\varphi_*$. □

Definition 6.1.7. *For a Real graded D-D bundle $\mathcal{A}$ over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$, we define the* gauge group $Aut_{\mathcal{G}}(\mathcal{A})$ *of $\mathcal{A}$ to be the set of automorphisms $\varphi : \mathcal{A}\longrightarrow\mathcal{A}$ in the category $\widehat{\mathfrak{BrR}}(\mathcal{G})$, where the group operation is composition of automorphisms.*

An immediate consequence of property 3 of the Theorem 6.1.6 is the following:

Corollary 6.1.8. *Let $\mathcal{A}\in\widehat{\mathfrak{BrR}}(\mathcal{G})$. The group $KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet)$ has the structure of $Aut_{\mathcal{G}}(\mathcal{A})$-module via the map*

$$Aut_{\mathcal{G}}(\mathcal{A})\times KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet)\longrightarrow KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet), (\varphi,x)\longmapsto\varphi_*x.$$

Notice that our theory is a generalization of the twisted KO-theory of locally compact spaces developped successively by Donovan-Karoubi [28], J. Rosenberg [78], and Mathai-Murray-Stevenson [58]. Indeed, when the Real structure of $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ is trivial, we have already seen that $\widehat{\mathrm{BrR}}(\mathcal{G})\cong\widehat{\mathrm{BrO}}(\mathcal{G})$ via the homomorphism $[\mathcal{A}]\longmapsto[\mathcal{A}_{\mathbb{R}}]$, where $\mathcal{A}_{\mathbb{R}}$ is the real graded D-D bundle whose fibre $(\mathcal{A}_{\mathbb{R}})_x$ is the subalgebra $(\mathcal{A}_x)_{\mathbb{R}}$ of the fixed points of $\mathcal{A}_x$ under the involution (cf. Chapter ???).

Proposition 6.1.9. *Assume the Real structure of the groupoid $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ is trivial. Then, for all $[\mathcal{A}]\in\widehat{\mathrm{BrR}}(\mathcal{G})$, there is a canonical isomorphism*

$$KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet)\cong KO_j(\mathcal{A}_{\mathbb{R}}\rtimes_r\mathcal{G}) = KKO_j(\mathbb{R},\mathcal{A}_{\mathbb{R}}\rtimes_r\mathcal{G}).$$

Proof. First, note that if A and B are Real graded C^*-algebras, there is a canonical isomorphism

$$KKR(A,B)\xrightarrow{\cong}KKO(A_{\mathbb{R}},B_{\mathbb{R}}), \tag{6.3}$$

given as follows. If $(E,\varphi,F)\in ER(A,B)$ (see [46] for the definitions), then define the "realification" $(E,\varphi,F)_{\mathbb{R}}$ of (E,φ,F) as $(E_{\mathbb{R}},\varphi_{\mathbb{R}},F_{\mathbb{R}})$, where $E_{\mathbb{R}}$ is the Real part of E, $\varphi_{\mathbb{R}}$ is the restriction of the homomorphism of Real graded C^*-algebras $\varphi : A\longrightarrow\mathcal{L}_B(E)$ to $A_{\mathbb{R}}$, and $F_{\mathbb{R}}$ is the restrction of the degree 1 operator $F : E\longrightarrow E\in\mathcal{L}_B(E)_{\mathbb{R}}$ to the real graded Hilbert

$B_{\mathbb{R}}$-module $E_{\mathbb{R}}$. The triple $(E_{\mathbb{R}},\varphi_{\mathbb{R}},F_{\mathbb{R}})$ obviously belongs to $EO(A_{\mathbb{R}},B_{\mathbb{R}})$. It is clear that $(E_{\mathbb{R}},\varphi_{\mathbb{R}},F_{\mathbb{R}})$ is degenerate if (E,φ,F) is. Furthermore, using the same construction, we see that if (E,φ,F) is a homotopy between (E_0,φ_0,F_0) and (E_1,φ_1,F_1) in $ER(A,B)$, then the "realification" of (E,φ,F) connects homotopically $((E_0)_{\mathbb{R}},(\varphi_0)_{\mathbb{R}},(F_0)_{\mathbb{R}})$ to $((E_1)_{\mathbb{R}},(\varphi_1)_{\mathbb{R}},(F_1)_{\mathbb{R}})$ in $EO(A_{\mathbb{R}},B_{\mathbb{R}})$. We then get a map $KKR(A,B)\longrightarrow KKO(A_{\mathbb{R}},B_{\mathbb{R}})$ which, by construction, is a homomorphism. Conversely, we obtain a homomorphism in the other way by sending every $(E,\varphi,F)\in EO(A_{\mathbb{R}},B_{\mathbb{R}})$ to its "complexification" $(E,\varphi,F)_{\mathbb{C}}:=(E_{\mathbb{C}},\varphi_{\mathbb{C}},F_{\mathbb{C}})\in ER(A,B)$ defined in the following way: $E_{\mathbb{C}}=E\dotplus iE$ is the usual complexification of E (see Appendix A); $\varphi_{\mathbb{C}}:A=A_{\mathbb{R}}\dotplus iA_{\mathbb{R}}\longrightarrow \mathcal{L}_B(E_{\mathbb{C}})$ is $\varphi_{\mathbb{C}}(a+ib):=\varphi(a)+i\varphi(b)$, and $F_{\mathbb{C}}:E_{\mathbb{C}}\longrightarrow E_{\mathbb{C}}$ is given by $F_{\mathbb{C}}:=F+iF\in\mathcal{L}_{B_{\mathbb{R}}\dotplus iB_{\mathbb{R}}}(E\dotplus iE)^1$. Observe that $((E,\varphi,F)_{\mathbb{C}})_{\mathbb{R}}\cong(E,\varphi,F)$ for all $(E,\varphi,F)\in EO(A_{\mathbb{R}},B_{\mathbb{R}})$, and $((E,\varphi,F)_{\mathbb{R}})_{\mathbb{C}}\cong(E,\varphi,F)$ for all $(E,\varphi,F)\in ER(A,B)$; hence, we have proved the isomorphism 6.3.

Now, to prove the proposition, it suffices to show that if the Real structure of $\mathcal{G}$ is trivial $\mathcal{E}\longrightarrow\mathcal{G}$ is a Real graded Fell bundle, we have an isomorphism of real graded C^*-algebras

$$C_r^*(\mathcal{G};\mathcal{E})_{\mathbb{R}}\cong C_r^*(\mathcal{G};\mathcal{E}_{\mathbb{R}}), \tag{6.4}$$

where $\mathcal{E}_{\mathbb{R}}\longrightarrow\mathcal{G}$ is the "realification" of $\mathcal{E}$. To see this, observe first that $\xi\in\mathcal{C}_c(\mathcal{G};\mathcal{E})_{\mathbb{R}}$ is and only if $\bar{\xi}(g)=\overline{\xi(g)}=\xi(g)$ for all $g\in\mathcal{G}$; hence, the real graded algebra $\mathcal{C}_(\mathcal{G};\mathcal{E})_{\mathbb{R}}$ identifies to $\mathcal{C}_c(\mathcal{G};\mathcal{E}_{\mathbb{R}})$, so that $\mathcal{C}_c(\mathcal{G};\mathcal{E})$ is a complexification of $\mathcal{C}_c(\mathcal{G};\mathcal{E}_{\mathbb{R}})$. Moreover, since the norm of $L^2(\mathcal{G};\mathcal{E}_{\mathbb{R}})$ is the one induced from that of $L^2(\mathcal{G};\mathcal{E})\cong L^2(\mathcal{G};\mathcal{E})_{\mathbb{R}}\dotplus iL^2(\mathcal{G};\mathcal{E})_{\mathbb{R}}$, the real graded Hilbert $\mathcal{C}_0(X;\mathcal{E}^{(0)})_{\mathbb{R}}$-modules $L^2(\mathcal{G};\mathcal{E})_{\mathbb{R}}$ and $L^2(\mathcal{G};\mathcal{E}_{\mathbb{R}})$ are isomorphic. Notice also that $\mathcal{C}_0(X;\mathcal{E}^{(0)})_{\mathbb{R}}\cong\mathcal{C}_0(C;\mathcal{E}^{(0)}_{\mathbb{R}})$ as real graded C^*-algebras. Now let $T\in\mathcal{L}_{\mathcal{C}_0(X;\mathcal{E}^{(0)})}(L^2(\mathcal{G};\mathcal{E}))$. Then, T is Real if and only if $T(\bar{\xi})=\overline{T(\xi)}$ for all $\xi\in L^2(\mathcal{G};\mathcal{E})$, if and only if there exists a unique $T_{\mathbb{R}}\in\mathcal{L}_{\mathcal{C}_0(X;\mathcal{E}^0)_{\mathbb{R}}}(L^2(\mathcal{G};\mathcal{E})_{\mathbb{R}})\cong\mathcal{L}_{\mathcal{C}_0(X;\mathcal{E}^{(0)}_{\mathbb{R}})}(L^2(\mathcal{G};\mathcal{E}_{\mathbb{R}}))$ such that $T(\xi)=T_{\mathbb{R}}(\xi_1)+iT_{\mathbb{R}}(\xi_2)$ for all $\xi=\xi_1+i\xi_2\in L^2(\mathcal{G};\mathcal{E})$. The map

$$\mathcal{L}_{\mathcal{C}_0(X;\mathcal{E}^{(0)})}(L^2(\mathcal{G};\mathcal{E}))_{\mathbb{R}}\ni T\longmapsto T_{\mathbb{R}}\in\mathcal{L}_{\mathcal{C}_0(X;\mathcal{E}^{(0)}_{\mathbb{R}})}(L^2(\mathcal{G};\mathcal{E}_{\mathbb{R}}))$$

is actually an isomorphism of real graded C^*-algebras. To complete the proof of 6.4, we just have to check that via this isomorphism, $(\pi_l(\mathcal{C}_c(\mathcal{G};\mathcal{E})))_{\mathbb{R}}\cong\pi_l(\mathcal{C}_c(\mathcal{G};\mathcal{E}_{\mathbb{R}}))$, which is straightforward. □

Definition 6.1.10. *Let* $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ *be given the trivial Real structure. Let* $[\mathcal{A}]\in\widehat{Br\mathrm{O}}(\mathcal{G})$ *and* $[\mathcal{A}_{\mathbb{C}}]\in\widehat{Br\mathrm{R}}(\mathcal{G})$ *its complexification. We define the* twisted orthogonal K-theory *of* $\mathcal{G}$ *by*

$$KO_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet):=KR_{\mathcal{A}_{\mathbb{C}}}^{-j}(\mathcal{G}^\bullet).$$

Example 6.1.11. *Suppose* X *is a locally compact space equipped with the trivial Real structure. We have* $\widehat{Br\mathrm{R}}(X)\cong\widehat{Br\mathrm{O}}(X)\cong\check{H}^0(X,\mathbb{Z}_8)\times\check{H}^1(X,\mathbb{Z}_2)\times\check{H}^2(X,\mathbb{Z}_2)$. *Then, if* $\alpha\in\widehat{Br\mathrm{O}}(X)$, *we have*

$$KR_\alpha^{-j}(X^\bullet)=KR_\alpha^{-j}(X)=KKO_j(\mathbb{R},\mathcal{C}_0(X,\mathcal{A})),$$

where $\mathcal{A} \in \widehat{Br\mathrm{O}}(X)$ *is any real graded D-D bundle over* X *realizing* α*. In particular, if* $\mathcal{A}$ *is of type* 0 *and trivially graded (i.e.,* $\alpha = (0,0,\mathbf{c})$*), then we recover Mathai-Murray-Stevenson's and Rosenberg's twisted* KO*-theory (see [58] and [78, §3]):*

$$KR_\alpha^{-j}(X) \cong KO_{\mathcal{A}}^{-j}(X) = KO_j(\mathcal{C}_0(X,\mathcal{A})).$$

As for twisted K-theory of topological spaces, there is an extension map in twisted Real K-theory of Real groupoids. Recall (cf. [90, p.868]) that a subgroupoid $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ of $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ is said to be *saturated* if Y is an invariant subset of X (i.e. $\mathcal{G}_Y^Y = \mathcal{G}_Y = \mathcal{G}^Y$) such that $\Gamma = \mathcal{G}_Y^Y$.

Proposition 6.1.12 (Extension map)**.** *Let* $[\mathcal{A}] \in \widehat{Br\mathrm{R}}(\mathcal{G})$*. Suppose that* $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ *is an open saturated Real subgroupoid of* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *(i.e.* $\bar{\gamma} \in \Gamma, \forall \gamma \in \Gamma$*). Then, the inclusion* $i : \Gamma \hookrightarrow \mathcal{G}$ *induces a canonical map*

$$i_* : KR^*_{\mathcal{A}_{|Y}}(\Gamma^\bullet) \longrightarrow KR^*_{\mathcal{A}}(\mathcal{G}^\bullet).$$

Another way to formulate Proposition 6.1.12 is as follows.

Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ be a Real groupoid with Real Haar system. Suppose that $U \subset X$ is a Real $\mathcal{G}$-invariant open subset of X. Then for $\mathcal{A} \in \widehat{\mathfrak{Br}\mathfrak{R}}(\mathcal{G})$, the inclusion map $i_U : \mathcal{G}_U \hookrightarrow \mathcal{G}$ induces an extension homomorphism

$$(i_U)_* : KR^*_{\mathcal{A}_{|U}}((\mathcal{G}_U)^\bullet) \longrightarrow KR^*_{\mathcal{A}}(\mathcal{G}^\bullet). \tag{6.5}$$

Proof. The proof is almost the same as the one of [90, Proposition 3.8]: we show that $\mathcal{A}_{|Y} \rtimes_r \Gamma$ is a Real graded ideal of $\mathcal{A} \rtimes_r \mathcal{G}$. Recall (cf. Chapter ??) that $\mathrm{supp}(\xi * \eta) \subset (\mathrm{supp}\,\xi)\cdot(\mathrm{supp}\,\eta)$, and $\mathrm{supp}\,\xi^* = (\mathrm{supp}\,\xi)^{-1}$. Thus, $\mathcal{C}_c(\Gamma; s^*\mathcal{A}_{|Y}) \subset \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$ is stable under the convolution and the adjoint. Further, since Γ is saturated, we have $\mathrm{supp}(\xi * \eta) \subset \Gamma$, and $\mathrm{supp}(\eta' * \xi') \subset \Gamma$ for all $\xi, \xi' \in \mathcal{C}_c(\Gamma; s^*\mathcal{A}_{|Y}), \eta, \eta' \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$. Now, using again the fact that $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ is saturated, we have

$$\begin{aligned}
\|\xi\|_{\mathcal{A}_{|\Gamma} \rtimes_r \Gamma} &= \sup_{y \in Y} \sup_{\|\eta\| \le 1, \eta \in L^2(\Gamma_y; s^*\mathcal{A}_{|\Gamma})} \|\xi * \eta\|_{L^2(\Gamma_y; s^*\mathcal{A}_{|\Gamma})} \\
&= \sup_{y \in Y} \sup_{\|\zeta\| \le 1, \zeta \in L^2(\mathcal{G}_y; s^*\mathcal{A})} \|\xi * \zeta\|_{L^2(\mathcal{G}_y; s^*\mathcal{A})} \\
&= \sup_{x \in X} \|\pi_x^{\mathcal{G}}(\xi)\| \\
&= \|\xi\|_{\mathcal{A} \rtimes_r \mathcal{G}},
\end{aligned}$$

and thus $\mathcal{A}_{|Y} \rtimes_r \Gamma$ is a sub-C^*-algebra of $\mathcal{A} \rtimes_r \mathcal{G}$. Moreover, $\mathcal{C}_c(\Gamma; s^*\mathcal{A}_{|Y})$ is obviously stable under the grading of $\mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$, and since Γ is invariant under the Real structure of $\mathcal{G}$, $\mathrm{supp}\,\bar{\xi} = \{\bar{\gamma}; \gamma \in \mathrm{suppp}\,\xi\} \subset \Gamma$, so that $\bar{\xi} \in \mathcal{C}_c(\Gamma; s^*\mathcal{A}_{|Y})$, for all $\xi \in \mathcal{C}_c(\Gamma; s^*\mathcal{A}_{|Y})$. Hence, $\mathcal{A}_{|Y} \rtimes \Gamma$ is a Real graded ideal of $\mathcal{A} \rtimes_r \mathcal{G}$. $\square$

Corollary 6.1.13. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be an* étale *Real groupoid. Then for* $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$, *there is an extension homomorphism*

$$KR^*_{\mathcal{A}}(X) \longrightarrow KR^*_{\mathcal{A}}(\mathcal{G}^\bullet),$$

induced by the canonical Real inclusion $X \hookrightarrow \mathcal{G}$, *where in the left hand side* $\mathcal{A}$ *is considered as a Real graded D-D bundle over the Real groupoid* $X \rightrightarrows X$ *(i.e., we just forget the* $\mathcal{G}$*-action).*

Proof. Recall [76] that a groupoid $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ is called *étale* (or r-discrete) if the unit space X is an open subset of $\mathcal{G}$. We then apply the above proposition to $X \rightrightarrows X$. □

Proposition 6.1.14. *(Compare with [90, Proposition 3.10]). Let* $[\mathcal{A}] \in \widehat{Br\mathbb{R}}(\mathcal{G})$. *Assume* $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ *is a closed saturated Real subgroupoid of* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$. *Then the Real inclusion* $\Gamma \hookrightarrow \mathcal{G}$ *induces a canonical extension map*

$$i^* : KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \longrightarrow KR^*_{\mathcal{A}_{|Y}}(\Gamma^\bullet).$$

Proof. The restriction map $\mathcal{C}_c(\mathcal{G}; s^*\mathcal{A}) \longrightarrow \mathcal{C}_c(\Gamma; s^*\mathcal{A}_{|Y})$ is evidently Real and graded, and is surjective since Γ is closed. It is moreover a *-homomorphism of convolution algebras, and by using the fact $\Gamma_y = \mathcal{G}_y$ for all $y \in Y$, we have

$$\|\xi_\Gamma\|_{\mathcal{A}_{|Y} \rtimes_r \Gamma} \leq \|\xi\|_{\mathcal{A} \rtimes_r \mathcal{G}}, \forall \xi \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A});$$

i.e. we have a sujective homomorphism of Real graded C^*-algebras $\mathcal{A} \rtimes \mathcal{G} \longrightarrow \mathcal{A}_{|Y} \rtimes \Gamma$, from which the result follows. □

To end this section, recall that if $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$ and $\mathcal{B} \in \widehat{\mathfrak{BrR}}(\Gamma)$ are such that the Fell systems $(\mathcal{G}, s^*\mathcal{A})$ and $(\Gamma, s^*\mathcal{B})$ are Morita equivalent, then the Real graded C^*-algebras $\mathcal{A} \rtimes_r \mathcal{G}$ and $\mathcal{B} \rtimes_r \Gamma$ are Morita equivalent. We thus have

Proposition 6.1.15. *Assume that* $(\mathcal{G}, s^*_{\mathcal{G}}\mathcal{A})$ *and* $(\Gamma, s^*_\Gamma\mathcal{B})$ *are Morita equivalent Real graded Fell systems, where* $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$ *and* $\mathcal{B} \in \widehat{\mathfrak{BrR}}(\Gamma)$. *Then,*

$$KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \cong KR^*_{\mathcal{B}}(\Gamma^\bullet).$$

In particular, twisted KR*-theory is invariant under Morita equivalences; i.e. if* $Z : \Gamma \longrightarrow \mathcal{G}$ *is a Real Morita equivalence, then*

$$KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \cong KR^*_{\mathcal{A}^Z}(\Gamma).$$

6.2 Relative twisted KR-groups

In this section we define the relative twisted KR-groups and establish some related exact sequences.

Definition 6.2.1 (The relative twisted KR-group). *Consider a pair* $(\mathcal{G},\Gamma)$ *consisting of a Real groupoid* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *and a closed saturated Real subgroupoid* $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$. *For* $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$, *we define the* relative twisted KR-groups *of the* $(\mathcal{G},\Gamma)$ *by*

$$KR^*_{\mathcal{A}}(\mathcal{G}^\bullet,\Gamma^\bullet) := KR^*_{\mathcal{A}_{|X\setminus Y}}((\mathcal{G}\setminus\Gamma)^\bullet).$$

Given such a pair $(\mathcal{G},\Gamma)$, we have that $\mathcal{G}\setminus\Gamma \rightrightarrows X\setminus Y$ is an open saturated Real subgroupoid of $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$. Denote by $i : \mathcal{G}\setminus\Gamma \hookrightarrow \mathcal{G}$ and $j : \Gamma \hookrightarrow \mathcal{G}$ the inclusions. Then for $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$, it is straightforward to check that the sequence

$$C^*_r(\mathcal{G}\setminus\Gamma, s^*\mathcal{A}_{|X\setminus Y}) \xrightarrow{i_*} C^*_r(\mathcal{G}; s^*\mathcal{A}) \xrightarrow{j^*} C^*_r(\Gamma; s^*\mathcal{A}_{|Y})$$

is exact in the middle. Hence we have

Proposition 6.2.2. *Let* $(\mathcal{G},\Gamma)$ *be as above, and let* $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$. *Then the Real inclusions* $i : \mathcal{G}\setminus\Gamma \hookrightarrow \mathcal{G}$ *and* $j : \Gamma \hookrightarrow \mathcal{G}$ *induces an exact sequence*

$$KR^*_{\mathcal{A}}(\mathcal{G}^\bullet,\Gamma^\bullet) \xrightarrow{i_*} KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \xrightarrow{j^*} KR^*_{\mathcal{A}_{|Y}}(\Gamma^\bullet). \tag{6.6}$$

Corollary 6.2.3. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a Real groupoid, and let* U *be an open Real* $\mathcal{G}$*-invariant subset of* X. *The for all* $\mathcal{A} \in \widehat{\mathfrak{BrR}}(\mathcal{G})$ *there is an exact sequence*

$$KR^{-j}_{\mathcal{A}_{|U}}((\mathcal{G}_U)^\bullet) \longrightarrow KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet) \longrightarrow KR^{-j}_{\mathcal{A}_{|F}}((\mathcal{G}_F)^\bullet), \tag{6.7}$$

where $F \subset X$ *is the complement of* U.

Remark 6.2.4. *From a result of J. Renault recorded as Proposition II.4.5 in [76], the sequence (6.7) can be written in terms of* C^**-algebraic* KR*-theory as*

$$KR_j(I(U)) \longrightarrow KR_j(\mathcal{A}\rtimes_r\mathcal{G}) \longrightarrow KR_j((\mathcal{A}\rtimes_r\mathcal{G})/I(U)),$$

where $I(U)$ *is the closure of the subspace* $\{\xi \in \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A}) \mid \xi(g) = 0 \text{ if } g \notin \mathcal{G}_U\} \subset \mathcal{C}_c(\mathcal{G}; s^*\mathcal{A})$ *with respect to the reduced norm.*

Lemma 6.2.5. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a Real groupoid such that the Real part* ${}^r\mathcal{G} \rightrightarrows {}^rX$ *is non-empty and saturated. Then for all* $\mathcal{A} \in \widehat{\mathrm{BrR}}(\mathcal{G})$, *the inclusion map* $j : {}^r\mathcal{G} \hookrightarrow \mathcal{G}$ *induces an exact sequence*

$$KR^*_{\mathcal{A}_{|{}^{\mathcal{I}}X}}({}^{\mathcal{I}}\mathcal{G}^\bullet) \xrightarrow{i_*} KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \xrightarrow{j^*} KO^*_{(\mathcal{A}_{|{}^rX})_{\mathbb{R}}}({}^r\mathcal{G}^\bullet).$$

Proof. The open Real subgroupoid ${}^{\mathcal{I}}\mathcal{G} \rightrightarrows {}^{\mathcal{I}}X$ of $\mathcal{G}$ is saturated. The result is then an immediate consequence of Proposition 6.1.9 and the exact sequence (6.7). □

6.3 Mayer-Vietoris exact sequence

In this section we establish the long exact sequence on twisted KR-theory associated with two open saturated Real subgroupoids of $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$. More specifically, we prove the following theorem (compare with [90, Proposition 3.9] in the ungraded complex case).

Theorem 6.3.1. *Suppose that $\mathcal{G}_i \rightrightarrows X_i$, $i = 1,2$ are open saturated Real subgroupoid of $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ such that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. Let $\mathcal{G}_{12} := \mathcal{G}_1 \cap \mathcal{G}_2$, and for $[\mathcal{A}] \in \widehat{\mathrm{BrR}}(\mathcal{G})$, let $[\mathcal{A}_1], [\mathcal{A}_2]$, and $[\mathcal{A}_{12}]$ be the classes of the obvious restrictions of $\mathcal{A}$. Then we have a 8-periodic long exact sequence of groups*

$$\begin{aligned}\ldots \xrightarrow{\partial} KR^{-j-1}_{\mathcal{A}_{12}}(\mathcal{G}^\bullet_{12}) \xrightarrow{j} KR^{-j-1}_{\mathcal{A}_1}(\mathcal{G}^\bullet_1) \oplus KR^{-j-1}_{\mathcal{A}_2}(\mathcal{G}^\bullet_2) \xrightarrow{i} \\ \xrightarrow{i} KR^{-j-1}_{\mathcal{A}}(\mathcal{G}^\bullet) \xrightarrow{\partial} KR^{-j}_{\mathcal{A}_{12}}(\mathcal{G}^\bullet_{12}) \xrightarrow{j} \ldots\end{aligned} \tag{6.8}$$

In view of the 8-periodicity of twisted KR-theory, the exact sequence (6.8) can be represented by the following 24-terms exact sequence

$$\begin{tikzcd}
KR^{-7}_{\mathcal{A}_{12}}(\mathcal{G}^\bullet_{12}) \arrow[rr, "j"] & & KR^{-7}_{\mathcal{A}_1}(\mathcal{G}^\bullet_1) \oplus KR^{-7}_{\mathcal{A}_2}(\mathcal{G}^\bullet_2) \arrow[r, "i"] & KR^{-7}_{\mathcal{A}}(\mathcal{G}^\bullet) \arrow[d, "\partial"] \\
 & KR^{-6}_{\mathcal{A}}(\mathcal{G}^\bullet) \arrow[d, "\partial"] & KR^{-6}_{\mathcal{A}_1}(\mathcal{G}^\bullet_1) \oplus KR^{-6}_{\mathcal{A}_2}(\mathcal{G}^\bullet_2) \arrow[l, "i"'] & KR^{-6}_{\mathcal{A}_{12}}(\mathcal{G}^\bullet_{12}) \arrow[l, "j"'] \\
 & KR^{-5}_{\mathcal{A}_{12}}(\mathcal{G}^\bullet_{12}) \arrow[r] & \cdots \arrow[r] & KR^{-1}_{\mathcal{A}}(\mathcal{G}^\bullet) \arrow[d, "\partial"'] \\
KR^{0}_{\mathcal{A}}(\mathcal{G}^\bullet) \arrow[uuu, "\partial"] & & KR^{0}_{\mathcal{A}_1}(\mathcal{G}^\bullet_1) \oplus KR^{0}_{\mathcal{A}_2}(\mathcal{G}^\bullet_2) \arrow[ll, "i"'] & KR^{0}_{\mathcal{A}_{12}}(\mathcal{G}^\bullet_{12}) \arrow[l, "j"']
\end{tikzcd} \tag{6.9}$$

In order to prove this theorem, we shall first give the "Real graded" analog of the long exact sequence of K-theory established in [35, p.90].

Lemma 6.3.2. *Let A be a Real graded C^*-algebra. Assume that I_1 and I_2 are two closed Real graded (two-sided) ideals of A such that $A = I_1 + I_2$. Then there is a 8-periodic long exact sequence*

$$\begin{aligned}\ldots \xrightarrow{\partial} KR_{p-q+1}(I_1 \cap I_2) \xrightarrow{j} KR_{p-q+1}(I_1) \oplus KR_{p-q+1}(I_2) \xrightarrow{i} \\ \xrightarrow{i} KR_{p-q+1}(A) \xrightarrow{\partial} KR_{p-q}(I_1 \cap I_2) \xrightarrow{j} \ldots\end{aligned} \tag{6.10}$$

Our proof is essentially an adaptation of the one of N. Higson, J. Roe, and G. Yu in the aforementioned refence.

Proof. Let the C^*-algebra $A[-1,1] := \mathcal{C}([-1,1],A)$ be equipped with the obvious grading and the Real involution given by $\bar{f}(t) := \overline{f(-t)}$. Next, form the graded sub-C^*-algebras of A

$$C := \{f \in A[-1,1] \mid f(-1) \in I_1, f(1) \in I_2\},$$
$$C^{-1} := \{f \in A[-1,1] \mid f(-1) \in I_2, f(1) \in I_1\},$$
$$S := \{f \in A[-1,1] \mid f(-1) = f(1) = 0\}.$$

The map sending $f \in C$ to the function $f^{\mathrm{op}} : [-1,1] \ni t \longmapsto f(-t) \in A$ is an isomorphism of graded C^*-algebras $C \cong C^{-1}$, and that C is not stable under the Real structure of $A[-1,1]$. However, since $\bar{f} \in C^{-1}$ and $\overline{f^{\mathrm{op}}} \in C$ for all $f \in C$, the "diagonal" of $C \oplus C^{-1}$ defined as

$$\Delta C := \{(f, f^{\mathrm{op}}) \in C \oplus C^{-1}\},$$

is a Real graded C^*-algebra under the involution given by $\overline{(f, f^{\mathrm{op}})} := (\overline{f^{\mathrm{op}}}, \bar{f})$. Notice that S is stable under the Real structure and the grading; it is in fact a Real graded ideal of ΔC by identifying each $f \in S$ with the pair $(f, f^{\mathrm{op}}) \in \Delta C$. Denote by $[(f, f^{\mathrm{op}})]$ the class of $(f, f^{\mathrm{op}}) \in \Delta C$ in the quotient space. Now consider the short exact sequence of Real graded C^*-algebras

$$0 \longrightarrow S \xrightarrow{i} \Delta C \xrightarrow{\pi} \Delta C/S \longrightarrow 0. \tag{6.11}$$

Then we claim that the projection π admits a cross-section which is a homomorphism of Real graded *-algebras. To see this, observe first that $\Delta C/S \cong I_1 \oplus I_2$ as Real graded C^*-algebras, via the map $[(f, f^{\mathrm{op}})] \longmapsto (f(-1), f(1))$, whose inverse is given by $(a,b) \longmapsto [(\gamma_{a,b}, \gamma_{a,b}^{\mathrm{op}})]$, where $\gamma_{a,b} \in C$ is given by

$$\gamma_{a,b}(t) := \frac{1-t}{2}a + \frac{1+t}{2}b \in I_1 + I_2 = A.$$

Then the map $s : I_1 \oplus I_2 \longrightarrow \Delta C$ given by $s(a,b) := (\gamma_{a,b}, \gamma_{a,b}^{\mathrm{op}})$ is easily seen to be a Real graded *-homomorphism which, via the identification $\Delta C/S \cong I_1 \oplus I$, verifies $\pi \circ s = \mathrm{Id}$. It then turns out that the sequence (6.11) satisfies to the conditions of Corollary 2.4 of [24] [2], so that we have a long exact sequence on KR-theory

$$\ldots \xrightarrow{i_*} KR_{p-q+1}(\Delta C) \xrightarrow{\pi_*} KR_{p-q+1}(I_1 \oplus I_2) \xrightarrow{\delta} KR_{p-q}(S) \xrightarrow{i_*} KR_{p-q}(\Delta C) \longrightarrow \ldots \tag{6.12}$$

(we refer to [84] for the details about the construction of the connecting map δ). Furthermore, we have $KR_j(I_1 \oplus I_2) \cong KR_j(I_1) \oplus KR_j(I_2)$ (cf. [46, Corollary 4.1]), and $S \cong S^{1,0}A = \mathcal{C}_0(\mathbb{R}^{1,0}; A)$ so that $KR_{p-q}(S) \cong KR_{p-q+1}(A)$. We thus deduce (6.10) by realizing that the obvious Real graded inclusion $(I_1 \cap I_2)[-1,1] \hookrightarrow \Delta C$ induces an isomorphism on the KR-theory level, and by defining the connecting ∂ to be the map $i_* : KR_{p-q+1}(A) \longrightarrow KR_{p-q}(I_1 \cap I_2)$ in (6.12). □

[2] As mentioned by the authors, this result holds also in the real case, and therefore it does hold in the Real case (thanks to the isomorphism (6.3) established in the proof of Proposition 6.1.9).

Proof of Theorem 6.3.1. The open subgroupoid $\mathcal{G}_{12} \rightrightarrows X_{12}$, with $X_{12} := X_1 \cap X_2$, is obviously saturated and Real since $\mathcal{G}_1$ and $\mathcal{G}_2$ are. Furthermore, as we have already seen in the proof of Lemma 6.1.12, $I_1 := \mathcal{A}_1 \rtimes_r \mathcal{G}_1$ and $I_2 := \mathcal{A}_2 \rtimes_r \mathcal{G}_2$ are Real graded closed ideals of $\mathcal{A} \rtimes_r \mathcal{G}$. Now in view of the long exact sequence (6.10), it thus suffices to show that $\mathcal{A} \rtimes_r \mathcal{G} = I_1 + I_2$ and $I_1 \cap I_2 = \mathcal{A}_{12} \rtimes_r \mathcal{G}_{12}$; but this is just an easy adaptation of the arguments used in the proof of Proposition 3.9 in [90]. □

6.4 Comparison with complex twisted K-theory

In this section we are comparing our Real twisted K-theory with the complex one (cf. [28, 30, 87]). Recall that for a complex graded Dixmier-Douady bundle $\mathcal{A}$ over a groupoid $\mathcal{G}$, J.-L. Tu [87] has defined the complex twisted K-theory $K_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet)$ to be the Kasparov complex KK-theory group $KK_n(\mathbb{C}, \mathcal{A} \rtimes_r \mathcal{G})$ (see also [43, §3] for a similar definition in term of the K-theory of graded Banach algebras). Now since an element $\mathcal{A} \in \widehat{\mathrm{BrR}}(\mathcal{G})$ cal also be viewed as a complex graded D-D bundle by forgetting the Real structures of $\mathcal{A}$ and $\mathcal{G}$, we can define complex twisted K-theory $K_{\mathcal{A}}^{*}(\mathcal{G}^\bullet)$ by Real graded twistings. It is then natural to compare the groups $KR_*^*(\mathcal{G}^\bullet)$ with $K_{\mathcal{A}}^{*}(\mathcal{G}^\bullet)$. We will prove for instance the following theorem.

Theorem 6.4.1. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a locally compact second countable Real groupoid with Real Haar system. Then for $\mathcal{A} \in \widehat{\mathrm{BrR}}(\mathcal{G})$, Real and complex (graded) twisted K-theories are related by the following isomorphism*

$$K_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet) \otimes \mathbb{Z}[1/2] \cong \left(KR_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet) \oplus KR_{\mathcal{A}}^{-j+2}(\mathcal{G}^\bullet)\right) \otimes \mathbb{Z}[1/2]$$

In order to prove this, we start with some observations about complex KK-theory of Rg C^*-algebras.

Let B be a Rg C^*-algebra, and denote as usual its Real structure by τ. There is an obvious homomorphism

$$c : KKR(\mathbb{C}, B) \longrightarrow KK(\mathbb{C}, B) \tag{6.13}$$

that consists of "forgetting the Real structures" (c is not injective nor surjective). Conversely, we want to construct a homomorphism $R : KK(\mathbb{C}, B) \longrightarrow KKR(\mathbb{C}, B)$.

Recall that τ induces an isomorphism of complex graded C^*-algebras

$$\tau^\flat : B \longrightarrow \bar{B}, b \longmapsto \tau(b)^\flat,$$

where $\bar{B}$ is the conjugate algebra of B. Hence we have an isomorphism

$$\tau_*^\flat : KK(\mathbb{C}, B) \longrightarrow KK(\mathbb{C}, \bar{B})$$

by cofunctoriality ([9, 46]). On the other hand, if E is a complex graded Hilbert $\bar{A}$-module, its conjugate algebra $\bar{E}$ is clearly a complex graded Hilbert A-module, and if $F \in \mathcal{L}_{\bar{B}}(E)$, then $\bar{F} \in \mathcal{L}_B(\bar{E})$, where $\bar{F}$ is defined by $\bar{F}(\xi^\flat) := (F(\xi))^\flat$ for all $\xi^\flat \in \bar{E}$, where $\flat : E \longrightarrow \bar{E}$ is the canonical map. Further, if $\varphi : \mathbb{C} \longrightarrow \mathcal{L}_{\bar{B}}(E)$, then we define $\bar{\varphi} : \mathbb{C} \longrightarrow \mathcal{L}_B(\bar{E})$ by $\bar{\varphi}(\lambda)\xi^\flat := (\varphi(\bar{\lambda})\xi)^\flat$. It is straightforward that $(\bar{E}, \bar{\varphi}, \bar{F}) \in \mathbf{E}(\mathbb{C}, B)$ if $(E, \varphi, F) \in \mathbf{E}(\mathbb{C}, \bar{B})$ and that this process respect homotopy and degeneracy. In other words, the canonical map $\flat : \bar{B} \longrightarrow B$ naturally induces a map

$$KK(\mathbb{C}, \bar{B}) \longrightarrow KK(\mathbb{C}, B), y \longmapsto \bar{y}.$$

Then we have

Lemma 6.4.2. *The maps $\tau^\flat : \bar{B} \longrightarrow B$ and $\flat : \bar{B} \longrightarrow B$ induce an involution on the Abelian group $KK(\mathbb{C}, B)$, that we also denote by τ;* i.e. *for $x = (E, \varphi, F) \in \mathbf{E}(\mathbb{C}, B)$, we have*

$$\tau(x) = (\overline{\tau^\flat_* E}, \overline{\tau^\flat_* \varphi}, \overline{\tau^\flat_* F}).$$

Now if $x = (E, \varphi, F)$ is a complex graded Kasparov module over the complex graded C^*-algebra B, we get a Real graded graded Kasparov module $R(x) \in \mathbf{ER}(\mathbb{C}, B)$ by setting

$$R(x) := x + \tau(x). \tag{6.14}$$

Lemma 6.4.3. *The formula (6.14) defines a homomorphism*

$$R : KK(\mathbb{C}, B) \longrightarrow KKR(\mathbb{C}, B).$$

Proof. The only thing that needs to be checked is that $R(x)$ admits a Real structure compatible with τ so that $R(x) \in \mathbf{ER}(\mathbb{C}, B)$. Note that $\tau^\flat_* E = E \hat{\otimes}_A \bar{B}$, so that $\overline{\tau^\flat_* E} \cong \bar{E} \hat{\otimes}_{\bar{B}} B$ as complex graded Hilbert B-modules (where the action of $\bar{B}$ on B is by the inverse of the automorphism τ^*). It then follows that $E \oplus \overline{\tau^\flat_* E}$ is isomorphic to its complex conjugate $\overline{E \oplus \overline{\tau^\flat_* E}} = \bar{E} \oplus \tau^\flat_* E$ via the map $(\xi, \sum \eta^\flat_i \hat{\otimes}_{\bar{B}} b_i) \longmapsto (\xi^\flat, \sum \eta_1 \hat{\otimes}_B b^\flat_i)$, which gives us the Real structure on the complex graded Hilbert A-module $E \oplus \overline{\tau^\flat_* E}$. The compatibility of this involution with τ and the inner product is easy to check. □

Definition 6.4.4. *For a Real graded C^*-algebra B, and $\epsilon \in \{+, -\}$, we denote by $KK(\mathbb{C}, B)^\epsilon$ the subgroup of $KK(\mathbb{C}, B)$ consisting of all elements x of $KK(\mathbb{C}, B)$ such that $\tau(x) = \epsilon x$, where $\tau : KK(\mathbb{C}, B) \longrightarrow KK(\mathbb{C}, B)$ is the involution defined in Lemma 6.4.2.*

Proposition 6.4.5. *Let B be a Rg C^*-algebra. Then the map R induces an isomorphism*

$$KK(\mathbb{C}, B)^+ \otimes \mathbb{Z}[1/2] \cong KKR(\mathbb{C}, B) \otimes \mathbb{Z}[1/2],$$

whose inverse is given by c.

In fact, in the special case where A is the trivially graded Real C^*-algebra $\mathcal{C}_0(X)$, where X is a locally compact space endowed with the trivial involution, this result is just Karoubi's [42, Corollary 2.9] since $KKR(\mathbb{C},\mathcal{C}_0(X)) = KR(X) \cong KO(X)$.

Proof. We have $\operatorname{Im} c \subset KKR(\mathbb{C},B)^+$. Indeed, if $x = (E,\varphi,F) \in \mathbf{ER}(\mathbb{C},B)$, then the Real structure of E induces an isomorphism of graded $\mathbb{C}$-algebras $E \cong \bar{E}$. Moreover, since E is a Real graded Hilbert B-module, then from the isomorphism $\tau^\flat : B \longrightarrow \bar{B}$ we get

$$\tau^\flat_* E = (\flat\circ\tau)_* E = \tau_*(\bar{E}\hat{\otimes}_{\bar{B}} B) = (\bar{E}\hat{\otimes}_{\bar{B}} B)\hat{\otimes}_B \bar{B} \cong \bar{E}\hat{\otimes}_{\bar{B}}\bar{B} \cong \bar{E}.$$

It follows that $\overline{\tau^\flat_* c(x)} = \tau(c(x)) = c(x) \in KK(\mathbb{C},B)^+$ for all $x \in KKR(\mathbb{C},B)$. Henceforth, $R(c(x)) = c(x) + \tau(c(x)) = 2c(x)$ for all $x \in KKR(\mathbb{C},B)$; and for any $y \in KK(\mathbb{C},B)^+$, one has $y = \tau(y)$, so that $2y = y + \tau(y) = R(y) \in \operatorname{Im} R$. Denoting by R' the restriction of R on $KK(\mathbb{C},B)^+ \subset KK(\mathbb{C},B)$, we get $c(R'(y)) = 2y$, hence if dividing by 2 in the groups involved is allowed, the maps $R' : KK(\mathbb{C},B)^+ \longrightarrow KKR(\mathbb{C},B)$ and $c : KKR(\mathbb{C},B) \longrightarrow KK(\mathbb{C},B)^+$ are inverse of each other; and this completes the proof. □

Lemma 6.4.6. *Let B and D be Real graded C^*-algebras. Then the Kasparov product in complex KK-theory $KK(\mathbb{C},B)\otimes KK(\mathbb{C},D) \longrightarrow KK(\mathbb{C},B\hat{\otimes}D)$ induces a bilinear map*

$$KK(\mathbb{C},B)^\epsilon \otimes KK(\mathbb{C},D)^\eta \longrightarrow KK(\mathbb{C},B\hat{\otimes}D)^{\epsilon\eta}, \tag{6.15}$$

where by convention $(+)(+) = (+), (+)(-) = (-)(+) = (-), (-)(-) = (+)$.

Proof. Recall that the Real structure on the graded tensor product $B\hat{\otimes}D$ is given on elementary tensors by $\overline{b\hat{\otimes}d} = \bar{b}\hat{\otimes}\bar{d}$. Hence, denoting by τ^B, τ^D ,and $\tau^{B\hat{\otimes}D}$ the involutions on $KK(\mathbb{C},B), KK(\mathbb{C},D)$ and $KK(\mathbb{C},B\hat{\otimes}D)$, respectively, we see that $\tau^{B\hat{\otimes}D}(x\hat{\otimes}_{\mathbb{C}} y) = \tau^B(x)\hat{\otimes}_{\mathbb{C}}\tau^D(y)$ for all $x \in KK(\mathbb{C},B), y \in KK(\mathbb{C},D)$; so that $\tau^{B\hat{\otimes}D}(x\hat{\otimes}_{\mathbb{C}} y) = (\epsilon\eta)(x\hat{\otimes}_{\mathbb{C}} y)$, where $\tau^B(x) = \epsilon x$ and $\tau^D(y) = \eta y$. □

Proposition 6.4.7. *The Kasparov product with the complex Bott element*

$$\beta_2 \in K(\mathcal{C}_0(\mathbb{R}^{0,2})) = KK(\mathbb{C},\mathcal{C}_0(\mathbb{R}^{0,2}))$$

induces an isomorphism

$$\beta_\epsilon : KK(\mathbb{C},B)^\epsilon \longrightarrow KK(\mathbb{C},B(\mathbb{R}^{0,2}))^{(-)\epsilon}.$$

Proof. Recall ([46, p.546]) that the Bott element $\beta_2 \in KK(\mathbb{C},\mathcal{C}_0(\mathbb{R}^{0,2}))$ is defined as the Kasparov module $(\mathcal{C}_0(\mathbb{R}^{0,2}), \mathbb{C}\cdot\mathrm{Id}, F)$, where $\mathbb{C}\cdot\mathrm{Id} \longrightarrow \mathcal{L}(\mathcal{C}_0(\mathbb{R}^{0,2}))$ is given by scalar multiplication, and $F \in \mathcal{L}(\mathcal{C}_0(\mathbb{R}^{0,2})) = \mathcal{M}(\mathcal{C}_0(\mathbb{R}^{0,2})) \cong \mathcal{C}_b(\mathbb{R}^{0,2})$ is defined to be the function $F(x) = x(1+\|x\|^2)^{-\frac{1}{2}}$. Since $F(x) \in \mathbb{R}, \forall x \in \mathbb{R}^{0,2}$, we have $\tau^\flat_* F(x) = F(-x) = -F(x)$, from which we deduce that $\tau(\beta_2) = -\beta_2 \in KK(\mathbb{C},\mathcal{C}_0(\mathbb{R}^{0,2}))^-$. However, the Kasparov product with β_2 induces an isomorphism $\hat{\otimes}_{\mathbb{C}}\beta_2 : KK(\mathbb{C},B) \longrightarrow KK(\mathbb{C},B(\mathbb{R}^{0,2}))$ ([46, Theorem 4.7]). We thus obtained the desired isomorphism by applying Lemma 6.4.6 to B and $D = \mathcal{C}_0(\mathbb{R}^{0,2})$. □

We now return to our story of twisted K-theory.

Proof of Theorem 6.4.1. Put $B := (\mathcal{A} \rtimes_r \mathcal{G}) \otimes \mathcal{C}_0(\mathbb{R}^{p,q})$ with its usual Real structure. By inverting by 2, the involution τ of $KK(\mathbb{C}, B)$ yields the decomposition $KK(\mathbb{C}, B) \otimes \mathbb{Z}[\frac{1}{2}] \cong (KK(\mathbb{C}, B)^+ \oplus KK(\mathbb{C}, B)^-) \otimes \mathbb{Z}[\frac{1}{2}]$. Now we conclude by Proposition 6.4.5 and Proposition 6.4.7. $\square$

Proposition 6.4.8. *Assume $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ is a Real groupoid with Real Haar system which is the disjoint union of two locally compact Hausdorff groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the involution $\tau : \mathcal{G} \longrightarrow \mathcal{G}$ consists of exchanging $\mathcal{G}_1$ with $\mathcal{G}_2$; i.e. $\bar{g}_1 = \tau(g_1) \in \mathcal{G}_2$ and $\bar{g}_2 = \tau(g_2) \in \mathcal{G}_1$ for all $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$. Then for all $\mathcal{A} \in \widehat{Br\mathrm{R}}(\mathcal{G})$, we have*

$$KR^*_{\mathcal{A}}(\mathcal{G}^\bullet) \otimes \mathbb{Z}[1/2] \cong K^*_{\mathcal{A}_1}(\mathcal{G}^\bullet_1) \otimes \mathbb{Z}[1/2] \cong K^*_{\mathcal{A}_2}(\mathcal{G}^\bullet_2) \otimes \mathbb{Z}[1/2],$$

where $\mathcal{A}_i, i = 1, 2$ is the restriction of $\mathcal{A}$ on X_i, and where the complex twisted K-theory used here is the graded one.

Proof. Denote by τ_i the restriction of τ on $\mathcal{G}_i, i = 1, 2$. Then under the decomposition C^*-algebra $\mathcal{A} \rtimes_r \mathcal{G} = \mathcal{A}_1 \rtimes_r \mathcal{G}_1 \oplus \mathcal{A}_2 \rtimes_r \mathcal{G}_2$, the Real structure of $\mathcal{A} \rtimes_r \mathcal{G}$ is given by the matrix

$$\tau = \begin{pmatrix} 0 & \tau_2 \\ \tau_1 & 0 \end{pmatrix},$$

where the maps $\tau_1 : \mathcal{C}_c(\mathcal{G}_1; s^*\mathcal{A}_1) \longrightarrow \mathcal{C}_c(\mathcal{G}_2; s^*\mathcal{A}_2)$, and $\tau_2 : \mathcal{C}_c(\mathcal{G}_2; s^*\mathcal{A}_2) \longrightarrow \mathcal{C}_c(\mathcal{G}_1; s^*\mathcal{A}_1)$ are the conjugate-linear homomorphism of graded convolution algebras given by $(\tau_1\xi_1)(g_2) := \overline{\xi_1(\bar{g}_2)}$ for $\bar{g}_2 \in \operatorname{supp}\xi_1$, and $(\tau_2\xi_2)(g_1) := \overline{\xi_2(\bar{g}_1)}$ for $\bar{g}_1 \in \operatorname{supp}\xi_2$, respectively. Notice that τ_1 and τ_2 are inverse of each other. Let $B_i := \mathcal{A}_i \rtimes_r \mathcal{G}_i, i = 1, 2$. Then $\tau_1^\flat : B_1 \longrightarrow \bar{B}_2, b_1 \longmapsto \tau_1(b)^\flat$ and $\tau_2^\flat : B_2 \longrightarrow \bar{B}_1, b_2 \longmapsto \tau_2(b_2)^\flat$ are isomorphisms of complex graded C^*-algebras. Now in the level of complex KK-theory, it turns out that via the isomorphism $KK(\mathbb{C}, \mathcal{A} \rtimes_r \mathcal{G}) \cong KK(\mathbb{C}, B_1) \oplus KK(\mathbb{C}, B_2)$ ([9, §17.7], [46, Corollary 1,§4]), the involution $\tau : KK(\mathbb{C}, \mathcal{A} \rtimes_r \mathcal{G}) \longrightarrow KK(\mathbb{C}, \mathcal{A} \rtimes_r \mathcal{G})$ (cf. Lemma 6.4.2) is given by

$$\tau = \begin{pmatrix} 0 & \tilde{\tau}_2 \\ \tilde{\tau}_1 & 0 \end{pmatrix} \tag{6.16}$$

where $\tilde{\tau}_1 : KK(\mathbb{C}, B_1) \longrightarrow KK(\mathbb{C}, B_2)$ and $\tilde{\tau}_2 : KK(\mathbb{C}, B_2) \longrightarrow KK(\mathbb{C}, B_1)$ are the isomorphism induced by the composites

$$KK(\mathbb{C}, B_1) \xrightarrow{\tau_1^\flat} KK(\mathbb{C}, \bar{B}_2) \xrightarrow{\flat} KK(\mathbb{C}, B_2), \text{ and}$$

$$KK(\mathbb{C}, B_2) \xrightarrow{\tau_2^\flat} KK(\mathbb{C}, \bar{B}_1) \xrightarrow{\flat} KK(\mathbb{C}, B_1),$$

respectively. Therefore,

$$\begin{aligned} KK(\mathbb{C},\mathcal{A}\rtimes_r\mathcal{G})^+ =&\{x\in KK(\mathbb{C},\mathcal{A}\rtimes_r\mathcal{G}) \mid \tau(x)=x\}\\ =&\{(x_1,\tilde{\tau}_1(x_1));\ x_1\in KK(\mathbb{C},B_1)\}\cong\{(\tilde{\tau}_2(x_2),x_2);\ x_2\in KK(\mathbb{C},B_2)\}\\ \cong& KK(\mathbb{C},B_1)\cong KK(\mathbb{C},B_2). \end{aligned}$$

The desired isomorphisms then result from Proposition 6.4.5. □

Remark 6.4.9. *Although we have the decomposition $K^*_{\mathcal{A}}(\mathcal{G}^\bullet)\cong K^*_{\mathcal{A}_1}(\mathcal{G}^\bullet_1)\oplus K^*_{\mathcal{A}_2}(\mathcal{G}^\bullet_2)$ in complex twisted K-theory, this is not true for twisted KR; indeed, the graded C^*-algebras B_1 and B_2 are not invariant under the Real structure of $\mathcal{A}\rtimes_r\mathcal{G}$ as we have seen in the proof above.*

We close this section with the following result whose proof can be copied from the one of Schick [80, Theorem 2.1] (see also Boersema [11]).

Proposition 6.4.10. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be a locally compact second-countable Hausdorff Real groupoid with Real Haar system. Let $\mathcal{A}\in\widehat{Br\mathrm{R}}(\mathcal{G})$. Then there is a long exact sequence relying complex twisted (graded) K-theory and Real twisted K-theory*

$$\cdots\longrightarrow KR^{-j+1}_{\mathcal{A}}(\mathcal{G}^\bullet)\xrightarrow{\chi}KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet)\xrightarrow{c}K^{j}_{\mathcal{A}}(\mathcal{G}^\bullet)\xrightarrow{\partial}KR^{-j+2}(\mathcal{G}^\bullet)\xrightarrow{\chi}\cdots \tag{6.17}$$

where χ is Kasparov product with the Bott element $\beta_{1,0}\in KR(\mathbb{R}^{1,0})\cong\mathbb{Z}_2$, c is the map defined by (6.13), and ∂ is the composite

$$KK(\mathbb{C},B)\xrightarrow{\beta_2}KK(\mathbb{C},B(\mathbb{R}^{0,2}))\xrightarrow{R}KKR(\mathbb{C},B(\mathbb{R}^{0,2})),$$

where $B=(\mathcal{A}\rtimes_r\mathcal{G})\hat{\otimes}\mathcal{C}_0(\mathbb{R}^{p,q})$ with $p-q=j$, and R is the homomorphism defined by (6.14).

6.5 4-periodicity theorem

Although twisted KR-theory is of period 8 (cf. Theorem 6.1.6(1)), we prove in this section that up to tensoring by $\mathbb{Q}$, the twisted KR-groups are 4-periodic.

Theorem 6.5.1. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be a Real groupoid with Real Haar system. Suppose $\Gamma\overset{r}{\underset{s}{\rightrightarrows}}Y$ is a closed saturated Real subgroupoid of $\mathcal{G}$. Then for $\mathcal{A}\in\widehat{\mathfrak{BrR}}(\mathcal{G})$, there is a canonical isomorphism*

$$KR^{-j}_{\mathcal{A}}(\mathcal{G}^\bullet,\Gamma^\bullet)\otimes\mathbb{Z}[1/2]\cong KR^{-j-4}_{\mathcal{A}}(\mathcal{G}^\bullet,\Gamma^\bullet)\otimes\mathbb{Z}[1/2].$$

Proof. It suffices to show the result for $j = 0$. For a Real graded C^*-algebra B, Proposition 6.4.7 applied to the Real graded C^*-algebras B and $B(\mathbb{R}^{0,2})$ gives us two isomorphisms

$$KK(\mathbb{C},B)^+ \xrightarrow{\beta_+} KK(\mathbb{C},B(\mathbb{R}^{0,2}))^- \xrightarrow{\beta_-} KK(\mathbb{C},B(\mathbb{R}^{0,4}))^+,$$

which yields the isomorphism

$$(\beta_-\otimes\mathbf{1})\circ(\beta_+\otimes\mathbf{1}) : KKR(\mathbb{C},B)\otimes\mathbb{Z}[1/2] \longrightarrow KKR(\mathbb{C},B(\mathbb{R}^{0,4}))\otimes\mathbb{Z}[1/2],$$

via the identifications (cf. Proposition 6.4.5)

$$\begin{aligned} KK(\mathbb{C},B)^+\otimes\mathbb{Z}[1/2] &\cong KKR(\mathbb{C},B)\otimes\mathbb{Z}[1/2], \\ KK(\mathbb{C},B(\mathbb{R}^{0,4}))^+\otimes\mathbb{Z}[1/2] &\cong KKR(\mathbb{C},B(\mathbb{R}^{0,4})). \end{aligned}$$

Now it suffices to take $B = \mathcal{A}_{|X\setminus Y} \rtimes_r (\mathcal{G}\setminus\Gamma)$. □

6.6 Computation of twisted KR-groups of $\mathbf{S}^{p,q}$

We start this section by the following simple examples.

Example 6.6.1 ($KR^0_{\mathbb{C}l_{p,q}}(pt)$). *An element of $n \in \widehat{Br}\mathrm{R}(*) = \mathbb{Z}_8$ is determined by a Rg elementary C^*-algebra of type $\widehat{\mathcal{K}}_n$. If $p,q\in\mathbb{N}$ is such that $p-q = n \mod 8$, then the Rg D-D bundle $\widehat{\mathcal{K}}_n \longrightarrow \cdot$ is Morita equivalent to $\mathbb{C}l_{p,q} \longrightarrow \cdot$. Recall that $(\mathbb{C}l_{p,q})_{\mathbb{R}} = Cl_{p,q}$, the latter being the real Clifford algebra (see Example A.5.6) It follows that*

$$KR^0_n(*) = KR^0_{\mathbb{C}l_{p,q}}(*) \cong KO^0_{Cl_{p,q}}(*) = KO^{p-q}(*).$$

Example 6.6.2 ($KR^*_{\mathcal{A}}(\mathbf{S}^{0,1})$). *Let $n \in \widehat{Br}\mathrm{R}(\mathbf{S}^{0,1}) = \mathbb{Z}_2$ (cf. Example 4.3.3). Then by Proposition 6.4.8,*

$$KR_n^{-j}(\mathbf{S}^{0,1}) \cong_{\mathbb{Q}} K^{-j-n}(\{pt\}) = \begin{cases} \mathbb{Z}, & \text{if } -j-n=0 \mod 2 \\ 0, & \text{if } -j-n=1 \mod 2 \end{cases}$$

The last example allows us to compute the twisted KR-theory o the Real space $\mathbf{S}^{0,q}$, for any $q\in\mathbb{N}$. Indeed, we have the following.

Proposition 6.6.3. *Let $q\in\mathbb{N}^*$. We have $\widehat{Br}\mathrm{R}(\mathbf{S}^{0,q}) \cong \mathbb{Z}_2^q$.*

Proof. We shall prove the isomorphism by induction with respect to q. For $q = 1$, this is already done in Example 4.3.3. For $q = 2$, let

$$S^2_+ := \{(x_1,x_2)\in S^{0,2} \mid x_2 > 0\}, \quad S^2_- := \{(x_1,x_2) \mid x_2 < 0\}$$

be the upper and the lower hemispheres. Let $S^2 = S^2_+ \cup S^2_-$. Then $S^2 \subset \mathbf{S}^{0,2}$ is invariant under the Real structure, and the induced involution consists of switching S^2_+ with S^2_-. Moreover, we have clearly $\mathbf{S}^{0,2} = S^2 \sqcup \mathbf{S}^{0,1}$. It rurns out that $\widehat{\mathrm{BrR}}(\mathbf{S}^{0,2}) = \widehat{\mathrm{BrR}}(S^2) \oplus \widehat{\mathrm{BrR}}(\mathbf{S}^{0,1})$. But from Proposition 4.3.1 and from the fact S^2_+ is contractible, we get $\widehat{\mathrm{BrR}}(S^2) \cong \widehat{\mathrm{Br}}(S^2_+) \cong \widehat{\mathrm{Br}}(\{pt\}) \cong \mathbb{Z}_2$. Therefore,

$$\widehat{\mathrm{BrR}}(\mathbf{S}^{0,2}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Now assume the result is true for any $r \leq q-1$, so that $\widehat{\mathrm{BrR}}(\mathbf{S}^{0,q-1}) \cong \mathbb{Z}_2^{q-1}$. Then, in the arguments above, by replacing $\mathbf{S}^{0,1}$ by $\mathbf{S}^{0,q-1}$, S^2_+ and S^2_- by the contractible spaces

$$S^q_+ := \{(x_1,\dots,x_2) \mid x_q > 0\}, \quad \text{and } S^q_- := \{(x_1,\dots,x_q) \mid x_q < 0\},$$

respectively, we obtain the result for q. □

Now in view of Example 6.6.2 we have:

Proposition 6.6.4. *Suppose a Rg D-D bundle $\mathcal{A}$ over $\mathbf{S}^{0,q}$ is represented by the q-tuple $(\epsilon_1,\dots,\epsilon_q) \in \widehat{BrR}(\mathbf{S}^{0,q})$. Then*

$$KR^{-j}_{(\epsilon_1,\dots,\epsilon_q)}(\mathbf{S}^{0,q}) \cong_{\mathbb{Q}} \bigoplus_{k=1}^{q} KR^{-j}_{\epsilon_k}(\mathbf{S}^{0,1}) \cong_{\mathbb{Q}} \bigoplus_{k=1}^{q} K^{-j-\epsilon_k}(\{pt\}).$$

Corollary 6.6.5. *Let $q \in \mathbb{N}^*$, and let $\mathcal{A}$ be a Rg D-D bundle represented by $(\epsilon_1,\dots,\epsilon_q) \in \widehat{BrR}(\mathbf{S}^{0,q})$. Then for $j \in \mathbb{Z}$, the twisted complex K-theory of the space $\mathbf{S}^{0,q}$ is given by*

$$K^{-j}_{\mathcal{A}}(\mathbf{S}^{0,q}) \cong_{\mathbb{Q}} \left(\bigoplus_{k=1}^{q} K^{-j-\epsilon_k}(\{pt\})\right)^2.$$

Proof. This an immediate application of Theorem 6.4.1 to the Real space $\mathbf{S}^{0,q}$, and the above proposition plus the fact that $K^{-j-2-\epsilon_k} \cong K^{-j-\epsilon_k}$. □

7

Fredholm picture of twisted KR-theory

7.1 Preliminaries: Fredholm picture of $KR_*(B)$

It is known ([9, 17.5.4], [93, Proposition 17.3.5]) that for a trivially graded C^*-algebra B, the group $K_i(B) \cong KK^i(\mathbb{C}, B)$ is isomorphic to the group $\pi_0(\mathcal{F}_B^j)$ of homotopy classes of elements of the space $\mathcal{F}_B^j$ defined as

$$\mathcal{F}_B^0 := \{T \in \mathcal{L}(\mathcal{H}_B) \mid T \text{ is invertible modulo } \mathcal{K}_B\},$$

where $\mathcal{H}_B = \mathcal{H} \otimes B$, and $\mathcal{K}_B = \mathcal{K}(\mathcal{H}_B)$, and $\mathcal{F}_B^1$ is the subspace of those elements of $\mathcal{F}_B^0$ which are self-adjoint. In this section, we give an analogous "Fredholm" interpretation of the groups $KR_{p-q}(B) = KKR^{q-p}(\mathbb{C}, B) \cong KR(B(\mathbb{R}^{p,q}))$ for a Real graded C^*-algebra B.

Let $\hat{\mathcal{H}}_B = \hat{\mathcal{H}} \hat{\otimes} B$ be endowed with its standard grading and Real structure (cf. Appendix A). Note that $(\hat{\mathcal{H}}_B)^i, i = 0, 1$ are Real Hilbert B^0-modules. Moreover, a Real operator $F \in \mathcal{L}(\hat{\mathcal{H}}_B)$ of degree 1 can be written as

$$F = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix},$$

where $T : (\hat{\mathcal{H}}_B)^0 \longrightarrow (\hat{\mathcal{H}}_B)^1$ and $S : (\hat{\mathcal{H}}_B)^1 \longrightarrow (\hat{\mathcal{H}}_B)^0$ are bounded Real B^0-linear operators (*i.e.*, compatible with the induced Real structures).

Definition 7.1.1. *A* generalized Fredholm operator *on B is a operator $F \in \mathcal{L}(\hat{\mathcal{H}}_B)$ of degree 1 such that*

$$ST - 1 \in \mathcal{K}((\hat{\mathcal{H}}_B)^0), TS - 1 \in \mathcal{K}((\hat{\mathcal{H}}_B)^1), \quad \text{and} \quad S - T^* \in \mathcal{K}((\hat{\mathcal{H}}_B)^1, (\hat{\mathcal{H}}_B)^0). \tag{7.1}$$

We denote by $\hat{\mathfrak{F}}_B$ the set all generalized Fredholm operators on B.

Alternatively, we may define $\hat{\mathfrak{F}}_B$ as the set of pairs

$$(S,T)\in\mathcal{L}((\hat{\mathcal{H}}_B)^1,(\hat{\mathcal{H}}_B)^0)\times\mathcal{L}((\hat{\mathcal{H}}_B)^0,(\hat{\mathcal{H}}_B)^1)$$

such that relations (7.1) hold. We specifically use this picture to define the topology of $\hat{\mathfrak{F}}_B$ as the one induced by the embedding

$$(S,T)\longmapsto(S,T,ST-1,TS-1,S-T^*) \tag{7.2}$$

of $\hat{\mathfrak{F}}_B$ in $\mathcal{L}((\hat{\mathcal{H}}_B)^1,(\hat{\mathcal{H}}_B)^0)\times\mathcal{L}((\hat{\mathcal{H}}_B)^0,(\hat{\mathcal{H}}_B)^1)\times\mathcal{K}((\hat{\mathcal{H}}_B)^0)\times\mathcal{K}((\hat{\mathcal{H}}_B)^1)\times\mathcal{K}((\hat{\mathcal{H}}_B)^1,(\hat{\mathcal{H}}_B)^0)$, where $\mathcal{L}((\hat{\mathcal{H}}_B)^1,(\hat{\mathcal{H}}_B)^0)$ and $\mathcal{L}((\hat{\mathcal{H}}_B)^0,(\hat{\mathcal{H}}_B)^1)$ are equipped with the compact-open topology (in the metrizable case, this is equivalent to *-strong operator topology as mentioned in [8, p.5]), while $\mathcal{K}((\hat{\mathcal{H}}_B)^0),\mathcal{K}((\hat{\mathcal{H}}_B)^1)$, and $\mathcal{K}((\hat{\mathcal{H}}_B)^1,(\hat{\mathcal{H}}_B)^0)$ are equipped with the norm-topology.

We now define the spaces $\hat{\mathfrak{F}}_B^{p,q}$ as follows.

Definition 7.1.2. *Let $p,q\in\mathbb{N}$, and let $e_1,\dots,e_p,\varepsilon_1,\dots,\varepsilon_q$ be a family of operators of degree 1 on $\hat{\mathcal{H}}_B$ subject to the conditions*

(i) $e_i^2=1, e_i^*=\bar{e}_i=e_i, i=1,\dots,p, \varepsilon_j^2=-1, \varepsilon_j^*=\bar{\varepsilon}_j=-\varepsilon_j, j=1,\dots q,$

(ii) $ee'=-e'e, \forall e\neq e'\in\{e_1,\dots,e_p,\varepsilon_1,\dots,\varepsilon_q\};$

(iii) (uniqueness) if $e'_i, i=1,\dots,p,\varepsilon'_j, j=1,\dots,q$ is another family of operators of degree 1 on $\hat{\mathcal{H}}_B$ satisfying (i) and (ii), then there exists a Real graded unitary u on $\hat{\mathcal{H}}_B$ such that $e'_i=ue_iu^*, i=1,\dots,p$, and $\varepsilon'_j=u\varepsilon_ju^*, j=1,\dots,q$.

By $\hat{\mathfrak{F}}_B^{p,q}$ we denote the subspace of $\hat{\mathfrak{F}}_B$ consisting of those F such that

$$Fe=eF, \quad \textit{for} \quad e=\varepsilon_1,\dots,\varepsilon_q,e_1,\dots,e_p. \tag{7.3}$$

$\hat{\mathfrak{F}}_B^{p,q}$ is equipped with topology induced from $\hat{\mathfrak{F}}_B$.

We should make some remarks about this definition.

Remark 7.1.3. *Observe that such a family always exists. Indeed, take for any $p,q\in\mathbb{N}$ a Real graded *-representation $\psi_{p,q}:\mathbb{C}l_{p,q}\longrightarrow\mathcal{L}(\hat{\mathcal{H}})$ with the property that if $\mathbb{C}l_{p,q}$ is not simple (so that it is the direct sum of two simple *-algebras) then each summand is represented with infinite multiplicity on $\hat{\mathcal{H}}$, so that $\psi_{p,q}$ is unique up to unitary equivalence [1]. Then, put $e_i:=\psi_{p,q}(\tilde{e}_i)\hat{\otimes}\mathbf{1}\in\mathcal{L}(\hat{\mathcal{H}}_B), i=1,\dots,p$, and $\varepsilon_j:=\psi_{p,q}(\tilde{\varepsilon}_j)\hat{\otimes}\mathbf{1}\in\mathcal{L}(\hat{\mathcal{H}}_B), j=1,\dots,q$, where $\tilde{e}_i,\tilde{\varepsilon}_j$ are the standard generators of $\mathbb{C}l_{p,q}$, and where the graded tensor product is given by the Real graded embedding $\mathcal{L}(\hat{\mathcal{H}})\hookrightarrow\mathcal{L}(\hat{\mathcal{H}}\hat{\otimes}B)$.*

[1] From a basic fact of representation theory of *-algebras, if $\mathbb{C}l_{p,q}$ is simple (*i.e.* is isomorphic to $\mathcal{K}(\hat{H})$, with $\hat{H}$ finite dimensional), then $\psi_{p,q}$ is unitarily equivalent to a multiple of the identity representation, and then is unique up to equivalence

Remark 7.1.4. *In fact, the family $e_1,...,e_p,\varepsilon_1,...,\varepsilon_q$ makes $\hat{\mathcal{H}}_B$ into a Real graded $\mathbb{C}l_{p,q}$-module. Moreover, it is easy to verify (by using the properties of the Clifford algebras) that the space $\hat{\mathfrak{F}}_B^{p,q}$ depends only on the difference $p-q$;* i.e., *if $p-q=p'-q'$, then the topological spaces $\hat{\mathfrak{F}}_B^{p,q}$ and $\hat{\mathfrak{F}}_B^{p',q'}$ are homeomorphic. We thus denote $\hat{\mathfrak{F}}_B^{p-q}:=\hat{\mathfrak{F}}_B^{p,q}$. Observe also that $\hat{\mathfrak{F}}_B^0\cong\hat{\mathfrak{F}}_B$.*

Remark 7.1.5. *For our convenience, we will usually suppose the family $e_1,...,e_p,\varepsilon_1,...,\varepsilon_q$ arises from the way described in Remark 7.1.3;* i.e. *it comes from a *-representation $\mathbb{C}l_{p,q}\longrightarrow\mathcal{L}(\hat{\mathcal{H}})$ (satisfying some conditions). Henceforth, the family $e_1,...,e_p,\varepsilon_1,...,\varepsilon_q$ are considered as degree 1 operators on $\hat{\mathcal{H}}$, and $\hat{\mathfrak{F}}_B^{p-q}$ is viewed as the space of $F\in\mathcal{L}(B\hat{\otimes}\hat{\mathcal{H}})$ such that*

- *F is of degree 1;*
- *$F(\mathbf{1}\hat{\otimes}e)=(\mathbf{1}\hat{\otimes}e)F,\forall e=e_1,...,e_p,\varepsilon_1,...,\varepsilon_q$; and*
- *F is* essentially an involution, i.e., $F^2-\mathbf{1}\in\mathcal{K}(B\hat{\otimes}\hat{\mathcal{H}})=B\hat{\otimes}\widehat{\mathcal{K}}_0$,
- *F is* essentially self-adjoint, i.e., $F-F^*\in B\hat{\otimes}\widehat{\mathcal{K}}_0$.

Notice that $\hat{\mathfrak{F}}_B^{p-q}$ is not stable under composition of operators; for, if $F,F'\in\hat{\mathfrak{F}}_B^{p-q}$, then condition (7.3) does not hold for the product FF'. We then give the following definition.

Definition 7.1.6. *The sum in $\hat{\mathfrak{F}}_B^{p-q}$ is defined the following way: from the stabilization theorem, there exists an isometric isomorphism of Real graded $\mathbb{C}l_{p,q}$-modules $u:\hat{\mathcal{H}}_B\oplus\hat{\mathcal{H}}_B\longrightarrow\hat{\mathcal{H}}_B$; we then set*

$$F+F':=u(F\oplus F')u^{-1}\in\hat{\mathfrak{F}}_B^{p-q},\quad \text{for}\quad F,F'\in\hat{\mathfrak{F}}_B^{p-q}.$$

We will often omit the isometry u and denote this sum simply by $F\oplus F'$.

Proposition 7.1.7. *Let B be a Rg C^*-algebra. Then*

$$KR_{p-q}(B)\cong\{[F]\mid F\in\hat{\mathfrak{F}}_{B,\tau}^{p-q}\},$$

where $\hat{\mathfrak{F}}_{B,\tau}$ denotes the invariant subspace of $\hat{\mathfrak{F}}_B^{p-q}$ of Real elements of $\hat{\mathfrak{F}}_B^{p-q}$, and $[F]$ denotes the homotopy class of F in $\hat{\mathfrak{F}}_{B,\tau}^{p-q}$.

Proof. This comes from the very definition of KKR-theory, the stabilization theorem and [46, Remark 4.2]. □

7.2 The C^*-algebra of a Rg Fell bundle over a Real proper groupoid

In this section we apply the result of the previous section to the reduced C^*-algebra $C^*_r(\mathcal{G};\mathscr{E})$ of a Rg u.s.c Fell bundle $\mathscr{E}=\coprod_{g\in\mathcal{G}}\mathscr{E}_g$ over the Real groupoid $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ with Real Haar system. For this, we follow [90, §4].

Recall that $L^2(\mathcal{G};\mathscr{E})$ is a Rg Hilbert $\mathscr{E}^{(0)}$-module bundle under the canonical projection

$$\coprod_{x\in X} L^2(\mathcal{G}_x;\mathscr{E}) \longrightarrow X,$$

where we have written $\mathscr{E}$ for $\iota_x^*\mathcal{E}\longrightarrow \mathcal{G}_x$ ($\iota_x:\mathcal{G}_x\hookrightarrow\mathcal{G}$ being the identity map), and where $\mathscr{E}^{(0)}:=\mathscr{E}_{|X}$. Now, consider the Rg C^*-bundles

$$\mathcal{L}(L^2(\mathcal{G};\mathscr{E})) := \coprod_{x\in X}\mathcal{L}_{A_x}(L^2(\mathcal{G}_x;\mathscr{E}))\longrightarrow X, \quad \text{and}$$

$$\mathcal{K}(L^2(\mathcal{G};\mathscr{E})) := \coprod_{x\in X}\mathcal{K}(L^2(\mathcal{G}_x;\mathscr{E}))\longrightarrow X.$$

They are in fact Rg C^*-$\mathcal{G}$-bundles under the continuous Real $\mathcal{G}$-action defined in the following way. For every $g\in\mathcal{G}$, let $R_{g^{-1}}:\mathcal{G}_{s(g)}\ni h\longmapsto hg^{-1}\in\mathcal{G}_{r(g)}$, and let $\mathscr{E}' := (R_{g^{-1}})^*(\iota^*_{r(g)}\mathscr{E})$; i.e. $\mathscr{E}'_h=\mathscr{E}_{hg^{-1}}\cong\mathscr{E}_h\hat{\otimes}_{A_{s(g)}}\mathscr{E}_{g^{-1}}$. Then, $L^2(\mathcal{G}_x;\mathscr{E}')\cong L^2(\mathcal{G}_x;\mathscr{E})\hat{\otimes}_{A_{s(g)}}L^2(\mathcal{G}_x;\mathcal{E}_{g^{-1}})$, and hence the map

$$\mathcal{L}(L^2(\mathcal{G}_{s(g)};\mathscr{E}))\ni T\longmapsto T\hat{\otimes}\mathbf{1}\in\mathcal{L}(L^2(\mathcal{G}_{s(g)};\mathscr{E}'))$$

is an isomorphism of graded C^*-algebras. But $L^2(\mathcal{G}_{s(g)};\mathscr{E}')\cong L^2(\mathcal{G}_{r(g)};\mathscr{E})$ under the map $\xi\longmapsto g\xi$ for $\xi\in L^2(\mathcal{G}_{s(g)};\mathscr{E}')$, where $(g\xi)(h):=\xi(hg)\in\mathscr{E}'_{hg}=\mathscr{E}_{hgg^{-1}}\cong\mathscr{E}_h$. Thus, we obtain an isomorphism of graded C^*-algebras

$$\alpha_g:\mathcal{L}(L^2(\mathcal{G}_{s(g)};\mathscr{E}))\longrightarrow\mathcal{L}(L^2(\mathcal{G}_{r(g)};\mathscr{E})).$$

Moreover, it is not hard to see from this construction that $\alpha=(\alpha_g)_{g\in\mathcal{G}}$ satisfies $\alpha_{\bar g}(\overline{T})=\overline{\alpha_g(T)}\in\mathcal{L}(L^2(\mathcal{G}_{r(\bar g)};\mathscr{E}))$ for all $T\in\mathcal{L}(L^2(\mathcal{G}_{s(g)};\mathscr{E}))$, and that $\mathcal{K}(L^2(\mathcal{G};\mathcal{E}))$ is stable under α. Denote by $\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))^{\mathcal{G}}$ the subspace of $\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$ consisting of all $\mathcal{G}$-invariant operators; *i.e.*

$$\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))^{\mathcal{G}} := \{T=(T_x)_{x\in X}\in\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))\mid \alpha_g(T_{s(g)})=T_{r(g)},\forall g\in\mathcal{G}\}.$$

Let $c:X\longrightarrow\mathbb{R}_+$ be a cutoff function for $\mathcal{G}$(cf. **??**). For every $T\in\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))$, we define an element $T^{\mathcal{G}}\in\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))^{\mathcal{G}}$ by setting

$$T^{\mathcal{G}}_x := \int_{\mathcal{G}^x}\alpha_g(T_{s(g)})c(s(g))d\mu^x(g),\ \text{for all } x\in X; \tag{7.4}$$

and hence a Rg projection $\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))\ni T\longmapsto T^{\mathcal{G}}\in\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))^{\mathcal{G}}$.

Definition 7.2.1. *We define the Rg C^*-algebra $\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E}))$ to be the image of $\mathcal{K}(L^2(\mathcal{G};\mathscr{E}))$ under the projection $\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))\longrightarrow\mathcal{L}(L^2(\mathcal{G};\mathscr{E}))^{\mathcal{G}}$ defined above; i.e.,*

$$\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E}))=\left\{T^{\mathcal{G}}\mid T\in\mathcal{K}(L^2(\mathcal{G};\mathscr{E}))\right\}.$$

Theorem 7.2.2. *(Compare [90, Theorem 4.6]). If $\mathcal{G}$ is a Real proper groupoid with Real Haar system and $\mathcal{E} \longrightarrow \mathcal{G}$ is a Rg u.s.c. Fell bundle, then*

$$KR_{p-q}(C_r^*(\mathcal{G};\mathscr{E})) = \left\{[F] \mid F \in \hat{\mathfrak{F}}^{p-q}(\mathcal{G},\mathcal{E})_\tau\right\},$$

where $\hat{\mathfrak{F}}^{p-q}(\mathcal{G},\mathscr{E})_\tau$ is the set consisting of all $F \in \mathcal{L}(L^2(\mathcal{G};\mathscr{E})\hat{\otimes}\hat{\mathcal{H}})^{\mathcal{G}}$ such that

- $F(\mathbf{1}\hat{\otimes}e) = (\mathbf{1}\hat{\otimes}e)F$, $\forall e = e_i, \varepsilon_j, i = 1,\dots,p, j = 1,\dots,q$;
- $F^2 - \mathbf{1}, F - F^* \in \mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E})\hat{\otimes}\hat{\mathcal{H}})$.

Proof. From slight modifications of [90, Proposition 4.3], the Real graded C^*-algebras $C_r^*(\mathcal{G};\mathscr{E})$ and $\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E}))$ are seen to be isomorphic. Therefore, from Proposition 7.1.7, we have

$$KR_j(C_r^*(\mathcal{G};\mathscr{E})) \cong \pi_0\left(\hat{\mathfrak{F}}^j_{\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E})),\tau}\right).$$

Moreover, the arguments used to prove [90, Corollary 4.5] are easily seen to also be valid in the Real graded case, specifically, the Rg C^*-algebras $M(\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E})\hat{\otimes}\hat{\mathcal{H}})$ and $\mathcal{L}(L^2(\mathcal{G};\mathscr{E})\hat{\otimes}\hat{\mathcal{H}})^{\mathcal{G}}$ are isomorphic. Thus, the result follows from the above group isomorphism and the isomorphism of Rg C^*-algebras $\mathcal{K}(\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E}))\hat{\otimes}\hat{\mathcal{H}}) \cong \mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{E})\hat{\otimes}\hat{\mathcal{H}})$. □

7.3 Twisted Real Fredholm operators

Our goal in this section is to give a give a Fredholm interpretation of the twisted KR-groups of a Real proper groupoid $\mathcal{G}$ with Real Haar system, when the twisting is of type 0. Our result will be the analog of [90, Theorem 3.15].

Let $\mathcal{A} \in \widehat{\mathrm{BrR}}_0(\mathcal{G})$ and let $\mathbb{E} = (\tilde{\Gamma},\Gamma,\delta,Z) \in \widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)$ such that $DD(\mathcal{A}) = -dd(\mathbb{E}) = \alpha$, where $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ is proper. Let $L = \tilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C}$ be the Rg Fell bundle over Γ associated to $\mathbb{E}$.

Consider again the Rg Hilbert bundle $\mathscr{H}_{\tilde{\Gamma}} \longrightarrow Y$ defined by (4.28), whose grading and Real structure are given by formulas (4.27) and (4.29), respectively. Recall that the Rg field $\hat{\mathcal{K}}(\mathscr{H}_{\tilde{\Gamma}}) := \coprod_{y\in Y}\hat{\mathcal{K}}(\mathscr{H}_{\tilde{\Gamma},y}) \longrightarrow Y$ is a Rg D-D bundle over the Real groupoid $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$.

Definition 7.3.1. *(Compare [90, p.872]). Denote by $\hat{\mathbb{K}}_\Gamma(\mathscr{H}_{\tilde{\Gamma}})$ the Rg space of norm-continuous Γ-invariant sections $\{\xi_y \mid y \in Y\}$ of $\hat{\mathcal{K}}(\mathscr{H}_{\tilde{\Gamma}}) \longrightarrow Y$ satisfying the boundary condition $\|\xi_y\| \to 0$ when $y \to \infty$ in Y/Γ.*

Now consider the Real field of $\mathbb{Z}_2$-graded C^*-algebras over Y with Rg Γ-action defined by unitary conjugation:

$$\mathcal{L}(\mathscr{H}_{\tilde{\Gamma}}) := \coprod_Y \mathcal{L}(\mathscr{H}_{\tilde{\Gamma},y}) \longrightarrow Y.$$

Then, by identifying $\mathcal{L}(\mathscr{H}_{\widetilde{\Gamma},y})$ with $M(\widehat{\mathcal{K}}(\mathscr{H}_{\widetilde{\Gamma},y}))$, form the Rg unital C^*-algebra $\mathcal{C}_b^{str}(Y;\mathcal{L}(\tilde{\mathscr{H}}_{\widetilde{\Gamma}}))$ as in Definition 5.6.4. We will denote by $\mathcal{C}_b^{str}(Y;\mathcal{L}(\tilde{\mathscr{H}}_{\widetilde{\Gamma}}))^{\Gamma}$ the subspace of $\mathcal{C}_b^{str}(Y;\mathcal{L}(\tilde{\mathscr{H}}_{\widetilde{\Gamma}}))$ consisting of Γ-invariant sections.

Definition 7.3.2. *Suppose we are given the above settings. An α-*twisted Fredholm operators *is a section $F \in \mathcal{C}_b^{str}(Y;\mathcal{L}(\mathscr{H}_{\widetilde{\Gamma}}))^{\Gamma}$ such that*

- *for all $y \in \Gamma$, F_y is of degree 1, $F_y(\mathrm{Id}_{\mathcal{L}_y^2}\hat{\otimes} e) = (\mathrm{Id}_{\mathcal{L}_y^2}\hat{\otimes} e)F_y$, $\forall e = e_i, \varepsilon_j, i = 1,\dots,p, j = 1,\dots,q$;*
- *the maps $y \longmapsto F_y^2 - \mathbf{1}$ and $y \longmapsto F_y - F_y^*$ are elements in $\hat{\mathbb{K}}_{\Gamma}(\mathscr{H}_{\widetilde{\Gamma}})$.*

The space $\widehat{\mathrm{Fred}}_{\alpha}^{p-q}$ of α-twisted Fredholm operators is endowed with the Real structure induced from $\mathcal{C}_b^{str}(Y;\mathcal{L}(\tilde{\mathscr{H}}_{\widetilde{\Gamma}}))$. Let $\widehat{\mathrm{Fred}}_{\alpha,\tau}^{p-q}$ denote the subspace of Real elements of $\widehat{\mathrm{Fred}}_{\mathcal{A}}$.

Theorem 7.3.3. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a Real proper groupoid, $\mathcal{A} \in \widehat{Br}\mathrm{R}_0(\mathcal{G})$, and $\mathbb{E} = (\widetilde{\Gamma},\Gamma,\delta) \in \widehat{Ext}\mathrm{R}(\mathcal{G},\mathbb{S}^1)$ such that $DD(\mathcal{A}) = -dd(\mathbb{E}) = \alpha \in \check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\times \check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)$. Then*

$$KR_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet) \cong \pi_0(\widehat{\mathrm{Fred}}_{\alpha,\tau}^{j}).$$

Proof. From the proof of Corollary 5.6.9 and the equivalence of Rg Fell systems (5.73) established in the proof of Theorem 5.7.13, we have an equivalence of Rg Fell systems

$$(\mathcal{G}, s^*\mathcal{A}) \sim (\Gamma, L).$$

Hence, the Rg C^*-algebras $\mathcal{A} \rtimes_r \mathcal{G} \cong \mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G}; s^*\mathcal{A}))$ and $C_r^*(\mathbb{E}) \cong \mathcal{K}_{\Gamma}(L^2(\Gamma;L))$ are Morita equivalent, so that $KR_{\mathcal{A}}^{-j}(\mathcal{G}^\bullet) = \pi_0(\hat{\mathfrak{F}}^j(\Gamma;L)_\tau)$, thanks to Theorem 7.2.2.

On the other hand, from the isometric isomorphism $\mathcal{C}_c(\Gamma;L) \cong \mathcal{C}_c(\widetilde{\Gamma})^{\mathbb{S}^1}$ of Lemma 5.7.9 we deduce that the Rg Hilbert bundles $\mathscr{H}_{\widetilde{\Gamma}} \longrightarrow Y$ and $\widetilde{L^2(\Gamma;L)}\hat{\otimes}\hat{\mathcal{H}} \longrightarrow Y$ are canonically isomorphic. Moreover, the induced isomorphisms of Real fields of graded C^*-algebras $\mathcal{L}(\mathscr{H}_{\widetilde{\Gamma}}) \cong \mathcal{L}(\widetilde{L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}})$ and $\widehat{\mathcal{K}}(\mathscr{H}_{\widetilde{\Gamma}}) \cong \widehat{\mathcal{K}}(\widetilde{L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}})$ are easily seen to be compatible with the Rg Γ-actions. Thus, we have isomorphisms of Rg C^*-algebras

$$\begin{aligned} \mathcal{L}(L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}) &\cong \mathcal{C}_b^{str}(Y;\mathcal{L}(\widetilde{L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}})) &&\cong \mathcal{C}_b^{str}(Y;\mathcal{L}(\mathscr{H}_{\widetilde{\Gamma}})), \text{ and}\\ \widehat{\mathcal{K}}(L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}) &\cong \mathcal{C}_0(Y;\widehat{\mathcal{K}}(\widetilde{L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}})) &&\cong \mathcal{C}_0(Y;\widehat{\mathcal{K}}(\mathscr{H}_{\widetilde{\Gamma}}))\end{aligned}$$

that are compatible with the Rg Γ-actions. Therefore, there is isomorphisms of Rg C^*-algebras $\hat{\mathbb{K}}_{\Gamma}(\mathscr{H}_{\widetilde{\Gamma}}) \cong \widehat{\mathcal{K}}_{\Gamma}(L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}})$ and $\mathcal{L}(L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}})^{\Gamma} \cong \mathcal{C}_b^{str}(Y;\mathcal{L}(\mathscr{H}_{\widetilde{\Gamma}}))^{\Gamma}$, and hence a Real homeomorphism $\hat{\mathfrak{F}}^j(\Gamma;L) \cong \widehat{\mathrm{Fred}}_{\alpha}^{j}$, which completes the proof. $\square$

7.4 Projective Real Fredholm operators

Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ be a Real proper groupoid, $\mathcal{A}$, $\mathbb{E}$, and α be as in the previous subsection.

For $p,q \in \mathbb{N}$, we define the Real subgroup $\widehat{\mathrm{PU}}_{p-q}(\hat{\mathfrak{H}})$ of $\widehat{\mathrm{PU}}(\hat{\mathfrak{H}})$ consisting of equivalence classes of homogeneous unitaries $u \in \hat{\mathrm{U}}(\hat{\mathfrak{H}})$ commuting with the operators $e_i, \varepsilon_j, i = 1,...,p, j = 1,...,q$. Recall that we have a generalized Real homomorphism

$$\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}} : \Gamma \longrightarrow \widehat{\mathrm{PU}}(\hat{\mathfrak{H}}),$$

defined by (4.30).

Then we obtain a generalized Real homomorphism

$$\mathbb{P}^{p-q} : \Gamma \longrightarrow \widehat{\mathrm{PU}}_{p-q}(\hat{\mathfrak{H}}) \tag{7.5}$$

by defining $\mathbb{P}^{p-q}$ to be the Real subspace of $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$ such that the fiber $\mathbb{P}_{p-q}\hat{\mathscr{U}}_{\tilde{\Gamma},y}$ over $y \in Y$ consists of equivalence classes of unitaries $u \in \hat{\mathscr{U}}_{\tilde{\Gamma},y}$ commuting with $e_i, \varepsilon_j, i = 1,...,p, j = 1,...,q$ in the sense that

$$ue_i = (\mathbf{1}_{L^2(\tilde{\Gamma}^y)^{\mathbb{S}^1}} \hat{\otimes} e_i)u, \quad u\varepsilon_j = (\mathrm{Id}_{L^2(\tilde{\Gamma}^y)^{\mathbb{S}^1}} \hat{\otimes} \varepsilon_j)u, i = 1,...,p, j = 1,...,q.$$

Notice that $\mathbb{P}^0 = \mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$.

Definition 7.4.1. *A continuous $\mathcal{G}$-invariant functions $F : \mathbb{P}^{p-q} \longrightarrow \hat{\mathfrak{F}}^{p-q}$ will be (abusively) called a* projective Fredholm operator *over $(\mathcal{G},\mathcal{A})$, where $\hat{\mathfrak{F}}^{p-q} := \hat{\mathfrak{F}}^{p-q}_{\mathbb{C}}$. Equivalently, a projective Fredholm operator over $(\mathcal{G},\mathcal{A})$ is a continuous function $F : \mathbb{P}^{p-q}/\Gamma \longrightarrow \hat{\mathfrak{F}}^{p-q}$.*

Theorem 7.4.2. *Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ be a Real proper groupoid. Let $\mathcal{A} \in \widehat{Br\mathrm{R}}_0(\mathcal{G})$, and suppose that $\mathbb{E} = (\tilde{\Gamma},\Gamma,\delta,Z) \in \widehat{Ext\mathrm{R}}(\mathcal{G},\mathbb{S}^1)$ is such that $DD(\mathcal{A}) = -dd(\mathbb{E}) = \alpha \in \check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)$. Then there is a bijection between elements of $KR^{-n}_{\mathcal{A}}(\mathcal{G}^\bullet)$ and homotopy classes of Real $\widehat{\mathrm{PU}}_n(\hat{\mathfrak{H}})$-equivariant projective Fredholm operators over $(\mathcal{G},\mathcal{A})$; in particular, we have*

$$KR^{-n}_{\mathcal{A}}(\mathcal{G}^\bullet) \cong \left[\mathbb{P}^n/\Gamma, \hat{\mathfrak{F}}^n\right]_R^{\widehat{\mathrm{PU}}_n(\hat{\mathfrak{H}})},$$

where the symbol $[\cdot,\cdot]_R^{\widehat{\mathrm{PU}}_n(\hat{\mathfrak{H}})}$ stands for homotopy classes of Real $\widehat{\mathrm{PU}}_n(\hat{\mathfrak{H}})$-equivariant continuous functions.

Proof. First of all notice that we have isomorphism of Rg C^*-Γ-bundles

$$\begin{aligned} \mathcal{L}(\hat{\mathscr{H}}_{\tilde{\Gamma}}) &\cong \mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}} \times_{\widehat{\mathrm{PU}}(\hat{\mathfrak{H}})} \mathcal{L}(\hat{\mathfrak{H}}) \longrightarrow Y \\ \hat{\mathfrak{K}}(\hat{\mathscr{H}}_{\tilde{\Gamma}}) &\cong \mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}} \times_{\widehat{\mathrm{PU}}(\hat{\mathfrak{H}})} \hat{\mathfrak{K}}_0 \longrightarrow Y \end{aligned}$$

induced by the map

$$\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma},y} \times_{\widehat{\mathrm{PU}}(\hat{\mathcal{H}})} \mathcal{L}(\hat{\mathcal{H}}) \ni [((y,[u]),T)] \longmapsto Ad_u(T) \in \mathcal{L}(\mathscr{H}_{\tilde{\Gamma},y}), \tag{7.6}$$

for $y \in Y$.

Let us consider the Real Γ-equivariant bundle

$$\hat{\mathfrak{F}}^n(\alpha) := \mathbb{P}^n \times_{\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})} \hat{\mathfrak{F}}^n \longrightarrow Y$$

with typical fibre $\hat{\mathfrak{F}}^n$. Then, using the map (7.6), it is straightforward to see that we may identify $\widehat{\mathrm{Fred}}^n_\alpha$ with the Real space $\mathcal{C}_b(Y;\hat{\mathfrak{F}}^n(\alpha))^\Gamma$ of norm-bounded continuous Γ-invariant sections $\xi : Y \ni y \longmapsto \xi_y \in \mathbb{P}^n_y \times_{\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})} \hat{\mathfrak{F}}^n$ with the property that $\xi^2 - \mathbf{1} : y \longmapsto \xi_y^2 - \mathbf{1}$, and $\xi - \xi^* : y \longmapsto \xi_y - \xi_y^*$ are in $\mathcal{C}_0(Y;\hat{\mathcal{K}}(\mathscr{H}_{\tilde{\Gamma}}))$. Now for $\xi \in \widehat{\mathrm{Fred}}^n_\alpha$, we define a projective Fredholm operator F over $(\mathcal{G},\mathcal{A})$ in the following manner. For $(y,[u]) \in \mathbb{P}^n$, let $F(y,[u])$ be the unique degree 1 element $\mathcal{L}(\hat{\mathcal{H}})$ such that

$$\xi_y = [((y,[u]),F(y,[u]))] \in \mathbb{P}^n_y \times_{\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})} \mathcal{L}(\hat{\mathcal{H}}).$$

Then we clearly have $F(y,[u]) \in \hat{\mathfrak{F}}^n$ thanks to the above interpretation of $\widehat{\mathrm{Fred}}^n_\alpha$, and F is a well-defined map from $\mathbb{P}^n$ to $\hat{\mathfrak{F}}^n$ since ξ depends only on $y \in Y$. Notice that F is Real if ξ is. To see that F is continuous, suppose $(y_i,[u_i]) \to (y,[u])$ in $\mathbb{P}^n$. Then from the definition of the topology of $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$, $y_i \to y$ in Y and $u_i \to u$ in $\widehat{\mathrm{PU}}(\hat{\mathcal{H}})$, and since ξ is continuous, $\xi_{y_i} \to \xi_y$ in $\mathbb{P}^n \times_{\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})} \mathcal{L}(\hat{\mathcal{H}})$, which shows that $F(y_i,[u_i]) \to F(y,[u])$ in $\hat{\mathfrak{F}}^n$. F is Γ-invariant; indeed, the Γ-invariance of ξ implies that if $\xi_{s(\gamma)} = [(s(\gamma),[u]),F(s(\gamma),[u]))]$, then $\xi_{r(\gamma)} = [((r(\gamma),[g\cdot u]),F(s(\gamma),[u]))]$, which shows that $F(r(\gamma),[\gamma\cdot u]) = F(s(\gamma),[u])$ for all $\gamma \in \Gamma$ and $(y,[u]) \in \mathbb{P}^n$ such that $y = s(\gamma)$. That F is $\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})$-equivariant is trivial, by definition of the quotient space $\mathbb{P}^n \times_{\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})} \hat{\mathfrak{F}}^n$.

Conversely, if F is a Real $\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})$-equivariant projective Fredholm operator over $(\mathcal{G},\mathcal{A})$, we evidently obtain an element $\xi \in \widehat{\mathrm{Fred}}^n_\alpha$ by setting $\xi_y := [((y,[u]),F(y,[u]))]$. This process is easily seen to provide a homeomorphism between $\widehat{\mathrm{Fred}}^n_{\alpha,\tau}$ and the space of Real $\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})$-equivariant projective Fredholm operators over $(\mathcal{G},\mathcal{A})$, and we are done. □

Example 7.4.3. *Let (X,τ) be a locally compact Hausdorff Real space, and let $\mathcal{A} \in \widehat{Br\mathrm{R}}_0(X)$. There is a Rg bundle gerbe $(\tilde{\Gamma},X[\mathfrak{U}],\delta)$ over X with Dixmier-Douady class $-DD(\mathcal{A})$, where $\mathfrak{U} = (U_i)_{i\in I}$ is a Real open cover of X. Then*

$$KR^{-n}_{\mathcal{A}}(X) = \left[\mathbb{P}^n/X[\mathfrak{U}],\hat{\mathfrak{F}}^n\right]_R^{\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})}.$$

More concretely, $\mathbb{P}^n$ is given by a Real cocycle $p^n : X[\mathfrak{U}] \longrightarrow \widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})$. Hence, the right hand side of the above equation is isomorphic to the set of Real homotopy classes of Real families $(F_i)_{i\in I}$ of continuous maps $F_i : U_i \longrightarrow \hat{\mathfrak{F}}^n$ with the property that $F_i(x_i) = p^n(x_{ij})F_j(x_j)$ for $x_i = (i,x) \in U_i$, $x_j = (j,x) \in U_j$, and $x_{ij} = (i,x,j) \in U_{ij}$.

Example 7.4.4. *A particular case of the previous example is when $\tau = id$. In that case, $\mathcal{A}$ is represented by a graded real bundle gerbe*

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r] & \tilde{\Gamma} \arrow[r] & Y^{[2]} \arrow[d, "\delta"] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

as noted in Remark 2.8.5. Denote by $\mathrm{PO}_n(\hat{\mathcal{H}}_{\mathbb{R}})$ be the subgroup of Real elements of $\widehat{\mathrm{PU}}_n(\hat{\mathcal{H}})$, where as usual $\hat{\mathcal{H}}_{\mathbb{R}}$ is the Real part of $\hat{\mathcal{H}}$. We define the $\mathrm{PO}_n(\hat{\mathcal{H}}_{\mathbb{R}})$-principal bundle $\mathbb{P}^n_R \longrightarrow Y$ as in the complex case, except that in the definition of $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}}$ we replace $\hat{\mathcal{H}}$ by $\hat{\mathcal{H}}_{\mathbb{R}}$, $\mathbb{C}$ by $\mathbb{R}$, and $\mathbb{S}^1$ by $\mathbb{Z}_2$. Let $\hat{\mathfrak{F}}^n_{\mathbb{R}}$ be the subspace of Real operators of $\hat{\mathfrak{F}}^n$. Thus

$$KR_{\mathcal{A}}^{-n}(X) = KO_{\mathcal{A}}^{-n}(X) \cong \left[\mathbb{P}^n_R/Y^{[2]}, \hat{\mathfrak{F}}^n_{\mathbb{R}}\right]^{\mathrm{PO}_n(\hat{\mathcal{H}}_{\mathbb{R}})}.$$

Moreover, notice that $\mathbb{P}^n/Y^{[2]}$ identifies with a $\mathrm{PO}_n(\hat{\mathcal{H}}_{\mathbb{R}})$-principal bundle $P^n \longrightarrow X$. Therefore,

$$KO_{\mathcal{A}}^{-n}(X) = \left[P^n, \hat{\mathfrak{F}}^n_{\mathbb{R}}\right]^{\mathrm{PO}_n(\hat{\mathcal{H}}_{\mathbb{R}})}. \tag{7.7}$$

Remark 7.4.5. *We shall point out that the spaces $\hat{\mathfrak{F}}^n_{\mathbb{R}}$ are the same as the spaces $Fred_n$ used by Mathai, Murray, and Stevenson in [58]. However, the Fredholm picture of twisted KO-theory given in Section 7.1 of the just mentioned article seems to be wrong; for instance, $PO(\mathcal{H})$ does not act on $Fred_n$.*

An immediate corollary of Theorem 7.4.2 is that in the proper case, the isomorphism established in Proposition 6.4.8 holds without tensoring with $\mathbb{Z}[1/2]$; in other words

Corollary 7.4.6. *Suppose $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$, where $\mathcal{G}_1, \mathcal{G}_2$ are proper Real groupoids such that τ exchanges $\mathcal{G}_1$ with $\mathcal{G}_2$. Then for $\mathcal{A} \in \widehat{\mathrm{BrR}}_0(\mathcal{G})$, we have*

$$KR_{\mathcal{A}}^*(\mathcal{G}^\bullet) \cong KR_{\mathcal{A}_i}^*(\mathcal{G}_i^\bullet).$$

7.5 The pairing $KR_\alpha^{-n}\otimes KR_\beta^{-m}\longrightarrow KR_{\alpha+\beta}^{-n-m}$

In this section we will construct a pairing

$$KR_\alpha^{-n}(\mathcal{G}^\bullet)\otimes KR_\beta^{-m}(\mathcal{G}^\bullet) \longrightarrow KR_{\alpha+\beta}^{-n-m}(\mathcal{G}^\bullet)$$

for a Real proper groupoid.

First of all, recall from [46, §1.16] that for any pairs $(p,q),(k,l)\in\mathbb{N}^2$, the isomorphism of Rg C^*-algebras

$$\mathbb{C}l_{p,q}\hat{\otimes}\mathbb{C}l_{k,l}\xrightarrow{\cong}\mathbb{C}l_{p+k,q+l}$$

is induced from the identifications:

$$\begin{array}{rcl} e_i\hat{\otimes}1 & \longmapsto & e_i,\ i\le p,\\ \varepsilon_j\hat{\otimes}1 & \longmapsto & \varepsilon_j,\ j\le q,\\ 1\hat{\otimes}e_i & \longmapsto & (-1)^q e_{i+p},\ i\le k,\\ 1\hat{\otimes}\varepsilon_j & \longmapsto & \varepsilon_{j+q},\ l\le l, \end{array} \tag{7.8}$$

where we have denoted indistinctly $e_i,\varepsilon_j, i=1,\dots,p, j=1,\dots,q$, $e_i,\varepsilon_j, i=1,\dots,k, j=1,\dots,l$, and $e_i,\varepsilon_j, i=1,\dots,p+k, j=1,\dots,q+l$, the generators of $\mathbb{C}l_{p,q}$, $\mathbb{C}l_{k,l}$, and $\mathbb{C}l_{p+k,q+l}$, respectively. Using those identifications, we get a continuous bilinear paring

$$\begin{array}{rcl} \hat{\mathfrak{F}}^{p-q}\times\hat{\mathfrak{F}}^{k-l} & \longrightarrow & \hat{\mathfrak{F}}^{(p+k)-(q+l)}\\ (F_1,F_2) & \longmapsto & F_1\hat{\boxtimes}F_2 := F_1\hat{\otimes}\mathbf{1}+\mathbf{1}\hat{\otimes}F_2. \end{array}$$

Now let $\mathcal{A},\mathcal{B}\in\widehat{\mathrm{BrR}}_0(\mathcal{G})$, $\mathbb{E}_1=(\tilde{\Gamma}_1,\Gamma_1,\delta_1),\mathbb{E}_2=(\tilde{\Gamma}_2,\Gamma_2,\delta_2)\in\widehat{\mathrm{ExtR}}(\mathcal{G},\mathbb{S}^1)=\alpha$ be such that $DD(\mathcal{A})=-dd(\mathbb{E}_1)$ and $DD(\mathcal{B})=-dd(\mathbb{E}_2)=\beta$. In view of Proposition 2.7.1, we may assume that $\Gamma_1=\Gamma_2=\Gamma$, where Γ is the Real Čech groupoid associated to some Real open cover of X. Hence $\mathbb{E}=\mathbb{E}_1\hat{\otimes}\mathbb{E}_2=(\tilde{\Gamma}\hat{\otimes}\tilde{\Gamma}_2,\Gamma,\delta_1+\delta_2)$ is such that $DD(\mathcal{A}+\mathcal{B})=-dd(\mathbb{E})=\alpha+\beta$. We will write $\mathbb{P}_1^{p-q}$ for $\mathbb{P}^{p-q}\hat{\mathscr{U}}_{\tilde{\Gamma}_1}$, $\mathbb{P}_2^{k-l}$ for $\mathbb{P}^{k-l}\hat{\mathscr{U}}_{\tilde{\Gamma}_2}$, and $\mathbb{P}^{(p+k)-(q+l)}$ for $\mathbb{P}^{(p+k)-(q+l)}\hat{\mathscr{U}}_{\tilde{\Gamma}_1\otimes\tilde{\Gamma}_2}$.

We have already seen in the proof of Lemma 4.9.10 that elements of $\mathbb{P}\hat{\mathscr{U}}_{\tilde{\Gamma}_1\otimes\tilde{\Gamma}_2}$ can be written as sum of elements of the form $[(y,[v\cdot(u_1\hat{\circ}u_2)])]$, for $u_i\in\hat{\mathscr{U}}_{\tilde{\Gamma}_i,y}, i=1,2$ and $(y,v)\in\mathscr{U}(\Gamma)$ (cf. isomorphism (4.32)), where the unitary $v\cdot(u_1\hat{\circ}u_2)$ is given by (4.33). Furthermore, from routine verifications using again the identifications (7.8), we see that if $[(y,v\cdot(u_1\hat{\circ}u_2))]\in\mathbb{P}^{(p+k)-(q+l)}$, then $[(y,u_1)]\in\mathbb{P}_1^{p-q}$ and $[(y,u_2)]\in\mathbb{P}_2^{k-l}$. Therefore, from the projective Fredhom picture of twisted KR-theory given by Theorem 7.4.2, we deduce the following

Proposition 7.5.1. *Let $\mathcal{G}\underset{s}{\overset{r}{\rightrightarrows}}X$ be a Real proper groupoid. Then for $\mathcal{A},\mathcal{B}\in\widehat{\mathrm{BrR}}_0(\mathcal{G})$, there is a bilinear pairing*

$$\begin{array}{rcl} KR_{\mathcal{A}}^{-n}(\mathcal{G}^\bullet)\times KR_{\mathcal{B}}^{-m}(\mathcal{G}^\bullet) & \longrightarrow & KR_{\mathcal{A}+\mathcal{B}}^{-n-m}(\mathcal{G}^\bullet)\\ ([F_1],[F_2]) & \longmapsto & [F_1\hat{\boxtimes}F_2], \end{array}$$

where if $[F_1],[F_2]$ are represented by a $\widehat{\mathrm{PU}}_{p-q}(\hat{\mathcal{H}})$-equivariant projective Real Fredholm operator $F_1:\mathbb{P}_1^{p-q}/\Gamma\longrightarrow\hat{\mathfrak{F}}^{p-q}$, and a $\widehat{\mathrm{PU}}_{k-l}(\hat{\mathcal{H}})$-equivariant projective Real Fredholm operator $F_2:\mathbb{P}_2^{k-l}/\Gamma\longrightarrow\hat{\mathfrak{F}}^{k-l}$, then the $\widehat{\mathrm{PU}}_{(p+k)-(q+l)}(\hat{\mathcal{H}})$-equivariant projective Real Fredholm operator $F_1\hat{\boxtimes}F_2:\mathbb{P}^{(p+k)-(q+l)}/\Gamma\longrightarrow\hat{\mathfrak{F}}^{(p+k)-(q+l)}$ is defined by

$$(F_1\hat{\boxtimes}F_2)([(y,v\cdot(u_1\hat{\circ}u_2))]) := F_1([(y,u_1)])\hat{\boxtimes}F_2([(y,u_2)]).$$

In particular, $KR_{\mathcal{A}}^*(\mathcal{G}^\bullet)$ has the structure of $KR^0(\mathcal{G}^\bullet)$-module.

8

The torsion case

In this chapter we want to express twisted KR-theory by geometric objects. Roughly speaking, we will see that as in the ungraded complex case, our twisted KR-theory groups may be expressed in terms of twisted vector bundles when the twisting is a torsion element of the Real graded Brauer group. We should point out that most of the constructions and results of this section are adaptation of the exposition of [90, §5].

8.1 Rg twisted vector bundles

Let X be a locally compact Hausdorff Real space. By a *Real graded hermitian vector bundle* over X we mean a Real vector bundle $V \longrightarrow X$ together with a $\mathbb{Z}_2$-grading $V = V^0 \oplus V^1$ compatible with the Real structure (i.e. each of $V^i \longrightarrow X, i = 0, 1$ is a Real vector bundle under the involution induced from that of V) and a hermitian metric $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}_+$ such that $\langle \bar{v}, \bar{w} \rangle = \langle v, w \rangle$ for all $v, w \in V$. Associated to such a $V \to X$, there is a canonical Real graded $\mathbf{S}^{1,1}$-twist

$$\begin{tikzcd} \mathbf{S}^{1,1} \arrow[r] & \mathbf{GL}^{gr}(V) \arrow[r, "pr"] & \mathbb{P}\mathbf{GL}^{gr}(V) \arrow[d, "deg"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{8.1}$$

where the Real groupoids $\mathbf{GL}^{gr}(V) \rightrightarrows X$ and $\mathbb{P}\mathbf{GL}^{gr}(V) \rightrightarrows X$ are defined as follows. For $i = 0, 1$, let

$$\mathbf{GL}^i(V) := \coprod_{x,y \in X} \mathrm{Isom}^{(i)}(V_y, V_x),$$

where, as usual, we have denoted by $\mathrm{Isom}^{(i)}(V_y, V_x)$ the set of all isomorphisms of degree $i \mod 2$ from $V_y^0 \oplus V_y^1$ to $V_x^0 \oplus V_x^1$. Next, define $\mathbf{GL}^{gr}(V)$ as the disjoint union

$$\mathbf{GL}^{gr}(V) := \mathbf{GL}^0(V) \sqcup \mathbf{GL}^1(V).$$

A morphism in $\mathbf{GL}^{gr}(V)$ is then an ismorphism $u_{x,y} : V_y \longrightarrow V_x$ which is homogeneous (i.e. of degree 0 or 1). Define the source and range maps by $s(u_{x,y}) = y$, and $r(u_{x,y}) = x$, respectively. Two morphisms $u_{x,y}$ and $u_{y',z}$ are composable if and only if $y = y'$, in which case the product is given by $u_{x,y} \cdot u_{y,z} := u_{x,y} \circ u_{y,z} : V_z \longrightarrow V_x$; the inverse of $u_{x,y}$ is $u_{x,y}^{-1} : V_x \longrightarrow V_y$. The involution of $\mathbf{GL}^{gr}(V)$ is given by $\overline{u_{x,y}} : V_{\bar{y}} \ni v \longmapsto \overline{u_{x,y}(\bar{v})} \in V_{\bar{x}}$, for all morphism $u_{x,y}$. Notice that $\mathbf{GL}^0(V) \rightrightarrows X$ is a full Real subgroupoid of $\mathbf{GL}^{gr}(X)$ while $\mathbf{GL}^1(V)$ is not stable under the product. It is easy to verify that $\mathbf{GL}^{gr}(V) \rightrightarrows X$, given the norm topology induced by the hermitian metric, is a locally compact Real groupoid. Now the Real group $\mathbf{S}^{1,1}$, identified with $\mathrm{U}(1) \subset \mathbb{C}^*$, acts by scalar multiplication on $\mathbf{GL}^{gr}(V)$, and the action is compatible with the Real structures. Set

$$\mathbb{P}\mathbf{GL}^{gr}(V) := \mathbf{GL}^{gr}(V)/\mathbb{S}^{1,1}.$$

Then $\mathbb{P}\mathbf{GL}^{gr}(V)$, equipped with the quotient topology and the involution $\overline{[u_{x,y}]} := [\overline{u_{x,y}}]$ is clearly a locally compact Real groupoid with base space X. The projection $pr : \mathbf{GL}^{gr}(V) \longrightarrow \mathbb{P}\mathbf{GL}^{gr}(V)$ is just the quotient map, and the grading of the twist $deg : \mathbb{P}\mathbf{GL}^{gr}(V) \longrightarrow \mathbb{Z}_2$ is the degree map.

We will use the following notations.

Notations 8.1.1. *Let $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^l$ be equipped with the grading $(z,w) \longmapsto (z,-w)$, and its usual Real structure given by the complex conjugation. Then we write $GL^{gr}_{k+l}(\mathbb{C})$ (resp. $PGL^{gr}_{k+l}(\mathbb{C})$) for $\mathbf{GL}^{gr}(\mathbb{C}^n) \rightrightarrows \cdot$ (resp. $\mathbb{P}\,\mathbf{GL}^{gr}(\mathbb{C}^n) \rightrightarrows \cdot$), where we consider $\mathbb{C}^n \longrightarrow \cdot$ as a Rg hermitian vector bundle (under its standard hermitian metric) over the point.*

Definition 8.1.2. *Let $(\widetilde{\Gamma},\delta)$ be a Real graded $\mathbb{S}^1$-groupoid over $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$. By a* Rg $(\widetilde{\Gamma},\delta)$-module *over $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ we mean a triple (V,h,ρ) consisting of a Rg hermitian vector bundle $V = V^0 \oplus V^1 \longrightarrow Y$, a Real hermitian endomorphism $h : V \longrightarrow V$ of degree 1, and a Real groupoid morphism*

$$\rho : \widetilde{\Gamma} \longrightarrow \mathbf{GL}^{gr}(V), \tilde{\gamma} \longmapsto \rho_{\tilde{\gamma}},$$

such that

(i) $\deg \rho_{\tilde{\gamma}} = \delta(\pi(\tilde{\gamma})), \forall \tilde{\gamma} \in \widetilde{\Gamma}$;

(ii) $\rho_{\lambda\tilde{\gamma}} = \lambda \rho_{\tilde{\gamma}}$, *for all* $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$, *and* $\tilde{\gamma} \in \widetilde{\Gamma}$;

(iii) $\rho_{\tilde{\gamma}} \circ h = h \circ \rho_{\tilde{\gamma}}, \forall \tilde{\gamma} \in \widetilde{\Gamma}$.

Such settings will often be represented by pair (V,h), and we say that ρ is a Real $(\widetilde{\Gamma},\delta)$-module structure on (V,h).

Lemma 8.1.3. *Let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Rg $\mathbb{S}^1$-twist. If there exists a Rg $(\widetilde{\Gamma},\delta)$-module (V,h) of rank 1, then $[\widetilde{\Gamma},\delta] = 0$ in $\widehat{ext}\mathrm{R}(\Gamma,\mathbb{S}^1)$.*

Proof. Since $V \longrightarrow Y$ is a line bundle, the grading of V is trivial; i.e., $V \cong V^0$. It then follows from (i) of Definition 8.1.2 that $\delta = 0$. Let $PV := V/\mathbb{S}^1$ be endowed with the obvious Real structure, where $\mathbb{S}^1 \subset \mathbb{C}$ acts by scalar multiplication on the fibres. Denote by $[v]$ the class of $v \in V$ in PV. Using the hermitian metric of V, we put

$$S(PV) := \{[v] \in PV \mid \|v\| = 1\}.$$

Then, the canonical projection $S(PV) \longrightarrow Y$ defines a Real $\mathbb{S}^1$-principal bundle. Moreover, $S(PV)$ defines an element of the Real Picard group $\mathrm{PicR}(\Gamma)$ under the Real Γ-action given by $\gamma \cdot [v] := [\rho_{\tilde{\gamma}}(v)]$, for $\gamma \in \Gamma$, $v \in PV_{s(\gamma)}$, and $\tilde{\gamma}$ any lift of γ through the projection $\pi : \widetilde{\Gamma} \longrightarrow \Gamma$. It turns out that there is an isomorphism of Real $\mathbb{S}^1$-principal bundles

$$\widetilde{\Gamma} \hat{\otimes} s^*(S(PV)) \xrightarrow{\cong} r^*(S(PV))$$

given by $\tilde{\gamma} \otimes [v] \longmapsto [\rho_{\tilde{\gamma}}(v)]$. We thus have $(\widetilde{\Gamma}, \delta) \cong (r^*(S(PV)) \otimes \overline{s^*(S(PV))}, 0)$. $\square$

Corollary 8.1.4. *Assume there is a Rg* $(\widetilde{\Gamma}, \delta)$*-module of finite rank. Then* $[(\widetilde{\Gamma}, \delta)]$ *is a torsion element in* $\widehat{\mathrm{ext}\mathrm{R}}(\Gamma, \mathbf{S}^{1,1})$.

Proof. Let (V, h, ρ) be a Rg $(\widetilde{\Gamma}, \delta)$-module of rank n. Then the determinant line bundle $\Lambda^n V \longrightarrow Y$, with the Real structure $v_1 \wedge \ldots \wedge v_n \longmapsto \bar{v}_1 \wedge \ldots \wedge \bar{v}_n$, is a Real line bundle, and the Real morphism $\rho : \widetilde{\Gamma} \longrightarrow \mathbf{GL}^{gr}(V)$ induces a Real groupoid $\mathbf{S}^{1,1}$-equivariant morphism

$$\rho^{\otimes n} : \widetilde{\Gamma}^{\hat{\otimes} n} \longrightarrow \mathbf{GL}^{gr}(\Lambda^n V),$$

by setting:

$$\rho^{\otimes n}_{(\tilde{\gamma}, \lambda_1, \cdots, \lambda_{n-1})}(v_1 \wedge \ldots \wedge v_n) := \lambda_1 \cdots \lambda_{n-1} \rho_{\tilde{\gamma}}(v_1) \wedge \ldots \wedge \rho_{\tilde{\gamma}}(v_n),$$

for all $(\tilde{\gamma}, \lambda_1, \ldots, \lambda_{n-1}) \in \widetilde{\Gamma} \times \mathbf{S}^{n-1,n-1} \cong \widetilde{\Gamma}^{\hat{\otimes} n}$, and $v_1 \wedge \ldots \wedge v_n \in \Lambda^n V_{s(\tilde{\gamma})}$. The result follows from Lemma 8.1.3. $\square$

Proposition 8.1.5. *Let* $(\widetilde{\Gamma}, \Gamma, \delta)$ *be a Rg* $\mathbb{S}^1$*-twist. Consider the following properties:*

(i) there exists a rank n Rg $(\widetilde{\Gamma}, \delta)$*-module* (V, h) *over* Γ*;*

(ii) there exists a generalized morphism $P : (\widetilde{\Gamma}, \Gamma, \delta) \longrightarrow (GL^{gr}_{k+l}(\mathbb{C}), PGL^{gr}_{k+l}(\mathbb{C}), deg)$*, where* $k + l = n$ *with* $k = dimV^0$ *and* $l = dimV^1$*;*

(iii) the class of $(\widetilde{\Gamma}, \delta)$ *is a torsion element in* $\widehat{\mathrm{ext}\mathrm{R}}(\Gamma, \mathbb{S}^1)$ *of degree n;*

(iv) there exists a Real open cover $\mathfrak{U}$ *of* Y *and a Real groupoid morphism* $\varphi : \widetilde{\Gamma}[\mathfrak{U}] \longrightarrow \mathbb{C}^*$ *such that* $\varphi(\lambda\tilde{\gamma}) = \lambda^n \varphi(\tilde{\gamma})$ *for all* $\lambda \in \widetilde{\Gamma}[\mathfrak{U}]$ *and* $\tilde{\gamma} \in \widetilde{\Gamma}[\mathfrak{U}]$*;*

(v) *there exists a Real open cover* $\mathcal{U}$ *of* Y *and a Real graded* $\mathbb{Z}_n$*-central extension*

$$\begin{tikzcd} \mathbb{Z}_n \arrow[r] & \widetilde{\Gamma}'[\mathcal{U}] \arrow[r] & \Gamma[\mathcal{U}] \arrow[d, "\delta'"] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

such that $\widetilde{\Gamma}[\mathcal{U}] \cong \widetilde{\Gamma}' \times_{\mathbb{Z}_n} \mathbf{S}^{1,1}$ *under an isomorphism compatible with the grading, where* $\mathbb{Z}_n$ *is identified with the group of* $n-th$ *roots of unity in* $\mathbb{S}^1 \subset \mathbb{C}$ *equipped with the Real structure given by complex conjugation.*

Then $(i) \Longleftrightarrow (ii) \Longrightarrow (iii) \Longleftrightarrow (iv) \Longleftrightarrow (v)$.

Proof. The proof is a slight modification of the one of [90, Proposition 5.5]. □

Remark 8.1.6. *It will prove useful to describe how the morphism*

$$P : (\widetilde{\Gamma}, \Gamma, \delta) \longrightarrow (GL^{gr}_{k+l}(\mathbb{C}), PGL^{gr}_{k+l}(\mathbb{C}), deg)$$

is constructed. In the (complex) ungraded case, $P \longrightarrow Y$ *is just the frame bundle of* $V \longrightarrow Y$. *However, in our context we have to take into account the involutions involved. Spelled out,* P *has fibre* $P_y = \mathrm{Isom}^{(0)}(\mathbb{C}^{k+l}, V_y) \sqcup \mathrm{Isom}^{(1)}(\mathbb{C}^{k+l}, V_y)$ *over* $y \in Y$, *with the obvious* $GL^{gr}_{k+l}(\mathbb{C})$*- and* $\mathbf{S}^{1,1}$*-actions (viewing* $\mathbf{S}^{1,1}$ *as a subset of* $\mathbb{C}$*). The Real structure of* P *is given by* $\bar{f}(z) := \overline{f(\bar{z})} \in V_{\bar{y}}$, *for* $f \in P_y, z \in \mathbb{C}^{k+l}$. *The* $\widetilde{\Gamma}$*-action on* P *is defined as follows: for* $\tilde{\gamma} \in \widetilde{\Gamma}, f \in P_{s(\tilde{\gamma})}$, *the isomorphism* $\tilde{\gamma} \cdot f \in P_{r(\tilde{\gamma})}$ *is given by* $(\tilde{\gamma} \cdot f)(z) := \rho_{\tilde{\gamma}}(f(z))$. *Now it is easy to check that as Rg vector bundles over* Y, $V \cong P \times_{GL^{gr}_{k+l}(\mathbb{C})} \mathbb{C}^{k+l}$.

Definition 8.1.7. *Let* (V, h_V) *and* (W, h_W) *be Rg* $(\widetilde{\Gamma}, \delta)$*-modules (of finite ranks). A morphism from* (V, h_V) *to* (W, h_W) *is a morphism of Rg vector bundles* $\varphi : V \longrightarrow W$ *such that*

(i) φ *is compatible with the hermitian metrics;*

(ii) $\varphi \circ h_V = h_W \circ \varphi$;

(iii) $\varphi_{r(\tilde{\gamma})} \circ \rho^V_{\tilde{\gamma}} = \rho^W_{\tilde{\gamma}} \circ \varphi_{s(\tilde{\gamma})}$, *for all* $\tilde{\gamma} \in \widetilde{\Gamma}$, *where* $\rho^V : \widetilde{\Gamma} \longrightarrow \boldsymbol{GL}^{gr}(V)$ *and* $\rho^W : \widetilde{\Gamma} \longrightarrow \boldsymbol{GL}^{gr}(W)$ *are the Real* $(\widetilde{\Gamma}, \delta)$*-module structures.*

Definition 8.1.8. *If* (V, h_V) *and* (W, h_W) *are Rg* $(\widetilde{\Gamma}, \delta)$*-modules over* Γ, *with hermitian metrics* $\langle\cdot,\cdot\rangle_V$ *and* $\langle\cdot,\cdot\rangle_W$, *respectively, then the direct sum* $V \oplus W$ *of the Real hermitian vector bundle* $V \longrightarrow Y$ *and* $W \longrightarrow Y$ *carries the hermitian metric* $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_V + \langle\cdot,\cdot\rangle_W$, *the Real* $(\widetilde{\Gamma}, \delta)$*-module structure* $\rho_{V\oplus W} : \widetilde{\Gamma} \longrightarrow \boldsymbol{GL}^{gr}(V \oplus W)$ *given by* $\rho^{V\oplus W}_{\tilde{\gamma}} := \rho^V_{\tilde{\gamma}} \oplus \rho^W_{\tilde{\gamma}}$, *and the degree* 1 *hermitian map* $h_{V\oplus W} := h_V \oplus h_W$. *The Rg* $(\widetilde{\Gamma}, \delta)$*-module* $(V \oplus W, h_{V\oplus W})$ *over* Γ *thus obtained is called the direct sum of* (V, h_V) *and* (W, h_W).

Definition 8.1.9. *Let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Rg $\mathbb{S}^1$-twist. By $\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma)$ we denote the Abelian monoid consisting of all isomorphism classes of Real graded $(\widetilde{\Gamma},\delta)$-modules over Γ, in which the sum of two classes $[(V,h_V)]$ and $[(W,h_W)]$ is*

$$[(V,h_V)]+[(W,h_W)]:=[(V\oplus W,h_{V\oplus W})].$$

Lemma 8.1.10. *The assignment $(\widetilde{\Gamma},\Gamma,\delta)\longmapsto \widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma)$ is a contravariant functor from the category of Rg $\mathbf{S}^{1,1}$-twists $\mathfrak{Twist}_{\mathfrak{R}}$ to the category of Abelian monoids. In particular, if $(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)$ and $(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$ are Morita equivalent Rg $\mathbf{S}^{1,1}$-twists. Then*

$$\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma}_1,\delta_1)}(\Gamma_1)\cong\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma}_2,\delta_2)}(\Gamma_2).$$

Proof. Let $Z:(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\longrightarrow(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$ be a morphism in $\mathfrak{Twist}_{\mathfrak{R}}$. If (V,h) is a Rg $(\widetilde{\Gamma}_2,\delta_2)$-module over $\Gamma_2 \rightrightarrows Y_2$, then by Proposition 8.1.5 (ii) there exists a morphism

$$P_2:(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)\longrightarrow(GL^{gr}_{k+l}(\mathbb{C}),PGL^{gr}_{k+l}(\mathbb{C}),deg),$$

and hence a morphism

$$P_1:(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\longrightarrow(GL^{gr}_{k_l}(\mathbb{C}),PGL^{gr}_{k+l}(\mathbb{C}),deg),\text{ where } P_1=P_2\circ Z.$$

We obtain the Rg $(\widetilde{\Gamma}_1,\delta_1)$-module (V^Z,h^Z) by setting:

$$V^Z:=P_1\times_{GL^{gr}_{k+l}(\mathbb{C})}\mathbb{C}^{k+l},$$

with the hermitian metric defined fiberwise as the standard metric of $\mathbb{C}^n$, and the Real structure $[p,z]\longmapsto[\bar{p},\bar{z}]$. The hermitian map $h^Z\in\mathrm{End}(V^Z)$ is defined fiberwise as the hermitian matrix of degree 1 obtained from $h\in\mathrm{End}(V)^1$ via the isomorphism $P_2\times_{GL^{gr}_{k_l}(\mathbb{C})}\mathbb{C}^{k+l}\xrightarrow{\cong}V$ given by $[p_y,z]\longmapsto p_y(z)\in V_y$. Suppose that $f:(V,h_V)\longrightarrow(W,h_W)$ is an isomorphism of Real graded $(\widetilde{\Gamma}_2,\delta_2)$-modules of rank $n=k+l$. Then f induces a Real $\mathcal{G},GL^{gr}_{k+l}(\mathbb{C})$-equivariant isomorphism $f:P_2\xrightarrow{\cong}Q_2$, where P_2 and Q_2 is the frame bundles of $V\longrightarrow Y_2$ and $W\longrightarrow Y_2$, respectively; hence f induces a 2-isomorphism

$$P_2\cong Q_2:(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)\longrightarrow(GL^{gr}_{k+l}(\mathbb{C}),PGL^{gr}_{k+l}(\mathbb{C}),deg).$$

The map

$$\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma}_2,\delta_2)}(\Gamma_2)\longrightarrow\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma}_1,\delta_1)}(\Gamma_1),[(V,h)]\longmapsto[(V^Z,h^Z)],\tag{8.2}$$

is thus well defined. Observe that up to isomorphism, this map depends only on the 2-isomorphism class of Z. Indeed, it is straightforward from the construction that if

$$Z':(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\longrightarrow(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$$

is another morphism in $\mathfrak{Twist}_{\mathfrak{R}}$, then every 2-morphism $f:Z\longrightarrow Z'$ naturally induces a morphism $(V^Z,h^Z)\longrightarrow(V^{Z'},h^{Z'})$ of Real graded $(\widetilde{\Gamma}_1,\delta_1)$-modules. In particular, if $Z\cong$

Z', then $(V^Z,h^Z)\cong(V^{Z'},h^{Z'})$. If Z is actually an isomorphism, then the map (8.2) is a bijection whose inverse is given by $[(V_1,h_1)]\longmapsto[(V_1^{Z^{-1}},h_1^{Z^{-1}})]$. Now we verify that (8.2) is a homomorphism of Abelian monoids. Let $(V,h_V),(W,h_W),P_2$, and Q_2 be as above. Then

$$\begin{aligned}V^Z\oplus W^Z &=\left(P_1\times_{GL^{gr}_{k+l}(\mathbb{C})}\mathbb{C}^{k+l}\right)\oplus\left(Q_1\times_{GL^{gr}_{k+l}(\mathbb{C})}\mathbb{C}^{k+l}\right)\\ &=\left(P_1\times_{GL^{gr}_{k+l}(\mathbb{C})}\mathbb{C}^{k+l}\right)\times_{Y_1}\left(Q_1\times_{GL^{gr}_{k+l}(\mathbb{C})}\mathbb{C}^{k+l}\right)\\ &\cong(P_1\times_{Y_1}Q_1)\times_{GL^{gr}_{k+l}\mathbb{C}}\mathbb{C}^{k+l}\\ &\cong(V\oplus W)^Z.\end{aligned}$$

$\square$

Definition 8.1.11. *Let* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *be a Real groupoid. Then for* $[(\widetilde{\Gamma},\delta)]\in\widehat{ext\mathrm{R}}(\Gamma,\mathbf{S}^{1,1})$, *we define the* K-theory of Rg $(\widetilde{\Gamma},\delta)$*-modules over* Γ *to be the Grothendieck group* $\widetilde{KR}_{(\widetilde{\Gamma},\delta)}(\Gamma^\bullet)$ *of* $\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma)$.

In view of the preceding lemma we have

Lemma 8.1.12. *(Compare with [90, Corollary 5.7]). The assignment*

$$(\widetilde{\Gamma},\Gamma,\delta)\longmapsto\mathbf{KR}(\widetilde{\Gamma},\Gamma,\delta):=\widetilde{KR}_{(\widetilde{\Gamma},\delta)}(\Gamma^\bullet)$$

is a contravariant functor from the category $\mathfrak{Twist}_{\mathfrak{R}}$ *to the category* $\mathfrak{Ab}$. *In particular, if* $(\widetilde{\Gamma}_1,\Gamma_1,\delta_1)\cong(\widetilde{\Gamma}_2,\Gamma_2,\delta_2)$ *then*

$$\widetilde{KR}_{(\widetilde{\Gamma}_1,\delta_1)}(\Gamma_1^\bullet)\cong\widetilde{KR}_{(\widetilde{\Gamma}_2,\delta_2)}(\Gamma_2^\bullet).$$

Let $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ be a Real groupoid. Let $(\widetilde{\Gamma}_i,\delta_i), i=1,2$ be Real graded $\mathbf{S}^{1,1}$-groupoids over $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$. Suppose that (V_i,h_i) is a Rg (Γ_i,δ_i)-module over $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$, with module structure $\rho_i:\widetilde{\Gamma}_i\longrightarrow\mathbf{GL}^{gr}(V_i), i=1,2$. Then the graded tensor product hermitian bundle $V_1\hat{\otimes}V_2\longrightarrow Y$ together with obvious Real structure and hermitian map $h=h_1\otimes h_2$, and equipped with the $(\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2,\delta_1\otimes\delta_2)$-module structure

$$\rho_1\hat{\otimes}\rho_2:\widetilde{\Gamma}_1\hat{\otimes}\widetilde{\Gamma}_2\longrightarrow\mathbf{GL}^{gr}(V_1\hat{\otimes}V_2),\ [(\tilde{\gamma}_1,\tilde{\gamma}_2)]\longmapsto(\rho_1)_{\tilde{\gamma}_1}\hat{\otimes}(\rho_2)_{\tilde{\gamma}_2},$$

where $((\rho_1)_{\tilde{\gamma}_1}\hat{\otimes}(\rho_2)_{\tilde{\gamma}_2})(v_1\hat{\otimes}v_2):=(\rho_1)_{\tilde{\gamma}_1}(v_1)\hat{\otimes}(\rho_2)_{\tilde{\gamma}_2}(v_2)$ for $v_1\hat{\otimes}v_2\in V_1\hat{\otimes}V_2$, is a Real graded $(\widetilde{\Gamma}_1\hat{\otimes}\Gamma_2,\delta_1\otimes\delta_2)$-module over $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$. From this construction we easyli show the following.

Lemma 8.1.13. *Suppose* $(\widetilde{\Gamma}_i,\delta_i), i=1,2$ *are Rg* $\mathbb{S}^1$*-groupoids over the Real groupoid* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$. *Then there is a homomorphism*

$$\widetilde{KR}_{(\widetilde{\Gamma}_1,\delta_1)}(\Gamma^\bullet)\otimes\widetilde{KR}_{(\widetilde{\Gamma}_2,\delta_2)}(\Gamma^\bullet)\longrightarrow\widetilde{KR}_{(\widetilde{\Gamma}_1,\delta_1)\hat{\otimes}(\widetilde{\Gamma}_2,\delta_2)}(\Gamma^\bullet). \tag{8.3}$$

We now define Rg twisted vector bundles over a Real groupoid.

Definition 8.1.14. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a Real groupoid with Real Haar system, and let* $\alpha \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$. *By a* Rg α-twisted vector bundle *we mean a Real graded* $(\widetilde{\Gamma}, \delta)$*-module* (V, h) *over a Real groupoid* $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ *Morita equivalent to* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *such that* $(\widetilde{\Gamma}, \Gamma, \delta)$ *is a Real graded* $\mathbb{S}^1$*-central extension over* $\mathcal{G}$ *whose Dixmier-Douady class is* α.

Definition 8.1.15. *Let* $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ *be a Real groupoid with Real Haar system. Given a class* $\alpha \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$, *we define the* K*-theory of Rg* α*-twisted vector bundles over* $\mathcal{G}$ *by*

$$KR_{\alpha,vb}(\mathcal{G}^\bullet) := \mathbf{KR}(\widetilde{\Gamma}, \Gamma, \delta) = \mathbf{KR}(\mathbb{E}),$$

where the extension $[\mathbb{E}] = [(\widetilde{\Gamma}, \Gamma, \delta)] \in \widehat{Ext}\mathrm{R}(\mathcal{G}, \mathbb{S}^1)$ *is such that* $DD(\mathbb{E}) = \alpha$.

Forgetting the Real structures, the following proposition generalizes the homomorphism in twisted complex K-theory of spaces $K^0_{\mathcal{A}}(X) \longrightarrow \mathrm{Pic}(X)$ constructed in [55, §6].

Proposition 8.1.16. *The assignment* $\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma) \ni (V, h) \longmapsto \Lambda^n V \in Pic\mathrm{R}(\Gamma)$ *induces a group homomorphism*

$$\det : KR_{\alpha,vb}(\mathcal{G}^\bullet) \longrightarrow \check{H}R^1(\mathcal{G}_\bullet, \mathbb{S}^1).$$

Proof. We showed in the proof of Lemma 8.1.3 that a Rg $(\widetilde{\Gamma}, \delta)$-module of rank 1 defines an element of the Real Picard group of Γ. Further, we saw in Corollary 8.1.4 that the determinant line bundle $\Lambda^n V$ of a Rg $(\widetilde{\Gamma}, \delta)$-module (V, h) was a Rg $(\widetilde{\Gamma}, \delta)$-module of rank 1, and hence an element of $\mathrm{PicR}(\Gamma) \cong \check{H}R^1(\Gamma, \mathbb{S}^1)$. It moreover is straightforward to verify that the hermitian line bundles over $\Gamma \overset{r}{\underset{s}{\rightrightarrows}} Y$ associated to isomorphic Rg $(\widetilde{\Gamma}, \delta)$-modules are isomorphic, and hence define the same class in $\mathrm{PicR}(\Gamma)$. The assignment $\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma) \ni [(V, h)] \longmapsto [\Lambda^n V] \in \mathrm{PicR}(\Gamma)$ is a homomorphism of Abelian monoids from the properties of the functor Λ^*. It then extends to a group homomorphism $\det : KR_{\alpha,vb}(\mathcal{G}^\bullet) \longrightarrow \check{H}R^1(\Gamma_\bullet, \mathbb{S}^1) \cong \check{H}R^1(\mathcal{G}_\bullet, \mathbb{S}^1)$. □

Consider the Real groupoid $\mathcal{G} \times \mathbf{S}^{p,q} \rightrightarrows X \times \mathbf{S}^{p,q}$. Then by cofunctoriality, the inclusion map $i_{p+q} : \mathcal{G} \times \{pt\} \hookrightarrow \mathcal{G} \times \mathbf{S}^{p,q}$ induces a homomorphism of groups

$$i^*_{p+q} : KR_{\alpha_{p+q},vb}((\mathcal{G} \times \mathbf{S}^{p,q})^\bullet) \longrightarrow KR_{\alpha,vb}(\mathcal{G}^\bullet),$$

where $\alpha_{p+q} := i^\alpha_{p+q} \in \check{H}R^1((\mathcal{G} \times \mathbf{S}^{p,q})_\bullet, \mathbb{Z}_2) \times \check{H}R^2((\mathcal{G} \times \mathbf{S}^{p,q})_\bullet, \mathbb{S}^1)$.

Definition 8.1.17. *We define the higher* $KR_{\alpha,vb}$*-groups by*

$$KR^{p-q}_{\alpha,vb}(\mathcal{G}^\bullet) := \ker(i^*_{p+q}),$$

where i^*_{p+q} *is the homomorphism defined above.*

Definition 8.1.18. *Define the* KR-theory of Real vector bundles over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *as*

$$KR^*_{vb}(\mathcal{G}^\bullet) := KR^*_{[1],vb}(\mathcal{G}^\bullet),$$

where $[1]$ *is the trivial class* $(0,1) \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$.

Note that if the Rg $\mathbb{S}^1$-central extension $\mathbb{E} = (\widetilde{\Gamma}, \Gamma, \delta)$ of $\mathcal{G}$ is such that $DD(\mathbb{E}) = \alpha$, then $DD(\mathbb{E}^{p,q}) = \alpha'_{p+q}$, where $\mathbb{E}^{p,q}$ is the Rg $\mathbb{S}^1$-central extension of the Real groupoid

$$\mathcal{G} \times \mathbb{R}^{p,q} \rightrightarrows X \times \mathbb{R}^{p,q}$$

given by

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \widetilde{\Gamma} \times \mathbb{R}^{p,q} \arrow[r, "\pi \times \mathrm{Id}"] & \Gamma \times \mathbb{R}^{p,q} \arrow[d, "pr_\Gamma \circ \delta"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{8.4}$$

where $pr_\Gamma : \Gamma \times \mathbb{R}^{p,q} \longrightarrow \Gamma$ is the canonical projection. Furthermore, it is not hard to prove the

Lemma 8.1.19. *Let* $\alpha \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$*, and let* $\mathbb{E} \in \widehat{Ext\mathrm{R}}(\mathcal{G}, \mathbb{S}^1)$ *such that* $DD(\mathbb{E}) = \alpha$*. Then for all* $p, q \in \mathbb{N}$*,*

$$KR^{p-q}_{\alpha,vb}(\mathcal{G}^\bullet) \cong KR_{\alpha'_{p+q},vb}((\mathcal{G} \times \mathbb{R}^{p,q})^\bullet) = \mathbf{KR}(\mathbb{E}^{p,q}).$$

Now as a consequence of Lemma 8.1.13 we get a multiplicative structure on K-theory of Rg twisted vector bundles.

Proposition 8.1.20 (Multiplicative structure)**.** *Let* $\alpha, \beta \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$*. Then there is a multiplicative structure*

$$KR^{-i}_{\alpha,vb}(\mathcal{G}^\bullet) \otimes KR^{-j}_{\beta,vb}(\mathcal{G}^\bullet) \longrightarrow KR^{-i-j}_{\alpha+\beta,vb}(\mathcal{G}^\bullet).$$

Definition 8.1.21. *(Compare [90, Definition 5.13]). Let* $\mathcal{F}$ *be a Real graded u.s.c Fell bundle over a Real groupoid* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *and let* $A = \mathcal{C}_0(Y; \mathcal{E})$*. A* $(\Gamma, \mathcal{E})$*-equivariant Real graded Hilbert* A*-module is a Real graded Hilbert* A*-module* E *together with isomorphisms of graded Hilbert* $A_{s(\gamma)}$*-modules*

$$E_{r(\gamma)} \hat{\otimes}_{A_{r(\gamma)}} \mathcal{E}_\gamma \longrightarrow E_{s(\gamma)}$$

such that

(i) over all $\gamma \in \Gamma$ *the following diagram commutes*

$$\begin{tikzcd} E_{r(\gamma)} \hat{\otimes}_{A_{r(\gamma)}} \mathcal{E}_\gamma \arrow[r] \arrow[d] & E_{s(\gamma)} \arrow[d] \\ E_{r(\bar{\gamma})} \hat{\otimes}_{A_{r(\bar{\gamma})}} \mathcal{E}_{\bar{\gamma}} \arrow[r] & E_{s(\bar{\gamma})} \end{tikzcd}$$

where the vertical arrows are induced by the Real structures;

(ii) over all $(\gamma_1,\gamma_2)\in\Gamma^{(2)}$ there is a commutative diagram

$$\begin{tikzcd} E_{r(\gamma_1)}\hat{\otimes}_{A_{r(\gamma_1)}}\mathcal{E}_{\gamma_1}\hat{\otimes}_{A_{s(\gamma_1)}}\mathcal{E}_{\gamma_2} \arrow[r] \arrow[d] & E_{r(\gamma_1)}\hat{\otimes}_{A_{r(\gamma_1\gamma_2)}}\mathcal{E}_{\gamma_1\gamma_2} \arrow[d] \\ E_{r(\gamma_2)}\hat{\otimes}_{A_{r(\gamma_2)}}\mathcal{E}_{\gamma_2} \arrow[r] & E_{s(\gamma_2)} \end{tikzcd}$$

The product is required to be continuous in the following sense: for all $\xi\in E$ and $\eta\in\mathcal{C}_0(\Gamma;\mathcal{E})$, the function $\gamma\longmapsto\xi(r(\gamma))\eta(\gamma)$ belongs to s^*E.

Remark 8.1.22. *Recall that E can be canonically identified with $\mathcal{C}_0(Y;\tilde{E})$ where*

$$\tilde{E}:=\coprod_{y\in Y}E\hat{\otimes}_A A_y\longrightarrow Y$$

is a Real field of graded Banach bundles such that each fibre $\tilde{E}_y$ is a graded Hilbert A_y-module. Hence E being $(\Gamma,\mathcal{E})$-equivariant means that $(s^*\tilde{E},\mathcal{E})$ is a right Real graded Fell Γ-pair over Γ (cf. Chapter 5).

Example 8.1.23. *Let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Real graded $\mathbb{S}^1$-twist, and let $L=\widetilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C}\longrightarrow\Gamma$ be the associated Real graded Fell bundle. Then by using the construction of the proof of Theorem 5.7.13, we see that $L^2(\Gamma;L)\hat{\otimes}\hat{\mathcal{H}}\cong L^2(\widetilde{\Gamma};\hat{\mathcal{H}})^{\mathbb{S}^1}$ is a (Γ,L)-equivariant Real graded Hilbert A-module, where $A=\mathcal{C}_0(Y;L)$.*

Definition 8.1.24. *Suppose that $\mathcal{E}\longrightarrow\Gamma$ is a Real graded u.s.c Fell bundle and E is a Real graded Hilbert A-module, where as usual $A:=\mathcal{C}_0(Y;\mathcal{E})$. Then we set*

$$\mathcal{C}(E):=\{T\in\mathcal{L}(E)\mid\varphi\cdot T\in\mathcal{K}(E),\forall\varphi\in A\}.$$

If E is $(\Gamma,\mathcal{E})$-equivariant, le $\mathcal{C}(E)^\Gamma$ denote the subspace of all Γ-invariant elements of $\mathcal{C}(E)$. Then we define $\mathcal{K}_\Gamma(E)$ to be the Real graded C^*-algebra

$$\mathcal{K}_\Gamma(E):=\{T\in\mathcal{C}(E)^\Gamma\mid\|T_x\|\longrightarrow0,x\longrightarrow\infty,\text{ in }Y/\Gamma\}.$$

Example 8.1.25. *Let $\Gamma\underset{s}{\overset{r}{\rightrightarrows}}Y$ be a Real proper groupoid with Real Haar system, and let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Real graded $\mathbb{S}^1$-twist. As usual we denote by $L:=\widetilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C}$ the Real graded Fell bundle over Γ associated to $(\widetilde{\Gamma},\Gamma,\delta)$. Suppose (V,h) is a Real graded $(\widetilde{\Gamma},\delta)$-module over $\Gamma\underset{s}{\overset{r}{\rightrightarrows}}Y$ with action $\rho:\widetilde{\Gamma}\longrightarrow\boldsymbol{GL}^{gr}(V)$. Then $E^V:=\mathcal{C}_0(Y;V)$ is a Real graded Hilbert $\mathcal{C}_0(Y)$-module with respect to the inner product given by*

$$\langle\varphi,\psi\rangle_{\mathcal{C}_0(Y)}(y):=\langle\varphi(y),\psi(y)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the Hermitian metric of V, and the obvious Real $\mathcal{C}_0(Y)$-action. It is clear that E^V is (Γ,L)-equivariant with respect to the isomorphisms

$$V_{r(\gamma)}\hat{\otimes}_{\mathcal{C}_0(Y)}(\widetilde{\Gamma}_\gamma\times_{\mathbb{S}^1}\mathbb{C})\longrightarrow V_{s(\gamma)},\quad v\hat{\otimes}[\tilde{\gamma},t]\longmapsto\rho_{\tilde{\gamma}^{-1}}(tv).\tag{8.5}$$

Note that the grading of E^V is induced from that of V. Moreover, by [90, Lemma 5.22], $\mathrm{Id}_{E^V} \in \mathcal{C}(E^V)$, and it is easily seen to be Γ-invariant. In particular, if Y/Γ is compact, then $\mathrm{Id}_{E^V} \in \mathcal{K}_\Gamma(E^V)$.

Definition 8.1.26. *(Cf. [90, Definition 5.14 & Proposition 5.16]). Let $\mathcal{E}$ be a Real graded u.s.c Fell bundle over a Real proper groupoid $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ with Real Haar system, $A = \mathcal{C}_0(Y;\mathcal{E})$ and E a $(\Gamma,\mathcal{E})$-equivariant Real graded Hilbert A-module. We say that E is* approximately finitely generated projective *(AFGP) if (and only if) the Real graded C^*-algebra $\mathcal{K}_\Gamma(E)$ has an aproximate unit consisting of Real projections.*

In the sequel we will use this

Definition 8.1.27. *A Real graded C^*-algebra A is said to be* stably unital *is the Real graded C^*-algebra $A\hat{\otimes}\widehat{\mathcal{K}}_0$ has an approximate unit (p_i) consisting of Real projections (i.e. $\bar{p}_i = p_i = p_i^* = p_i^2 \in (A\hat{\otimes}\widehat{\mathcal{K}}_0)^0$).*

Lemma 8.1.28. *(Compare [90, Corollary 5.17]). Let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Real graded $\mathbf{S}^{1,1}$-twist where $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ is a Real proper groupoid, and let $L = \widetilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C} \longrightarrow \Gamma$ be the associated Real graded Fell bundle. Then the (Γ,L)-equivariant Real graded Hilbert A-module $L^2(\widetilde{\Gamma};\hat{\mathcal{H}})^{\mathbb{S}^1}$ is AFGP if and only if the Real graded C^*-algebra $C^*_r(\Gamma;L)$ is stably unital.*

Proof. $L^2(\widetilde{\Gamma};\hat{\mathcal{H}})^{\mathbb{S}^1}$ is AFGP if and only if $\mathcal{K}_\Gamma(L^2(\widetilde{\Gamma};\hat{\mathcal{H}})^{\mathbf{S}^{1,1}})$ has an approximate unit consisting of Real projections. But one has

$$\mathcal{K}_\Gamma(L^2(\widetilde{\Gamma};\hat{\mathcal{H}})^{\mathbb{S}^1}) \cong \mathcal{K}_\Gamma(L^2(\widetilde{\Gamma})^{\mathbb{S}^1}\hat{\otimes}\hat{\mathcal{H}}) \cong \mathcal{K}_\Gamma(L^2(\Gamma;L))\hat{\otimes}\widehat{\mathcal{K}}_0,$$

and as we already pointed it out in the proof of Theorem 7.2.2, the isomorphism

$$\mathcal{K}_\Gamma(L^2(\Gamma;L))\hat{\otimes}\widehat{\mathcal{K}}_0 \cong C^*_r(\Gamma;L)\hat{\otimes}\widehat{\mathcal{K}}_0$$

of [90, Proposition 4.3] is easily seen to respect the gradings and the Real structures. □

Corollary 8.1.29. *Let $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ be a Real proper groupoid. Then $L^2(\Gamma)\hat{\otimes}\hat{\mathcal{H}}$ is AFGP if and only if the Real (trivially graded) C^*-algebra $C^*_r(\Gamma)$ is stably unital.*

Proof. Consider the trivial Real twist $(\Gamma\times\mathbf{S}^{1,1},\Gamma,0)$. Then as a particular case of Example 8.1.23, $L^2(\Gamma)\otimes\hat{\mathcal{H}}$ is a $(\Gamma,\mathbb{C})$-equivariant Real graded Hilbert $\mathcal{C}_0(Y)$-module, where the grading is that of $\hat{\mathcal{H}}$, and we have abusively denoted $\mathbb{C}$ for the trivial line bundle $\Gamma\times\mathbb{C}\longrightarrow\mathbb{C}$. It then suffices to apply the previous lemma to $L^2(\Gamma)\otimes\hat{\mathcal{H}}$. □

8.2 Case of oriented twistings

It is natural to expect the groups $KR^*_{\alpha,vb}(\mathcal{G}^\bullet)$ to be linked to $KR^*_\alpha(\mathcal{G}^\bullet)$. What we are doing first is to construct a homomorphism from the K-theory groups of Real graded α-twisted vector bundles over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ to $KR^*_\alpha(\mathcal{G}^\bullet)$.

Let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Real graded $\mathbf{S}^{1,1}$-twist and let $L := \widetilde{\Gamma}\times_{\mathbf{S}^{1,1}}\mathbb{C} \longrightarrow \Gamma$ be the associated Real graded Fell bundle. Suppose that $\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma)$ is not the empty set. Let (V,h) be a Real graded $(\widetilde{\Gamma},\delta)$-module over $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$. Then by (8.5), (s^*V,L) is a Real graded (right) Fell Γ-pair over Γ (where the latter is considered as a right Real Γ-space) by setting for all $(\gamma_1,\gamma_2)\in\Gamma^{(2)}$,

$$\begin{array}{ccc} V_{s(\gamma_1)}\times(\widetilde{\Gamma}_{\gamma_1}\times_{\mathbf{S}^{1,1}}\mathbb{C}) & \longrightarrow & V_{s(\gamma_1\gamma_2)} \\ (v,[\tilde{\gamma}_2,t]) & \longmapsto & v\cdot[\tilde{\gamma}_2,t] := \rho_{\tilde{\gamma}_2^{-1}}(tv). \end{array} \tag{8.6}$$

These maps are clearly compatible with the gradings (since $deg(\rho_{\tilde{\gamma}}) = \delta(\pi(\tilde{\gamma}))$ for all $\tilde{\gamma}\in\widetilde{\Gamma}$) and the Real structures (since ρ is a Real homomorphism). Next, we define a Real graded L-valued inner product on s^*V (cf. Definition 5.3.8) by setting for all $\gamma_1,\gamma_2\in\Gamma$ such that $r(\gamma_1)=r(\gamma_2)$:

$$\begin{array}{ccc} \overline{s^*V}_{\gamma_1}\times s^*V_{\gamma_2} & \longrightarrow & L_{<\gamma_1,\gamma_2>_\Gamma} = \widetilde{\Gamma}_{\gamma_1^{-1}\gamma_2}\times_{\mathbf{S}^{1,1}}\mathbb{C} \\ (v^\flat,w) & \longmapsto & \langle v,w\rangle_L := [\tilde{\gamma}_1^{-1}\tilde{\gamma}_2,\langle\rho_{\tilde{\gamma}_2^{-1}\tilde{\gamma}_1}(v),w\rangle] \end{array} \tag{8.7}$$

It turns out that $\mathcal{C}_c(\Gamma;s^*V) = \mathcal{C}_c(\Gamma;s^*V^0)\oplus\mathcal{C}_c(\Gamma;s^*V^1)$ is a Real graded pre-inner product $\mathcal{C}_c(\Gamma;L)$-module with respect to the Real graded $\mathcal{C}_c(\Gamma;L)$-action and $\mathcal{C}_c(\Gamma;L)$-valued inner product $\langle\cdot\,,\cdot\rangle_\star$ induced by (8.6) and (8.7) on $\mathcal{C}_c(\Gamma;s^*V)$ (cf. Proposition 5.4.10).

Definition 8.2.1. *Let $(\widetilde{\Gamma},\Gamma,\delta)$, $L\longrightarrow\Gamma$, and (V,h) be as above. Then by $E(V)$ we denote the Real graded Hilbert $C^*_r(\Gamma;L)$-module defined as the completion of $\mathcal{C}_c(\Gamma;s^*V)$ with respect to the norm $\|\phi\|_\star := \|\langle\phi\,,\phi\rangle_\star\|^{\frac{1}{2}}_{C^*_r(\Gamma;L)}$ for $\phi\in\mathcal{C}_c(\Gamma;s^*V)$.*

Definition 8.2.2. *If $W\longrightarrow X$ is a Real graded vector bundle over the Real space X, then for all Real graded homomorphism $f:W\longrightarrow W$, we define the Real graded linear operator $F_f:\mathcal{C}_c(X;W)\longrightarrow\mathcal{C}_c(X;W)$ by the formula*

$$F_f(\phi)(x) := f(\phi(x)),\quad \forall\phi\in\mathcal{C}_c(X;W), x\in X.$$

Lemma 8.2.3. *Let $(\widetilde{\Gamma},\Gamma,\delta)$ be a Real graded $\mathbf{S}^{1,1}$-twist, $L\longrightarrow\Gamma$ its associated Real graded Fell bundle, and let (V,h) be a Real graded $(\widetilde{\Gamma},\delta)$-module over $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$. Then the map $F_h:\mathcal{C}_c(\Gamma;s^*V)\longrightarrow\mathcal{C}_c(\Gamma;s^*V)$ extends to a Real degree 1 bounded self-adjoint operator $F_h\in\mathcal{L}(E(V))$ verifying*

$$F_h^2 - 1 \in \mathcal{K}(E(V)). \tag{8.8}$$

In other words, the triple $(E(V),\mathbb{C}\cdot\mathrm{Id},F_h)$ *is a Real graded Kasparov* $\mathbb{C}$*-*$C_r^*(\Gamma;L)$*-bimodule; i.e.* $(E(V),\mathbb{C}\cdot\mathrm{Id},F_h)\in ER(\mathbb{C},C_r^*(\Gamma;L))$.

Proof. For $(v,[\tilde{\gamma}_2,t])\in V_{s(\gamma_1)}\times L_{\gamma_2}$, we have $h(v\cdot[\tilde{\gamma}_2,t])=h(\rho_{\tilde{\gamma}_2^{-1}}(tv))=\rho_{\tilde{\gamma}_2^{-1}}(th(v))=h(v)\cdot[\tilde{\gamma}_2,t]$, thanks to Definition 8.1.2 (iii). We hence may remark that the map h is invariant with the respect to the L-action (8.6) on s^*V. Thus, for all $\phi\in\mathcal{C}_c(\Gamma;s^*V),\xi\in\mathcal{C}_c(\Gamma;L)$, we have

$$\begin{aligned}F_h(\phi\cdot\xi)(\gamma)=&h\left(\int_{\Gamma^{s(\gamma)}}\phi(\gamma\cdot g)\xi(g^{-1})d\mu^{s(\gamma)}(g)\right)\\=&\int_{\Gamma^{s(\gamma)}}h(\phi(\gamma\cdot g))\xi(g^{-1})d\mu^{s(\gamma)}(g)\\=&(F_h(\phi)\cdot\xi)(\gamma),\quad\forall\gamma\in\Gamma;\end{aligned}$$

which shows that F_h is $\mathcal{C}_c(\Gamma;L)$-linear.

To see that F_h is adjointable, notice that for all $(v^\flat,w)\in\overline{s^*V}_{\gamma_1}\times s^*V_{\gamma_2}$ where $r(\gamma_1)=r(\gamma_2)$, one has $\langle h(v),w\rangle_L=\langle v,h^*(w)\rangle_L=\langle v,h(w)\rangle$ since h is hermitian, where $h^*:V\longrightarrow V$ is the adjoint of h with respect to the fibrewise the scalar product $\langle\cdot,\cdot\rangle$ of V. Now if $\phi,\psi\in\mathcal{C}_c(\Gamma;s^*V)$, then for all $\gamma\in\Gamma$:

$$\begin{aligned}\langle F_h\phi\,,\psi\rangle_\star(\gamma)=&\int_{\Gamma^{s(\gamma)}}\langle h(\phi(g^{-1}\cdot\gamma^{-1})),\psi(g^{-1})\rangle_L d\mu^{s(\gamma)}(g)\\=&\int_{\Gamma^{s(\gamma)}}\langle\phi(g^{-1}\cdot\gamma^{-1}),h^*(\psi(g^{-1}))\rangle_L d\mu^{s(\gamma)}(g)\\=&\langle\phi\,,F_{h^*}\psi\rangle_\star(\gamma).\end{aligned}$$

Hence $F_h^*=F_{h^*}=F_h$, so that F_h is bounded and extends to a self-adjoint operator $F_h\in\mathcal{L}(E(V))$. Moreover, F_h is Real and of degree 1 since h is. Indeed, if writting h as a matrix $h=\begin{pmatrix}0&h_1\\h_1^*&0\end{pmatrix}$, where $h_1:V^1\longrightarrow V^0$ and $h_1^*:V^0\longrightarrow V^1$ is the (fibrewise) adjoint of h_1, we see that

$$F_h:=\begin{pmatrix}0&F_{h_1}\\F_{h_1^*}&0\end{pmatrix}=\begin{pmatrix}0&F_{h_1}\\F_{h_1}^*&0\end{pmatrix},\tag{8.9}$$

where $F_{h_1}:\mathcal{C}_c(\Gamma;s^*V^1)\longrightarrow\mathcal{C}_c(\Gamma;s^*V^0),F_{h_1}^*:\mathcal{C}_c(\Gamma;s^*V^0)\longrightarrow\mathcal{C}_c(\Gamma;s^*V^1)$ are the obvious $\mathcal{C}_c(\Gamma;L)^0$-linear operators.

Now to verify (8.8), since V is of finite rank n, we may, without loss of generality, assume that V is trivial (and moreover $\widetilde{\Gamma}\cong\Gamma\times\mathbf{S}^{1,1}$). Then h expresses into a hermitian $n\times n$-matrix, and $E(V)$ identifies to the Real (trivially graded) projective Hilbert (right) $C_r^*(\Gamma)$-module $C_r^*(\Gamma)^n$. It turns out that $\mathcal{L}(E(V))$ consists of compact operators. □

Proposition 8.2.4. *Let* $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ *be a Real groupoid with Real Haar system. Let* $\alpha\in\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\times\check{H}R^2(\mathcal{G}_\bullet,\mathbf{S}^{1,1})$*, and* $\mathbb{E}=(\widetilde{\Gamma},\Gamma,\delta)\in\widehat{Ext\mathrm{R}}(\mathcal{G},\mathbf{S}^{1,1})$ *such that* $DD(\mathbb{E})=\alpha$*. Suppose that* $\widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma)$

is not empty. Then the process consisting of associating to each Real graded $(\widetilde{\Gamma},\delta)$-module (V,h) the Kasparov module $(E(V),\mathbb{C}\cdot\mathrm{Id},F_h)$ induces a homomorphism of abelian groups

$$\omega_\alpha : KR^0_{\alpha,vb}(\mathcal{G}^\bullet) \longrightarrow KR^0_\alpha(\mathcal{G}^\bullet).$$

Proof. In view of Lemma 8.2.3, associated to every Real graded $(\widetilde{\Gamma},\delta)$-module (V,h), there is a Kasparov module $(E(V),\mathbb{C}\cdot\mathrm{Id},F_h)\in ER(\mathbb{C},C^*_r(\Gamma;L))$. Further, it is not hard to show that every isomorphism of Real graded $(\widetilde{\Gamma},\delta)$-modules $u:(V_1,h_1)\longrightarrow(V_2,h_2)$ over $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ induces a unitary equivalence between

$$(E(V_1),\mathbb{C}\cdot\mathrm{Id},F_{h_1}) \sim (E(V_2),\mathbb{C}\cdot\mathrm{Id},F_{h_2})$$

in $ER(\mathbb{C},C^*_r(\Gamma;L))$. We then have a well defined map

$$\tilde{\omega}_\alpha : \widehat{\mathbf{Mod}\mathrm{R}}_{(\widetilde{\Gamma},\delta)}(\Gamma) \longrightarrow KKR(\mathbb{C},C^*_r(\Gamma;L)) \cong KKR(\mathbb{C},C^*_r(\mathbb{E})).$$

That $\tilde{\omega}_\alpha$ is a homomorphism of abelian monoids is also obvious; for if (V,h^V) and (W,h^W) are Real graded α-twisted vector bundles relative to $(\widetilde{\Gamma},\Gamma,\delta)$, then $\mathcal{C}_c(\Gamma;s^*(V\oplus W))\cong\mathcal{C}_c(\Gamma;s^*V)\oplus\mathcal{C}_c(\Gamma;s^*W)$ and with respect to this decomposition we have $E(V\oplus W)\cong E(V)\oplus E(W)$ are Real graded Hilbert $C^*_r(\Gamma;L)$-module. Now from the universality of the Grothendieck group (see [42, §1.1, Chap.II]), $\tilde{\omega}_\alpha$ induces a homomorphism $\omega_\alpha : KR^0_{\alpha,vb}(\mathcal{G}^\bullet)\longrightarrow KR^0_\alpha(\mathcal{G}^\bullet)$. □

Now we are concerned with proving the following result which is the "Real graded" version of [90, Theorem 5.28].

Theorem 8.2.5. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a locally compact Hausdorff second-countable Real proper groupoid such that $X/\mathcal{G}$ is compact. Let $\mathcal{A}$ be an oriented Real graded D-D bundle of type 0 over $\mathcal{G}$ with Dixmier-Douady class $\alpha=(0,\mathfrak{c})\in \check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\times\check{H}R^2(\mathcal{G}_\bullet,\mathbf{S}^{1,1})$. Let $\mathbb{E}=(\widetilde{\Gamma},\Gamma,0)$ be a Real (trivially graded) $\mathbf{S}^{1,1}$-extension of $\mathcal{G}$ such that $DD(\mathbb{E})=\alpha$. Suppose that*

(a) $L^2(\Gamma)\otimes\hat{\mathcal{H}}$ is AFGP;

(b) there is exists a Real graded α-twisted vector bundle (relative to $\mathbb{E}$).

Then the map $\omega_\alpha : KR^0_{\alpha,vb}(\mathcal{G}^\bullet)\longrightarrow KR^0_\alpha(\mathcal{G}^\bullet)$ is an isomorphism.

Remark 8.2.6. *Let $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ be a Real proper groupoid such that Y/Γ is compact. Then $C^*_r(\Gamma)\otimes\hat{\mathcal{K}}_0\cong\mathcal{K}_\Gamma(L^2(\Gamma)\otimes\hat{\mathcal{H}})\cong\mathcal{C}_0(Y;\mathcal{K}(\widetilde{L^2(\Gamma)}\otimes\hat{\mathcal{H}}))^\Gamma$. It follows that the $(\Gamma,\mathbb{C})$-equivariant Real graded Hilbert $\mathcal{C}_0(Y)$-module $L^2(\Gamma)\otimes\hat{\mathcal{H}}$ being AFGP means that there exists a sequence $(p_n)_n$ such that*

(i) $p_n=(p_n(y))_{y\in Y}$ is a Γ-equivariant continuous section of the field of compact operators $\mathcal{K}(\widetilde{L^2(\Gamma)}\otimes\hat{\mathcal{H}})\longrightarrow Y$;

(ii) *for all* n, $y \longmapsto p_n(y)$ *is Real; i.e.* $p_n(\bar{y}) = \overline{p_n(y)}$ *for all* $y \in Y$;

(iii) $p_n(y)$ *is a finite rank projection for all* $y \in Y$,

(iv) *for every* $\xi \in \mathcal{C}_c(Y; \widetilde{L^2(\Gamma)} \otimes \hat{\mathcal{H}})$, $(p_n\xi)(y) \longrightarrow \xi(y)$ *uniformly on* Y *when* $n \longrightarrow \infty$.

In the proof of Theorem 8.2.5 we will use the following result which is in some sense a generalisation of the well known Serre-Swan theorem.

Proposition 8.2.7. *Assume* $(\widetilde{\Gamma}, \Gamma, \delta)$ *is a Real graded* $\mathbf{S}^{1,1}$*-twist where* $\Gamma \underset{s}{\overset{r}{\rightrightarrows}} Y$ *is a Real proper groupoid such that* Y/Γ *is compact. Let* $L := \widetilde{\Gamma} \times_{\mathbf{S}^{1,1}} \mathbb{C} \longrightarrow \Gamma$ *be the associated Real graded Fell bundle. Consider the category* $\mathfrak{P}(\Gamma, L)$ *of pairs* (E, F) *consisting of a* (Γ, L)*-equivariant Real graded Hilbert* $\mathcal{C}_0(Y)$*-modules* E *such that* $\mathrm{Id}_E \in \mathcal{K}_\Gamma(E)$ *and a degree* 1 *self-adjoint Real operator* $F \in \mathcal{L}(E)$. *Then the functor from the category of Real graded* $(\widetilde{\Gamma}, \delta)$*-modules to the category* $\mathfrak{P}(\Gamma, L)$, *defined by*

$$\Phi : (V, h) \longmapsto (\mathcal{C}_0(Y; V), F),$$

where $F(\phi)(y) := h(\phi(y))$ *for* $\phi \in \mathcal{C}_0(Y; V)$, *is an equivalence of categories. Therefore,* $KR^0_{\alpha,vb}(\mathcal{G}^\bullet)$ *is isomorphic to the Grothendieck group of the category* $\mathfrak{P}(\Gamma, L)$.

Proof. This is but the Real graded analogue of Proposition 5.27 in [90]. We however should show how the inverse functor is defined in our context. Suppose E is a (Γ, L)-equivariant Real graded Hilbert $\mathcal{C}_0(Y)$-module such that $\mathrm{Id}_E \in \mathcal{K}_\Gamma(E)$, and $F \in \mathcal{L}(E)$[1] is self-adjoint. Consider the graded Hilbert space $V_y := E \hat{\otimes}_{ev_y} \mathbb{C}$ (whose grading is given by that of E), and the continuous field of graded Hilbert spaces $V := \coprod_Y V_y \longrightarrow Y$ equipped with the Real structure $V_y \longrightarrow V_{\bar{y}}$ induced by $\xi \hat{\otimes}_{ev_y} \lambda \longmapsto \bar{\xi} \hat{\otimes}_{ev_y} \bar{\lambda}$, where $\bar{\xi}$ is the image of $\xi \in E$ by the Real structure of E. Then $E \cong \mathcal{C}_0(Y; V)$ as Real graded Hilbert $\mathcal{C}_0(Y)$-module. Moreover, by applying Swan theorem to E on each compact Real subspace $K \subset Y$, it is easy to see that $V \longrightarrow Y$ is in fact a Real graded hermitian vector bundle. Now we define the hermitian degree 1 endomorphism h by setting for all $y \in Y$, $h_y(\xi \hat{\otimes}_{ev_y} \lambda) := F(\xi) \hat{\otimes}_{ev_y} \lambda$, so that through the isomorphism $E \cong \mathcal{C}_0(Y; V)$, we have $F(\phi)(y) = h_y(\phi(y))$, for all $\phi \in \mathcal{C}_0(Y; V)$. The Real graded $(\widetilde{\Gamma}, \delta)$-module structure $\rho : \widetilde{\Gamma} \longrightarrow \mathbf{GL}^{gr}(V)$ over (V, h) is induced by the isomorphisms $E_{s(\gamma)} \hat{\otimes} (\widetilde{\Gamma}_{\gamma^{-1}} \times_{\mathbf{S}^{1,1}} \mathbb{C}) \xrightarrow{\cong} E_{r(\gamma)}$; i.e. for $\tilde{\gamma} \in \widetilde{\Gamma}$, we set

$$\rho_{\tilde{\gamma}}(\xi \hat{\otimes} \lambda) := (\xi \hat{\otimes} [\tilde{\gamma}^{-1}, 1]) \hat{\otimes} \lambda,$$

for $\xi \hat{\otimes} \lambda \in V_{s(\gamma)}$ where $\gamma \in \Gamma$ is such that $\pi(\tilde{\gamma}) = \gamma$. □

Proof of Theorem 8.2.5. Observe that since $C^*_r(\Gamma; L)$ is trivially graded, then using the Fredholm picture of $KR_0(\mathcal{A} \rtimes_r \mathcal{G}) = KR_0(C^*_r(\Gamma; L))$, $KR_\alpha(\mathcal{G}^\bullet)$ is isomorphic to the group of homotopy classe of degree 1 Real operators $F = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \in \mathcal{L}(C^*_r(\Gamma; L) \otimes \mathcal{H})$ such that $ST - 1, TS - 1 \in$

$\mathcal{K}(C^*_r(\Gamma;L)\otimes\mathcal{H})$, where the ungraded infinite dimensional Hilbert space $\mathcal{H}$ is equipped with a Real structure of the type $J_{\mathbb{R}}$ (cf. Section **??** in Appendix A). Hence since any approximate unit $(p_n)_n$ consisting of Real projections of the Real graded C^*-algebra $C^*_r(\Gamma;L)\hat{\otimes}\widehat{\mathcal{K}}_0$ is in fact an approximate unit consisting of projections of the ungraded Real C^*-algebra $C^*_r(\Gamma;L)\otimes\mathcal{K}(\mathcal{H})$, we obtain an inverse map

$$\omega'_\alpha : KR_\alpha(\mathcal{G}^\bullet)\longrightarrow KR^0_{\alpha,vb}(\mathcal{G}^\bullet)$$

for ω_α from the same constructions as that of [46, p.556] and [90, p.888] (note that the picture of $KR^0_{\alpha,vb}(\mathcal{G}^\bullet)$ used here is the one given by Proposition 8.2.7). □

8.3 Twisted equivariant KR-theory, and Twisted representation rings

In this section we focus on the study of the twisted KR_{vb}-theory of transformation Real groupoids. Recall that if G is a Real group acting (on the right, say) on a Real space X, then the Real groupoid $X\rtimes G \rightrightarrows X$ has unit Real space X, morphisms $(x,g)\in X\times G$, source and range maps $s(x,g):=xg, r(x,g):=x$, inverse $(x,g)^{-1}:=(xg,g^{-1})$, partial product $(x,g)(xg,h):=(x,gh)$, and Real structure $\overline{(x,g)}:=(\bar{x},\bar{g})$.

Definition 8.3.1. *Let G be a compact Real group acting on a locally compact Hausdorff Real space X. Let $\alpha\in \check{H}R^1(G_\bullet,\mathbb{Z}_2)\times\check{H}R^3(G_\bullet,\mathbb{Z}^{0,1})$. Let $\tilde{\alpha}\in\check{H}R^1((X\rtimes G)_\bullet,\mathbb{Z}_2)\times\check{H}R^3((X\rtimes G)_\bullet,\mathbb{Z}^{0,1})$ be the pull-back of α along the projection $\pi_G: X\rtimes G\longrightarrow G$. Then we define the* twisted equivariant KR-theory *of X by*

$$KR^*_{G,\alpha}(X):=KR^*_{\tilde{\alpha},vb}((X\rtimes G)^\bullet).$$

Similarly, when G and X are equipped with the trivial involutions, one defines twisted equivariant KO-theory $KR^*_{G,\alpha}(X)$, *for $\alpha\in\check{H}R^1(G_\bullet,\mathbb{Z}_2)\times\check{H}R^2(G_\bullet,\mathbb{Z}_2)$.*

Notice that this definition is the Real version of Adem-Ruan's twisted equivariant K-theory (see [1, Section 7] or [2, §3.6]); it also agrees, in the complex case, with the Freed-Hopkins-Teleman's twisted form of equivariant K-theory (see [30, Examples 1.10, 1.12, & 1.13]).

Definition 8.3.2. *(Compare [1, Definition 7.1]). A Real graded $\tilde{\alpha}$-twisted vector bundle over $X\rtimes G\rightrightarrows X$ will be simply called an α-*twisted Real G-bundle *over X.*

Remark 8.3.3. *Suppose that $\alpha=0$. Then it is easy to see that there is bijection between α-twisted Real G-bundles over X and Rg hermitian G-equivariant vector bundles over X. In other words, $KR^*_G(X):=KR^*_{G,0}(X)$ is the equivariant analog of Atiyah's KR-theory.*

Definition 8.3.4. *Let G be a compact Real group. Let $\alpha \in \check{H}R^1(G_\bullet,\mathbb{Z}_2)\times \check{H}R^3(G_\bullet,\mathbb{Z}^{0,1})$. We define the α-*twisted Real representation ring *of G as*

$$RR_\alpha(G) := KR_{G,\alpha}(\{pt\}).$$

Similarly, one defines $RO_\alpha(G)$, when G is endowed with the trivial involution.

We will write $RR(G)$ (resp. $RO(G)$) instead of $RR_\alpha(G)$ (resp. $RO_\alpha(G)$) if α is trivial.

Remark 8.3.5. *We shall remark that $RR(G)$ is nothing but the subgroup of the representation ring $R(G)$ consisting of isomorphism classes of graded complex G-modules M (with G-invariant hermitian forms) that are the complexifications of graded real G-modules with G-invariant metric. Hence, by [38, Theorem 11.4], we have $RR(G)\cong RO(G)$.*

The following is the analog of [1, Lemma 7.3] whose proof is almost the same as in the untwisted complex case (see [82, Proposition 2.2]); we then omit the proof.

Proposition 8.3.6. . *Suppose the compact Real group G acts trivially on the Real space X. Then for $\alpha \in \check{H}R^1(G_\bullet,\mathbb{Z}_2)\times \check{H}R^3(G_\bullet,\mathbb{Z}^{0,1})$, there is a natural isomorphism*

$$RR_\alpha(G)\otimes KR(X) \xrightarrow{\cong} KR_{G,\alpha}(X).$$

Corollary 8.3.7. *We have $KR^*_{G,\alpha}(\{pr\})\cong RR_\alpha(G)\otimes KR^*(\{pt\})$.*

Proof. Consider the Real space $\mathbb{R}^{p,q}$ acted upon by G by the trivial Real action. Then we have $KR_{G,\alpha}(\mathbb{R}^{p,q})\cong RR_\alpha(G)\otimes KR(\mathbb{R}^{p,q})$, by Proposition 8.3.6. Note that $KR_{G,\alpha}(\mathbb{R}^{p,q}) = KR^{p-q}_{G,\alpha}(\{pt\})$, thanks to Lemma 8.1.19; we deduce the result from the fact that $KR(\mathbb{R}^{p,q}) = KR^{p-q}(\{pt\})$ by Bott periodicity in KR-theory. □

We are now going to study periodicity in twisted equivariant KR-theory.

Suppose that α corresponds to a Rg $\mathbb{S}^1$-central extension

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & \tilde{G}^\alpha & \longrightarrow & G \\ & & & & \downarrow{\scriptstyle\delta} \\ & & & & \mathbb{Z}_2 \end{array}$$

An α-twisted G-bundle V over X is in fact a Rg $\tilde{G}^\alpha$-equivariant hermitian vector bundle over X, and hence defines an element in $KR_{\tilde{G}^\alpha}(X)$. As in the complex case ([30, Example 1.10], and [1, p.23]), $KR_{G,\alpha}(X)$ may then be seen as the subgroup of $KR_{\tilde{G}^\alpha}(X)$ generated by isomorphism classes of Rg $\tilde{G}^\alpha$-equivariant vector bundles that restrict to multiplication by scalars on the center $\mathbb{S}^1\subset \tilde{G}^\alpha$. Thus,

Proposition 8.3.8 (Bott periodicity)**.** *Suppose X is a compact Real space acted upon by a compact Real group G. Let α be as previously. Then*

$$KR^{-n-8}_{G,\alpha}(X)\cong KR^{-n}_{G,\alpha}(X).$$

Proof. The compact Real group $\tilde{G}^\alpha$ acts on X through the projection $\tilde{G}^\alpha \longrightarrow G$ (this does make sense since $\mathbb{S}^1$ acts trivially on X). Denote by Γ the transformation Real groupoid $X \rtimes \tilde{G}^\alpha \longrightarrow X$. Then $L^2(\Gamma)\hat{\otimes}\hat{\mathcal{H}}$ is AFGP, thanks to [90, Corollary 5.21]. Thus, from Theorem 8.2.5, we have $KR^*_{\tilde{G}^\alpha}(X) = KR^*(\Gamma^\bullet)$. Therefore, $KR^{*-8}_{\tilde{G}^\alpha}(X) \cong KR^*_{\tilde{G}^\alpha}(X)$. The result then follows from the 8-periodicity of twisted groupoid KR-theory (cf. Chapter 6) and the observations made just before the proposition. $\square$

9

Real $KK_{\mathcal{G}}$-theory via correspondences and the Thom Isomorphism

A generalization of Le Gall's groupoid equivariant KK-theory has arisen in [90] to the larger framework of groupoid actions by Morita equivalences. This chapter is devoted to the study of that theory in the category of Rg C^*-algebras acted upon by Real groupoids. Then we will establish the *Thom isomorphism* in groupoid twisted KR-theory.

9.1 C^*-correspondences and generalized actions

This section is essentially an adaptation of [90, §6.2] to Real graded C^*-algebras.

Definition 9.1.1. *Let A and B be Real graded C^*-algebras. A C^*-*correspondence *from A to B is a pair $(\mathcal{E},\varphi)$ where $\mathcal{E}$ is a Real graded Hilbert (right) B-module, and $\varphi : A \longrightarrow \mathcal{L}(\mathcal{E})$ is a non-degenerate homomorphism of Real graded C^*-algebras. We then view $\mathcal{E}$ as a Real graded left A-module by $a\cdot e := \varphi(a)e$. When there is no risk of confusion we will write ${}_A\mathcal{E}_B$ for the C^*-correspondence $(\mathcal{E},\varphi)$. We also say that $\mathcal{E}$ is a Real graded A,B-correspondence.*

Definition 9.1.2. *If $(\mathcal{E},\varphi)$ and $(\mathcal{F},\psi)$ are C^*-correspondences from A to B and from B to C, respectively, we define the C^*-correspondence $(\mathcal{F},\psi)\circ(\mathcal{E},\varphi)$ from A to C by $(\mathcal{E}\hat{\otimes}_\psi\mathcal{F},\varphi\hat{\otimes}\mathbf{1})$; this C^*-correspondence is called the* composition of $(\mathcal{F},\psi)$ by $(\mathcal{E},\varphi)$.

Definition 9.1.3. *An isomorphism of C^*-correspondences from ${}_A\mathcal{E}_B$ to ${}_A\mathcal{F}_B$ is a Real degree* 0 *unitary $u\in\mathcal{L}_B(\mathcal{E},\mathcal{F})$ such that $u\circ\varphi_{\mathcal{E}}(a)=\varphi_{\mathcal{F}}(a)\circ u$ for all $a\in A$.*

Definition 9.1.4. *Let $\mathcal{G}$ be a Real groupoid, and A a Real graded C^*-algebra. A* generalized $\mathcal{G}$-action on *A consists of a Real graded u.s.c. Fell bundle $\mathscr{A}\longrightarrow\mathcal{G}$ such that $A\cong\mathcal{C}_0(\mathcal{G}^{(0)};\mathscr{A})$, where, as usual, we have denoted $\mathcal{C}_0(\mathcal{G}^{(0)};\mathscr{A})$ for $\mathcal{C}_0(\mathcal{G}^{(0)};\mathscr{A}_{|\mathcal{G}^{(0)}})$.*

Example 9.1.5. *If $\mathcal{A}\longrightarrow\mathcal{G}^{(0)}$ is a Real u.s.c. $\mathcal{G}$-field of graded C^*-algebras, then the Real graded u.s.c. Fell bundle $s^*\mathcal{A}\longrightarrow\mathcal{G}$ is a generalized $\mathcal{G}$-action on $A=\mathcal{C}_0(\mathcal{G}^{(0)};\mathcal{A})$.*

Remark 9.1.6. *As mentioned in [90, §6.2], a generalized action is in fact an action by Morita equivalences, which justifies the terminology. Indeed, if $\mathscr{A}$ is a generalized $\mathcal{G}$-action on A, then from the properties of Fell bundles we see that for $g \in \mathcal{G}$, $\mathscr{A}_{g^{-1}}$ is a graded $\mathscr{A}_{s(g)}$-$\mathscr{A}_{r(g)}$-Morita equivalence.*

Denote by $i : \mathcal{G} \longrightarrow \mathcal{G}$ the inversion map. If A is a Real graded C^*-algebra endowed with a generalized $\mathcal{G}$-action $\mathscr{A}$, we define $\mathcal{F}_b(i^*\mathscr{A})$ as the Real graded Banach algebra of norm-bounded continuous functions vanishing at infinity $a' : \mathcal{G} \ni g \longmapsto a'_g \in \mathscr{A}_{g^{-1}}$; the Real structure is given by $(\bar{a'})_g := \overline{(a'_{\bar{g}})}$, and the grading is inherited from that of $\mathscr{A}$. Observe that $\mathcal{F}_b(i^*\mathscr{A})$ is naturally a Real graded (right) Hilbert r^*A-module under the module structure

$$(a' \cdot a)_g := a'_g \cdot a_g, \quad \text{for } a' \in \mathcal{F}_b(i^*\mathscr{A}), a \in r^*A = \mathcal{C}_0(\mathcal{G}; r^*(\mathscr{A}_X)),$$

and the graded scalar product

$$\langle a', a"\rangle_g := (a'_g)^* \cdot a"_g, \quad a', a" \in \mathcal{F}_b(i^*\mathscr{A}).$$

Also, $\mathcal{F}_b(i^*\mathscr{A})$ has the structure of Real graded s^*A-module by setting

$$(\xi \cdot a')_g := \xi(g) \cdot a'_g, \quad \text{for } \xi \in s^*A, a' \in \mathcal{F}_b(i^*\mathscr{A}), \text{ and } g \in \mathcal{G}.$$

Moreover, we have the following straightforward lemma.

Lemma 9.1.7. *Let $(\mathcal{E}, \varphi)$ be a C^*-correspondence from A to B, and let $\mathscr{A}$ and $\mathscr{B}$ be generalized $\mathcal{G}$-actions on the Real graded C^*-algebras A and B, respectively. Then, we have two C^*-correspondences ${}_{s^*A}(\mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A} r^*\mathcal{E})_{r^*B}$ and ${}_{s^*A}(s^*\mathcal{E}\hat{\otimes}_{s^*B}\mathcal{F}_b(i^*\mathscr{B}))_{r^*B}$ with respect to the maps*

$$\mathrm{Id}\hat{\otimes} r^*\varphi : s^*A \longrightarrow \mathcal{L}_{r^*B}(\mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A} r^*\mathcal{E}),$$

and

$$s^*\varphi\hat{\otimes}\mathrm{Id} : s^*A \longrightarrow \mathcal{L}_{r^*B}(s^*\mathcal{E}\hat{\otimes}_{s^*B}\mathcal{F}_b(i^*\mathscr{B})),$$

respectively.

We now give the definition of equivariant C^*-correspondence.

Definition 9.1.8. *Let $A, B, \mathscr{A}, \mathscr{B}, \mathcal{E}$ be as above. A C^*-correspondence ${}_A\mathcal{E}_B$ is said $\mathcal{G}$-*equivariant *if there is an isomorphism of C^*-correspondences*

$$W \in \mathcal{L}(s^*\mathcal{E}\hat{\otimes}_{s^*B}\mathcal{F}_b(i^*\mathscr{B}), \mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A} r^*\mathcal{E}),$$

such that for every $(g,h) \in \mathcal{G}^{(2)}$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{E}_{s(h)}\hat{\otimes}_{B_{s(h)}}\mathscr{B}_{h^{-1}}\hat{\otimes}_{B_{r(h)}}\mathscr{B}_{g^{-1}} \arrow[rr, "W_h\hat{\otimes}_{B_{r(h)}}\mathrm{Id}_{\mathscr{B}_{g^{-1}}}"] \arrow[d, phantom, sloped, "\cong"] & & \mathscr{A}_{h^{-1}}\hat{\otimes}_{A_{r(h)}}\mathcal{E}_{s(g)}\hat{\otimes}_{B_{s(g)}}\mathscr{B}_{g^{1}} \arrow[d, "\mathrm{Id}_{\mathscr{A}_{h^{-1}}}\hat{\otimes}_{A_{r(g)}}W_g"] \\
\mathcal{E}_{s(gh)}\hat{\otimes}_{B_{s(gh)}}\mathscr{B}_{h^{-1}g^{-1}} \arrow[dr, "W_{gh}"] & & \mathscr{A}_{h^{-1}}\hat{\otimes}_{A_{r(h)}}\mathscr{A}_{g^{-1}}\hat{\otimes}_{A_{r(g)}}\mathcal{E}_{r(g)} \arrow[dl, phantom, sloped, "\cong"] \\
& \mathscr{A}_{h^{-1}g^{-1}}\hat{\otimes}_{A_{r(gh)}}\mathcal{E}_{r(gh)} &
\end{tikzcd} \tag{9.1}$$

where the isomorphisms $\mathscr{A}_{h^{-1}}\hat{\otimes}_{A_{r(h)}}\mathscr{A}_{g^{-1}} \cong \mathscr{A}_{h^{-1}g^{-1}}$ and $\mathscr{B}_{h^{-1}}\hat{\otimes}_{B_{s(g)}}\mathscr{B}_{g^{-1}} \cong \mathscr{B}_{h^{-1}g^{-1}}$ come from the properties of Fell bundles.

Lemma 9.1.9. *Let A, B, and C be Real graded C^*-algebras endowed with generalized $\mathcal{G}$-actions $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, respectively. If ${}_A\mathcal{E}_B$, and ${}_B\mathcal{F}_C$ are $\mathcal{G}$-equivariant C^*-correspondences, their composition $\mathcal{F}\circ\mathcal{E}$ is a $\mathcal{G}$-equivariant Real graded A, C-correspondence. Therefore, there is a category $\mathfrak{Corr}_{\mathcal{G}}$ whose objects are Real graded C^*-algebras endowed with generalized $\mathcal{G}$-actions, and whose morphisms are isomorphism classes of equivariant correspondences.*

Proof. Suppose $W' \in \mathcal{L}_{r^*B}(s^*\mathcal{E}\hat{\otimes}_{s^*B}\mathcal{F}_b(i^*\mathscr{B}), \mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A}r^*\mathcal{E})$ is an isomorphism of Real graded s^*A, r^*B-correspondences and $W'' \in \mathcal{L}_{r^*C}(s^*\mathcal{F}\hat{\otimes}_{s^*C}\mathcal{F}_b(i^*\mathscr{C}), \mathcal{F}_b(i^*\mathscr{B})\hat{\otimes}_{r^*B}r^*\mathcal{F})$ is an isomorphism of s^*B, r^*C-correspondences implementing $\mathcal{G}$-equivarience. We define the isomorphism of Real graded s^*A, r^*C-correspondences

$$W : s^*(\mathcal{E}\hat{\otimes}_{\psi}\mathcal{F})\hat{\otimes}_{s^*C}\mathcal{F}_b(i^*\mathscr{C}) \longrightarrow \mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A}r^*(\mathcal{E}\hat{\otimes}_{\psi}\mathcal{F})$$

by setting $W := (W'\hat{\otimes}_{r^*B}\mathrm{Id}_{r^*\mathcal{F}})\circ(\mathrm{Id}_{s^*\mathcal{E}}\hat{\otimes}_{s^*B}W'')$, via the identification

$$s^*(\mathcal{E}\hat{\otimes}_{\psi}\mathcal{F}) \cong s^*\mathcal{E}\hat{\otimes}_{s^*B}s^*\mathcal{F}$$
$$r^*(\mathcal{E}\hat{\otimes}_{\psi}\mathcal{F}) \cong r^*\mathcal{E}\hat{\otimes}_{r^*B}r^*\mathcal{F}.$$

Now it is straightforward that commutativity of the diagram (9.1) holds for W.

□

9.2 The $KKR_{\mathcal{G}}$-bifunctor

To define the equivariant KKR-groups, we also need some more notions (cf. [24, Appendix A], [90, Definition 6.5]). Let A, B be Real graded C^*-algebras. Let $\mathcal{E}_1$ be Real graded Hilbert A-module, and $\mathcal{E}_2$ a Real graded A, B-correspondence. Put $\mathcal{E} = \mathcal{E}_1\hat{\otimes}_A\mathcal{E}_2$. For $\xi \in \mathcal{E}_1$, let $T_\xi \in \mathcal{L}_B(\mathcal{E}_2, \mathcal{E})$ be given by $T_\xi(\eta) := \xi\hat{\otimes}_A\eta$ (with adjoint given by $T_\xi^*(\xi_1\hat{\otimes}_A\eta) := \langle\xi, \xi_1\rangle\eta$). Observe that $\overline{T_\xi} = T_{\bar{\xi}}$, so that T_ξ is Real if and only if ξ is.

Definition 9.2.1. *Let $A, B, \mathcal{E}_1$, and $\mathcal{E}_2$ be as above. Let $F_2 \in \mathcal{L}(\mathcal{E}_2)$, and $F \in \mathcal{L}(\mathcal{E})$. We say that F is an F_2-*connection for $\mathcal{E}_1$ *if for every $\xi \in \mathcal{E}_1$:*

$$T_\xi F_2 - (-1)^{|\xi|\cdot|F_2|} F T_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}),$$
$$F_2 T_\xi^* - (-1)^{|\xi|\cdot|F_2|} T_\xi^* F \in \mathcal{K}(\mathcal{E}, \mathcal{E}_2).$$

Remark 9.2.2. *It is easy to check that F is an F_2-connection for $\mathcal{E}_1$ if and only if for every $\xi \in \mathcal{E}_1$,*

$$[\theta_\xi, F_2 \oplus F] \in \mathcal{K}(\mathcal{E}_2 \oplus \mathcal{E}),$$

where $\theta_\xi := \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix} \in \mathcal{L}_B(\mathcal{E}_2 \oplus \mathcal{E})$ ([85, Definition 8]).

Definition 9.2.3. *Let A and B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. A $\mathcal{G}$-*equivariant (or just equivariant if $\mathcal{G}$ is understood) Kasparov A,B-correspondence *is a pair $(\mathcal{E}, F)$ where $\mathcal{E}$ is a $\mathcal{G}$-equivariant Rg A,B-correspondence, F is a Real operator of degree 1 in $\mathcal{L}(\mathcal{E})$ such that for all $a \in A$,*

(i) $a(F - F^*) \in \mathcal{K}(\mathcal{E})$;

(ii) $a(F^2 - 1) \in \mathcal{K}(\mathcal{E})$;

(iii) $[a, F] \in \mathcal{K}(\mathcal{E})$;

(iv) $W(s^*F \hat{\otimes}_{s^*B} \mathrm{Id})W^* \in \mathcal{L}(\mathcal{F}_b(i^*\mathscr{A}) \hat{\otimes}_{r^*A} r^*\mathcal{E})$ *is an r^*F-connection for $\mathcal{F}_b(i^*\mathscr{A})$.*

We say that $(\mathcal{E}, F)$ is degenerate *if the elements*

$$a(F - F^*), a(F^2 - 1), [a, F], \text{ and } [\theta_\xi, r^*F \oplus W(s^*F \hat{\otimes}_{s^*B} \mathrm{Id})W^*]$$

are 0 *for all $a \in A, \xi \in \mathcal{F}_b(i^*\mathscr{A})$.*

Remark 9.2.4. *Suppose $\mathcal{G}$ acts on A and B by automorphisms and ${}_A\mathcal{E}_B$ is a $\mathcal{G}$-equivariant Rg A,B-correspondence. Then the isomorphism W induces a continuous family of graded isomorphisms $\widetilde{W}_g : \mathcal{E}_{s(g)} \longrightarrow \mathcal{E}_{r(g)}$ via the identifications*

$$\mathcal{E}_{s(g)} \hat{\otimes}_{\mathscr{B}_{s(g)}} \mathscr{B}_{r(g)} \cong \mathcal{E}_{s(g)} \hat{\otimes}_{\mathscr{B}_{s(g)}} \mathscr{B}_{s(g)} \cong \mathcal{E}_{s(g)}, \quad \text{and} \quad \mathcal{A}_{r(g)} \hat{\otimes}_{\mathscr{A}_{r(g)}} \mathcal{E}_{r(g)} \cong \mathcal{E}_{r(g)}.$$

Note that from the commutativity of (9.1), *$\widetilde{W}$ verifies $\widetilde{W}_{gh} = \widetilde{W}_g \circ \widetilde{W}_h$ for all $(g,h) \in \mathcal{G}^{(2)}$; so that $\widetilde{W}$ is a Rg $\mathcal{G}$-action by automorphisms on $\mathcal{E}$. Moreover, it is straightforward to see that the map $\varphi : A \longrightarrow \mathcal{L}(\mathcal{E})$ is $\mathcal{G}$-equivariant;* i.e. *$\widetilde{W}_g \varphi(a) \widetilde{W}_g^* = \varphi(\alpha_g(a))$ for all $g \in \mathcal{G}$ and $a \in A_{s(g)}$. Now, if $(\mathcal{E}, F)$ is a $\mathcal{G}$-equivariant Kasparov A,B-correspondence, then condition (iv) of Definition 9.2.3 implies that $\widetilde{W}_g F_{s(g)} \widetilde{W}_g^* - F_{r(g)} \in \mathcal{K}(\mathcal{E}_{r(g)})$ for all $g \in \mathcal{G}$ (take $\xi = 0$), so that we recover Le Gall's definition of an equivariant Kasparov bimodule ([52, Definition 4.1.2]). We will then refer to condition (iv) as the condition of invariance modulo compacts.*

Definition 9.2.5. *Two equivariant Kasparov A,B-correspondences $(\mathcal{E}_i,F_i), i=1,2$ are* unitarily equivalent *if there exists an isomorphism of Real graded A,B-correspondences $u\in\mathcal{L}(\mathcal{E}_1,\mathcal{E}_2)$ such that $F_2=uF_1u^*$; in this case we write $(\mathcal{E}_1,F_1)\sim_u(\mathcal{E}_2,F_2)$. The set of unitarily equivalence classes of equivariant Kasparov A,B-correspondences is denoted by $\mathbf{ER}_{\mathcal{G}}(A,B)$.*

Let the C^*-algebra A be endowed with a generalized $\mathcal{G}$-action $\mathscr{A}$. Then the Real graded C^*-algebra $A[0,1]:=\mathcal{C}([0,1],A)$ (with the gradding $(A[0,1])^i=A^i[0,1], i=0,1$, and Real structure $\bar{f}(t):=\overline{f(t)}$, for $f\in A[0,1], t\in[0,1]$) is equipped with the generalized $\mathcal{G}$-action given by the Real graded u.s.c Fell bundle $\mathscr{A}[0,1]\longrightarrow\mathcal{G}$ with $(\mathscr{A}[0,1])_g=\mathscr{A}_g[0,1]$.

Definition 9.2.6. *Let A and B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. A* homotopy *in $\mathbf{ER}_{\mathcal{G}}(A,B)$ is an element $(\mathcal{E},F)\in\mathbf{ER}_{\mathcal{G}}(A,B[0,1])$. Two elements $(\mathcal{E}_i,F_i), i=0,1$ of $\mathbf{ER}_{\mathcal{G}}(A,B)$ are said to be* homotopically equivalent *if there is a homotopy $(\mathcal{E},F)$ such that $(\mathcal{E}\hat{\otimes}_{ev_0}B,F\hat{\otimes}_{ev_0}\mathrm{Id})\sim_u(\mathcal{E}_0,F_0)$, and $(\mathcal{E}\hat{\otimes}_{ev_1}B,F\hat{\otimes}_{ev_1}\mathrm{Id})\sim_u(\mathcal{E}_1,F_1)$, where for all $t\in[0,1]$, the evolution map $ev_t:B[0,1]\longrightarrow B$ is the sujective Rg *-homomorphism $ev_t(f):=f(t)$. The set of homotopy classes of elements of $\mathbf{ER}_{\mathcal{G}}(A,B)$ is denoted by $KKR_{\mathcal{G}}(A,B)$.*

Example 9.2.7. *Let A be a Rg C^*-algebra equipped with a generalized Real $\mathcal{G}$-action. Then there is a canonical element $\mathbf{1}_A\in KKR_{\mathcal{G}}(A,A)$ given by the class of the equivariant Kasparov A,A-correspondence $(A,0)$, where A is naturally viewed as a Rg A,A-correspondence via the homomorphism $A\longrightarrow\mathcal{L}_A(A)=A$ defined by left multiplication by elements of A.*

Definition 9.2.8. *Given two elements $x_1,x_2\in KKR_{\mathcal{G}}(A,B)$, their sum is given by $x_1\oplus x_2=(\mathcal{E}_1\oplus\mathcal{E}_2,F_1\oplus F_2)$, where $(\mathcal{E}_i,F_i), i=1,2$ is any representative of x_i.*

Let A,B, and D be Rg C^*-algebras endowed with the generalized Real $\mathcal{G}$-actions $\mathscr{A},\mathscr{B}$, and $\mathscr{D}$, respectively. Then the Rg C^*-algebras $A\hat{\otimes}_{\mathcal{C}_0(X)}D\cong\mathcal{C}_0(X;\mathscr{A}\hat{\otimes}_X\mathscr{D})$ and $B\hat{\otimes}_{\mathcal{C}_0(X)}D\cong\mathcal{C}_0(X;\mathscr{B}\hat{\otimes}_X\mathscr{D})$ are provided with the generalized Real $\mathcal{G}$-actions given by the Rg u.s.c. Fell bundles $\mathscr{A}\hat{\otimes}_{\mathcal{G}}\mathscr{D}\longrightarrow\mathcal{G}$ and $\mathscr{B}\hat{\otimes}_{\mathcal{G}}\mathscr{D}\longrightarrow\mathcal{G}$. Now if ${}_A\mathcal{E}_B$ is a $\mathcal{G}$-equivariant C^*-correspondence via the non-degenerate homomorphism $\varphi:A\longrightarrow\mathcal{L}(\mathcal{E})$, it is easy to see that $\mathcal{E}\hat{\otimes}_AA\hat{\otimes}_{\mathcal{C}_0(X)}D$ is a $\mathcal{G}$-equivariant $A\hat{\otimes}_{\mathcal{C}_0(X)}D,B\hat{\otimes}_{\mathcal{C}_0(X)}D$-correspondence via the map

$$\varphi\hat{\otimes}\mathrm{Id}_A\hat{\otimes}\mathrm{Id}_D:A\hat{\otimes}_{\mathcal{C}_0(X)}D\longrightarrow\mathcal{L}_{B\hat{\otimes}_{\mathcal{C}_0(X)}D}(\mathcal{E}\hat{\otimes}_AA\hat{\otimes}_{\mathcal{C}_0(X)}D).$$

We therefore may give the same construction as in [52, Définition 4.2.1].

Definition 9.2.9. *Let A,B, and D be as above. We define a group homomorphism*

$$\tau_D:KKR_{\mathcal{G}}(A,B)\longrightarrow KKR_{\mathcal{G}}(A\hat{\otimes}_{\mathcal{C}_0(X)}D,B\hat{\otimes}_{\mathcal{C}_0(X)}D)$$

by setting

$$\tau_D([\mathcal{E},F]):=[(\mathcal{E}\hat{\otimes}_AA\hat{\otimes}_{\mathcal{C}_0(X)}D,F\hat{\otimes}\mathbf{1}_A\hat{\otimes}\mathbf{1}_D)],\ \text{for}\ [(\mathcal{E},F)]\in KKR_{\mathcal{G}}(A,B).$$

The following can be proven as in the standard case where no generalized actions are involved (see [46, §4]).

Proposition 9.2.10. *Under the operations of direct sum, $KKR_{\mathcal{G}}(A,B)$ is an abelian group. Moreover, the assignement $(A,B) \longmapsto KKR_{\mathcal{G}}(A,B)$ is a bifunctor, covariant in B and contravariant in A, from the category $\mathfrak{Corr}_{\mathcal{G}}$ to the category $\mathfrak{Ab}$ of abelian groups*

Note that the inverse of an element $x \in KKR_{\mathcal{G}}(A,B)$ is given by the class of $(-\mathcal{E},-F)$, where $(\mathcal{E},F)$ is a representative of x, $-\mathcal{E}$ is the Real graded A,B-correspondence given by $\mathcal{E}$ with the opposite grading (i.e. $(-\mathcal{E})^i = \mathcal{E}^{1-i}, i = 0,1$) and the same Real structure, and the non-degenerate homomorphism of Real graded C^*-algebras $-\varphi : A \to \mathcal{L}(-\mathcal{E})$ defined by

$$-\varphi(a) := \begin{pmatrix} 0 & \mathbf{1}_{\mathcal{E}^1} \\ \mathbf{1}_{\mathcal{E}^0} & 0 \end{pmatrix} \varphi(a), \ \forall a \in A.$$

Also, as in the usual case, degenerate elements are homotopically equivalent to $(0,0)$, so that they represent the zero element of $KKR_{\mathcal{G}}(A,B)$.

Remark 9.2.11. *One recovers Kasparov's $KKR(A,B)$ of [46] by taking the Real groupoid $\mathcal{G}$ to be the point. Indeed, in this case we may omit condition (iv) of Definition 9.2.3 since, thanks to (9.1), the automorphism $W : \mathcal{E} \longrightarrow \mathcal{E}$ is in fact equal to the identity.*

Higher $KKR_{\mathcal{G}}$-groups are defined in an obvious way. Given a Rg C^*-algebra A endowed with a generalized $\mathcal{G}$-action $\mathscr{A} \longrightarrow \mathcal{G}$, the Rg u.s.c. Fell bundle

$$\mathscr{A} \hat{\otimes} \mathbb{C}l_{p,q} := \coprod_{g \in \mathcal{G}} \mathscr{A}_g \hat{\otimes} \mathbb{C}l_{p,q}$$

over $\mathcal{G}$ is a generalized $\mathcal{G}$-action on the the Rg C^*-algebra

$$A_{p,q} := A \hat{\otimes} \mathbb{C}l_{p,q} \cong \mathfrak{C}_0(X; \mathscr{A} \hat{\otimes} \mathbb{C}l_{p,q}).$$

Definition 9.2.12. *Let A,B be Rg C^*-algebras endowed with generalized $\mathcal{G}$-actions. Then, the higher $KKR_{\mathcal{G}}$-groups $KKR_{\mathcal{G},j}(A,B)$ are defined by*

$$KKR_{\mathcal{G},j}(A,B) = KKR_{\mathcal{G}}^{-j}(A,B) := \begin{cases} KKR_{\mathcal{G}}(A_{j,0},B) \cong KKR_{\mathcal{G}}(A,B_{0,j}), & \text{if } j \geq 0 \\ KKR_{\mathcal{G}}(A_{-j,0},B) \cong KKR_{\mathcal{G}}(A,B_{0,-j}), & \text{if } j \leq 0 \end{cases}$$

Let us outline the construction of the Kasparov product in groupoid equivariant KKR-theory. To do this, we need the following

Theorem 9.2.13. *(cf. [90, Theorem 6.9]). Let A,D and B be separable Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. Let $(\mathcal{E}_1,F_1) \in \mathbf{ER}_{\mathcal{G}}(A,D), (\mathcal{E}_2,F_2) \in \mathbf{ER}_{\mathcal{G}}(D,B)$. Denote by $\mathcal{E}$ the Rg $\mathcal{G}$-equivariant A,B-correspondence $\mathcal{E} = \mathcal{E}_1 \hat{\otimes}_D \mathcal{E}_2$. Then the set $F_1 \hat{\#} F_2$ of Real operators $F \in \mathcal{L}(\mathcal{E})$ such that*

- $(\mathcal{E}, F) \in \mathbf{ER}_{\mathcal{G}}(A, B)$;
- *F is a F_2-connection for $\mathcal{E}_1$;*
- $\forall a \in A, a[F_1 \hat{\otimes}_D \mathbf{1}, F]a^* \geq 0$ *modulo* $\mathcal{K}(\mathcal{E})$

is non-empty.

Now from this theorem, the *Kasparov product*

$$\hat{\otimes}_{\mathcal{G},D} : KKR_{\mathcal{G}}(A, D) \otimes_D KKR_{\mathcal{G}}(D, B) \longrightarrow KKR_{\mathcal{G}}(A, B) \tag{9.2}$$

of $[(\mathcal{E}_1, F_1)] \in KKR_{\mathcal{G}}(A, D)$ and $[(\mathcal{E}_2, F_2)] \in KKR_{\mathcal{G}}(D, B)$ is defined by

$$[(\mathcal{E}_1, F_1)]\hat{\otimes}_D[(\mathcal{E}_2, F_2)] := [(\mathcal{E}, F)], \tag{9.3}$$

where $\mathcal{E} := \mathcal{E}_1 \hat{\otimes} \mathcal{E}_2$ and $F \in F_1 \hat{\#} F_2$. It is not hard to see that as in the complex case where the C^*-algebras are equipped with $\mathcal{G}$-actions by automorphisms (see [52]), this product is well-defined, bilinear, associative, homotopy-invariant, covariant with respect to B and contravariant with respect to A.

More generally, we have

Theorem 9.2.14. *Let $A_1, A_2 B_1, B_2$ and D be separable Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. Then, the product* (9.2) *induces an associative product*

$$KKR_{\mathcal{G},i}(A_1, B_1 \hat{\otimes}_{\mathcal{C}_0(X)} D) \otimes_D KKR_{\mathcal{G},j}(D \hat{\otimes}_{\mathcal{C}_0(X)} A_2, B_2) \longrightarrow KKR_{\mathcal{G},i+j}(A_1 \hat{\otimes}_{\mathcal{C}_0(X)} A_2, B_1 \hat{\otimes}_{\mathcal{C}_0(X)} B_2).$$

Proof. The proof is almost the same as that of [46, Theorem 5.6]. □

Moreover, it is not hard to check that as the usual case ([47]), there are *descent morphisms*

$$\begin{aligned} j_{\mathcal{G}} : KKR_{\mathcal{G}}(A, B) &\longrightarrow KKR(C^*(\mathcal{G}; \mathscr{A}), C_r^*(\mathcal{G}; \mathscr{B})); \\ j_{\mathcal{G},red} : KKR_{\mathcal{G}}(A, B) &\longrightarrow KKR(C_r^*(\mathcal{G}, \mathscr{A}), C_r^*(\mathcal{G}, \mathscr{B})), \end{aligned}$$

compatible with the Kasparov product.

9.3 Functoriality in the category $\mathfrak{RG}$

Let $f : \Gamma \longrightarrow \mathcal{G}$ be a strict morphism of Real groupoids and A a Rg C^*-algebra endowed with the generalized Real $\mathcal{G}$-action $\mathscr{A} \longrightarrow \mathcal{G}$. Then the pull-back $f^*\mathscr{A} \longrightarrow \Gamma$ defines a generalized Real Γ-action on the Rg C^*-algebra $f^*A = \mathcal{C}_0(Y; f^*\mathscr{A})$. Let B be another Rg C^*-algebra together with a generalized Real $\mathcal{G}$-action $\mathscr{B}$. Suppose ${}_A\mathcal{E}_B$ is a Rg C^*-correspondence. Then under the identifications

$$f^*A = A \hat{\otimes}_{\mathcal{C}_0(X)} \mathcal{C}_0(Y), f^*B = B \hat{\otimes}_{\mathcal{C}_0(X)} \mathcal{C}_0(Y), \text{ and } f^*\mathcal{E} = \mathcal{E} \hat{\otimes}_{\mathcal{C}_0(X)} \mathcal{C}_0(Y), \tag{9.4}$$

we see that $f^*\mathcal{E}$ is a Rg f^*A, f^*B-correspondence. Further, assume that ${}_A\mathcal{E}_B$ is $\mathcal{G}$-equivariant with respect to the isomorphism $W : s_{\mathcal{G}}^*\mathcal{E}\hat{\otimes}_{s_{\mathcal{G}}^*B}\mathcal{F}(i_{\mathcal{G}}^*\mathscr{B}) \xrightarrow{\cong} \mathcal{F}_b(i_{\mathcal{G}}^*\mathscr{A})\hat{\otimes}_{r_{\mathcal{G}}^*A} r_{\mathcal{G}}^*\mathcal{E}$. Then, by using the following identifications (compare with [52, p.65])

$$\begin{array}{ll} i_\Gamma^*(f^*\mathscr{A}) = f^*(i_{\mathcal{G}}^*\mathscr{A}), & i_\Gamma^*(f^*\mathscr{B}) = f^*(i_{\mathcal{G}}^*\mathscr{B}),\\ s_\Gamma^*(f^*A) = s_{\mathcal{G}}^*A\hat{\otimes}_{\mathcal{C}_0(\mathcal{G})}\mathcal{C}_0(\Gamma), & r_\Gamma^*(f^*A) = r_{\mathcal{G}}^*A\hat{\otimes}_{\mathcal{C}_0(\mathcal{G})}\mathcal{C}_0(\Gamma)\\ s_\Gamma^*(f^*\mathcal{E}) = r_{\mathcal{G}}^*\mathcal{E}\hat{\otimes}_{\mathcal{C}_0(\mathcal{G})}\mathcal{C}_0(\Gamma), & r_\Gamma^*(f^*\mathcal{E}) = r_{\mathcal{G}}^*\mathcal{E}\hat{\otimes}_{\mathcal{C}_0(\mathcal{G})}\mathcal{C}_0(\Gamma), \end{array} \tag{9.5}$$

where the Real action of $\mathcal{C}_0(\mathcal{G})$ on $\mathcal{C}_0(\Gamma)$ is induced by f in an obvious way, we get an isomorphism

$$f^*W : s_\Gamma^*(f^*\mathcal{E})\hat{\otimes}_{s_\Gamma^*f^*B}\mathcal{F}_b(i_\Gamma^*f^*\mathscr{B}) \longrightarrow \mathcal{F}_b(i_\Gamma^*f^*\mathcal{A})\hat{\otimes}_{r_\Gamma^*f^*A} r_\Gamma^*f^*\mathcal{E},$$

making $f^*\mathcal{E}$ into a Rg $\mathcal{G}$-equivariant f^*A, f^*B-correspondence. Hence equivariant KKR-theory has a functorial property in the category $\mathfrak{RG}_s$.

Definition and Lemma 9.3.1. *Let $f : \Gamma \longrightarrow \mathcal{G}$ be a strict morphism of Real groupoids. Let A and B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. Then we define a group homomorphism*

$$f^* : KKR_{\mathcal{G}}(A,B) \longrightarrow KKR_\Gamma(f^*A, f^*B)$$

by assigning to a $\mathcal{G}$-equivariant Kasparov A,B-correspondence $(\mathcal{E},F)\in \mathbf{ER}_{\mathcal{G}}(A,B)$ the pair

$$f^*(\mathcal{E},F) := (f^*\mathcal{E}, f^*F)\in \mathbf{ER}_\Gamma(f^*A, f^*B),$$

where under the identifications (9.4)*, we put $f^*F = F\hat{\otimes}_{\mathcal{C}_0(X)}\mathrm{Id}_{\mathcal{C}_0(Y)}$.*
Moreover, the map f^* is natural with respect to the Kasparov product (9.2) *in the sense that if D is another Rg C^*-algebra equipped with a generalized Real $\mathcal{G}$-action, then*

$$f^*(x)\hat{\otimes}_{f^*D} f^*(y) = f^*(x\hat{\otimes}_D y),\ \forall x\in KKR_{\mathcal{G}}(A,D), y\in KKR_{\mathcal{G}}(D,B).$$

Proof. The proof is the same as those of Lemma 6.1.1 and Proposition 6.1.3 in [52]. □

More generally, in order to establish functoriality in the category $\mathfrak{RG}$ we need the following proposition which is a generalisation of Le Gall's [52, Proposition 3.1.3].

Proposition 9.3.2. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a locally compact second-countable Real groupoid, and let $\mathcal{U} = (U_j)_{j\in J}$ a Real open cover of X. Denote by $\iota : \mathcal{G}[\mathcal{U}] \longrightarrow \mathcal{G}$ the canonical inclusion. For all Rg Fell bundle (resp. u.s.c. Fell bundle) $\mathscr{A} \longrightarrow \mathcal{G}[\mathcal{U}]$, there exist a Rg Fell bundle (resp. u.s.c. Fell bundle) $\iota_!\mathscr{A} \longrightarrow \mathcal{G}$ and an isomorphism of Rg Fell bundles (resp. of u.s.c. Fell bundles)*

$$\iota^*(\iota_!\mathscr{A}) \xrightarrow{\cong} \mathscr{A}$$

over $\mathcal{G}[\mathcal{U}] \rightrightarrows \coprod_{j\in J} U_j$.

Proof. We may suppose $\mathcal{U}$ is locally finite. We use the following notations: as usual, we write $g_{j_0j_1}$ for $(j_0,g,j_1)\in\mathcal{G}[\mathcal{U}]$, and x_j for $(j,x)\in\coprod_{j\in J}U_j$; let $\pi:\mathscr{A}\longrightarrow\mathcal{G}[\mathcal{U}]$ be the projection of the given Rg Fell bundle (resp. u.s.c. Fell bundle); an element $a\in\pi^{-1}(g_{j_0j_1})$ will be written $a_{j_0j_1}$.

For $x\in X$ we denote by I_x the finite subset of $j\in J$ such that $x\in U_j$, and for $g\in\mathcal{G}$ let I_g the finite subset of pairs $(j_0,j_1)\in J\times J$ such that $g\in\mathcal{G}_{U_{j_1}}^{U_{j_0}}$. Put

$$\iota_!\mathscr{A}_g:=\bigoplus_{(j_0,j_1)\in I_g}\mathscr{A}_{g_{j_0j_1}},\quad\text{and }\widetilde{\iota_!\mathscr{A}}:=\coprod_{g\in\mathcal{G}}\iota_!\mathscr{A}.$$

Then $\widetilde{\iota_!\mathscr{A}}\longrightarrow\mathcal{G}$ is a Rg Banach bundle (resp. u.s.c. Banach bundle), with the projection $\iota_!\pi$ defined by

$$\iota_!\pi((g,a_{j_0j_1})_{(j_0,j_1)\in I_g}):=g.$$

Moreover, for $(g_1,g_2)\in\mathcal{G}^{(2)}$, the pairing

$$\begin{array}{ccc}I_{g_1}\times_{I_{s(g_1)}}I_{g_2} & \longrightarrow & I_{g_1g_2}\\ ((j_0,j_1),(j_1,j_2)) & \longmapsto & (j_0,j_2)\end{array}$$

where $I_{g_1}\times_{I_{s(g_1)}}I_{g_2}:=\{((j_0,j_1),(j_1,j_2))\in I_{g_1}\times I_{g_2}\}$, enables us to define a multiplication on $\widetilde{\iota_!\mathscr{A}}$

$$\iota_!\mathscr{A}_{g_1}\hat{\otimes}_{\iota_!\mathscr{A}_{s(g_1)}}\iota_!\mathscr{A}_{g_2}\longrightarrow\iota_!\mathscr{A}_{g_1g_2}$$

generated by $a_{j_0j_1}\hat{\otimes}b_{j_1j_2}\longmapsto(ab)_{j_0j_2}$. One verifies easily that together with this multiplication, $\iota_!\mathscr{A}\longrightarrow\mathcal{G}$ is a Rg Fell bundle (resp. u.s.c Fell bundle) that satisfies to the desired isomorphism. □

Definition 9.3.3. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ and $\mathcal{U}$ be as above. Given a Rg C^*-algebra A with a generalized Real $\mathcal{G}[\mathcal{U}]$-action $\mathscr{A}$, we denote by $\iota_!A$ the Rg C^*-algebra $\mathcal{C}_0(X;\iota_!\mathscr{A})$ endowed with the generalized Real $\mathcal{G}$-action $\iota_!\mathscr{A}\longrightarrow\mathcal{G}$.*

Proposition 9.3.4. *Let A,B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}[\mathcal{U}]$-actions $\mathscr{A}$ and $\mathscr{B}$, respectively. Assume $\mathcal{E}$ is a $\mathcal{G}[\mathcal{U}]$-equivariant Rg A,B-correspondence. Then there exists a Rg $\iota_!A,\iota_!B$-correspondence $\iota_!\mathcal{E}$ and an isomorphism of Rg A,B-correspondences $\iota^*\iota_!\mathcal{E}\cong\mathcal{E}$.*

Proof. The map $\iota_!:B\longrightarrow\iota_!B$ sending $\phi\in\mathcal{C}_0(\coprod_jU_j;\mathscr{B})$ to the function $\iota_!\phi\in\mathcal{C}_0(X;\iota_!\mathscr{B})$ defined by

$$\iota_!\phi(x):=(\phi((j,x)))_{j\in I_x},$$

is a surjective Rg *-homomorphism. We then can define the pushout $\iota_!\mathcal{E}$ of the Rg Hilbert B-module $\mathcal{E}$ via $\iota_!$. Let us recall (see [39, §1.2.2.]) the definition of the Rg Hilbert $\iota_!B$-module

$\iota_!\mathcal{E}$. Let $N_{\iota_!} = \{\xi \in \mathcal{E} \mid \iota_!(\langle \xi, \xi\rangle_B) = 0\}$; denote by $\dot{\xi}$ the image of $\xi \in \mathcal{E}$ in the quotient space $\mathcal{E}/_{N_{\iota_!}}$, the latter being a Rg $\iota_!B$-module by setting

$$\dot{\xi}\cdot \iota_!(b) := \dot{\widehat{\xi b}}, \quad \dot{\xi} \in \mathcal{E}/_{N_{\iota_!}}, \iota_!(b) \in \iota_!B.$$

Then define the Rg Hilbert $\iota_!B$-module $\iota_!\mathcal{E}$ as the completion of $\mathcal{E}/_{N_{\iota_!}}$ with respect to the Rg $\iota_!B$-valued pre-inner product

$$\langle \dot{\xi}, \dot{\eta}\rangle_{\iota_!B} := \iota_!(\langle \xi, \eta\rangle), \quad \xi, \eta \in \mathcal{E}.$$

For $T \in \mathcal{L}(\mathcal{E})$, let $\iota_!T$ be the unique operator in $\mathcal{L}(\iota_!\mathcal{E})$ making the following diagram commute

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "T"] & \iota_!\mathcal{E} \arrow[d, "\iota_!T"] \\ \mathcal{E} \arrow[r] & \iota_!\mathcal{E} \end{tikzcd}$$

where the horizontal arrows are the quotient map; *i.e.*,

$$\iota_!T(\dot{\xi}) = \dot{\widehat{T(\xi)}}, \quad \dot{\xi} \in \iota_!\mathcal{E}. \tag{9.6}$$

Hence, the map $\varphi : A \longrightarrow \mathcal{L}(\mathcal{E})$ implementing the A,B-correspondence gives rise to a non-degenerate *-homomorphism $\iota_!\varphi : \iota_!A \longrightarrow \mathcal{L}(\iota_!\mathcal{E})$ such that

$$(\iota_!\varphi)(\iota_!(a)) := \iota_!(\varphi(a)), \quad \forall a \in A. \tag{9.7}$$

Therefore, $\iota_!\mathcal{E}$ is a Rg $\iota_!A, \iota_!B$-correspondence. It is not hard to check that $\iota_!\mathcal{E}$ is isomorphic to $\mathcal{C}_0(X; \widetilde{\iota_!\mathscr{E}})$, where $\widetilde{\iota_!\mathscr{E}} \longrightarrow X$ is the unique Rg u.s.c. field of Banach algebras with fibre $(\iota_!\mathscr{E})_x = \bigoplus_{j\in I_x} \mathcal{E}\hat{\otimes}_B B_{(j,x)}$ (cf. Appendix C ???); indeed, since $\mathcal{E}$ is a Rg Hilbert B-module, recall that there is a unique topology on the Rg u.s.c. field $\tilde{\mathscr{E}} = \coprod_{(j,x)} \mathcal{E}\hat{\otimes}_B B_{(j,x)}$ such that $\mathcal{E} \cong \mathcal{C}_0(\coprod_j U_j; \tilde{\mathscr{E}})$; so, by Proposition 9.3.2, we get the Rg u.s.c. field $\widetilde{\iota_{\mathscr{E}}}$.

Let $W : s^*\mathcal{E}\hat{\otimes}_{s^*B}\mathcal{F}_b(i^*\mathscr{B}) \longrightarrow \mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A} r^*\mathcal{E}$ be the isomorphism of C^*-correspondences implementing the $\mathcal{G}[\mathcal{U}]$-equivariance. Then from the trivial identifications

$$\begin{aligned} s^*(\iota_!A) &= \iota_!(s^*A), & r^*(\iota_!A) &= \iota_!(r^*A),\\ s^*(\iota_!B) &= \iota_!(s^*B), & r^*(\iota_!B) &= \iota_!(r^*B),\\ i^*(\iota_!\mathscr{A}) &= \iota_!(i^*\mathscr{A}), & i^*(\iota_!\mathscr{B}) &= \iota_!(i^*\mathscr{B}),\\ s^*(\iota_!\mathcal{E}) &= \iota_!(s^*\mathcal{E}), & r^*(\iota_!\mathcal{E}) &= \iota_!(r^*\mathcal{E}), \end{aligned}$$

we get

$$\begin{aligned} s^*(\iota_!\mathcal{E})\hat{\otimes}_{s^*(\iota_!B)}\mathcal{F}_b(i^*(\iota_!\mathscr{B})) &\cong \iota_!\left(s^*\mathcal{E}\hat{\otimes}_{s^*B}\mathcal{F}_b(i^*\mathscr{B})\right), \text{ and}\\ \mathcal{F}_b(i^*(\iota_!\mathscr{A}))\hat{\otimes}_{r^*(\iota_!A)} r^*(\iota_!\mathcal{E}) &\cong \iota_!\left(\mathcal{F}_b(i^*\mathscr{A})\hat{\otimes}_{r^*A} r^*\mathcal{E}\right). \end{aligned}$$

Thus, W induces an isomorphism of Rg $s^*(\iota_! A), r^*(\iota_! B)$-correspondences

$$\iota_! W : s^*(\iota_!\mathcal{E})\hat{\otimes}_{s^*(\iota_! B)}\mathcal{F}_b(i^*(\iota_!\mathscr{B})) \longrightarrow \mathcal{F}_b(i^*(\iota_!\mathscr{A}))\hat{\otimes}_{r^*(\iota_! A)} r^*(\iota_!\mathcal{E}),$$

defined in a similar fashion as (9.7); so that $\iota_! W$ is compatible with the partial product of $\mathcal{G}$ in the sense of the commutative diagram (9.1). That the $\mathcal{G}[\mathcal{U}]$-equivariant A, B-correspondences $\mathcal{E}$ and $\iota^*\iota_!\mathcal{E}$ are isomorphic is a mere consequence of the construction of $\iota_!\mathcal{E}$ and $\iota_! W$. □

Lemma 9.3.5. *Suppose $x = (\mathcal{E}, F) \in \mathbf{ER}_{\mathcal{G}[\mathcal{U}]}(A, B)$. Then $\iota_! x := (\iota_!\mathcal{E}, \iota_! F) \in \mathbf{ER}_{\mathcal{G}}(\iota_! A, \iota_! B)$, where the operator $\iota_! F \in \mathcal{L}(\iota_!\mathcal{E})$ is given by* (9.6).

Proof. This results from very easy algebraic verifications. For instance, the map $\iota_! : \mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{L}(\iota_!\mathcal{E})$ respects the degree and the Real structures. Moreover $\iota_!$ sends $\mathcal{K}(\mathcal{E})$ onto $\mathcal{K}(\iota_!\mathcal{E})$ because $\iota_!(\theta_{\xi,\eta}) = \theta_{\dot{\xi},\dot{\eta}}$ for all $\xi, \eta \in \mathcal{E}$, $(\iota_! T_1)(\iota_! T_1) = \iota_!(T_1 T_2)$, and $[\iota_! T_1, \iota_! T_2] = \iota_![T_1, T_2], , \forall T_1, T_2 \in \mathcal{L}(\mathcal{E})$; thus conditions (i)-(iii) in Definition 9.2.3 are satisfied by the pair $(\iota_!\mathcal{E}, \iota_! F)$. To verify condition (iv), let us put $\mathcal{E}_1 = \mathcal{F}_b(i^*\mathscr{A}), \mathcal{E}_2 = r^*\mathcal{E}$, and $\mathcal{E} = \mathcal{E}_1\hat{\otimes}_{r^*A}\mathcal{E}_2$. Then $\mathcal{K}(\iota_!\mathcal{E}_2 \oplus \iota_!\mathcal{E}) = \mathcal{K}(\iota_!(\mathcal{E}_1 \oplus \mathcal{E}))$, and we have seen in the proof of Proposition 9.3.2 that $\iota\mathcal{E}_1\hat{\otimes}_{\iota_! r^* A}\iota_!\mathcal{E} = \iota_!(\mathcal{E}_1\hat{\otimes}_{r^*A}\mathcal{E})$. It follows that for $\xi \in \mathcal{E}_1$ and $\eta \in \mathcal{E}_2$, one has

$$\iota_!(T_\xi) = \dot{\widehat{T_\xi(\eta)}} = \dot{\widehat{\xi\hat{\otimes}_{r^*A}\eta}} = \dot{\xi}\hat{\otimes}_{\iota_! r^* A}\dot{\eta},$$

which means that $\iota_!(T_\xi) = T_{\dot{\xi}} \in \mathcal{L}(\iota_!\mathcal{E}_2, \iota_!\mathcal{E})$, and hence $\iota_!(\theta_\xi) = \theta_{\dot{\xi}}$ (recall notations used in Remark 9.2.2). Then,

$$\begin{aligned} [\theta_{\dot{\xi}}, \iota_!(r^*F) \oplus \iota_!(W(s^*F\hat{\otimes}_{s^*B}\mathrm{Id})W^*)] &= \iota_!([\theta_\xi, r^*F \oplus W(s^*F\hat{\otimes}_{s^*B}\mathrm{Id})W^*] \\ &\in \mathcal{K}(\iota_!(\mathcal{E}_2 \oplus \mathcal{E})) = \mathcal{K}(\iota_!\mathcal{E}_2 \oplus \iota_!\mathcal{E}); \end{aligned}$$

therefore, $\iota_! W(s^*(\iota_! F)\hat{\otimes}_{s^*\iota_! B}\mathrm{Id})\iota_! W^* = \iota_!(W(s^*F\hat{\otimes}_{s^*B}\mathrm{Id})W^*)$ is an $r^*\iota_! F$-connection for $\iota_!\mathcal{E}_1 = \mathcal{F}_b(i^*(\iota_!\mathscr{A}))$. □

The following result can be proved with similar arguments as in [52, Théorème 6.2.1], so we omit the proof.

Theorem 9.3.6. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, $\mathcal{U}$, A and B be as previously. Then the canonical Real inclusion $\mathcal{G}[\mathcal{U}] \hookrightarrow \mathcal{G}$ induces a group isomorphism*

$$\iota^* : KKR_{\mathcal{G}}(\iota_! A, \iota_! B) \longrightarrow KKR_{\mathcal{G}[\mathcal{U}]}(A, B),$$

whose inverse is

$$\iota_! : KKR_{\mathcal{G}[\mathcal{U}]}(A, B) \ni [(\mathcal{E}, F)] \longmapsto [(\iota_!\mathcal{E}, \iota_! F)] \in KKR_{\mathcal{G}}(\iota_! A, \iota_! B).$$

Corollary 9.3.7. *([52, Corollaire 6.2.1]) The isomorphism $\iota_! : KKR_{\mathcal{G}[\mathcal{U}]}(A, B) \longrightarrow KKR_{\mathcal{G}}(\iota_! A, \iota_! B)$ is natural with respect to Kasparov product;* i.e.,

$$\iota_! x\hat{\otimes}_{\iota_! D}\iota_! y = \iota_!(x\hat{\otimes}_D y), \forall x \in KKR_{\mathcal{G}[\mathcal{U}]}(A, D), y \in KKR\mathcal{G}[\mathcal{U}](D, B).$$

Now Theorem 9.3.6 enables us to define the pull-back of a Rg C^*-algebra endowed with a generalized action via a morphism in $\mathfrak{RG}$.

Definition 9.3.8. *Let $Z : \Gamma \longrightarrow \mathcal{G}$ be a generalized Real homomorphism, A and B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. Let $(\mathfrak{U}, f)$ be a representative of a morphism in $\mathfrak{RG}_\Omega$ realizing Z. Let $\iota : \Gamma[\mathfrak{U}] \longrightarrow \Gamma$ be the canonical Real inclusion. Define the pull-back of A and B via Z by*

$$Z^*B := \iota_! f^*A, \quad \text{and } Z^*B := \iota_! f^*B.$$

Then we define the pull-back homomorphism

$$Z^* : KKR_{\mathcal{G}}(A,B) \longrightarrow KKR_\Gamma(Z^*A, Z^*B),$$

as the composite

$$KKR_{\mathcal{G}}(A,B) \xrightarrow{f^*} KKR_{\Gamma[\mathfrak{U}]}(f^*A, f^*B) \xrightarrow{\iota_!} KKR_\Gamma(\iota_! f^*A, \iota_! f^*B),$$

Once again, by using similar arguments as in [52] (Corollaire 3.2.1), one can see that the definition of Z^* does not depend on the choice of the pair $[(\mathfrak{U}, f)]$, is functorial in $\mathfrak{RG}$ and natural with respect to Kasparov product ([52, Théorème 6.2.2]). In particular, if Z is a Morita equivalence, then Z^* is an isomorphism.

9.4 $KKR_{\mathcal{G}}$-equivalence

Definition 9.4.1. *(Compare with [9, Definition 19.1.1]). Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, be a Real groupoid, A and B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. An element $x \in KKR_{\mathcal{G}}(A,B)$ is a $KKR_{\mathcal{G}}$-*equivalence *if there is $y \in KKR_{\mathcal{G}}(B,A)$ such that*

$$x \hat{\otimes}_{\mathcal{G},B} y = \mathbf{1}_A, \quad \text{and } y \hat{\otimes}_{\mathcal{G},A} x = \mathbf{1}_B.$$

*A and B are said $KKR_{\mathcal{G}}$-*equivalent *if there exists a $KKR_{\mathcal{G}}$-equivalence in $KKR_{\mathcal{G}}(A,B)$.*

As in the usual case ([9, §19.1]), we have

Lemma 9.4.2. *Assume $x \in KKR_{\mathcal{G}}(A,B)$ is a $KKR_{\mathcal{G}}$-equivalence. Then for any Rg C^*-algebra D endowed with a generalized Real $\mathcal{G}$-action, the maps*

$$x \hat{\otimes}_{\mathcal{G},B} (\cdot) : KKR_{\mathcal{G}}(B,D) \longrightarrow KKR_{\mathcal{G}}(A,D), \quad \text{and } (\cdot) \hat{\otimes}_{\mathcal{G},A} x : KKR_{\mathcal{G}}(D,A) \longrightarrow KKR_{\mathcal{G}}(D,B),$$

are isomorphisms which are natural in D by associativity.

The proof is almost the same as that of [46, Theorem 4.6]. For instance, the map

$$KKR_{\mathcal{G}}(A,D) \ni z \longmapsto y\hat{\otimes}_{\mathcal{G},A} z \in KKR_{\mathcal{G}}(B,D)$$

is an inverse of the first homomorphism.

Proposition 9.4.3. *$KKR_{\mathcal{G}}$-equivalence is functorial in $\mathfrak{RG}$ in the following sense: if $Z : \Gamma \longrightarrow \mathcal{G}$ is a generalized Real homomorphism, and if A and B are $KKR_{\mathcal{G}}$-equivalent, then Z^*A and Z^*B are KKR_{Γ}-equivalent.*

Proof. By naturality of Z^* with respect to Kasparov product, we have

$$\mathbf{1}_{Z^*A} = Z^*(x\hat{\otimes}_{\mathcal{G},A} y) = Z^*(x)\hat{\otimes}_{\Gamma,Z^*A} Z^*(y),$$

and

$$\mathbf{1}_{Z^*B} = Z^*(y\hat{\otimes}_{\mathcal{G},B} x) = Z^*(y)\hat{\otimes}_{\Gamma,Z^*B} Z^*(x);$$

therefore $Z^*(x) \in KKR_{\Gamma}(Z^*A, Z^*B)$ is a KKR_{Γ}-equivalence. □

9.5 Bott periodicity

Definition 9.5.1. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a Real groupoid. A* Real Euclidean vector bundle of type *$p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ is a Euclidean vector bundle $\pi : E \longrightarrow X$ of rank $p+q$ equipped with a Real $\mathcal{G}$-action (with respect to π) such that the Euclidean metric is $\mathcal{G}$-invariant and the Real space E is locally homeomorphic to $\mathbb{R}^{p,q}$; that is to say, for every $x \in X$, there is a Real open neighborhood U of x and a Real homeomorphism $h_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^{p,q}$, where $U \times \mathbb{R}^{p,q}$ is provided with the Real structure $(x,t) \longmapsto (\bar{x},\bar{t})$. This is equivalent to the existence of a Real open cover $\mathcal{U} = (U_j)_{j\in J}$ and a family of homeomorphisms $h_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{R}^{p+q}$ such that the following diagram commute*

$$\begin{array}{ccc} \pi^{-1}(U_j) & \xrightarrow{h_j} & U_j \times \mathbb{R}^{p+q} \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau\times(\mathbf{1}_p - \mathbf{1}_q)} \\ \pi^{-1}(U_{\bar{j}}) & \xrightarrow{h_{\bar{j}}} & U_{\bar{j}} \times \mathbb{R}^{p+q} \end{array} \qquad (9.8)$$

For $p,q \in \mathbb{N}$ with $n = p+q \neq 0$, we define the Real group $O(p+q)$ to be the orthogonal group $O(n)$ equipped with the involution induced from $\mathbb{R}^{p,q}$ (we identify $M_{p+q}(\mathbb{R})$ with $\mathcal{L}(\mathbb{R}^{p,q})$,he latter is then a Real space). Similarly one defines the Real group $SO(p+q)$

Definition 9.5.2. *Associated to any Real Euclidean vector bundle E of type $p-q$ over the Real groupoid $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$, there is a generalized Real homomorphism*

$$\mathbb{F}(E) : \mathcal{G} \longrightarrow O(p+q),$$

where $\mathbb{F}(E)$ is the frame bundle of $E \longrightarrow X$.

Remark 9.5.3. *The above definition does make sense, for the fibre of the $O(p+q)$-principal bundle $\mathbb{F}(E) \longrightarrow X$ at a point $x \in X$ identifies to the $\mathbb{R}$-linear space $\mathrm{Isom}(\mathbb{R}^{p+q}, E_x)$ is $\mathbb{R}$-linear isomorphisms; so that $\mathcal{G}$ acts on $\mathbb{F}(E)$ by $g\cdot(s(g),\varphi) \longmapsto (r(g), g\cdot\varphi)$, for $\varphi \in \mathrm{Isom}(\mathbb{R}^{p+q}, E_{s(g)})$, where $(g\cdot\varphi)(t) := g\cdot\varphi(t)$, $t \in \mathbb{R}^{p+q}$. $\mathbb{F}(E)$ is equipped with the Real structure $(x,\varphi) \longmapsto (\bar{x}, \bar{\varphi})$, where $\bar{\varphi}(t) := \overline{\varphi(\bar{t})}$, $t \in \mathbb{R}^{p+q}$. It is clear that the actions by $\mathcal{G}$ and $O(p+q)$ are compatible with this involution.*

Example 9.5.4. *The trivial bundle $= X \times \mathbb{R}^{p,q} \longrightarrow X$ is a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ with respect to the Real $\mathcal{G}$-action $g\cdot(s(g),t) \longmapsto (r(g),t)$. The associated generalized Real homomorphism $\mathbb{F}(X \times \mathbb{R}^{p,q})$ from $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ to $O(p+q) \rightrightarrows \cdot$ is isomorphic to the trivial Real $O(p+q)$-principal $\mathcal{G}$-bundle $X \times O(p+q) \longrightarrow X$; we denote it by $\mathbb{F}_{p,q}$.*

Recall that $\mathbb{C}l_{p,q} := \mathbb{C}l(\mathbb{R}^{p,q}) := Cl_{p,q} \otimes_{\mathbb{R}} \mathbb{C} = Cl(\mathbb{R}^{p,q}) \otimes_{\mathbb{R}} \mathbb{C}$ is the Real graded Clifford C^*-algebra, with the Real structure is $x \otimes_{\mathbb{R}} \lambda \longmapsto \bar{x} \otimes_{\mathbb{R}} \bar{\lambda}$ (see Appendix A), where the involution "bar" in $Cl(\mathbb{R}^{p,q})$ is induced from $\mathbb{R}^{p,q}$. The Real action of $O(p+q)$ on $\mathbb{R}^{p,q}$ induces a Real $O(p+q)$-action on $\mathbb{C}l_{p,q}$.

Recall that Kasparov has defined in [46, §5] a $KKR_{O(p+q)}$-equivalence

$$\alpha_{p,q} \in KKR_{O(p+q)}\left(\mathcal{C}_0(\mathbb{R}^{p,q}), \mathbb{C}l_{p,q}\right), \quad \text{and } \beta_{p,q} \in KKR_{O(p+q)}\left(\mathbb{C}l_{p,q}, \mathcal{C}_0(\mathbb{R}^{p,q})\right),$$

that define a $KKR_{O(p+q)}$-equivalence between.

These important elements enable us to establish *Bott periodicity* in "generalized" Real groupoid KK-theory as well as the *Thom isomorphism* in twisted KR-theory which will be discussed in the next section.

Theorem 9.5.5 (Bott periodicity)**.** *Let $\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} X$ be a locally compact second-countable Real groupoid, and let A and B be Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. Then the Kasparov product with $\mathbb{F}^*_{p,q}\alpha_{p,q} \in KKR_{\mathcal{G}}\left(\mathcal{C}_0(X) \otimes \mathcal{C}_0(\mathbb{R}^{p,q}), \mathcal{C}_0(X)_{p,q}\right)$ defines an isomorphism*

$$KKR_{\mathcal{G},i+p-q}(A,B) \quad \cong \quad KKR_{\mathcal{G},i}(A(\mathbb{R}^{p,q}),B), \tag{9.9}$$

where $A(\mathbb{R}^{p,q}) = \mathcal{C}_0(\mathbb{R}^{p,q};A) = \mathcal{C}_0(\mathbb{R}^{p,q}) \otimes A$.

Proof. First of all notice that the pullbacks $\mathbb{F}^*_{p,q}(\mathcal{C}_0(\mathbb{R}^{p,q}))$ and $\mathbb{F}^*_{p,q}(\mathbb{C}l_{p,q})$ via the generalized Real homomorphism $\mathbb{F}_{p,q}:\mathcal{G}\longrightarrow O(p+q)$ are isomorphic to $\mathcal{C}_0(X;\mathcal{C}_0(\mathbb{R}^{p,q}))=\mathcal{C}_0(X)\otimes\mathcal{C}_0(\mathbb{R}^{p,q})$ and $\mathcal{C}_0(X)\otimes\mathbb{C}l_{p,q}=\mathcal{C}_0(X)_{p,q}$, respectively. These are then Rg C^*-algebras endowed with generalized Real $\mathcal{G}$-actions. Since $\alpha_{p,q}\in KKR_{O(p+q)}(\mathcal{C}_0(\mathbb{R}^{p,q}),\mathbb{C}l_{p,q})$ is a $KKR_{O(p+q)}$-equivalence, its pullback $\mathbb{F}^*_{p,q}\alpha_{p,q}\in KKR_{\mathcal{G}}(\mathcal{C}_0(X)\otimes\mathcal{C}_0(\mathbb{R}^{p,q}),\mathcal{C}_0(X)_{p,q})$ is a $KKR_{\mathcal{G}}$-equivalence, thanks to Proposition 9.4.3. Hence, from Lemma 9.4.2, the Kasparov product

$$\begin{array}{c}
KKR_{\mathcal{G}}\left(\mathcal{C}_0(X)\hat{\otimes}\mathcal{C}_0(\mathbb{R}^{p,q}),\mathcal{C}_0(X)\otimes\mathbb{C}l_{p,q}\right)\otimes_{\mathcal{C}_0(X;\mathbb{C}l_{p,q})}KKR_{\mathcal{G},i}\left(\mathcal{C}_0(X)\otimes_{\mathcal{C}_0(X)}\mathbb{C}l_{p,q}\hat{\otimes}A,B\right)\\
\Big\downarrow{\scriptstyle \mathbb{F}^*_{p,q}\alpha_{p,q}\hat{\otimes}_{\mathcal{G},\mathcal{C}_0(X;\mathbb{C}l_{p,q})}(\cdot)}\\
KKR_{\mathcal{G},i}\left(A(\mathbb{R}^{p,q}),B\hat{\otimes}_{\mathcal{C}_0(X)}\mathcal{C}_0(X)\right)
\end{array}$$

is an isomorphism. We therefore have the desired isomorphism since $\mathcal{C}_0(X)\otimes_{\mathcal{C}_0(X)}\mathbb{C}l_{p,q}\hat{\otimes}A\cong A\hat{\otimes}\mathbb{C}l_{p,q}$. □

9.6 Multiplicative structure in twisted KR-theory

In this section we are using generalized Real groupoid equivariant KK-theory to display the pairing

$$KR^{-i}_{\alpha}(\mathcal{G}^\bullet)\otimes KR^{-j}_{\beta}(\mathcal{G}^\bullet)\longrightarrow KR^{-i-j}_{\alpha+\beta}(\mathcal{G}^\bullet), \tag{9.10}$$

where $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ is a locally compact second-countable Real proper groupoid with Real Haar system such that $X/\mathcal{G}$, and $\alpha,\beta\in\check{H}R^1(\mathcal{G}_\bullet,\mathbb{Z}_2)\ltimes\check{H}R^2(\mathcal{G}_\bullet,\mathbb{S}^1)$.

To do this, we need the following result which is an obvious adaptation of [90, Proposition 6.11].

Proposition 9.6.1. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be a locally compact second-countable Real proper groupoid with Real Haar system such that $X/\mathcal{G}$ is compact. Let A be a Rg C^*-algebra endowed with a generalized Real $\mathcal{G}$-action $\mathscr{A}\longrightarrow\mathcal{G}$. Then*

$$KKR_{\mathcal{G},i}(\mathcal{C}_0(X),A)\cong KKR_i(\mathbb{C},C^*_r(\mathcal{G};\mathscr{A}))\cong KKR_i(\mathbb{C},\mathcal{K}_{\mathcal{G}}(L^2(\mathcal{G};\mathscr{A}))).$$

Corollary 9.6.2. *Let $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ be a locally compact second-countable Real proper groupoid with Real Haar system such that $X/\mathcal{G}$ is compact. Let $\mathcal{A}\in\widehat{Br\mathrm{R}}_0(\mathcal{G})$, and $[\mathbb{E}]=[(\tilde{\Gamma},\Gamma,\delta)]\in\widehat{Ext\mathrm{R}}(\mathcal{G},\mathbb{S}^1)$ such that $dd(\mathbb{E})=-DD(\mathcal{A})$. Let $L=\tilde{\Gamma}\times_{\mathbb{S}^1}\mathbb{C}$ be the corresponding Rg Fell bundle over the proper Real groupoid $\Gamma\overset{r}{\underset{s}{\rightrightarrows}}Y$. Then $L\to\Gamma$ defines a generalized Real $\mathcal{G}$-action on the Real (trivially graded) C^*-algebra*

$$A_{\mathbb{E}}:=\mathcal{C}_0(Y;L)=\mathcal{C}_0(Y).$$

Moreover, for all $i \in \mathbb{Z}$, we have

$$KR_{\mathcal{A}}^{-i}(\mathcal{G}^\bullet) \cong KKR_{\mathcal{G},i}(\mathcal{C}_0(X), A) \cong KKR_{\Gamma,i}(\mathcal{C}_0(Y), A_{\mathbb{E}}).$$

Proof. We have $KR_{\mathcal{A}}^{-i}(\mathcal{G}^\bullet) = KKR_i(\mathbb{C}, \mathcal{A} \rtimes_r \mathcal{G}) \cong KKR_i(\mathbb{C}, C_r^*(\mathbb{E}))$ (cf. Proposition 6.1.4). Then we conclude by Proposition 9.6.1. □

We then define the pairing (9.10) by the Kasparov product: let $\mathcal{A}, \mathcal{B} \in \widehat{\mathrm{BrR}}_0(\mathcal{G})$ such that $DD(\mathcal{A}) = \alpha$ and $DD(\mathcal{B}) = \beta$; then $DD(\mathcal{A} + \mathcal{B}) = DD(\mathcal{A} \hat{\otimes}_X \mathcal{B}) = \alpha + \beta$; by the identifications $A = A \hat{\otimes}_{\mathcal{C}_0(X)} \mathcal{C}_0(X)$ and $B = B \hat{\otimes}_{\mathcal{C}_0(X)} \mathcal{C}_0(X)$, where $A = \mathcal{C}_0(X; \mathcal{A}), B = \mathcal{C}_0(X; \mathcal{B})$, Kasparov product gives the coupling

$$KKR_{\mathcal{G},i}(\mathcal{C}_0(X), A) \otimes_{\mathcal{C}_0(X)} KKR_{\mathcal{G},j}(\mathcal{C}_0(X), B) \longrightarrow KKR_{\mathcal{G},i+j}\left(\mathcal{C}_0(X), A \hat{\otimes}_{\mathcal{C}_0(X)} B\right);$$

and hence the map (9.10) since $A \hat{\otimes}_{\mathcal{C}_0(X)} B \cong \mathcal{C}_0(X; \mathcal{A} \hat{\otimes}_X \mathcal{B})$.

9.7 Twistings by *Real* Clifford bundles, Stiefel-Whitney classes

Let $n = p + q \in \mathbb{N}^*$. Recall ([42, §IV.4]) that the group $\mathrm{Pin}(p+q)$ is defined as

$$\mathrm{Pin}(p+q) := \left\{\gamma \in Cl_{p,q} \mid \varepsilon(\gamma) v \gamma^* \in \mathbb{R}^{p,q}, \forall v \in \mathbb{R}^{p,q}, \text{ and } \gamma\gamma^* = 1\right\},$$

where ε is the canonical $\mathbb{Z}_2$-grading of $CL_{p,q}$. It is known that

$$\mathrm{Pin}(p+q) \cong \{\gamma = x_1 \cdots x_k \in Cl_{p,q} \mid x_i \in \mathbf{S}^{p,q}, 1 \le k \le 2n\}.$$

It is then a Real group with respect to the involution induced from $\mathbf{S}^{p,q}$; *i.e.*, $\bar{\gamma} = \bar{x}_1 \cdots \bar{x}_k$, for $\gamma \in \mathrm{Pin}(p+q)$. Of course, this involution is equivalent to that induced from $Cl_{p,q}$. Moreover, the surjective homomorphism $\pi : \mathrm{Pin}(p+q) \longrightarrow O(p+q), \gamma \longmapsto \pi_\gamma$, where $\pi_\gamma(v) := \varepsilon(\gamma) v \gamma^*$ for $v \in \mathbb{R}^{p,q}$, is clearly Real. Recall that $\ker \pi = \{\pm 1\} = \mathbb{Z}_2$. Hence, there is a Real graded $\mathbb{Z}_2$-central extension of the Real groupoid $O(p+q) \rightrightarrows \cdot$

$$\begin{array}{ccccc} \mathbb{Z}_2 & \longrightarrow & \mathrm{Pin}(p+q) & \xrightarrow{\pi} & O(p+q) \\ & & & & \downarrow{\scriptstyle\delta} \\ & & & & \mathbb{Z}_2 \end{array}$$

where the homomorphism $\delta : O(p+q) \longrightarrow \mathbb{Z}_2$ is the map such that $\det A = (-1)^{\delta(A)}$ (compare with [87, §2.5]). Let

$$\mathrm{Pin}^c(p+q) := \mathrm{Pin}(p+q) \times_{\{\pm 1\}} \mathbb{S}^1,$$

be endowed with the Real structure $[(\gamma,\lambda)]\longmapsto[(\bar{\gamma},\bar{\lambda})]$, where as usual, the "bar" operation in $\mathbb{S}^1$ is the complex conjugation. Then the above Rg $\mathbb{Z}_2$-central extension induces Rg $\mathbb{S}^1$-central extension $\mathcal{T}_{p,q}$

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \mathrm{Pin}^c(p+q) \arrow[r, "\pi"] & O(p+q) \arrow[d, "\delta"] \\ & & \mathbb{Z}_2 \end{tikzcd} \tag{9.11}$$

of $O(p+q) \rightrightarrows \cdot$.

Let V be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$. Then there is a Rg $\mathbb{S}^1$-central extension $\mathbb{E}^V$ obtained by pulling back $\mathcal{T}_{p,q}$ via the generalized Real homomorphism $\mathbb{F}(V):\mathcal{G}\longrightarrow O(p+q)$.

Definition 9.7.1. *Let V be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$. We define its* associated Rg D-D bundle *as the Rg D-D bundle $\mathcal{A}_V$ of type* 0 *over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ whose image in $\widehat{Ext}\mathrm{R}(\mathcal{G},\mathbb{S}^1)$ is $[\mathbb{E}^V]$ via the isomorphism $\widehat{Br}\mathrm{R}_0(\mathcal{G})\longrightarrow\widehat{Ext}\mathrm{R}(\mathcal{G},\mathbb{S}^1)$.*

Lemma 9.7.2. *Let V and V' be Real Euclidean vector bundles of type $p-q$ and $p'-q'$, respectively, over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$. Then the Rg D-D bundles $\mathcal{A}_{V\oplus V'}$ and $\mathcal{A}_V\hat{\otimes}_X\mathcal{A}_{V'}$ are Morita equivalent.*

Proof. Considering the Real homomorphisms $\mathrm{Pin}^c(p+q)\times\mathrm{Pin}^c(p'+q')\longrightarrow\mathrm{Pin}^c((p+p')+(q+q'))$ and $O(p+q)\times O(p'+q')\longrightarrow O((p+p')+(q+q'))$ (cf. [46]) and a Real open cove of X trivializing both V and V' (and hence the direct sum $V\oplus V'$), one easily shows that $(\mathbb{F}(V)^*\mathcal{T}_{p,q})\hat{\otimes}(\mathbb{F}(V')^*\mathcal{T}_{p',q'})\sim(\mathbb{F}(V\oplus V'))^*\mathcal{T}_{p+p',q+q'}$. □

Lemma 9.7.3. *Let $\mathbf{1}^{p,q}$ be the trivial Real Euclidean vector bundle $X\times\mathbb{R}^{p,q}\longrightarrow X$ of type $p-q$ over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$, where the Real $\mathcal{G}$-action is $g\cdot(s(g),t)=(r(g),t)$. Then $\mathcal{A}_{\mathbf{1}^{p,q}}=0$ in $\widehat{Br}\mathrm{R}_0(\mathcal{G})$.*

Proof. The generalized Real homomorphism $\mathbb{F}(\mathbf{1}^{p,q}):\mathcal{G}\longrightarrow O(p+q)$ is nothing but the generalized homomorphism induced by the strict Real homomorphism $\mathbf{1}:\mathcal{G}\ni g\longmapsto\mathbf{1}\in O(p+q)$. Hence $\mathbb{E}^{\mathbf{1}^{p,q}}=(\mathbf{1}^{p,q})^*\mathcal{T}_{p,q}$, and since the kernel of the projection $\mathrm{Pin}^c(p+q)\longrightarrow O(p+q)$ is $\mathbb{S}^1$, $\mathbb{E}^{\mathbf{1}^{p,q}}$ is isomorphic to the trivial extension $(\mathcal{G}\times\mathbb{S}^1,\mathcal{G},0)$. □

On the other hand, the action of $\mathcal{G}$ on the complexified bundle $V_{\mathbb{C}}:=V\otimes_{\mathbb{R}}\mathbb{C}\longrightarrow X$ induces a Real $\mathcal{G}$-action by graded automorphisms on the complex Clifford bundle

$$\mathbb{C}l(V):=Cl(V_{\mathbb{C}})\longrightarrow X,$$

making it a Rg D-D bundle of type $q-p \mod 8$ over $\mathcal{G}\overset{r}{\underset{s}{\rightrightarrows}}X$ (recall Example A.5.6).

We want to compare the Rg D-D bundles $\mathcal{A}_V$ and $\mathbb{C}l(V)$. In particular, we want to show the following.

Theorem 9.7.4. *Let V be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$. Then for all $\mathcal{A} \in \widehat{Br}\mathrm{R}(\mathcal{G})$, we have*

$$KR^*_{\mathcal{A}+\mathbb{C}l(V)}(\mathcal{G}^\bullet) \cong KR^{*+q-p}_{\mathcal{A}+\mathcal{A}_V}(\mathcal{G}^\bullet).$$

We shall mention that this result is already known in the complex case ([87, Proposition 2.5]). The approach we are using here to prove it is however very different from that used in the just cited reference.

Our proof requires the notion of *generalized Stiefel-Whitney classes* of a Real vector bundles over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$. Recall that associated to any real vector bundle V over a locally compact paracompact space X, there are cohomology classes $w_i(V) \in H^i(X,\mathbb{Z}_2)$ called the i^{th} *Stiefel-Whitney classes* of V (see for instance [38, Chap.17 §2]). For instance $w_1(V)$ is the constraint for V being *oriented*, and $w_2(V)$ is the constraint for V being Spin^c (we will say more about that later).

We have already seen that a Real Euclidean vector bundle V of type $p-q$ gives rise to a generalized Real homomorphism $\mathbb{F}(V) : \mathcal{G} \longrightarrow O(p+q)$. In fact, Real Euclidean vector bundles arise this way: given $P : \mathcal{G} \longrightarrow O(p+q)$, $V := P \times_{O(p+q)} \mathbb{R}^{p,q} \longrightarrow X$ is a Real Euclidean vector bundle of type $p-q$. There is then a bijection between the set $\mathrm{Vect}_{p+q}(\mathcal{G})$ of isomorphism classes of Real Euclidean vector bundles of type $p-q$ and the set $\mathrm{Hom}_{\mathfrak{RG}}(\mathcal{G}, O(p+q))$, and hence with $\check{H}R^1(\mathcal{G}_\bullet, O(p+q))$.

Let $\mathfrak{c}$ be a Real $O(p+q)$-valued 1-cocycle over $\mathcal{G}$ realizing $\mathbb{F}(V)$. This can be considered as a Real family of continuous maps $\mathfrak{c}_{(j_0,j_1)} : U^1_{(j_0,j_1)} \longrightarrow O(p+q)$ such that

$$\mathfrak{c}_{(j_0,j_1)}(\gamma_1)\mathfrak{c}_{(j_1,j_2)}(\gamma_2) = \mathfrak{c}_{(j_0,j_2)}(\gamma_1\gamma_2), \quad (\gamma_1,\gamma_2) \in U^2_{(j_0,j_1,j_2)}, \tag{9.12}$$

where $\mathfrak{U} = \{U_j\}_{j\in J}$ is a Real open cover of X (indeed, if $f : \mathcal{G}[\mathfrak{U}] \longrightarrow O(p+q)$ is a Real homomorphism realizing $\mathbb{F}(V)$, then one can take $\mathfrak{c}_{(j_0,j_1)}(g_{(j_0,j_1)}) := f(g_{j_0 j_1})$). We may suppose that the simplicial Real cover $\mathfrak{U}_\bullet$ of $\mathcal{G}_\bullet$ is "small" enough so that we can pick a Real family of continuous maps $\tilde{\mathfrak{c}}_{(j_0,j_1)} : U^1_{(j_0,j_1)} \longrightarrow \mathrm{Pin}^c(p+q)$ which are a $\mathrm{Pin}^c(p+q)$-lifting of $(\mathfrak{c}_{(j_0,j_1)})$ through the Real projection $\pi : \mathrm{Pin}^c(p+q) \longrightarrow O(p+q)$; *i.e.*, $\pi(\tilde{\mathfrak{c}}_{(j_0,j_1)}(\gamma)) = \mathfrak{c}_{(j_0,j_1)}(\gamma), \forall \gamma \in U^1_{(j_0,j_1)}$. In view of equation (9.12), we have

$$\tilde{\mathfrak{c}}_{(j_0,j_1)}(\gamma_1)\tilde{\mathfrak{c}}_{(j_1,j_2)}(\gamma_2) = \omega_{(j_0,j_1,j_2)}(\gamma_1,\gamma_2)\tilde{\mathfrak{c}}_{(j_0,j_2)}(\gamma_1\gamma_2), \forall (\gamma_1,\gamma_2) \in U^2_{(j_0,j_1,j_2)}, \tag{9.13}$$

for some $\omega_{(j_0,j_1,j)}(\gamma_1,\gamma_2) \in \mathbb{S}^1$. The elements $\omega_{(j_0,j_1,j)}(\gamma_1,\gamma_2)$ clearly define a Real family of continuous functions $\omega_{(j_0,j_1,j)} : U^2_{(j_0,j_1,j_2)} \longrightarrow \mathbb{S}^1$ which are easily checked to be an element of $ZR^2_{ss}(\mathfrak{U}_\bullet, \mathbb{S}^1)$.

Definition 9.7.5. *Let V be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$. Let $\mathfrak{c}$ be the class of $\mathbb{F}(V)$ in $\check{H}R^1(\mathcal{G}_\bullet, O(p+q))$.*

(a) *The* first generalized Stiefel-Whitney class $w_1(V) \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2)$ *as* $w_1(V) := \delta \circ \mathfrak{c}$, *where* $\delta : O(p+q) \longrightarrow \mathbb{Z}_2$ *is the homomorphism defined in* (9.11).

(b) *The* second generalized Stiefel-Whitney class $w_2(V)$ *is the class in* $\check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$ *of the Real* 2*-cocycle* ω *uniquely determined by equation* (9.13).

We define $w(V) := (w_1(V), w_2(V)) \in \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)$.

Remark 9.7.6. *Note that* $w_1(V) = 0$ *implies that* $\mathbb{F}(V)$ *is actually a Real* $SO(p+q)$*-principal bundle over* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *, which means that* V *is* oriented. *Moreover, the Real family* $(\omega_{(j_0,j_1)})$ *is nothing but the obstruction for the Real* $O(p+q)$*-valued* 1*-cocyle* $\mathfrak{c}$ *to lift to a Real* $\mathrm{Pin}^c(p+q)$*-valued* 1*-cocyle* $\tilde{\mathfrak{c}}$*; or in other words, it is the obstruction for* $\mathbb{F}(V)$ *to lift to a Real* $\mathrm{Pin}^c(p+q)$*-principal bundle over* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *.*

Example 9.7.7. *Denote by* $\theta^{p,q}$ *the trivial Euclidean vector bundle* $\mathbb{R}^{p,q}$ *over* $O(p+q) \rightrightarrows \cdot$ *. Then* $w_1(\theta^{p,q}) = \delta$ *and* $w_2(\theta^{p,q}) = c_{p,q}$ *is the Real* $\mathbb{S}^1$*-valued* 2*-cocycle corresponding to the Rg* $\mathbb{S}^1$*-central extension* $\mathcal{T}_{p,q} \in \widehat{\mathrm{ExtR}}(O(p+q), \mathbb{S}^1)$ *of* (9.11)*. Moreover,* $w(\theta^{p,q}) = dd(\mathcal{T}_{p,q})$*.*

Definition 9.7.8. *A Real Euclidean vector bundle* V *of type* $p-q$ *over* $\mathcal{G}$ *admits a* Spin^c-structure *if* $w(V) = 0$*; in this case we say that* V *is* Spin^c*. We also say that* V *is* KR*-oriented (following Hilsum-Skandalis terminology [36], see also H. Schröder [81]).*

Taking the involution of $\mathcal{G}$ to be the trivial one, the next result is actually the groupoid equivariant analogue of Plymen's [74, Theorem 2.8].

Proposition 9.7.9. *Let* V *be a Real Euclidean vector bundle of type* $p-q$ *over* $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ *.*

1. *We have* $DD(\mathcal{A}_V) = w(V)$*. Hence,* V *is* KR*-oriented if and only if* $\mathcal{A}_V$ *is trivial.*

2. *If* $p = q \in \mathbb{N}^*$*, then* $DD(\mathbb{C}l(V)) = DD(\mathcal{A}_V) = w(V)$*. Therefore, if* $p = q$*, the following statements are equivalent:*

   (i) V *is* KR*-oriented.*

   (ii) *The Rg D-D bundle* $\mathbb{C}l(V) \longrightarrow X$ *is trivial.*

   (iii) *The Rg D-D bundle* $\mathcal{A}_V \longrightarrow X$ *is trivial.*

Proof. 1. Since the map sending a Rg $\mathbb{S}^1$-central extension to its Dixmier-Douady class is a natural isomorphism (Theorem **??**), we have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{ExtR}}(O(p+q), \mathbb{S}^1) \arrow[r, "\mathbb{F}(V)^*"] \arrow[d, "dd"', "\cong"] & \widehat{\mathrm{ExtR}}(\mathcal{G}, \mathbb{S}^1) \arrow[d, "\cong"', "dd"] \\
\check{H}R^1(O(p+q)_\bullet, \mathbb{Z}_2) \times \check{H}R^2(O(p+q)_\bullet, \mathbb{S}^1) \arrow[r, "\mathbb{F}(V)^* \times \mathbb{F}(F)^*"] & \check{H}R^1(\mathcal{G}_\bullet, \mathbb{Z}_2) \times \check{H}R^2(\mathcal{G}_\bullet, \mathbb{S}^1)
\end{tikzcd}$$

Hence, if $f : \mathcal{G}[\mathfrak{U}] \longrightarrow O(p+q)$ is a Real homomorphism realizing $\mathbb{F}(V)$, then

$$DD(\mathcal{A}_V) = dd(f^*\mathcal{T}_{p,q}) = (f^* w_1(\theta^{p,q}), f^* w_2(\theta^{p,q})) = w(V),$$

where the first equality comes from the very definition of the Rg D-D bundle $\mathcal{A}$, the second one follows from Example 9.7.7, and the last one is a simple interpretation on the construction of w_1 and w_2.

2. If $p = q$, then $\mathbb{C}l_{p,p}$ is the type 0 Rg elementary C^*-algebra $\mathcal{K}(\hat{H})$, where

$$\hat{H} = \mathbb{C}^{2^{p-1}} \oplus \mathbb{C}^{2^{p-1}}.$$

Identifying the Real space $\mathbb{R}^{p,p}$ with $\mathbb{C}^p$ endowed with the coordinatewise complex conjugation, there is a degree conserving Real representation

$$\lambda : \mathrm{Pin}^c(p+p) \to \widehat{\mathbb{U}}(\hat{H}),$$

induced by the Real map $\mathbb{C}^p \longrightarrow \mathcal{L}(\Lambda^*\mathbb{C}^p) \cong \mathcal{L}(\hat{H})$ given by exterior multiplication (cf. [46]). This gives us a projective Real representation

$$Ad_\lambda : O(p+p) \longrightarrow \widehat{\mathbb{PU}}(\hat{H})$$

given by $Ad_\lambda(\gamma)(T) := \lambda(\tilde{\gamma})T\lambda(\tilde{\gamma})^{-1}$, for $T \in \mathcal{K}(\hat{H})$, where $\tilde{\gamma} \in \mathrm{Pin}^c(2p)$ is an arbitrary lift of $\gamma \in O(p+p)$, and where we have used the identification $\widehat{\mathbb{PU}}(\hat{H}) \cong \mathrm{Aut}^{(0)}(\mathcal{K}(\hat{H}))$. We thus have commutative diagrams

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r] \arrow[d, equal] & \mathrm{Pin}^c(p+p) \arrow[r] \arrow[d, "\lambda"] & O(p+p) \arrow[d, "Ad_\lambda"] \arrow[dr] & \\
 & & & \mathbb{Z}_2 \\
\mathbb{S}^1 \arrow[r] & \widehat{\mathbb{U}}(\hat{H}) \arrow[r] & \widehat{\mathbb{PU}}(\hat{H}) \arrow[ur] &
\end{tikzcd}$$

which yields to an equivalence of Rg $\mathbb{S}^1$-central extensions

$$(Ad_\lambda)^*\mathbb{E}_{\mathcal{K}(\hat{H})} \sim \mathcal{T}_{p,p} \in \widehat{\mathrm{ExtR}}(O(p+q), \mathbb{S}^1) \tag{9.14}$$

where $\mathbb{E}_{\mathcal{K}(\hat{H})}$ is the Rg $\mathbb{S}^1$-central extension $(\widehat{\mathbb{U}}(\hat{H}), deg)$ of the Real groupoid $\widehat{\mathbb{PU}}(\hat{H}) \rightrightarrows \cdot$.

Now let (U_i, h_i) be a Real trivialization of V, with transition functions $\alpha_{ij} : U_{ij} \longrightarrow O(p+p)$. Then V is isomorphic to the Euclidean Real vector bundle over $\mathcal{G}$

$$\coprod_i U_i \times \mathbb{C}^p /_\sim \longrightarrow X,$$

where $(x,t)_i \sim (x, \alpha_{ij}(x)t)_j$, for $x \in U_{ij}$, endowed with the Real $\mathcal{G}$-action

$$g \cdot [(s(g),t)]_{j_1} := [(r(g), \mathfrak{c}_{(j_1,j_0)}(g)t)]_{j_0}, g \in U^1_{(j_0,j_1)}.$$

Here $[(x,t)]_i$ denotes the class of $(x,t)_i \in U_i \times \mathbb{C}^p$ in $\coprod_i U_i \times \mathbb{C}^p/_\sim$.

Moreover, by universality of Clifford algebras (cf. [42, Chap.IV, §4]), the Real family of homeomorphisms $h_i : V_{|U_i} \longrightarrow U_i \times \mathbb{C}^p$ induces a Real family of homeomorphisms

$$\mathbb{C}l(V)_{|U_i} \longrightarrow U_i \times \mathbb{C}l_{p,p} = U_i \times \mathcal{K}(\hat{H})$$

with transition functions $Ad_\lambda \alpha_{ij}(\cdot) : U_{ij} \longrightarrow \widehat{\mathrm{PU}}(\hat{H})$. Since the Real action of $\mathcal{G}$ on the Rg D-D bundle $\mathbb{C}l(V) \longrightarrow X$ is induced from the Real action of $\mathcal{G}$ on $V \longrightarrow X$, it follows that $\mathbb{C}l(V)$ is isomorphic to the Rg D-D bundle

$$\coprod_i U_i \times \mathcal{K}(\hat{H})/_\sim$$

where the equivalence relation is $(x,a)_i \sim (x, Ad_\lambda(\alpha_{ij}(x))a)_j$ for $x \in U_{ij}$, with the Real $\mathcal{G}$-action by graded automorphisms

$$g \cdot [(s(g),a)]_{j_1} := [(r(g), Ad_\lambda(\mathfrak{c}_{(j_0,j_1)})a)]_{j_0}, g \in U^1_{(j_0,j_1)}.$$

Therefore, if $P : \mathcal{G} \longrightarrow \widehat{\mathrm{PU}}(\hat{H})$ is the generalized classifying morphism for $\mathbb{C}l(V)$, it corresponds to the class of $Ad_\lambda \mathfrak{c}$ in $\check{H}R^1(\mathcal{G}_\bullet, \widehat{\mathrm{PU}}(\hat{\mathcal{H}}))$ (cf. Proposition 4.6.4). Putting this in terms of generalized Real homomorphisms, there is a commutative diagram in the category $\mathfrak{RG}$

$$\begin{tikzcd}
 & O(p+p) \arrow[dd, "Ad_\lambda"] \\
\mathcal{G} \arrow[ur, "\mathbb{F}(V)"] \arrow[dr, "P"'] & \\
 & \widehat{\mathrm{PU}}(\hat{H})
\end{tikzcd}$$

This combined with (9.14) implies

$$P^*\mathbb{E}_{\mathcal{K}(\hat{H})} \sim \mathbb{F}(V)^*\mathfrak{T}_{p,p}.$$

Hence, $DD(\mathbb{C}l(V)) = dd([P^*\mathbb{E}_{\mathcal{K}(\hat{H})}]) = dd([\mathbb{F}(V)^*\mathfrak{T}_{p,p}]) = dd(\mathcal{A}_V) = w(V)$, where the third and fourth equalities come from the first statement of the proposition. □

To see how things work in the general case, observe first that if V_1, V_2 are Real Euclidean vector bundles of types $p_1 - q_1$ and $p_2 - q_2$, respectively, then $\mathbb{C}l(V_1 \oplus V_2) \longrightarrow X$ is a Rg D-D bundle of type $(q_1+q_2)-(p_1+p_2)$ because $\mathbb{C}l(V_1 \oplus V_2) \cong \mathbb{C}l(V_1)\hat{\otimes}_X \mathbb{C}l(V_2)$ (cf. [46, §2.15]).

The next definition is adapted from [4, §3.5].

Definition 9.7.10. *Let V be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$. Then we define $\tilde{V} := V \oplus \mathbf{1}^{q,p}$, and $\widetilde{\mathbb{C}l}(V) \in \widehat{Br\mathrm{R}}_0(\mathcal{G})$ as the Rg D-D bundle of type 0 defined by*

$$\widetilde{\mathbb{C}l}(V) := \mathbb{C}l(\tilde{V}) \longrightarrow X,$$

with the obvious Real $\mathcal{G}$-action.

Theorem 9.7.11. *Let V be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$. Then*

$$DD(\mathbb{C}l(V)) \quad = \quad (q-p, w_1(V), w_2(V)). \tag{9.15}$$

Proof. We have $DD(\widetilde{\mathbb{C}l}(V)) = DD(\mathcal{A}_{\tilde{V}})$, thanks to Proposition 9.7.9 2). Applying Lemma 9.7.2 and Lemma 9.7.3, we get $DD(\widetilde{\mathbb{C}l}(V)) = DD(\mathcal{A}_V)$. Furthermore, $\mathbb{C}l(V)$ is clearly Morita equivalent to $\widetilde{\mathbb{C}l}(V)\hat{\otimes}_X \mathbb{C}l(\mathbf{1}^{p,q})$. Therefore,

$$DD(\mathbb{C}l(V)) = DD(\widetilde{\mathbb{C}l}(V)) + DD(\mathbb{C}l(\mathbf{1}^{p,q})) = DD(\mathcal{A}_V) + (q-p, 0, 0).$$

We conclude by applying Proposition 9.7.9 1). □

By using the fact DD is a group homomorphism, we immediately deduce from the above theorem that

Corollary 9.7.12. *If V and V' are Real Euclidean vector bundles over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ then*

$$w_1(V \oplus V') = w_1(V) + w_1(V'), \quad \text{and } w_2(V \oplus V') = (-1)^{w_1(V)+w_1(V')} w_2(V)\cdot w_2(V').$$

Proof of Theorem 9.7.4. As a consequence of Theorem 9.7.11, one has

$$KR^*_{\mathbb{C}l(V)}(\mathcal{G}^\bullet) \cong KR^*_{\mathcal{A}_V + \mathbb{C}l_{p,q}}(\mathcal{G}^\bullet),$$

so we conclude by using Kasparov product in KKR-theory (recall that by definition we have $KR^*_{\mathcal{A}+\mathcal{B}}(\mathcal{G}^\bullet) = KKR_{-*}(\mathbb{C}, (\mathcal{A}\hat{\otimes}_X \mathcal{B}) \rtimes_r \mathcal{G})$). □

9.8 Thom isomorphism in twisted KR-theory

We start this section by some observations about Spin^c Real Euclidean vector bundles. Let

$$\mathrm{Spin}(p+q) := \mathrm{Pin}(p+q) \cap \mathbb{C}l^0_{p,q}$$

(cf. [7, 42, 46]). The restriction of the projection $\mathrm{Pin}(p+q) \longrightarrow O(p+q)$ induces a surjective Real homomorphism

$$\mathrm{Spin}(p+q) \longrightarrow SO(p+q)$$

with kernel $\mathbb{Z}_2$, where $SO(p+q)$ is equipped with the Real structure induced from $O(p+q)$. Moreover, there is a Real (trivially) graded $\mathbb{S}^1$-central extension $\mathcal{T}'_{p,q}$

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \mathrm{Spin}^c(p+q) \arrow[r] & SO(p+q) \arrow[d] \\ & & \mathbb{Z}_2 \end{tikzcd}$$

over the Real groupoid $SO(p+q) \rightrightarrows \cdot$, where $SO(p+q) \longrightarrow \mathbb{Z}_2$ is the zero map, and where

$$\mathrm{Spin}^c(p+q) := \mathrm{Spin}(p+q) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

Now suppose V is a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$. If $w_1(V) = 0$, then $\mathbb{F}(V)$ reduces to a generalized Real homomorphism from $\mathcal{G}$ to $SO(p+q) \rightrightarrows \cdot$. So, $\mathcal{A}_V$ comes from the Rg $\mathbb{S}^1$-central extension $\mathbb{F}(V)^*\mathcal{T}'_{p,q}$. Moreover, V being KR-oriented means that $\mathbb{F}(V)$ is actually a Real Spin^c-principal bundle over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, hence a generalized Real homeomorphism from $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ to $\mathrm{Spin}^c(p+q) \rightrightarrows \cdot$.

The following result generalizes the Thom isomorphism theorem in twisted orthogonal K-theory already known in the case of topological spaces (see Karoubi and Donovan [28] and Karoubi [43]).

Theorem 9.8.1. *Let $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ be a locally compact Hausdorff second-countable Real groupoid with Real Haar system. Let $\pi : V \longrightarrow X$ be a Real Euclidean vector bundle of type $p-q$ over $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$, and let $\mathcal{A} \in \widehat{\mathrm{BrR}}(\mathcal{G})$. Then there is a canonical group isomorphism*

$$KR^*_{\pi^*\mathcal{A}}((\pi^*\mathcal{G})^\bullet) \quad \cong \quad KR_{\mathcal{A}+\mathbb{C}l(V)}(\mathcal{G}^\bullet). \tag{9.16}$$

Furthermore, if V is KR-oriented, then there is a canonical isomorphism

$$KR^*_{\pi^*\mathcal{A}}((\pi^*\mathcal{G})^\bullet) \quad \cong \quad KR^{*-p+q}_{\mathcal{A}}(\mathcal{G}^\bullet), \tag{9.17}$$

where as usual, the Real groupoid $\pi^*\mathcal{G} \rightrightarrows V$ is the pullback of $\mathcal{G} \underset{s}{\overset{r}{\rightrightarrows}} X$ via the projection π.

Proof. From Theorem 9.7.4 we have $KR^*_{\mathcal{A}+\mathbb{C}l(V)}(\mathcal{G}^\bullet) \cong KR^{*-p+q}_{\mathcal{A}+\mathcal{A}_V}$; in particular if V is Spin^c, $\mathcal{A}_V = 0$ and $KR^*_{\mathcal{A}+\mathbb{C}l(V)} \cong KR^{*-p+q}_{\mathcal{A}}(\mathcal{G}^\bullet)$, which implies that the isomorphism (9.17) deduces from isomorphism (9.16). Let us show the latter. The $KKR_{O(p+q)}$-equivalence

$$\alpha_{p,q} \in KKR(\mathcal{C}_0(\mathbb{R}^{p,q}), \mathbb{C}l(\mathbb{R}^{p,q}))$$

induces by functoriality in the category $\mathfrak{RG}$ a $KKR_{\mathcal{G}}$-equivalence

$$\mathbb{F}(V)^*\alpha_{p,q} \in KKR_{\mathcal{G},*}(\mathcal{C}_0(V)\hat{\otimes}\mathcal{C}_0(X), \mathcal{C}_0(X;\mathbb{C}l(V))).$$

Thus, by the identifications of C^*-algebras with generalized Real $\mathcal{G}$-actions

$$\mathcal{C}_0(X;\mathcal{A}) \cong \mathcal{C}_0(X)\hat{\otimes}_{\mathcal{C}_0(X)}\mathcal{C}_0(X;\mathcal{A}), \quad \text{and}$$

$$\mathcal{C}_0(V;\pi^*\mathcal{A}) \cong \mathcal{C}_0(V)\hat{\otimes}_{\mathcal{C}_0(X)}\mathcal{C}_0(X;\mathcal{A}),$$

we get a $KKR_{\mathcal{G}}$-equivalence

$$\tilde{\alpha}_V \in KKR_{\mathcal{G},*}(\mathcal{C}_0(V;\pi^*\mathcal{A}), \mathcal{C}_0(X;\mathcal{A}\hat{\otimes}_X\mathbb{C}l(V))),$$

by taking α_V to be the Kasparov product of $\mathbb{F}(V)^*\alpha_{p,q}$ with the canonical $KKR_{\mathcal{G}}$-equivalence

$$\mathbf{1}_{\mathcal{C}_0(X;\mathcal{A})} \in KKR_{\mathcal{G}}(\mathcal{C}_0(X;\mathcal{A}), \mathcal{C}_0(X;\mathcal{A})).$$

Therefore, we obtain a KKR-equivalence

$$\alpha_V := j_{\mathcal{G},red}(\tilde{\alpha}_V) \in KKR_*(\mathcal{C}_0(V;\pi^*\mathcal{A}) \rtimes_r \mathcal{G}, \mathcal{C}_0(X;\mathcal{A}\hat{\otimes}_X\mathbb{C}l(V)) \rtimes_r \mathcal{G}),$$

where $j_{\mathcal{G},red}$ is the descent morphism; and we are done. $\square$

A

Classification of Real graded elementary C^*-algebras

A.1 Rg C^*-algebras

Recall that *a complexification* of a real Banach space $(E, \|.\|)$ is a complex Banach space $(E_{\mathbb{C}}, \|.\|_c)$ such that $E_{\mathbb{C}} = E + iE$ as a complex linear space, the norm $\|.\|_c$ restricts to $\|.\|$ on E, and $\|\eta + i\xi\| = \|\eta - i\xi\|$ for all $\eta, \xi \in E$ (i.e., $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$). Moreover, for any real Banach space E , there is a unique (up to equivalence) complexification of it. We refer to [53] for a general theory of real Banach spaces and real Banach (*-)algebras.
In this way, any real Banach (*-)algebra A is associated to a complex Banach (*-)algebra $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$. In particular, if A is a real C^*-algebra (see [81, chap.1] for the definition of real C^*-algebra), then $A_{\mathbb{C}}$ admits a structure of a complex C^*-algebra. It is however natural to ask the following question

Question A.1.1. *Let B be a complex C^*-algebra. Does there exist a closed real C^*-subalgebra B_r of B such that $B \cong B_r \otimes_{\mathbb{R}} \mathbb{C}$?*

Although it was mentioned in [53] that this question reminds open, the answer is in fact "no". Indeed, as we will see later, the existence of B_r is equivalent to the existence of a conjugate-linear involution on B, which is also equivalent to B being isomorphic to its conjugate algebra via a 2-periodic isomorphism). But such an involution induces an involutory anti-automorphism $\varphi : B \longrightarrow B$ (i.e. φ verifies $\varphi(ab) = \varphi(b)\varphi(a), \forall a, b \in B$ and $\varphi^2 = \mathbf{1}$). On the other hand, A. Connes [21] and T. Giordano [31] have constructed examples of von Neumann algebras that are not anti-isomorphic to themselves. Recently, other explicite examples of C^*-algebras not isomorphic to their conjugate algebras have been constructed (see N.C. Phillips [72], and N.C. Phillips and M.G. Viola [73]).

We should however point out that being anti-isomorphic to itself is not sufficient for a C^*-algebra B to admit a conjugate-linear involution, as it was proved by V. Jones in [40].

In this section we are concerned with those (C^*-)algebras that for which Question A.1.1 has a positive answer.

Definition A.1.2. *A* Real ($\mathbb{Z}_2$)-graded C^**-algebra consists of a* C^**-algebra* A *together with*

(i) an involutive **-homomorphism* $\alpha : A \longrightarrow A$ *with* $\alpha^2 = \mathbf{1}$*;* α *is called the* grading*;*

(ii) an involutive **-automorphism* $\sigma_A : A \longrightarrow A$ *which is antilinear, such that* $\sigma_A^2 = \mathbf{1}$*, and* $\sigma_A \circ \alpha = \alpha \circ \sigma_A$*.* σ_A *is called the* Real structure *of* A*.*

We will say that A *is a Rg* C^**-algebra, for short.*

We will often write (A, σ_A) for such a Rg C^*-algebra and we decompose A as a the direct sum $A = A^0 \oplus A^1$ where $A^0 = \mathrm{Ker}(\frac{1-\alpha}{2})$ and $A^1 = \mathrm{Ker}(\frac{1+\alpha}{2})$. We write $|a|$ for the degree of an element $a \in A$. However, it is easy to see that A^0 is a C^*-subalgebra of A while A^1 is not.

An element $a \in A$ is called homogenous of degree i, for $i = 0, 1 \mod 2$, if $a \in A^i$. a is said to be *invariant* if it is of degree 0 and $\sigma_A(a) = a$.

Example A.1.3. *Let* $A = A^0 \oplus A^1$ *be a graded real* C^**-algebra. Then its complexification* $A_{\mathbb{C}}$ *is also graded. Indeed, we have* $A_{\mathbb{C}} = A^0_{\mathbb{C}} \oplus A^1_{\mathbb{C}}$*. Now the* bar operation $^- : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ *given by* $\overline{a + ib} := a - ib$ *defines a Real structure on* $A_{\mathbb{C}}$*. For instance, any real* C^**-algebra* A *gives rise to a Rg* C^**-algebra by taking* $A^1 = 0$*.*

Example A.1.4. *Given a real* C^**-algebra* A*, the direct sum* $A \oplus A$ *admits a canonical grading given by* $(a, b) \longmapsto (b, a)$*; then* $(A \oplus A)^0 = \{(a, a) \mid a \in A\}$ *and* $(A \oplus A)^1 = \{(a, -a) \mid a \in A\}$*. This induces a grading on the complex* C^**-algebra* $A_{\mathbb{C}} \oplus A_{\mathbb{C}}$ *which becomes a Real graded* C^**-algebra. This grading is called* the standard odd grading*. In particular, the complex clifford algebra* $\mathbb{C}l_1 = \mathbb{C} \oplus \mathbb{C}$ *is a Rg* C^**-algebra with its canonical Real structure given by the complex conjugation.*

Definition A.1.5. *Let* (A, σ_A) *and* (B, σ_B) *be Rg* C^**-algebras. A* Real graded homomorphism *between* A *and* B *is a homomorphism of* C^**-algebras* $\varphi : A \longrightarrow B$ *that intertwines the Real structures and the gradings.*

In particular, we say that (A, σ_A) and (B, σ_B) are isomorphic as Rg C^*-algebras, and we write $(A, \sigma_A) \cong (B, \sigma_B)$, if there exists a Rg isomorphism between them.

If (A, σ_A) is a Rg C^*-algebra, then the multiplier algebra $\mathcal{M}(A)$ has also a structure of Real graded C^*-algebra. Indeed, if ϵ is the grading on A and $(T_1, T_2) \in \mathcal{M}(A)$, we put

$Ad_\epsilon(T_1,T_2) := (\epsilon T_1\epsilon, \epsilon T_2\epsilon)$ and it is easy to see that this defines a grading on $\mathcal{M}(A)$ with $\mathcal{M}(A)^{(i)} = \{(T_1,T_2)\in\mathcal{M}(A) \mid \epsilon T_k\epsilon = (-1)^i T_k,\ k=1,2\}$; moreover the Real structure is given by

$$\sigma_A(T_1,T_2) := (\sigma_A T_1\sigma_A, \sigma_A T_2\sigma_2).$$

A subspace B of A is *Real graded* if it is invariant under σ_A and if it is the direct sum of the intersections $B\cap A^i$ (or equivalently, if it is invariant under the grading of A). For instance, it is easy to check that the center of any Rg C^*-algebra is Rg.

Le I be a Real graded ideal in (A,σ). Let $[a]$ denotes the class of a in A/I, then we can show that the maps $\sigma([a]) := [\sigma(a)]$ and $\epsilon([a]) := [\epsilon(a)]$, are well defined from A/I to A/I, giving us a grading and a Real structure on the quotient C^*-algebra A/I.

Now let us give the following simple characterization of Rg C^*-algebras.

Lemma A.1.6. *Let (A,σ_A) be a Rg C^*-algebra. Then there exists a real $\mathbb{Z}_2$-graded C^*-algebra $A_\mathbb{R}$ such that $(A,\sigma_A)\cong(A_\mathbb{R}\otimes_\mathbb{R}\mathbb{C},\bar{\ })$, where $(\bar{\ })$ is the bar operation.*

Mainly speaking, a Rg C^*-algebra is just a graded C^*-algebra which is the complexification of a graded real C^*-algebra, together with the bar operation. This justifies the terminology of *Real*.

Proof. Put $A_\mathbb{R} := \{a\in A \mid \sigma_A(a) = a\}$. Then $A_\mathbb{R}$ is a real graded C^*-algebra. Moreover, it is very easy to check that the map $A\longrightarrow A_\mathbb{R} + iA_\mathbb{R},\ a\longmapsto \frac{a+\sigma_A(a)}{2} + i(\frac{a-\sigma_A(a)}{2i})$ extends to an isomorphism of complex C^*-algebras intertwining the Real structures and the gradings. $\square$

Remark A.1.7. *Similarly, we will call* Rg Banach space *any complex graded Banach space which is the complexification of a Banach space over $\mathbb{R}$.*

Example A.1.8. *Let (X,τ) be a (Hausdorff and locally compact) Real space. Then τ induces a Real structure, also denoted by τ, on the C^*-algebra $\mathcal{C}_0(X)$ of complex values functions on X vanishing at infinity, given by $\tau(f)(x) = \overline{f(\tau(x))}$, for $f\in\mathcal{C}_0(X),\ x\in X$. Therefore, from Lemma A.1.6 we have $(\mathcal{C}_0(X),\tau)\cong(\mathcal{C}_0(X,\tau)\otimes_\mathbb{R}\mathbb{C},\bar{\ })$ where $\mathcal{C}_0(X,\tau) := \{f\in\mathcal{C}_0(X) \mid f(\tau(x)) = \overline{f(x)},\ \forall x\in X\}$ is the real C^*-algebra of invariant elements of $(\mathcal{C}_0(X),\tau)$.*

We must also say something about the tensor product of two Real graded C^*-algebras. This paragraph is a direct adaptation of [70] to the Real case. Let (A,σ) be a Real graded C^*-algebra. A *Real graded linear functional* on A is a linear functional $f: A\longrightarrow\mathbb{C}$ such that $f_{|_{A^1}} = 0$ and $f(\sigma(a)) = \overline{f(a)}$ for all $a\in A$. A Real graded state on A is a positive linear functional s on A such that $\|s\| = 1$. Suppose that (A,σ) and (B,ς) are separable, Real graded C^*-algebras, then $(A\hat{\odot}B, \sigma\hat{\odot}\varsigma)$ denotes the algebraic Real graded tensor product of

A and B, where elements are graded be $|a\hat{\odot}b| = |a| + |b|$, and the Real structure is given by $\sigma\hat{\odot}\varsigma(a\hat{\odot}b) := \sigma(a)\hat{\odot}\varsigma(b)$. The product and involutions are defined by

$$(a\hat{\odot}b)(a'\hat{\odot}b') := (-1)^{|b||a'|}(aa'\hat{\odot}bb'),$$

$$(a\hat{\odot}b)^* := (-1)^{|a||b|}(a^*\hat{\odot}b^*).$$

Now if s and t are Real graded states on A and B respectively, let

$$(s\hat{\odot}t)(c^*c) := \sum_{i,j=1}^{n} s(a_i^*a_j)t(b_i^*b_j),$$

for $c = \sum_{i=1}^n a_i\hat{\odot}b_i \in A\hat{\odot}B$. Then $s\hat{\odot}t$ is a Real graded state on $A\hat{\odot}B$. We define a C^*-norm on $A\hat{\odot}B$ by

$$\|c\| := \sup_{s,t,d}\frac{(s\hat{\odot}t)(d^*c^*cd)}{(s\hat{\odot}t)(d^*d)},$$

where the supremum is taken over all Real graded states s on A, t on B, and over all $d \in A\hat{\odot}B$ with $(s\hat{\odot}t)(d^*d) \neq 0$. The completion of $A\hat{\odot}B$ with respect to this norm is graded C^*-algebra denoted by $A\hat{\otimes}B$; moreover, $\sigma\hat{\odot}\varsigma$ extends to a Real involution on $A\hat{\otimes}B$ which gives a Real graded C^*-algebra $(A\hat{\otimes}B, \sigma\hat{\otimes}\varsigma)$ called the `(Real graded) tensor product of` (A,σ) `and` (B,ς).

A.2 Elementary Rg C^*-algebras

We are interested in the study of Real structures on graded C^*-algebras of compact operators.

Definition A.2.1. *A complex graded C^*-algebra A is called* elementary of parity 0 *(resp.* of parity 1*) if it isomorphic as a graded C^*-algebra to $\mathcal{K}(\hat{\mathcal{H}})$ (resp. to $\mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})$), where $\hat{\mathcal{H}}$ (resp. $\mathcal{H}$) is a complex graded Hilbert space (resp. a complex Hilbert space), and $\mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})$ is equipped with the standard odd grading.*

Example A.2.2 (The complex Clifford C^*-algebras)**.** *The complex Clifford C^*-algebras $\mathbb{C}l_p$ can be defined as graded C^*-algebras of compact operators in the following way. If $p = 2m$, $\mathbb{C}l_p$ is $\mathbb{C}l_{2m} := \mathcal{K}(\mathbb{C}^{2^{m-1}} \oplus \mathbb{C}^{2^{m-1}})$ equipped with the standard even grading Ad_ϵ, where $\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; if $p = 2m+1$ is odd, then $\mathbb{C}l_{2m+1} := \mathcal{K}(\mathbb{C}^{2^m}) \oplus \mathcal{K}(\mathbb{C}^{2^m})$ with the standard odd grading. We then see that the $\mathbb{C}l_{2m}$'s are graded elementary C^*-algebras of parity 0, while the $\mathbb{C}l_{2m+1}$'s are graded elementary C^*-algebras of parity 1. Moreover, these algebras verify $\mathbb{C}l_p\hat{\otimes}\mathbb{C}l_q \cong \mathbb{C}l_{p+q}$ as graded C^*-algebras ([9, §.14.5]).*

For the sake of simplicity, we assume in what follows that $\mathcal{H}$ is a complex separable infinite-dimensional Hilbert space. Then, by choosing an isomorphism $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$, we have a complex graded Hilbert space $\hat{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H} = (\mathcal{H} \oplus \mathcal{H})^0 \oplus (\mathcal{H} \oplus \mathcal{H})^1$, where the grading is given by $(x, y) \longmapsto (y, x)$. We thus obtain a complex graded elementary C^*-algebra $\widehat{\mathcal{K}}_{ev} := \mathcal{K}(\hat{\mathcal{H}})$ of parity 0 (here "*ev*" stands for *even*) whose grading automorphism is the unitary $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We also get a Real graded elementary C^*-algebra $\widehat{\mathcal{K}}_{odd} := \mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})$ with the standard odd grading. The next subsections are aimed at describing the Real structures of $\widehat{\mathcal{K}}_{ev}$ and $\widehat{\mathcal{K}}_{odd}$.

A.3 Real structures on $\widehat{\mathcal{K}}_{ev}$

Definition A.3.1. *A* `Real structure` *(resp.* `quaternionic structure`*) on $\widehat{\mathcal{H}}$ is a homogenous anti-unitary $J : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$ such that $J^2 = 1$ (resp. such that $J^2 = -1$).*
Real structures on $\widehat{\mathcal{H}}$ will be denoted as $J_{\mathbb{R}}$, or as $J_{i,\mathbb{R}}, i = 0, 1$ if we need to emphasize the degree i of $J_{\mathbb{R}}$. Similarly, quaternionic structures will be denoted as $J_{\mathbb{H}}$, or $J_{i,\mathbb{H}}, i = 0, 1$.

Given a Real structure $J_{\mathbb{R}} : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$, its $(+1)$-eigenspace $\widehat{\mathcal{H}}_{J_{\mathbb{R}}} := \{x \in \widehat{\mathcal{H}} \mid J_{\mathbb{R}}(x) = x\}$ (that we will also denote by $\widehat{\mathcal{H}}_{\mathbb{R}}$ if there is no risk of confusion) is a real graded separable infinite-dimensional Hilbert space such that $\widehat{\mathcal{H}} \cong \widehat{\mathcal{H}}_{J_{\mathbb{R}}} \otimes_{\mathbb{R}} \mathbb{C}$. Furthermore, there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $\widehat{\mathcal{H}}$, unique up to conjugation with homogenous elements in the orthogonal group $O(\widehat{\mathcal{H}}_{J_{\mathbb{R}}})$, such that $J_{\mathbb{R}}$ is given by $J_{\mathbb{R}}(x) := \sum_n \bar{x}_n e_n$ for all $x = \sum_n x_n e_n \in \widehat{\mathcal{H}}$. Writting $J_{\mathbb{R}}$ in this form, we get the following straightforward lemma.

Lemma A.3.2. *Let $J_{\mathbb{R}}$ be as above. Define $\sigma_{\mathbb{R}} : \widehat{\mathcal{K}}_{ev} \longrightarrow \widehat{\mathcal{K}}_{ev}$ by $\sigma_{\mathbb{R}}(T) := J_{\mathbb{R}} T J_{\mathbb{R}}$. Then $\sigma_{\mathbb{R}}$ is a Real structure on $\widehat{\mathcal{K}}_{ev}$ such that $(\widehat{\mathcal{K}}_{ev})_{\sigma_{\mathbb{R}}} \cong \mathcal{K}_{\mathbb{R}}(\widehat{\mathcal{H}}_{\mathbb{R}})$ as real graded C^*-algebras.*

Now suppose $J_{\mathbb{H}} : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$ is a quaternionic structure. Define the degree 0 operator $I : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$ by $Ix := ix$. Then $I^2 = -1$, and $IJ = -JI$. Thus, we can define the operator $K := IJ : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$ which has the same degree as J and is such that $K^2 = -1 = IJK$. It turns out that there exists a graded action of the quaternions $\mathbb{H}$ on $\widehat{\mathcal{H}}$ given by $(i, x) \longmapsto ix, (j, x) \longmapsto jx := Jx$, and $(k, x) \longmapsto kx := Kx = IJx$, where $\{1, i, j, k\}$ is the usual basis of the division ring $\mathbb{H}$. Let $\widehat{\mathcal{H}}_{J_{\mathbb{H}}}$ (or just $\mathcal{H}_{\mathbb{H}}$ if there is no risk of confusion) be the quaternionic graded Hilbert space, where the $\mathbb{H}$-valued inner product is given by $\langle x, y\rangle_{\mathbb{H}} := \langle x, y\rangle + \langle x, Jy\rangle j$ if $\langle\cdot,\cdot\rangle$ denotes the complex scalar product of $\widehat{\mathcal{H}}$.

Lemma A.3.3. *Let $J_{\mathbb{H}}$ be as above. Define $\sigma_{\mathbb{H}} : \widehat{\mathcal{K}}_{ev} \longrightarrow \widehat{\mathcal{K}}_{ev}$ by $\sigma_{\mathbb{H}}(T) := -J_{\mathbb{H}} T J_{\mathbb{H}}$. Then $\sigma_{\mathbb{H}}$ is a Real structure on $\widehat{\mathcal{K}}_{ev}$ such that $(\widehat{\mathcal{K}}_{ev})_{\sigma_{\mathbb{H}}}$ is isomorphic, under a graded isomorphism, to the real graded C^*-algebra $\mathcal{K}_{\mathbb{H}}(\widehat{\mathcal{H}}_{\mathbb{H}})$ of the compact $\mathbb{H}$-linear operators on the quaternionic graded Hilbert space $\widehat{\mathcal{H}}_{\mathbb{H}}$.*

Proof. The only thing we need to show is the graded isomorphism. Suppose that $T \in (\widehat{\mathcal{K}}_{ev})_{\sigma_{\mathbb{H}}}$. Then, $TJ_{\mathbb{H}} = J_{\mathbb{H}}T$, so that T extends uniquely to a compact $\mathbb{H}$-linear operator $\widetilde{T} : \widehat{\mathcal{H}}_{\mathbb{H}} \longrightarrow \widehat{\mathcal{H}}_{\mathbb{H}}$ through the formula $\widetilde{T}(jx) := J_{\mathbb{H}}(Tx)$ for $x \in \widehat{\mathcal{H}}$. This provides a homomorphism of real graded C^*-algebras $(\widehat{\mathcal{K}}_{ev})_{\sigma_{\mathbb{H}}} \longrightarrow \mathcal{K}_{\mathbb{H}}(\widehat{\mathcal{H}}_{\mathbb{H}}), T \longmapsto \widetilde{T}$. Conversely, any $\widetilde{T} \in \mathcal{K}_{\mathbb{H}}(\widehat{\mathcal{H}}_{\mathbb{H}})$ induces a unique $T \in \mathcal{L}(\widehat{\mathcal{H}})$ such that $Tx = \widetilde{T}x$ for all $x \in \widehat{\mathcal{H}}$. Then $T \in \widehat{\mathcal{K}}_{ev}$. Moreover, one has $(TJ_{\mathbb{H}})x = T(J_{\mathbb{H}}x) = \widetilde{T}(jx) = j\widetilde{T}x = (J_{\mathbb{H}}T)x$; hence, $TJ_{\mathbb{H}} = J_{\mathbb{H}}T$, and then $\sigma_{\mathbb{H}}(T) = T$. We then get a homomorphism of real graded C^*-algebras $\mathcal{K}_{\mathbb{H}}(\widehat{\mathcal{H}}_{\mathbb{H}}) \longrightarrow (\widehat{\mathcal{K}}_{ev})_{\sigma_{\mathbb{H}}}, \widetilde{T} \longmapsto T$. It is easy to check that these two homomorphisms are inverses of each other. □

The following result classifies all the Real structures on $\widehat{\mathcal{K}}_{ev}$.

Proposition A.3.4. *Suppose that σ is a Real structure on $\widehat{\mathcal{K}}_{ev}$. Then, σ is either of the form $\sigma_{\mathbb{R}}$, or of the form $\sigma_{\mathbb{H}}$.*

Proof. Choose an orthonormal basis $\{e_n\}$ of $\widehat{\mathcal{H}}$, and for $T \in \widehat{\mathcal{K}}_{ev}$, define $\overline{T} \in \widehat{\mathcal{K}}_{ev}$ by $\overline{T}(x) := \overline{T(\bar{x})}$, where if $x = \sum_n x_n e_n$, we set $\bar{x} := \sum_n \bar{x}_n e_n$. Then $\overline{T} = vTv$, where $v : \widehat{\mathcal{H}} \longrightarrow \widehat{\mathcal{H}}$ is the anti-unitary define by the complex conjugation with respect to the basis $\{e_n\}$. Moreover, $v^2 = 1$. Now, define $\bar{\sigma} \in \mathrm{Aut}^{(0)}(\widehat{\mathcal{K}}_{ev})$ by $\bar{\sigma}(T) := \sigma(\overline{T})$. Then, there exists a homogenous unitary $u \in \widehat{\mathbb{U}}(\widehat{\mathcal{H}})$ such that $\bar{\sigma} = \mathrm{Ad}_u$. Whence, $\sigma(T) = \bar{\sigma}(\overline{T}) = uvTvu^{-1} = JTJ^{-1}$, where $J := uv$. Observe that J is a homogenous anti-unitary since v is. Furthermore, for all $T \in \widehat{\mathcal{K}}_{ev}$, we have $T = \sigma^2(T) = J^2T(J^{-1})^2$; therefore $J^2 = \pm 1$. □

Definition A.3.5. *We say that a Rg elementary C^*-algebra (A, σ) of parity 0 is (of type) $[0;\varepsilon,\eta]$, where $\varepsilon = 0, 1$, $\eta = \pm$, if its Real structure is induced by an anti-unitary J of degree ε such that $J^2 = \eta 1$.*

Remark A.3.6. *According to Proposition A.3.4, there are four types of Rg elementary C^*-algebras of parity 0: $[0;0,+]$, $[0;0,-]$, $[0;1,+]$, and $[0;1,-]$.*

Remark A.3.7. *Regarding $\mathcal{K}(\mathcal{H})$ as of parity 0 (with the trivial grading of $\mathcal{H}$), there any Real structure on $\mathcal{K}(\mathcal{H})$ is given by the conjugation with anti-unitary $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = \pm 1$. Thus, such a Real graded C^*-algebra is either a $[0;0,+]$ or $[0;0,-]$.*

Example A.3.8 (Real strucutres on $\mathbb{C}l_2$). *Consider the second Clifford algebra $\mathbb{C}l_2 = \mathcal{K}(\mathbb{C} \oplus \mathbb{C}) = M_2(\mathbb{C})$, equipped with the standard even grading. There is a canonical Real structure $J_{\mathbb{R}}$ of degree 0 on the graded Hilbert space $\mathbb{C} \oplus \mathbb{C}$ given by the complex conjugation, and a canonical quaternionic Real structure of degree 0 $J_{0,\mathbb{H}} = iJ_{0,\mathbb{R}}$, which induce the same Real structure $cl_{0,2}$ on $\mathbb{C}l_2$ such that $(\mathbb{C}l_2)_{\sigma_{\mathbb{R}}} \cong (\mathbb{C}l_2)_{cl_{0,2}} \cong M_2(\mathbb{R}) \cong Cl_{0,2}$. In other words, $\mathbb{C}l_2$ is the complexification of the second real Clifford algebra $Cl_{0,2}$ (see [7] for more details on the real Clifford algebras $Cl_{p,q}$).*

However, $\mathbb{C}l_2$ is also the complexification of the quaternions $\mathbb{H}$ as follows. Define the quaternionic structure $J_{1,\mathbb{H}}:\mathbb{C}\oplus\mathbb{C}\longrightarrow\mathbb{C}\oplus\mathbb{C}$ of degree 1 by $(x,y)\longmapsto(\bar{y},-\bar{x})$. The graded quaternionic Hilbert space obtained is $\mathbb{H}$; the Real structure induced by $J_{1,\mathbb{H}}$ is denoted by $cl_{2,0}$. Observe that $(\mathbb{C}l_2)_{cl_{2,0}}=\mathcal{K}_{\mathbb{H}}(\mathbb{H})=\mathbb{H}\cong Cl_{2,0}$. Note that this Real structure is equivalent to that induced by the anti-unitary $J_{1,\mathbb{R}}(x,y):=(\bar{y},\bar{x})$. These two Real structures will play a central role in the classification of elementary Rg C^*-algebras in Section A.5.

A.4 Real structures on $\widehat{\mathcal{K}}_{odd}$

In this section we describe the Real structures on $\widehat{\mathcal{K}}_{odd}$. We start by some usefull observations. Suppose we are given a trivially graded C^*-algebra A. Then, any Real structure σ on A defines the two different Real structures $\sigma\oplus\sigma$ and $\sigma\oplus(-\sigma)$ on the graded C^*-algebra $A\oplus A$ (with the standard odd grading), resepctively given by $(a,b)\longmapsto(\sigma(a),\sigma(b))$ and $(a,b)\longmapsto(\sigma(a),-\sigma(b))$. Notice that the latter Real structure is equivalent to $(a,b)\longmapsto(\sigma(b),\sigma(a))$. Furthermore, if we denote $A_{\mathbb{R}}:=A_\sigma$, then on the one hand, we get $(A\oplus A)_{\sigma\oplus\sigma}$ is the real graded C^*-algebra $A_{\mathbb{R}}\oplus A_{\mathbb{R}}$ with the standard odd grading, and on the other hand, $(A\oplus A)_{\sigma\oplus(-\sigma)}=A_{\mathbb{R}}\oplus iA_{\mathbb{R}}$ is isomorphic to the real graded C^*-algebra A_{real} which is the underlying $\mathbb{R}$-algebra of A. It is easy to see that the grading of A_{real} is given by $A^0_{real}=A_{\mathbb{R}}$ and $A^1_{real}=iA_{\mathbb{R}}$. Conversely, we have the following.

Proposition A.4.1. *Let A be a complex C^*-algebra, and let $A\oplus A$ be equipped with the standard odd grading $(a,b)\longmapsto(b,a)$. Suppose τ is a Real structure on $A\oplus A$. Then, τ is either of the form $(a,b)\longmapsto(\sigma(a),\sigma(b))$ or $(a,b)\longmapsto(\sigma(b),\sigma(a))$, where $\sigma:A\longrightarrow A$ is a Real structure on the ungraded C^*-algebra A.*

Proof. Since τ is of degree 0, it can be written in the form $\tau=\begin{pmatrix}\tau^+ & 0\\ 0 & \tau^-\end{pmatrix}$ with respect to the decomposition $A\oplus A=(A\oplus A)^0\oplus(A\oplus A)^1$, where $\tau^+:(A\oplus A)^0\longrightarrow(A\oplus A)^0$ is a Real structure on the C^*-subalgebra $(A\oplus A)^0$ of $A\oplus A$, and $\tau^-:(A\oplus A)^1\longrightarrow(A\oplus A)^1$ is an anti-linear isomorphism of vector space. For all $(a,a)\in(A\oplus A)^0$, $\tau^+(a,a)\in(A\oplus A)^0$, so that it is of the form $(\sigma(a),\sigma(a))$. If $(a_i,a_i)\longrightarrow(a,a)\in(A\oplus A)^0$, then $(\sigma(a_i),\sigma(a_i))=\tau^+(a_i,a_i)\longrightarrow\tau^+(a,a)=(\sigma(a),\sigma(a))$, and then $\sigma(a_i)\longrightarrow\sigma(a)$ in A. Furthermore, it is straightforward that $\sigma(ab)=\sigma(a)\sigma(b)$, $\sigma(\lambda a)=\bar{\lambda}\sigma(a)$ for all $\lambda\in\mathbb{C}, a\in A$, and that $\sigma^2=\mathbf{1}$, so that σ is a Real structure on A. Now, for all $(b,-b)\in(A\oplus A)^1$, $(b,-b)\cdot(b,-b)=(b^2,b^2)\in(A\oplus A)^0$; thus,

$$(\tau^-(b,-b))^2=\tau(b^2,b^2)=\tau^+(b^2,b^2)=(\sigma(b)^2,\sigma(b)^2).$$

Hence, since this is true for all $b\in A$, we obtain $\tau^-(b,-b)=(\pm\sigma(b),\mp\sigma(b))$. If $\tau^-(b,-b)=(\sigma(b),-\sigma(b))$, then τ is given by $\tau(a,b)=(\sigma(a),\sigma(b))$, for all $(a,b)\in a\oplus a$, and if $\tau^-(b,-b)=(-\sigma(b),\sigma(b))$, then for all $(a,b)\in A\oplus A$, $\tau(a,b)=(\sigma(b),\sigma(a))$. $\square$

Definition A.4.2. *A Real structure τ on $A \oplus A$ is called* `even` *if it is of the form $(a,b) \longmapsto (\sigma(a),\sigma(b))$, it is* `odd` *if it is of the form $(a,b) \longmapsto (\sigma(b),\sigma(a))$, where σ is a Real structure on the ungraded C^*-algebra A.*

Proposition A.4.3. *Assume $\tau : A \oplus A \longrightarrow A \oplus A$ is a Real structure. Then*

(a) $(A \oplus A, \tau) \cong (A \hat{\otimes} \mathbb{C}l_1, \sigma \hat{\otimes} cl_{0,1})$*, if τ is even, and*

(b) $(A \oplus A, \tau) \cong (A \hat{\otimes} \mathbb{C}l_1, \sigma \hat{\otimes} cl_{1,0})$*, if τ is odd.*

Proof. As graded complex C^*-algebras, $A \oplus A \cong A \hat{\otimes} \mathbb{C}l_1 \cong A \otimes \mathbb{C}l_1$ (cf. [9, Corollary 14.5.3]). If τ is even, then as real graded C^*-algebras, $(A\oplus A)_{\mathbb{R}} \cong A_{\mathbb{R}} \oplus A_{\mathbb{R}} \cong (A_{\mathbb{R}} \hat{\otimes} Cl_{0,1}) = (A \hat{\otimes} \mathbb{C}l_1)_{\sigma \hat{\otimes} cl_{0,1}}$, where $A_{\mathbb{R}} := A_\sigma$, and $(A \oplus A)_{\mathbb{R}} := (A \oplus A)_\tau$; this establishes (a). If τ is odd, then $(A \oplus A)_{\mathbb{R}} \cong A_{real} \cong A_{\mathbb{R}} \hat{\otimes} \mathbb{C} \cong A_{\mathbb{R}} \hat{\otimes} Cl_{1,0} \cong (A \hat{\otimes} \mathbb{C}l_1)_{\sigma \hat{\otimes} cl_{1,0}}$, which establishes (b). □

Corollary A.4.4. *Suppose σ is a Real structure on $\widehat{\mathcal{K}}_{odd}$. Then, there exists an anti-unitary $J : \mathcal{H} \longrightarrow \mathcal{H}$ with $J^2 = \pm 1$, such that either $(\widehat{\mathcal{K}}_{odd}, \sigma) \cong (\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathbb{C}l_1, Ad_J \hat{\otimes} cl_{0,1})$, or $(\widehat{\mathcal{K}}_{odd}, \sigma) \cong (\mathcal{K}(\mathcal{H}) \hat{\otimes} \mathbb{C}l_1, Ad_J \hat{\otimes} cl_{1,0})$.*

Definition A.4.5. *We say that a Rg elementary C^*-algebra $(\widehat{\mathcal{K}}_{odd}, \sigma)$ of parity 1 is (of type) $[1;\varepsilon,\eta]$, if the Real structure is of parity ε (i.e., ε is 0 if σ is even, and 1 if σ is odd), and if the anti-unitary J of Corollary A.4.4 is such that $J^2 = \eta 1$, where $\eta = \pm$.*

Remark A.4.6. *Notice that there are four of such types:* $[1;0,+]$, $[1;0,-]$, $[1;1,+]$*, and* $[1;1,-]$.

Example A.4.7. $(\mathbb{C}l_1, cl_{0,1})$ *and* $(\mathbb{C}l_1, cl_{1,0})$ *are of types* $[1;0,+]$ *and* $[1;1,+]$*, respectively.*

A.5 The classification table

We start this section with the following lemma.

Lemma A.5.1. *Let $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ be two complex graded Hilbert spaces, and let J_i, $i = 1,2$ be an anti-unitary of degree ε_i on $\hat{\mathcal{H}}_i$ such that $J_i^2 = \pm 1$. Denote by $g_i, i = 1,2$ the grading automorphism of $\mathcal{K}(\hat{\mathcal{H}}_i)$. Then, there is an isomorphism of Real graded (elementary) C^*-algebras*

$$(\mathcal{K}(\hat{\mathcal{H}}_1) \hat{\otimes} \mathcal{K}(\hat{\mathcal{H}}_2), Ad_{J_1} \hat{\otimes} Ad_{J_2}) \cong (\mathcal{K}(\hat{\mathcal{H}}_1 \hat{\otimes} \hat{\mathcal{H}}_2), Ad_J),$$

where $J := J_1 g_1^{\varepsilon_2} \hat{\otimes} J_2 g_2^{\varepsilon_2}$.

Proof. The isomorphism of graded C^*-algebras $\mathcal{K}(\hat{\mathcal{H}}_1) \hat{\otimes} \mathcal{K}(\hat{\mathcal{H}}_2) \longrightarrow \mathcal{K}(\hat{\mathcal{H}}_1 \hat{\otimes} \hat{\mathcal{H}}_2)$ is given on homogenous tensors by

$$(T_1 \hat{\otimes} T_2)(x_1 \hat{\otimes} x_2) = (-1)^{|T_2|\cdot|x_1|} T_1(x_1) \hat{\otimes} T_2(x_2).$$

Moreover, a simple calculation shows that this is actually a Real isomorphism, when $\mathcal{K}(\hat{\mathcal{H}}_1\hat{\otimes}\hat{\mathcal{H}}_2)$ is equipped with the Real structure Ad_J; indeed,

$$\begin{aligned}\mathrm{Ad}_J(T_1\hat{\otimes}T_2) &= \left(J_1 g_1^{\varepsilon_2}\hat{\otimes}J_2 g_2^{\varepsilon_1}\right)\left(T_1\hat{\otimes}T_2\right)\left(J_1 g_1^{\varepsilon_2}\hat{\otimes}J_2 g_2^{\varepsilon_1}\right)^*\\ &= (-1)^{\varepsilon_1\varepsilon_2+\varepsilon_2|T_1|}\left(J_1 g_1^{\varepsilon_2}T_1\hat{\otimes}J_2 g_2^{\varepsilon_1}T_2\right)\left((g_1^*)^{\varepsilon_2}J_1^*\hat{\otimes}(g_2^*)^{\varepsilon_1}J_2^*\right)\\ &= (-1)^{\varepsilon_2|T_1|+\varepsilon_1|T_2|}\left((J_1 g_1^{\varepsilon_2}T_1(g_1^*)^{\varepsilon_2}J_1^*)\hat{\otimes}(J_2 g_2^{\varepsilon_1}T_2(g_2^*)^{\varepsilon_1}J_2^*)\right)\\ &= \mathrm{Ad}_{J_1}(T_1)\hat{\otimes}\mathrm{Ad}_{J_2}(T_2).\end{aligned}$$

□

A particular case of this lemma is the following.

Corollary A.5.2. *Le J be an anti-unitary on the ungraded Hilbert space $\mathcal{H}$ such that $J^2=\eta 1$, where as usual $\eta=\pm$. Let $cl_{0,2}$ and $cl_{2,0}$ be the Real structures of $\mathbb{C}l_2$ defined in Example A.3.8. Then,*

- $[0;0,\eta]\cong(\mathcal{K}(\mathcal{H}),Ad_J)\hat{\otimes}(\mathbb{C}l_2,cl_{0,2})$*, where $J:\mathcal{H}\longrightarrow\mathcal{H}$ is such that $J^2=\eta 1$, and*
- $[0;1,\eta]\cong(\mathcal{K}(\mathcal{H}),Ad_J)\hat{\otimes}(\mathbb{C}l_2,cl_{2,0})$*, where $J:\mathcal{H}\longrightarrow\mathcal{H}$ is such that $J^2=-\eta 1$.*

The next theorem can be viewed as a generalisation of Wall's result, [92, Theorem 3], to the infinite dimensional case.

Theorem A.5.3. *The type of the Real graded tensor product of two Real graded elementary C^*-algebras (A,σ_A) and (B,σ_B) depends only on those of (A,σ_A) and (B,σ_B). Moreover, we have the formulae*

$$[0;\varepsilon_1,\eta_1]\hat{\otimes}[0;\varepsilon_2,\eta_2]=[0;\varepsilon_1+\varepsilon_2,(-)^{\varepsilon_1\varepsilon_2}\eta_1\eta_2] \tag{1.1}$$

$$[0;\varepsilon_1,\eta_1]\hat{\otimes}[1;\varepsilon_2,\eta_2]=[1;\varepsilon_1+\varepsilon_2,(-)^{\varepsilon_1+\varepsilon_1\varepsilon_2}\eta_1\eta_2] \tag{1.2}$$

$$[1;\varepsilon_1,\eta_1]\hat{\otimes}[1;\varepsilon_2,\eta_2]=[0;1+\varepsilon_1+\varepsilon_2,(-)^{\varepsilon_1\varepsilon_2}\eta_1\eta_2]\,, \tag{1.3}$$

where the sum of degrees is mod 2.

Proof. The formula (1.1) is nothing more than Lemma A.5.1. Indeed, we have seen that the Real structure on $\mathcal{K}(\hat{\mathcal{H}}_1\hat{\otimes}\hat{\mathcal{H}}_2)$ is defined by the anti-unitary $J=J_1g_1^{\varepsilon_2}\hat{\otimes}J_2g_2^{\varepsilon_1}$. The degree of J is then $\varepsilon=\varepsilon_1+\varepsilon_2$, and $J^2=(-1)^{\varepsilon_1\varepsilon_2}J_1^2\hat{\otimes}J_2^2=(-1)^{\varepsilon_1\varepsilon_2}\eta_1\eta_2 1\hat{\otimes}1$.
Also, combining Corollary A.4.4, Corollary A.5.2, we get (1.2), by considering the isomorphism of Rg C^*-algebras

$$(\mathcal{K}(\hat{\mathcal{H}}_1),\mathrm{Ad}_{J_1})\hat{\otimes}(\mathcal{K}(\mathcal{H}_2)\hat{\otimes}\mathbb{C}l_1,\mathrm{Ad}_{J_2}\hat{\otimes}\tau_1)\cong(\mathcal{K}(\hat{\mathcal{H}}_1\hat{\otimes}\mathcal{H}_2)\hat{\otimes}\mathbb{C}l_1,\mathrm{Ad}_J\hat{\otimes}\tau_1),$$

where $J=J_1\hat{\otimes}J_2$, and τ_1 is either $cl_{0,1}$ or $cl_{1,0}$.
Finally, the equality (1.3) follows from Corollary A.5.2 and the following isomorphisms of

Rg C^*-algebras, which can be established by merely using the properties of the real Clifford algebras (see [7]):

$$\begin{aligned}
(\mathbb{C}l_1, cl_{0,1})\hat{\otimes}(\mathbb{C}l_1, cl_{0,1}) &\cong (\mathbb{C}l_2, cl_{0,2})\\
(\mathbb{C}l_1, cl_{0,1})\hat{\otimes}(\mathbb{C}l_1, cl_{1,0}) &\cong (\mathbb{C}l_2, cl_{0,2})\\
(\mathbb{C}l_1, cl_{1,0})\hat{\otimes}(\mathbb{C}l_1, cl_{1,0}) &\cong (\mathbb{C}l_2, cl_{2,0}).
\end{aligned}$$

□

We summarise all the preceding discussions by the following result.

Definition and Proposition A.5.4. *Denote by $\widehat{\mathcal{K}}_0$, the Rg elementary C^*-algebra $(\widehat{\mathcal{K}}_{ev}, Ad_{J_{\mathbb{R}}})$, where $J_{\mathbb{R}}$ is the anti-unitary of degree 0 on $\hat{\mathcal{H}}$ defined by $(x,y) \longmapsto (\bar{x},\bar{y})$ ("$^-$" is the complex conjugation with respect to an arbitrary orthonormal basis of $\hat{\mathcal{H}}$). Then $\widehat{\mathcal{K}}_0$ is of type $[0;0,+]$.*

Say that two Rg elementary C^*-algebras A and B are stably isomorphic *if $A\hat{\otimes}\widehat{\mathcal{K}}_0 \cong B\hat{\otimes}\widehat{\mathcal{K}}_0$, as Rg C^*-algebras.*

Slable isomorphism classes of Rg elementary C^*-algebras form an abelian group of order 8 under Rg tensor products, denoted by $\widehat{Br\mathrm{R}}(*)$, and called the Rg Brauer group of the point. *The unit element of $\widehat{Br\mathrm{R}}(*)$ is the element $\widehat{\mathcal{K}}_0$.*

Furthermore, elements of $\widehat{Br\mathrm{R}}(*)$ are, up to stable isomorphisms, classified by the following 8-periodic table

Parity 0	Parity 1
$\widehat{\mathcal{K}}_0 := [0;0,+]$	$\widehat{\mathcal{K}}_1 := [1;0,+]$
$\widehat{\mathcal{K}}_2 := [0;1,+]$	$\widehat{\mathcal{K}}_3 := [1;1,-]$
$\widehat{\mathcal{K}}_4 := [0;0,-]$	$\widehat{\mathcal{K}}_5 := [1;0,-]$
$\widehat{\mathcal{K}}_6 := [0;1,-]$	$\widehat{\mathcal{K}}_7 := [1;1,+]$

Table A.5.1: Classification of Rg elementary C^*-algebras

Remark A.5.5. *Under the notations of Table A.5.1, we set for all $n \in \mathbb{N}^*$:*

$$\widehat{\mathcal{K}}_n := \underbrace{\widehat{\mathcal{K}}_1\hat{\otimes}\cdots\hat{\otimes}\widehat{\mathcal{K}}_1}_{n-times}.$$

Then $\widehat{\mathcal{K}}_p\hat{\otimes}\widehat{\mathcal{K}}_q \cong \widehat{\mathcal{K}}_{p+q}$, and from the theorem, $\widehat{\mathcal{K}}_n \cong \widehat{\mathcal{K}}_{n+8}$ for all $n \in \mathbb{N}$. Now, define $\widehat{\mathcal{K}}_{-n}$ as the inverse of $\widehat{\mathcal{K}}_n$ in $\widehat{Br\mathrm{R}}(*)$. Then $\widehat{\mathcal{K}}_{-n} = \widehat{\mathcal{K}}_{8-n}$

Example A.5.6. *(Cf. [81]). One can determine the Real structures of the graded Clifford C^*-algebras $\mathbb{C}l_n$ (recall Example A.2.2), for $n \in \mathbb{N}^*$, in the following way: decompose n into a sum $p+q$, and consider the Real space $\mathbb{R}^{p,q} \otimes_{\mathbb{R}} \mathbb{C}$, with the obvious involution; this latter* *induces a Real structure* $cl_{p,q}$ *on the graded* C^**-algebra* $\mathbb{C}l_n = Cl(\mathbb{R}^{p,q} \otimes_{\mathbb{R}} \mathbb{C})$*, such that the Real part is isomorphic to the graded real Clifford algebra* $Cl_{p,q}$*. For this reason, we denote the thus obtained Real graded* C^**-algebra by* $\mathbb{C}l_{p,q}$*. Indeed, for every decomposition* $n = p+q$*, it is not hard to check that* $\mathbb{C}l_{p,q}$ *is a Rg elementary* C^**-algebra of type* $q-p \mod 8$ *(see for instance [28]).*

B

GNS-construction for Rg C^*-algebras. Rg $\mathcal{C}_0(X)$-algebras

B.1 The GNS-construction for Real graded C^*-algebras

Our goal here in this section to fit the *GNS-construction* techniques on the framework of Real graded C^*-algebras. For this, we have to give some basic definitions.

Definition B.1.1. *A* Real graded representation *of* (A,σ) *is a Real graded* **-homomorphism* $\pi : A \longrightarrow \mathcal{L}(H_\pi)$ *where* H_π *is a Real graded Hilbert space.*

If $H = H^0 \oplus H^1$ and $H' = H'^0 \oplus H'^1$ are Real graded Hilbert spaces, we have already seen that $\mathcal{L}(H,H')$ is Real via $\bar{T}(h) = \overline{T(\bar{h})}$. Furthermore, operators in $\mathcal{L}(H,H')$ can be represented by matrices of the form

$$T = \begin{pmatrix} T^+ & D^\mp \\ D^\pm & T^- \end{pmatrix},$$

where $T^+ : H^0 \longrightarrow H'^0$, $T^- : H^1 \longrightarrow H'^1$, and $D^\mp : H^1 \longrightarrow H'^0$, $D^\pm : H^0 \longrightarrow H'^1$. Note that if $D^\mp = D^\pm = 0$, then T is of order 0, while T is of order 1 if $T^+ = T^- = 0$. Now, let

$$\mathrm{U}_r^0(H,H') := \left\{ u = \begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix} \in \mathrm{U}(H,H') \mid \bar{u} = u \right\},$$

$$\mathrm{U}_r^1(H,H') = \left\{ u = \begin{pmatrix} 0 & u^\mp \\ u^\pm & 0 \end{pmatrix} \in \mathrm{U}(H,H') \mid \bar{u} = u \right\},$$

and let

$$\widehat{\mathrm{U}}_r(H,H') := \mathrm{U}_r^0(H,H') \sqcup \mathrm{U}_r^1(H,H').$$

Then $\hat{\mathrm{U}}_r$ is by construction a closed (Real) subgroup of $\mathrm{U}(H,H')$ which inherits the strong operator topology from $\mathrm{U}(H,H')$. Elements of $\widehat{\mathrm{U}}_r(H,H')$ are called *Real graded unitaries.*

Definition B.1.2. *Two Real graded representations $\pi : A \longrightarrow \mathcal{L}(H)$ and $\pi' : A \longrightarrow \mathcal{L}(H')$ of a Real graded C^*-algebra (A,σ) are* (unitarily) equivalent*, and we write $\pi \sim \pi'$, if there exists a Real graded unitary isomorphism $u \in \widehat{\mathfrak{U}}_r(H,H')$ such that*

$$\pi(a) = u\pi'(a)u^*,\ \forall a \in A.$$

As in the general theory of representations of C^*-algebras (ungraded case without considering Real structures), it is easy to check that equivalence of Real graded representations is an equivalence relation.

Also as in the usual case, a closed subspace $K \in H_\pi$ is called *invariant subspace of π* if $\pi(A)K \subset K$. A Real graded representation (π, H_π) is *irreducible* if it has no proper invariant subspaces.

Lemma B.1.3. *Let f be a Real graded positive functional on (A,σ). Then for all $a,b \in A$ we have*

1. *$f(\sigma(a^*b)) = f(b^*a)$;*
2. *(**Cauchy-Schwarz inequality**) $|f(b^*a)| \le f(b^*b)f(a^*a)$.*

Proof. From [75, Lemma A.4] we have $f(b^*a) = \overline{f(a^*b)}$ for any positive linear function f; now since f is Real, we get (1). The second point is the usual Cauchy-Schwarz inequality. □

Let $\hat{S}_r(A)$ be the convex set of Real graded states on (A,σ). Then an extreme point [1] of $\hat{S}_r(A)$ is called a *pure Real graded state* on (A,σ). The proof of the following lemma is the the same as in the usual case ([75, Lemma A.13]).

Lemma B.1.4. *For every element of a Real graded C^*-algebra (A,σ), there is a Real graded pure state s on (A,σ) such that $s(a^*a) = \|a\|^2$.*

Now we come to the GNS-construction for Real graded C^*-algebras. Let us start with a Real graded state s on (A,σ) and let

$$N_s := \left\{a \in A \mid s(a^*a) = 0\right\}.$$

We can see directly that N_s is invariant with respect to σ. Moreover, if $a = a^0 + a^1 \in N_s$, then $a^*a = (a^0)^*a^0 + (a^1)^*a^1 + (a^0)^*a^1 + (a^1)^*a^0$, and since $((a^0)^*a^1)$ and $(a^1)^*a^0$ belong to A^1, we have $\sigma(a^*a) = \sigma((a^0)^*a^0) + \sigma((a^1)^*a^1) = 0$, hence $\sigma((a^0)^*a^0) = \sigma((a^1)^*a^1) = 0$ which

[1] Recall that in general, if S is a convex subset of a vector space V, then $v \in S$ is an extreme point of S if $v = tw + (1-t)z$ for some $w,z \in S$ and $t \in [0,1]$ implies $v = w = z$

shows that $N_s = N_s \cap A^0 \oplus N_s \cap A^1$ is graded. It turns out that the quotient space A/N_s is Real graded (its Real involution, designed by the *bar*, is $\overline{a+N_s} := \sigma(a)+N_s$). However, it follows from Lemma B.1.3 that $s(b^*a) = 0$ if either a or b lies in N_s. Thus there is a well-defined inner product on A/N_s such that

$$\langle a+N_s, b+N_s\rangle = s(b^*a).$$

It is also easy to check that this inner product is Real graded. Indeed, if we consider A/N_s as a $\mathbb{C}$-module with $\mathbb{C}$ being given the trivial grading and its canonical Real structure, then $\langle a^0+N_s, b^1+N_s\rangle = \langle a^1+N_s, b^0+N_s\rangle = 0$ for $a^i \in A^1, b^i \in A^i$, $i = 0,1$; the compatibility of this inner product with the Real structures comes from the fact that s is Real. Let $\mathcal{H}_s$ be the completion of A/N_s with respect to this Real graded inner product. Then $\mathcal{H}_s$ is a Real graded Hilbert space. Now from Lemma B.1.3, N_s is a Real graded left ideal in A for if $a \in A, b \in N_s$ we have $|s((ab)^*ab)| \le s(b^*b)s((a^*ab)^*a^*ab) = 0$. So A acts by left multiplication on A/N_s, i.e., $a.(b+N_s) := ab+N_s$. This action is Real and graded. Moreover, since $b^*a^*ab \le \|a\|^2b^*b$, we have that

$$\|a.(b+N_s)\| \le \|a\|^2 s(b^*b) = \|a\|^2\|b+N_s\|,$$

so the elements of A act as bounded Real graded operators on A/N_s and extends to bounded Real graded operators $\pi_s(a)$ on the completion $\mathcal{H}_s$. The obtained Real graded representation $\pi_s : A \longrightarrow \mathcal{L}(\mathcal{H}_s)$ is called *the GNS-representation* of the Real graded C^*-algebra (A,σ). Then we have the following proposition:

Proposition B.1.5. *Every Real graded state s on (A,σ) gives rise to a (nondegenerate) Real graded representation π_s on a Real graded Hilbert space $\mathcal{H}_s$ given by the GNS-construction.*

Lemma B.1.6. *Let s be a Real graded state on (A,σ). Then the GNS-representation $(\pi_s,\mathcal{H}_s)$ is irreducible if and only if s is a Real graded pure state.*

Proof. Forgetting the gradings and the Real structures, this lemma is just [75, Lemma A.12]. Now it remains to manage with the gradings and the Real structures which is not difficult to do. □

Now, if we combine this lemma to B.1.4, we come out to the conclusion that every Real graded C^*-algebra (A,σ) admits Real graded representations.

B.2 The spectrum as a Real space

In this section we deal with the set of equivalence classes of Real graded representations of a Real graded C^*-algebra, and show how this can naturally be provided with the structure of Real space.

Definition B.2.1. *The spectrum* $(\hat{A},\tau)$ *of* (A,σ) *consists of the set of equivalence classes of Real graded representations of* A *and the Real structure* $\tau:\hat{A}\longrightarrow\hat{A}$ *defined by* $\tau([\pi]):=[\tau(\pi)]$*, for a class* $[\pi]$ *of a Real graded representation* $(\pi,\mathcal{H}_\pi)$*, where* $\tau(\pi)(a):=\overline{\pi(\sigma(a))}$.

Example B.2.2. *Let* (X,τ) *be a Real locally compact and Hausdorff space. Then the spectrum of the Real* C^**-algebra* $(\mathcal{C}_0(X),\tau)$ *is identified with* (X,τ).

Given a Real graded irreducible representation $(\pi,\mathcal{H}_\pi)$ of (A,σ), it is easy to check that $Ker\ \pi$ is a closed Real graded ideal in A.

A closed Real graded ideal I in (A,σ) is said *primitive* if it is the kernel of a Real graded irreducible representation of (A,σ). If $I=Ker\ \pi$ is a Real graded primitive ideal in (A,σ), then

$$\tau(I):=Ker\ \tau(\pi)$$

is a Real graded primitive ideal, where τ is the Real structure defined in Definition B.2.1, and where we consider π as the representative of the class $[\pi]\in\hat{A}$.

Definition B.2.3 (The Real graded primitive ideal space). *The* Real graded primitive ideal space $(\widehat{\text{Prim}}_R A,\tau)$ *of* (A,σ) *is the set of Real graded primitive ideals of* (A,σ) *endowed with the* Jacobson topology *([75, A.2]), together with the Real structure* τ *defined above .*

$(\widehat{\text{Prim}}_R A,\tau)$ is then a topological Real space, and the spectrum $(\hat{A},\tau)$ inherits the topology from $(\widehat{\text{Prim}}_R A,\tau)$ (through the Real map $\hat{A}\longrightarrow\widehat{\text{Prim}}_R A$, $\pi\longmapsto\ker\pi$) by considering a subset $U\in\hat{A}$ as open if $\{Ker\ \pi\mid\pi\in U\}$ is open in $\widehat{\text{Prim}}_R A$. In this topology, each set of the form $\{\pi\in\hat{A}\mid\|\pi(a)\|>k\}$ is open, and each set of the form $\{\pi\in\hat{A}\mid\|\pi(a)\|\geq k\}$ is compact ([75, Lemma A.30]). It turns out that the spectrum is always locally compact. Moreover, we have the following simple lemma.

Lemma B.2.4. *Suppose that the specrtum* $(\hat{A},\tau)$ *of* (A,σ) *is Hausdorff. Then the map* $\pi\longmapsto\ker\pi$ *is a Real homeomorphism between* $(\hat{A},\tau)$ *and* $(\widehat{\text{Prim}}_r A,\tau)$.

As in the ungraded complex case, we can show that any Real graded primitive ideal is prime. However, we have also the following lemma which will play an important role in the sequel.

Lemma B.2.5. *If* (A,σ) *is separable then any Real graded prime ideal is primitive.*

Now we can give the *Dauns-Hoffmann Theorem* analog for Real graded C^*-algebras, which can be proved through an easy adaptation of [75, A.3].

Theorem B.2.6 (Dauns-Hoffmann). *Let* (A,σ) *be a Real graded* C^**-algebra. For each* $I\in\widehat{\text{Prim}}_R A$*, let* $p_I:A\longrightarrow A/I$ *be the quotient map. Then there is a Real isomorphism* $\Psi:\mathcal{C}_b(\widehat{\text{Prim}}_R A)\longrightarrow\mathcal{ZM}(A)^{(o)}$ *such that for all* $f\in\mathcal{C}_b(\widehat{\text{Prim}}_R A)$ *and* $a\in A$*,*

$$p_I(\Psi(f)a)=f(I)p_I(a),\ \forall I\in\widehat{\text{Prim}}_R A. \tag{2.1}$$

B.3 *Real* graded $\mathcal{C}_0(X)$-algebras

Definition B.3.1. *Let (A,σ) be a Real graded C^*-algebra and (X,τ) be a Real locally compact Hausdorff space. Then (A,σ) is a Real graded $\mathcal{C}_0(X)$-algebra if there is a Real homomorphism*

$$\Phi_A : \mathcal{C}_0(X) \longrightarrow \mathcal{ZM}(A)^{(0)}$$

which is nondegerate in the sense that the (Real graded) ideal

$$\Phi_A(\mathcal{C}_0(X))A := \{\Phi_A(f)a \mid f \in \mathcal{C}_0(X),\ a \in A\} \tag{2.2}$$

is dense in A.

Proposition B.3.2. *Suppose (A,σ) is a Real graded $\mathcal{C}_0(X)$-algebra and J is an ideal in $\mathcal{C}_0(X)$. Then $\overline{\Phi_A(J).A}$ is a graded ideal in A. Moreover, if J is invariant in $\mathcal{C}_0(X)$ (with respect to the involution of $\mathcal{C}_0(X)$, also denoted by τ, induced by τ), then $\Phi_A(J).A$ is Real graded in A.*

Proof. The closure of $\Phi_A(J).A$ is by definition the closed linear span of $I := \{\Phi_A(f)a \mid f \in J, a \in A\}$. Therefore we have to show that given $a \in A$ and $\Phi_A(f)b \in I$ then $a(\Phi_A(f)b)$ and $(\Phi_A(f)b)a$ belong to I. Let us consider the pair $(L,R) \in \mathcal{M}(A)$ consisting of left and right translation in A. Then, since $\Phi_A(f) \in \mathcal{ZM}(A)$, we have

$$a(\Phi_A(f)b) = L_a(\Phi_A(f)b) = \Phi_A(f)(L_a(b)) = \Phi_A(f)(ab) \in I,$$

and

$$(\Phi_A(f)b)a = R_a(\Phi_A(f)b) = \Phi_A(f)(R_a(b)) = \Phi_A(f)(ba) \in I.$$

Now, since $\Phi_A(f)$ is of order 0 in $\mathcal{ZM}(A)$, then $\Phi_A(f)a = \Phi_A(f)a^0 + \Phi_A(f)a^1$ with $\Phi_A(f)a^i \in A^i$; but this means that $I = I \cap A^0 \oplus I \cap A^1$.

For the second part of the proposition, observe that we have a commutative diagram

$$\begin{tikzcd} \mathcal{C}_0(X) \arrow[r, "\Phi_A"] \arrow[d, "\tau"] & \mathcal{ZM}(A)^{(0)} \arrow[d, "\sigma"] \\ \mathcal{C}_0(X) \arrow[r, "\Phi_A"] & \mathcal{ZM}(A)^{(0)} \end{tikzcd} .$$

Then if J is invariant in $\mathcal{C}_0(X)$ (i.e., $\tau(f) \in J$, $\forall f \in J$) and $f \in J$, we get

$$\sigma(\Phi_A(f)a) = \sigma(\Phi_A(f)(\sigma^2(a))) = (\sigma(\Phi_A(f))(\sigma(a)) = (\Phi_A(\tau(f))(\sigma(a)).$$

□

Let (A,σ) be a Real graded $\mathcal{C}_0(X)$-algebra and let the C^*-algebra $\mathbb{C}$ be endowed with its complex conjugation as Real structure. For $x \in X$, we denote by J_x the ideal in $\mathcal{C}_0(X)$ defined by the kernel of the evaluation map $ev_x : \mathcal{C}_0(X) \longrightarrow \mathbb{C}$ (not necessarily a Real map). Then we denote the ideal $\overline{\Phi_A(J_x).A}$ by I_x and the quotient A/I_x by A_x. We think of A_x as the *fibre of A over x* and the class in I_x of an element $a \in A$ is denoted by a_x. In this way, we think of each $a \in A$ as a function from X to $\coprod_{x\in X} A_x$.

Remark B.3.3. *If $x \in X$ is invariant with respect to τ then the ideal J_x is invariant in $\mathcal{C}_0(X)$; thus from Proposition B.3.2, $\overline{\Phi_A(J_x).A}$ is Real graded in A and σ induces a Real structure on the graded C^*-algebra A_x. In particular, if τ is trivial then every fibre A_x is a Real graded C^*-algebra with respect to σ.*

Remark B.3.4. *Notice that τ induces a map $J_x \longrightarrow J_{\tau(x)}$. Indeed, given an $f \in J_x$, we have*

$$\tau(f)(\tau(x)) = \overline{f(\tau^2(x))} = \overline{f(x)} = 0,$$

thus $\tau(f) \in J_{\tau(x)}$.
Since $\Phi_A \circ \rho = \sigma \circ \Phi_A$, it then follows that σ induces a map $\Phi_A(J_x) \longrightarrow \Phi_A(J_{\tau(x)})$ and hence a graded map $\sigma' : I_x \longrightarrow I_{\tau(x)}$ given by

$$\sigma'(\Phi_A(f)a) := \sigma(\Phi_A(f)).\sigma(a) = \Phi_A(\tau(f)).\sigma(a).$$

It turns out that for any $x \in X$, there is a map $A_x \longrightarrow A_{\tau(x)}$ given by

$$a_x \longmapsto \sigma(a)_{\tau(x)}$$

which is obviously anti-linear and it intertwines the gradings.

Definition B.3.5 (**Homomorphism of Real graded $\mathcal{C}_0(X)$-algebras**)**.** *Let (A,σ) and (B,ν) be Real graded $\mathcal{C}_0(X)$-algebras. A C^*-homomorphism $\varphi : A \longrightarrow B$ is called* homomorphism of Real graded $\mathcal{C}_0(X)$-algebras *if it is Real, graded, and $\mathcal{C}_0(X)$-linear in the sense that for all $f \in \mathcal{C}_0(X)$ and $a \in A$, one has $\varphi(\sigma(a)) = \nu(\varphi(a))$ and $\varphi(\Phi_A(f)a) = \Phi_B(f)\varphi(a)$.*

Example B.3.6. *Let (D,ς) be a Real graded C^*-algebra and let (X,ρ) be a Real space. The Real structure on the C^*-algebra $A := \mathcal{C}_0(X,D)$ is given by*

$$\sigma(a)(x) := \varsigma(a(\rho(x))),\ \forall y \in A, x \in X,$$

while the grading is $\epsilon(a)(x) := \epsilon'(a(x))$ if ϵ' is the grading in D. Define $\Phi : \mathcal{C}_0(X) \longrightarrow \mathcal{ZM}(A)^{(0)}$ by

$$\Phi(f)(a)(x) := f(x)a(x),\ \textit{for } f \in \mathcal{C}_0(X),\ x \in X \textit{ and } a \in A.$$

In this way, (A,σ) is a Real graded $\mathcal{C}_0(X)$-algebra. The only thing that we have to show here is that Φ commutes with the Real involutions. Let $f \in \mathcal{C}_0(X)$, $a \in A$ and $x \in X$, then we have

$$\sigma(\Phi(f)(a)(x)) = \sigma(\Phi(f)(\sigma(a)))(x) = \varsigma(\Phi(f)(\sigma(a))(\tau(x)))$$

$$= \varsigma(f(\tau(x))\varsigma(a(x))) = \overline{f(\tau(x))}.a(x)$$
$$= (\tau(f))(x).a(x) = \Phi(\tau(f))(a)(x).$$

However, the map $A_x \ni a.I_x \longmapsto a(x) \in D$ is well defined (and is graded), and it identifies each fibre A_x with D (the identification of elements of D with elements of A being the obvious one).

Example B.3.7. *Suppose that (X,τ) and (Y,τ') are Real locally compact Hausdorff spaces and that $p: Y \longrightarrow X$ is a continuous Real map. Then $(\mathcal{C}_0(Y),\tau')$ is a Real graded $\mathcal{C}_0(X)$-algebra with respect to the Real map $\Phi_{\mathcal{C}_0(Y)} : \mathcal{C}_0(X) \longrightarrow \mathcal{C}_b(Y) = \mathcal{M}(\mathcal{C}_0(Y))$ given by*

$$\Phi_{\mathcal{C}_0(Y)}(f)(g)(y) := f(p(y))g(y),$$

and the trivial grading. Each fibre $\mathcal{C}_0(Y)_x$ is isomorphic to $\mathcal{C}_0(p^{-1}(x))$ (under a Real isomorphism) and if $f \in \mathcal{C}_0(Y)$ then f_x is just $f_{|p^{-1}(x)}$.

Example B.3.8. *Let (A,σ) be a Real graded C^*-algebra. Then the Dauns-Hoffmann theorem can also be characterized in terms of Real graded irreducible representations. If $(\pi,\mathcal{H}_\pi) \in (\hat{A},\tau)$, then by taking $I = \ker\pi$ in Theorem B.2.6, we have*

$$\pi(\Psi(f)a) = f(\ker\pi)\pi(a),\ \forall f \in \mathcal{C}_b(\widehat{\mathrm{Prim}_R} A), a \in A.$$

Thus, if (X,τ') is a Real space and if $\varphi_A : \widehat{\mathrm{Prim}_R} A \longrightarrow X$ is any Real continuous map, we get a Real homomorphism $\Phi_A : \mathcal{C}_0(X) \longrightarrow \mathcal{C}_b(\widehat{\mathrm{Prim}_R} A) \cong^{\varphi_A} \mathcal{ZM}(A)^{(0)}$ by defining

$$\Phi_A(f) := f \circ \varphi_A,$$

where we identify $f\circ\varphi_A$ with its image $\Psi(f\circ\varphi_A)$ in $\mathcal{ZM}(A)^{(0)}$. However, Φ_A is nondegenerate (see [94, p.355]); hence (A,σ) is Real graded $\mathcal{C}_0(X)$-algebra.

Through this example we have proved the first statement of the following proposition ([94, Proposition C.5]).

Proposition B.3.9. *Suppose (X,τ') is Real locally compact space. If there is a Real continuous map $\varphi_A : \widehat{\mathrm{Prim}_R} A \longrightarrow X$, then (A,σ) is a Real graded $\mathcal{C}_0(X)$-algebra with*

$$\Phi_A(f) = f \circ \varphi_A,\ \forall f \in \mathcal{C}_0(X). \tag{2.3}$$

Conversely, if $\Phi_A : \mathcal{C}_0(X) \longrightarrow \mathcal{C}_b(\widehat{\mathrm{Prim}_R} A) \cong \mathcal{ZM}(A)^{(0)}$ is a Real graded $\mathcal{C}_0(X)$-algebra, then there is a Real continuous map $\varphi_A : \widehat{\mathrm{Prim}_R} A \longrightarrow X$ such that (2.3) holds.

In particular, every Real graded irreducible representation of (A,σ) is lifted from a fibre A_x for some $x \in X$. More precisely, if $\pi \in (\hat{A},\tau)$, then the Real graded ideal $I_{\varphi_A(\ker\pi)}$ is contained in $\ker\pi$, and π is lifted from an irreducible representation of the C^*-algebra $A_{\varphi_A(\ker\pi)}$. Furthermore, if π is invariant with respect to τ, then it is lifted from a Real graded irreducible representation of $(A_{\varphi_A(\ker\pi)},\sigma)$.

Example B.3.10 ([32]). *Let (A,σ) be a separable Rg C^*-algebra with Hausdorff spectrum. If we identify $(\widehat{\mathrm{Prim}}_R A,\tau)$ with $(\hat{A},\tau)$, then the identity map $\varphi_A = id : \widehat{\mathrm{Prim}}_R A \longrightarrow \hat{A}$ allows us to view (A,σ) as a Rg $\mathcal{C}_0(\hat{A})$-algebra with*

$$(\Phi_A(f)a)_\pi = f(\pi)a_\pi,$$

where we identify any Rg primitive ideal $\ker\pi$ with $\pi \in \hat{A}$. Moreover, each fibre A_π can be identified with $A/\ker\pi$, and since A is separable, A_π is an elementary (Real if π in invariant under τ) graded C^*-algebra.

C

Real fields of graded C^*-algebras

Definition C.0.11. *Let (X,τ) be a locally compact Hausdorff Real space. A* continuous (resp. upper semicontiniuous) Real field of graded Banach spaces $\mathcal{A}$ *over X consists of a family $(\mathcal{A}_x)_{x\in X}$ of graded Banach spaces together with a topology on $\tilde{\mathcal{A}} = \coprod_{x\in X}\mathcal{A}_x$ and a un involution $\sigma : \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}$ such that*

(i) the topology on $\mathcal{A}_x$ induced from that on $\tilde{\mathcal{A}}$ is the norm-topology;

(ii) the projection $p : \tilde{\mathcal{A}} \longrightarrow X$ is Real, continuous, and open;

(iii) the map $a \longmapsto \|a\|$ is continuous (u.s.c) from $\tilde{\mathcal{A}}$ to $\mathbb{R}^+$, and $\|\sigma(a)\| = \|a\|$, $\forall a \in A$.

(iv) the map $(a,b) \longmapsto a+b$ is continuous from $\tilde{\mathcal{A}} \times_X \tilde{\mathcal{A}}$ to $\tilde{\mathcal{A}}$;

(v) the scalar multiplication $(\lambda, a) \longmapsto \lambda a$ is continuous from $\mathbb{C}\times\tilde{\mathcal{A}}$ to $\tilde{\mathcal{A}}$;

(vi) the induced bijection $\sigma_x : \mathcal{A}_x \longrightarrow \mathcal{A}_{\tau(x)}$ is an anti-linear isomorphism of graded Banach spaces for every $x \in X$, i.e. the diagram

$$\begin{tikzcd} \mathbb{C}\times\mathcal{A}_x \arrow[r] \arrow[d] & \mathcal{A}_x \arrow[d] \\ \mathbb{C}\times\mathcal{A}_{\tau(x)} \arrow[r] & \mathcal{A}_{\tau(x)} \end{tikzcd} \tag{3.1}$$

commutes, where the horizontal arrows are the action of $\mathbb{C}$ on the fibres and the vertical ones are the Real involutions ($\mathbb{C}$ being endowed with the complex conjugation), and $\sigma\circ\epsilon_x = \epsilon_{\tau(x)}\circ\sigma_x$.

(vii) if $\{a_i\}$ is a net in $\tilde{\mathcal{A}}$ such that $\|a_i\| \to 0$ and $p(a_i) \to x \in X$, then $a_i \to 0_x$, where 0_x is the zero element in $\mathcal{A}_x$.

We also say that $(\mathcal{A},\sigma)$ *is a* Real graded Banach bundle (resp. u.s.c. bundle) over (X,τ).

Definition C.0.12. *A* Real graded Hilbert bundle (resp. u.s.c. bundle) *over* (X,τ) *is a Real graded Banach bundle (resp. u.s.c. bundle)* $(\mathcal{A},\sigma)$ *over* (X,τ) *each fibre* $\mathcal{A}_x$ *is a graded Hilbert space with such that the fibrewise scalar products verify*

$$\langle\sigma_x(\xi),\sigma_x(\eta)\rangle = \overline{\langle\xi,\eta\rangle}$$

for every $\xi,\eta\in\mathcal{A}_x$.

Definition C.0.13. *A* Real graded C^*-bundle *(resp. u.s.c. C^*-bundle) over* (X,τ) *is a Real graded Banach bundle (resp. u.s.c. Banach bundle)* $(\mathcal{A},\sigma)$ *such that each fibre is a graded C^*-algebra satisfying the following axioms*

(a) *the map* $(a,b)\longmapsto ab$ *is continuous from* $\tilde{\mathcal{A}}\times_X\tilde{\mathcal{A}}$ *to* $\tilde{\mathcal{A}}$*;*

(b) $\sigma(ab)=\sigma(a)\sigma(b)$ *for all* $(a,b)\in\tilde{\mathcal{A}}\times_X\tilde{\mathcal{A}}$*;*

(c) *for* $x\in X$, $\sigma_x(a^*)=\sigma_x(a)^*$ *for all* $a\in\mathcal{A}_x$.

Homomorphism of Real graded u.s.c. Banach bundles and of u.s.c. C^*-bundles are defined in an obvious way.

Example C.0.14 (Trivial bundles)**.** *If* $(\mathfrak{A},^-)$ *is any graded Real Banach algebra (resp. C^*-algebra), then the first projection* $pr_1:(X\times\mathfrak{A},\tau\times{}^-)\longrightarrow(X,\tau)$ *defines a Real graded Banach bundle (resp. C^*-bundle) with fibre* $\mathfrak{A}$*. A Real graded Banach bundle (resp. C^*-bundle) of this form is called* trivial.

Definition C.0.15. *A u.s.c. field of graded Banach spaces* $\tilde{\mathcal{A}}\longrightarrow X$ *(without Real structure) is said to be* locally trivial *if for every* $x\in X$*, there exists a neighborhood* $U\ni x$ *such that* $\tilde{\mathcal{A}}_{|U}$ *is isomorphic (under a graded isomorphism) to a trivial field* $U\times B$*, where* B *is a graded Banach space.*
Similarly, we talk about locally trivial field of graded Banach algebras, graded Hilbert algebras, and graded C^*-algebras.

Unless otherwise stated, all of the graded Banach bundles and C^*-bundles we are dealing with are assumed locally trivial.

We shall however point out that the above notion of local triviality is not sufficient in the category of Real bundles. Roughly speaking, suppose (X,τ) is a Real space and $(\mathcal{A},\sigma)\longrightarrow(X,\tau)$ is a u.s.c. Real field of graded Banach spaces which is locally trivial in the sense of Definition C.0.15. Then it is not true that there exists a Rg Banach space A such that the Real space $\mathcal{A}$ locally behaves like A in the sense that there would exists, for all $x \in X$, an open Real neighborhood U of x (*i.e.* $\tau(U) = U$) and and a Real homeomorphism $h : p^{-1}(U) \longrightarrow U \times A$; or equivalently, there would exists a Real open cover $\{U_i\}$ of X and a trivialization $h_i : p^{-1}(U_i) \longrightarrow U_i \times A$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "h_i"] \arrow[d, "\tau_{|U_i}"] & U_i \times A \arrow[d, "\tau\times bar"] \\ p^{-1}(U_{\bar{i}}) \arrow[r, "h_{\bar{i}}"] & U_{\bar{i}} \times A \end{tikzcd} \tag{3.2}$$

where as usual we have written "bar" for the Real structure of A.

Definition C.0.16. *A Rg Banach bundle (resp. C^*-bundle, Hilbert bundle, etc.) $\mathcal{A} \longrightarrow X$ is said* locally trivial in the category of Real spaces *(LTCR, for short) if there exists a Rg Banach space (resp. C^*-algebra, Hilbert space, etc.) A and a Real local trivialization $(U_i, h_i)_{i\in I}$ such that the diagram (3.2) commutes.*

Example C.0.17. *Let A be a simple separable stably finite unital C^*-algebra that is not the complexification of any real C^*-algebra ([72, Corollary 4.1]). Define a continuous Real field of (trivially) graded C^*-algebras $\tilde{\mathcal{A}}$ over the Real space $\mathbf{S}^{0,1} = \{+1, -1\}$ by setting*

$$\mathcal{A}_{-1} := A, \quad \text{and } \mathcal{A}_{+1} := \overline{A},$$

where $\overline{A}$ is the complex conjugate of A, together with the Real structure $\sigma : \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}$ given by the conjugate linear *-isomorphism $\flat : A \longrightarrow \overline{A}$ (the identity map). Then $\mathcal{A}$ is not LTCR since $A \ncong \overline{A}$.

Definition C.0.18. *An* even (resp. odd) elementary graded C^*-bundle $\mathcal{A}$ over X *is a locally trivial field of graded C^*-algebras $\tilde{\mathcal{A}} \longrightarrow X$ such that every fibre $\mathcal{A}_x$ is isomorphic to $\mathcal{K}(\hat{H}_x)$ (resp. $\mathcal{K}(H_x) \oplus \mathcal{K}(H_x)$), where $\hat{H}_x$ (resp. H_x) is a graded Hilbert space (resp. is a Hilbert space).*

Definition C.0.19 (**Pull-backs**). *If $(\mathcal{A}, \sigma)$ is a graded Real C^*-bundle over (X, ρ) and $\varphi : (Y, \varrho) \longrightarrow (X, \rho)$ is a continuous Real map, then the* `pull-back of` $(\mathcal{A}, \sigma)$ `along` *φ is the graded Real C^*-bundle $(\varphi^*\mathcal{A}, \varphi^*\sigma) \longrightarrow (Y, \varrho)$, where $\varphi^*\mathcal{A} := Y \times_{\varphi, Y, p} \mathcal{A}$, and $\varphi^*\sigma(y, a) := (\varrho(x), \sigma(a))$, $\forall (y, a) \in \varphi^*\mathcal{A}$. Each fibre $(\varphi^*\mathcal{A})_y$ can be identified with $\mathcal{A}_{\varphi(y)}$ and then inherits the grading of the latter.*

Remark C.0.20. *For any graded Real Banach (resp. C^*-) bundle $(\mathcal{A}, \sigma) \longrightarrow (X, \rho)$, $\mathcal{C}_0(X, \mathcal{A})$ is a graded Real Banach (resp. C^*-) algebra with respect to the obvious pointwise operations and norm $\|s\| := \sup_{x\in X} \|s(x)\|$; the grading and the Real structure are given by $\epsilon(s)(x) := \epsilon_x(s(x))$ and $\sigma(s)(x) := \sigma_{\rho(x)}(s(\rho(x)))$. It is straightforward that $\sigma_x \circ \sigma_{\rho(x)} = \mathrm{Id}_{\mathcal{A}_{\rho(x)}}$, $\sigma_{\rho(x)} \circ \sigma_x = \mathrm{Id}_{\mathcal{A}_x}$. In particular, for a Real point $x \in X$, $\mathcal{A}_x$ is a Real graded Banach (resp. C^*-) algebra.*

Note that if $p:(\mathcal{A},\sigma)\longrightarrow(X,\rho)$ is a graded Real C^*-bundle, then $\mathcal{C}_0(X)$ acts by multiplication on $\mathcal{C}_0(X,\mathcal{A})$. Moreover, this action is Real and graded. Indeed, for $f\in\mathcal{C}_0(x)$ and $s\in\mathcal{C}_0(X,\mathcal{A})$, we put $\sigma(f.s)(x):=\overline{f(\rho(x))}\sigma(s(\rho(x)))=\rho(f)(x).\sigma(s)(x)$. Thus, (A,σ), where $A=\mathcal{C}_0(X,\mathcal{A})$, is a graded Real $\mathcal{C}_0(X)$-module.

If $(\mathcal{A},\sigma)$ is a graded Real Banach bundle over (X,ρ), then a continuous function $s: X\longrightarrow\mathcal{A}$ such that $p\circ s=\mathrm{Id}_X$ is called a `section` of $\mathcal{A}$. Note that if s is a section of $\mathcal{A}$, then for any $x\in X$, $s(\rho(x))$ and $\sigma(s(x))$ are in the same fibre $\mathcal{A}_{\rho(x)}$. We say that s is Real if $s(\rho(x))=\sigma(s(x))$. The collection of sections s for which $x\longmapsto\|s(x)\|$ is in $\mathcal{C}_0(X)$ is denoted by $\mathcal{C}_0(X,\mathcal{A})$.

Definition C.0.21. *A Real graded Banach bundle $p:\mathcal{A}\longrightarrow X$ has* enough sections *if given any $x\in X$ and any $a\in\mathcal{A}_x$, there is a continuous section $s\in\mathcal{C}_0(X,\mathcal{A})$ such that $s(x)=a$.*

Actually the following result assures us that all our Rg Banach bundles have enough sections (see [29, Appendix C] for a detailed proof).

Theorem C.0.22 (Douady - dal Soglio-Hérault). *Any Banach bundle over a paracompact or locally compact space has enough sections.*

Corollary C.0.23. *Suppose (X,ρ) is a locally compact Hausdorff Real space. Then, if $p:(\mathcal{A},\sigma)\longrightarrow(X,\rho)$ is a Tg Banach bundle, Real sections always exist.*

Proof. Let $x\in X, a\in\mathcal{A}_x$; then by Theorem C.0.22 there exists $s\in\mathcal{C}_0(X,\mathcal{A})$ such that $s(x)=a$. Since for every $x\in X$, $s(x)$ and $\sigma_{\rho(x)}(s(\rho(x)))$ belong to the Banach algebra $\mathcal{A}_x$, the map $\tilde{s}:=\frac{1}{2}(s+\sigma(s))$ is a well-defined section in $\mathcal{C}_0(X,\mathcal{A})$ which verifies $\sigma(\tilde{s})=\tilde{s}$. □

Remark C.0.24. *Let $p:(\mathcal{A},\sigma)\longrightarrow(X,\rho)$ be a graded Real C^*-bundle. Then, the Definition C.0.11 can be interpreted by the fact that there exists a graded Real C^*-algebra $(\mathfrak{A},^-)$ (the `fibre of` $(\mathcal{A},\sigma)$), together with a family of graded isomorphisms of C^*-algebras $m_x:\mathcal{A}_x\longrightarrow\mathfrak{A}$, $x\in X$ (or in other words, $m_x\in Isom^{(0)}(\mathcal{A}_x,\mathfrak{A})$), such that*

$$\overline{m_x(a)}=m_{\rho(x)}(\sigma_x(a)),\ \forall x\in X, a\in\mathcal{A}_x. \tag{3.3}$$

Definition C.0.25 (Elementary Rg C^*-bundle). *A Rg C^*-bundle $(\mathcal{A},\sigma)\longrightarrow(X,\rho)$ is called* elementary *if each fibre $\mathcal{A}_x$ is isomorphic to a graded elementary C^*-algebra.*

Definition C.0.26. *We say that a Rg elementary C^*-bundle $p:(\mathcal{A},\sigma)\longrightarrow(X,\rho)$ satisfies* Fell's condition *if (and only if) $(\mathcal{C}_0(X,\mathcal{A}),\sigma)$ is continuous-trace.*

Note that if $(\mathcal{A},\sigma)\longrightarrow(X,\rho)$ is a Rg elementary C^*-bundle, the spectrum of (A,σ) is naturally identified with (X,ρ).

In the sequel, we will write A for $\mathcal{C}_0(X,\mathcal{A})$ and if $\varphi:(Y,\varrho)\longrightarrow(X,\rho)$ is a continuous Real map, we write φ^*A for $\mathcal{C}_0(Y,\varphi^*\mathcal{A})$.

Definition C.0.27 (**Morita equivalence**). *Suppose that* $p_{\mathcal{A}} : (\mathcal{A},\sigma_{\mathcal{A}}) \longrightarrow (X,\rho)$ *and* $p_{\mathcal{B}} : (\mathcal{A},\sigma_{\mathcal{B}}) \longrightarrow (X,\rho)$ *are graded Real* C^**-bundles. Then a Rg Banach bundle* $q : (\mathcal{E},\sigma_{\mathcal{A}}) \longrightarrow (X,\rho)$ *is called a* Rg $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule *if each fibre* $\mathcal{E}_x$ *is a graded* $\mathcal{A}_x$-$\mathcal{B}_x$*-imprimitivity bimodule such that*

(a) *the natural maps* $(\mathcal{A}\times_X\mathcal{E},\sigma_{\mathcal{A}}\times\sigma_{\mathcal{E}}) \longrightarrow (\mathcal{E},\sigma_{\mathcal{E}})$, $(a,\xi) \longrightarrow a\cdot\xi$ *and* $(\mathcal{E}\times_X\mathcal{B},\sigma_{\mathcal{E}}\times\sigma_{\mathcal{A}}) \longrightarrow (\mathcal{E},\sigma_{\mathcal{E}})$, $(b,\xi) \longmapsto b\cdot\xi$ *are Real and continuous;*

(b) $(\sigma_{\mathcal{A}})_x({}_{\mathcal{A}_x}\langle\xi,\eta\rangle) = {}_{\mathcal{A}_{\rho(x)}}\langle(\sigma_{\mathcal{E}})_x(\xi),(\sigma_{\mathcal{E}})_x(\eta)\rangle$ *and* $(\sigma_{\mathcal{B}})_x(\langle\xi,\eta\rangle_{\mathcal{B}_x}) = \langle(\sigma_{\mathcal{B}})_x(\xi),(\sigma_{\mathcal{E}})_x(\eta)\rangle_{\mathcal{B}_{\rho(x)}}$.

If such a Rg $\mathcal{A}$-$\mathcal{B}$*-imprimitivity bimodule exists, we say that* $(\mathcal{A},\sigma_{\mathcal{A}})$ *and* $(\mathcal{B},\sigma_{\mathcal{B}})$ *are* Morita equivalent.

Let $(\mathcal{A},\sigma_{\mathcal{A}})$ and $(\mathcal{B},\sigma_{\mathcal{B}})$ be elementary Rg C^*-bundles over (X,ρ). Then, there is a unique elementary Rg C^*-bundle $\mathcal{A}\hat{\otimes}\mathcal{B}$ over $X\times X$ with fibre $\mathcal{A}_x\hat{\otimes}\mathcal{B}_y$ over (x,y) and such that $(x,y) \longmapsto f(x)\hat{\otimes}g(y)$ is a section for all $f\in A=\mathcal{C}_0(X,\mathcal{A})$ and $g\in B=\mathcal{C}_0(X,\mathcal{B})$. The Real structure is given by $(\sigma_{\mathcal{A}})_x\hat{\otimes}(\sigma_{\mathcal{B}})_y$ over (x,y). By this construction, the elementary Rg C^*-bundle $(\mathcal{A}\hat{\otimes}\mathcal{B},\sigma_{\mathcal{A}}\hat{\otimes}\sigma_{\mathcal{B}})$ satisfies Fell's condition if $(\mathcal{A},\sigma_{\mathcal{A}})$ and $(\mathcal{B},\sigma_{\mathcal{B}})$ do, as does its restriction $(\mathcal{A}\hat{\otimes}_X\mathcal{B},\sigma_{\mathcal{A}}\hat{\otimes}_X\sigma_{\mathcal{B}})$ to the diagonal $\Delta=\{(x,x)\in X\times X\}$.

Definition C.0.28. *Let* $(\mathcal{A},\sigma_{\mathcal{A}})$ *and* $(\mathcal{B},\sigma_{\mathcal{B}})$ *be Rg elementary* C^**-bundles over* (X,ρ). *Then, their* tensor product *is defined to be the Rg elementary* C^**-bundle* $(\mathcal{A}\hat{\otimes}_X\mathcal{B},\sigma_{\mathcal{A}}\hat{\otimes}_X\sigma_{\mathcal{B}})$ *over the Real space* (X,ρ) *which is identified with the diagonal* (Δ,ρ) *of* $(X\times X,\rho\times\rho)$.

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